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OPTIMALITY CONDITIONS FOR A CLASS OF MULTI-OBJECTIVE CONTROL PROBLEMS

Yanming Dong[1] and Xu Liu[2,*]

Abstract. This paper is devoted to studying a class of infinite-dimensional multi-objective control problems. First, an abstract linear system is driven by a leader control and two follower controls. For each leader control, a pair of follower controls is searched for as a Nash equilibrium of the game problem, while the aim of leader controls is to solve a constrained infinite-dimensional optimization problem. The first-order necessary condition for solutions to this optimization problem is given through a finite codimensionality condition. As applications, some multi-objective control problems for wave equations and quasi-linear parabolic equations are studied, respectively.

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1. Introduction

Let Y, U, V_1, V_2 and W be Hilbert spaces, and E be a nonempty closed convex subset of Y. For $T > 0$, set $\mathcal{U} = L^2(0,T;U)$, $\mathcal{V}_1 = L^2(0,T;V_1)$, $\mathcal{V}_2 = L^2(0,T;V_2)$ and $\mathcal{H} = C([0,T];Y)$. Assume that $f_0 : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{R}$ and $f : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \to Y$ are continuous. For two Banach spaces Z_1 and Z_2, we denote by $\mathcal{L}(Z_1;Z_2)$ the set of all bounded linear operators from Z_1 to Z_2, and denote by Z^* the dual space of a Banach space Z.

First, we consider the following controlled linear system:

$$\begin{cases} y_t(t) = Ay(t) + Bu(t) + C_1v_1(t) + C_2v_2(t) \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $(u, v_1, v_2) \in \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2$ is a triple of control variables, $y \in \mathcal{H}$ is the state variable, $y_0 \in Y$ is an initial value, $A : \mathcal{D}(A) \subseteq Y \to Y$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on Y, $B \in \mathcal{L}(U;Y)$ and $C_i \in \mathcal{L}(V_i;Y)$ for $i = 1, 2$. For each fixed $u \in \mathcal{U}$ and $i = 1, 2$, consider the following functionals:

$$J_i(v_1, v_2; u) = \frac{1}{2}\int_0^T \left|D_i[y(t) - y_{i,d}(t)]\right|_W^2 \mathrm{d}t + \frac{\mu_i}{2}\int_0^T |v_i(t)|_{V_i}^2 \mathrm{d}t, \quad \forall\, v_i \in \mathcal{V}_i, \tag{1.2}$$

where $\mu_1, \mu_2 > 0$, $y_{1,d}, y_{2,d} \in L^2(0,T;Y)$ and $D_1, D_2 \in \mathcal{L}(Y;W)$.

Keywords and phrases: Multi-objective control problem, necessary condition, Nash equilibrium, finite codimensionality.

[1] School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China.
[2] Key Laboratory of Applied Statistics of MOE, School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China.

[*] Corresponding author: liux216@nenu.edu.cn

For any given $u \in \mathcal{U}$, a pair $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is called a Nash equilibrium for J_1 and J_2, if

$$J_1(\bar{v}_1, \bar{v}_2; u) \leq J_1(v_1, \bar{v}_2; u), \ \forall \ v_1 \in \mathcal{V}_1 \ \text{ and } \ J_2(\bar{v}_1, \bar{v}_2; u) \leq J_2(\bar{v}_1, v_2; u), \ \forall \ v_2 \in \mathcal{V}_2. \tag{1.3}$$

Note that a pair $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is a Nash equilibrium for J_1 and J_2, if and only if

$$J_{1,v_1}(\bar{v}_1, \bar{v}_2; u)v_1 \triangleq \lim_{r \to 0} \frac{J_1(\bar{v}_1 + rv_1, \bar{v}_2; u) - J_1(\bar{v}_1, \bar{v}_2; u)}{r} = 0, \ \forall \ v_1 \in \mathcal{V}_1, \tag{1.4}$$

and

$$J_{2,v_2}(\bar{v}_1, \bar{v}_2; u)v_2 \triangleq \lim_{r \to 0} \frac{J_2(\bar{v}_1, \bar{v}_2 + rv_2; u) - J_2(\bar{v}_1, \bar{v}_2; u)}{r} = 0, \ \forall \ v_2 \in \mathcal{V}_2. \tag{1.5}$$

It will be shown that for any given $u \in \mathcal{U}$, there exists a unique Nash equilibrium $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ for the functionals J_1 and J_2.

Next, we study the following infinite-dimensional optimization problem:

$$(\mathbf{P}) \quad \min \left\{ f_0(u, L_1(u), L_2(u)) \in \mathbb{R} \;\middle|\; f(u, L_1(u), L_2(u)) \in E \text{ and } u \in \mathcal{K} \right\},$$

where $\mathcal{K} \subseteq \mathcal{U}$ is a closed convex subset, and $(L_1(u), L_2(u))$ denotes the Nash equilibrium for the functionals J_1 and J_2 associated to $u \in \mathcal{K}$. Suppose that $\bar{u} \in \mathcal{K}$ is a solution to the problem $(\mathbf{P})$, that is,

$$f_0(\bar{u}, L_1(\bar{u}), L_2(\bar{u})) = \min \left\{ f_0(u, L_1(u), L_2(u)) \in \mathbb{R} \;\middle|\; f(u, L_1(u), L_2(u)) \in E \text{ and } u \in \mathcal{K} \right\}.$$

The purpose of this paper is to present a first-order necessary condition for the solution $\bar{u}$ to $(\mathbf{P})$.

The problem proposed above is a multi-objective control problem. In the system (1), the triple (u, v_1, v_2) of controls may be divided into two levels. The leader control u is dominant, and decisions of follower controls (v_1, v_2) are influenced by it. For any leader control, a pair of follower controls is searched for such that the functionals J_1 and J_2 have a Nash equilibrium. Then, the leader control is characterized as a solution to a constrained infinite-dimensional optimization problem. As an example in practice, strategies of government in a city may be regarded as leader controls and different decisions of some real estate agencies serve as follower controls. This multi-objective control problem is to find a strategy of the government, according to which, the decisions of these agencies can reach a Nash equilibrium in the game. Meanwhile, the behaviors of the government and agencies keep housing price as close to expected and reduce cost as possible.

Up to now, there are numerous works on multi-objective control problems of partial differential equations. We refer to [1–12] and references therein for some known results in this respect. In the above existing works, the existence of hierarchical controls was discussed and some of them may be regarded as a special type of the problem $(\mathbf{P})$. For example, in [3], Stackelberg-Nash strategies to control linear parabolic equations were studied. For any leader control, a couple of follower controls is given as a Nash equilibrium for the functionals J_1 and J_2. The leader control is responsible for null controllability with minimum control cost, that is, in the optimization problem $(\mathbf{P})$, $E = \{0\}$, $f(u, L_1(u), L_2(u)) = y(\cdot, T)$ and $f_0(u, L_1(u), L_2(u)) = |u|^2_{L^2(\mathcal{O} \times (0,T))}$ with $\mathcal{O}$ being a nonempty open subset of $\mathbb{R}^n$, where y is the solution to a linear parabolic equation with the leader control u and the follower controls $(L_1(u), L_2(u))$. This paper is mainly concerned with first-order necessary conditions for solutions to the problem $(\mathbf{P})$.

The rest of this paper is organized as follows. In Section 2, we characterize a Nash equilibrium for the functionals J_1 and J_2 for any given leader control. Section 3 is devoted to giving a first-order necessary condition for solutions to the optimization problem $(\mathbf{P})$ by finite codimensionality. In Section 4, as applications, two examples on wave and quasi-linear parabolic equations are given, respectively.

2. Characterization on nash equilibrium

For a linear densely defined operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq W_1 \to W_2$ with two Hilbert spaces W_1 and W_2, we denote by $\mathcal{A}' : \mathcal{D}(\mathcal{A}') \subseteq W_2 \to W_1$ its adjoint operator, and

$$(\mathcal{A}z, w)_{W_2} = (z, \mathcal{A}'w)_{W_1}, \quad \forall\, w \in \mathcal{D}(\mathcal{A}') \ \text{ and } \ z \in \mathcal{D}(\mathcal{A}).$$

Further, we denote by $\mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \subseteq W_2^* \to W_1^*$ its conjugate operator, and

$$\langle w, \mathcal{A}z\rangle_{W_2^*,W_2} = \langle \mathcal{A}^*w, z\rangle_{W_1^*,W_1}, \quad \forall\, w \in \mathcal{D}(\mathcal{A}^*) \ \text{ and } \ z \in \mathcal{D}(\mathcal{A}).$$

Here and hereafter, we use $(\cdot,\cdot)_Z$ to denote the inner product of a Hilbert space Z, and $\langle\cdot,\cdot\rangle_{Z^*,Z}$ to denote the dual product between Z^* and Z.

Since W_1 and W_2 are two Hilbert spaces, there is an isometric isomorphism between W_i and W_i^* for $i = 1, 2$. Hence, $\mathcal{A}^*$ and $\mathcal{A}'$ only differ by an isometric isomorphism. The notation of adjoint operators for linear densely defined operators is used in this and subsequent sections, while the notation of conjugate operators will be used in Sections 3–4.

As usual, for any given $u \in \mathcal{U}$, a Nash equilibrium for the functionals J_1 and J_2 may be represented as solutions to a coupled linear system.

Proposition 2.1. *For any given $u \in \mathcal{U}$, a Nash equilibrium $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ for the functionals J_1 and J_2 is given by*

$$\bar{v}_i(\cdot) = -\frac{1}{\mu_i} C_i' p_i(\cdot), \qquad i = 1, 2, \tag{2.1}$$

where $(y, p_1, p_2) \in \mathcal{H}^3$ satisfy the following coupled system:

$$\begin{cases} y_t(t) = Ay(t) + Bu(t) - \frac{1}{\mu_1} C_1 C_1' p_1(t) - \frac{1}{\mu_2} C_2 C_2' p_2(t) & t \in (0,T), \\ p_{1,t}(t) = -A' p_1(t) - D_1' D_1 [y(t) - y_{1,d}(t)] & t \in (0,T), \\ p_{2,t}(t) = -A' p_2(t) - D_2' D_2 [y(t) - y_{2,d}(t)] & t \in (0,T), \\ y(0) = y_0, \ \ p_1(T) = p_2(T) = 0. \end{cases} \tag{2.2}$$

Proof. First, by (1.4), for a Nash equilibrium $(\bar{v}_1, \bar{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ for J_1 and J_2, it holds that

$$\begin{aligned} & J_{1,v_1}(\bar{v}_1, \bar{v}_2; u) v_1 \\ &= \int_0^T \big(D_1[y(t) - y_{1,d}(t)], D_1 y_1(t)\big)_W \mathrm{d}t + \mu_1 \int_0^T (\bar{v}_1(t), v_1(t))_{V_1} \mathrm{d}t = 0, \quad \forall\, v_1 \in \mathcal{V}_1, \end{aligned} \tag{2.3}$$

where $y \in \mathcal{H}$ satisfies

$$\begin{cases} y_t(t) = Ay(t) + Bu(t) + C_1 \bar{v}_1(t) + C_2 \bar{v}_2(t) \ t \in (0,T), \\ y(0) = y_0, \end{cases}$$

and $y_1 \in \mathcal{H}$ satisfies

$$\begin{cases} y_{1,t}(t) = Ay_1(t) + C_1 v_1(t) \ t \in (0,T), \\ y_1(0) = 0. \end{cases} \tag{2.4}$$

In the same way, by (1.5), for any $v_2 \in \mathcal{V}_2$, it holds that

$$\int_0^T \big(D_2[y(t)-y_{2,d}(t)], D_2 y_2(t)\big)_W \mathrm{d}t + \mu_2 \int_0^T (\bar{v}_2(t), v_2(t))_{V_2} \mathrm{d}t = 0, \tag{2.5}$$

where $y_2 \in \mathcal{H}$ satisfies

$$\begin{cases} y_{2,t}(t) = Ay_2(t) + C_2 v_2(t) \ t \in (0,T), \\ y_2(0) = 0. \end{cases} \tag{2.6}$$

Taking the inner product between p_1 and the first equation of (2.4), and between p_2 and the first equation of (2.6), respectively, and integrating the associated equalities in $[0,T]$, we get that

$$(y_i(T), p_i(T))_Y - \int_0^T (y_i(t), p_{i,t}(t))_Y \mathrm{d}t = \int_0^T \Big[(y_i(t), A' p_i(t))_Y + (v_i(t), C_i' p_i(t))_{V_i}\Big] \mathrm{d}t,\ i=1,2,$$

which, together with (2.2), implies that

$$\int_0^T (v_i(t), C_i' p_i(t))_{V_i} \mathrm{d}t = \int_0^T (y_i(t), D_i' D_i[y(t)-y_{i,d}(t)])_Y \mathrm{d}t, \quad i=1,2.$$

By (2.3) and (2.5), (2.1) holds and the proof is completed. □

In the following, we apply Proposition 2.1 to give the following examples on heat and wave equations.

Example 2.2. First, denote by Ω a nonempty bounded open subset of $\mathbb{R}^n$ with smooth boundary Γ. Let $\mathcal{O}$, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ be open subsets of Ω. Set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $Y = U = W = L^2(\Omega)$ and $V_i = L^2(\mathcal{O}_i)$ for $i = 1,2$. In the system (1), we set $A = \Delta$ with $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $B = \chi_{\mathcal{O}}$, $C_i = \chi_{\mathcal{O}_i}$, $D_i = \sqrt{\alpha_i}\chi_{\mathcal{O}_{i,d}}$ and $y_{i,d} \in L^2(Q)$ for $i = 1,2$, where α_1 and α_2 are any given positive constants, and χ_ω denotes the characteristic function of the set ω in $\mathbb{R}^n$. Then $A' = A$, $C_i' = C_i$ for $i = 1,2$, (1) becomes the following heat equation:

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{O}} u + \chi_{\mathcal{O}_1} v_1 + \chi_{\mathcal{O}_2} v_2 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{2.7}$$

and the functionals J_i are as follows:

$$J_i(v_1, v_2; u) = \frac{\alpha_i}{2} \int_0^T \int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 \mathrm{d}x\mathrm{d}t + \frac{\mu_i}{2} \int_0^T \int_{\mathcal{O}_i} |v_i|^2 \mathrm{d}x\mathrm{d}t, \quad i=1,2, \tag{2.8}$$

where y is the solution to (2.7).

By Proposition 2.1, for any $u \in L^2(Q)$, the associated Nash equilibrium for J_1 and J_2 is given by

$$\bar{v}_i = -\frac{1}{\mu_i} \chi_{\mathcal{O}_i} p_i, \quad i=1,2,$$

where p_1 and p_2 satisfy the following coupled system:

$$\begin{cases} y_t - \Delta y = \chi_{\mathcal{O}} u - \frac{1}{\mu_1}\chi_{\mathcal{O}_1} p_1 - \frac{1}{\mu_2}\chi_{\mathcal{O}_2} p_2 & \text{in } Q, \\ p_{1,t} + \Delta p_1 = -\alpha_1 \chi_{\mathcal{O}_{1,d}}(y - y_{1,d}) & \text{in } Q, \\ p_{2,t} + \Delta p_2 = -\alpha_2 \chi_{\mathcal{O}_{2,d}}(y - y_{2,d}) & \text{in } Q, \\ y = 0, \ p_1 = p_2 = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \ p_1(x,T) = p_2(x,T) = 0 & \text{in } \Omega. \end{cases}$$

On the other hand, set $Y = W = H_0^1(\Omega) \times L^2(\Omega)$, $U = L^2(\Omega)$ and $V_i = L^2(\mathcal{O}_i)$ for $i = 1,2$. In the system (1), we set $\mathcal{D}(A) = \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$,

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ \chi_{\mathcal{O}} \end{pmatrix}, \ C_i = \begin{pmatrix} 0 \\ \chi_{\mathcal{O}_i} \end{pmatrix} \ \text{ and } \ D_i = \begin{pmatrix} 0 & 0 \\ \sqrt{\alpha_i}\chi_{\mathcal{O}_{i,d}} & 0 \end{pmatrix}.$$

Then,

$$A' = -A = -\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad C_i' = (0 \ \ \chi_{\mathcal{O}_i}), \quad D_i' = \begin{pmatrix} 0 & -\sqrt{\alpha_i}\Delta^{-1}(\chi_{\mathcal{O}_{i,d}}\cdot) \\ 0 & 0 \end{pmatrix},$$

(1) becomes the following wave equation:

$$\begin{cases} y_{tt} - \Delta y = \chi_{\mathcal{O}} u + \chi_{\mathcal{O}_1} v_1 + \chi_{\mathcal{O}_2} v_2 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0^1(x), \ y_t(x,0) = y_0^2(x) & \text{in } \Omega, \end{cases} \tag{2.9}$$

with $(y_0^1, y_0^2)^\top \in Y$, and the functionals J_1 and J_2 are as follows:

$$J_i(v_1, v_2; u) = \frac{\alpha_i}{2}\int_0^T \int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 \mathrm{d}x\mathrm{d}t + \frac{\mu_i}{2}\int_0^T \int_{\mathcal{O}_i} |v_i|^2 \mathrm{d}x\mathrm{d}t, \quad i = 1,2,$$

where y is the solution to (2.9) and $y_{i,d} \in L^2(0,T; H_0^1(\Omega))$.

By Proposition 2.1, for any $u \in L^2(Q)$, the associated Nash equilibrium for J_1 and J_2 is given by

$$\bar{v}_i = -\frac{1}{\mu_i}\chi_{\mathcal{O}_i} p_i, \quad i = 1,2,$$

where p_1 and p_2 satisfy the following coupled system:

$$\begin{cases} y_{tt} - \Delta y = \chi_{\mathcal{O}} u - \frac{1}{\mu_1}\chi_{\mathcal{O}_1} p_1 - \frac{1}{\mu_2}\chi_{\mathcal{O}_2} p_2 & \text{in } Q, \\ p_{1,tt} - \Delta p_1 = \alpha_1 \chi_{\mathcal{O}_{1,d}}(y - y_{1,d}) & \text{in } Q, \\ p_{2,tt} - \Delta p_2 = \alpha_2 \chi_{\mathcal{O}_{2,d}}(y - y_{2,d}) & \text{in } Q, \\ y = 0, \ p_1 = p_2 = 0 & \text{on } \Sigma, \\ y(x,0) = y_0^1(x), \ y_t(x,0) = y_0^2(x) & \text{in } \Omega, \\ p_1(x,T) = p_{1,t}(x,T) = p_2(x,T) = p_{2,t}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

The above example presents the expressions of Nash equilibrium for linear heat and wave equations, which have been given in the existing works, respectively.

3. Necessary condition for solutions to an optimization problem

In this section, we use a finite codimensionality condition to establish a first-order necessary condition for solutions to the problem (**P**). Then, some verification inequalities for this condition are given.

3.1. First-order necessary condition

First, suppose that $f_0 : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{R}$ and $f : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \to Y$ are continuously Fréchet differentiable. Assume that $\bar{u} \in \mathcal{K}$ is a solution to the problem (**P**), that is,

$$f_0(\bar{u}, L_1(\bar{u}), L_2(\bar{u})) = \min \Big\{ f_0(u, L_1(u), L_2(u)) \in \mathbb{R} \ \Big|\ f(u, L_1(u), L_2(u)) \in E \text{ and } u \in \mathcal{K} \Big\},$$

where $(L_1(u), L_2(u))$ is the Nash equilibrium for the functionals J_1 and J_2 corresponding to $u \in \mathcal{K}$. By Proposition 2.1, $L_1(u) = -\frac{1}{\mu_1} C_1' p_1(\cdot)$ and $L_2(u) = -\frac{1}{\mu_2} C_2' p_2(\cdot)$, where $p_1(\cdot)$ and $p_2(\cdot)$ satisfy the coupled system (2.2).

Consider the following controlled linear system:

$$\begin{cases} \tilde{y}_t(t) = A\tilde{y}(t) + B[v(t) - \bar{u}(t)] - \frac{1}{\mu_1} C_1 C_1' \tilde{p}_1(t) - \frac{1}{\mu_2} C_2 C_2' \tilde{p}_2(t) & t \in (0,T), \\ \tilde{p}_{1,t}(t) = -A'\tilde{p}_1(t) - D_1' D_1 \tilde{y}(t) & t \in (0,T), \\ \tilde{p}_{2,t}(t) = -A'\tilde{p}_2(t) - D_2' D_2 \tilde{y}(t) & t \in (0,T), \\ \tilde{y}(0) = \tilde{p}_1(T) = \tilde{p}_2(T) = 0, \end{cases} \tag{3.1}$$

where $A, B, C_1, C_2, D_1, D_2, \mu_1$ and μ_2 are the same as those in (1) and (1.2). Define the set

$$M = \Big\{ \eta^1 \in Y \ \Big|\ \eta^1 = f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v - \bar{u}) - \sum_{i=1}^{2} \frac{1}{\mu_i} f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(C_i' \tilde{p}_i), \\ \text{for some } v \in \mathcal{K} \text{ satisfying } |v - \bar{u}|_{\mathcal{U}} \le 1 \Big\}, \tag{3.2}$$

where f_u, f_{v_1} and f_{v_2} denote the Fréchet derivative operators of $f = f(u, v_1, v_2)$ with respect to u, v_1 and v_2, respectively.

Next, we recall some concepts concerning finite codimensionality (see [13], Def. 1.5, P. 134). For a subset D_0 of Y, denote by $\overline{\text{co}}\,(D_0)$ and $\overline{\text{span}}\{D_0\}$, respectively, the closed convex hull of D_0 and closure of the linear subspace $\text{span}\{D_0\}$ spanned by D_0.

Definition 3.1. A linear subspace Y_1 of Y is called finite codimensional, if there exists $m \in \mathbb{N}$ and linearly independent $y_1, y_2, \cdots, y_m \in Y \setminus Y_1$, such that

$$\overline{Y_1} + \text{span}\{y_1, y_2, \cdots, y_m\} = Y.$$

Definition 3.2. A subset D of Y is called finite codimensional in Y, if there exists a $\tilde{y}_* \in \overline{\text{co}}\, D$, such that

(**i**) $\overline{\text{span}}\{D - \tilde{y}_*\}$ is a finite codimensional subspace of Y; and

(**ii**) $\overline{\text{co}}\,(D - \tilde{y}_*)$ has at least one interior point in $\overline{\text{span}}\{D - \tilde{y}_*\}$.

The main result of this section is the following necessary condition on the solution $\bar{u}$ to (**P**).

Theorem 3.3. *Assume that the set $M - E$ is finite codimensional in Y. Then there exists a non-zero pair $(\psi_0, \psi) \in \mathbb{R} \times Y^*$ and a sufficiently large positive constant μ^*, such that for any $\mu_1, \mu_2 \ge \mu^*$, the following* *necessary condition for the solution $\bar{u}$ to the problem* $(\mathbf{P})$ *holds:*

$$\psi_0\eta^0 + \langle\psi, \eta^1\rangle_{Y^*,Y} \geq 0, \quad \forall\ v \in \mathcal{K} \tag{3.3}$$

with

$$\begin{cases} \eta^0 = f_{0,u}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-\bar{u}) - \sum\limits_{i=1}^{2} \dfrac{1}{\mu_i} f_{0,v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\big(C_i'\tilde{p}_i\big), \\ \eta^1 = f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-\bar{u}) - \sum\limits_{i=1}^{2} \dfrac{1}{\mu_i} f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\big(C_i'\tilde{p}_i\big), \end{cases}$$

where $(\tilde{p}_1, \tilde{p}_2) \in \mathcal{H}^2$ *are the solutions to the system* (3.1). *In addition, it holds that*

$$\langle\psi,\ e - f(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\rangle_{Y^*,Y} \leq 0, \quad \forall\ e \in E. \tag{3.4}$$

Proof. Similar to the arguments in [13, 14] and [15], we give a sketch of the proof and it is divided into three parts.

Step 1. For any $\varepsilon \in (0,1)$ and $u \in \mathcal{K}$, set

$$\hat{f}_0(u, L_1(u), L_2(u)) = f_0(u, L_1(u), L_2(u)) - f_0(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))$$

and

$$J_\varepsilon(u) = \Big[\big|\mathrm{dist}\big(f(u, L_1(u), L_2(u)), E\big)\big|^2 + \big|\hat{f}_0(u, L_1(u), L_2(u)) + \varepsilon\big|^2\Big]^{\frac{1}{2}},$$

where $\mathrm{dist}(\cdot, E)$ denotes the distance function, and for any $y_* \in Y$, $\mathrm{dist}(y_*, E) = \inf\big\{\ |y_* - e|_Y \in \mathbb{R} \mid e \in E\ \big\}$. Then $J_\varepsilon(\cdot)$ is continuous on $\mathcal{K}$ and $J_\varepsilon(\bar{u}) = \varepsilon \leq \inf\limits_{u\in\mathcal{K}} J_\varepsilon(u) + \varepsilon$.

By the Ekeland variational principle, there exists a $u_\varepsilon \in \mathcal{K}$, such that $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\bar{u})$, $|u_\varepsilon - \bar{u}|_{\mathcal{U}} \leq \sqrt{\varepsilon}$, and

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(v) + \sqrt{\varepsilon}|u_\varepsilon - v|_{\mathcal{U}}, \quad \forall\ v \in \mathcal{K}. \tag{3.5}$$

For any $\rho \in (0,1)$ and $v \in \mathcal{K}$, set $u_\varepsilon^\rho = u_\varepsilon + \rho(v - u_\varepsilon)$. Then $u_\varepsilon^\rho \in \mathcal{K}$, and by the definition of J_ε, it holds that

$$\begin{aligned} &J_\varepsilon(u_\varepsilon^\rho) - J_\varepsilon(u_\varepsilon) \\ &= \frac{\big|\hat{f}_0(u_\varepsilon^\rho, L_1(u_\varepsilon^\rho), L_2(u_\varepsilon^\rho)) + \varepsilon\big|^2 - \big|\hat{f}_0(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)) + \varepsilon\big|^2}{J_\varepsilon(u_\varepsilon^\rho) + J_\varepsilon(u_\varepsilon)} \\ &\quad + \frac{\big|\mathrm{dist}\big(f(u_\varepsilon^\rho, L_1(u_\varepsilon^\rho), L_2(u_\varepsilon^\rho)), E\big)\big|^2 - \big|\mathrm{dist}\big(f(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)), E\big)\big|^2}{J_\varepsilon(u_\varepsilon^\rho) + J_\varepsilon(u_\varepsilon)}. \end{aligned} \tag{3.6}$$

Notice that for $i = 1, 2$,

$$\lim_{\rho\to 0} \frac{1}{\rho}\big[L_i(u_\varepsilon^\rho) - L_i(u_\varepsilon)\big] = \lim_{\rho\to 0} \frac{1}{\rho}\Big[\Big(-\frac{1}{\mu_i}C_i' p_i^{\varepsilon,\rho}\Big) - \Big(-\frac{1}{\mu_i}C_i' p_i^{\varepsilon}\Big)\Big] = -\frac{1}{\mu_i}C_i'\tilde{p}_i^{\varepsilon},$$

where $p_i^{\varepsilon,\rho}$ and p_i^ε denote the solutions to (2.2) with $u=u_\varepsilon^\rho$ and $u=u_\varepsilon$, respectively, and $(\tilde p_1^\varepsilon,\tilde p_2^\varepsilon)\in\mathcal{H}^2$ satisfy

$$\begin{cases}\tilde y_t^\varepsilon(t)=A\tilde y^\varepsilon(t)+B[v(t)-u_\varepsilon(t)]-\sum_{i=1}^{2}\dfrac{1}{\mu_i}C_iC_i'\tilde p_i^\varepsilon(t) & t\in(0,T),\\ \tilde p_{1,t}^\varepsilon(t)=-A'\tilde p_1^\varepsilon(t)-D_1'D_1\tilde y^\varepsilon(t) & t\in(0,T),\\ \tilde p_{2,t}^\varepsilon(t)=-A'\tilde p_2^\varepsilon(t)-D_2'D_2\tilde y^\varepsilon(t) & t\in(0,T),\\ \tilde y^\varepsilon(0)=\tilde p_1^\varepsilon(T)=\tilde p_2^\varepsilon(T)=0.\end{cases}\tag{3.7}$$

Hence, we have that

$$\begin{aligned}&\lim_{\rho\to0}\frac{1}{\rho}\Big[\hat f_0(u_\varepsilon^\rho,L_1(u_\varepsilon^\rho),L_2(u_\varepsilon^\rho))-\hat f_0(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\Big]\\&=\hat f_{0,u}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)-\sum_{i=1}^{2}\frac{1}{\mu_i}\hat f_{0,v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big).\end{aligned}$$

Next, by Lipschitz continuity of distance function and weak* compactness of the subdifferential $\partial\mathrm{dist}\big(f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon)),E\big)$ for $\mathrm{dist}(\cdot,E)$ at $f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))$, there is a $\psi_\varepsilon\in Y^*$ with $|\psi_\varepsilon|_{Y^*}=1$, such that

$$\begin{aligned}&\lim_{\rho\to0}\frac{1}{\rho}\Big[\mathrm{dist}\big(f(u_\varepsilon^\rho,L_1(u_\varepsilon^\rho),L_2(u_\varepsilon^\rho)),E\big)-\mathrm{dist}\big(f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon)),E\big)\Big]\\&=\langle\psi_\varepsilon,f_u(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)-\sum_{i=1}^{2}\frac{1}{\mu_i}f_{v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big)\rangle_{Y^*,Y}.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\lim_{\rho\to0}\frac{J_\varepsilon(u_\varepsilon^\rho)-J_\varepsilon(u_\varepsilon)}{\rho}\\&=\frac{\hat f_0(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))+\varepsilon}{J_\varepsilon(u_\varepsilon)}\cdot\Big[\hat f_{0,u}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)\\&\qquad\qquad-\sum_{i=1}^{2}\frac{1}{\mu_i}\hat f_{0,v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big)\Big]\\&\quad+\frac{\mathrm{dist}\big(f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon)),E\big)}{J_\varepsilon(u_\varepsilon)}\cdot\langle\psi_\varepsilon,f_u(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)\\&\qquad\qquad-\sum_{i=1}^{2}\frac{1}{\mu_i}f_{v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big)\rangle_{Y^*,Y}.\end{aligned}\tag{3.8}$$

Write

$$\begin{aligned}&a_\varepsilon=\frac{\hat f_0(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))+\varepsilon}{J_\varepsilon(u_\varepsilon)}\in\mathbb{R},\quad b_\varepsilon=\frac{\mathrm{dist}\big(f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon)),E\big)\psi_\varepsilon}{J_\varepsilon(u_\varepsilon)}\in Y^*,\\&\eta_\varepsilon^0=\hat f_{0,u}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)-\sum_{i=1}^{2}\frac{1}{\mu_i}\hat f_{0,v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big)\in\mathbb{R},\\&\text{and}\quad\eta_\varepsilon^1=f_u(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)-\sum_{i=1}^{2}\frac{1}{\mu_i}f_{v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\big(C_i'\tilde p_i^\varepsilon\big)\in Y.\end{aligned}$$

From (3.5) and (3.8), we conclude that

$$a_\varepsilon\eta_\varepsilon^0+\langle b_\varepsilon,\eta_\varepsilon^1\rangle_{Y^*,Y}\ge-\sqrt{\varepsilon}|v-u_\varepsilon|_{\mathcal{U}}\quad\text{with}\quad |a_\varepsilon|+|b_\varepsilon|_{Y^*}=1. \tag{3.9}$$

Meanwhile, since $\psi_\varepsilon\in\partial\text{dist}\big(f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon)),E\big)$, it holds that

$$\langle b_\varepsilon,\ e-f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\rangle_{Y^*,Y}\le 0,\quad \forall\, e\in E. \tag{3.10}$$

Step 2. Note that $|a_\varepsilon|+|b_\varepsilon|_{Y^*}=1$. Hence, there exists a subsequence of $\{(a_\varepsilon,b_\varepsilon)\}_{\varepsilon>0}$ (still denoted by itself) and a pair $(\psi_0,\psi)\in\mathbb{R}\times Y^*$, such that as $\varepsilon\to 0$,

$$a_\varepsilon\to\psi_0\ \text{ in }\ \mathbb{R}\quad\text{ and }\quad b_\varepsilon\to\psi\text{ weakly}^*\text{ in }Y^*.$$

Furthermore, since $f_0:\mathcal{U}\times\mathcal{V}_1\times\mathcal{V}_2\to\mathbb{R}$ and $f:\mathcal{U}\times\mathcal{V}_1\times\mathcal{V}_2\to Y$ are continuously Fréchet differentiable, and $u_\varepsilon\to\bar u$ in $\mathcal{U}$ as $\varepsilon\to 0$, there exists a pair $(\eta^0,\eta^1)\in\mathbb{R}\times Y$, such that

$$\eta_\varepsilon^0\to\eta^0\ \text{ in }\ \mathbb{R}\quad\text{ and }\quad \eta_\varepsilon^1\to\eta^1\text{ in }Y,$$

where

$$\begin{aligned}\eta^0&=f_{0,u}(\bar u,L_1(\bar u),L_2(\bar u))(v-\bar u)-\sum_{i=1}^2\frac{1}{\mu_i}f_{0,v_i}(\bar u,L_1(\bar u),L_2(\bar u))\big(C_i'\tilde p_i\big),\\ \eta^1&=f_u(\bar u,L_1(\bar u),L_2(\bar u))(v-\bar u)-\sum_{i=1}^2\frac{1}{\mu_i}f_{v_i}(\bar u,L_1(\bar u),L_2(\bar u))\big(C_i'\tilde p_i\big),\end{aligned}$$

and $(\tilde p_1,\tilde p_2)\in\mathcal{H}^2$ satisfy the coupled system (3.1) for $v\in\mathcal{K}$.

By (3.9) and (3.10), we obtain that

$$\psi_0\eta^0+\langle\psi,\eta^1\rangle_{Y^*,Y}\ge 0$$

and

$$\langle\psi,\ e-f(\bar u,L_1(\bar u),L_2(\bar u))\rangle_{Y^*,Y}\le 0,\quad\forall\, e\in E.$$

Step 3. Finally, we show $(\psi_0,\psi)\ne(0,0)$ by finite codimensionality of the set $M-E$.

If $\psi_0=0$, there exists a $\delta_0>0$, such that for sufficiently small $\varepsilon>0$, $|b_\varepsilon|_{Y^*}\ge\delta_0$. For any $e\in E$ and $v\in\mathcal{K}$ with $|v-u_\varepsilon|_{\mathcal{U}}\le 1$, by (3.9) and (3.10), it holds that

$$\begin{aligned}&\langle b_\varepsilon,\eta^1-(e-f(\bar u,L_1(\bar u),L_2(\bar u)))\rangle_{Y^*,Y}\\&=\langle b_\varepsilon,\eta_\varepsilon^1\rangle_{Y^*,Y}+\langle b_\varepsilon,\eta^1-\eta_\varepsilon^1\rangle_{Y^*,Y}+\langle b_\varepsilon,f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))-e\rangle_{Y^*,Y}\\&\quad-\langle b_\varepsilon,f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))-f(\bar u,L_1(\bar u),L_2(\bar u))\rangle_{Y^*,Y}\\&\ge-\sqrt{\varepsilon}|v-u_\varepsilon|_{\mathcal{U}}-a_\varepsilon\eta_\varepsilon^0+\langle b_\varepsilon,\eta^1-\eta_\varepsilon^1\rangle_{Y^*,Y}\\&\quad-\langle b_\varepsilon,f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))-f(\bar u,L_1(\bar u),L_2(\bar u))\rangle_{Y^*,Y}\\&\ge-\sqrt{\varepsilon}-|a_\varepsilon\eta_\varepsilon^0|-|\eta^1-\eta_\varepsilon^1|_Y-|f(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))-f(\bar u,L_1(\bar u),L_2(\bar u))|_Y.\end{aligned} \tag{3.11}$$

In the following, we give estimates with respect to $|\eta_\varepsilon^0|$ and $|\eta^1-\eta_\varepsilon^1|_Y$, respectively. First,

$$\begin{aligned}|\eta_\varepsilon^0|&\le|f_{0,u}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(v-u_\varepsilon)|+\sum_{i=1}^2\frac{1}{\mu_i}\Big|f_{0,v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))(C_i'\tilde p_i^\varepsilon)\Big|\\&\le|f_{0,u}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))|_{\mathcal{U}^*}+\sum_{i=1}^2\frac{1}{\mu_i}\Big|f_{0,v_i}(u_\varepsilon,L_1(u_\varepsilon),L_2(u_\varepsilon))\Big|_{\mathcal{V}_i^*}|C_i'\tilde p_i^\varepsilon|_{\mathcal{V}_i}.\end{aligned}$$

Notice that $u_\varepsilon \to \bar{u}$ in $\mathcal{U}$ as $\varepsilon \to 0$, and f_0 is continuously Fréchet differentiable. Hence, we have that $|f_{0,u}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))|_{\mathcal{U}^*}$ and $\big|f_{0,v_i}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))\big|_{\mathcal{V}_i^*}$ are bounded for sufficiently small ε. Further, for the mild solution $(\tilde{y}^\varepsilon, \tilde{p}_1^\varepsilon, \tilde{p}_2^\varepsilon)$ to (3.7), it holds that

$$\begin{aligned}
&|\tilde{p}_1^\varepsilon|_{C([0,T];Y)} + |\tilde{p}_2^\varepsilon|_{C([0,T];Y)} \le C|\tilde{y}^\varepsilon|_{L^1(0,T;Y)},\\
&\text{and } |\tilde{y}^\varepsilon|_{C([0,T];Y)} \le C\big(|v-u_\varepsilon|_{L^2(0,T;U)} + \sum_{i=1}^{2}\frac{1}{\mu_i}|\tilde{p}_i^\varepsilon|_{L^1(0,T;Y)}\big),
\end{aligned}$$

here and hereafter, C denotes a positive constant, depending only on T, B, C_1, C_2, D_1 and D_2, which may be different from one place to another. Hence, when μ_1 and μ_2 are sufficiently large,

$$|\tilde{p}_1^\varepsilon|_{C([0,T];Y)} + |\tilde{p}_2^\varepsilon|_{C([0,T];Y)} \le C|v-u_\varepsilon|_{\mathcal{U}} \le C. \tag{3.12}$$

This implies the boundedness of $\sum_{i=1}^{2}|C_i'\tilde{p}_i^\varepsilon|_{\mathcal{V}_i}$ and therefore, $|\eta_0^\varepsilon|$ is bounded for sufficiently small ε.

On the other hand, similar to (3.12), we get that

$$|\tilde{p}_1^\varepsilon - \tilde{p}_1|_{C([0,T];Y)} + |\tilde{p}_2^\varepsilon - \tilde{p}_2|_{C([0,T];Y)} \le C|\bar{u} - u_\varepsilon|_{\mathcal{U}}.$$

By the definitions of η^1 and η_ε^1,

$$\begin{aligned}
&|\eta^1 - \eta_\varepsilon^1|_Y\\
&\le |f_u(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))(v-u_\varepsilon) - f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-\bar{u})|_Y\\
&\quad + \sum_{i=1}^{2}\frac{1}{\mu_i}\Big|f_{v_i}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))(C_i'\tilde{p}_i^\varepsilon) - f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(C_i'\tilde{p}_i)\Big|_Y\\
&\le |f_u(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))(v-u_\varepsilon) - f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-u_\varepsilon)|_Y\\
&\quad + |f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-u_\varepsilon) - f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(v-\bar{u})|_Y\\
&\quad + \sum_{i=1}^{2}\frac{1}{\mu_i}\Big|f_{v_i}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon))(C_i'\tilde{p}_i^\varepsilon) - f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(C_i'\tilde{p}_i^\varepsilon)\Big|_Y\\
&\quad + \sum_{i=1}^{2}\frac{1}{\mu_i}\Big|f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(C_i'\tilde{p}_i^\varepsilon) - f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))(C_i'\tilde{p}_i)\Big|_Y\\
&\le |f_u(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)) - f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))|_{\mathcal{L}(\mathcal{U};Y)}|v-u_\varepsilon|_{\mathcal{U}}\\
&\quad + |f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))|_{\mathcal{L}(\mathcal{U};Y)}|u_\varepsilon - \bar{u}|_{\mathcal{U}}\\
&\quad + \sum_{i=1}^{2}\frac{1}{\mu_i}\big|f_{v_i}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)) - f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\big|_{\mathcal{L}(\mathcal{V}_i;Y)}\big|C_i'\tilde{p}_i^\varepsilon\big|_{\mathcal{V}_i}\\
&\quad + \sum_{i=1}^{2}\frac{1}{\mu_i}\big|f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\big|_{\mathcal{L}(\mathcal{V}_i;Y)}\big|C_i'(\tilde{p}_i^\varepsilon - \tilde{p}_i)\big|_{\mathcal{V}_i}.
\end{aligned}$$

This implies that

$$
\begin{aligned}
&|\eta^1 - \eta^1_\varepsilon|_Y \\
&\le |f_u(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)) - f_u(\bar u, L_1(\bar u), L_2(\bar u))|_{\mathcal{L}(\mathcal{U};Y)} + |f_u(\bar u, L_1(\bar u), L_2(\bar u))|_{\mathcal{L}(\mathcal{U};Y)}|u_\varepsilon - \bar u|_{\mathcal{U}} \\
&\quad + C\sum_{i=1}^{2}\frac{1}{\mu_i}\big|f_{v_i}(u_\varepsilon, L_1(u_\varepsilon), L_2(u_\varepsilon)) - f_{v_i}(\bar u, L_1(\bar u), L_2(\bar u))\big|_{\mathcal{L}(\mathcal{V}_i;Y)} \\
&\quad + C\sum_{i=1}^{2}\frac{1}{\mu_i}\big|f_{v_i}(\bar u, L_1(\bar u), L_2(\bar u))\big|_{\mathcal{L}(\mathcal{V}_i;Y)}|u_\varepsilon - \bar u|_{\mathcal{U}}.
\end{aligned}
$$

Consequently, by (3.11) and the above estimates, we obtain that

$$
\langle b_\varepsilon, \eta^1 - (e - f(\bar u, L_1(\bar u), L_2(\bar u)))\rangle_{Y^*,Y} \ge -\delta_\varepsilon,
$$

where $\delta_\varepsilon > 0$ satisfies that $\delta_\varepsilon \to 0$, as $\varepsilon \to 0$.

By [13], Lemma 3.6, p. 142, since $M - E$ is finite codimensional in Y, we have that $\psi \neq 0$. The proof is finished. □

3.2. Finite codimensionality condition

The key of applying Theorem 3.3 is to verify finite codimensionality of the set $M - E$. If the constraint set E is finite codimensional in Y, $M - E$ is also finite codimensional. Otherwise, we give some equivalent conditions on finite codimensionality of the set M in the case of $\mathcal{K} = \mathcal{U}$. For this purpose, define the linear operator $F : \mathcal{U} \to Y$ as follows:

$$
F(w) = f_u(\bar u, L_1(\bar u), L_2(\bar u))w - \sum_{i=1}^{2}\frac{1}{\mu_i}f_{v_i}(\bar u, L_1(\bar u), L_2(\bar u))\big(C_i'\tilde p_i\big), \tag{3.13}
$$

where $\tilde p_1$ and $\tilde p_2$ satisfy the following coupled system:

$$
\begin{cases}
\tilde y_t(t) = A\tilde y(t) + Bw(t) - \dfrac{1}{\mu_1}C_1C_1'\tilde p_1(t) - \dfrac{1}{\mu_2}C_2C_2'\tilde p_2(t) & t \in (0,T),\\
\tilde p_{1,t}(t) = -A'\tilde p_1(t) - D_1'D_1\tilde y(t) & t \in (0,T),\\
\tilde p_{2,t}(t) = -A'\tilde p_2(t) - D_2'D_2\tilde y(t) & t \in (0,T),\\
\tilde y(0) = \tilde p_1(T) = \tilde p_2(T) = 0,
\end{cases} \tag{3.14}
$$

with μ_1 and μ_2 being sufficiently large.

It is easy to check that $F \in \mathcal{L}(\mathcal{U};Y)$ and in (3.2), $M = \big\{\ F(w) \in Y \ \big|\ |w|_{\mathcal{U}} \le 1\ \big\}$.

By [15, 16] and [17], we have the following equivalent assertions on finite codimensionality.

Lemma 3.4. *The following statements are equivalent:*

1. *The set M is finite codimensional in Y;*
2. *The range Im (F) of F is a finite codimensional closed subspace of Y;*
3. *There exists a Banach space X and a compact operator $\mathcal{G}$ from Y^* to X, such that*

$$
|z|_{Y^*} \le C\big(|F^*(z)|_{\mathcal{U}^*} + |\mathcal{G}(z)|_X\big), \quad \forall\ z \in Y^*, \tag{3.15}
$$

where C is a positive constant independent of $z \in Y^*$.

By the definition of the bounded linear operator F, the conjugate operator $F^* \in \mathcal{L}(Y^*;\mathcal{U}^*)$ of F satisfies that for any $z \in Y^*$ and $w \in \mathcal{U}$,

$$\begin{aligned}&\langle F^*(z), w\rangle_{\mathcal{U}^*,\,\mathcal{U}} = \langle z, F(w)\rangle_{Y^*,Y}\\&= \langle z, f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))w - \sum_{i=1}^{2}\frac{1}{\mu_i} f_{v_i}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))\big(C_i'\tilde{p}_i\big)\rangle_{Y^*,Y}.\end{aligned}$$

Hence, we get

$$F^*(z) = f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^* z + B^*\eta,$$

where $\eta \in C([0,T];Y^*)$ satisfies

$$\begin{cases}\eta_t(t) + A^*\eta(t) = -D_1^*(D_1')^*w_1(t) - D_2^*(D_2')^*w_2(t) & t\in(0,T),\\ w_{1,t}(t) - (A')^*w_1(t) = -\frac{1}{\mu_1}(C_1')^*C_1^*\eta(t) - \frac{1}{\mu_1}(C_1')^* f_{v_1}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^* z & t\in(0,T),\\ w_{2,t}(t) - (A')^*w_2(t) = -\frac{1}{\mu_2}(C_2')^*C_2^*\eta(t) - \frac{1}{\mu_2}(C_2')^* f_{v_2}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^* z & t\in(0,T),\\ \eta(T) = w_1(0) = w_2(0) = 0.\end{cases} \tag{3.16}$$

By (3.15) in Lemma 3.4, we have the following equivalence results.

Corollary 3.5. *The following statements are equivalent:*

(1) *The set M is finite codimensional in Y;*
(2) *There exists a Banach space X and a compact operator $\mathcal{G}$ from Y^* to X, such that*

$$|z|_{Y^*} \le C\Big(\big|\big[f_u(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^* + B^*\widetilde{K}\big]z\big|_{L^2(0,T;U^*)} + |\mathcal{G}(z)|_X\Big), \quad \forall\, z \in Y^*, \tag{3.17}$$

where $\widetilde{K} : Y^* \to L^2(0,T;Y^*)$ is defined by $\widetilde{K}(z) = \eta(\cdot)$ for the solution η to (3.16).

Notice that the corresponding conclusions in this section still hold true, in the case that (1) has a bounded potential. In such a case, (1) becomes

$$\begin{cases} y_t(t) = Ay(t) + E(t)y(t) + Bu(t) + C_1v_1(t) + C_2v_2(t) & t\in(0,T),\\ y(0) = y_0,\end{cases} \tag{3.18}$$

where $E(\cdot) \in L^\infty(0,T;\mathcal{L}(Y;Y))$, and (3.1) becomes

$$\begin{cases}\tilde{y}_t(t) = A\tilde{y}(t) + E(t)\tilde{y}(t) + B[v(t) - \bar{u}(t)] - \frac{1}{\mu_1}C_1C_1'\tilde{p}_1(t) - \frac{1}{\mu_2}C_2C_2'\tilde{p}_2(t) & t\in(0,T),\\ \tilde{p}_{1,t}(t) = -A'\tilde{p}_1(t) - E(t)'\tilde{p}_1(t) - D_1'D_1\tilde{y}(t) & t\in(0,T),\\ \tilde{p}_{2,t}(t) = -A'\tilde{p}_2(t) - E(t)'\tilde{p}_2(t) - D_2'D_2\tilde{y}(t) & t\in(0,T),\\ \tilde{y}(0) = \tilde{p}_1(T) = \tilde{p}_2(T) = 0.\end{cases} \tag{3.19}$$

Further, (3.16) becomes

$$\begin{cases} \eta_t(t) + A^*\eta(t) = -E(t)^*\eta - D_1^*(D_1')^*w_1(t) - D_2^*(D_2')^*w_2(t) & t \in (0,T), \\ w_{1,t}(t) - (A')^*w_1(t) = (E(t)')^*w_1 - \frac{1}{\mu_1}(C_1')^*C_1^*\eta(t) & \\ \qquad\qquad -\frac{1}{\mu_1}(C_1')^*f_{v_1}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^*z & t \in (0,T), \\ w_{2,t}(t) - (A')^*w_2(t) = (E(t)')^*w_2 - \frac{1}{\mu_2}(C_2')^*C_2^*\eta(t) & \\ \qquad\qquad -\frac{1}{\mu_2}(C_2')^*f_{v_2}(\bar{u}, L_1(\bar{u}), L_2(\bar{u}))^*z & t \in (0,T), \\ \eta(T) = w_1(0) = w_2(0) = 0. \end{cases} \tag{3.20}$$

In this case, the associated results in Theorem 3.3 and Corollary 3.5 may be applied to some controlled systems with lower-order terms, in which the coefficients in potential may depend the spacial and time variables.

4. Applications of main results

As applications of Theorem 3.3 and Corollary 3.5, in this section, we give two examples on multi-objective control problems.

4.1. Example 4.1

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty bounded open set with smooth boundary Γ for $n \in \mathbb{N}$. Set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $Y = W = H_0^1(\Omega) \times L^2(\Omega)$, $U = L^2(\Omega)$ and $V_i = L^2(\mathcal{O}_i)$ for $i = 1,2$. In the system (3.18), we set $\mathcal{D}(A) = \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$,

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \chi_{\mathcal{O}} \end{pmatrix}, \quad C_i = \begin{pmatrix} 0 \\ \chi_{\mathcal{O}_i} \end{pmatrix},$$

$$E(t) = \begin{pmatrix} 0 & 0 \\ -a(\cdot,t) & 0 \end{pmatrix} \quad \text{and} \quad D_i = \begin{pmatrix} 0 & 0 \\ \sqrt{\alpha_i}\chi_{\mathcal{O}_{i,d}} & 0 \end{pmatrix},$$

where $\mathcal{O}$, $\mathcal{O}_i$, $\mathcal{O}_{i,d}$ and α_i are the same as those in Example 2.2, and $a \in C([0,T]; L^\infty(\Omega))$. Then, $Y^* = H^{-1}(\Omega) \times L^2(\Omega)$, $U^* = L^2(\Omega)$, $C_i' = C_i^* = (0 \ \ \chi_{\mathcal{O}_i})$,

$$D_i' = \begin{pmatrix} 0 & -\sqrt{\alpha_i}\Delta^{-1}(\chi_{\mathcal{O}_{i,d}}\cdot) \\ 0 & 0 \end{pmatrix}, \quad E(t)' = \begin{pmatrix} 0 & \Delta^{-1}(a(\cdot,t)\cdot) \\ 0 & 0 \end{pmatrix},$$

$$(D_i')^* = \begin{pmatrix} 0 & 0 \\ -\sqrt{\alpha_i}\chi_{\mathcal{O}_{i,d}}\Delta^{-1} & 0 \end{pmatrix} \quad \text{and} \quad (E(t)')^* = \begin{pmatrix} 0 & 0 \\ a(\cdot,t)\Delta^{-1} & 0 \end{pmatrix}.$$

Similar to Example 2.2, (3.18) becomes

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = \chi_{\mathcal{O}}u + \chi_{\mathcal{O}_1}v_1 + \chi_{\mathcal{O}_2}v_2 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0^1(x), \ \ y_t(x,0) = y_0^2(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

where (u, v_1, v_2) is a triple of control variables, (y, y_t) is the state variable and $(y_0^1, y_0^2)^\top \in Y$. The functionals J_i are as follows:

$$J_i(v_1, v_2; u) = \frac{\alpha_i}{2}\int_0^T \int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 \mathrm{d}x\mathrm{d}t + \frac{\mu_i}{2}\int_0^T \int_{\mathcal{O}_i} |v_i|^2 \mathrm{d}x\mathrm{d}t, \quad i = 1, 2,$$

where y is the solution to (4.1), $y_{1,d}, y_{2,d} \in L^2(0, T; H_0^1(\Omega))$ and $\mu_1, \mu_2 > 0$. Moreover, for any $u \in L^2(Q)$, the Nash equilibrium $(L_1(u), L_2(u))$ for the functionals J_1 and J_2 is given by

$$L_1(u) = -\frac{1}{\mu_1}\chi_{\mathcal{O}_1} p_1(\cdot) \quad \text{and} \quad L_2(u) = -\frac{1}{\mu_2}\chi_{\mathcal{O}_2} p_2(\cdot),$$

where p_1 and p_2 satisfy the following coupled system:

$$\begin{cases} y_{tt} - \Delta y + a(x,t)y = \chi_{\mathcal{O}} u - \frac{1}{\mu_1}\chi_{\mathcal{O}_1} p_1 - \frac{1}{\mu_2}\chi_{\mathcal{O}_2} p_2 & \text{in } Q, \\ p_{1,tt} - \Delta p_1 + a(x,t)p_1 = \alpha_1 \chi_{\mathcal{O}_{1,d}}(y - y_{1,d}) & \text{in } Q, \\ p_{2,tt} - \Delta p_2 + a(x,t)p_2 = \alpha_2 \chi_{\mathcal{O}_{2,d}}(y - y_{2,d}) & \text{in } Q, \\ y = 0, \ p_1 = p_2 = 0 & \text{on } \Sigma, \\ y(x,0) = y_0^1(x), \ y_t(x,0) = y_0^2(x) & \text{in } \Omega, \\ p_1(x,T) = p_{1,t}(x,T) = p_2(x,T) = p_{2,t}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

Furthermore, consider the following optimization problem:

$$f_0(u, L_1(u), L_2(u)) = \int_Q g\big(x, t, y(x,t), u(x,t)\big)\mathrm{d}x\mathrm{d}t, \quad \forall\, u \in L^2(Q),$$

where y is the solution to (4.1) associated to $(u, v_1, v_2) = (u, L_1(u), L_2(u))$ or equivalently, the solution to (4.2) associated to u, and $g : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function, such that for any $u \in L^2(Q)$, $g(\cdot, \cdot, y(\cdot, \cdot; u), u(\cdot, \cdot)) \in L^1(Q)$. Set

$$f(u, L_1(u), L_2(u)) = \big(y(\cdot, T), y_t(\cdot, T)\big)^\top$$

and $E = \{(y_0^d, y_1^d)\}$ for any given $(y_0^d, y_1^d)^\top \in Y$.

The above optimization problem $(\mathbf{P})$ is indeed an optimal control problem for the coupled system (4.2) with terminal state constraints. In this optimal control problem, the set of admissible controls is

$$\mathcal{U}_{ad} = \Big\{ \ u \in L^2(Q) \ \Big| \ \text{the solution component } y \text{ to (4.2) satisfies } (y(\cdot, T), y_t(\cdot, T)) = (y_0^d, y_1^d) \ \Big\}.$$

Assume that $\bar{u} \in \mathcal{U}_{ad}$ is an optimal control and $(\bar{y}, \bar{p}_1, \bar{p}_2)$ are the associated solutions to (4.2).

By Theorem 3.3, under the assumption on finite codimensionality of the set M (see (3.2)), there exists a non-zero triple of $(\psi_0, \psi^1, \psi^2) \in \mathbb{R} \times H^{-1}(\Omega) \times L^2(\Omega)$, such that for sufficiently large μ_1 and μ_2, it holds that

$$\psi_0 \eta^0 + \langle \psi^1, \eta_1^1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\psi^2, \eta_2^1)_{L^2(\Omega)} \geq 0, \quad \forall\, v \in L^2(Q), \tag{4.3}$$

where

$$\eta^0 = \int_Q \Big\{ g_u(x, t, \bar{u}(x,t), \bar{y}(x,t))[v(x,t) - \bar{u}(x,t)] + g_y(x, t, \bar{u}(x,t), \bar{y}(x,t))\tilde{y}(x,t) \Big\}\mathrm{d}x\mathrm{d}t,$$

$$\eta_1^1 = \tilde{y}(\cdot, T) \qquad \text{and} \qquad \eta_2^1 = \tilde{y}_t(\cdot, T),$$

and $\tilde{y}$ satisfies the following coupled system:

$$\begin{cases} \tilde{y}_{tt} - \Delta \tilde{y} + a(x,t)\tilde{y} = \chi_{\mathcal{O}}(v - \bar{u}) - \frac{1}{\mu_1}\chi_{\mathcal{O}_1}\tilde{p}_1 - \frac{1}{\mu_2}\chi_{\mathcal{O}_2}\tilde{p}_2 & \text{in } Q, \\ \tilde{p}_{1,tt} - \Delta \tilde{p}_1 + a(x,t)\tilde{p}_1 = \alpha_1 \chi_{\mathcal{O}_{1,d}}\tilde{y} & \text{in } Q, \\ \tilde{p}_{2,tt} - \Delta \tilde{p}_2 + a(x,t)\tilde{p}_2 = \alpha_2 \chi_{\mathcal{O}_{2,d}}\tilde{y} & \text{in } Q, \\ \tilde{y} = 0,\ \tilde{p}_1 = \tilde{p}_2 = 0 & \text{on } \Sigma, \\ \tilde{y}(x,0) = \tilde{y}_t(x,0) = 0 & \text{in } \Omega, \\ \tilde{p}_1(x,T) = \tilde{p}_{1,t}(x,T) = \tilde{p}_2(x,T) = \tilde{p}_{2,t}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.4}$$

Furthermore, introduce the following coupled system:

$$\begin{cases} \tilde{\psi}_{tt} - \Delta \tilde{\psi} + a(x,t)\tilde{\psi} = -\psi_0 g_y(x,t,\bar{u}(x,t),\bar{y}(x,t)) + \sum\limits_{i=1}^{2} \alpha_i \chi_{\mathcal{O}_{i,d}}\tilde{\gamma}_i & \text{in } Q, \\ \tilde{\gamma}_{1,tt} - \Delta \tilde{\gamma}_1 + a(x,t)\tilde{\gamma}_1 = -\frac{1}{\mu_1}\chi_{\mathcal{O}_1}\tilde{\psi} & \text{in } Q, \\ \tilde{\gamma}_{2,tt} - \Delta \tilde{\gamma}_2 + a(x,t)\tilde{\gamma}_2 = -\frac{1}{\mu_2}\chi_{\mathcal{O}_2}\tilde{\psi} & \text{in } Q, \\ \tilde{\psi} = 0,\ \tilde{\gamma}_1 = \tilde{\gamma}_2 = 0 & \text{on } \Sigma, \\ \tilde{\psi}(x,T) = -\psi^2,\ \tilde{\psi}_t(x,T) = \psi^1 & \text{in } \Omega, \\ \tilde{\gamma}_1(x,0) = \tilde{\gamma}_{1,t}(x,0) = \tilde{\gamma}_2(x,0) = \tilde{\gamma}_{2,t}(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

By (4.3)–(4.5), we obtain that

$$\int_Q \Big\{ [\psi_0 g_u(x,t,\bar{u}(x,t),\bar{y}(x,t)) - \chi_{\mathcal{O}}\tilde{\psi}(x,t)][v(x,t) - \bar{u}(x,t)] \Big\} \mathrm{d}x\mathrm{d}t \geq 0,\ \forall\ v \in L^2(Q),$$

which implies

$$\psi_0 g_u(x,t,\bar{u}(x,t),\bar{y}(x,t)) = \chi_{\mathcal{O}}\tilde{\psi}(x,t) \quad \text{a. e. in } Q. \tag{4.6}$$

This is a necessary condition for the optimal control $\bar{u}$.

On the other hand, by the definition of F in (3.13),

$$F(u) = (\hat{y}(\cdot,T), \hat{y}_t(\cdot,T))^{\top}, \quad \forall\ u \in L^2(Q),$$

where $\hat{y}$ satisfies the following coupled system:

$$\begin{cases} \hat{y}_{tt} - \Delta \hat{y} + a(x,t)\hat{y} = \chi_{\mathcal{O}}u - \frac{1}{\mu_1}\chi_{\mathcal{O}_1}\hat{p}_1 - \frac{1}{\mu_2}\chi_{\mathcal{O}_2}\hat{p}_2 & \text{in } Q, \\ \hat{p}_{1,tt} - \Delta \hat{p}_1 + a(x,t)\hat{p}_1 = \alpha_1 \chi_{\mathcal{O}_{1,d}}\hat{y} & \text{in } Q, \\ \hat{p}_{2,tt} - \Delta \hat{p}_2 + a(x,t)\hat{p}_2 = \alpha_2 \chi_{\mathcal{O}_{2,d}}\hat{y} & \text{in } Q, \\ \hat{y} = 0,\ \hat{p}_1 = \hat{p}_2 = 0 & \text{on } \Sigma, \\ \hat{y}(x,0) = \hat{y}_t(x,0) = 0 & \text{in } \Omega, \\ \hat{p}_1(x,T) = \hat{p}_{1,t}(x,T) = \hat{p}_2(x,T) = \hat{p}_{2,t}(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.7}$$

By Corollary 3.5, the finite codimensionality of the set M is equivalent to the estimate (3.17). In this example, for any $z=(\phi_2,\phi_1)^\top\in Y^*=H^{-1}(\Omega)\times L^2(\Omega)$, (3.20) becomes

$$\begin{cases}\hat\phi_{tt}-\Delta\hat\phi+a(x,t)\hat\phi=\sum_{i=1}^{2}\alpha_i\chi_{\mathcal{O}_{i,d}}q_i & \text{in } Q,\\ q_{1,tt}-\Delta q_1+a(x,t)q_1=-\frac{1}{\mu_1}\chi_{\mathcal{O}_1}\hat\phi-\frac{1}{\mu_1}\chi_{\mathcal{O}_1}\bar\phi & \text{in } Q,\\ q_{2,tt}-\Delta q_2+a(x,t)q_2=-\frac{1}{\mu_2}\chi_{\mathcal{O}_2}\hat\phi-\frac{1}{\mu_2}\chi_{\mathcal{O}_2}\bar\phi & \text{in } Q,\\ \bar\phi_{tt}-\Delta\bar\phi+a(x,t)\bar\phi=0 & \text{in } Q,\\ \hat\phi=\bar\phi=0,\ q_1=q_2=0 & \text{on } \Sigma,\\ \hat\phi(x,T)=\hat\phi_t(x,T)=0 & \text{in } \Omega,\\ q_1(x,0)=q_{1,t}(x,0)=q_2(x,0)=q_{2,t}(x,0)=0 & \text{in } \Omega,\\ \bar\phi(x,T)=\phi_1,\ \bar\phi_t(x,T)=\phi_2 & \text{in } \Omega,\end{cases}$$

and in the expression of F^*, $f_u(\bar u,L_1(\bar u),L_2(\bar u))^*z=\chi_{\mathcal{O}}\bar\phi$ and $B^*\eta=\chi_{\mathcal{O}}\hat\phi$.

Set $\phi=\hat\phi+\bar\phi$. Then ϕ satisfies the following coupled system:

$$\begin{cases}\phi_{tt}-\Delta\phi+a(x,t)\phi=\sum_{i=1}^{2}\alpha_i\chi_{\mathcal{O}_{i,d}}q_i & \text{in } Q,\\ q_{1,tt}-\Delta q_1+a(x,t)q_1=-\frac{1}{\mu_1}\chi_{\mathcal{O}_1}\phi & \text{in } Q,\\ q_{2,tt}-\Delta q_2+a(x,t)q_2=-\frac{1}{\mu_2}\chi_{\mathcal{O}_2}\phi & \text{in } Q,\\ \phi=0,\ q_1=q_2=0 & \text{on } \Sigma,\\ \phi(x,T)=\phi_1,\ \phi_t(x,T)=\phi_2 & \text{in } \Omega,\\ q_1(x,0)=q_{1,t}(x,0)=q_2(x,0)=q_{2,t}(x,0)=0 & \text{in } \Omega,\end{cases}\tag{4.8}$$

and for any $(\phi_2,\phi_1)^\top\in H^{-1}(\Omega)\times L^2(\Omega)$,

$$F^*(\phi_2,\phi_1)=\chi_{\mathcal{O}}\phi.$$

Note that here

$$\langle(\phi_2,\phi_1)^\top,(\hat\psi_1,\hat\psi_2)^\top\rangle_{Y^*,Y}\stackrel{\Delta}{=}(\phi_1,\hat\psi_2)_{L^2(\Omega)}-\langle\phi_2,\hat\psi_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},\ \forall\ (\hat\psi_1,\hat\psi_2)^\top\in H_0^1(\Omega)\times L^2(\Omega).$$

The estimate (3.17) becomes

$$\begin{aligned}&|(\phi_2,\phi_1)|^2_{H^{-1}(\Omega)\times L^2(\Omega)}\\ &\le C\int_0^T\int_{\mathcal{O}}\phi^2(x,t)\mathrm{d}x\mathrm{d}t+C|\mathcal{G}(\phi_2,\phi_1)|_X^2,\qquad\forall\ (\phi_2,\phi_1)^\top\in H^{-1}(\Omega)\times L^2(\Omega),\end{aligned}\tag{4.9}$$

for a compact operator $\mathcal{G}$ from Y^* to some Banach space X.

In the following, we show that if $(\Omega,\mathcal{O},T)$ satisfies geometric control condition, (4.9) holds true and therefore, M is finite codimensional. In fact, set $\phi=\xi_1+\xi_2$, where ξ_1 and ξ_2 satisfy, respectively,

$$\begin{cases}\xi_{1,tt}-\Delta\xi_1=0 & \text{in } Q,\\ \xi_1=0 & \text{on } \Sigma,\\ \xi_1(x,T)=\phi_1(x),\ \xi_{1,t}(x,T)=\phi_2(x) & \text{in } \Omega,\end{cases}\tag{4.10}$$

and

$$\begin{cases} \xi_{2,tt} - \Delta\xi_2 + a(x,t)\phi = \alpha_1 q_1 \chi_{\mathcal{O}_{1,d}} + \alpha_2 q_2 \chi_{\mathcal{O}_{2,d}} & \text{in } Q, \\ q_{1,tt} - \Delta q_1 + a(x,t)q_1 = -\frac{1}{\mu_1}\phi\chi_{\mathcal{O}_1} & \text{in } Q, \\ q_{2,tt} - \Delta q_2 + a(x,t)q_2 = -\frac{1}{\mu_2}\phi\chi_{\mathcal{O}_2} & \text{in } Q, \\ \xi_2 = 0,\ q_1 = q_2 = 0 & \text{on } \Sigma, \\ \xi_2(x,T) = \xi_{2,t}(x,T) = 0 & \text{in } \Omega, \\ q_1(x,0) = q_{1,t}(x,0) = q_2(x,0) = q_{2,t}(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.11}$$

Define the following linear operators G_1, G_2 and G_3:

$$\begin{cases} \text{the continuous operator } G_1 : H^{-1}(\Omega) \times L^2(\Omega) \to (L^2(Q))^3, \\ G_1(\phi_2, \phi_1) = (\phi, q_1, q_2), \qquad \forall\ (\phi_2, \phi_1)^\top \in H^{-1}(\Omega) \times L^2(\Omega), \end{cases}$$

where (ϕ, q_1, q_2) are the mild solutions to (4.8);

$$\begin{cases} \text{the continuous operator } G_2 : (L^2(Q))^3 \to C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ G_2(\phi, q_1, q_2)^\top = \xi_2, \qquad \forall\ (\phi, q_1, q_2)^\top \in (L^2(Q))^3, \end{cases}$$

where ξ_2 satisfies the first, the fourth and the fifth equations of (4.11); and

$$\begin{cases} \text{the compact operator } G_3 : C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \to L^2(Q), \\ G_3(\xi_2) = \xi_2, \qquad \forall\ \xi_2 \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \end{cases}$$

Let $\mathcal{G} = G_1 G_2 G_3$. Then $\mathcal{G}$ is compact and

$$\mathcal{G} : H^{-1}(\Omega) \times L^2(\Omega) \to L^2(Q) \overset{\Delta}{=} X, \quad \mathcal{G}(\phi_2, \phi_1) = \xi_2.$$

By [18], if $(\Omega, \mathcal{O}, T)$ satisfies geometric control condition, it holds that

$$|(\phi_2, \phi_1)|^2_{H^{-1}(\Omega)\times L^2(\Omega)} \le C \int_0^T \int_{\mathcal{O}} \xi_1^2 \mathrm{d}x\mathrm{d}t, \quad \forall\ (\phi_2, \phi_1)^\top \in H^{-1}(\Omega) \times L^2(\Omega).$$

Therefore,

$$|(\phi_2, \phi_1)|^2_{H^{-1}(\Omega)\times L^2(\Omega)} \le C \int_0^T \int_{\mathcal{O}} (\phi^2 + \xi_2^2) \mathrm{d}x\mathrm{d}t \le C \int_0^T \int_{\mathcal{O}} \phi^2 \mathrm{d}x\mathrm{d}t + C|\mathcal{G}(\phi_2, \phi_1)|^2_{L^2(Q)}.$$

This implies that if $(\Omega, \mathcal{O}, T)$ satisfies geometric control condition, the necessary condition (4.6) holds.

4.2. Example 4.2

For $j = 0, 1, 2$, assume that ω_j and $\tilde{\omega}_j$ are nonempty open subsets of Ω such that $\overline{\tilde{\omega}_j} \subseteq \omega_j$. Denote by $\zeta_j \in C_0^\infty(\omega_j)$ with $\zeta_j(x) = 1$ in $\tilde{\omega}_j$ and $0 \le \zeta_j(x) \le 1$ in ω_j. Set $Y = U = W = L^2(\Omega)$ and $V_j = L^2(\omega_j)$ for $j = 1, 2$. In the system (1), we set $A = \Delta$ with $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $B = \zeta_0$, $C_1 = \zeta_1$, $C_2 = \zeta_2$ and $D_j = \sqrt{\alpha_j \zeta_{j,d}}$ for $\alpha_j > 0$ and $j = 1, 2$, where $\zeta_{j,d} \in C_0^\infty(\omega_{j,d})$ and $\zeta_{j,d} = 1$ in $\omega'_{j,d}$ with $\overline{\omega'_{j,d}} \subseteq \omega_{j,d}$, and $\omega_{j,d}$ and $\omega'_{j,d}$ being nonempty open subsets of Ω. Then, $Y^* = Y$, $A' = A$, $C'_j = C_j$ and $D'_j = D_j$ for $j = 1, 2$.

Similar to Example 2.2, (1) becomes

$$\begin{cases} y_t - \Delta y = \zeta_0 u + \zeta_1 v_1 + \zeta_2 v_2 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{4.12}$$

where (u, v_1, v_2) is a triple of control variables, y is the state variable and $y_0 \in \tilde{B}_{\rho_1} \triangleq \{ v \in C_0^{2+\theta}(\Omega) \mid |v|_{C^{2+\theta}(\overline{\Omega})} \leq \rho_1 \}$ for $\theta \in (0,1)$ and $\rho_1 > 0$ (ρ_1 will be specified later). The functionals J_j are as follows:

$$J_j(v_1, v_2; u) = \frac{\alpha_j}{2} \int_Q \zeta_{j,d} |y - \hat{y}_{j,d}|^2 \mathrm{d}x\mathrm{d}t + \frac{\mu_j}{2} \int_0^T \int_{\omega_j} |v_j|^2 \mathrm{d}x\mathrm{d}t, \quad j = 1, 2,$$

where y is the solution to (4.12) and we require $\hat{y}_{j,d} \in C^{\theta, \frac{\theta}{2}}(\overline{Q})$. Moreover, for any given $u \in C^{\theta, \frac{\theta}{2}}(\overline{Q})$, the Nash equilibrium $(L_1(u), L_2(u))$ for the functionals J_1 and J_2 is given by

$$L_j(u) = -\frac{1}{\mu_j} \zeta_j p_j, \quad j = 1, 2,$$

where p_1 and p_2 satisfy the following coupled system:

$$\begin{cases} y_t - \Delta y = \zeta_0 u - \frac{1}{\mu_1} \zeta_1^2 p_1 - \frac{1}{\mu_2} \zeta_2^2 p_2 & \text{in } Q, \\ p_{1,t} + \Delta p_1 = -\alpha_1 \zeta_{1,d}(y - \hat{y}_{1,d}) & \text{in } Q, \\ p_{2,t} + \Delta p_2 = -\alpha_2 \zeta_{2,d}(y - \hat{y}_{2,d}) & \text{in } Q, \\ y = 0, \ p_1 = p_2 = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \ p_1(x,T) = p_2(x,T) = 0 & \text{in } \Omega. \end{cases} \tag{4.13}$$

Furthermore, consider the following optimization problem:

$$f_0(u, L_1(u), L_2(u)) = \int_Q g(x, t, \hat{Y}(x,t), u(x,t)) \mathrm{d}x\mathrm{d}t,$$
$$\forall\ u \in \mathcal{K} = B_{\rho_1} \triangleq \{ v \in C^{\theta, \frac{\theta}{2}}(\overline{Q}) \mid |v|_{C^{\theta, \frac{\theta}{2}}(\overline{Q})} \leq \rho_1 \},$$

where $\hat{Y}$ satisfies the following quasi-linear parabolic equation:

$$\begin{cases} \hat{Y}_t - \sum\limits_{i,j=1}^n (a^{ij}(\hat{Y}, \nabla \hat{Y}) \hat{Y}_{x_i})_{x_j} = \zeta_0 u + \zeta_1 L_1(u) + \zeta_2 L_2(u) & \text{in } Q, \\ \hat{Y} = 0 & \text{on } \Sigma, \\ \hat{Y}(x,0) = \hat{y}_0(x) & \text{in } \Omega. \end{cases} \tag{4.14}$$

In (4.14), u is the control variable, $\hat{Y}$ is the state variable and $\hat{y}_0$ is an initial value. Let $a^{ij} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be C^3 functions with $a^{ij} = a^{ji}$ in $\mathbb{R}^{1+n}$ for $i, j = 1, \ldots, n$. Moreover, for a constant $\rho_0 > 0$,

$$\sum_{i,j=1}^n a^{ij}(s, \tau) \nu_i \nu_j \geq \rho_0 |\nu|^2, \quad \forall\ (s, \tau, \nu) = (s, \tau^1, \cdots, \tau^n, \nu_1, \cdots, \nu_n) \in \mathbb{R}^{1+2n}.$$

By well-posedness results for quasi-linear parabolic equations, when μ_1 and μ_2 are sufficiently large, there exists a $\rho_1 > 0$ such that for any $y_0, \hat{y}_0 \in \tilde{B}_{\rho_1}$ and $u \in B_{\rho_1}$, the equation (4.14) admits a unique solution $\hat{Y} \in C^{2+\theta,1+\frac{\theta}{2}}(\overline{Q})$.

Further, $g : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function, such that for any $u \in B_{\rho_1}$, $g(\cdot,\cdot,\hat{Y}(\cdot,\cdot), u(\cdot,\cdot)) \in L^1(Q)$. Set $f(u, L_1(u), L_2(u)) = \hat{Y}(\cdot, T)$ and assume that E is an open subset of $L^2(\Omega)$. Assume that $\bar{u} \in B_{\rho_1}$ is a solution to the above optimization problem and $\hat{Y}^*$ is the associated solution to (4.14).

By Theorem 3.3, since E is finite codimensional in $L^2(\Omega)$, there exists a non-zero pair $(\psi_0, \psi) \in \mathbb{R} \times L^2(\Omega)$, such that when μ_1 and μ_2 are sufficiently large, the following condition holds:

$$\psi_0 \eta^0 + \int_\Omega \psi(x)\eta^1(x)\mathrm{d}x \geq 0, \quad \forall\ v \in B_{\rho_1}, \tag{4.15}$$

where

$$\begin{aligned}
&\eta^0 = \int_Q \Big\{ g_u(x,t,\bar{u}(x,t),\hat{Y}^*(x,t))[v(x,t) - \bar{u}(x,t)] + g_y(x,t,\bar{u}(x,t),\hat{Y}^*(x,t))Z(x,t) \Big\}\mathrm{d}x\mathrm{d}t,\\
&\eta^1 = Z(\cdot, T),
\end{aligned}$$

and Z satisfies the following coupled system:

$$\begin{cases}
Z_t - \sum\limits_{i,j=1}^n (a^{ij}(\hat{Y}^*, \nabla\hat{Y}^*)Z_{x_i})_{x_j} - \sum\limits_{i,j=1}^n (a^{ij}_\tau(\hat{Y}^*, \nabla\hat{Y}^*)\cdot\nabla Z\hat{Y}^*_{x_i})_{x_j} & \\
\qquad - \sum\limits_{i,j=1}^n (a^{ij}_s(\hat{Y}^*, \nabla\hat{Y}^*)\hat{Y}^*_{x_i}Z)_{x_j} = \zeta_0(v-\bar{u}) - \frac{1}{\mu_1}\zeta_1^2\tilde{p}_1 - \frac{1}{\mu_2}\zeta_2^2\tilde{p}_2 & \text{in } Q,\\
\tilde{y}_t - \Delta\tilde{y} = \zeta_0(v-\bar{u}) - \frac{1}{\mu_1}\zeta_1^2\tilde{p}_1 - \frac{1}{\mu_2}\zeta_2^2\tilde{p}_2 & \text{in } Q,\\
\tilde{p}_{1,t} + \Delta\tilde{p}_1 = -\alpha_1\zeta_{1,d}\tilde{y} & \text{in } Q,\\
\tilde{p}_{2,t} + \Delta\tilde{p}_2 = -\alpha_2\zeta_{2,d}\tilde{y} & \text{in } Q,\\
Z = \tilde{y} = 0,\ \tilde{p}_1 = \tilde{p}_2 = 0 & \text{on } \Sigma,\\
Z(x,0) = \tilde{y}(x,0) = \tilde{p}_1(x,T) = \tilde{p}_2(x,T) = 0 & \text{in } \Omega.
\end{cases} \tag{4.16}$$

Here, a^{ij}_s and $a^{ij}_{\tau_k}$ denote partial derivatives of $a^{ij} = a^{ij}(s,\tau) = a^{ij}(s,\tau_1,\cdots,\tau_n)$ with respect to the variables s and τ_k for $k = 1,\cdots,n$, respectively, and a^{ij}_τ denotes gradient of a^{ij} with respect to $\tau = (\tau_1,\cdots,\tau_n)$. Note that

$$\sum_{i,j=1}^n \big(a^{ij}_\tau(\hat{Y}^*, \nabla\hat{Y}^*)\cdot\nabla Z\hat{Y}^*_{x_i}\big)_{x_j} = \sum_{i,j=1}^n \big(\sum_{k=1}^n a^{kj}_{\tau_i}(\hat{Y}^*, \nabla\hat{Y}^*)\hat{Y}^*_{x_k}Z_{x_i}\big)_{x_j}.$$

Set

$$c^{ij}(\hat{Y}^*, \nabla\hat{Y}^*) = \frac{1}{2}\sum_{k=1}^n \Big[a^{kj}_{\tau_i}(\hat{Y}^*, \nabla\hat{Y}^*)\hat{Y}^*_{x_k} + a^{ki}_{\tau_j}(\hat{Y}^*, \nabla\hat{Y}^*)\hat{Y}^*_{x_k}\Big].$$

Then $c^{ij} = c^{ji}$ and the first equation of (4.16) becomes

$$\begin{aligned}
&Z_t - \sum_{i,j=1}^n (a^{ij}(\hat{Y}^*, \nabla\hat{Y}^*)Z_{x_i})_{x_j} - \sum_{i,j=1}^n (c^{ij}(\hat{Y}^*, \nabla\hat{Y}^*)Z_{x_i})_{x_j} - \sum_{i,j=1}^n (a^{ij}_s(\hat{Y}^*, \nabla\hat{Y}^*)\hat{Y}^*_{x_i}Z)_{x_j}\\
&= \zeta_0(v-\bar{u}) - \frac{1}{\mu_1}\zeta_1^2\tilde{p}_1 - \frac{1}{\mu_2}\zeta_2^2\tilde{p}_2 \quad \text{in } Q.
\end{aligned}$$

When ρ_1 is sufficiently small, $\sum_{i,j=1}^{n} |c^{ij}(\hat{Y}^*, \nabla \hat{Y}^*)| \leq \frac{\rho_0}{2}$ and the above partial differential equation is a linear parabolic equation with respect to Z. Further, introduce the following coupled system:

$$\begin{cases} \tilde{W}_t + \sum_{i,j=1}^{n} (a^{ij}(\hat{Y}^*, \nabla \hat{Y}^*) \tilde{W}_{x_i})_{x_j} + \sum_{i,j=1}^{n} (c^{ij}(\hat{Y}^*, \nabla \hat{Y}^*) \tilde{W}_{x_i})_{x_j} \\ \qquad - \sum_{i,j=1}^{n} a_s^{ij}(\hat{Y}^*, \nabla \hat{Y}^*) \hat{Y}^*_{x_i} \tilde{W}_{x_j} = -\psi_0 g_y(x, t, \bar{u}(x,t), \hat{Y}^*(x,t)) & \text{in } Q, \\ \tilde{\psi}_t + \Delta \tilde{\psi} = \alpha_1 \zeta_{1,d} \tilde{\gamma}_1 + \alpha_2 \zeta_{2,d} \tilde{\gamma}_2 & \text{in } Q, \\ \tilde{\gamma}_{1,t} - \Delta \tilde{\gamma}_1 = \frac{1}{\mu_1} \zeta_1^2 (\tilde{\psi} + \tilde{W}) & \text{in } Q, \\ \tilde{\gamma}_{2,t} - \Delta \tilde{\gamma}_2 = \frac{1}{\mu_2} \zeta_2^2 (\tilde{\psi} + \tilde{W}) & \text{in } Q, \\ \tilde{W} = \tilde{\psi} = 0, \ \tilde{\gamma}_1 = \tilde{\gamma}_2 = 0 & \text{on } \Sigma, \\ \tilde{W}(x,T) = \psi(x), \ \tilde{\psi}(x,T) = 0, \ \tilde{\gamma}_1(x,0) = \tilde{\gamma}_2(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.17}$$

where $(\tilde{W}, \tilde{\psi}, \tilde{\gamma}_1, \tilde{\gamma}_2)^\top \in \big(L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))\big)^4$ for sufficiently large μ_1 and μ_2. By (4.15)–(4.17), we obtain that

$$\zeta_0 \tilde{W}(x,t) + \zeta_0 \tilde{\psi}(x,t) = \psi_0 g_u(x, t, \bar{u}(x,t), \hat{Y}^*(x,t)) \quad \text{a. e. in } Q.$$

This is the necessary condition on $\bar{u}$.

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Data availability statement

The research data associated with this article are included in the article.

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