ESAIM: COCV 31 (2025) 97
https://doi.org/10.1051/cocv/2025083

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

THREE-LEVEL MULTI-LEADER-FOLLOWER INCENTIVE STACKELBERG DIFFERENTIAL GAME WITH H_∞ CONSTRAINT

Na Xiang and Jingtao Shi*

Abstract. This paper is concerned with a three-level multi-leader-follower incentive Stackelberg game with H_∞ constraint. Based on H_2/H_∞ control theory, we firstly obtain the worst-case disturbance and the team-optimal strategy by finding a closed-loop Nash equilibrium of the corresponding nonzero-sum stochastic differential game. The main objective is to establish an incentive Stackelberg strategy set of the three-level hierarchy in which the whole system achieves the top leader's team-optimal solution and attenuates the external disturbance under H_∞ constraint. On the other hand, followers on the bottom two levels in turn attain their state feedback Nash equilibrium, ensuring incentive Stackelberg strategies while considering the worst-case disturbance. By convex analysis theory, maximum principle and decoupling technique, the three-level incentive Stackelberg strategy set is obtained. Finally, a numerical example is given to illustrate the existence of the proposed strategy set.

Mathematics Subject Classification. 91A15, 91A65, 93E20.

Received December 23, 2024. Accepted September 29, 2025.

1. Introduction

The Stackelberg game, pioneered by von Stackelberg [1] in 1952, has important applications in various fields, such as economics, engineering management and computer science, *etc.* Bagchi and Başar [2] initially considered *stochastic linear-quadratic* (SLQ) nonzero-sum Stackelberg differential game and established existence and uniqueness of the Stackelberg solution, where the diffusion term of the state equation did not contain the state and control variables. Yong [3] discussed the open-loop solution of SLQ nonzero-sum Stackelberg differential game, where the coefficients of the system with both state-dependent and control-dependent noise are random, and the weight matrices for the controls in the cost functionals are not necessarily positive definite, and obtained the feedback representation of the open-loop equilibrium *via* a new stochastic Riccati equation. In recent two decades, stochastic Stackelberg differential game has been extensively investigated and there has been a great deal of literature around it. Bensoussan *et al.* [4] derived the maximum principle for the stochastic Stackelberg differential game with the control-independent diffusion term under different information structures. Shi *et al.* [5, 6] studied SLQ Stackelberg differential games with asymmetric information. Li *et al.* [7] discussed the SLQ Stackelberg differential game under asymmetric information by a layered calculation method. Li and Yu [8] considered an SLQ generalized Stackelberg game with the multilevel hierarchy. Moon and Başar [9] studied SLQ Stackelberg *mean field game* (MFG) with the adapted open-loop information structure of the leader,

Keywords and phrases: Incentive Stackelberg differential game, H_∞ constraint, H_2/H_∞ control, Nash equilibrium, team-optimal solution.

School of Mathematics, Shandong University, Jinan 250100, PR China.

* Corresponding author: shijingtao@sdu.edu.cn

where one leader and arbitrarily large number of followers was considered, and solved by fixed-point approach, while Wang [10] tackled it by a direct approach. Zheng and Shi [11] discussed a Stackelberg stochastic differential game with asymmetric noisy observations. Kang and Shi [12] studied a three-level SLQ Stackelberg differential game with asymmetric information. Feng *et al.* [13] established a unified two-person differential decision setup, and studied the relationships between zero-sum SLQ Nash and Stackelberg differential game, local versus global information. Wang and Wang [14] considered SLQ mean-field type partially observed Stackelberg differential game with two followers. Li and Shi [15] studied closed-loop solvability of SLQ mean-field type Stackelberg differential games.

In the Stackelberg game, the leader's strategy can induce the decision or action of the followers such that the leader's team-optimal solution can be achieved, which is called the *incentive* Stackelberg game, and this has been extensively studied for over 40 years. Ho *et al.* [16] investigated the deterministic and stochastic versions of the incentive problem, and their relationship to economic literature is discussed. Zheng and Başar [17] studied the existence and derivation of optimal affine incentive schemes for Stackelberg games with partial information by a geometric approach. Zheng *et al.* [18] discussed applicability and appropriateness of a function-space approach in the derivation of causal real-time implementable optimal incentive Stackelberg strategies under various information patterns. Mizukami and Wu [19, 20] considered the derivation of the sufficient conditions for the LQ incentive Stackelberg game with multi-players in a two-level hierarchy. Ishida and Shimemura [21] investigated the three-level incentive Stackelberg strategy in a nonlinear differential game, and derived the sufficient condition for a linear quadratic differential game, while Ishida [22] considered different incentive strategies. Li *et al.* [23] studied the team-optimal state feedback Stackelberg strategy of a class of discrete-time two-person nonzero-sum LQ dynamic games. Mukaidani *et al.* [24] discussed the incentive Stackelberg game for a class of Markov jump linear stochastic systems with multiple leaders and followers. Lin *et al.* [25] investigated the incentive feedback strategy for a class of stochastic Stackelberg games in finite and infinite horizon. Gao *et al.* [26] considered the incentive feedback Stackelberg strategy for discrete-time stochastic systems, then the incentive Stackelberg strategy for discrete-time stochastic systems with mean-field terms is discussed in [27].

The incentive Stackelberg game is often combined with robust control theory. The incentive Stackelberg games under the H_∞ constraint is based on H_2/H_∞ control theory to solve. Ahmed and Mukaidani [28] studied the incentive Stackelberg game for a class of deterministic discrete-time system with a deterministic external disturbance. For stochastic version, Mukaidani *et al.* [29] considered the incentive Stackelberg game with one leader and one follower subject to external disturbance by means of static output-feedback. [30–34] discussed the incentive Stackelberg game with one leader and multiple non-cooperative followers subjected to the H_∞ constraint. Ahmed *et al.* [35] investigated multi-leader-follower incentive Stackelberg games for SLQ systems with H_∞ constraint. Mukaidani *et al.* [36] considered the incentive Stackelberg game for a class of Markov jump SLQ systems with multi-leader-follower under H_∞ constraint, where followers attain their state feedback Nash equilibrium and Pareto optimality. Mukaidani *et al.* [37] studied a robust static output feedback incentive Stackelberg game for a Markov jump SLQ system with multi-leader-follower, where the Pareto optimal strategies of the followers as cooperative strategy is chosen. Mukaidani *et al.* [38] considered the static output feedback strategy for robust incentive Stackelberg games with a large population for mean-field stochastic systems.

In this paper, we study a three-level multi-leader-follower incentive Stackelberg game with H_∞ constraint, which has practical significance, especially in corporate governance. The first level is specified as the Decision-making Level 1 with one person; the second level is lower than the first level and is specified as the Managerial Level 2 with two people; the third level is the lowest level of the whole system and is specified as the Executive Level 3 with three people (see Fig. 1a). Considering practical examples such as company management (Chairman – General Manager – Department Head) and national governance (Central Government – Local Government – Subdistrict Office), they are all multi-level. Based on the pyramid principle, the number of people at the upper level is always relatively small. These two reasons prompt us to consider a three-level multi-leader-follower incentive Stackelberg game. For the convenience of research, we assume that there are i players in level i, and there exists a special one-way influence relationship, which also reflects the asymmetry of the hierarchical structure. The arrows in the Figure 1a indicate the leader's incentives for followers. Since Decision-making Level 1 is in the highest position, who can incentivize each player in Managerial Level 2 and Executive Level 3

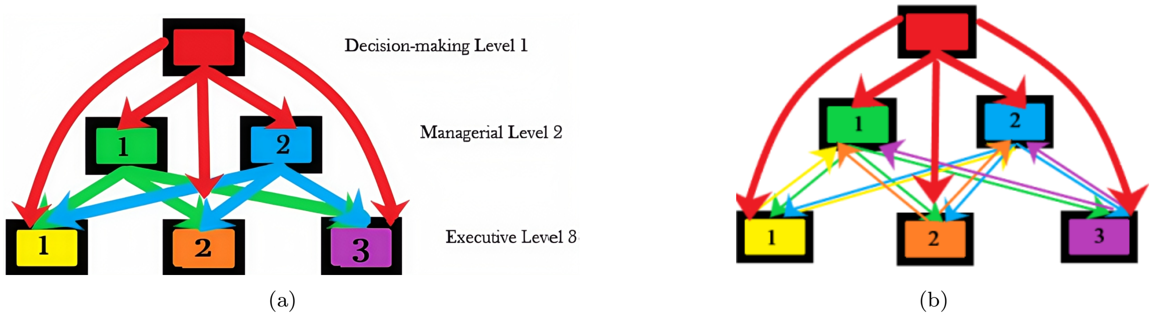

FIGURE 1. The three-level incentive Stackelberg game.

accordingly. Two people in Managerial Level 2 can only make incentives to the three people in the next level (Executive Level 3). For three people of Executive Level 3, they can only respond to the incentives of all the leaders (Decision-making Level 1 and Managerial Level 2). Figure 1b depicts the incentives and responses of the three-level Stackelberg game system in detail. In this hierarchical incentive mechanism (see Fig. 1b), Decision-making Level 1 allocates individualized incentives to all players. Subsequently, Managerial Level 2 and Executive Level 3, after internalizing the incentive signals, engage in Nash equilibrium to achieve Decision-making Level 1's team optima. Two people in Managerial Level 2 act as the leaders of Executive Level 3, also need to incentivize each player in Executive Level 3 in order to achieve the team optimum of Decision-making Level 1. Each individual in Executive Level 3 engages in a Nash game to respond to Managerial Level 2's incentive. Since there is a significant difference in hierarchy between Decision-making Level 1 and Executive Level 3, the actions of Executive Level 3 can only affect Managerial Level 2 and cannot directly influence Decision-making Level 1.

Besides, due to the limited abilities and complicated information environment, the whole company doesn't have access to full information about the market. So it is quite natural to consider information uncertainty. Based on this, the drift term of the state equation contains the external stochastic disturbance $v(\cdot)$, which is unknown to all players in the three-level incentive Stackelberg game system. We address incentive and model uncertainty primarily through H_2/H_∞ control, viewing the disturbance as a new player and finding a closed-loop Nash equilibrium solution of the corresponding two-person nonzero-sum Nash game based on the definition of team-optimal strategy. Then, according to the worst-case disturbance, the incentives between each level are carried out, and finally the team-optimal strategy of the game is reached. Our main contributions are summarized as follows.

- We investigate a class of three-level multi-leader-follower Stackelberg games with H_∞ constraint, where the control variables and the external disturbance enter the diffusion term and drift term of the state equation, respectively.
- We derive sufficient conditions for the three-level incentive Stackelberg game using information on follower's strategies, and prove that three-level incentive matrices depend on an initial state value x_0.
- *via* convex optimization approach, we can achieve the open-loop Nash equilibrium for Managerial Level 2 and Executive Level 3 with corresponding incentives from leaders.
- By H_2/H_∞ control theory, the team-optimal strategy and the worst-case disturbance, which are the outcomes of a closed-loop Nash equilibrium are derived at the same time, based on which the three-level incentive Stackelberg strategy is obtained.

The rest of this paper is organized as follows. Section 2 introduces some preliminary notations and formulations of the team-optimal solution and the finite horizon stochastic H_2/H_∞ control problem. Section 3 discusses incentive Stackelberg equilibrium strategies. In Section 4, an algorithm procedure and a numerical example are used to elaborate the effectiveness of the proposed results. Section 5 concludes the paper.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t), 0 \le t < \infty\}$ is defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}$ is the natural filtration of $W(\cdot)$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with standard Euclidean norm $|\cdot|$ and standard Euclidean inner product $\langle\cdot,\cdot\rangle$. The transpose of a vector (or matrix) x is denoted by $x^\top$. $\mathrm{Tr}(A)$ denotes the trace of a square matrix A. Let $\mathbb{R}^{n\times m}$ be the Hilbert space consisting of all $n \times m$-matrices with the inner product $\langle A, B\rangle := \mathrm{Tr}(AB^\top)$ and the norm $|A| := \langle A, A\rangle^{\frac{1}{2}}$. Denote the set of symmetric $n \times n$ matrices with real elements by $\mathbb{S}^n$. If $M \in \mathbb{S}^n$ is positive (semi-) definite, we write $M > (\ge) 0$. If there exists a constant $\delta > 0$ such that $M \ge \delta I$, we write $M \gg 0$.

Consider a finite time horizon $[0, T]$ for $T > 0$. Let $\mathbb{H}$ be a given Hilbert space, we denote

$$L^2_{\mathbb{F}}(0,T;\mathbb{H}) := \left\{ \phi : [0,T]\times\Omega \mapsto \mathbb{H} \,\middle|\, \phi \text{ is } \mathbb{F}\text{-progressively measurable}, \mathbb{E}\int_0^T |\phi(s)|^2 \mathrm{d}s < \infty \right\}.$$

We consider the following controlled linear *stochastic differential equation* (SDE):

$$\begin{cases} \mathrm{d}x(t) = \Bigg[A(t)x(t) + \sum_{i=1}^{2} B_{1i}(t)u_{1i}(t) + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}(t)u_{2ij}(t) + \sum_{j=1}^{3}\sum_{i=1}^{2} B_{3ji}(t)u_{3ji}(t) + E(t)v(t)\Bigg]\mathrm{d}t \\ \qquad + \Bigg[C(t)x(t) + \sum_{i=1}^{2} D_{1i}(t)u_{1i}(t) + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}(t)u_{2ij}(t) + \sum_{j=1}^{3}\sum_{i=1}^{2} D_{3ji}(t)u_{3ji}(t)\Bigg]dW(t), \quad t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $A(\cdot), B_{1i}(\cdot), B_{2ij}(\cdot), B_{3ji}(\cdot), C(\cdot), D_{1i}(\cdot), D_{2ij}(\cdot), D_{3ji}(\cdot), E(\cdot)$ are deterministic and uniformly bounded functions on $[0, T]$ of proper dimensions. $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})$ represents the external unknown disturbance, $x(\cdot) \in \mathbb{R}^n$ the state process, $x_0 \in \mathbb{R}^n$ the initial state. $u_{1i}(\cdot) \in \mathbb{R}^{m_{1i}}$ represents Decision-making Level 1's control input for the ith of Managerial Level 2; $u_{2ij}(\cdot) \in \mathbb{R}^{m_{2ij}}$ represents the ith of Managerial Level 2's control input for the jth of Executive Level 3; $u_{3ji}(\cdot) \in \mathbb{R}^{m_{3ji}}$ represents the jth of Executive Level 3's control input according to the ith of Managerial Level 2 in the sense of incentive strategy. Moreover, the index $i = 1, 2$ and $j = 1, 2, 3$ denote the ith player of Managerial Level 2 and the jth player of Executive Level 3.

For the sake of simplicity, for $i = 1, 2$, $j = 1, 2, 3$, let

$$\begin{aligned} u_1(t) &:= \mathbf{col}\big[u_{11}(t) \;\; u_{12}(t)\big], \quad u_{2i}(t) := \mathbf{col}\big[u_{2i1}(t) \;\; u_{2i2}(t) \;\; u_{2i3}(t)\big], \\ u_{3j}(t) &:= \mathbf{col}\big[u_{3j1}(t) \;\; u_{3j2}(t)\big], \quad u_{ci}(t) := \mathbf{col}\big[u_{2i1}(t) \;\; u_{2i2}(t) \;\; u_{2i3}(t) \;\; u_{31i}(t) \;\; u_{32i}(t) \;\; u_{33i}(t)\big]. \end{aligned}$$

The admissible control set $\mathcal{U}_1$ of Decision-making Level 1 is defined as follows:

$$\mathcal{U}_1 := \Bigg\{ u_1 : [0,T]\times\Omega \mapsto \mathbb{R}^{m_1} \,\Big|\, u_1 \text{ is } \mathbb{F}\text{-progressively measurable}, \text{ such that } \left(\mathbb{E}\int_0^T |u_1(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} < +\infty, \text{ with } m_1 = \sum_{i=1}^{2} m_{1i} \Bigg\},$$

the admissible control set $\mathcal{U}_{2i}$ of the ith of Managerial Level 2 is

$$\mathcal{U}_{2i} := \Bigg\{ u_{2i} : [0,T]\times\Omega \mapsto \mathbb{R}^{m_{2i}} \,\Big|\, u_{2i} \text{ is } \mathbb{F}\text{-progressively measurable, such that } \left(\mathbb{E}\int_0^T |u_{2i}(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} < +\infty \text{ with } m_{2i} = \sum_{j=1}^{3} m_{2ij} \Bigg\},$$

the admissible control set $\mathcal{U}_{3j}$ of the jth of Executive Level 3 is

$$\mathcal{U}_{3j} := \Bigg\{ u_{3j} : [0,T]\times\Omega \mapsto \mathbb{R}^{m_{3j}} \,\Big|\, u_{3j} \text{ is } \mathbb{F}\text{-progressively measurable, such that } \left(\mathbb{E}\int_0^T |u_{3j}(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} < +\infty \text{ with } m_{3j} = \sum_{i=1}^{2} m_{3ji} \Bigg\}.$$

Under some mild conditions on the coefficients, for any $(x_0, u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v) \in \mathbb{R}^n \times \mathcal{U}_1 \times \mathcal{U}_{21} \times \mathcal{U}_{22} \times \mathcal{U}_{31} \times \mathcal{U}_{32} \times \mathcal{U}_{33} \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})$, there exists a unique (strong) solution $x(\cdot) \equiv x(\cdot; x_0, u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ to (2.1). Thus, we can define the cost functionals as follows.

For Decision-making Level 1:

$$\begin{aligned} J^1(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v; x_0) = \mathbb{E}\Bigg\{ \int_0^T \Bigg[\langle Q_1(t)x(t), x(t)\rangle + \sum_{i=1}^{2} \langle R_i(t)u_{1i}(t), u_{1i}(t)\rangle \\ + \sum_{i=1}^{2}\sum_{j=1}^{3} \langle R_{1ij}(t)u_{2ij}(t), u_{2ij}(t)\rangle + \sum_{j=1}^{3}\sum_{i=1}^{2} \left\langle \bar{R}_{1ji}(t)u_{3ji}(t), u_{3ji}(t)\right\rangle \Bigg]\mathrm{d}t + \langle G_1 x(T), x(T)\rangle \Bigg\}, \end{aligned} \tag{2.2}$$

for the ith of Managerial Level 2:

$$\begin{aligned} J^2_i(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v; x_0) = \mathbb{E}\Bigg\{ \int_0^T \Bigg[\langle Q_{2i}(t)x(t), x(t)\rangle + \langle R_{2i}(t)u_{1i}(t), u_{1i}(t)\rangle \\ + \sum_{j=1}^{3} \langle R_{2ij}(t)u_{2ij}(t), u_{2ij}(t)\rangle + \sum_{j=1}^{3} \left\langle \bar{R}_{2ji}(t)u_{3ji}(t), u_{3ji}(t)\right\rangle \Bigg]\mathrm{d}t + \langle G_{2i} x(T), x(T)\rangle \Bigg\}, \end{aligned} \tag{2.3}$$

for the jth player of Executive Level 3:

$$\begin{aligned} J^3_j(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v; x_0) = \mathbb{E}\Bigg\{ \int_0^T \Bigg[\langle Q_{3j}(t)x(t), x(t)\rangle + \sum_{i=1}^{2} \langle R_{3ij}(t)u_{2ij}(t), u_{2ij}(t)\rangle \\ + \sum_{i=1}^{2} \left\langle \bar{R}_{3ji}(t)u_{3ji}(t), u_{3ji}(t)\right\rangle \Bigg]\mathrm{d}t + \langle G_{3j} x(T), x(T)\rangle \Bigg\}, \end{aligned} \tag{2.4}$$

where $Q_1(\cdot), R_i(\cdot), R_{1ij}(\cdot), \bar{R}_{1ji}(\cdot), Q_{2i}(\cdot), R_{2i}(\cdot), R_{2ij}(\cdot), \bar{R}_{2ji}(\cdot), Q_{3j}(\cdot), R_{3ij}(\cdot), \bar{R}_{3ji}(\cdot)$ are deterministic and uniformly bounded symmetric matrix-valued functions on $[0,T]$ of proper dimensions. G_1, G_{2i}, G_{3j} are $n\times n$ symmetric matrices. In addition, the weighting coefficients of cost functionals (2.2)–(2.4) satisfy the following:

(A1) $Q_1(\cdot) \geq 0$, $Q_{2i}(\cdot) \geq 0$, $Q_{3j}(\cdot) \geq 0$, $G_1 \geq 0$, $G_{2i} \geq 0$, $G_{3j} \geq 0$.

(A2) $R_i(\cdot) \gg 0$, $R_{1ij}(\cdot) \gg 0$, $\bar{R}_{1ji}(\cdot) \gg 0$, $R_{2i}(\cdot) \gg 0$, $R_{2ij}(\cdot) \gg 0$, $\bar{R}_{2ji}(\cdot) \gg 0$, $R_{3ij}(\cdot) \gg 0$, $\bar{R}_{3ji}(\cdot) \gg 0$.

Next, we will introduce some definitions. First, we introduce the team-optimal solution concept (see [39]), which is an important concept in this paper.

Definition 2.1. Let $J^1(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}; x_0)$ be a given cost functional of the leader (Decision-making Level 1), where $u_1(\cdot)$ denotes the leader's control, and $(u_{21}(\cdot), u_{22}(\cdot), u_{31}(\cdot), u_{32}(\cdot), u_{33}(\cdot))$ denotes the followers' controls (Managerial Level 2 and Executive Level 3). A control set $(u_1^*, u_{21}^*, u_{22}^*, u_{31}^*, u_{32}^*, u_{33}^*)$ is known as the *team-optimal solution* of this game if

$$J^1(u_1^*, u_{21}^*, u_{22}^*, u_{31}^*, u_{32}^*, u_{33}^*; x_0) \leq J^1(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}; x_0),$$

for any $u_1 \in \mathcal{U}_1$, $u_{2i} \in \mathcal{U}_{2i}$, $u_{3j} \in \mathcal{U}_{3j}$.

If J^1 is quadratic and strictly convex on the product space $\mathcal{U}_1 \times \mathcal{U}_{21} \times \mathcal{U}_{22} \times \mathcal{U}_{31} \times \mathcal{U}_{32} \times \mathcal{U}_{33}$, then a unique team-optimal solution exists.

Second, we introduce the finite horizon stochastic H_2/H_∞ problem. Consider the following stochastic linear system:

$$\begin{cases} \mathrm{d}x(t) = \big[A(t)x(t) + B(t)u(t) + E(t)v(t)\big]\mathrm{d}t + A_1(t)x(t)dW(t), \\ x(0) = x_0, \\ z(t) = \mathbf{col}\big[C(t)x(t)\ D(t)u(t)\big], \quad D^\top(t)D(t) = I_m, \end{cases} \tag{2.5}$$

where all coefficient matrices are continuous functions of time with suitable dimensions. $x_0 \in \mathbb{R}^n$ denotes the initial state, $u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ denotes the control input, $v(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})$ denotes the external disturbance, and $W(t) \in \mathbb{R}$ is a 1-dimensional standard Brownian motion defined in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. $z(\cdot) \in \mathbb{R}^{n_z}$ represents the controlled output. Then, the finite horizon stochastic H_2/H_∞ problem of (2.5) can be stated as follows (see [40]).

Definition 2.2. Given the disturbance attenuation $\gamma > 0$, $0 \leq T < \infty$, to find a state feedback control $u^*(t,x) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ and the worst case disturbance $v^*(t,x) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})$, such that

1)

$$\begin{aligned} \|\mathcal{L}\|_{[0,T]} &= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{\|z\|_{[0,T]}}{\|v\|_{[0,T]}} \\ &= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{\Big\{\mathbb{E}\int_0^T [x^\top C^\top C x + {u^*}^\top u^*]\mathrm{d}t\Big\}^{\frac{1}{2}}}{\Big\{\mathbb{E}\int_0^T v^\top v \mathrm{d}t\Big\}^{\frac{1}{2}}} < \gamma. \end{aligned}$$

2) When the worst case disturbance $v^*(t,x) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})$, if it exists, is applied to (2.5), $u^*(\cdot)$ minimizes the output energy

$$J_u(u, v^*) = \mathbb{E}\int_0^T [x^\top C^\top C x + u^\top u]\mathrm{d}t. \tag{2.6}$$

Here, the so-called worst case disturbance v^* means that

$$v^*(t,x) = \underset{v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})}{\arg\min}\ J_v(u^*, v) = \underset{v\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})}{\arg\min}\ \mathbb{E}\int_0^T [\gamma^2 v^\top v - z^\top z]\mathrm{d}t, \quad \forall x_0 \in \mathbb{R}^n. \tag{2.7}$$

If the previous (u^*, v^*) exists, then we say that the finite horizon H_2/H_∞ control has *Nash equilibrium solution* (u^*, v^*), *i.e.*,

$$J_u(u^*, v^*) \le J_u(u, v^*), \quad J_v(u^*, v^*) \le J_v(u^*, v),$$

u^* is a solution to the stochastic H_2/H_∞ control, and v^* is the corresponding worst-case disturbance.

Remark 2.3. $J_v(u_c^*, v^*; 0) \ge 0$ if and only if $\|\mathcal{L}\|_{[0,T]} \le \gamma$.

In [41], the finite horizon stochastic H_2/H_∞ control problem can be formulated as a stochastic LQ nonzero-sum game.

Three-level incentive Stackelberg game under the H_∞ constraint with multiple leaders and multiple followers, in this paper, is formulated as follows.

1) Given the disturbance attenuation level $\gamma > 0$, find the team-optimal strategy of Decision-making Level 1 with H_∞ constraint:

$$\begin{aligned} J^1(u_1^*, u_{21}^*, u_{22}^*, u_{31}^*, u_{32}^*, u_{33}^*, v^*; x_0) &\le J^1(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v^*; x_0), \\ 0 \le J_v(u_1^*, u_{21}^*, u_{22}^*, u_{31}^*, u_{32}^*, u_{33}^*, v^*; x_0) &\le J_v(u_1^*, u_{21}^*, u_{22}^*, u_{31}^*, u_{32}^*, u_{33}^*, v; x_0), \end{aligned} \tag{2.8}$$

for any $(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}) \in \mathcal{U}_1 \times \mathcal{U}_{21} \times \mathcal{U}_{22} \times \mathcal{U}_{31} \times \mathcal{U}_{32} \times \mathcal{U}_{33}$, $v \ne 0 \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n_v})$, where

$$\begin{aligned} &J_v(u_1, u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v; x_0) \\ := \mathbb{E}\Bigg\{ &- \langle G_1 x(T), x(T)\rangle + \int_0^T \Bigg[\gamma^2\|v\|^2 - \langle Q_1(t)x(t), x(t)\rangle - \sum_{i=1}^2 \langle R_i(t)u_{1i}(t), u_{1i}(t)\rangle \\ &- \sum_{i=1}^2\sum_{j=1}^3 \langle R_{1ij}(t)u_{2ij}(t), u_{2ij}(t)\rangle - \sum_{j=1}^3\sum_{i=1}^2 \big\langle \bar{R}_{1ji}(t)u_{3ji}(t), u_{3ji}(t)\big\rangle\Bigg]\mathrm{d}t\Bigg\}. \end{aligned} \tag{2.9}$$

2) Decision-making Level 1 announces the following feedback strategy in advance to Managerial Level 2:

$$u_{1i}(t) = \Gamma_{1i}\big(x(t), u_{2i}(t), u_{31i}(t), u_{32i}(t), u_{33i}(t), t\big). \tag{2.10}$$

The parameters in (2.10) can be determined in the next step.

3) Find the Nash equilibrium strategies of Managerial Level 2:

$$\bar{J}_1^2(u_{c1}^+, u_{c2}^+; x_0) \le \bar{J}_1^2(u_{c1}, u_{c2}^+; x_0), \quad \bar{J}_2^2(u_{c1}^+, u_{c2}^+; x_0) \le \bar{J}_2^2(u_{c1}^+, u_{c2}; x_0), \tag{2.11}$$

where for $i = 1, 2$,

$$\bar{J}_i^2(u_{c1}, u_{c2}; x_0) := J_i^2\big(\Gamma_1(x, , u_{2i}, u_{31i}, u_{32i}, u_{33i}), u_{21}, u_{22}, u_{31}, u_{32}, u_{33}, v^*(x); x_0\big), \tag{2.12}$$

and $u_{ci} = \mathbf{col}\big[u_{2i1}\ u_{2i2}\ u_{2i3}\ u_{31i}\ u_{32i}\ u_{33i}\big]$, $\Gamma_1 = \mathbf{col}\big[\Gamma_{11}, \Gamma_{12}\big]$, $v^*(x)$ is the closed-loop outcome worst-case disturbance solved in the first step. Decide parameters in (2.10) such that $u_{ci}^+(t) = u_{ci}^*(t)$, then the corresponding incentive strategy is denoted as Γ_{1i}^*.

4) Managerial Level 2 announce the following feedback strategy in advance to Executive Level 3:

$$u_{2ij}(t) = \Gamma_{2ij}\big(x(t), u_{3ji}(t), t\big). \tag{2.13}$$

The parameters in (2.13) can be determined in the next step.

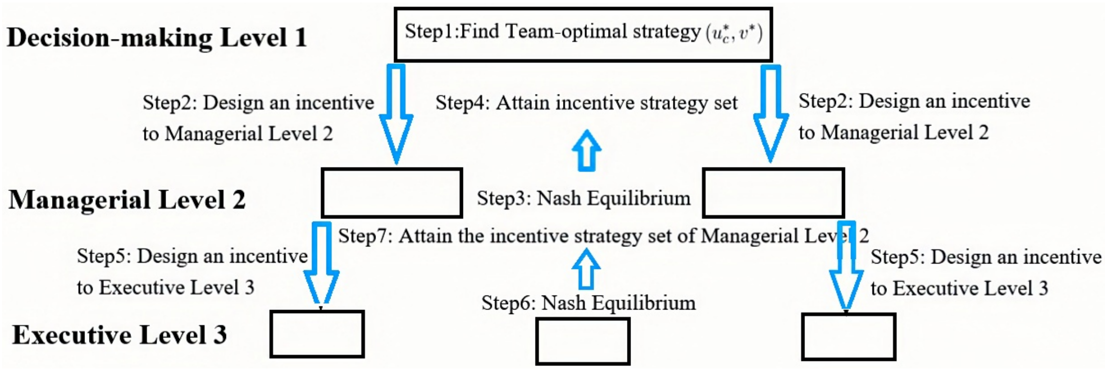

FIGURE 2. The procedure of three-level incentive Stackelberg differential game with H_∞ constraint.

5) Find the Nash equilibrium strategies of Executive Level 3:

$$\bar{J}_j^3(\bar{u}_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0) \leq \bar{J}_j^3\big(\bar{u}_{-3j}(u_{3j}); x_0\big), \tag{2.14}$$

where for $j = 1, 2, 3$,

$$\bar{J}_j^3(u_{31}, u_{32}, u_{33}; x_0) := J_j^3\big(\Gamma_{11}^*, \Gamma_{12}^*, (\Gamma_{2ij})_{i,j}, u_{31}, u_{32}, u_{33}, v^*(x); x_0\big), \tag{2.15}$$

and $u_{3j} = \mathbf{col}\big[u_{3j1}\ u_{3j2}\big]$, $\bar{u}_{-3j}(u_{3j})$ denotes the control profile which include u_{3j} and $\bar{u}_{3k}$ $(k = 1, 2, 3,\ k \neq j)$. Decide parameters in (2.13) such that $\bar{u}_{3j}(t) = u_{3j}^*(t)$, then the corresponding incentive strategy is denoted as Γ_{2ij}^*.

We provide a flowchart (see Fig. 2) to facilitate a more intuitive understanding of the three-level incentive Stackelberg differential game with H_∞ constraint.

In this article, we may suppress time index t if it causes no confusion.

3. Incentive Stackelberg equilibrium strategies

3.1. Team-optimal strategy

As mentioned above, the state equation is (2.1), and the cost functional of Decision-making Level 1 is (2.2). Due to the definition of team-optimal solution, for simplicity, we introduce the following notation, centralizing the control inputs in the stochastic system (2.1) and (2.2). The following centralized stochastic system can be introduced:

$$\begin{cases} \mathrm{d}x(t) = \big[A(t)x(t) + B_c(t)u_c(t) + E(t)v(t)\big]\mathrm{d}t + \big[C(t)x(t) + D_c(t)u_c(t)\big]dW(t), \\ x(0) = x_0, \end{cases} \tag{3.1}$$

together with

$$J^1(u_c, v; x_0) = \mathbb{E}\bigg\{ \int_0^T \big[\, \langle Q_1(t)x(t), x(t)\rangle + \langle R_c(t)u_c(t), u_c(t)\rangle \,\big]\mathrm{d}t + \langle G_1 x(T), x(T)\rangle \bigg\}, \tag{3.2}$$

$$J_v(u_c, v; x_0) = \mathbb{E}\bigg\{ \int_0^T \big[\gamma^2\|v\|^2 - \langle Q_1(t)x(t), x(t)\rangle - \langle R_c(t)u_c(t), u_c(t)\rangle \,\big]\mathrm{d}t - \langle G_1 x(T), x(T)\rangle \bigg\}, \tag{3.3}$$

where

$$\begin{aligned}
B_c &:= \big[B_{11}\ B_{12}\ B_{211}\ B_{212}\ B_{213}\ B_{221}\ B_{222}\ B_{223}\ B_{311}\ B_{312}\ B_{321}\ B_{322}\ B_{331}\ B_{332}\big],\\
D_c &:= \big[D_{11}\ D_{12}\ D_{211}\ D_{212}\ D_{213}\ D_{221}\ D_{222}\ D_{223}\ D_{311}\ D_{312}\ D_{321}\ D_{322}\ D_{331}\ D_{332}\big],\\
R_c &:= \textbf{block diag}\big(R_1\ R_2\ R_{111}\ R_{112}\ R_{113}\ R_{121}\ R_{122}\ R_{123}\ \bar{R}_{111}\ \bar{R}_{112}\ \bar{R}_{121}\ \bar{R}_{122}\ \bar{R}_{131}\ \bar{R}_{132}\big),\\
u_c &:= \textbf{col}\big[u_1\ u_{21}\ u_{22}\ u_{31}\ u_{32}\ u_{33}\big].
\end{aligned}$$

Theorem 3.1. *For system (3.1), the following three statements are equivalent:*
(i) The finite horizon H_2/H_∞ control has a solution $(u_c^*(\cdot), v^*(\cdot))$ with

$$u_c^*(t) = \Theta_1^*(t)x(t), \quad v^*(t) = \Theta_2^*(t)x(t).$$

(ii) There exists a closed-loop Nash equilibrium $(\Theta_1^*, 0; \Theta_2^*, 0)$.
(iii) The coupled GDREs (generalized differential Riccati equations)

$$\begin{cases}\dot{P}_1 + P_1(A + E\Theta_2^*) + (A + E\Theta_2^*)^\top P_1 + C^\top P_1 C + Q_1 \\ \quad - (P_1 B_c + C^\top P_1 D_c)(R_c + D_c^\top P_1 D_c)^{-1}(B_c^\top P_1 + D_c^\top P_1 C) = 0, \\ \Theta_2^* = -\gamma^{-2}E^\top P_2, \quad P_1(T) = G_1, \quad R_c + D_c^\top P_1 D_c > 0.\end{cases} \tag{3.4}$$

$$\begin{cases}\dot{P}_2 + P_2(A + B_c\Theta_1^*) + (A + B_c\Theta_1^*)^\top P_2 + (C + D_c\Theta_1^*)^\top P_2(C + D_c\Theta_1^*) \\ \quad - Q_1 - (\Theta_1^*)^\top R_c\Theta_1^* - \gamma^{-2}P_2EE^\top P_2 = 0, \\ \Theta_1^* = -(R_c + D_c^\top P_1 D_c)^{-1}(B_c^\top P_1 + D_c^\top P_1 C), \quad P_2(T) = -G_1,\end{cases} \tag{3.5}$$

have a solution $(P_1(\cdot), P_2(\cdot))$ on $[0, T]$. If the solution of GDREs exists, then

1) *The optimal strategies are given by*

$$u_c^*(t) = -(R_c + D_c^\top P_1 D_c)^{-1}(B_c^\top P_1 + D_c^\top P_1 C)x(t), \quad v^*(t) = -\gamma^{-2}E^\top P_2 x(t). \tag{3.6}$$

2) *The cost functionals at equilibrium are*

$$J^1(u_c^*, v^*; x_0) = x_0^\top P_1(0)x_0, \quad J_v(u_c^*, v^*; x_0) = x_0^\top P_2(0)x_0.$$

3) *The following inequalities hold:*

$$P_1(t) \geq 0, \quad P_2(t) \leq 0, \quad t \in [0, T].$$

Proof. For the implication (iii)⇒(ii), applying the standard completion of squares argument and Itô's formula to $s \mapsto \langle P_2(s)x(s), x(s)\rangle$, under the system (3.1), we have

$$\begin{aligned}\mathbb{E}\langle G_1 x(T), x(T)\rangle = -\mathbb{E}\int_0^T &\big[\big\langle\big(\dot{P}_2 + P_2A + A^\top P_2 + C^\top P_2 C\big)x, x\big\rangle + 2\big\langle\big(P_2B_c + C^\top P_2 D_c\big)u_c, x\big\rangle \\ &+ 2\langle P_2Ev, x\rangle + \big\langle D_c^\top P_2 D_c u_c, u\big\rangle\big]\mathrm{d}t - \langle P_2(0)x_0, x_0\rangle.\end{aligned}$$

Substituting for $\mathbb{E}\langle G_1 x(T), x(T)\rangle$ in $J_v(u_c, v; x_0)$ gives

$$J_v(u_c, v; x_0) = \mathbb{E}\int_0^T \big[\gamma^2|v - \Theta_2^* x|^2 + \big\langle\big(-R_c + D_c^\top P_2 D_c\big)u_c, u_c\big\rangle - \big\langle\big(-R_c + D_c^\top P_2 D_c\big)u_c^*, u_c^*\big\rangle \\ + 2\big\langle\big(P_2 B_c + C^\top P_2 D_c\big)(u_c - u_c^*), x\big\rangle\big]\mathrm{d}t + \langle P_2(0)x_0, x_0\rangle, \tag{3.7}$$

which implies that

$$J_v(u_c^*, v; x_0) = \langle P_2(0)x_0, x_0\rangle + \mathbb{E}\int_0^T \gamma^2|v - \Theta_2^* x|^2\mathrm{d}t \geq \langle P_2(0)x_0, x_0\rangle = J_v(u_c^*, v^*; x_0). \tag{3.8}$$

Accordingly, the second Nash inequality is derived. In addition, when $v^*(t) = -\gamma^{-2}E^\top P_2 x(t)$ is implemented in the state equation, it becomes

$$\begin{cases} \mathrm{d}x(t) = \big[(A - \gamma^{-2}EE^\top P_2)x + B_c u_c\big]\mathrm{d}t + [Cx + D_c u_c]dW, \\ x(0) = x_0, \end{cases}$$

Thus, minimizing

$$J^1(u_c, v^*; x_0) = \mathbb{E}\left\{\int_0^T \big[\langle Q_1 x, x\rangle + \langle R_c u_c, u_c\rangle\big]\mathrm{d}t + \langle G_1 x(T), x(T)\rangle\right\}$$

is a standard stochastic LQ optimization problem. By Theorem 5.2 of Sun *et al.* [42] and GDRE (3.4), applying Itô's formula to $s \mapsto \langle P_1(s)x(s), x(s)\rangle$, we have

$$\mathbb{E}\langle G_1 x(T), x(T)\rangle = \langle P_1(0)x_0, x_0\rangle + \mathbb{E}\int_0^T \big[\big\langle\big(\dot{P}_1 + P_1(A + E\Theta_2^*) + (A + E\Theta_2^*)^\top P_1 + C^\top P_1 C\big)x, x\big\rangle \\ + 2\langle(P_1 B_c + C^\top P_1 D_c)u_c, x\rangle + \langle D_c^\top P_1 D_c u_c, u_c\rangle\big]\mathrm{d}t.$$

Substituting $\mathbb{E}\langle G_1 x(T), x(T)\rangle$ into $J^1(u_c, v^*; x_0)$ gives

$$J^1(u_c, v^*; x_0) = \langle P_1(0)x_0, x_0\rangle + \mathbb{E}\int_0^T \big\langle\big(R_c + D_c^\top P_1 D_c\big)(u_c - \Theta_1^* x), u_c - \Theta_1^* x\big\rangle\mathrm{d}t \\ \geq \langle P_1(0)x_0, x_0\rangle = J^1(u_c^*, v^*; x_0). \tag{3.9}$$

Hence, the first Nash inequality is derived.

The implication (ii)⇒(iii) follows from Theorem 5.2 of Sun *et al.* [42].

For the implication (ii)⇒(i), the operator $\mathcal{L}$ associated with the system

$$\begin{cases} \mathrm{d}x(t) = [Ax + B_c u_c^* + Ev]\mathrm{d}t + [Cx + D_c u_c^*]dW, \\ x(0) = x_0, \end{cases}$$

is defined as

$$\mathcal{L} : L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n_v}) \to \mathbb{R}, \qquad \mathcal{L}(v(t)) = J^1(u_c^*, v; x_0), \quad t \in [0, T].$$

From Definition 2.2 and (3.8), $\|\mathcal{L}\|_{[0,T]}$ is

$$\begin{aligned}\|\mathcal{L}\|^2_{[0,T]} &= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{\mathbb{E}\left\{\int_0^T \left[\langle Q_1 x, x\rangle + \langle R_c u_c^*, u_c^*\rangle\right]\mathrm{d}t + \langle G_1 x(T), x(T)\rangle\right\}}{\mathbb{E}\int_0^T |v(t)|^2 \mathrm{d}t}\\
&= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{-J_v(u_c^*, v; x_0) + \gamma^2\mathbb{E}\int_0^T |v(t)|^2\mathrm{d}t}{\mathbb{E}\int_0^T |v(t)|^2 \mathrm{d}t}\\
&= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{-\langle P_2(0)x_0, x_0\rangle - \gamma^2\mathbb{E}\int_0^T |v-\Theta_2^* x|^2\mathrm{d}t + \gamma^2\mathbb{E}\int_0^T |v|^2\mathrm{d}t}{\mathbb{E}\int_0^T |v|^2 \mathrm{d}t}\\
&= \sup_{v(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}),\, v\neq 0,\, x_0=0} \frac{-\gamma^2\mathbb{E}\int_0^T |v-\Theta_2^* x|^2\mathrm{d}t}{\mathbb{E}\int_0^T |v|^2 \mathrm{d}t} + \gamma^2 < \gamma^2.\end{aligned}$$

Thus, we get

$$\|\mathcal{L}\|_{[0,T]} \leq \gamma.$$

From the previous proof process, (ii) implies that

$$\|\mathcal{L}\|_{[0,T]} \leq \gamma, \quad J^1(u_c^*, v^*; x_0) \leq J^1(u_c, v^*; x_0), \tag{3.10}$$

where u_c^*, v^* are defined by (iii).

In order to obtain the implication (ii)⇒(i), it suffices to show that $\|\mathcal{L}\|_{[0,T]} \leq \gamma$ in (3.10) can be replaced by $\|\mathcal{L}\|_{[0,T]} < \gamma$.

Following a similar line of argument as in [41]. Define an operator

$$\begin{aligned}\mathcal{L}_0 : L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v}) &\to L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n_v})\\
v(t) &\mapsto v(t) - \Theta_2^*(t)x(t; u_c^*, v; 0)\end{aligned}$$

i.e.,

$$\mathcal{L}_0 v(t) = v(t) - \Theta_2^*(t)x(t; u_c^*, v; 0),$$

where $x(\cdot; u_c^*, v; 0)$ denotes the solution of

$$\begin{cases}\mathrm{d}x(t) = \left[(A + B_c\Theta_1^*)x + Ev\right]\mathrm{d}t + (C + D_c\Theta_1^*)x dW(t),\\ x(0) = 0.\end{cases}$$

It is easy to see that $\mathcal{L}_0$ is a bounded linear operator and bijective, and that its inverse $\mathcal{L}_0^{-1}$ is given by

$$\mathcal{L}_0^{-1} v(t) = v(t) + \Theta_2^*(t)\tilde{x}^{(v)}(t),$$

where $\tilde{x}^{(v)}(\cdot)$ is the solution of

$$\begin{cases}\mathrm{d}\tilde{x}^{(v)}(t) = \left[(A + B_c\Theta_1^* + E\Theta_2^*)\tilde{x}^{(v)} + Ev\right]\mathrm{d}t + (C + D_c\Theta_1^*)\tilde{x}^{(v)} dW(t),\\ \tilde{x}^{(v)}(0) = 0.\end{cases}$$

By the bounded inverse theorem, $\mathcal{L}_0^{-1}$ is bounded with $\|\mathcal{L}_0^{-1}\|>0$. Thus from (3.8), we have

$$\begin{aligned}
\gamma^2\|v\|^2_{[0,T]} - J^1(u_c^*, v; 0) &= J_v(u_c^*, v; 0) = \gamma^2\mathbb{E}\int_0^T |v - \Theta_2^* x(t; u_c^*, v; 0)|^2 \mathrm{d}t \\
&= \gamma^2\|\mathcal{L}_0 v\|^2 \geq \frac{\gamma^2}{\|\mathcal{L}_0^{-1}\|^2}\mathbb{E}\int_0^T |v(t)|^2\mathrm{d}t \geq \epsilon\|v\|^2_{[0,T]},
\end{aligned}$$

for some sufficiently small $\epsilon > 0$, which yields $\|\mathcal{L}\|_{[0,T]} < \gamma$. The implication (ii)⇒(i) is complete.
For the implication (i)⇒(ii), by Definition 2.2, (i)⇒(ii) is obvious. □

By R_c, B_c, D_c and u_c, we can display and represent all elements in $u_c^*(\cdot)$ as follows:

$$\begin{aligned}
u_c^* &= \mathbf{col}\big[u_1^*\ u_{21}^*\ u_{22}^*\ u_{31}^*\ u_{32}^*\ u_{33}^*\big], \\
&= \mathbf{col}\big[u_{11}^*\ u_{12}^*\ u_{211}^*\ u_{212}^*\ u_{213}^*\ u_{221}^*\ u_{222}^*\ u_{223}^*\ u_{311}^*\ u_{312}^*\ u_{321}^*\ u_{322}^*\ u_{331}^*\ u_{332}^*\ \big], \\
u_{1i}^*(t) &= -(R_i + D_{1i}^\top P_1 D_{1i})^{-1}(B_{1i}^\top P_1 + D_{1i}^\top P_1 C)x(t), \\
u_{2ij}^*(t) &= -(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C)x(t), \\
u_{3ji}^*(t) &= -(\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)x(t).
\end{aligned} \tag{3.11}$$

In other words, (3.11) is the team-optimal solution with H_∞ constraint in three-level incentive system, and the worst-case disturbance is $v^*(t) = -\gamma^{-2}E^\top P_2 x(t)$.
For the sake of the discussion that follows, we set

$$\begin{aligned}
R_{1i} &:= \mathbf{block\ diag}(R_{1i1}\ R_{1i2}\ R_{1i3}\ \bar{R}_{11i}\ \bar{R}_{12i}\ \bar{R}_{13i}), \\
B_i &:= \big[B_{2i1}\ B_{2i2}\ B_{2i3}\ B_{31i}\ B_{32i}\ B_{33i}\ \big], \quad D_i := \big[D_{2i1}\ D_{2i2}\ D_{2i3}\ D_{31i}\ D_{32i}\ D_{33i}\ \big],
\end{aligned}$$

then

$$u_{ci}^*(t) = -(R_{1i} + D_i^\top P_1 D_i)^{-1}(B_i^\top P_1 + D_i^\top P_1 C)x(t). \tag{3.12}$$

3.2. Nash equilibrium of Managerial Level 2

Based on the incentive given by Decision-making Level 1 ahead of time, two people in Managerial Level 2 determine their strategies to achieve a Nash equilibrium by responding to the announced strategy of

Decision-making Level 1. Assume incentive form of Decision-making Level 1 is the following feedback strategy:

$$
\begin{aligned}
u_{1i}(t) = \Gamma_{1i}\big(x(t), u_{2i}(t), u_{31i}(t), u_{32i}(t), u_{33i}(t), t\big) &= -(R_i + D_{1i}^\top P_1 D_{1i})^{-1}(B_{1i}^\top P_1 + D_{1i}^\top P_1 C)x \\
&\quad + \sum_{j=1}^{3} \eta_{1ij}\big[u_{2ij} + (R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C)x\big] \\
&\quad + \sum_{j=1}^{3} \zeta_{1ij}\big[u_{3ji} + (\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)x\big] \\
&= \Big[-(R_i + D_{1i}^\top P_1 D_{1i})^{-1}(B_{1i}^\top P_1 + D_{1i}^\top P_1 C) + \sum_{j=1}^{3} \eta_{1ij}(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C) \\
&\quad + \sum_{j=1}^{3} \zeta_{1ij}(\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)\Big]x + \sum_{j=1}^{3}\eta_{1ij}u_{2ij} + \sum_{j=1}^{3}\zeta_{1ij}u_{3ji} \\
&= \sum_{j=1}^{3} \xi_{1ij}(t)x(t) + \sum_{j=1}^{3}\eta_{1ij}(t)u_{2ij}(t) + \sum_{j=1}^{3}\zeta_{1ij}(t)u_{3ji}(t),
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
\xi_{1ij} :=& -(R_i + D_{1i}^\top P_1 D_{1i})^{-1}(B_{1i}^\top P_1 + D_{1i}^\top P_1 C) + \eta_{1ij}(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1} \\
&\times (B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C) + \zeta_{1ij}(\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C),
\end{aligned}
$$

and $\xi_{1ij}(t) \in \mathbb{R}^{m_{1i}\times n}$, $\eta_{1ij}(t) \in \mathbb{R}^{m_{1i}\times m_{2ij}}$, $\zeta_{1ij}(t) \in \mathbb{R}^{m_{1i}\times m_{3ji}}$ are (unknown) strategy parameter matrices whose components are continuous functions of t on the interval $[0,T]$ respectively. Notice that

$$
\begin{aligned}
u_{1i}^*(t) &= \Gamma_{1i}\big(x(t), u_{2i}^*(t), u_{31i}^*(t), u_{32i}^*(t), u_{33i}^*(t), t\big) \\
&= \sum_{j=1}^{3}\xi_{1ij}(t)x(t) + \sum_{j=1}^{3}\eta_{1ij}(t)u_{2ij}^*(t) + \sum_{j=1}^{3}\zeta_{1ij}(t)u_{3ji}^*(t).
\end{aligned}
\tag{3.14}
$$

The worst-case disturbance $v^*(\cdot)$ has the following closed-loop form

$$
v^*(x)(t) = -\gamma^{-2}(t)E^\top(t)P_2(t)x(t). \tag{3.15}
$$

Then we substitute (3.13) and (3.15) into state equation (2.1) and cost functional for the ith of Managerial Level 2 (2.3). By simplification, we have

$$
\begin{cases}
\mathrm{d}x = \Big[\bar{A}x + \sum_{i=1}^{2}\sum_{j=1}^{3}\bar{B}_{2ij}u_{2ij} + \sum_{j=1}^{3}\sum_{i=1}^{2}\bar{B}_{3ji}u_{3ji}\Big]\mathrm{d}t + \Big[\bar{C}x + \sum_{i=1}^{2}\sum_{j=1}^{3}\bar{D}_{2ij}u_{2ij} + \sum_{j=1}^{3}\sum_{i=1}^{2}\bar{D}_{3ji}u_{3ji}\Big]dW \\
\quad\;\; = \Big[\bar{A}x + \sum_{i=1}^{2}\bar{B}_{ci}u_{ci}\Big]\mathrm{d}t + \Big[\bar{C}x + \sum_{i=1}^{2}\bar{D}_{ci}u_{ci}\Big]dW, \\
x(0) = x_0,
\end{cases}
\tag{3.16}
$$

$$\bar{J}_i^2(u_{c1},u_{c2};x_0)=\mathbb{E}\bigg\{\int_0^T\Big[\big\langle\bar{Q}_{2i}(t)x(t),x(t)\big\rangle+\,2\,\langle x(t),S_{2i}(t)u_{ci}(t)\rangle \\ +\langle R_{ci}(t)u_{ci}(t),u_{ci}(t)\rangle\Big]\mathrm{d}t+\langle G_{2i}x(T),x(T)\rangle\bigg\}, \tag{3.17}$$

where

$$\begin{aligned}
u_{ci}(t) &:= \mathbf{col}\big[u_{2i1}(t)\ \ u_{2i2}(t)\ \ u_{2i3}(t)\ \ u_{31i}(t)\ \ u_{32i}(t)\ \ u_{33i}(t)\big],\\
\bar{A} &:= A-\gamma^{-2}EE^\top P_2+\sum_{i=1}^{2}\sum_{j=1}^{3}B_{1i}\xi_{1ij},\quad \bar{C}:=C+\sum_{i=1}^{2}\sum_{j=1}^{3}D_{1i}\xi_{1ij},\\
\bar{B}_{2ij} &:= B_{2ij}+B_{1i}\eta_{1ij},\quad \bar{B}_{3ji}:=B_{3ji}+B_{1i}\zeta_{1ij},\quad \bar{B}_{ci}:=\big[\bar{B}_{2i1}\ \bar{B}_{2i2}\ \bar{B}_{2i3}\ \bar{B}_{31i}\ \bar{B}_{32i}\ \bar{B}_{33i}\big],\\
\bar{D}_{2ij} &:= D_{2ij}+D_{1i}\eta_{1ij},\quad \bar{D}_{3ji}:=D_{3ji}+D_{1i}\zeta_{1ij},\quad \bar{D}_{ci}:=\big[\bar{D}_{2i1}\ \bar{D}_{2i2}\ \bar{D}_{2i3}\ \bar{D}_{31i}\ \bar{D}_{32i}\ \bar{D}_{33i}\big],\\
\bar{Q}_{2i} &:= Q_{2i}+\Big(\sum_{j=1}^{3}\xi_{1ij}^\top\Big)R_{2i}\Big(\sum_{j=1}^{3}\xi_{1ij}\Big),\quad S_{2i}:=\sum_{j=1}^{3}\xi_{1ij}^\top R_{2i}\big[\eta_{1i1}\ \eta_{1i2}\ \eta_{1i3}\ \zeta_{1i1}\ \zeta_{1i2}\ \zeta_{1i3}\big],\\
R_{ci}^1 &:= \begin{pmatrix} R_{2i1}+\eta_{1i1}^\top R_{2i}\eta_{1i1} & \eta_{1i1}^\top R_{2i}\eta_{1i2} & \eta_{1i1}^\top R_{2i}\eta_{1i3}\\ \eta_{1i2}^\top R_{2i}\eta_{1i1} & R_{2i2}+\eta_{1i2}^\top R_{2i}\eta_{1i2} & \eta_{1i2}^\top R_{2i}\eta_{1i3}\\ \eta_{1i3}^\top R_{2i}\eta_{1i1} & \eta_{1i3}^\top R_{2i}\eta_{1i2} & R_{2i3}+\eta_{1i3}^\top R_{2i}\eta_{1i3}\end{pmatrix},\\
R_{ci}^2 &:= \begin{pmatrix} \eta_{1i1}^\top R_{2i}\zeta_{1i1} & \eta_{1i1}^\top R_{2i}\zeta_{1i2} & \eta_{1i1}^\top R_{2i}\zeta_{1i3}\\ \eta_{1i2}^\top R_{2i}\zeta_{1i1} & \eta_{1i2}^\top R_{2i}\zeta_{1i2} & \eta_{1i2}^\top R_{2i}\zeta_{1i3}\\ \eta_{1i3}^\top R_{2i}\zeta_{1i1} & \eta_{1i3}^\top R_{2i}\zeta_{1i2} & \eta_{1i3}^\top R_{2i}\zeta_{1i3}\end{pmatrix},\qquad R_{ci}=\begin{pmatrix} R_{ci}^1 & R_{ci}^2\\ R_{ci}^3 & R_{ci}^4\end{pmatrix},\\
R_{ci}^3 &:= \begin{pmatrix} \zeta_{1i1}^\top R_{2i}\eta_{1i1} & \zeta_{1i1}^\top R_{2i}\eta_{1i2} & \zeta_{1i1}^\top R_{2i}\eta_{1i3}\\ \zeta_{1i2}^\top R_{2i}\eta_{1i1} & \zeta_{1i2}^\top R_{2i}\eta_{1i2} & \zeta_{1i2}^\top R_{2i}\eta_{1i3}\\ \zeta_{1i3}^\top R_{2i}\eta_{1i1} & \zeta_{1i3}^\top R_{2i}\eta_{1i2} & \zeta_{1i3}^\top R_{2i}\eta_{1i3}\end{pmatrix},\\
R_{ci}^4 &:= \begin{pmatrix} \bar{R}_{21i}+\zeta_{1i1}^\top R_{2i}\zeta_{1i1} & \zeta_{1i1}^\top R_{2i}\zeta_{1i2} & \zeta_{1i1}^\top R_{2i}\zeta_{1i3}\\ \zeta_{1i2}^\top R_{2i}\zeta_{1i1} & \bar{R}_{22i}+\zeta_{1i2}^\top R_{2i}\zeta_{1i2} & \zeta_{1i2}^\top R_{2i}\zeta_{1i3}\\ \zeta_{1i3}^\top R_{2i}\zeta_{1i1} & \zeta_{1i3}^\top R_{2i}\zeta_{1i2} & \bar{R}_{23i}+\zeta_{1i3}^\top R_{2i}\zeta_{1i3}\end{pmatrix}.
\end{aligned} \tag{3.18}$$

Next, Managerial Level 2 need to find the Nash equilibrium

$$(u_{c1}^+(\cdot),u_{c2}^+(\cdot))\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^{\sum_{j=1}^3(m_{21j}+m_{3j1})})\times L_{\mathbb{F}}^2(0,T;\mathbb{R}^{\sum_{j=1}^3(m_{22j}+m_{3j2})}),$$

such that

$$\bar{J}_1^2(u_{c1}^+,u_{c2}^+;x_0)\le\bar{J}_1^2(u_{c1},u_{c2}^+;x_0),\quad \bar{J}_2^2(u_{c1}^+,u_{c2}^+;x_0)\le\bar{J}_2^2(u_{c1}^+,u_{c2};x_0), \tag{3.19}$$

for any $(u_{c1}(\cdot),u_{c2}(\cdot))\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^{\sum_{j=1}^3(m_{21j}+m_{3j1})})\times L_{\mathbb{F}}^2(0,T;\mathbb{R}^{\sum_{j=1}^3(m_{22j}+m_{3j2})})$.

Definition 3.2. Let $F(g)$ be a *real-valued functional* of $g\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^n)$. If $F(g)\ge 0$ for all $g\in L_{\mathbb{F}}^2(0,T;\mathbb{R}^n)$, $F(\cdot)$ is said to be *positive semidefinite*. If furthermore, $F(g)>0$ for all $g\neq 0$, F is said to be *positive definite*.

Lemma 3.3. *Let (A1)–(A2) hold. For any $x_0 \in \mathbb{R}^n$, $\bar{J}_i^2(u^+_{-ci}(u_{ci}); x_0)$ is convex (resp., strictly convex) in $u_{ci}(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{\sum_{j=1}^3(m_{2ij}+m_{3ji})})$ if and only if $J_i'(g_i(\cdot))$ is positive semidefinite (resp., positive definite), where*

$$J_i'(g_i(\cdot)) := \mathbb{E}\left\{\int_0^T \left[\left\langle \bar{Q}_{2i}z_i', z_i'\right\rangle + 2\left\langle z_i', S_{2i}g_i\right\rangle + \left\langle R_{ci}g_i, g_i\right\rangle\right]\mathrm{d}t + \left\langle G_{2i}z_i'(T), z_i'(T)\right\rangle\right\},$$

and $z_i'(\cdot)$ satisfies:

$$\begin{cases}\mathrm{d}z_i'(t) = [\bar{A}z_i' + \bar{B}_{ci}g_i]\mathrm{d}t + [\bar{C}z_i' + \bar{D}_{ci}g_i]dW,\\ \ z_i'(0) = 0.\end{cases} \tag{3.20}$$

Proof. We consider the case $i=1$, and the proof is the same when $i=2$. For any $x_0 \in \mathbb{R}^n$, fix u^+_{c2}, let $x_1(\cdot)$, $x_2(\cdot)$ be the states of (3.16) corresponding $u^1_{c1}(\cdot)$, $u^2_{c1}(\cdot)$, respectively. Taking any $\lambda_1 \in [0,1]$ and denoting $\lambda_2 := 1-\lambda_1$, we get

$$\begin{aligned}&\lambda_1\bar{J}_1^2(u^1_{c1}, u^+_{c2}; x_0) + \lambda_2\bar{J}_1^2(u^2_{c1}, u^+_{c2}; x_0) - \bar{J}_1^2(\lambda_1u^1_{c1} + \lambda_2u^2_{c1}, u^+_{c2}; x_0)\\ &= \lambda_1\lambda_2\mathbb{E}\bigg\{\int_0^T \left[\left\langle \bar{Q}_{21}(x_1-x_2), x_1-x_2\right\rangle + 2\left\langle x_1-x_2, S_{21}(u^1_{c1}-u^2_{c1})\right\rangle\right.\\ &\qquad\qquad \left.+\left\langle R_{c1}(u^1_{c1}-u^2_{c1}), u^1_{c1}-u^2_{c1}\right\rangle\right]\mathrm{d}t + \left\langle G_{21}(x_1-x_2)(T), (x_1-x_2)(T)\right\rangle\bigg\}.\end{aligned}$$

Denote $g_1 := u^1_{c1} - u^2_{c1}$, $z_1' := x_1 - x_2$. Therefore, $z_1'(\cdot)$ is deterministic and satisfies (3.20). Hence

$$\lambda_1\bar{J}_1^2(u^1_{c1}, u^+_{c2}; x_0) + \lambda_2\bar{J}_1^2(u^2_{c1}, u^+_{c2}; x_0) - \bar{J}_1^2(\lambda_1u^1_{c1} + \lambda_2u^2_{c1}, u^+_{c2}; x_0) = \lambda_1\lambda_2J_1'(g_1(\cdot)),$$

and the lemma follows. ◻

Remark 3.4. When the coefficient matrix $R_{ci}(\cdot)$ is sufficiently large, the convexity of $\bar{J}_1^2(u_{c1}, u^+_{c2}; x_0)$ and $\bar{J}_2^2(u^+_{c1}, u_{c2}; x_0)$ can usually be ensured.

We give the following assumption:
(A3) For $i=1,2$, $R_{ci}(\cdot) \gg 0$.
For Managerial Level 2, in order to obtain the ith player's Nash equilibrium strategies, the following result can be derived through by Theorem 2.2.1 of [43].

Proposition 3.5. *Let (A1)-(A3) hold. Suppose the attenuation index γ is sufficiently large. Then, $(u^+_{c1}(\cdot), u^+_{c2}(\cdot))$ is an open-loop Nash equilibrium if and only if the stationary condition is satisfied:*

$$\begin{cases}S_{21}^\top(t)\tilde{x}(t) + R_{c1}(t)u^+_{c1}(t) + \bar{B}_{c1}^\top(t)\tilde{y}(t) + \bar{D}_{c1}^\top(t)\tilde{z}(t) = 0, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.},\\ S_{22}^\top(t)\tilde{x}(t) + R_{c2}(t)u^+_{c2}(t) + \bar{B}_{c2}^\top(t)\tilde{Y}(t) + \bar{D}_{c2}^\top(t)\tilde{Z}(t) = 0, \quad t\in[0,T], \quad \mathbb{P}\text{-a.s.},\end{cases}$$

where $(\tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}(\cdot))$ and $(\tilde{x}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot))$ satisfy the following FBSDEs respectively:

$$\begin{cases}\ \mathrm{d}\tilde{x} = \left[\bar{A}\tilde{x} + \bar{B}_{c1}u^+_{c1} + \bar{B}_{c2}u^+_{c2}\right]\mathrm{d}t + \left[\bar{C}\tilde{x} + \bar{D}_{c1}u^+_{c1} + \bar{D}_{c2}u^+_{c2}\right]dW,\\ \ \mathrm{d}\tilde{y} = -\left[\bar{A}^\top\tilde{y} + \bar{C}^\top\tilde{z} + \bar{Q}_{21}\tilde{x} + S_{21}u^+_{c1}\right]\mathrm{d}t + \tilde{z}dW,\\ \tilde{x}(0) = x_0, \quad \tilde{y}(T) = G_{21}\tilde{x}(T),\end{cases} \tag{3.21}$$

$$\begin{cases} \mathrm{d}\tilde{x} = \big[\bar{A}\tilde{x} + \bar{B}_{c1}u_{c1}^{+} + \bar{B}_{c2}u_{c2}^{+}\big]\mathrm{d}t + \big[\bar{C}\tilde{x} + \bar{D}_{c1}u_{c1}^{+} + \bar{D}_{c2}u_{c2}^{+}\big]dW, \\ \mathrm{d}\tilde{Y} = -\big[\bar{A}^{\top}\tilde{Y} + \bar{C}^{\top}\tilde{Z} + \bar{Q}_{22}\tilde{x} + S_{22}u_{c2}^{+}\big]\mathrm{d}t + \tilde{Z}dW, \\ \tilde{x}(0) = x_0, \quad \tilde{Y}(T) = G_{22}\tilde{x}(T). \end{cases} \tag{3.22}$$

Moreover, the Nash equilibrium becomes

$$\begin{cases} u_{c1}^{+}(t) = -R_{c1}^{-1}(t)\big[S_{21}^{\top}(t)\tilde{x}(t) + \bar{B}_{c1}^{\top}(t)\tilde{y}(t) + \bar{D}_{c1}^{\top}(t)\tilde{z}(t)\big], \quad t\in[0,T], \quad \mathbb{P}\text{-}a.s., \\ u_{c2}^{+}(t) = -R_{c2}^{-1}(t)\big[S_{22}^{\top}(t)\tilde{x}(t) + \bar{B}_{c2}^{\top}(t)\tilde{Y}(t) + \bar{D}_{c2}^{\top}(t)\tilde{Z}(t)\big], \quad t\in[0,T], \quad \mathbb{P}\text{-}a.s. \end{cases} \tag{3.23}$$

To obtain the closed-loop representation of Nash equilibrium, plugging (3.23) into (3.21) and (3.22), we have the following Hamiltonian system:

$$\begin{cases} \mathrm{d}\tilde{x} = \big[(\bar{A} - \bar{B}_{c1}R_{c1}^{-1}S_{21}^{\top} - \bar{B}_{c2}R_{c2}^{-1}S_{22}^{\top})\tilde{x} - \bar{B}_{c1}R_{c1}^{-1}\bar{B}_{c1}^{\top}\tilde{y} - \bar{B}_{c1}R_{c1}^{-1}\bar{D}_{c1}^{\top}\tilde{z} \\ \qquad - \bar{B}_{c2}R_{c2}^{-1}\bar{B}_{c2}^{\top}\tilde{Y} - \bar{B}_{c2}R_{c2}^{-1}\bar{D}_{c2}^{\top}\tilde{Z}\big]\mathrm{d}t \\ \qquad + \big[(\bar{C} - \bar{D}_{c1}R_{c1}^{-1}S_{21}^{\top} - \bar{D}_{c2}R_{c2}^{-1}S_{22}^{\top})\tilde{x} - \bar{D}_{c1}R_{c1}^{-1}\bar{B}_{c1}^{\top}\tilde{y} - \bar{D}_{c1}R_{c1}^{-1}\bar{D}_{c1}^{\top}\tilde{z} \\ \qquad - \bar{D}_{c2}R_{c2}^{-1}\bar{B}_{c2}^{\top}\tilde{Y} - \bar{D}_{c2}R_{c2}^{-1}\bar{D}_{c2}^{\top}\tilde{Z}\big]dW, \\ \mathrm{d}\tilde{y} = -\big[(\bar{A} - \bar{B}_{c1}R_{c1}^{-1}S_{21}^{\top})^{\top}\tilde{y} + (\bar{C} - \bar{D}_{c1}R_{c1}^{-1}S_{21}^{\top})^{\top}\tilde{z} + (\bar{Q}_{21} - S_{21}R_{c1}^{-1}S_{21}^{\top})\tilde{x}\big]\mathrm{d}t + \tilde{z}dW, \\ \mathrm{d}\tilde{Y} = -\big[(\bar{A} - \bar{B}_{c2}R_{c2}^{-1}S_{22}^{\top})^{\top}\tilde{Y} + (\bar{C} - \bar{D}_{c2}R_{c2}^{-1}S_{22}^{\top})^{\top}\tilde{Z} + (\bar{Q}_{22} - S_{22}R_{c2}^{-1}S_{22}^{\top})\tilde{x}\big]\mathrm{d}t + \tilde{Z}dW, \\ \tilde{x}(0) = x_0, \quad \tilde{y}(T) = G_{21}\tilde{x}(T), \quad \tilde{Y}(T) = G_{22}\tilde{x}(T). \end{cases} \tag{3.24}$$

Define

$$\begin{aligned} &\mathbb{Y} := \begin{pmatrix} \tilde{y} \\ \tilde{Y} \end{pmatrix}, \quad \mathbb{Z} := \begin{pmatrix} \tilde{z} \\ \tilde{Z} \end{pmatrix}, \quad \mathbb{B}_1 := \begin{pmatrix} -\bar{B}_{c1}R_{c1}^{-1}\bar{B}_{c1}^{\top} & -\bar{B}_{c2}R_{c2}^{-1}\bar{B}_{c2}^{\top} \end{pmatrix}, \\ &\mathbb{B}_2 := \begin{pmatrix} -\bar{B}_{c1}R_{c1}^{-1}\bar{D}_{c1}^{\top} & -\bar{B}_{c2}R_{c2}^{-1}\bar{D}_{c2}^{\top} \end{pmatrix}, \quad \mathbb{D}_1 := \begin{pmatrix} -\bar{D}_{c1}R_{c1}^{-1}\bar{B}_{c1}^{\top} & -\bar{D}_{c2}R_{c2}^{-1}\bar{B}_{c2}^{\top} \end{pmatrix}, \\ &\mathbb{D}_2 := \begin{pmatrix} -\bar{D}_{c1}R_{c1}^{-1}\bar{D}_{c1}^{\top} & -\bar{D}_{c2}R_{c2}^{-1}\bar{D}_{c2}^{\top} \end{pmatrix}, \quad \mathbb{A} := \begin{pmatrix} \bar{A} - \bar{B}_{c1}R_{c1}^{-1}S_{21}^{\top} & 0 \\ 0 & \bar{A} - \bar{B}_{c2}R_{c2}^{-1}S_{22}^{\top} \end{pmatrix}, \\ &\mathbb{C} := \begin{pmatrix} \bar{C} - \bar{D}_{c1}R_{c1}^{-1}S_{21}^{\top} & 0 \\ 0 & \bar{C} - \bar{D}_{c2}R_{c2}^{-1}S_{22}^{\top} \end{pmatrix}, \quad \mathbb{Q} := \begin{pmatrix} \bar{Q}_{21} - S_{21}R_{c1}^{-1}S_{21}^{\top} \\ \bar{Q}_{22} - S_{22}R_{c2}^{-1}S_{22}^{\top} \end{pmatrix}, \quad \mathbb{G} := \begin{pmatrix} G_{21} \\ G_{22} \end{pmatrix}. \end{aligned} \tag{3.25}$$

Then (3.24) can be rewrite as follows:

$$\begin{cases} \mathrm{d}\tilde{x} = \big[(\bar{A} - \bar{B}_{c1}R_{c1}^{-1}S_{21}^{\top} - \bar{B}_{c2}R_{c2}^{-1}S_{22}^{\top})\tilde{x} + \mathbb{B}_1\mathbb{Y} + \mathbb{B}_2\mathbb{Z}\big]\mathrm{d}t \\ \qquad + \big[(\bar{C} - \bar{D}_{c1}R_{c1}^{-1}S_{21}^{\top} - \bar{D}_{c2}R_{c2}^{-1}S_{22}^{\top})\tilde{x} + \mathbb{D}_1\mathbb{Y} + \mathbb{D}_2\mathbb{Z}\big]dW, \\ \mathrm{d}\mathbb{Y} = -[\mathbb{A}^{\top}\mathbb{Y} + \mathbb{C}^{\top}\mathbb{Z} + \mathbb{Q}\tilde{x}]\mathrm{d}t + \mathbb{Z}dW, \\ \tilde{x}(0) = x_0, \quad \mathbb{Y}(T) = \mathbb{G}\tilde{x}(T). \end{cases} \tag{3.26}$$

Proposition 3.6. *Suppose that the following equation*

$$\begin{cases}\dot{\Pi}+\Pi\Big(\bar{A}-\sum_{i=1}^{2}\bar{B}_{ci}R_{ci}^{-1}S_{2i}^{\top}\Big)+\Pi\mathbb{B}_1\Pi+\mathbb{A}^{\top}\Pi+\mathbb{Q}+(\mathbb{C}^{\top}+\Pi\mathbb{B}_2)(I-\Pi\mathbb{D}_2)^{-1}\\ \quad\times\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)=0,\quad \Pi(T)=\mathbb{G},\end{cases} \tag{3.27}$$

admits a solution $\Pi(\cdot)$ such that $I-\Pi\mathbb{D}_2$ has bounded inverse, then the Nash equilibrium (3.23) has the following representation:

$$\begin{aligned}u_{c1}^{+}(t)&=-R_{c1}^{-1}\left[S_{21}^{\top}+\begin{pmatrix}\bar{B}_{c1}^{\top} & 0\end{pmatrix}\Pi+\begin{pmatrix}\bar{D}_{c1}^{\top} & 0\end{pmatrix}(I-\Pi\mathbb{D}_2)^{-1}\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)\right]\tilde{x}(t),\\ u_{c2}^{+}(t)&=-R_{c2}^{-1}\left[S_{22}^{\top}+\begin{pmatrix}0 & \bar{B}_{c2}^{\top}\end{pmatrix}\Pi+\begin{pmatrix}0 & \bar{D}_{c2}^{\top}\end{pmatrix}(I-\Pi\mathbb{D}_2)^{-1}\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)\right]\tilde{x}(t),\end{aligned} \tag{3.28}$$

where $\tilde{x}(\cdot)$ satisfies

$$\begin{cases}\mathrm{d}\tilde{x}=\left[\bar{A}-\bar{B}_{c1}R_{c1}^{-1}S_{21}^{\top}-\bar{B}_{c2}R_{c2}^{-1}S_{22}^{\top}+\mathbb{B}_1\Pi+\mathbb{B}_2(I-\Pi\mathbb{D}_2)^{-1}\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)\right]\tilde{x}\mathrm{d}t\\ \qquad+\left[\bar{C}-\bar{D}_{c1}R_{c1}^{-1}S_{21}^{\top}-\bar{D}_{c2}R_{c2}^{-1}S_{22}^{\top}+\mathbb{D}_1\Pi+\mathbb{D}_2(I-\Pi\mathbb{D}_2)^{-1}\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)\right]\tilde{x}dW,\\ \tilde{x}(0)=x_0.\end{cases} \tag{3.29}$$

Proof. Let $\mathbb{Y}=\Pi\tilde{x}$ with $\Pi=\begin{pmatrix}\Pi_1\\ \Pi_2\end{pmatrix}$, we have

$$\mathbb{Z}=(I-\Pi\mathbb{D}_2)^{-1}\left(\Pi\bar{C}-\sum_{i=1}^{2}\Pi\bar{D}_{ci}R_{ci}^{-1}S_{2i}^{\top}+\Pi\mathbb{D}_1\Pi\right)\tilde{x}=\Sigma\tilde{x},$$

with $\Sigma=\begin{pmatrix}\Sigma_1\\ \Sigma_2\end{pmatrix}$, and Π satisfies (3.27), the proof is omitted here. □

For (3.13) to become the incentive strategy of Decision-making Level 1, (3.28) must be matched with the corresponding team-optimal strategies of Managerial Level 2 and Executive Level 3 from some conditions. We assume that the following equation holds:

$$u_{ci}^{+}(t)=u_{ci}^{*}(t),\qquad t\in[0,T]. \tag{3.30}$$

Proposition 3.7. *When the relation (3.30) holds, $\tilde{x}(t)=\bar{x}(t)$, where $\bar{x}(\cdot)$ is the closed-loop state process corresponding to $(x_0,\Theta_1^*,0,\Theta_2^*,0)$ in Theorem 3.1.*

Proof. By the closed-loop system, let $\bar{x}(\cdot) = x(\cdot; u_c^*, v^*)$ be the closed-loop state process corresponding to $(x_0, \Theta_1^*, 0, \Theta_2^*, 0)$:

$$\begin{aligned} \mathrm{d}\bar{x} &= [A\bar{x} + B_c u_c^* + Ev^*]\mathrm{d}t + [C\bar{x} + D_c u_c^*]dW \\ &= \Bigg[(A - \gamma^{-2}EE^\top P_2)\bar{x} + \sum_{i=1}^{2} B_{1i}u_{1i}^* + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}u_{2ij}^* + \sum_{j=1}^{3}\sum_{i=1}^{2} B_{3ji}u_{3ji}^*\Bigg]\mathrm{d}t \\ &\quad + \Bigg[C\bar{x} + \sum_{i=1}^{2} D_{1i}u_{1i}^* + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}u_{2ij}^* + \sum_{j=1}^{3}\sum_{i=1}^{2} D_{3ji}u_{3ji}^*\Bigg]dW. \end{aligned} \tag{3.31}$$

Plugging (3.14) into (3.31), we get

$$\begin{cases} \mathrm{d}\bar{x} = \Bigg[\Bigg(A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}\xi_{1ij}\Bigg)\bar{x} + \sum_{i=1}^{2}\sum_{j=1}^{3}(B_{2ij} + B_{1i}\eta_{1ij})u_{2ij}^* + \sum_{j=1}^{3}\sum_{i=1}^{2}(B_{3ji} + B_{1i}\zeta_{1ij})u_{3ji}^*\Bigg]\mathrm{d}t \\ \qquad + \Bigg[\Bigg(C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}\xi_{1ij}\Bigg)\bar{x} + \sum_{i=1}^{2}\sum_{j=1}^{3}(D_{2ij} + D_{1i}\eta_{1ij})u_{2ij}^* + \sum_{j=1}^{3}\sum_{i=1}^{2}(D_{3ji} + D_{1i}\zeta_{1ij})u_{3ji}^*\Bigg]dW, \\ \bar{x}(0) = x_0. \end{cases} \tag{3.32}$$

By Proposition 3.5 and notations (3.18), we have

$$\begin{cases} \mathrm{d}\tilde{x} = \Bigg[\Bigg(A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}\xi_{1ij}\Bigg)\tilde{x} + \sum_{i=1}^{2}\sum_{j=1}^{3}(B_{2ij} + B_{1i}\eta_{1ij})u_{2ij}^+ + \sum_{j=1}^{3}\sum_{i=1}^{2}(B_{3ji} + B_{1i}\zeta_{1ij})u_{3ji}^+\Bigg]\mathrm{d}t \\ \qquad + \Bigg[\Bigg(C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}\xi_{1ij}\Bigg)\tilde{x} + \sum_{i=1}^{2}\sum_{j=1}^{3}(D_{2ij} + D_{1i}\eta_{1ij})u_{2ij}^+ + \sum_{j=1}^{3}\sum_{i=1}^{2}(D_{3ji} + D_{1i}\zeta_{1ij})u_{3ji}^+\Bigg]dW, \\ \tilde{x}(0) = x_0, \end{cases} \tag{3.33}$$

then the subtraction of these two equations yields the following equation:

$$\begin{cases} d(\tilde{x} - \bar{x}) = \Bigg(A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}\xi_{1ij}\Bigg)(\tilde{x} - \bar{x})\mathrm{d}t + \Bigg(C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}\xi_{1ij}\Bigg)(\tilde{x} - \bar{x})dW, \\ (\tilde{x} - \bar{x})(0) = 0. \end{cases} \tag{3.34}$$

Obviously, $\tilde{x}(\cdot) - \bar{x}(\cdot) \equiv 0$ is the solution to the above equation. Therefore,

$$\tilde{x}(t) = \bar{x}(t), \qquad t \in [0, T], \qquad \mathbb{P}\text{-}a.s.$$

The proof is complete. □

When the relation (3.30) holds, for $i = 1, 2$, it can be expressed briefly as follows:

$$\begin{aligned} \Big[R_{ci}^{-1}\big(S_{2i}^\top + \bar{B}_{ci}^\top\Pi_i + \bar{D}_{ci}^\top\Sigma_i\big) - (R_{1i} + D_i^\top P_1 D_i)^{-1}(B_i^\top P_1 + D_i^\top P_1 C)\Big]\bar{x}(t) = 0, \\ t \in [0, T],\ \mathbb{P}\text{-}a.s. \end{aligned} \tag{3.35}$$

Remark 3.8. In fact, equation (3.35) contains twelve equations, in which η_{1ij}, ζ_{1ij} and the solution $\Pi(\cdot)$ of equation (3.27) are mutually coupled.

Summarizing what is stated above, we obtain the theorem with the incentive strategy of Decision-making Level 1 under the additional condition.

Theorem 3.9. *Let (A1)-(A3) hold, suppose the attenuation γ is sufficiently large. If there exist matrix-valued functions η^*_{1ij} and ζ^*_{1ij} such that (3.27) admits solution $\Pi^*(\cdot)$, satisfying equation (3.35), then there exists the incentive strategy set of Decision-making Level 1 in the three-level incentive Stackelberg game under the H_∞ constraint with multiple leaders and multiple followers (2.1)–(2.4), given by*

$$\begin{aligned}&\Gamma^*_{1i}\big(x(t), u_{2i}(t), u_{31i}(t), u_{32i}(t), u_{33i}(t), t\big)\\&= \sum_{j=1}^{3}\xi^*_{1ij}(t)x(t) + \sum_{j=1}^{3}\eta^*_{1ij}(t)u_{2ij}(t) + \sum_{j=1}^{3}\zeta^*_{1ij}(t)u_{3ji}(t),\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\xi^*_{1ij} :=& -(R_i + D_{1i}^\top P_1 D_{1i})^{-1}(B_{1i}^\top P_1 + D_{1i}^\top P_1 C) + \eta^*_{1ij}(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}\\&\times (B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C) + \zeta^*_{1ij}(\bar R_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C).\end{aligned}$$

Proof. The proof is obvious from what is stated prior to Theorem 3.9, we omit it here. □

Remark 3.10. We notice that the following equation holds, which will be used next:

$$\begin{aligned}u^*_{1i}(t) &= \Gamma^*_{1i}(x(t), u^*_{2i}(t), u^*_{31i}(t), u^*_{32i}(t), u^*_{33i}(t), t)\\&= \sum_{j=1}^{3}\xi^*_{1ij}(t)x(t) + \sum_{j=1}^{3}\eta^*_{1ij}(t)u^*_{2ij}(t) + \sum_{j=1}^{3}\zeta^*_{1ij}(t)u^*_{3ji}(t).\end{aligned} \tag{3.37}$$

ξ^*_{1ij}, η^*_{1ij} and ζ^*_{1ij} depend on the initial state value x_0, because the equation (3.35) includes the closed-loop team-optimal trajectory $\bar x(\cdot)$, which depends on x_0.

3.3. Nash equilibrium of Executive Level 3

Since a sufficient condition for the incentive strategy for Decision-making Level 1 has been obtained by Theorem 3.9, we will derive one for Managerial Level 2 next. Decision-making Level 1 announces the incentive strategy (3.36) and Managerial Level 2 announces the incentive strategy ahead of time, three people in Executive Level 3 determine their strategies to achieve a Nash equilibrium by responding to the announced strategy of Decision-making Level 1 and Managerial Level 2.

We consider the following announced strategy of the ith of Managerial Level 2:

$$\begin{aligned}u_{2ij}(t) &= \Gamma_{2ij}\big(x(t), u_{3ji}(t), t\big) = -(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C)x\\&\quad + \rho_{2ij}\big[u_{3ji} + (\bar R_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)x\big]\\&= \big[-(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C) + \rho_{2ij}(\bar R_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}\\&\quad \times (B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)\big]x + \rho_{2ij}u_{3ji} := \theta_{2ij}(t)x(t) + \rho_{2ij}(t)u_{3ji}(t),\end{aligned} \tag{3.38}$$

and $\theta_{2ij}(t) \in \mathbb{R}^{m_{2ij}\times n}$, $\rho_{2ij}(t) \in \mathbb{R}^{m_{2ij}\times m_{3ji}}$ are (unknown) strategy parameter matrices whose components are continuous functions of t on the interval $[0,T]$ respectively. Notice that

$$u^*_{2ij}(t) = \Gamma_{1i}(x(t), u^*_{3ji}(t), t) = \theta_{2ij}(t)x(t) + \rho_{2ij}(t)u^*_{3ji}(t). \tag{3.39}$$

Substituting (3.36), (3.38) and (3.15) into state equation (2.1) and cost functional for the jth member of Executive Level 3 (2.4), we will get

$$\begin{cases} \mathrm{d}x = \Big[\hat{A}x + \sum_{j=1}^{3}\sum_{i=1}^{2}\hat{B}_{3ji}u_{3ji}\Big]\mathrm{d}t + \Big[\hat{C}x + \sum_{j=1}^{3}\sum_{i=1}^{2}\hat{D}_{3ji}u_{3ji}\Big]dW \\ \quad\;\, = \Big[\hat{A}x + \sum_{j=1}^{3}\hat{B}_{3j}u_{3j}\Big]\mathrm{d}t + \Big[\hat{C}x + \sum_{j=1}^{3}\hat{D}_{3j}u_{3j}\Big]dW, \\ x(0) = x_0, \end{cases} \tag{3.40}$$

$$\begin{aligned} \bar{J}^3_j(u_{31}, u_{32}, u_{33}; x_0) = \mathbb{E}\bigg\{ \int_0^T \big[\langle \hat{Q}_{3j}(t)x(t), x(t)\rangle + 2\,\langle x(t), S_{3j}(t)u_{3j}(t)\rangle \\ + \langle R_{3j}(t)u_{3j}(t), u_{3j}(t)\rangle\big]\mathrm{d}t + \langle G_{3j}x(T), x(T)\rangle \bigg\}, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned}
&\hat{A} := A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}\theta_{2ij},\\
&\hat{C} := C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}\theta_{2ij}, \quad u_{3j}(t) := \mathbf{col}\big[u_{3j1}(t)\ u_{3j2}(t)\big],\\
&\hat{B}_{3ji} := B_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) + B_{2ij}\rho_{2ij} + B_{3ji}, \quad \hat{D}_{3ji} := D_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) + D_{2ij}\rho_{2ij} + D_{3ji},\\
&\hat{B}_{3j} := \begin{pmatrix}\hat{B}_{3j1} & \hat{B}_{3j2}\end{pmatrix}, \quad \hat{D}_{3j} := \begin{pmatrix}\hat{D}_{3j1} & \hat{D}_{3j2}\end{pmatrix}, \quad \hat{Q}_{3j} := Q_{3j} + \sum_{i=1}^{2}\theta^\top_{2ij}R_{3ij}\theta_{2ij},\\
&S_{3j} := \begin{pmatrix}\theta^\top_{21j}R_{31j}\rho_{21j} & \theta^\top_{22j}R_{32j}\rho_{22j}\end{pmatrix}, \quad R_{3j} := \begin{pmatrix}\bar{R}_{3j1} + \rho^\top_{21j}R_{31j}\rho_{21j} & 0\\ 0 & \bar{R}_{3j2} + \rho^\top_{22j}R_{32j}\rho_{22j}\end{pmatrix}.
\end{aligned}$$

Next, Executive Level 3 need to find the Nash equilibrium $(\bar{u}_{31}(\cdot), \bar{u}_{32}(\cdot), \bar{u}_{33}(\cdot)) \in \mathcal{U}_{31}\times\mathcal{U}_{32}\times\mathcal{U}_{33}$, such that

$$\begin{aligned} \bar{J}^3_1(\bar{u}_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0) \le \bar{J}^3_1(u_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0),\\ \bar{J}^3_2(\bar{u}_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0) \le \bar{J}^3_2(\bar{u}_{31}, u_{32}, \bar{u}_{33}; x_0),\\ \bar{J}^3_3(\bar{u}_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0) \le \bar{J}^3_3(\bar{u}_{31}, \bar{u}_{32}, u_{33}; x_0), \end{aligned} \tag{3.42}$$

i.e., for $j = 1, 2, 3$,

$$\bar{J}^3_j(\bar{u}_{31}, \bar{u}_{32}, \bar{u}_{33}; x_0) \le \bar{J}^3_j(\bar{u}_{-3j}(u_{3j}); x_0),$$

for any $(u_{31}(\cdot), u_{32}(\cdot), u_{33}(\cdot)) \in \mathcal{U}_{31}\times\mathcal{U}_{32}\times\mathcal{U}_{33}$.

Lemma 3.11. *Let (A1)-(A3) hold. For any $x_0 \in \mathbb{R}^n$, $\bar{J}_j^3(\bar{u}_{-3j}(u_{3j}); x_0)$ is convex (resp., strictly convex) in $u_{3j}(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{\sum_{i=1}^2 m_{3ji}})$ if and only if $J_j''(h_j(\cdot))$ is positive semidefinite (resp., positive definite), where*

$$J_j''(h_j(\cdot)) := \mathbb{E}\left\{\int_0^T \left[\left\langle \hat{Q}_{3j} z_j'', z_j''\right\rangle + 2\left\langle z_j'', S_{3j} h_j\right\rangle + \left\langle R_{3j} h_j, h_j\right\rangle\right] \mathrm{d}t + \left\langle G_{3j} z_j''(T), z_j''(T)\right\rangle\right\},$$

and $z_j''(\cdot)$ satisfies:

$$\begin{cases} \mathrm{d}z_j''(t) = [\hat{A} z_j'' + \hat{B}_{3j} h_j]\mathrm{d}t + [\hat{C} z_j'' + \hat{D}_{3j} h_j]dW, \\ z_j''(0) = 0. \end{cases} \tag{3.43}$$

Proof. Similar to the proof of Lemma 3.3, we omit it. □

Remark 3.12. When $R_{3j}(\cdot)$ is sufficiently large, the convexity of $\bar{J}_j^3(\bar{u}_{-3j}(u_{3j}); x_0)$ can usually be ensured.

We introduce the following assumption:
(A4) The map $h_j \mapsto J_j''(h_j)$ is uniformly positive definite.

Proposition 3.13. *Let (A1)–(A4) hold. Suppose the attenuation index γ is sufficiently large. Then, $(\bar{u}_{31}(\cdot), \bar{u}_{32}(\cdot), \bar{u}_{33}(\cdot))$ is an open-loop Nash equilibrium if and only if stationary condition is satisfied:*

$$R_{3j}(t)\bar{u}_{3j}(t) + S_{3j}^\top(t)\hat{x}(t) + \hat{B}_{3j}^\top(t)\hat{y}_j(t) + \hat{D}_{3j}^\top(t)\hat{z}_j(t) = 0, \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s.,$$

where $(\hat{x}(\cdot), \hat{y}_j(\cdot), \hat{z}_j(\cdot))$ satisfies the following FBSDE:

$$\begin{cases} d\hat{x} = \left[\hat{A}\hat{x} + \sum_{j=1}^3 \hat{B}_{3j}\bar{u}_{3j}\right]\mathrm{d}t + \left[\hat{C}\hat{x} + \sum_{j=1}^3 \hat{D}_{3j}\bar{u}_{3j}\right]dW, \\ \mathrm{d}\hat{y}_j = -\left[\hat{A}^\top \hat{y}_j + \hat{C}^\top \hat{z}_j + \hat{Q}_{3j}\hat{x} + S_{3j}\bar{u}_{3j}\right] + \hat{z}_j dW, \\ \hat{x}(0) = x_0, \quad \hat{y}_j(T) = G_{3j}\hat{x}(T). \end{cases} \tag{3.44}$$

Moreover, the Nash equilibrium becomes

$$\bar{u}_{3j}(t) = -R_{3j}^{-1}(t)\left[S_{3j}^\top(t)\hat{x}(t) + \hat{B}_{3j}^\top(t)\hat{y}_j(t) + \hat{D}_{3j}^\top(t)\hat{z}_j(t)\right], \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s. \tag{3.45}$$

Proof. The details are omitted here. □

To obtain the state feedback representation of Nash equilibrium, plugging (3.45) into (3.44), we have the following Hamiltonian system:

$$
\begin{cases}
\mathrm{d}\hat{x} = \Big[\Big(\hat{A} - \sum_{j=1}^{3} \hat{B}_{3j} R_{3j}^{-1} S_{3j}^{\top}\Big)\hat{x} - \sum_{j=1}^{3} \hat{B}_{3j} R_{3j}^{-1} \hat{B}_{3j}^{\top} \hat{y}_j - \sum_{j=1}^{3} \hat{B}_{3j} R_{3j}^{-1} \hat{D}_{3j}^{\top} \hat{z}_j\Big]\mathrm{d}t \\
\qquad + \Big[\Big(\hat{C} - \sum_{j=1}^{3} \hat{D}_{3j} R_{3j}^{-1} S_{3j}^{\top}\Big)\hat{x} - \sum_{j=1}^{3} \hat{D}_{3j} R_{3j}^{-1} \hat{B}_{3j}^{\top} \hat{y}_j - \sum_{j=1}^{3} \hat{D}_{3j} R_{3j}^{-1} \hat{D}_{3j}^{\top} \hat{z}_j\Big] dW, \\
\mathrm{d}\hat{y}_1 = -\Big[(\hat{A} - \hat{B}_{31} R_{31}^{-1} S_{31}^{\top})^{\top} \hat{y}_1 + (\hat{C} - \hat{D}_{31} R_{31}^{-1} S_{31}^{\top})^{\top} \hat{z}_1 + (\hat{Q}_{31} - S_{31} R_{31}^{-1} S_{31}^{\top})\Big]\mathrm{d}t + \hat{z}_1 dW, \\
\mathrm{d}\hat{y}_2 = -\Big[(\hat{A} - \hat{B}_{32} R_{32}^{-1} S_{32}^{\top})^{\top} \hat{y}_2 + (\hat{C} - \hat{D}_{32} R_{32}^{-1} S_{32}^{\top})^{\top} \hat{z}_2 + (\hat{Q}_{32} - S_{32} R_{32}^{-1} S_{32}^{\top})\Big]\mathrm{d}t + \hat{z}_2 dW, \\
\mathrm{d}\hat{y}_3 = -\Big[(\hat{A} - \hat{B}_{33} R_{33}^{-1} S_{33}^{\top})^{\top} \hat{y}_3 + (\hat{C} - \hat{D}_{33} R_{33}^{-1} S_{33}^{\top})^{\top} \hat{z}_3 + (\hat{Q}_{33} - S_{33} R_{33}^{-1} S_{33}^{\top})\Big]\mathrm{d}t + \hat{z}_3 dW, \\
\hat{x}(0) = x_0, \quad \hat{y}_1(T) = G_{31}\hat{x}(T), \quad \hat{y}_2(T) = G_{32}\hat{x}(T), \quad \hat{y}_3(T) = G_{33}\hat{x}(T).
\end{cases}
\tag{3.46}
$$

Set

$$
\mathbf{Y} := \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{pmatrix}, \quad \mathbf{Z} := \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{pmatrix}, \quad \mathbf{B}_1 := \begin{pmatrix} -\hat{B}_{31} R_{31}^{-1} \hat{B}_{31}^{\top} & -\hat{B}_{32} R_{32}^{-1} \hat{B}_{32}^{\top} & -\hat{B}_{33} R_{33}^{-1} \hat{B}_{33}^{\top} \end{pmatrix},
$$
$$
\mathbf{B}_2 := \begin{pmatrix} -\hat{B}_{31} R_{31}^{-1} \hat{D}_{31}^{\top} & -\hat{B}_{32} R_{32}^{-1} \hat{D}_{32}^{\top} & -\hat{B}_{33} R_{33}^{-1} \hat{D}_{33}^{\top} \end{pmatrix},
$$
$$
\mathbf{D}_1 := \begin{pmatrix} -\hat{D}_{31} R_{31}^{-1} \hat{B}_{31}^{\top} & -\hat{D}_{32} R_{32}^{-1} \hat{B}_{32}^{\top} & -\hat{D}_{33} R_{33}^{-1} \hat{B}_{33}^{\top} \end{pmatrix},
$$
$$
\mathbf{D}_2 := \begin{pmatrix} -\hat{D}_{31} R_{31}^{-1} \hat{D}_{31}^{\top} & -\hat{D}_{32} R_{32}^{-1} \hat{D}_{32}^{\top} & -\hat{D}_{33} R_{33}^{-1} \hat{D}_{33}^{\top} \end{pmatrix},
$$
$$
\mathbf{A} := \begin{pmatrix} \hat{A} - \hat{B}_{31} R_{31}^{-1} S_{31}^{\top} & 0 & 0 \\ 0 & \hat{A} - \hat{B}_{32} R_{32}^{-1} S_{32}^{\top} & 0 \\ 0 & 0 & \hat{A} - \hat{B}_{33} R_{33}^{-1} S_{33}^{\top} \end{pmatrix}, \quad \mathbf{Q} := \begin{pmatrix} \hat{Q}_{31} - S_{31} R_{31}^{-1} S_{31}^{\top} \\ \hat{Q}_{32} - S_{32} R_{32}^{-1} S_{32}^{\top} \\ \hat{Q}_{33} - S_{33} R_{33}^{-1} S_{33}^{\top} \end{pmatrix},
$$
$$
\mathbf{C} := \begin{pmatrix} \hat{C} - \hat{D}_{31} R_{31}^{-1} S_{31}^{\top} & 0 & 0 \\ 0 & \hat{C} - \hat{D}_{32} R_{32}^{-1} S_{32}^{\top} & 0 \\ 0 & 0 & \hat{C} - \hat{D}_{33} R_{33}^{-1} S_{33}^{\top} \end{pmatrix}, \quad \mathbf{G} := \begin{pmatrix} G_{31} \\ G_{32} \\ G_{33} \end{pmatrix}.
$$

We can express the Hamiltonian system (3.46) more compactly as follows:

$$
\begin{cases}
\mathrm{d}\hat{x} = \Big[\Big(\hat{A} - \sum_{j=1}^{3} \hat{B}_{3j} R_{3j}^{-1} S_{3j}^{\top}\Big)\hat{x} + \mathbf{B}_1 \mathbf{Y} + \mathbf{B}_2 \mathbf{Z}\Big]\mathrm{d}t + \Big[\Big(\hat{C} - \sum_{j=1}^{3} \hat{D}_{3j} R_{3j}^{-1} S_{3j}^{\top}\Big)\hat{x} + \mathbf{D}_1 \mathbf{Y} + \mathbf{D}_2 \mathbf{Z}\Big] dW, \\
\mathrm{d}\mathbf{Y} = -\big[\mathbf{A}^{\top} \mathbf{Y} + \mathbf{C}^{\top} \mathbf{Z} + \mathbf{Q}\hat{x}\big]\mathrm{d}t + \mathbf{Z} dW, \\
\hat{x}(0) = x_0, \quad \mathbf{Y}(T) = \mathbf{G}\hat{x}(T).
\end{cases}
\tag{3.47}
$$

Proposition 3.14. *Suppose that the following equation*

$$
\begin{cases}
\dot{\Phi} + \Phi\Big(\hat{A} - \sum_{j=1}^{3} \hat{B}_{3j} R_{3j}^{-1} S_{3j}^{\top}\Big) + \mathbf{A}^{\top}\Phi + \Phi \mathbf{B}_1 \Phi + \mathbf{Q} + (\mathbf{C}^{\top} + \Phi \mathbf{B}_2)(I - \Phi \mathbf{D}_2)^{-1} \\
\qquad \times \Big(\Phi\hat{C} - \sum_{j=1}^{3} \Phi \hat{D}_{3j} R_{3j}^{-1} S_{3j}^{\top} + \Phi \mathbf{D}_1 \Phi\Big) = 0, \quad \Phi(T) = \mathbf{G},
\end{cases}
\tag{3.48}
$$

admits a solution $\Phi(\cdot)$ such that $I-\Phi\mathbf{D}_2$ has bounded inverse, then the Nash equilibrium (3.45) has the following representation:

$$\begin{aligned}
\bar{u}_{31}(t) = -R_{31}^{-1}\Big[&S_{31}^\top + \begin{pmatrix}\hat{B}_{31}^\top & 0 & 0\end{pmatrix}\Phi + \begin{pmatrix}\hat{D}_{31}^\top & 0 & 0\end{pmatrix}(I-\Phi\mathbf{D}_2)^{-1} \\
&\times\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\Big]\hat{x}(t), \\
\bar{u}_{32}(t) = -R_{32}^{-1}\Big[&S_{32}^\top + \begin{pmatrix}0 & \hat{B}_{32}^\top & 0\end{pmatrix}\Phi + \begin{pmatrix}0 & \hat{D}_{32}^\top & 0\end{pmatrix}(I-\Phi\mathbf{D}_2)^{-1} \\
&\times\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\Big]\hat{x}(t),
\end{aligned}$$

$$\begin{aligned}
\bar{u}_{33}(t) = -R_{33}^{-1}\Big[&S_{33}^\top + \begin{pmatrix}0 & 0 & \hat{B}_{33}^\top\end{pmatrix}\Phi + \begin{pmatrix}0 & 0 & \hat{D}_{33}^\top\end{pmatrix}(I-\Phi\mathbf{D}_2)^{-1} \\
&\times\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\Big]\hat{x}(t),
\end{aligned} \tag{3.49}$$

where $\hat{x}(\cdot)$ satisfies

$$\begin{cases}
\mathrm{d}\hat{x} = \Big[\hat{A} - \sum_{j=1}^{3}\hat{B}_{3j}R_{3j}^{-1}S_{3j}^\top + \mathbf{B}_1\Phi + \mathbf{B}_2(I-\Phi\mathbf{D}_2)^{-1}\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\Big]\hat{x}\mathrm{d}t \\
\qquad + \Big[\hat{C} - \sum_{j=1}^{3}\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \mathbf{D}_1\Phi + \mathbf{D}_2(I-\Phi\mathbf{D}_2)^{-1}\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\Big]\hat{x}dW, \\
\hat{x}(0) = x_0.
\end{cases} \tag{3.50}$$

Proof. Let $\mathbf{Y} = \Phi\hat{x}$ with $\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix}$, we have

$$\mathbf{Z} = (I-\Phi\mathbf{D}_2)^{-1}\Big(\Phi\hat{C} - \sum_{j=1}^{3}\Phi\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi\mathbf{D}_1\Phi\Big)\hat{x} = \Psi\hat{x},$$

with $\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix}$, and Φ satisfies (3.48). Then the Nash equilibrium (3.45) has the above representation (3.49). □

For (3.38) to become the incentive strategy of Managerial Level 2, (3.49) must be matched with the corresponding team-optimal strategies of Executive Level 3 from some conditions. We give the following assumption:

$$\bar{u}_{3j}(t) = u_{3j}^*(t), \qquad t\in[0,T], \tag{3.51}$$

i.e.,

$$\bar{u}_{3ji}(t) = u^*_{3ji}(t), \qquad t \in [0,T], \qquad j = 1,2,3,\ i = 1,2. \tag{3.52}$$

Proposition 3.15. *When the relation (3.51) holds, $\hat{x}(t) = \bar{x}(t)$.*

Proof. In fact, substituting (3.39) into (3.37), we have

$$u^*_{1i} = \sum_{j=1}^{3} \left[(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij})x + (\zeta^*_{1ij} + \eta^*_{1ij}\rho_{2ij})u^*_{3ji} \right]. \tag{3.53}$$

By (3.53), (3.39) and (3.31), we achieve

$$\begin{cases} \mathrm{d}\bar{x} = \Bigg[\Bigg(A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}\theta_{2ij} \Bigg)\bar{x} \\ \qquad + \sum_{j=1}^{3}\sum_{i=1}^{2} \Big(B_{3ji} + B_{2ij}\rho_{2ij} + B_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) \Big) u^*_{3ji} \Bigg] \mathrm{d}t \\ \quad + \Bigg[\Bigg(C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}\theta_{2ij} \Bigg)\bar{x} \\ \qquad + \sum_{j=1}^{3}\sum_{i=1}^{2} \Big(D_{3ji} + D_{2ij}\rho_{2ij} + D_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) \Big) u^*_{3ji} \Bigg] dW, \\ \bar{x}(0) = x_0. \end{cases} \tag{3.54}$$

Moreover, $\hat{x}(\cdot)$ satisfies

$$\begin{cases} \mathrm{d}\hat{x} = \Bigg[\Bigg(A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}\theta_{2ij} \Bigg)\hat{x} \\ \qquad + \sum_{j=1}^{3}\sum_{i=1}^{2} \Big(B_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) + B_{2ij}\rho_{2ij} + B_{3ji} \Big) \bar{u}_{3ji} \Bigg] \mathrm{d}t \\ \quad + \Bigg[\Bigg(C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}(\xi^*_{1ij} + \eta^*_{1ij}\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}\theta_{2ij} \Bigg)\hat{x} \\ \qquad + \sum_{j=1}^{3}\sum_{i=1}^{2} \Big(D_{1i}(\eta^*_{1ij}\rho_{2ij} + \zeta^*_{1ij}) + D_{2ij}\rho_{2ij} + D_{3ji} \Big) \bar{u}_{3ji} \Bigg] dW, \\ \hat{x}(0) = x_0. \end{cases} \tag{3.55}$$

Subtracting the above two equations under the constraint (3.52), we get

$$\begin{cases} d(\bar{x}-\hat{x}) = \Big[A - \gamma^{-2}EE^\top P_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{1i}(\xi_{1ij}^* + \eta_{1ij}^*\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} B_{2ij}\theta_{2ij}\Big](\bar{x}-\hat{x})\mathrm{d}t \\ \qquad + \Big[C + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{1i}(\xi_{1ij}^* + \eta_{1ij}^*\theta_{2ij}) + \sum_{i=1}^{2}\sum_{j=1}^{3} D_{2ij}\theta_{2ij}\Big](\bar{x}-\hat{x})dW, \\ (\bar{x}-\hat{x})(0) = 0. \end{cases} \tag{3.56}$$

Then, $\bar{x}(\cdot) - \hat{x}(\cdot) \equiv 0$ is the solution of this equation, we have

$$\hat{x}(t) = \bar{x}(t), \qquad t \in [0,T], \qquad \mathbb{P}\text{-}a.s.$$

The proof is complete. □

When the relation (3.52) holds, it can be expressed briefly as follows,

$$\Big[-(\bar{R}_{3ji} + \rho_{2ij}^\top R_{3ij}\rho_{2ij})^{-1}(\rho_{2ij}^\top R_{3ij}\theta_{2ij} + \hat{B}_{3ji}^\top \Phi_j + \hat{D}_{3ji}^\top \Psi_j) + (\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1} \\ \times (B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C)\Big]\bar{x}(t) = 0. \tag{3.57}$$

From the above analysis, we obtain the following theorem.

Theorem 3.16. *Let (A1)–(A4) hold, and suppose the attenuation γ is sufficiently large. If there exist matrix-valued functions ρ_{2ij}^* and θ_{2ij}^* such that (3.48) admits solution $\Phi^*(\cdot)$, satisfying equation (3.57), then there exists the incentive strategy set of Managerial Level 2 in the three-level incentive Stackelberg game under the H_∞ constraint with multiple leaders and multiple followers (2.1)–(2.4), given by*

$$\Gamma_{2ij}^*(x(t), u_{3ji}(t), t) = \theta_{2ij}^*(t)x(t) + \rho_{2ij}^*(t)u_{3ji}(t), \tag{3.58}$$

where

$$\theta_{2ij}^* := -(R_{1ij} + D_{2ij}^\top P_1 D_{2ij})^{-1}(B_{2ij}^\top P_1 + D_{2ij}^\top P_1 C) + \rho_{2ij}^*(\bar{R}_{1ji} + D_{3ji}^\top P_1 D_{3ji})^{-1}(B_{3ji}^\top P_1 + D_{3ji}^\top P_1 C),$$
$$\Psi^* := (I - \Phi^* \mathbf{D}_2)^{-1}\Big(\Phi^*\hat{C} - \sum_{j=1}^{3}\Phi^*\hat{D}_{3j}R_{3j}^{-1}S_{3j}^\top + \Phi^*\mathbf{D}_1\Phi^*\Big).$$

Proof. The proof is obvious from what is stated prior to Theorem 3.16, we omit the details. □

Remark 3.17. Notice that the following equation holds:

$$u_{2ij}^*(t) = \Gamma_{2ij}^*(x(t), u_{3ji}^*(t), t) = \theta_{2ij}^*(t)x(t) + \rho_{2ij}^*(t)u_{3ji}^*(t). \tag{3.59}$$

ρ_{2ij}^* and θ_{2ij}^* depend on the initial state value x_0, because the equation (3.57) includes the closed-loop team-optimal trajectory $\bar{x}(\cdot)$, which depends on x_0.

4. A numerical example

In this section, we give a numerical example with three-level multi-leader-follower, to demonstrate the effectiveness of our proposed incentive Stackelberg strategy set. Due to the close relationship between H_2/H_∞ control and the closed-loop Nash equilibrium of the corresponding two-person nonzero-sum game, the H_2/H_∞ optimal control and the worst-case disturbance are the outcomes of the closed-loop Nash equilibrium rather than the closed-loop representation of the open-loop Nash equilibrium.

Firstly, from the previous analysis in Section 3, we give an algorithm procedure to illustrate the process of solving incentive Stackelberg equilibrium strategies.

Algorithm 1: Algorithm procedure of the incentive Stackelberg equilibrium strategies.

Input: Choose coefficients of the stochastic system (2.1)–(2.4).
Solve $P_1(\cdot)$, $P_2(\cdot)$ from equation (3.4) and (3.5).
Obtain the closed-loop state trajectory $\bar{x}(\cdot)$ by (3.31).
Obtain the team-optimal strategy $u_c^*(\cdot)$ and the worst-case disturbance $v^*(\cdot)$ by (3.6).
Input: Equal division N, terminal values of η_{1ij} and ζ_{1ij}: $\eta_{1ij}(N+1)$, $\zeta_{1ij}(N+1)$.
for *k=1 to N* **do**
  Calculate $\xi_{1ij}(N-k+2)$ *via* (3.13).
  Calculate the values of all matrices in (3.18) and (3.25) at time $N-k+2$.
  Solve $\Pi(N-k+1)$ from equation (3.27), and calculate $\Sigma(N-k+2)$.
  Obtain $\eta_{1ij}(N-k+1)$ and $\zeta_{1ij}(N-k+1)$ by (3.35), and calculate $\xi_{1ij}(N-k+1)$.
Output: Incentive strategy parameters η_{1ij}^*, ζ_{1ij}^*, and ξ_{1ij}^* for Decision-making Level 1.

Input: Terminal values of ρ_{2ij}: $\rho_{2ij}(N+1)$.
for *k=1 to N* **do**
  Calculate $\theta_{2ij}(N-k+2)$ *via* (3.38).
  Calculate the values of all matrices mentioned in Section 3.3 at time $N-k+2$.
  Solve $\Phi(N-k+1)$ from equation (3.48), and calculate $\Psi(N-k+2)$.
  Obtain $\rho_{2ij}(N-k+1)$, and calculate $\theta_{2ij}(N-k+1)$.
Output: Incentive strategy parameters ρ_{2ij}^*, θ_{2ij}^* for Managerial Level 2.
Obtain the incentive strategy sets of Decision-making Level 1 and Managerial Level 2 *via* (3.36) and (3.58), respectively.

Example 4.1. The system coefficients are given as follows: $A = 0.01$, $C = 0.01$, $E = -0.1$, $B_{11} = 0.65$, $B_{12} = 0.8$, $D_{11} = 0.4$, $D_{12} = 0.9$, $B_{211} = -0.1$, $B_{212} = 0.2$, $B_{213} = 0.5$, $B_{221} = 0.2$, $B_{222} = -0.1$, $B_{223} = 0.2$, $B_{311} = 0.5$, $B_{312} = 0.1$, $B_{321} = 0.5$, $B_{322} = 0.2$, $B_{331} = 0.5$, $B_{332} = 0.1$, $D_{211} = -0.1$, $D_{212} = 0.1$, $D_{213} = 0.2$, $D_{221} = 0.5$, $D_{222} = -0.2$, $D_{223} = -0.9$, $D_{311} = 0.2$, $D_{312} = 0.1$, $D_{321} = 0.2$, $D_{322} = 0.1$, $D_{331} = 0.5$, $D_{332} = 0.1$, $Q_1 = 1$, $Q_{21} = 8$, $Q_{22} = 0.4$, $Q_{31} = 5$, $Q_{33} = 3$, $Q_{32} = 0.2$, $R_1 = R_2 = 1$, $R_{111} = R_{112} = 5$, $R_{113} = 1$, $R_{121} = 5$, $R_{122} = 1$, $R_{123} = 5$, $\bar{R}_{111} = 5$, $\bar{R}_{112} = 1$, $\bar{R}_{121} = 5$, $\bar{R}_{122} = 1$, $\bar{R}_{131} = 1$, $\bar{R}_{132} = 1$, $R_{221} = 9$, $R_{212} = 8$, $R_{213} = 1$, $R_{221} = 9$, $R_{222} = R_{223} = 1$, $\bar{R}_{211} = 3$, $\bar{R}_{212} = 2$, $\bar{R}_{221} = 4$, $\bar{R}_{222} = 1$, $\bar{R}_{231} = \bar{R}_{232} = 2$, $R_{311} = R_{321} = R_{322} = R_{323} = 1$, $R_{312} = 0.2$, $R_{313} = 0.3$, $\bar{R}_{311} = \bar{R}_{321} = \bar{R}_{322} = \bar{R}_{332} = 1$, $\bar{R}_{312} = 2$, $\bar{R}_{331} = 0.5$, $G_1 = G_{21} = G_{22} = 1$, $G_{31} = G_{33} = 0.5$, $G_{32} = 0.2$, $x_0 = 0.5$, $T = 0.8$, and the disturbance attenuation level $\gamma = 10$.

Consider the three-level multi-leader-follower incentive Stackelberg game with H_∞ constraint by the algorithm. We can give plots of the optimal state trajectory $\bar{x}(\cdot)$ and the team-optimal strategy $u_c^* = \mathbf{col}\big[u_{11}^*\ u_{12}^*\ u_{211}^*\ u_{212}^*\ \ u_{213}^*\ u_{221}^*\ u_{222}^*\ u_{223}^*\ u_{311}^*\ u_{312}^*\ u_{321}^*\ u_{322}^*\ u_{331}^*\ u_{332}^*\ \big]$ in Figures 3 and 4, respectively.

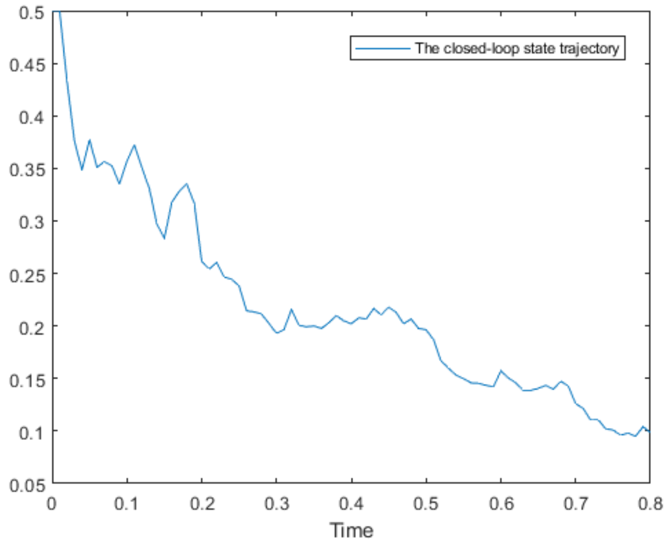

FIGURE 3. The closed-loop state trajectory $\bar{x}(\cdot)$.

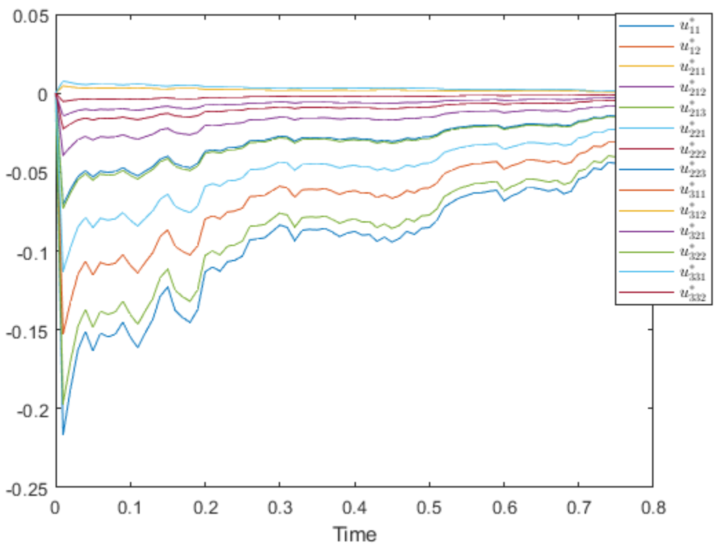

FIGURE 4. The team-optimal strategy $u_c^*(\cdot)$.

In the meanwhile, the worst-case disturbance $v^*(\cdot)$ is plotted in Figure 5.

Avoiding complex calculations, we only discuss the incentive for Decision-making Level 1, Managerial Level 2 being the same as it. Set $N = 80$, terminal values $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$ are given as follows: $\eta_{111}(N+1) = 50$, $\eta_{112}(N+1) = 30$, $\eta_{113}(N+1) = -20$, $\eta_{121}(N+1) = -10$, $\eta_{122}(N+1) = 40$, $\eta_{123}(N+1) = 10$, $\zeta_{111}(N+1) = 10$, $\zeta_{112}(N+1) = -10$, $\zeta_{113}(N+1) = 10$, $\zeta_{121}(N+1) = 10$, $\zeta_{122}(N+1) = 20$, $\zeta_{123}(N+1) = -10$. Through the algorithm above, we can derive plots of the moduli of incentive strategy parameters η^*_{1ij}, ζ^*_{1ij} and ξ^*_{1ij} in Figures 6–8 under the disturbance attenuation level $\gamma = 10$. (Since these parameters are complex numbers, for convenience, we only provide the images of their moduli.)

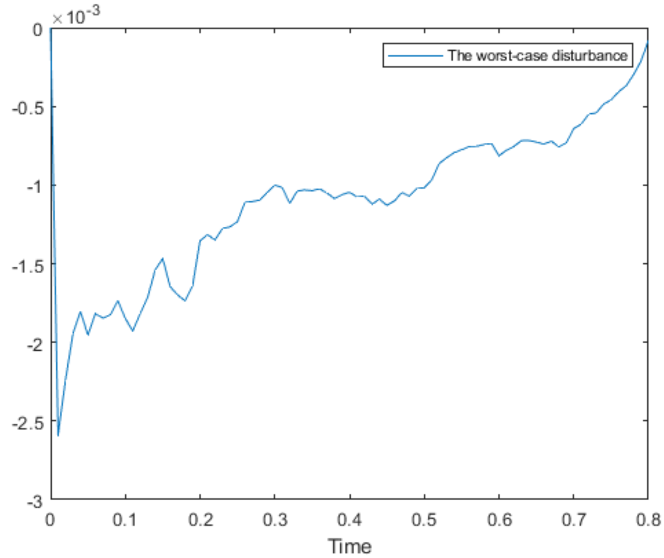

FIGURE 5. The worst-case disturbance $v^*(\cdot)$.

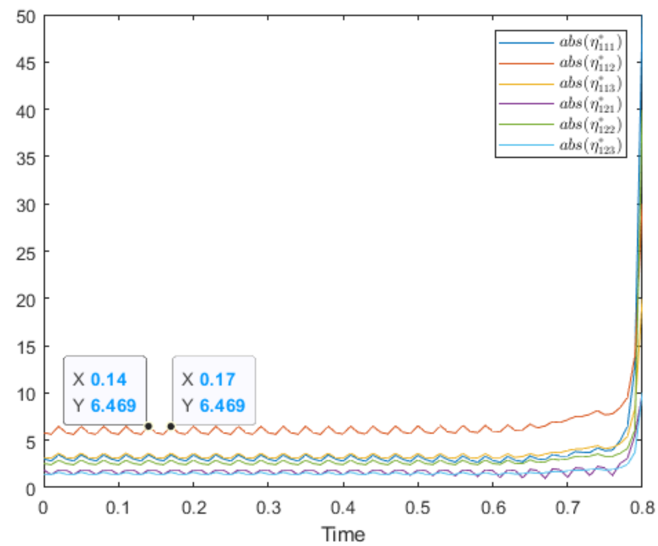

FIGURE 6. The moduli of incentive strategy parameters $\eta^*_{1ij}(\cdot)$.

The incentive intensities are very regular in the early stage. It can be roughly seen from Figures 6–8 that the incentive cycle of Decision-making Level 1 is 0.03, but in the final stage, a drastic adjustment occurs to achieve the team-optimal strategy. The figures and results are consistent with the theory of digital signal processing, where the measured excitation signal is a periodic signal, and non-sinusoidal periodic signal excitation in radio and electronic engineering ([44], [45]).

Next, we investigate the impacts of the disturbance attenuation level γ on the closed-loop state trajectory $\bar{x}(\cdot)$ and team-optimal strategy $u^*_c(\cdot)$. When the disturbance attenuation level γ takes these values in $\{1, 10, 100\}$, the change of the worst-case disturbance $v^*(\cdot)$ is plotted in Figure 9, and Figures 10 and 11 depict the changes of the closed-loop state trajectory $\bar{x}(\cdot)$ and team-optimal strategy $u^*_c(\cdot)$.

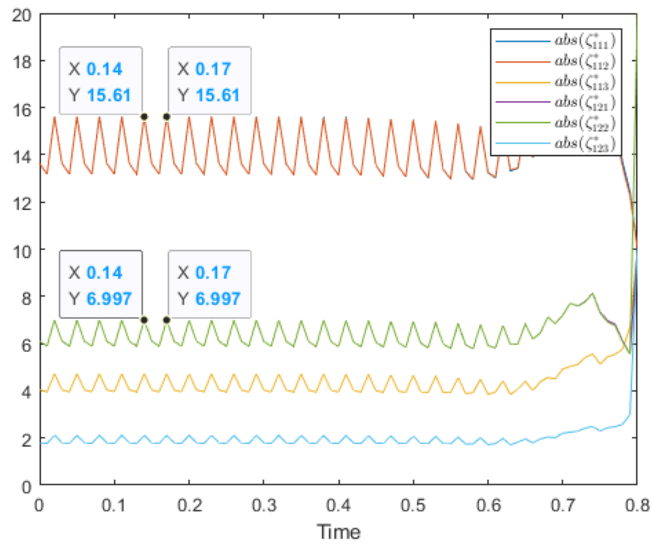

FIGURE 7. The moduli of incentive strategy parameters $\zeta_{1ij}^*(\cdot)$.

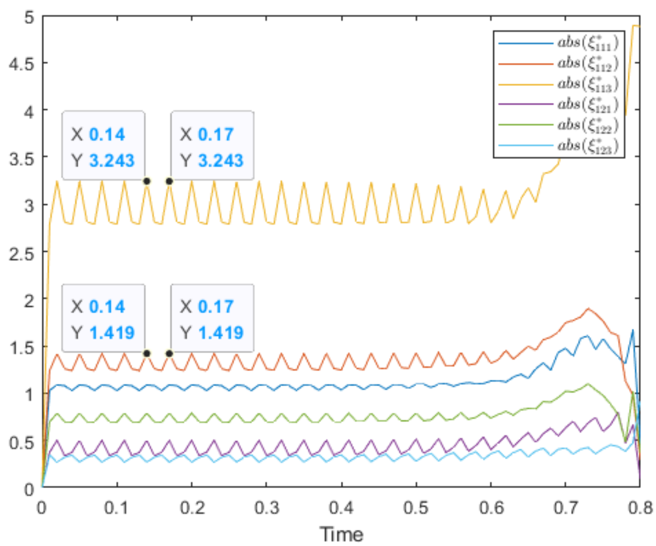

FIGURE 8. The moduli of incentive strategy parameters $\xi_{1ij}^*(\cdot)$.

In Figure 9, the higher disturbance attenuation level means that the measure of the worst possible impact of stochastic disturbance on the system that players can accept is larger. In other words, players are not conservative and remain optimistic about unknown risks. As a result, the worst-case disturbance $v^*(\cdot)$ they consider will be smaller. So the intensity of the team-optimal strategy $u_c^*(\cdot)$ they give will also be relatively small (Fig. 11).

Finally, we consider the impacts of terminal values of η_{1ij} and ζ_{1ij} on the moduli of incentive strategy parameters η_{1ij}^*, ζ_{1ij}^* and ξ_{1ij}^*. Consider the intensities (absolute values) of terminal values $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$ become 50% of the original under the same disturbance attenuation level $\gamma = 10$, which are given as follows: $\eta_{111}(N+1) = 25$, $\eta_{112}(N+1) = 15$, $\eta_{113}(N+1) = -10$, $\eta_{121}(N+1) = -5$, $\eta_{122}(N+1) = 20$, $\eta_{123}(N+1) = 5$, $\zeta_{111}(N+1) = 5$, $\zeta_{112}(N+1) = -5$, $\zeta_{113}(N+1) = 5$, $\zeta_{121}(N+1) = 5$, $\zeta_{122}(N+1) = 10$,

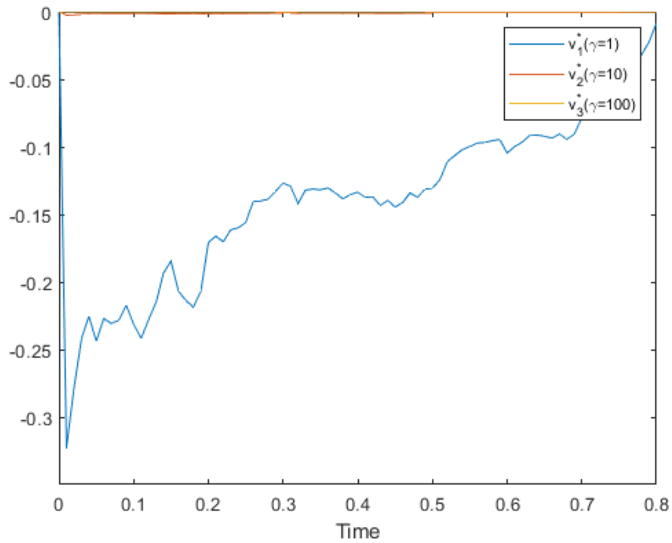

FIGURE 9. The impact of γ on the worst-case disturbance $v^*(\cdot)$.

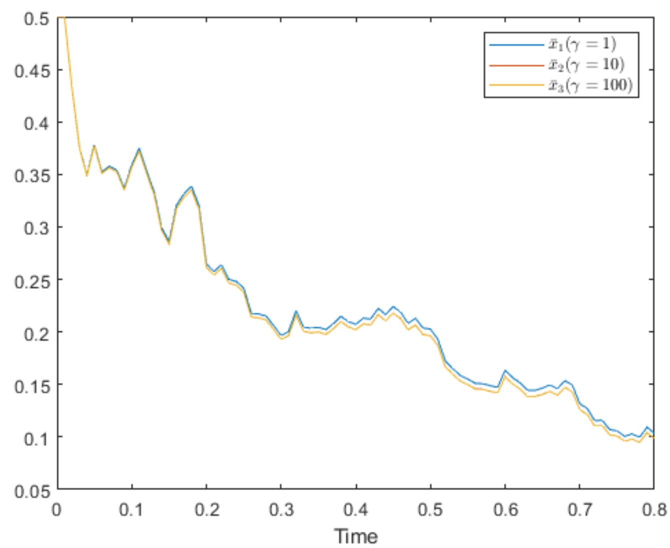

FIGURE 10. The impact of γ on the closed-loop state trajectory $\bar{x}(\cdot)$.

$\zeta_{123}(N+1) = -5$. By the algorithm procedure, we get plots of the corresponding moduli of incentive strategy parameters η_{1ij}^*, ζ_{1ij}^* and ξ_{1ij}^* in Figures 12–14.

Compared to Figures 6–8, to make the intensities of terminal incentive parameters smaller, the incentive strategy parameters' intensities at the initial stage must be larger to achieve the same objective under the same system, while the incentive cycle of Decision-making Level 1 remains unchanged (see Figs. 15–17).

Furthermore, as shown in Figures 15–17, when the terminal values of the incentive intensities becomes 50% of the original values, the incentive intensities in the early stage are actually consistent with the original incentive intensities (about the original terminal values) delayed by 0.01. In Figures 18–20, the cyan dashed lines are

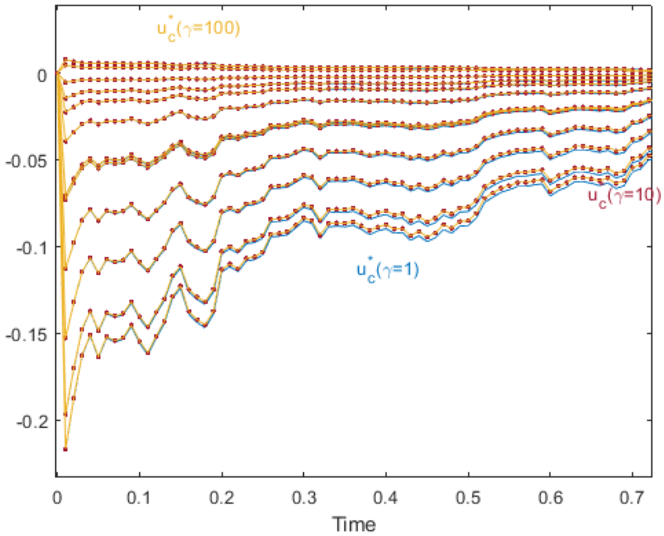

FIGURE 11. The impact of γ on the team-optimal strategy $u_c^*(\cdot)$.

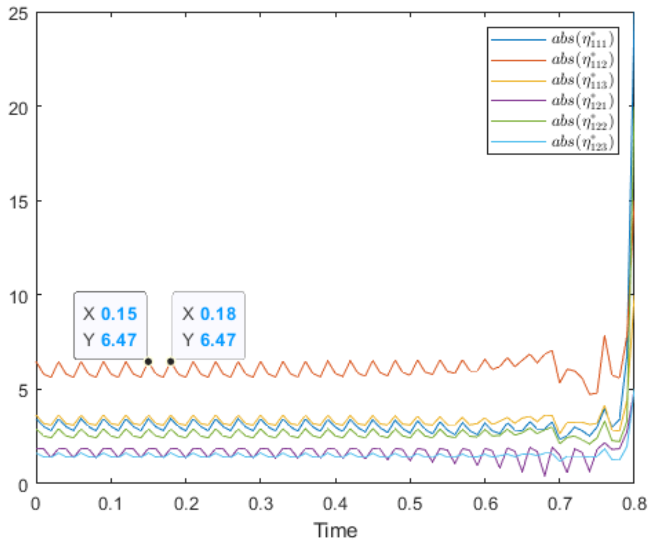

FIGURE 12. The impacts of $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$ on the moduli of $\eta^*_{1ij}(\cdot)$.

obtained by translating the blue lines (original) to the right by 0.01 units. In the early stage, they completely coincide with the red lines (new).

Remark 4.2. We have to emphasise that the algorithm procedure of solving incentive Stackelberg equilibrium strategies depends on the terminal values of incentive strategy parameters $\eta_{1ij}(\cdot)$, $\zeta_{1ij}(\cdot)$ and $\rho_{2ij}(\cdot)$ to be determined, which is a drawback of the algorithm.

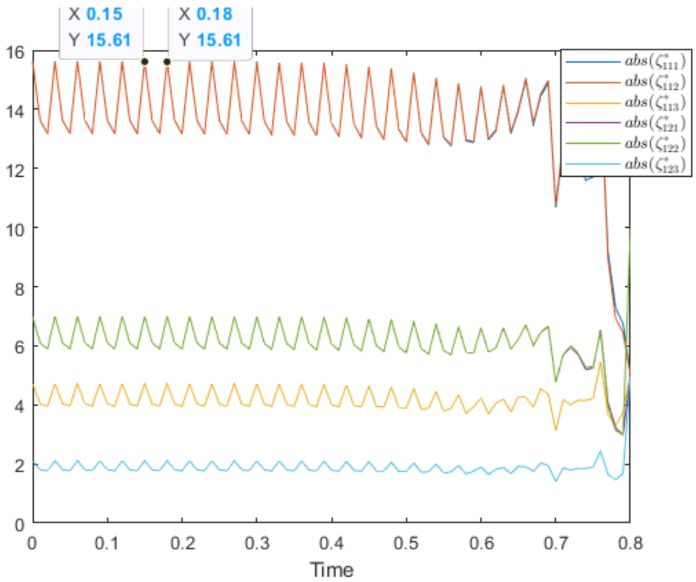

FIGURE 13. The impacts of $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$ on the moduli of $\zeta^*_{1ij}(\cdot)$.

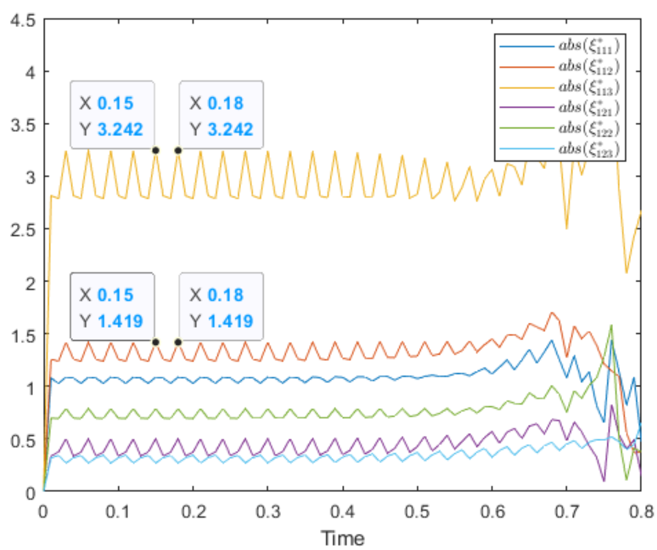

FIGURE 14. The impacts of $\xi_{1ij}(N+1)$ and $\xi_{1ij}(N+1)$ on the moduli of $\xi^*_{1ij}(\cdot)$.

5. CONCLUDING REMARKS

In this paper, we have studied a three-level multi-leader-follower incentive Stackelberg game with H_∞ constraint, where the control variables and the external disturbance enter the diffusion term and drift term of the state equation, respectively. *Via* H_2/H_∞ control theory, convex analysis theory, maximum principle and

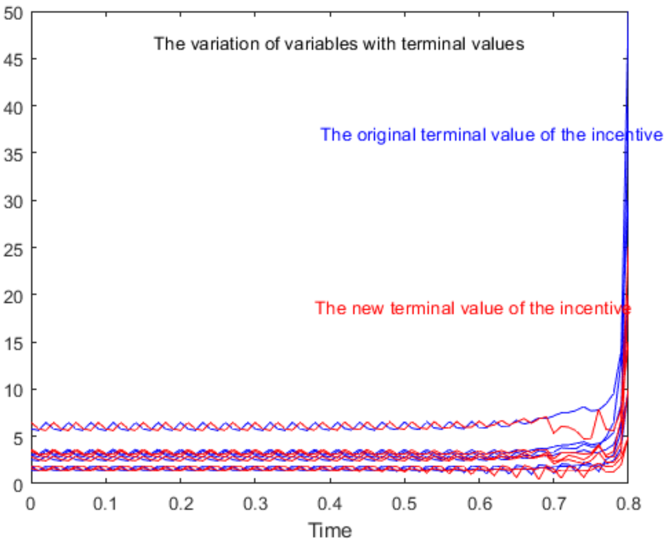

FIGURE 15. The variation of η with the terminal value $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$.

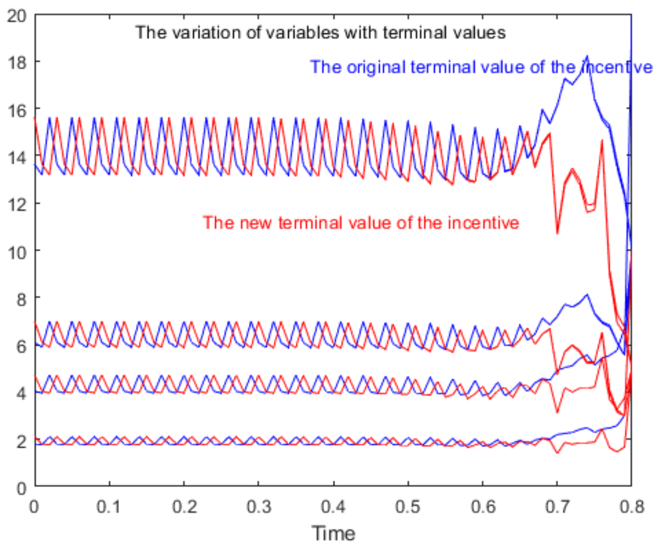

FIGURE 16. The variation of ζ with the terminal value $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$.

decoupling technique, the three-level incentive Stackelberg strategy set is given. We derive sufficient conditions for the three-level incentive Stackelberg game, and show that three-level incentive matrices depend on an initial state value x_0 and the corresponding trajectory is equivalent to the optimal-team trajectory $\bar{x}(\cdot)$ after achieving incentive. Numerical simulations are also given.

In the future, we will consider multi-leader-follower incentive Stackelberg differential game with partial observation, where both the leader and the followers have their own observation equations. Linear-quadratic mean

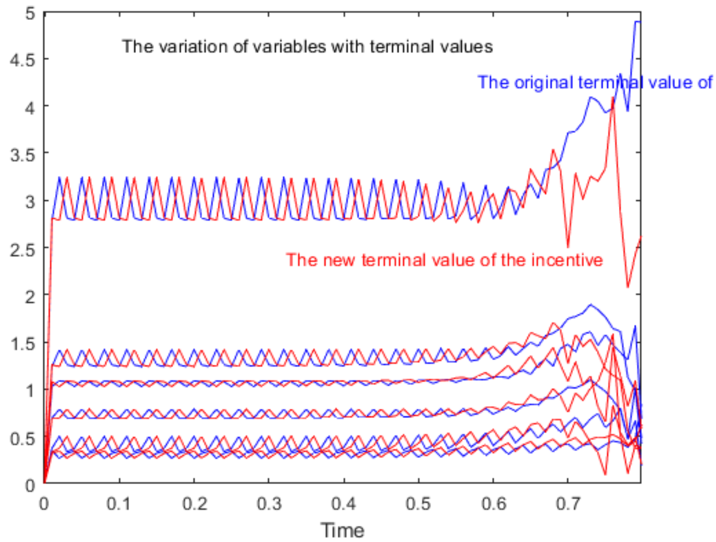

FIGURE 17. The variation of ξ with the terminal value $\eta_{1ij}(N+1)$ and $\zeta_{1ij}(N+1)$.

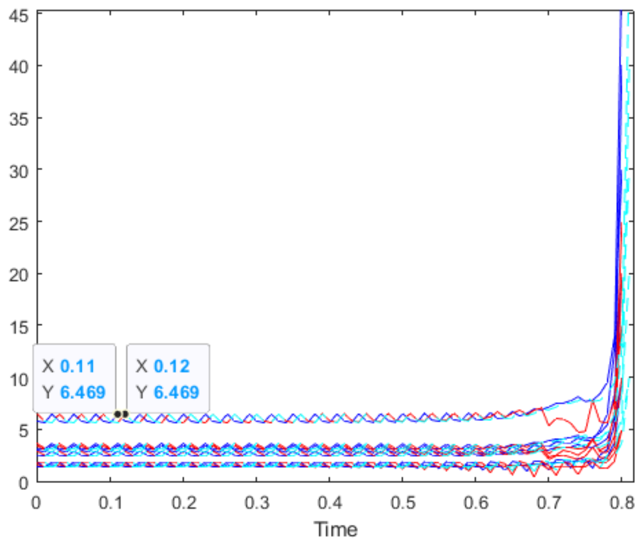

FIGURE 18. The relationship between the original incentive intensities η (blue) and the new incentive intensities η (red).

field type multi-leader-follower incentive Stackelberg differential game is also an interesting and challenging topic.

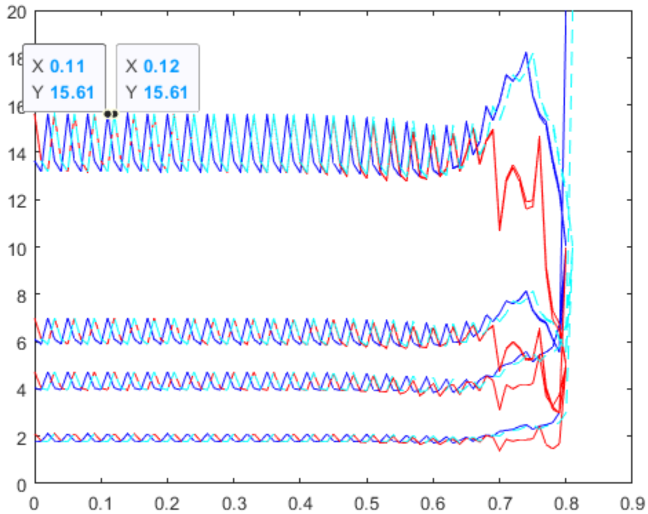

FIGURE 19. The relationship between the original incentive intensities ζ (blue) and the new incentive intensities ζ (red).

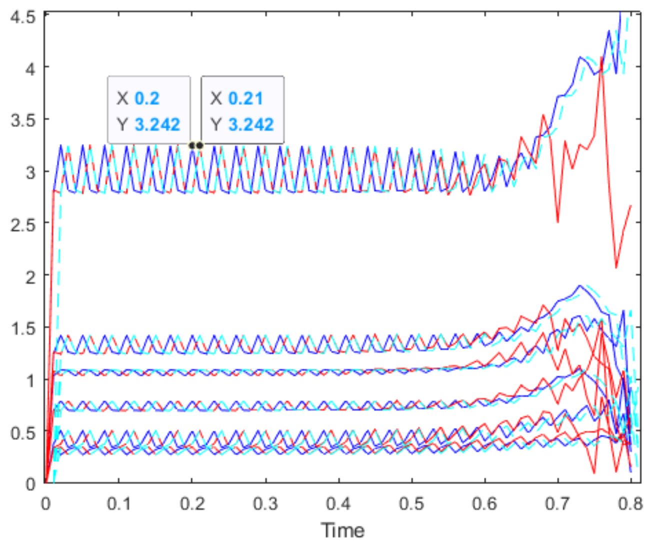

FIGURE 20. The relationship between the original incentive intensities ξ (blue) and the new incentive intensities ξ (red).

FUNDING

This work is supported by National Key R&D Program of China (2022YFA1006104), National Natural Science Foundations of China (12471419, 12271304), and Shandong Provincial Natural Science Foundations (ZR2024ZD35, ZR2022JQ01).

DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

References

[1] H. von Stackelberg, *Marktform und Gleichgewicht.* Springer, Vienna (1934). (An English translation appeared in *The Theory of The Market Economy*, Oxford University Press, UK (1952).)

[2] A. Bagchi and T. Başar, Stackelberg strategies in linear-quadratic stochastic differential games. *J. Optim. Theory Appl.* **35** (1981) 443–464.

[3] J. Yong, A leader-follower stochastic linear quadratic differential game. *SIAM J. Control Optim.* **41** (2002) 1015–1041.

[4] A. Bensoussan, S. Chen and S.P. Sethi, The maximum principle for global solutions of stochastic Stackelberg differential games. *SIAM J. Control Optim.* **53** (2015) 1956–1981.

[5] J. Shi, G. Wang and J. Xiong, Leader-follower stochastic differential game with asymmetric information and applications. *Automatica* **63** (2016) 60–73.

[6] J. Shi, G. Wang and J. Xiong, Linear-quadratic stochastic Stackelberg differential game with asymmetric information. *Sci. China Infor. Sci.* **60** (2017) 092202.

[7] Z. Li, D. Marelli, M. Fu, Q. Cai and W. Meng, Linear quadratic Gaussian Stackelberg game under asymmetric information patterns. *Automatica* **125** (2021) 109406.

[8] N. Li and Z. Yu, Forward–backward stochastic differential equations and linear-quadratic generalized Stackelberg games. *SIAM J. Control Optim.* **56** (2018) 4148–4180.

[9] J. Moon and T. Başar, Linear quadratic mean field Stackelberg differential games. *Automatica* **97** (2018) 200–213.

[10] B. Wang, Leader-follower mean field LQ games: a direct approach. *Asian J. Control* **26** (2024) 617–625.

[11] Y. Zheng and J. Shi, Stackelberg stochastic differential game with asymmetric noisy observations. *Int. J. Control* **95** (2022) 2510–2530.

[12] K. Kanga and J. Shi, A three-level stochastic linear-quadratic Stackelberg differential game with asymmetric information. *Proc. 35th Chinese Control and Decision Conference*, May 20–22, Yichang, China (2023) 5629–5636.

[13] X. Feng, Y. Hu and J. Huang, Linear-quadratic two-person differential game: Nash game versus Stackelberg game, local information versus global information. *ESAIM Control Optim. Calc. Var.* **30** (2024) 47.

[14] Y. Wang and W. Wang, Partially observed mean-field Stackelberg stochastic differential game with two followers. *Int. J. Control* **97** (2024) 1999–2008.

[15] Z. Li and J. Shi, Closed-loop solvability of linear quadratic mean-field type Stackelberg stochastic differential games. *Appl. Math. Optim.* **90** (2024) 22.

[16] Y. Ho, P. Luh and G.J. Olsder, A control-theoretic view of incentives. *Automatica* **18** (1982) 167–180.

[17] Y. Zheng and T. Başar, Existence and derivations of optimal affine incentive schemes for Stackelberg games with partial information: a geometric approach. *Int. J. Control* **35** (1982) 997–1011.

[18] Y. Zheng, T. Başar and J.B. Cruz Jr., Stackelberg strategies and incentives in multiperson deterministic decision problems. *IEEE Trans. Syst. Man Cybernet.* **14** (1984) 10–24.

[19] K. Mizukami and H. Wu, Two-level incentive Stackelberg strategies in LQ differential games with two noncooperative leaders and one follower. *Trans. Soc. Instrum. Control Eng.* **23** (1987) 625–632.

[20] K. Mizukami and H. Wu, Incentive Stackelberg strategies in linear quadratic differential games with two noncooperative followers. System Modelling and Optimization, vol. 113. *Lecture Notes in Control and Information Sciences.* Springer, Berlin Heidelberg (1988) 436–445.

[21] T. Ishida and E. Shimemura, Three-level incentive strategies in differential games. *Int. J. Control* **38** (1983) 1135–1148.

[22] T. Ishida, Three-level incentive schemes using follower's strategies in differential games. *Int. J. Control* **46** (1987) 1739–1750.

[23] M. Li, J.B. Cruz Jr. and M.A. Simaan, An approach to discrete-time incentive feedback Stackelberg games. *IEEE Trans. Syst. Man Cybernet.* **32** (2002) 10–24.

[24] H. Mukaidani, H. Xu, T. Shima and V. Dragan, A stochastic multiple-leader-follower incentive Stackelberg strategy for Markov jump linear systems. *IEEE Control Syst. Lett.* **1** (2017) 250–255.

[25] Y. Lin, W. Gao and W. Zhang, Incentive feedback Stackelberg strategy for stochastic system with state-dependent noise. *J. Frankl. Inst.* **359** (2022) 2058–2072.

[26] W. Gao, Y. Lin and W. Zhang, Incentive feedback Stackelberg strategy for the discrete-time stochastic systems. *J. Frankl Inst.* **360** (2023) 2404–2420.

[27] W. Gao, Y. Lin and W. Zhang, Incentive feedback Stackelberg strategy in mean-field type stochastic difference games. *J. Syst. Sci. Complex.* **37** (2024) 1425–1445.

[28] M. Ahmed and H. Mukaidani, H_∞-constrained incentive Stackelberg game for discrete time systems with multiple non-cooperative followers. *IFAC-PapersOnLine* **49** (2016) 262–267.

[29] H. Mukaidani, H. Xu and V. Dragan, Static output-feedback incentive Stackelberg game for discrete-time Markov jump linear stochastic systems with external disturbance. *IEEE Control Syst. Lett.* **2** (2018) 701–706.

[30] M. Ahmed, H. Mukaidani and T. Shima, H_∞-constrained incentive Stackelberg games for discrete-time stochastic systems with multiple followers. *IET Control Theory Appl.* **11** (2017) 2475–2485.

[31] K. Kawakami, H. Mukaidani, H. Xu and Y. Tanaka, Incentive Stackelberg–Nash strategy with disturbance attenuation for stochastic LPV systems. *Proc. 2018 IEEE International Conference on Systems, Man, and Cybernetics*, October 7–10, Miyazaki, Japan (2018) 3951–3955.

[32] H. Mukaidani, T. Shima, M. Unno, H. Xu and V. Dragan, Team-optimal incentive Stackelberg strategies for Markov jump linear stochastic systems with H_∞ constraint. *IFAC-PapersOnLine* **50** (2017) 3780–3785.

[33] H. Mukaidani and H. Xu, Robust incentive Stackelberg games for stochastic LPV systems, *Proc. 2018 IEEE Conference on Decision and Control*, December 17–19, Miami Beach, USA (2018) 1059–1064.

[34] H. Mukaidani and H. Xu, Incentive Stackelberg games for stochastic linear systems with H_∞ constraint. *IEEE Trans. Cybernet.* **49** (2019) 1463–1474.

[35] M. Ahmed, H. Mukaidani and T. Shima, Infinite-horizon multi-leader-follower incentive Stackelberg games for linear stochastic systems with H_∞ constraint, *Proc. SICE Annual Conference 2017*, September 19–22, Kanazawa, Japan (2017) 1202–1207.

[36] H. Mukaidani, H. Xu, T. Shima and M. Ahmed, Multi-leader-follower incentive Stackelberg game for infinite-horizon Markov jump linear stochastic systems with H_∞ constraint. *Proc. 2018 IEEE International Conference on Systems, Man, and Cybernetics*, October 7–10, Miyazaki, Japan (2018) 3956–3963.

[37] H. Mukaidani, R. Saravanakumar and H. Xu, Robust incentive Stackelberg strategy for Markov jump linear stochastic systems via static output feedback. *IET Control Theory Appl.* **14** (2020) 1246–1254.

[38] H. Mukaidani, S. Irie, H. Xu and W. Zhang, Robust incentive Stackelberg games with a large population for stochastic mean-field systems. *IEEE Control Syst. Lett.* **6** (2022) 1934–1939.

[39] T. Başar and G.J. Olsder, Dynamic Noncooperative Game Theory. SIAM, Philadelphia (1999).

[40] B.-S. Chen and W. Zhang, Stochastic H_2/H_∞ control with state-dependent noise. *IEEE Trans. Autom. Control.* **49** (2004) 45–57.

[41] D.J.N. Limebeer, B.D.O. Anderson and B. Hendel, A Nash game approach to mixed H_2/H_∞ control. *IEEE Trans. Autom. Control.* **39** (1994) 69–82.

[42] J. Sun and J. Yong, Linear quadratic stocahastic two-person nonzero-sum differential games: Open-loop and closed-loop Nash equilibria. *Stoch. Proc. Appl.* **129** (2019) 381–418.

[43] J. Sun and J. Yong, Stochastic linear-quadratic optimal control theory: differential games and mean-field problems. *Springer Briefs in Mathematics* (2020).

[44] A. Nowicki, Z. Klimonda, M. Lewandowski, J. Litniewski, P.A. Lewin and I. Trots, Comparison of sound fields generated by different coded excitations – experimental results. *Ultrasonics* **44** (2006) 121–129.

[45] M. Ricci, L. Senni and P. Burrascano, Exploiting pseudorandom sequences to enhance noise immunity for air-coupled ultrasonic nondestructive testing. *IEEE Trans. Instrum. Meas.* **61** (2012) 2905–2915.