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# ON NON-AUTONOMOUS PARABOLIC EQUATIONS WITH MEASURE-VALUED RIGHT HAND SIDES AND APPLICATIONS TO OPTIMAL CONTROL

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**Abstract.** The main aim of this paper is to develop a theory for non-autonomous parabolic equations with time-dependent measures on the spatial domain appearing as right hand sides. Restricting these measures to ones which have their supports on 'curves' or 'surfaces' – the latter understood in the sense of geometric measure theory – we succeed in interpreting them as distributional objects from a (negative indexed) Sobolev–Slobodetskii space $W^{s,2}(\Omega)$ with $s$ close to $-1$. For these indices $s$ a tailor suited parabolic theory is established, based on results of Disser *et al.* [*Ann. Sci. Norm. Super. Pisa, Cl. Sci.* **17** (2017) 65–79] and Haller-Dintelmann *et al.* [*Ann. Mat. Pura Appl.* **198** (2019) 1227–1241]. The proposed frame work is well-suited for optimal control problems with controls acting on sub-manifolds.

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## 1. Introduction

In this paper we study non-autonomous parabolic equations

$$\frac{\partial u}{\partial t} + \mathcal{A}(\cdot)u = \varrho, \quad u(0) = 0 \tag{1.1}$$

with right hand sides including measures. For every time point $t$ the operator $\mathcal{A}(t)$ is a second order divergence operator, and the setting has the following peculiarities:
• $\mathcal{A}(t)$ is combined with mixed boundary conditions, *i.e.* (homogenous) Dirichlet on one part of the boundary $\partial\Omega$ and Neumann on the complementary part,
• the coefficient function of the operators are spatially and temporarily dependent, and it is allowed that they are discontinuous in both,
• the underlying spatial domain $\Omega$ is non-smooth. The investigation of such equations has been carried out in the pioneering paper [1] – as long as the spatial geometry and the time-dependent coefficients are assumed to be smooth – and mixed boundary conditions were not included. It is our aim here to include a broader admissible framework in order to admit as many real world problems as possible.

*Keywords and phrases:* Non-autonomous evolution equations, parabolic initial boundary value problems, maximal parabolic regularity, measure-valued right hand sides, optimal control.

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Concerning the measure $\rho$ on the right hand side of (1.1), in the literature typically on the space time cylinder it is considered to be of the form

$$C_0(]0,T[\times\Omega) \ni f \mapsto \int_0^T \int_\Omega f(t,x)\, d\mu_t(x)\, \mathrm{d}t, \tag{1.2}$$

with $\{\mu_t\}_{t\in]0,T]}$ a suitable family of Radon measures on $\Omega$, which is – in its dependence on $t$ – weak$^*$ measurable. The procedure how to treat such parabolic equations is widely common: embed $\mathcal{M}(\Omega)$, the space of bounded Radon measures on $\Omega$, into a space $W^{-1,q}_{\mathfrak{D}}(\Omega)$ and identify the r.h.s. in this manner with a function $f \in L^r(]0,T[,W^{-1,q}_{\mathfrak{D}}(\Omega))$ ($W^{1,q}_{\mathfrak{D}}(\Omega)$ denoting the usual Sobolev space which includes a trace-zero condition on $\mathfrak{D} \subset \partial\Omega$ and $W^{-1,q}_{\mathfrak{D}}(\Omega)$ being the space of continuous antilinear forms on $W^{1,q'}_{\mathfrak{D}}(\Omega)$). For such right hand sides one may – under mild conditions – apply maximal parabolic regularity of the second order divergence operators involved to get a solution which belongs to the maximal parabolic regularity space (see (3.13) below). Unfortunately, this has two serious drawbacks: In order to catch *all* bounded Radon measures on $\Omega$, one has to chose $q$'s which lie definitely below $\frac{d}{d-1}$, $d$ being the space dimension. Therefore the domain of the elliptic second order divergence operator can be at best $W^{1,q}_{\mathfrak{D}}(\Omega)$ – with this limitation of $q$. This is more irregular than $W^{1,2}_{\mathfrak{D}}(\Omega)$. But even worse: in general it is extremely delicate – in view of pathologies which were discovered already by Serrin in [2] – to give the divergence operators on $W^{-1,q}_{\mathfrak{D}}(\Omega)$ a precise meaning at all, if $q < 2$ is far from 2. Secondly, for non-autonomous, second order parabolic equations no results on maximal parabolic regularity in the $W^{-1,q}$ scale are at hand if the geometry of the spatial domain is really non-smooth and the time dependence of the coefficients is 'wild' – unless $q$ is close to 2.

Consequently, in this paper we go another way: We consider sets of codimension 1 or 2 in the spatial domain $\Omega$. If focused on dimensions 2 and 3 this results in measures which live on 'curves' or 'surfaces' and are absolute continuous with respect to the induced Hausdorff measures. 'Curves ' and 'surfaces' are to be understood here in an extremely broad sense – based on the concept of upper $l$-sets from geometric measure theory, developped by Jonsson and Wallin (see [3]) in the seventies. These sets $M$ are characterized by the condition

$$\mathcal{H}_l(M \cap B(x,r)) \le \mathfrak{c}\, r^l, \quad x \in M,\ r \in ]0,1], \tag{1.3}$$

for some positive constant $\mathfrak{c}$, where $l \in \{1,\dots,d-1\}$ and $\mathcal{H}_l$ is the $l$-dimensional Hausdorff measure. In order to avoid trivial cases, we additionally demand that $\mathcal{H}_l(M) > 0$ without mentioning this explicitly in the sequel. Thus, $M$ being an upper $l$-set, we consider measures $\sigma\mathcal{H}_l|_M$, with $\sigma$ a function from $L^2(M;\mathcal{H}_l)$. Fortunately, the pioneering results of Jonsson/Wallin admit in our case embeddings

$$L^2(M;\mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M \in W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega), \tag{1.4}$$

where, in our context, $\epsilon > 0$ may be taken arbitrarily small. Even more: one gets uniform boundedness for norms of the mappings (1.4), if $M$ runs through a class of subsets in $\Omega$ admitting a *uniform* upper $l$ estimate, *i.e.* the constant $\mathfrak{c}$ in (1.3) can be chosen uniformly for all sets under consideration.

All of this provides a constellation which is quite comfortable concerning the investigation of the parabolic equation in the context of (non-autonomous) maximal parabolic regularity, namely: in [4] an elliptic extrapolation theorem was established which asserts that the operator

$$-\nabla\cdot\mu\nabla + 1 : W^{1\pm\epsilon,2}_{\mathfrak{D}}(\Omega) \to W^{-1\pm\epsilon,2}_{\mathfrak{D}}(\Omega), \tag{1.5}$$

as a topological isomorphism for $\epsilon = 0$ by Lax-Milgram, extends to a (consistent) isomorphism for small $\epsilon > 0$ under very general assumptions. Having this at hand, it is not too difficult to show that (the negative of) these extrapolated operators indeed generate analytic semigroups on the corresponding Hilbert spaces $W^{1\pm\epsilon,2}_{\mathfrak{D}}(\Omega)$, see Thm. 3.6 below. The next step is easy: it has been established since long that the negative of a generator of an analytic semigroup satisfies maximal parabolic regularity – if the underlying Banach space is topologically a Hilbert space. Knowing this, we deduce from these foregoing insights and the central result of [5], that the mapping

$$\begin{aligned}&w \mapsto w' - div\, \hat{\mu}\, grad\, w \\ &\text{from } W_0^{1,q}(J; W_{\mathfrak{D}}^{-1\pm\epsilon,2}(\Omega)) \cap L^q(J; W_{\mathfrak{D}}^{1\pm\epsilon,2}(\Omega)) \text{ to } L^q(J; W_{\mathfrak{D}}^{-1\pm\epsilon,2}(\Omega)),\end{aligned} \tag{1.6}$$

which is a topological isomorphism by the classical Lions' result ([6], Sect. XVIII.3, Rem. 9) for $\epsilon = 0$ and $q = 2$, extrapolates to an isomorphism for $q \sim 2$ and small $\epsilon$. Fitting everything together: the embedding (1.4) – including the control over the embedding constants – with the non-autonomous parabolic regularity result, one gets as much regularity for the solution as one can realistically expect: maximal parabolic regularity. Astonishingly, $q$ and $s = 1 \pm \epsilon$ in their inter-relation cleverly chosen, one can achieve that the space of solutions, namely the left hand side of (1.6), even embeds *compactly* in the usual trace space $C(\overline{J}; L^2(\Omega))$.

In recent years also the numerical analysis of problems (1.1) has been treated, see [7], [8], and also [9], [10], [11]. In the first paper it is reflected that discontinuous diffusion coefficients allow the treatment of moving interfaces – a property which is clearly required in applications. In [8] and [11] practical problems are discussed, where the – time dependent – measures on the right hand side of the parabolic equation are concentrated on hypersurfaces.

In the last section of this paper our analysis of (1.1) will be used in the context of optimal control problems. The analysis for these problems is typically carried out for the cases where the control acts on subdomains of $\Omega$ or $\partial\Omega$. The situation where the support of the control has no interior in $\Omega$ or $\partial\Omega$ has received surprisingly little attention. In [12] problems with point control in the interior of the domain are considered. Optimal control problems with controls as measures were extensively investigated, see *e.g.* [13], Chapter 4 and the literature cited there. It should, however, be noticed that formulating optimal control problems over the whole space of measures favors minimizers which are pointwise source functions, see eg. [14], [15], [16]. The optimal controls obtained in this manner are typically not concentrated on lower dimensional manifolds. The present paper aims at providing necessary prerequisites for optimal control on possibly time-dependent lower dimensional manifolds of codimension 1 and 2, and first steps towards exploiting these results are taken in section 4. There are only few other publications which also focus on such control problems. All of them consider the problem under investigation in a Banach (Sobolev-) space setting, while in the present paper we favor a Hilbert space frame-work, as much as this is possible. Our coefficient functions may depend *discontinuously* on space *and* time, and in fact *non-autonomous* parabolic equations are one of the main subjects of this paper. Likely the paper most closely related to ours is [17], where also a convection term is admitted in the equation. The cost functional there includes a gradient term which leads to an adjoint equation which is much less regular than in our case. In [10, 18] finite element approximation of the optimal control problems with controls on manifolds are investigated. Let us also mention [11] where approximate controllability of the heat equation by controls acting on a lower dimensional manifold is investigated.

## 2. Preliminaries

### 2.1. Definitions, assumptions and basic facts

Throughout this paper we denote by $d$ the dimension of the domain $\Omega$ and by $\mathcal{H}_l$ the $l$-dimensional Hausdorff measure, where $l \in \{1, \ldots, d-1\}$. We recall that on smooth and Lipschitzian submanifolds of $\mathbb{R}^d$ the Hausdorff measure is identical with the measure defined by parametrizations on this manifold, see [19], Chapter 3.3/3.4. Moreover, if $M \subset \Omega \subset \mathbb{R}^d$ we abbreviate $L^p(M; \mathcal{H}_l|_M)$ by $L^p(M; \mathcal{H}_l)$ in all what follows. For $\Omega \subset \mathbb{R}^d$ a bounded domain, we denote by $\mathcal{M}(\Omega)$ the space of finite Radon measures on $\Omega$ – in subscripts abbreviated by $\mathcal{M}$. For two Banach spaces $X, Y$, with $Y$ continuously embedded into $X$, we denote by $(X, Y)_{\theta,r}$ the usual real interpolation space and by $[X, Y]_\theta$ the corresponding complex interpolation space (see [20], Ch. I), with $\theta \in ]0, 1[$ . Finally $B(x, r)$ denotes the ball in $\mathbb{R}^d$ with centre $x$ and radius $r$.

We generally admit *complex* coefficients in this paper. In the sequel we need the following measure theoretic notion of sets.

**Definition 2.1.** Let $M \subset \mathbb{R}^d$ be a bounded Borel set. We call $M$ an upper $l$-set if (1.3) holds for a positive constant $\mathfrak{c}$. On the other hand, $M$ is called a *lower* $l$-set if

$$\mathfrak{c}_\bullet \, r^{d-1} \le \mathcal{H}_{d-1}(M \cap B(x,r)), \quad x \in M, \; r \in ]0,1]$$

for some positive constant $\mathfrak{c}_\bullet$. If $M$ is an upper $l$-set and, simultaneously, a lower $l$-set, we call it an $l$-set. In case of $l = d-1$ an $l$-set is said to satisfy the *Ahlfors–David condition*.

In all what follows we assume that the subsequent assumption holds.

**Assumption 2.2.** $\Omega \subset \mathbb{R}^d$ is a bounded domain $d \ge 2$.

(a) $\mathfrak{D}$ is a closed subset of $\partial\Omega$, which satisfies the *Ahlfors–David condition*.
(b) For every $x \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there exists an open neighbourhood $U_x$ of $x$ and a bi-Lipschitz map $\Phi_x$ from $U_x$ onto the cube $K := ]-1,1[^d$, such that the following three conditions are satisfied:

$$\begin{aligned}\Phi_x(x) &= 0,\\ \Phi_x(U_x \cap \Omega) &= \{x \in K : x_d < 0\},\\ \Phi_x(U_x \cap \partial\Omega) &= \{x \in K : x_d = 0\},\end{aligned}$$

(c)
$$|\Omega \cap B(x,r)| \ge c\, r^d, \quad x \in \Omega, \quad r \in ]0,1] \tag{2.1}$$

for some positive constant $c$.

Unless indicated otherwise, the integrability index $p$ is always assumed to be in $]1,\infty[$ in the sequel.

**Definition 2.3** (Sobolev–Slobodetskii spaces)**.** $W^{1,p}(\mathbb{R}^d)$ is the usual Sobolev space. For $s \in ]0, 1+\frac{1}{p}[\setminus\{1\}$ write $s = k + \sigma$ with $k \in \{0,1\}$ and $\sigma \in ]0,1[$. Then the space $W^{s,p}(\mathbb{R}^d)$ is given by the normed vector space of functions $\psi \in L^2(\mathbb{R}^d)$ for which

$$\|\psi\|_{W^{s,p}(\mathbb{R}^d)} := \|\psi\|_{W^{k,p}(\mathbb{R}^d)} + \left( \sum_{i=1}^d \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\partial_i^k \psi(x) - \partial_i^k \psi(y)|^p}{|x-y|^{d+p\sigma}} \mathrm{d}x\mathrm{d}y \right)^{1/p} < \infty.$$

For further purpose we also need Sobolev–Slobodetskii spaces on upper $l$-sets.

**Definition 2.4** (Sobolev–Slobodetskii spaces on singular sets)**.** Let $M \subset \mathbb{R}^d$ be an upper $l$-set, $s \in ]0,1[$. Define, for $\psi \in L^p(M;\mathcal{H}_l)$

$$\|\psi\|_{W^{s,p}(M)} := \|\psi\|_{L^p(M;\mathcal{H}_l)} + \left( \iint_{M \times M} \frac{|\psi(x) - \psi(y)|^p}{|x-y|^{d+ps}} \, \mathrm{d}\mathcal{H}_l(x)\, \mathrm{d}\mathcal{H}_l(y) \right)^{1/p}, \tag{2.2}$$

finite or infinite. We introduce $W^{s,p}(M)$ as the space of functions on $M$, for which (2.2) is finite.

**Proposition 2.5.** *Let $E \subset \mathbb{R}^d$ be an $l$-set, $l \in \{d-2, d-1\}$. Assume $p \in ]1,\infty[$ and $s \in ]\frac{1}{p}, 1+\frac{1}{p}[$ such that $\beta = s - \frac{d-l}{p} > 0$. Then, for $\psi \in W^{s,p}(\mathbb{R}^d)$, the limit*

$$(\mathrm{tr}_E\,\psi)(x) := \lim_{r \searrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \psi, \quad x \in E \tag{2.3}$$

*exists for $\mathcal{H}_l$-almost all $x \in E$ and the resulting operator $tr_E$ maps $W^{s,p}(\mathbb{R}^d)$ continuously* onto $W^{s-\frac{d-l}{p},p}(E)$. *Conversely, for $tr_E$ exists a continuous right inverse $\mathfrak{F}_E : W^{s-\frac{d-l}{p},p}(E) \to W^{s,p}(\mathbb{R}^d)$, such that every function $\mathfrak{F}_E u$ is smooth on $\mathbb{R}^d \setminus E$. Moreover, in case of $l = d-1$, $\mathfrak{F}_E$ maps the Lipschitzian functions on $E$ into the set of Lipschitzian functions on $\mathbb{R}^d$. Finally, the extension operators are* consistent *for all $s \in ]\frac{1}{p}, 1+\frac{1}{p}, [$.*

*Proof.* For the first two statements see [3], Theorem VI.1. The assertion on Lipschitz continuity of the extension is proved in [21]. ☐

**Definition 2.6** (Function spaces with zero trace)**.** Let $E \subset \mathbb{R}^d$ be a $(d-1)$-set and let $s \in ]\frac{1}{p}, 1+\frac{1}{p}[$. Then we define $W^{s,p}_E(\mathbb{R}^d) := \ker \mathrm{tr}_E$ in $W^{s,p}(\mathbb{R}^d)$.

The analogues of the spaces $W^{s,p}(\mathbb{R}^d)$ and $W^{s,p}_E(\mathbb{R}^d)$ on $\Omega$ are defined as quotient spaces corresponding to restriction to $\Omega$ of their $\mathbb{R}^d$ versions as follows:

**Definition 2.7** (Function spaces on $\Omega$)**.** Let $p \in ]1,\infty[$ and $s \in ]0, 1+\frac{1}{p}[$.

(i) We define $W^{s,p}(\Omega)$ to be the factor space of restrictions of $W^{s,p}(\mathbb{R}^d)$ to $\Omega$, equipped with the natural quotient norm. Moreover, $W^{-s,p}(\Omega) := \big(W^{s,p'}(\Omega)\big)^\star$.

(ii) Let $E \subseteq \overline{\Omega}$ be a $(d-1)$-set and $s \in ]\frac{1}{p}, 1+\frac{1}{p}[$. Then, as before, we define $W^{s,p}_E(\Omega)$ as the factor space of restrictions to $\Omega$ of $W^{s,p}_E(\mathbb{R}^d)$, equipped with the natural quotient norm. Moreover, $W^{-s,p}_E(\Omega) := \big(W^{s,p'}_E(\Omega)\big)^\star$.

**Lemma 2.8.** *Let $\Lambda = \mathbb{R}^d$ or $\Lambda = \Omega$ and define*

$$C^\infty_{\mathfrak{D}}(\Lambda) = \{\psi|_\Lambda : \psi \in C^\infty_0(\mathbb{R}^d), supp\,\psi \cap \mathfrak{D} = \emptyset\},$$

*and let $\widetilde{W}^{1,p}_{\mathfrak{D}}(\Lambda)$ be the closure of $C^\infty_{\mathfrak{D}}(\Lambda)$ in $W^{1,p}(\Lambda)$.*

*i) Assumption. 2.2 (b) implies the existence of a continuous extension operator $\mathcal{E} : \widetilde{W}^{1,p}_{\mathfrak{D}}(\Omega) \to \widetilde{W}^{1,p}_{\mathfrak{D}}(B)$, $B \subset \mathbb{R}^d$ being an (open) ball containing $\overline{\Omega}$.*

*ii) Since $\mathfrak{D}$ is Ahlfors-David regular, one has $\widetilde{W}^{1,p}_{\mathfrak{D}}(\mathbb{R}^d) = W^{1,p}_{\mathfrak{D}}(\mathbb{R}^d)$. So*

$$\widetilde{W}^{1,p}_{\mathfrak{D}}(\Omega) = W^{1,p}_{\mathfrak{D}}(\Omega), \tag{2.4}$$

*as sets and topologically.*

*iii) If Assumption 2.2 (a) and (b) hold, then $W^{1,p}_{\mathfrak{D}}(\Omega)$ can be intrinsically characterized as*

$$W^{1,p}_{\mathfrak{D}}(\Omega) = W^{1,p}(\Omega) \cap \{\psi \in L^p(\Omega) : \int_\Omega \Big|\frac{\psi}{dist_{\mathfrak{D}}}\Big|^p < \infty\}.$$

*Proof.* (i) is proved in [22], Lemma 3.2. (ii) The equality $\widetilde{W}^{1,p}_{\mathfrak{D}}(\mathbb{R}^d) = W^{1,p}_{\mathfrak{D}}(\mathbb{R}^d)$ is shown in [21], Theorem 3.7. According to (i), $\widetilde{W}^{1,p}_{\mathfrak{D}}(\Omega)$ is the set of restrictions from $\widetilde{W}^{1,p}_{\mathfrak{D}}(\mathbb{R}^d)$ and, evidently, $W^{1,p}_{\mathfrak{D}}(\Omega)$ is the set of restrictions from $W^{1,p}_{\mathfrak{D}}(\mathbb{R}^d)$. This gives (2.4) as sets. Moreover, one has the estimate

$$\|\psi\|_{\widetilde{W}^{1,p}_{\mathfrak{D}}(\Omega)} \le \|\psi\|_{W^{1,p}_{\mathfrak{D}}(\Omega)}, \qquad \psi \in W^{1,p}_{\mathfrak{D}}(\Omega).$$

This, together with the open mapping theorem shows the topological coincidence. iii) This follows from (2.4) and [23], Theorem 2.1, the latter is based on [24] in an essential manner. □

**Remark 2.9.** i) The definition of the spaces $W^{s,p}(\Omega)$ as factor spaces of restrictions implies that these spaces inherit the usual Sobolev-type embedding results between them when defined on balls.

ii) Suppose $s \in ]0,1[$. Then condition (2.1) implies that the factor space $W^{s,p}(\Omega)$ agrees with the space $W^{s,p}_*(\Omega)$ defined intrinsically by the set of all functions $u \in L^p(\Omega)$ such that

$$\|\psi\|_{W^{s,p}_*(\Omega)} := \|\psi\|_{L^p(\Omega)} + \left( \iint_{\Omega\times\Omega} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{d+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p} < \infty \tag{2.5}$$

up to equivalent norms (see [3], Thm. V.1). Moreover, very recently it was shown in [25] that, if $E \subseteq \partial\Omega$ is a $(d-1)$-set and $\Omega$ satisfies the interior thickness condition (2.1) for $x \in \partial\Omega \setminus E$, then $W^{s,p}_E(\Omega)$ coincides with the intrinsically given $W^{s,p}_*(\Omega) \cap L^p(\Omega, \mathrm{dist}_E^{-ps})$, also up to equivalent norms.

iii) The reader should carefully notice that, in case of $p = 2$ the so defined Sobolev–Slobodetskii spaces on $\mathbb{R}^d$ are identical with the corresponding Bessel potential spaces, *i.e.* $H^{s,2}(\mathbb{R}^d)$, see [20], Chapter 2.3.2. Up to now, we decided to maintain the notation $W^{s,2}/W^{s,2}_{\mathfrak{D}}$ by the following reason: at least for $s \in ]0,1[$ one has, due to ii), an explicit description of the spaces also on $\Omega$ and hence a more clear perception.

**Agreement**

a) For all Sobolev–Slobodetskii (Bessel Potential) spaces with integrability index 2 we henceforth write $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$ and $H^s_{\mathfrak{D}}(\Omega)$ instead of $W^{s,2}_{\mathfrak{D}}(\Omega)$ .

b) Since the domain $\Omega$ is fixed through the whole paper, we mostly omit the notation $'\Omega'$ from now on - writing *e.g.* $H^1_{\mathfrak{D}}$ instead of $H^1_{\mathfrak{D}}(\Omega)$. For the following density result we require yet another definition.

**Definition 2.10.** The space of infinitely differentiable functions with bounded gradient in $\Omega$ is denoted by $C^\infty_b(\Omega)$.

**Lemma 2.11.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and $E \subset \partial\Omega$ be a closed $(d-1)$-set.*

*i) Assume $s \in ]\frac{1}{2}, 1]$. Then $C^\infty_E(\Omega) \subseteq H^s_E(\Omega) \cap C^\infty(\Omega)$ is dense in $H^s_E(\Omega)$.*
*ii) Assume $s \in ]1, \frac{3}{2}[$. Then $H^s_E(\Omega) \cap C^\infty_b(\Omega)$ is dense in $H^s_E(\Omega)$.*

*Proof.* (i) is proved in [26], Proposition 3.7. (ii) We first prove that the statement is correct if $\Omega$ is replaced by $\mathbb{R}^d$. Let $\tilde{\psi} \in W^{s,2}_E(\mathbb{R}^d) \subset W^{s,2}(\mathbb{R}^d)$. Then there is a sequence $\{\psi_n\}$ in $C^\infty_0(\mathbb{R}^d)$ converging towards $\tilde{\psi}$ in the $W^{s,2}$ topology, see [20], Chapter 2.3.2. Let $tr_E$ and $\mathfrak{F}_E$ be the restriction/extension operators from Proposition 2.5. Since $tr_E$ is a left inverse of $\mathfrak{F}_E$, the operator $\mathfrak{P} := 1 - \mathfrak{F}_E tr_E$ is a continuous projection in $W^{s,2}(\mathbb{R}^d)$, called the Jonsson/Wallin projection. Recall that, for $\phi \in C^\infty_0(\mathbb{R}^d)$, $\mathfrak{P}\phi$ is smooth on $\mathbb{R}^d \setminus E$. Moreover, it is also Lipschitzian on $\mathbb{R}^d$, so that the gradient is globally bounded, in particular bounded on $\Omega$ (*cf.* Proposition 2.5). Applying the projector $\mathfrak{P}$ and taking into account $\mathfrak{P}\tilde{\psi} = \tilde{\psi}$, one gets $\lim_{n\mapsto\infty} \mathfrak{P}\psi_n = \mathfrak{P}\tilde{\psi} = \tilde{\psi}$. But $\mathfrak{P}\psi_n$ is $C^\infty$ in $\mathbb{R}^d \setminus E$, and this is in particular true in $\Omega$. Now assume $\psi \in H^s_E(\Omega)$. Take $\tilde{\psi} \in W^{s,2}_E(\mathbb{R}^d)$ as any extension of $\psi$. Then it is clear that $\psi - \mathfrak{P}\psi_n|_\Omega = (\tilde{\psi} - \mathfrak{P}\psi_n)|_\Omega$ converges to zero in $H^s_E(\Omega)$ in the factor topology. □

## 2.2. Some interpolation results

We shall require several interpolation results – mainly in order to obtain further regularity results for the solutions of the parabolic equations. For convenience of the reader we collected most of them in this subsection.

**Proposition 2.12.** *i) Assume that* $\tau_0, \tau_1 \in ]\frac{1}{2}, \frac{3}{2}[$ *and put* $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $\theta \in ]0,1[$. *Then*

$$(H_{\mathfrak{D}}^{\tau_0}, H_{\mathfrak{D}}^{\tau_1})_{\theta,2} = [H_{\mathfrak{D}}^{\tau_0}, H_{\mathfrak{D}}^{\tau_1}]_\theta = H_{\mathfrak{D}}^{\tau}. \tag{2.6}$$

*ii) Let* $\tau \in ]\frac{1}{2}, \frac{3}{2}[$. *Then we have*

$$[L^2, H_{\mathfrak{D}}^{\tau}]_\theta = (L^2, H_{\mathfrak{D}}^{\tau})_{\theta,2} = \begin{cases} H^{\tau\theta} & \text{if } \tau\theta \in ]0, \frac{1}{2}[ \\ H_{\mathfrak{D}}^{\tau\theta} & \text{if } \tau\theta > \frac{1}{2}. \end{cases} \tag{2.7}$$

*Proof.* i) is proved in [26] as Thm. 7.1. ii) is Prop. 7.8 in [26]. □

**Lemma 2.13.** *Assume that* $s \in [0, \frac{1}{2}[$. *Then any interpolation space* $[L^2, H^s]_\theta$ *compactly embeds into* $L^2$. *For* $s \in ]\frac{1}{2}, \frac{3}{2}[$ *the interpolation space* $[L^2, H_{\mathfrak{D}}^s]_\theta$ *compactly embeds into* $L^2$.

*Proof.* Both, $L^2 = L^2(\Omega)$ and $H^s = H^s(\Omega)$ are retracts from $L^2(B)$ and $H^s(B)$, respectively – in the corresponding factor topology – if $B$ is a sufficiently large ball in $\mathbb{R}^d$. So, since $H^s(B)$ compactly embeds into $L^2(B)$, $H^s(\Omega)$ compactly embeds into $L^2(\Omega)$. Hence, the first claim follows from [20], Chapter 1.16.4. The second assertion follows from the first and the embedding $H_{\mathfrak{D}}^s \hookrightarrow H^{\frac{1}{4}}$. □

**Proposition 2.14.** *Let* $V$ *be a reflexive Banach space and* $H$ *a Hilbert space with continuous, dense injection* $V \hookrightarrow H$. *Then one has the (complex) interpolation identity* $[V, V^*]_{\frac{1}{2}} = H$.

*Proof.* The result dates back to [27], compare also [28] and [29]. □

**Lemma 2.15.** *Assume that* $s \in ]0, \frac{1}{2}[$. *If* $\theta \in ]\frac{1+s}{2}, 1[$, *then one has the interpolation identity*

$$[H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1-s}]_\theta = [L^2, H_{\mathfrak{D}}^{1+s}]_{\frac{2\theta-1-s}{1+s}}, \tag{2.8}$$

*and, hence, the embedding*

$$(H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1-s})_{\theta,1} \hookrightarrow [L^2, H_{\mathfrak{D}}^{1+s}]_{\frac{2\theta-1-s}{1+s}}. \tag{2.9}$$

*Proof.* Proposition 2.14 implies that $L^2 = [H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{1}{2}}$, from which we make repeated use. By Prop. 2.12 and re-iteration we have

$$H_{\mathfrak{D}}^{1-s} = [L^2, H_{\mathfrak{D}}^{1+s,2}]_{\frac{1-s}{1+s}} = \big[[H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{1}{2}}, H_{\mathfrak{D}}^{1+s}\big]_{\frac{1-s}{1+s}} = [H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{1}{1+s}}, \tag{2.10}$$

and thus

$$[H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1-s}]_\theta = \big[H_{\mathfrak{D}}^{-1-s}, [H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{1}{1+s}}\big]_\theta = [H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{\theta}{1+s}}. \tag{2.11}$$

Since $\frac{\theta}{1+s} > \frac{1}{2}$, (2.11) may be written by re-iteration as

$$\big[[H_{\mathfrak{D}}^{-1-s}, H_{\mathfrak{D}}^{1+s}]_{\frac{1}{2}}, H_{\mathfrak{D}}^{1+s}\big]_{\frac{2\theta-1-s}{1+s}} = [L^2, H_{\mathfrak{D}}^{1+s}]_{\frac{2\theta-1-s}{1+s}},$$

which provides the desired result. □

**Lemma 2.16.** *Assume that* $s \in ]0, \frac{1}{2}[$. *For* $\theta \in ]\frac{1-s}{2}, \frac{1}{2}]$ *we have*

$$[H^{-1+s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_\theta = [L^2, H^{1+s}_{\mathfrak{D}}]_{\frac{2\theta-1+s}{1+s}} = H^{2\theta-1+s}. \tag{2.12}$$

*Proof.* From (2.10) and duality (see [20], Ch. 1.11.3) we obtain

$$H^{-1+s}_{\mathfrak{D}} = \left(H^{1-s}_{\mathfrak{D}}\right)^* = [H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]^*_{\frac{1}{1+s}} = [H^{1+s}_{\mathfrak{D}}, H^{-1-s}_{\mathfrak{D}}]_{\frac{1}{1+s}} = [H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_{\frac{s}{1+s}}.$$

Exploiting this, we get by re-iteration

$$[H^{-1+s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_\theta = \big[[H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_{\frac{s}{1+s}}, H^{1+s}_{\mathfrak{D}}\big]_\theta$$

$$= [H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_\kappa \quad \text{with } \kappa = (1-\theta)\frac{s}{1+s} + \theta = \frac{\theta+s}{1+s}. \tag{2.13}$$

Our choice of $\theta$ implies that $\kappa > \frac{1}{2}$. So we may again apply re-iteration in order to write

$$[H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_\kappa = \big[[H^{-1-s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}}]_{\frac{1}{2}}, H^{1+s}_{\mathfrak{D}}\big]_{2\kappa-1} = \big[L^2, H^{1+s}_{\mathfrak{D}}\big]_{2\kappa-1} = \big[L^2, H^{1+s}_{\mathfrak{D}}\big]_{\frac{2\theta-1+s}{1+s}}. \tag{2.14}$$

Thus, the first equality in (2.12) is proved. The second follows from this one by means of Proposition 2.12. ☐

## 3. Elliptic and parabolic regularity in the $H^{-s}_{\mathfrak{D}}$ scale

### 3.1. Elliptic operators

**Definition 3.1.** For $\mu \in L^\infty(\Omega; \mathbb{C}^{d\times d})$, we define the operator

$$-\nabla\cdot\mu\nabla + 1 : H^1_{\mathfrak{D}} \to H^{-1}_{\mathfrak{D}} \tag{3.1}$$

by

$$\langle -\nabla\cdot\mu\nabla\psi + \psi, \varphi\rangle_{H^{-1}_{\mathfrak{D}}\times H^1_{\mathfrak{D}}} = \int_\Omega \mu\nabla\psi\cdot\nabla\overline{\varphi} + \psi\overline{\varphi}, \quad \psi,\varphi \in H^1_{\mathfrak{D}}. \tag{3.2}$$

If $\mu$ satisfies the strong ellipticity condition

$$\Re(\mu(x)\xi,\xi)_{\mathbb{C}^d} \geq m|\xi|^2, \quad \xi\in\mathbb{C}^d \tag{3.3}$$

for some $m > 0$, uniformly for almost all $x \in \Omega$, then (3.1) is a topological isomorphism by the Lax-Milgram theorem.

**Definition 3.2** (Multiplier)**.** Let $X$ be a Banach space of functions $\Omega \to \mathbb{C}$. A bounded function $\zeta\colon \Omega \to \mathbb{C}$ is a *multiplier on* $X$ if the multiplication operator $M_\zeta$ defined by $(M_\zeta f)(x) := \zeta(x)f(x)$ maps $X$ continuously into itself. We write $\zeta \in \mathfrak{M}(X)$ and the multiplier norm is given by $\|\zeta\|_{\mathfrak{M}(X)} := \|M_\zeta\|_{X\to X}$.

**Assumption 3.3.** There exits $\delta \in ]0, \frac{1}{2}[$, such that all components $\mu_{i,j}$ are multipliers on the space $H^s$, $s \in ]0, \delta]$

**Remark 3.4.** i) Trivially, any $\omega \in L^\infty$ is a multiplier on $L^2$. So, if $\zeta$ is multiplier on $H^\epsilon$ and, additionally, $\zeta \in L^\infty(\Omega)$, then one deduces from Proposition 2.12 and interpolation that $\zeta$ is a also multiplier for all $H^s$ for $s \in ]0,\epsilon[$.

ii) If $s \in ]0, \frac{1}{2}[$ and $\sigma > s$, then every $\zeta \in C^\sigma(\Omega)$ is a multiplier on $H^s$.

iii) The class of Hölder functions does not exhausts the set of multipliers. On the contrary, every indicator function $\chi_\Lambda$ is a multiplier for each space $H^s$, $s \in ]0, \frac{1}{2}[$ as long as $\Lambda \subset \Omega$ is a set of locally finite perimeter ([30], p. 214ff). An advanced criterion for 'set of finite perimeter' is given in [19], Theorem 5.23.

iv) Further remarks on the multipliers can be found in [4], Chapter 5 and references therein.

**Proposition 3.5.** *Let the coefficient function $\mu$ be bounded and strongly elliptic, and let Assumption 3.3 hold. Then there is $\iota \in ]0, \frac{1}{2}[$ such that (3.1) consistently extrapolates for $s \in [-\iota, \iota]$ to a topological isomorphism*

$$-\nabla \cdot \mu \nabla + 1 : H^{1+s}_{\mathfrak{D}} \to H^{-1+s}_{\mathfrak{D}}. \tag{3.4}$$

*Further, both $\iota$ and the norms of the inverse operators $(-\nabla \cdot \mu \nabla + 1)^{-1}$ for $s \in [-\iota, \iota]$ can be estimated uniformly in the norm of the multiplier norms $\|\mu_{i,j}\|_{\mathfrak{M}(H^\iota)}$ and the ellipticity constant $m$ in (3.3).*

*Proof.* see [4], Theorem 1/Corollary 1. □

In all what follows the letter $\iota$ has the meaning of characterizing a *closed* interval of numbers $s$, such that (3.4) is a topological isomorphim.

In the context of this proposition we always use the same symbol $-\nabla \cdot \mu \nabla$ irrespective what $s$ is. Unfortunately, one cannot expect in general that $\iota$ significantly differs from zero as is known since long (see [31], compare also [32], Ch. 4). The following result asserts that $\nabla \cdot \mu \nabla - 1$ generates an analytic semigroup on $H^{-s}_{\mathfrak{D}}$ for $s \in ]-\frac{3}{2}, -\frac{1}{2}[$.

**Theorem 3.6.** *i) For $s \in ]\frac{1}{2}, 1[$ the part of*

$$\nabla \cdot \mu \nabla - 1 : H^1_{\mathfrak{D}} \to H^{-1}_{\mathfrak{D}}$$

*in $H^{-1+s}_{\mathfrak{D}}$ generates an analytic semigroup on that space. In particular, $-\nabla \cdot \mu \nabla + 1$ is a positive operator in the sense of Triebel, see [20], Chapter 1.14 on these spaces.*

*ii) Assume $s \in ]0, \iota[$. Then $\nabla \cdot \mu \nabla - 1$ admits the following resolvent estimate*

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{\mathcal{L}(H^{-1-s}_{\mathfrak{D}})} \leq \frac{c}{1 + |\lambda|}, \quad \Re\lambda \geq 0, \tag{3.5}$$

*and, hence, it generates an analytic semigroup on $H^{-1-s}_{\mathfrak{D}}(\Omega)$. In particular, $-\nabla \cdot \mu \nabla + 1$ is a* positive *operator on $H^{-1-s}_{\mathfrak{D}}$ in the sense of Triebel.*

*Proof.* i) $\nabla \cdot \mu \nabla - 1$ generates analytic semigroups on both, $L^2$ and $H^{-1}_{\mathfrak{D}}$, see [33], Chapter 1.4.2. In particular, it satisfies, for $X = L^2$ or $X = H^{-1}_{\mathfrak{D}}$, resolvent estimates like

$$\|(-\nabla \cdot \mu \nabla + 1 + \lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{1 + |\lambda|}, \quad \Re\lambda \geq 0. \tag{3.6}$$

Applying the well-known characterization of an analytic generator property via the resolvent decay, it is clear that $\nabla \cdot \mu \nabla - 1$ generates an analytic semigroup also on every (real or complex) interpolation space between $L^2$ and $H^{-1}_{\mathfrak{D}}$ . Taking into account (2.7) and applying the duality formula for complex interpolation (see [20], Ch. 1.11.3) one gets $[L^2, H^{-1}_{\mathfrak{D}}]_{1-s} = H^{-1+s}_{\mathfrak{D}}$ .

ii) It is well-known that the $L^2$ realization of $-\nabla \cdot \mu \nabla + 1$ admits a bounded holomorphic calculus (see [34], Cor. 7.1.17, compare also [35], Ch. 2.3) and, hence, bounded purely imaginary powers (see [35], Ch. 2.3). Secondly, the positive resolved Kato square root problem in [26] (see also [36]) tells us

$$\mathrm{Dom}\Big(\big((-\nabla\cdot\mu\nabla+1)|_{L^2}\big)^{\frac{1}{2}}\Big)=H^1_{\mathfrak{D}}. \tag{3.7}$$

Together, this gives, according to [20], Chapter 1.15.3,

$$\mathrm{Dom}\Big(\big((-\nabla\cdot\mu\nabla+1)|_{L^2}\big)^{\frac{1-s}{2}}\Big)=\big[L^2,\mathrm{Dom}\Big(\big((-\nabla\cdot\mu\nabla+1)|_{L^2}\big)^{\frac{1}{2}}\Big)\big]_{1-s}=[L^2,H^1_{\mathfrak{D}}]_{1-s}=H^{1-s}_{\mathfrak{D}}, \tag{3.8}$$

thanks to (2.7). In other words,

$$\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1-s}{2}}:H^{1-s}_{\mathfrak{D}}\to L^2$$

is a topological isomorphism. Combining this with (3.4) one also gets the isomorphism property

$$\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1-s}{2}}(-\nabla\cdot\mu\nabla+1)^{-1}:H^{-1-s}_{\mathfrak{D}}\to L^2. \tag{3.9}$$

Let us now prove that the – well-known – resolvent decay on $L^2$ implies the asserted resolvent decay on $H^{-1-s}_{\mathfrak{D}}$: due to the consistency of $-\nabla\cdot\mu\nabla+1$ on different spaces under consideration, one has for $\psi\in L^2$ and $\lambda\in\mathbb{C}$ with $\Re\lambda\geq 0$

$$\|(-\nabla\cdot\mu\nabla+1+\lambda)^{-1}\psi\|_{H^{-1-s}_{\mathfrak{D}}}=\|\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1+s}{2}}(-\nabla\cdot\mu\nabla+1+\lambda)^{-1}\big(-\nabla\cdot\mu\nabla+1\big)^{-\frac{1+s}{2}}\psi\|_{H^{-1-s}_{\mathfrak{D}}}$$

$$\leq\|(-\nabla\cdot\mu\nabla+1)\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{s-1}{2}}\|_{\mathcal{L}(L^2;H^{-1-s}_{\mathfrak{D}})}\quad\|(-\nabla\cdot\mu\nabla+1+\lambda)^{-1}\|_{\mathcal{L}(L^2)}$$

$$\|\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1-s}{2}}(-\nabla\cdot\mu\nabla+1)^{-1}\|_{\mathcal{L}(H^{-1-s}_{\mathfrak{D}};L^2)}\|\psi\|_{H^{-1-s}_{\mathfrak{D}}}.$$

Since $L^2$ is dense in $H^{-1-s}_{\mathfrak{D}}$ this resolvent estimate extends to *all* $\psi\in H^{-1-s}_{\mathfrak{D}}$, and the theorem is proved. □

**Remark 3.7.** The reader should carefully notice that now, *knowing* that $-\nabla\cdot\mu\nabla+1$ in fact is a positive operator on $H^{-1-s}_{\mathfrak{D}}$, one indeed has the operator identity

$$\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1-s}{2}}(-\nabla\cdot\mu\nabla+1)^{-1}=\big(-\nabla\cdot\mu\nabla+1\big)^{-\frac{1+s}{2}} \tag{3.10}$$

as an operator equality on $H^{-1-s}_{\mathfrak{D}}$. So (3.9) can be read as

$$\mathrm{Dom}_{H^{-1-s}_{\mathfrak{D}}}\big((-\nabla\cdot\mu\nabla+1)^{\frac{s+1}{2}}\big)=L^2. \tag{3.11}$$

### 3.2. Maximal parabolic regularity: Definition and results

Throughout the rest of this paper let $T > 0$ and set $J = ]0, T[$. Let us start by recalling the following (standard) definition.

**Definition 3.8.** If $X$ is a Banach space and $q \in ]1, \infty[$, then we denote by $L^q(J; X)$ the space of $X$-valued functions $f$ on $J$ which are Bochner-measurable and for which $\int_J \|f(t)\|^q \, dt$ is finite. We define $W^{1,q}(J; X) := \{u \in L^q(J; X) : \frac{\partial u}{\partial t} \in L^q(J; X)\}$, where $\frac{\partial u}{\partial t}$ is to be understood as the time derivative of $u$ in the sense of $X$-valued distributions (*cf.* [37], Sect. III.1). Moreover, we introduce the subspace

$$W_0^{1,q}(J; X) := \{u \in W^{1,q}(J; X) : u(0) = 0\}.$$

We equip this subspace always with the norm $u \mapsto \|\frac{\partial u}{\partial t}\|_{L^q(J;X)}$.

**Definition 3.9.** Let $X$, $D$ be Banach spaces with $D$ densely and continuously embedded in $X$. Let $J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ be a bounded and strongly measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in $X$ for all $t \in J$. Let $q \in ]1, \infty[$. Then we say that the family $\{\mathcal{A}(t)\}_{t \in J}$ satisfies **(non-autonomous) maximal parabolic $L^q(J; X)$-regularity**, if for each $f \in L^q(J; X)$ there is a unique function $u \in L^q(J; D) \cap W_0^{1,q}(J; X)$ which satisfies

$$\frac{\partial u}{\partial t} + \mathcal{A}(t)u(t) = f(t) \tag{3.12}$$

for almost all $t \in J$. We write

$$\mathrm{MR}_0^q(J; D, X) := L^q(J; D) \cap W_0^{1,q}(J; X) \tag{3.13}$$

for the space of maximal parabolic regularity. Introducing the norm

$$\|u\|_{\mathrm{MR}_0^q(J;D,X)} = \|u\|_{L^q(J;D)} + \|\frac{\partial u}{\partial t}\|_{L^q(J;X)},$$

makes $\mathrm{MR}_0^q(J; D, X)$ a Banach space.

We emphasize that $\mathrm{Dom}(\mathcal{A}(t)) = D$ for all $t \in J$ in Definition 3.9. In particular, all operators $\mathcal{A}(t)$ have the same domain. If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator $\mathcal{A}_0$, and there exists an $q_0 \in ]1, \infty[$ such that $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^{q_0}(J; X)$-regularity, then $\{\mathcal{A}(t)\}_{t \in J}$ satisfies maximal parabolic $L^q(J; X)$-regularity for all $q \in ]1, \infty[$ and we say that $\mathcal{A}_0$ satisfies **maximal parabolic regularity on $X$**.

Let us recall some abstract embedding properties of the space of maximal parabolic regularity which we will need later.

**Proposition 3.10.** *Let $X, Y$ be Banach spaces with $Y$ is continuously embedded into $X$. Let $T > 0$ and set $J = ]0, T[$.*

*i) If $q \in ]1, \infty[$, then*

$$W^{1,q}(J; X) \cap L^q(J; Y) \hookrightarrow C(\overline{J}; (X, Y)_{1-\frac{1}{q},q}), \tag{3.14}$$

*(see [37], Ch. III, Thm. 4.10).*

*ii) If $q \in ]1, \infty[$ and $\theta \in ]0, 1 - \frac{1}{q}[$, then*

$$W^{1,q}(J; X) \cap L^q(J; Y) \hookrightarrow C^\beta(J; (X, Y)_{\theta,1}) \hookrightarrow C^\beta(J; [X, Y]_\theta), \tag{3.15}$$

*where $\beta = 1 - \frac{1}{q} - \theta$, ( see [5], Lem. 2.11).*

Our next aim is to show that the second order divergence operators indeed satisfy maximal parabolic regularity on the spaces $H^{-s}_{\mathfrak{D}}$ – starting with the *autonomous* case. We recall that Assumptions 2.2 and 3.3 are supposed to hold throughout.

**Theorem 3.11.** *i) For all $s \in [-\iota, \iota]$, the domain of the operator $-\nabla \cdot \mu\nabla + 1$, when considered in $H^{-1+s}_{\mathfrak{D}}$, is $H^{1+s}_{\mathfrak{D}}$.*

*ii) For $s \in [-\iota, \frac{1}{2}[$, $-\nabla \cdot \mu\nabla + 1$ satisfies maximal parabolic regularity in $H^{-1+s}_{\mathfrak{D}}$.*

*Proof.* Property i) follows directly from Theorem 3.5. ii) It is well-known that every negative of a generator of an analytic semigroup on a Hilbert space satisfies maximal parabolic regularity there. So, for $s \in [0, \frac{1}{2}[$ the claim follows from Theorem 3.6 i). For $s \in [-\iota, 0[$ the result is obtained by Theorem 3.6 ii). □

Recall that all these considerations are *uniformly* valid for a family of coefficient functions $\{\mu_\tau\}_\tau$ which admits a uniform in $\tau$ ellipticity constant and uniform in $\tau$ multiplier norms.

Now we pass to *non-autonomous* parabolic operators on the $H^{-s}_{\mathfrak{D}}$ scale.

**Assumption 3.12.** i) Let $\hat{\mu} \colon J \times \Omega \to \mathbb{C}^{d\times d}$ be a bounded mapping, such that

$$J \in t \mapsto \mu(t, \cdot) \in L^1(\Omega, \mathbb{C}^{d\times d}), \tag{3.16}$$

is measurable.

ii) The coeffcient functions $\mu(t, \cdot)$ are elliptic, and the ellipticity constants may be taken uniform with respect to $t \in J$, see (3.3).

iii) For every $t \in J$ the coefficient function $\mu(t, \cdot) =: \mu$ satisfies Assumption 3.3 The corresponding norms as multipliers are uniformly (in $t \in J$ and $s \in ]0, \delta]$) bounded.

We consider the coefficient function $\hat{\mu}$ satisfying this assumption as fixed.

Note that the set of points in $\Omega$ where $\mu(t, \cdot)$ is discontinuous may depend on $t$. Consequently, we admit the situation that the mapping $t \mapsto \mu(t, \cdot)$ from $J$ into $L^\infty(\Omega; \mathbb{C}^{d\times d})$ is discontinuous at every time point $t$. In this case it cannot be measurable, see the example in [38], Chapter 7.1. For this reason we only demand $L^1$-measurability in Assumption 3.12.

**Definition 3.13.** Let $q \in ]1, \infty[$ and $s \in ] - \delta, \delta[$, with $\delta$ as in Assumption 3.12. Define the mapping $\mathcal{A} : L^q(J; H^{1+s}_{\mathfrak{D}}) \to L^q(J, H^{-1+s}_{\mathfrak{D}})$ by

$$\big(\mathcal{A}u\big)(t) = -\nabla \cdot \hat{\mu}(t, \cdot)\nabla(u(t)), \quad u \in L^q(J; H^{1+s}_{\mathfrak{D}}). \tag{3.17}$$

**Lemma 3.14.** *Let $q \in ]1, \infty[$ and $s \in ] - \delta, \delta[$, with $\delta$ as in Assumption 3.12. Then the following properties hold.*

*i) For every $\psi \in H^{1+s}_{\mathfrak{D}}$, the mapping*

$$t \mapsto -\nabla \cdot \hat{\mu}(t, \cdot)\nabla\psi \tag{3.18}$$

*is (strongly) measurable from $J$ into $H^{-1+s}_{\mathfrak{D}}$.*

*ii)*

$$\mathrm{MR}^q_0(J; H^{1+s}_{\mathfrak{D}}, H^{-1+s}_{\mathfrak{D}}) \ni u \mapsto \frac{\partial u}{\partial t} - \mathcal{A}\,u$$

*is a bounded linear map into $L^q(J; H^{-1+s}_{\mathfrak{D}})$ with a norm which depends on the multiplier norms of the coefficient functions $\hat{\mu}(t, \cdot)$.*

*Proof.* i) We start with the case $s > 0$. One first considers

$$J \ni t \mapsto \langle -\nabla \cdot \mu(t,\cdot)\nabla\psi, \varphi\rangle_{(H_{\mathfrak{D}}^{-1+s}; H_{\mathfrak{D}}^{1-s})} = \langle \mu(t,\cdot)\nabla\psi, \nabla\varphi\rangle_{(H_{\mathfrak{D}}^{s}; H_{\mathfrak{D}}^{-s})}. \tag{3.19}$$

Taking $\psi \in H_{\mathfrak{D}}^{1+s} \cap C_b^\infty$ (see Def. 2.10) and $\varphi$ from the set $C_{\mathfrak{D}}^\infty(\Omega)$, the right hand side in (3.19) equals $\int_\Omega \mu(t,\cdot)\nabla\psi \cdot \nabla\overline{\varphi}$. Since $\nabla\psi, \nabla\varphi \in L^\infty(\Omega)$, the measurability in $t$ follows from the asserted measurability for $\hat{\mu}$. But $H_{\mathfrak{D}}^{1+s} \cap C_b^\infty(\Omega)$ is dense in $H_{\mathfrak{D}}^{1+s}$ and $C_{\mathfrak{D}}^\infty(\Omega)$ is dense in $H_{\mathfrak{D}}^{1-s}$ by Lemma 2.11. So the measurability for general $\psi \in H_{\mathfrak{D}}^{1+s}$ and $\varphi \in H_{\mathfrak{D}}^{1-s}$ follows by taking the limit in (3.19). Thus, we have proved *weak* measurability of (3.18). But this implies strong measurability since $H_{\mathfrak{D}}^{1-s}$ is separable and reflexive – and so is $H_{\mathfrak{D}}^{-1+s}$ . The case $s < 0$ can be treated analogously, identifying $\langle -\nabla \cdot \mu(t,\cdot)\nabla\psi, \varphi\rangle_{H_{\mathfrak{D}}^{-1+s} \times H_{\mathfrak{D}}^{1-s}}$ as $\langle \nabla\psi, \big(\mu(t,\cdot)\big)^*\nabla\varphi\rangle_{H_{\mathfrak{D}}^{s} \times H_{\mathfrak{D}}^{-s}}$ (see [4], p. 1237 for more details) and then arguing as before.

Property ii) is easily deduced from Proposition 3.5. □

**Proposition 3.15.** *There is an open interval $\mathfrak{I} \ni 2$, such that for each $q \in \mathfrak{I}$*

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; H_{\mathfrak{D}}^1, H_{\mathfrak{D}}^{-1}) \to L^q(J; H_{\mathfrak{D}}^{-1}) \tag{3.20}$$

*is a topological isomorphism i.e. $-\mathcal{A}$ satisfies non-autonomous maximal parabolic regularity on $L^q(J, H_{\mathfrak{D}}^{-1})$.*

*Proof.* The result is proved in [38], Theorem 7.2. The reader should notice that here *no* geometric suppositions on $\Omega$ are required and that for $\mathfrak{D}$ (relative) closedness in $\partial\Omega$ suffices. □

**Theorem 3.16.** *For every $q \in \mathfrak{I}$, there is an open interval $\mathfrak{J}(q) \ni 0$ so that for $s \in \mathfrak{J}(q)$*

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; H_{\mathfrak{D}}^{1+s}, H_{\mathfrak{D}}^{s-1}) \to L^q(J; H_{\mathfrak{D}}^{s-1}) \tag{3.21}$$

*is a topological isomorphism. Thus $-\mathcal{A}$ satisfies non-autonomous maximal parabolic regularity on $L^q(J, H_{\mathfrak{D}}^{s-1})$ .*

*Proof.* Let $q \in \mathfrak{I}$. Assumption 3.12 implies the existence of an $\iota \in ]0, \frac{1}{2}[$ such that, for every $s \in [-\iota, \iota]$

$$-\nabla \cdot \mu(t,\cdot)\nabla + 1 : H_{\mathfrak{D}}^{1+s} \to H_{\mathfrak{D}}^{-1+s}, \qquad t \in J$$

is a toplogical isomorphisms (see Thm. 3.5) and, additionally, each of these operators satisfies maximal (autonomous) maximal parabolic regularity, see Theorem 3.16. Moreover, we already know that (3.20) is a topological isomorphism, which is (3.21) for $s = 0$.

Thanks to (2.6), one may write, for $\epsilon \in ]-\frac{1}{2}, \frac{1}{2}[$,

$$H_{\mathfrak{D}}^1 = [H_{\mathfrak{D}}^{1+\epsilon}, H_{\mathfrak{D}}^{1-\epsilon}]_{\frac{1}{2}},$$

and, by duality (see [20], Ch. 1.11.3)

$$H_{\mathfrak{D}}^{-1} = [H_{\mathfrak{D}}^{-1-\epsilon}, H_{\mathfrak{D}}^{-1+\epsilon}]_{\frac{1}{2}} = [H_{\mathfrak{D}}^{-1+\epsilon}, H_{\mathfrak{D}}^{-1-\epsilon}]_{\frac{1}{2}}$$

Hence, (3.20) can be interpreted in the sense that

$$\frac{\partial}{\partial t} - \mathcal{A} : \mathrm{MR}_0^q(J; [H_{\mathfrak{D}}^{1+\epsilon}, H_{\mathfrak{D}}^{1-\epsilon}]_{\frac{1}{2}}, [H_{\mathfrak{D}}^{-1+\epsilon}, H_{\mathfrak{D}}^{-1-\epsilon}]_{\frac{1}{2}}) \mapsto L^q(J; [H_{\mathfrak{D}}^{-1+\epsilon}, H_{\mathfrak{D}}^{-1-\epsilon}]_{\frac{1}{2}}) \tag{3.22}$$

provides a topological isomorphism. But then [38], Theorem 3.4 tells us that this remains a topological isomorphism, if the interpolation index $\frac{1}{2}$ is replaced by indices $\theta$ sufficiently close to $\frac{1}{2}$. Re-identifying, by (2.6), $[H^{1+\epsilon}_{\mathfrak{D}}, H^{1-\epsilon}_{\mathfrak{D}}]_\theta$ as $H^{1+\epsilon(1-2\theta)}_{\mathfrak{D}}$ and, by duality,

$$[H^{-1+\epsilon}_{\mathfrak{D}}, H^{-1-\epsilon}_{\mathfrak{D}}]_\theta = [H^{1-\epsilon}_{\mathfrak{D}}, H^{1+\epsilon}_{\mathfrak{D}}]^*_\theta = \big(H^{1-\epsilon(1-2\theta)}_{\mathfrak{D}}\big)^* = H^{-1+\epsilon(1-2\theta)}_{\mathfrak{D}},$$

one obtains the assertion. □

Up to now we considered the initial value problem with initial value 0. We will now pass to initial values $u_0 \neq 0$.

**Lemma 3.17.** *Let $X$ be a Banach space and $B$ the generator of an analytic semigroup on $X$. Then the following identity of sets holds*

$$(X, D(B))_{1-\frac{1}{p},p} = \{x \in X : B\mathrm{e}^{-tB}x \in L^p(]0,1[;X)\}. \tag{3.23}$$

*Moreover, both spaces in (3.23) also coincide topologically.*

*Proof.* See [39], Proposition 2.2.2 □

**Theorem 3.18.** *Choose $q \in \mathfrak{I}$ and $s \in \mathfrak{J}(q)$ as in Theorem 3.16. Then for every $u_0 \in (H^{-1+s,}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}})_{1-\frac{1}{q},q}$ and $f \in L^q(J; H^{-1+s}_{\mathfrak{D}})$ there exists a unique solution to*

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = f, \quad u(0) = u_0. \tag{3.24}$$

*This solution belongs to the maximal parabolic space*

$$W^{1,q}(J; H^{-1+s}_{\mathfrak{D}}) \cap L^q(J; H^{1+s}_{\mathfrak{D}}) =: X$$

*and admits the estimate*

$$\|u\|_X \le C(\|u_0\|_{(H^{-1+s,}_{\mathfrak{D}}, H^{1+s,}_{\mathfrak{D}})_{1-\frac{1}{q},q}} + \|f\|_{L^2(J;H^{-1+s,2}_{\mathfrak{D}})}), \tag{3.25}$$

*for a constant $C$ independent of $u_0$ and $f$.*

*Proof.* Using Theorem 3.16 and Lemma 3.17 the existence of a unique solution to follows (3.24) follows from [37], Proposition 2.1. The a-priori estimate (3.25) is obtained by following the proof of the cited proposition. □

Having now the inclusion of the solution in the maximal parabolic regularity space at hand, our next aim are some embedding results for the space of maximal parabolic regularity in the $H^{-s}_{\mathfrak{D}}$ scale, based on Proposition 3.10.

**Theorem 3.19.** *For $s \in ]0, \frac{1}{2}[$ the space $L^2(J; H^{1+s}_{\mathfrak{D}}) \cap W^{1,2}(J; H^{-1+s}_{\mathfrak{D}}) =: X$ embeds compactly into $C(\overline{J}; L^2)$.*

*Proof.* According to Proposition 3.10 one has for $\theta \in ]0, \frac{1}{2}[$ the embedding $X \hookrightarrow C^\beta(J; (H^{-1+s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}})_{\theta,1})$, where $\beta = \frac{1}{2} - \theta$. Thus, by Arzela/Ascoli it is sufficient to show that, $\theta$ clever chosen, the space $(H^{-1+s}_{\mathfrak{D}}, H^{1+s}_{\mathfrak{D}})_{\theta,1}$ compactly embeds into $L^2$. But Lemma 2.16 in combination with Lemma 2.13 tells us that all $\theta \in ]\frac{1-s}{2}, \frac{1}{2}[$ do this job. □

**Theorem 3.20.** *Let $q > 2$, $s \in ]0, \frac{1}{2}[$ such that the interval $]\frac{1+s}{2}, 1 - \frac{1}{q}[$ is non-empty and $\theta \in ]\frac{1+s}{2}, 1 - \frac{1}{q}[$. Put $\beta = 1 - \frac{1}{q} - \theta$. Then one has the embedding*

$$W^{1,q}(J; H^{-1-s}_{\mathfrak{D}}) \cap L^q(J; H^{1-s}_{\mathfrak{D}}) \hookrightarrow C^\beta(J; [L^2, H^{1+s}_{\mathfrak{D}}]_{\frac{2\theta-1-s}{1+s}}), \tag{3.26}$$

*and the maximal parabolic space $\mathrm{MR}^q_0(J; H^{1-s}_{\mathfrak{D}}, H^{-1-s}_{\mathfrak{D}})$ embeds compactly into $C(\overline{J}; L^2)$.*

*Proof.* The first assertion is implied by a combination of (3.15) and (2.9). The second follows from (3.26), Lemma 2.13 and the Arzela/Ascoli theorem. □

## 4. Non-autonomous problems with measure-valued functions as right hand sides

### 4.1. Generalities

In this chapter we investigate non-autonomous parabolic problems like

$$\frac{\partial u}{\partial t} - \mathcal{A}u = \varrho, \quad u(0) = 0, \tag{4.1}$$

with $\varrho$ a function on $J$, taking values as bounded Radon measures $\rho_t$ on $\Omega$ at every $t \in J$.

It is important to consider mappings $J \ni t \mapsto \rho_t \in \mathcal{M}(\Omega)$, which are only weak* measurable, this means: mappings, such that

$$J \ni t \mapsto \langle \rho_t, \psi \rangle_{\mathcal{M} \times C(\Omega)} \tag{4.2}$$

is measurable for all bounded $\psi \in C(\Omega)$, (compare the discussion in [15], Ch. 2.1). Otherwise one would exclude examples as the following:

Let $J \ni t \mapsto x(t)$ be an injective curve in $\Omega$. If one defines $\rho_t := \delta_{x(t)}$ (the Dirac measure in the point $x(t)$), then the mapping $J \ni t \mapsto \delta_{x(t)}$ is in *every* point discontinuous, if one equips the space of (bounded) measures with the strong topology. Hence, it is not measurable if one defines the structure of measurability via this strong topology. On the contrary, if one considers the weak* topology and the induced concept of measurability, then the mapping $J \ni t \mapsto \delta_{x(t)}$ is at least measurable if the mapping $J \ni t \mapsto x(t)$ is measurable itself.

If $\mathcal{N}$ is a space of measures for which one knows an embedding $\mathcal{N} \hookrightarrow H^{-s}_{\mathfrak{D}}$, then the measurability of (4.2) is in particular true for functions $\psi \in C^\infty_b(\Omega) \cap H^s_{\mathfrak{D}}$, which are dense in $H^s_{\mathfrak{D}}$ (see Lem. 2.11). Hence, the measurability carries over to all functions $\psi \in H^s_{\mathfrak{D}}$ by density. But this means: the mapping $J \ni t \mapsto \rho_t \in H^{-s}_{\mathfrak{D}}$ is weakly measurable in this case. Then the separability of $H^{-s}_{\mathfrak{D}}$ implies, quite in contrast to the situation in $\mathcal{M}(\Omega)$, even the strong measurability. Indeed, by Pettis's measurability theorem, which will again be used several times below, a function $f : \mathcal{X} \to \mathcal{B}$ defined on a measure space $\mathcal{X}$ and taking values in a Banach space $\mathcal{B}$ is strongly measurable (that is, it equals a.e. the limit of a sequence of measurable countably-valued functions) if and only if it is both weakly measurable and almost surely separably valued. Thus in our context one is, via embeddding, in a situation in which rather general mappings $J \ni t \mapsto \rho_t$ are admissable and, additionally, suit in the context of maximal parabolic regularity– even in the non-autonomous case.

However, the reader should carefully notice: weak* limits of measures, these being possibly concentrated on sets of lower Hausdorff dimension, can be of entirely different nature. *E.g. every* Radon measure on $\Omega$ is the weak* limit of linear combinations of Dirac measures on $\Omega$. In other words: the affiliation of a measure to a class of measures, concentrated on lower dimensional objects, is by no means necessarily preserved for the weak* limit.

### 4.2. Interpretation of singular measures as elements from $H_{\mathfrak{D}}^{-s}$

Up to now we have established a parabolic theory for second order operators in the $H_{\mathfrak{D}}^{-s}$ scale. In order to treat parabolic second order equations with measure valued right hand side it is, in consequence, necessary to allow an interpretation for these objects as elements from $H_{\mathfrak{D}}^{-s}$. This will be delivered next.

**Lemma 4.1.** *Let $\Omega \subset \mathbb{R}^2$, $\mathfrak{D}$ be a closed subset of $\partial\Omega$. Then the space of bounded Radon measures on $\Omega$, $\mathcal{M}(\Omega)$, continuously embeds into any space $H_{\mathfrak{D}}^{-1-\epsilon}$ if $\epsilon > 0$.*

*Proof.* Let $\epsilon > 0$ . For every $u \in H_{\mathfrak{D}}^{1+\epsilon}$ there is a $\tilde{u} \in W^{1+\epsilon,2}(\mathbb{R}^d)$ the restriction of which is $u$ and, additionally, $\|\tilde{u}\|_{W^{1+\epsilon,2}(\mathbb{R}^d)} \le 2\|u\|_{H_{\mathfrak{D}}^{1+\epsilon}}$. So, for every $\epsilon > 0$ one has $H_{\mathfrak{D}}^{1+\epsilon} \hookrightarrow C(\overline{\Omega})$. Thus, one gets for every bounded Radon measure $\mathfrak{m}$ on $\Omega$

$$\|\mathfrak{m}\|_{H_{\mathfrak{D}}^{-1-\epsilon}} = \sup_{\|\psi\|_{H_{\mathfrak{D}}^{1+\epsilon}}=1} \Big| \int_\Omega \psi \, d\mathfrak{m} \Big| \le \sup_{\|\psi\|_{H_{\mathfrak{D}}^{1+\epsilon}}=1} \sup_{x\in\Omega} |\psi(x)| \, \|\mathfrak{m}\| \le c \, \|\mathfrak{m}\|_{\mathcal{M}},$$

where $\|\mathfrak{m}\|_{\mathcal{M}}$ denotes the total variational norm of $\mathfrak{m}$. □

So far, this affects *general* bounded Radon measures on $\Omega$ irrespective of their singularity – even Dirac measures are admitted, compare *e.g.* [12], [9], [11].

In the sequel we restrict the class of measures which are admitted. The reason is twofold: first the classes which we will consider are the most relevant ones in view of applications. Secondly, as we have seen, optimal elliptic and parabolic regularity are only available for second order divergence operators if the differentiability index $s$ of the corresponding Hilbert space is close to $-1$, see Theorems 3.16 and 3.18. Thus, concerning parabolic equations, one is restricted to measures which can be considered as elements of $H_{\mathfrak{D}}^{-1\pm\epsilon}$ with $\epsilon \sim 0$. In two space dimensions it turns out that – besides the class of all bounded Radon measures – the measures situated on sets of Hausdorff dimension 1 deserve special attention. In three space dimensions this affects the measures concentrated on 'surfaces' and 'curves' – in fact: sets of Hausdorff dimensions 1 or 2. In order to make this precise, we need some preparation. Recall first the definition of an upper $l$-set from the introduction.

**Lemma 4.2.** *If the closed set $M \subset \mathbb{R}^d$ is an upper $l$-set satisfying (1.3), and one defines the measure $\chi$ on $\mathbb{R}^d$ by $\chi(N) = \mathcal{H}_l(N \cap M)$ for every Borel set $N \subset \mathbb{R}^d$, then $\chi$ satisfies $\chi(B(x,r)) \le 2^l \mathfrak{c}\, r^l$ for $x \in \mathbb{R}^d$ and $r \le 1/2$. Hence, $\chi(M)$ is finite.*

*Proof.* For all $x \in \mathbb{R}^d$ with $\operatorname{dist}(x,M) > 1/2$ one has $B(x,r) \cap M = \emptyset$ for $r \le 1/2$, so that $\chi(B(x,r)) = 0$ for these $r$. If $\operatorname{dist}(x,M) = r \le 1/2$, then exists a $y \in M$ with $|x-y| = r \le 1/2$. But then $B(x,r) \subseteq B(y,2r)$ and the assertion follows. □

**Proposition 4.3.** *If $M \subset \Omega$ is a Borel set of finite Hausdorff measure $\mathcal{H}_l$, then the mapping*

$$C_0(\Omega) \ni v \mapsto \int_M v \, d\mathcal{H}_l$$

*is a bounded Radon measure on $\Omega$.*

*Proof.* Since $\mathcal{H}_l$ is a Borel measure on $\mathbb{R}^d$ and $\mathcal{H}_l(M)$ is finite, the restriction of $\mathcal{H}_l$ to $M$ is a (bounded) Radon measure on $\mathbb{R}^d$ (see [19], Ch. 2, Thm. 2.1). It is clear that the restriction of this to $\Omega$ remains a (bounded) Radon measure. □

From the previous two results we conclude that if $M \subset \Omega$ is a Borelian upper $l$-set, then $\mathcal{H}_l|_M$ is a bounded Radon measure on $\Omega$. Moreover, the total mass of $\overline{M} \supset M$ with respect to $\mathcal{H}_l$ can be estimated by $\mathfrak{c} \times \mathfrak{n}$, where $\mathfrak{n}$ is the number of (shifted) unit balls required for a covering of $\overline{M}$.

**Proposition 4.4.** *Let $M \subset \mathbb{R}^d$ be an upper $l$-set with $0 < l \in \{d-2, d-1\}$. If $l = d-2$, let $\alpha \in ]1, \frac{3}{2}[$, and if $l = d-1$ let $\alpha \in ]\frac{1}{2}, 1]$.*

*i) For $f \in L^2(\mathbb{R}^d)$ and $u = G_\alpha \star f$ one has*

$$\|u\|_{L^2(M;\mathcal{H}_l)} \leq c\|f\|_{L^2(\mathbb{R}^d)}, \tag{4.3}$$

*$G_\alpha$ being the Bessel potential (see [40], Ch. V.3) for the index $\alpha$. The constant $c$ can be chosen independent from $f$.*

*ii) The constant $c$ may be taken even uniform for sets $M$ obeying the right estimate (1.3) with a uniform $\mathfrak{c}$.*

*Proof.* (i) is a special case of [3], Chapter VI, Lemma 6. (ii) follows by a careful inspection of that proof. For the convenience of the reader we give some comments in the appendix how to read the proof of [3], Chapter VI, Lemma 6 in the special case under consideration here. □

**Corollary 4.5.** *Let $M \subset \mathbb{R}^d$ be a closed upper $l$-set with $0 < l \in \{d-2, d-1\}$. Let $\alpha$ be as in Proposition 4.4. Then one has a continuous trace mapping $W^{\alpha,2}(\mathbb{R}^d) = H^\alpha_2(\mathbb{R}^d)$ into $L^2(M;\mathcal{H}_l)$, $H^\alpha_2(\mathbb{R}^d)$ being the well-known Bessel potential space (see [20], Chapter 2.3.3, compare also [40], Ch. V.3).*

*The norms of these trace mappings are uniformly bounded for different sets $M$ obeying the right estimate (1.3) with a uniform $\mathfrak{c}$.*

*Proof.* As is well-known, the space $H^\alpha_2(\mathbb{R}^d)$ can be defined as the set $\{G_\alpha \star f : f \in L^2(\mathbb{R}^d)\}$, equipped with the corresponding graph norm (see [20], Ch. 2.3.4). So (4.3) can be interpreted as

$$\|u\|_{L^2(M;\mathcal{H}_l)} \leq c\|u\|_{H^\alpha(\mathbb{R}^d)}, \tag{4.4}$$

and the assertions follow. □

**Theorem 4.6.** *i) Suppose that $M$ is a closed subset of $\Omega$, which is an upper $(d-1)$-set. Then, for $s \in ]\frac{1}{2}, 1[$, $H^{1-s}_{\mathfrak{D}}$ admits a trace on $M$, this belonging to $L^2(M;\mathcal{H}_{d-1})$.*
*The norm of the mapping $H^{1-s}_{\mathfrak{D}} \hookrightarrow L^2(M;\mathcal{H}_{d-1})$ is finite. Moreover, these norms are uniformly bounded for different sets $M$ obeying the right estimate (1.3) with a uniform $\mathfrak{c}$.*

*ii) Let now $d = 3$. Suppose that $M$ is a closed subset of $\Omega$, which is an upper $1$-set. Then, for $s \in ]1, \frac{1}{2}[$, $H^{1+s}_{\mathfrak{D}}$ admits a trace on $M$, this belonging to $L^2(M;\mathcal{H}_1)$.*
*The norm for the mapping $H^{1+s}_{\mathfrak{D}} \hookrightarrow L^2(M;\mathcal{H}_1)$ is finite. Moeover, these norms are uniformly bounded for different sets $M$ obeying the right estimate (1.3) with a uniform $\mathfrak{c}$.*

*Proof.* Recall that $H^s_{\mathfrak{D}}$ is the space of restrictions of $W^{s,2}_{\mathfrak{D}}(\mathbb{R}^d)$ functions, equipped with the factor topology. i) In this spirit, let, for $u \in H^{1-s}_{\mathfrak{D}}$, let $\tilde{u} \in W^{1-s,2}_{\mathfrak{D}}(\mathbb{R}^d)$ be an extension of $u$ with $\|\tilde{u}\|_{W^{1-s,2}_{\mathfrak{D}}(\mathbb{R}^d)} \leq 2\|u\|_{H^{1-s}_{\mathfrak{D}}}$. Now one applies Corollary 4.5, which implies $tr_{\overline{M}}\tilde{u} \in L^2(\overline{M};\mathcal{H}_{d-1})$– inclusively a corresponding estimate. Evidently, then also

$$tr_M\tilde{u} = tr_{\overline{M}}\tilde{u}|_M \in L^2(M;\mathcal{H}_{d-1}).$$

Finally, one takes into account that forming the trace on $M$ is the same for the extended function $\tilde{u}$ and the function $u$ on $\Omega$ since $M \subset \Omega$ and $\Omega$ is open. ii) The proof proceeds along the same lines, again fundamentally resting on Corollary 4.5. □

Having this at hand, our next aim is to show, by duality, that (suitable) measures $\sigma\mathcal{H}_l|_M$ may be considered in a natural manner as elements from (suitable) spaces $H^{-s}_{\mathfrak{D}}$.

**Lemma 4.7.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ . Let $M$ be a closed subset of $\Omega$ of finite $\mathcal{H}_l$ measure, $l \in \{1, \dots, d\}$. Suppose that $H^\tau_{\mathfrak{D}}$ admits a continuously trace mapping into $L^2(M;\mathcal{H}_l)$ with norm $\mathfrak{c}$. Then, for all $\sigma \in L^2(M;\mathcal{H}_l)$, the measure $\sigma\mathcal{H}_l|_M$ belongs to $H^{-\tau}_{\mathfrak{D}}$, and the mapping*

$$L^2(M;\mathcal{H}_l) \ni \sigma \mapsto \sigma\mathcal{H}_l|_M =: \Psi \in H^{-\tau}_{\mathfrak{D}} \tag{4.5}$$

*is well defined and has a norm not larger than $\mathfrak{c}$.*

*Proof.* One has

$$|\langle \sigma\mathcal{H}_l|_M, \psi\rangle_{H^{-\tau}\times H^\tau}| \le \int_M |\psi||\sigma| \mathrm{d}\mathcal{H}_l \le \|\sigma\|_{L^2(M;\mathcal{H}_l)}\|\psi\|_{L^2(M;\mathcal{H}_l)} \le \mathfrak{c}\ \|\sigma\|_{L^2(M;\mathcal{H}_l)}\|\psi\|_{H^\tau_{\mathfrak{D}}}, \quad \psi \in H^\tau_{\mathfrak{D}}. \tag{4.6}$$

□

**Theorem 4.8.** *i) Adopt the assumptions of Theorem 4.6 i). Then, for $s \in ]\frac{1}{2}, 1[$, $L^2(M;\mathcal{H}_{d-1})$ continuously embeds into $H^{-1+s}_{\mathfrak{D}}$.*

*ii) Adopt the assumptions of Theorem 4.6 ii). Then, for $s \in ]0, \frac{1}{2}[$, $L^2(M;\mathcal{H}_{d-2})$ continuously embeds into $H^{-1-s}_{\mathfrak{D}}$.*

*The embedding constants for the two previous mappings are uniformly bounded for different sets $M$ obeying the right estimate (1.3) with uniform $\mathfrak{c}$.*

*Proof.* The claims follow from Theorem 4.6 and Lemma 4.7. □

Let us have a closer look at what kind of restriction the uniformity of the constant $\mathfrak{c}$ means in a simple example:

Consider a bounded domain $\Omega \subset \mathbb{R}^2$ which includes $0 \in \mathbb{R}^2$ and a closed ball $\overline{B(0, r_0)}$ around it. Take a sequence $\{\alpha_k\}_n$ from the interval $[0, \pi]$, which converges to zero. From this we form the set

$$\mathcal{N}_N := \cup_{k\le N}\{x \in \mathbb{R}^2 : x = r\mathrm{e}^{i\alpha_k}, r \in [0, r_0]\}.$$

Then condition (1.3), with $l = 1$, obviously gives a bound for the admissible $N$. In any case, the union over *all* $k$ is not admissible.

However: if one changes the above set to

$$\cup_k\{x \in \mathbb{R}^2 : x = r\mathrm{e}^{i\alpha_k}, r \in [0, r_k]\}$$

and chooses the $r_k$ suitably, then (1.3) can indeed be satisfied in this case. Very roughly speaking, one can say: only finitely many 'curves' of constant length are admissible, but if the lengths may shrink to zero, then infinitely many may be admissible and still satisfy (1.3).

Let us now take a function $\eta \in C^\infty_0(\Omega)$ which is identical 1 on $\overline{B(0, r_0)}$. Then $\|\eta\|_{L^2(\mathcal{N}_N;\mathcal{H}_1)} = N r_0^{1/2}$. This clearly shows that, in order to delimitate the embedding constant of $H^1_{\mathfrak{D}} \hookrightarrow L^2(\mathcal{N}_N;\mathcal{H}_1))$ one must delimitate $N$.

Clearly, one can construct analogous examples also in higher dimensions.

**Lemma 4.9.** *Assume $s \in ]-\frac{1}{2}, \frac{1}{2}[$. For every $t \in J$, let $\rho_t$ be a bounded Borel measure on $\Omega$, such that the mapping $J \ni t \mapsto \rho_t$ is weak$^*$ measurable. Suppose further that*

$$\sup_{\psi \in H^{1+s}_{\mathfrak{D}} \cap C^\infty(\Omega), \|\psi\|_{H^{1+s}_{\mathfrak{D}}}=1} \Big| \int_\Omega \psi \, d\rho_t \Big| < \infty \text{ for every } t \in J. \tag{4.7}$$

*Then the mapping*

$$H^{1+s}_{\mathfrak{D}} \cap C^\infty(\Omega) \ni \psi \mapsto \int_\Omega \overline{\psi} \, d\rho_t \tag{4.8}$$

*extends by density to an element $\Psi_t \in H^{-1-s}_{\mathfrak{D}}$ (see (4.5)). Moreover, the mapping $J \ni t \mapsto \Psi_t \in H^{-1-s}_{\mathfrak{D}}$ is* strongly *measurable.*

*Proof.* First, recall Lemma 2.11. Assumption (4.7) implies that (4.8) is a continuous (anti)linear functional on $H^{1+s}_{\mathfrak{D}} \cap C^\infty(\Omega)$ with respect to the induced $H^{1+s}_{\mathfrak{D}}$ topology. By density of $H^{1s}_{\mathfrak{D}} \cap C^\infty(\Omega)$ in $H^{1+s,}_{\mathfrak{D}}$ this can be extended to a continuous antilinear functional on the whole space $H^{1+s}_{\mathfrak{D}}$.

Secondly, from the supposed weak$^*$ measurability it follows that $J \ni t \mapsto \int_\Omega \overline{\psi} \, d\rho_t = \langle \Psi_t, \psi \rangle$ is measurable as long as $\psi \in H^{1+s}_{\mathfrak{D}} \cap C^\infty(\Omega)$. But this latter set is dense in $H^{1+s}_{\mathfrak{D}}$, so the measurability for general $\psi \in H^{1+s}_{\mathfrak{D}}$ follows. This implies *weak* measurability of $J \ni t \mapsto \Psi_t \in H^{-1-s}_{\mathfrak{D}}$. Since $H^{-1+-s}_{\mathfrak{D}}$ is separable and reflexive, the asserted strong measurability follows. □

**Remark 4.10.** It is not by accident that we consider (4.8) first only on $H^{1+s}_{\mathfrak{D}} \cap C^\infty(\Omega)$ since it is *not a priori* clear that all elements of $H^{1+s}_{\mathfrak{D}}$ are measurable with respect to $\rho_t$ – and, hence, that the mapping (4.8) is well defined for all $\psi \in H^{1+s}_{\mathfrak{D}}$.

Up to now we were primarily interested in individual measures $\sigma \mathcal{H}_l|_M$. Having parabolic equations with *varying in time* measures as right hand sides in our general focus, we must find a concept which allows to identify the *time dependent*, measure-valued function as one with values in the Bessel potential space $H^{-\tau}_{\mathfrak{D}}$ – including suitable measurability and integrability properties. This is achieved in the next theorem. It appears to be unavoidable here to exploit a classical instrument from measure theory, namely the 'upper integral', denoted by $\int_J^* f \mathrm{d}t$. We follow here entirely the fundamental work [41], see Section Ch. 13.5 there. What we finally use concerning this, is the following: If $f \le g$ on the interval $J$, then $\int_J^* f \, \mathrm{d}t \le \int_J^* g \, \mathrm{d}t$. This has the consequence

$$\int_J f \mathrm{d}t \le \int_J^* g \, \mathrm{d}t, \quad \text{if } f \text{ is measurable.} \tag{4.9}$$

**Theorem 4.11.** *Suppose $0 < l \in \{d-2, d-1\}$. For every $t \in J$, let $M_t$ be a closed subset of $\Omega$ which is an upper $l$-set, with uniform $\mathfrak{c}$ in (1.3).*

*Assume that for every $t \in J$ there exists $\sigma_t \in L^2(M_t; \mathcal{H}_l)$ such that*
*a) the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_l|_{M_t} \in \mathcal{M}(\Omega) \tag{4.10}$$

*is weak$^*$ measurable and*
*b) the upper integral (see [41], Ch. 13.5) $\int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \, \mathrm{d}t$ is finite.*

*i) Let $s \in ]0, \frac{1}{2}[$ $\sigma_t \mathcal{H}_l|_{M_t}$ by Theorem 4.8 i).*

*Then the mapping* $J \ni t \mapsto \Psi_t \in H_{\mathfrak{D}}^{-1+s}$ *is strongly measurable and one has*

$$\int_J \|\Psi_t\|^q_{H_{\mathfrak{D}}^{-1+s}} \, \mathrm{d}t \le \mathfrak{k} \int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \mathrm{d}t \tag{4.11}$$

*for some constant* $\mathfrak{k}$*. Moreover, the constant* $\mathfrak{k}$ *is uniform with respect to all families* $\{\sigma_t\}_{t\in J}$ *for which* $\int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \, \mathrm{d}t < \infty$.

*ii) Let* $s \in ]0, \frac{1}{2}[$ *and* $\Psi_t \in H_{\mathfrak{D}}^{-1-s}$ *be the element which is associated to the measure* $\sigma_t \mathcal{H}_l|_{M_t}$ *by Theorem 4.8 ii).*

*Then the mapping* $J \ni t \mapsto \Psi_t \in H_{\mathfrak{D}}^{-1-s}$ *is strongly measurable and one has*

$$\int_J \|\Psi_t\|^q_{H_{\mathfrak{D}}^{-1-s}} \, \mathrm{d}t \le \mathfrak{k} \int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \mathrm{d}t \tag{4.12}$$

*for some constant* $\mathfrak{k}$*. Moreover, the constant* $\mathfrak{k}$ *is uniform with respect to all families* $\{\sigma_t\}_{t\in J}$ *for which* $\int_J^* \|\sigma_t\|^q_{L^2(M_t,\mathcal{H}_l)} \, \mathrm{d}t < \infty$.

*Proof.* Note that we do not know – at this stage – that the function $J \ni t \mapsto \|\sigma_t\|_{L^2(M_t,\mathcal{H}_l)}$ is measurable, so we imposed the required integrability condition via the upper integral.
i) Applying Lemma 4.9 – the assumptions of which are fulfilled according to Theorems 4.6 and 4.7 – the asserted measurability follows. Moreover thanks to Theorems 4.6 and 4.7 the uniform upper $l$-estimate implies $\|\Psi_t\|_{H_{\mathfrak{D}}^{-s}} \le \mathfrak{l} \|\sigma_t\|_{L^2(M_t,\mathcal{H}_l)}$ with a *uniform* in $t$ constant $\mathfrak{l}$. Thanks to (4.9), this proves (4.12).
ii) is proved analogously.

This theorem will be instrumental to establish solutions to (4.1) in Theorems 4.13 and 4.15 below. □

**Remark 4.12.** Reconsidering the assumptions of the previous theorem, the reader should carefully notice that – besides the weak* measurablity of the function (4.10) – *no* measurability condition is supposed for the function $t \mapsto \sigma_t$ and even not for $t \mapsto \|\sigma_t\|_{L^2(M_t,\mathcal{H}_l)}$ . To make such a measurability precise would be a challing task – and not easy to control in examples. This is the reason why we posed the conditions on the densities $\sigma_t$ in form of the upper integral. For the finiteness of the upper integral a uniform boundedness condition for the functions $\sigma_t$, for example, is a sufficient one since the function $J \ni t \mapsto \mathcal{H}_l(M_t)$ is bounded by the (supposed) uniform (upper) $l$-property of the sets $M_t$.

## 4.3. Regularity for non-autonomous parabolic equations with measure-valued right hand sides

In this section we arrive at one of the final aims of this paper: namely to prove parabolic regularity results for equations with measure valued right hand sides. The crucial point is two-fold: on one hand, the results of the foregoing section allow to interpret suitable measures as elements from $H_{\mathfrak{D}}^{-s}$. Here in the two dimensional case there are no restrictions concerning the measures under consideration: all bounded Radon measures are admissable. In the higher dimensional cases one is restricted in this concept to measures which live on sets with Hausdorff dimension one or two and are, additionally, absolutely continuous with respect to the corresponding Hausdorff measure there. Subsequently we are in the position to apply the results of maximal *non-autonomous* parabolic regularity from Section 3.

**Theorem 4.13.** *Let* $\Omega \subset \mathbb{R}^2$ *and assume that* $\rho : J \to \mathcal{M}(\Omega)$ *is weakly* measurable with* $\int_J^* \|\rho_t\|^{q_0}_{\mathcal{M}} \mathrm{d}t < \infty$ *fo some* $q_0 > 2$.

*Then, there exists a $q > 2$ such that, for sufficiently small $\epsilon > 0$ the solution $u$ of the problem (4.1) exists. It lies in the space $W_0^{1,q}(J; H_{\mathfrak{D}}^{-1-\epsilon}) \cap L^q(J; H_{\mathfrak{D}}^{1-\epsilon}) = \mathrm{MR}_0^q(J, H_{\mathfrak{D}}^{1-\epsilon}, H_{\mathfrak{D}}^{-1-\epsilon})$ and satisfies an estimate*

$$\|u\|_{\mathrm{MR}_0^q(J, H_{\mathfrak{D}}^{1-\epsilon}, H_{\mathfrak{D}}^{-1-\epsilon})} \le c \int_J^* \|\rho_t\|_{\mathcal{M}}^{q_0} \mathrm{d}t \tag{4.13}$$

*for some constant $c$.*

*Proof.* In case of two space dimensions, the space of bounded Radon measures on $\Omega$ continuously embeds into *every* space $H_{\mathfrak{D}}^{-1-\epsilon}$ with $\epsilon \in ]0, \frac{1}{2}[$ – see Lemma 4.1. Associating therefore to the measure $\rho_t$ an element $\Psi(t) \in H_{\mathfrak{D}}^{-1-\epsilon}$, one can prove as in Theorem 4.11 the measurability of the mapping $J \ni t \mapsto \Psi(t) \in H_{\mathfrak{D}}^{-1-s}$ inclusively the estimate $\int_J \|\Psi(t)\|^{q_0} \mathrm{d}t \le c \int_J^* \|\rho_t\|_{\mathcal{M}}^{q_0} \mathrm{d}t$. Now the claim follows from Theorem 3.16. □

**Corollary 4.14.** *Under the assumptions of the previous theorem, the solution belongs to $C^\beta(J; H^\kappa)$, for some $\beta > 0$ and $\kappa > 0$.*

*Proof.* Given $q > 2$, there exists $\epsilon > 0$ sufficiently small such that the interval $]\frac{1+\epsilon}{2}, 1 - \frac{1}{q}[$ is non-empty. The claim then follows with Theorem 3.20. □

Unfortunately, the range of admissable integrability exponents $q$ with respect to time is restricted to be close to 2.

The next result shows that the solution is more regular with respect to the spatial variable, if one restricts the admissible measures to those living on lower dimensional sets. In particular, in dimension 2 the function $u(t, \cdot)$ is Hölderian on $\Omega$ for almost all $t \in J$. This is as special case of the following result which concentrates on subsets of codimension 1 in $\mathbb{R}^2$, respectively $\mathbb{R}^3$.

For the following theorems recall the definition of $\mathfrak{J}$ from Proposition 3.15.

**Theorem 4.15.** *Let $l = d - 1$, and suppose the following:*

(a) *For every $t \in J$, let $M_t \subset \Omega$ be a closed subset of $\Omega$, and assume that*

$$\mathcal{H}_{d-1}(\overline{M}_t \cap B(x, r)) \le \mathfrak{c}\, r^{d-1}, \quad x \in \overline{M}_t, \quad r \in ]0, 1] \tag{4.14}$$

*for a constant $\mathfrak{c}$ independent of $t \in J$.*

(b) *For every $t \in J$ there is a $\sigma_t \in L^2(M_t; \mathcal{H}_{d-1})$, such that the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_{d-1}|_{M_t} =: \rho(t) \in \mathcal{M}(\Omega) \tag{4.15}$$

*is weakly* continuous, and $\int_J^* \|\sigma_t\|_{L^2(M_t; \mathcal{H}_{d-1})}^{q_0} \mathrm{d}t < \infty$ holds for some $q_0 \ge 2$.*

*Then, for $s \in ]0, \frac{1}{2}[$, each $\rho(t)$ can be understood as an element in $H_{\mathfrak{D}}^{-1+s}$, and by Theorem 4.11 the function $\rho$ is in $L^{q_0}(J; H_{\mathfrak{D}}^{-1+s})$. Further the solution $u$ of*

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = \rho(\cdot), \quad u(0) = 0 \tag{4.16}$$

*belongs to the space of maximal parabolic regularity $\mathrm{MR}_0^q(J; H_{\mathfrak{D}}^{1+s}, H_{\mathfrak{D}}^{-1+s})$, for $s > 0$ sufficiently small, and $q = 2$, if $q_0 = 2$, else $2 < q \in \mathfrak{J}$.*

*Proof.* Thanks to Theorem 4.11, the function $t \mapsto \rho(t)$ can be interpreted with values in $H_{\mathfrak{D}}^{-1+s}$ and an integrability exponent $q \le q_0$ in time, as long as $s \in ]0, \frac{1}{2}[$. Possibly diminishing $s$ and $q$, one may now apply Theorem 3.16. □

We next address the situation of codimension 2 in $\mathbb{R}^d$, $d > 2$.

**Theorem 4.16.** *Let $l = d-2$, and suppose the following:*

(a) *For every $t \in J$, let $M_t \subset \Omega$ be a closed subset of $\Omega$ and assume*

$$\mathcal{H}_l(\overline{M}_t \cap B(x,r)) \le \mathfrak{c}\, r^l, \quad x \in \overline{M}_t, \quad r \in ]0,1] \tag{4.17}$$

*for a constant $\mathfrak{c}$ independent of $t \in J$.*

(b) *For every $t \in J$ there is a $\sigma_t \in L^2(M_t;\mathcal{H}_l)$, such that the mapping*

$$J \ni t \mapsto \sigma_t \mathcal{H}_l|_{M_t} =: \rho(t) \in \mathcal{M}(\Omega) \tag{4.18}$$

*is weakly* continuous and $\int_J^* \|\sigma_t\|^q_{L^2(M_t;\mathcal{H}_l)} \mathrm{d}t < \infty$ holds for some $q \ge 2$.*

*Then, for $s \in ]0, \frac{1}{2}[$ each $\rho(t)$ can be understood as an element in $H^{-1-s}_{\mathfrak{D}}$, and by Theorem 4.11 the function $\rho$ is in $L^q(J;H^{-1-s}_{\mathfrak{D}})$. Further, the solution $u$ of*

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = \rho(\cdot), \quad u(0) = 0 \tag{4.19}$$

*belongs to the space of maximal parabolic regularity $\mathrm{MR}^q_0(J;H^{1-s}_{\mathfrak{D}}, H^{-1-s}_{\mathfrak{D}})$, for $s > 0$ sufficiently small, and $q = 2$, if $q = 2$, else $2 < q \in \mathfrak{I}$.*

*Proof.* The proof again follows from Theorems 3.16 and 4.11. □

## 5. Optimal control

### 5.1. Measure theoretic preliminaries

Up to now we considered a parabolic equation with prescribed right hand side $\sigma_t \mathcal{H}_l|_{M_t}$ only demanding the finiteness of the upper integral

$$\int_J^* \|\sigma_t\|^q_{L^2(M_t;\mathcal{H}_l)} \mathrm{d}t < \infty \tag{5.1}$$

and, secondly, the measurability of the mappings

$$J \ni t \mapsto \langle \sigma_t \mathcal{H}_l, \psi \rangle_{H^{-s}_{\mathfrak{D}} \times H^s_{\mathfrak{D}}} = \int_{M_t} \sigma_t\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l, \quad \text{for all } \psi \in H^s_{\mathfrak{D}}. \tag{5.2}$$

Within this subsection we will investigate this second condition and introduce a specific construction for the $\sigma_t$'s. In order to illustrate the problem, consider first the case where all sets $M_t$ are identical, *i.e.* $M_t = M$ for a fixed upper $l$-set $M$. Then it is clear that the weak$^*$-measurability amounts to the measurability of the function $J \ni t \mapsto \sigma_t$. But, if the sets $M_t$ evolve in time, then this aspect becomes a non-trivial one. We investigate this in some particular setting by making the following assumption which is supposed to hold throughout the rest of the paper.

**Assumption 5.1.** Let $M$ be a closed subset of $\mathbb{R}^d$ and an upper $l$-set, such that, for all $t \in J$, there is a bi-Lipschitz diffeomorphism $\phi_t$ from $M$ onto $M_t$. The Lipschitz constants $l_t$ of the $\phi_t$'s and the Lipschitz constants $l_t^-$ of their inverses $\phi_t^{-1}$, are uniformly bounded in $t \in J$.

**Proposition 5.2.** *Suppose $M_\bullet, M_\blacktriangle \subset \mathbb{R}^d$ and let $\phi$ be bi-Lipschitzian from $M_\bullet$ onto $M_\blacktriangle$. Then*

$$\gamma \mathcal{H}_l(M_\blacktriangle)) \le \mathcal{H}_l(M_\bullet)) \le \gamma^{-1} \mathcal{H}_l(M_\blacktriangle)), \tag{5.3}$$

*$\gamma$ depending only on the Lipschitz constants of $\phi$ and $\phi^{-1}$.*

*Proof.* The left hand side of (5.3) is proved in [19], Theorem 2.8 for the case when $\phi$ is defined on whole $\mathbb{R}^d$. In our case this is not fulfilled, but $\phi$ may be extended to $\mathbb{R}^d$ as a Lipschitzian function into $\mathbb{R}^d$ with the *same* Lipschitz constant, see [19], Chapter 3.1.1. To this extended function the above quoted theorem applies. The right hand side of (5.3) is proved by the same arguments, this time applied to $\phi^{-1}$. □

**Lemma 5.3.** *The sets $M_t$, defined in Assumption 5.1, are upper $l$-sets as well. The corresponding constants $\mathfrak{c}_t$ in (1.3) may be taken uniform in $t$.*

*Proof.* Note first, that every $\phi_t$ extends to a bi-Lipschitzian mapping from $\overline{M}$ onto $\overline{M_t}$ – under preservation of the Lipschitz constants. The Lipschitz continuity of $\phi_t$ implies , for all $x \in \overline{M_t}$,

$$\overline{M_t} \cap B(x;r)) = \phi_t(\overline{M}) \cap B(x;r) \subseteq \phi_t\big(\overline{M} \cap B(\phi_t^{-1}x, \lambda r)\big), \tag{5.4}$$

where $\lambda$ may be chosen uniform in $t$. Take now in particular $M_\bullet = \overline{M} \cap B(\phi_t^{-1}x, \lambda r)$ for an $x \in \overline{M_t}$ and apply (5.3), to conclude the proof. □

**Lemma 5.4.** *Adopt the assumptions of Proposition 5.2. Consider the (pushforwarded) image, named $\varpi$, of the Hausdorff measure $\mathcal{H}_l$ on $M_\blacktriangle$ under $\phi$ on $M_t$. Then $\varpi$ is of the form $\varpi = \varsigma \mathcal{H}_l$, where $\varsigma$ is $\mathcal{H}_l$-measurable and is bounded from above and below by constants. These constants may be taken uniformly for bi-Lipschitz mappings $\phi$, admitting uniform Lipschitz constants for $\phi$ and $\phi^{-1}$.*

*Proof.* For each $\mathcal{H}_l$-measurable subset $N_\bullet \subset M_\bullet$ and $N_\blacktriangle = \phi(N_\bullet)$ one has – mutatis mutandis – (5.3). For mappings $\phi$ with the uniform Lipschitz constants for $\phi$ and $\phi^{-1}$, $\gamma$ may be taken uniform. This implies that $\varpi$ is absolutely continuous with respect to $\mathcal{H}_l$ on $M_\blacktriangle$ and, hence, admits a density $\varsigma$ by the Radon-Nikodym theorem. Further (5.3) implies uniform bounds for $\varsigma$'s from above and below for different mappings $\phi$ – as long they admit uniform Lipschitz constants for $\phi$ and $\phi^{-1}$. □

**Lemma 5.5.** *Let $\psi$ be uniformly continuous on $\Omega$ and assume that the mappings $J \ni t \mapsto \phi_t(x) \in \Omega$ are measurable for every $x \in M$. Then*

$$J \ni t \mapsto \psi(\phi_t(\cdot)) =: f_t \in L^2(M; \mathcal{H}_l) \tag{5.5}$$

*is measurable.*

*Proof.* First one observes that the system of functions $\{f_t\}_t$ is equicontinuous on $M$ according to the uniform continuity of $\psi$ and the (uniform) Lipschitz properties of the mappings $\phi_t$. Let $\{x_j\}_j$ be a countable, dense subset of $M$. Standard arguments (see [41], Ch. 13.9, 13.9.6) tell us that, for every $x \in M$, the function $J \ni t \mapsto f_t(x)$ is measurable. Let $\epsilon > 0$ be arbitrary. So, by Lusin's theorem, for every $j$ there is a compact set $\mathcal{K}_\epsilon^j \subset J$, such that $|J \setminus \mathcal{K}_\epsilon^j| \le \epsilon 2^{-j-1}$ and the mapping

$$\mathcal{K}_\epsilon^j \ni t \mapsto f_t(x_j)$$

is continuous (see [41], Ch. 13.9, 13.9.4). Define $\mathcal{K} = \cap_j \mathcal{K}_\epsilon^j$. We show:
For *every* $x \in M$, the mapping

$$\mathcal{K} \ni t \mapsto f_t(x) \tag{5.6}$$

is continuous. One has

$$|f_t(x) - f_s(x)| \le |f_t(x) - f_t(x_j)| + |f_t(x_j) - f_s(x_j)| + |f_s(x) - f_s(x_j)|,$$

and all three addends can be made arbitrarily small by taking $x_j$ close enough to $x$. Let $\varphi \in L^2(M; \mathcal{H}_l)$.

Knowing the continuity of (5.6), Lebesgue dominance tells us that

$$\mathcal{K} \ni t \mapsto \int_M f_t \, \varphi \, d\mathcal{H}_l \tag{5.7}$$

is continuous, see [41], Chapters 13.8 and 13.8.6. But the measure of $J \setminus \mathcal{K}$ is at most $\epsilon$. So Lusin's theorem again applies and tells us that

$$J \ni t \mapsto \int_M f_t \, \varphi \, d\mathcal{H}_l \tag{5.8}$$

is measurable. So (5.5) is weakly, and, hence, strongly measurable. □

Lastly, if one only aims at measurability of the mapping (5.2), then Assumption 5.1 can be relaxed as follows: divide the interval into intervals $J_1, J_2, \ldots$ and demand for every subinterval $J = J_k$ again Assumption 5.1. Subsequently on each of the subintervals $J_k$ the same calculus can be done as for $J$ now, and can subsequently be concatenated to again obtain functions on $J$.

Consider now, for every $t$, the mapping $V_t : L^2(M; \mathcal{H}_l) \to L^2(M_t; \mathcal{H}_l)$ defined by

$$\big(V_t(\varphi)\big)(x) = \varsigma_t(x)\varphi(\phi_t^{-1}x), \quad \varphi \in L^2(M; \mathcal{H}_l), \, x \in M_t. \tag{5.9}$$

Then the definition of the image of a measure together with Lemma 5.4 show that, for every $t \in J$, $V_t$ is a bounded linear mapping from $L^2(M; \mathcal{H}_l)$ onto $L^2(M_t; \mathcal{H}_l)$, the norms of which together with their inverses are uniformly bounded in $t \in J$. Our choice for $\sigma_t$ in (5.2) will be

$$\sigma_t = V_t(\varphi), \text{ for } \varphi \in L^2(M; \mathcal{H}_l).$$

With reference to Theorem 4.8 we introduce the embedding operators

$$\begin{aligned} \mathcal{I}_t : L^2(M_t; \mathcal{H}_l) \ni \sigma \to \sigma \mathcal{H}_l \in H^{-1+\tau}_{\mathfrak{D}}, \text{ for } \sigma \in L^2(M_t; \mathcal{H}_l), \\ \text{with } \tau \in (0, \tfrac{1}{2}) \text{ if } l = d-1, \text{ and } \tau \in (-\tfrac{1}{2}, 0) \text{ if } l = d-2. \end{aligned} \tag{5.10}$$

*Here and throughout the remainder the case $l = d-2$ is only considered for $d \ge 3$.*

For $v \in L^q(J; L^2(M; \mathcal{H}_l))$ and $l \in \{d-1, d-2\}$ the crucial point now is the measurability – or not – of the mappings

$$J \ni t \mapsto \langle \mathcal{I}_t V_t(v(t)), \psi \rangle_{H^{-1+\tau}_{\mathfrak{D}} \times H^{1-\tau}_{\mathfrak{D}}} = \langle V_t(v(t)) \mathcal{H}_l, \psi \rangle_{H^{-1+\tau}_{\mathfrak{D}} \times H^{1-\tau}_{\mathfrak{D}}} = \int_{M_t} V_t(v(t)) \, \overline{\psi}|_{M_t} \, d\mathcal{H}_l, \tag{5.11}$$

for every $\psi \in H^{1-\tau}$. In this respect we have the following

**Lemma 5.6.** *With Assumption 5.1 holding and $V, \mathcal{I}$ as defined in (5.9) and (5.10), the function*

$$J \ni t \mapsto \mathcal{I}_t V_t v(t) \in H^{-1+\tau}_{\mathfrak{D}} \tag{5.12}$$

*is measurable for every* $v \in L^q(M;\mathcal{H}_l)$*, with* $q \in ]1,\infty[$*,* $l \in \{d-1, d-2\}$*, if and only if*

$$J \ni t \mapsto \psi(\phi_t(\cdot)) \in L^2(M;\mathcal{H}_l) \tag{5.13}$$

*is measurable for every function* $\psi \in C_b^\infty(\Omega) \cap H_{\mathfrak{D}}^{1-\tau}$ *in case* $l = d-2$ *and* $\psi \in C^\infty \cap H_{\mathfrak{D}}^{1-\tau}$ *in case* $l = d-1$*.*

*Proof.* We utilize that by a well-known theorem of Pettis the measurability of $t \mapsto \mathcal{I}_t V_t v(t)$ follows from its weak measurability, thus from (5.2) with $s = 1-\tau$. We now turn to the case $l = d-2$. According to Lemma 2.11 we may restrict ourselves to $\psi \in C_b^\infty(\Omega) \cap H_{\mathfrak{D}}^{1-\tau}$. One calculates for $v \in L^q(J; L^2(M;\mathcal{H}_l))$

$$\begin{aligned} \langle \mathcal{I}_t V_t(v(t)), \psi\rangle_{H_{\mathfrak{D}}^{-1+\tau} \times H_{\mathfrak{D}}^{1+\tau}} &= \textstyle\int_{M_t} V_t(v(t))\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l = \int_{M_t} \varsigma_t\, v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\mathcal{H}_l \\ &= \textstyle\int_{M_t} v_t(\phi_t^{-1}(\cdot))\, \overline{\psi}|_{M_t}\, d\varpi_t = \int_M v_t\, \overline{\psi}(\phi_t(\cdot))\, d\mathcal{H}_l, \end{aligned} \tag{5.14}$$

where we used that $\varpi_t$ was the image of the measure $\mathcal{H}_l|_M$ under $\phi_t$.

The reader should notice that the function $M \ni x \mapsto \psi(\phi_t(x)) \to \mathbb{C}$ is bounded and continuous – hence measurable with respect to $\mathcal{H}_l$. Since $\mathcal{H}_l(M)$ is finite, the function $v_t\, \overline{\psi}(\phi_t(\cdot))$, consequently, belongs to $L^2(M;\mathcal{H}_l)$, and the last term in (5.14) is well defined – irrespective of the Hausdorff dimension of $M$.

Since the functions $v$ run through the whole space $L^q(J; L^2(M;\mathcal{H}_l))$, it is straight forward that the measurability of (5.14) is equivalent to the measurability of the function $t \mapsto \psi(\phi_t(\cdot)) \in L^2(M;\mathcal{H}_l)$ for every function $\psi \in C_b^\infty(\Omega)$, which is (5.13). For the case $l = d-1$ one chooses $\psi \in C^\infty \cap H_{\mathfrak{D}}^{1-\tau}$ and proceeds in the same manner. □

This lemma guarantees the measurability of the right hand side of equation (5.15) below. The latter is assured by the measurability of (5.13), which is addressed in Lemma 5.5 under a natural and extremely general condition.

## 5.2. An optimal control problem

For $u_d \in L^2(J; L^2)$ and $u_T \in L^2(\Omega)$ we consider the optimal control problem

$$\min_{\xi \in L^q(J; L^2(M;\mathcal{H}_l))} \mathcal{J}(\xi) = \frac{1}{2}\int_0^T \|u(\xi)(t) - u_d(t)\|_{L^2}^2\, \mathrm{d}t + \frac{\alpha}{2}\|u(\xi)(T) - u_T\|_{L^2}^2 + \frac{\beta}{q}\int_0^T \|\xi\|_{L^q(M;\mathcal{H}_l)}^q\, \mathrm{d}t, \tag{$\mathcal{P}$}$$

where $u(\xi)$ is the solution to

$$\frac{\partial u}{\partial t} - \mathcal{A}\, u = B\xi, \quad u(0) = u_0, \tag{5.15}$$

and $\alpha \geq 0$, $\beta > 0$. The control operator $B$ and $q \geq 2$ will be specified below. First it will be convenient to summarize the conditions which will henceforth be assumed.

**Assumption 5.7.** We assume that Assumptions 2.2, 3.12 and 5.1 are satisfied and that for every $x \in M$, the mapping $J \ni t \mapsto \phi_t(x) \in \mathbb{R}^d$ is measurable.

Thus the prerequisites established in Section 5.1 are at our disposal. We further recall the definition of the spaces $X = W^{1,q}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^q(J; H_{\mathfrak{D}}^{1+\tau})$, where $\tau$ will be chosen positive respectively negative as asked for in (5.16)–(5.18) below. We shall consider combinations of $(d,l)$ corresponding to the cases covered in Theorems 4.15

and 4.16, in particular

$$l = d - 1 \text{ with } \tau \in ]0, \epsilon[, \text{ and } q = 2, \tag{5.16}$$

or

$$l = d - 2 \text{ with } \tau \in ] - \epsilon, 0[, \text{ and } q = 2, \alpha = 0, \tag{5.17}$$

or

$$l = d - 2 \text{ with } \tau \in ] - \epsilon, 0[, \text{ and } \mathfrak{I} \ni q > 2, \tag{5.18}$$

with $\epsilon > 0$ sufficiently small, $\alpha$ to be introduced below, and $\mathfrak{I}$ from Proposition 3.15. The reason for considering different cases for $l = d-2$ results from the fact that $X = W^{1,q}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^q(J; H_{\mathfrak{D}}^{1+\tau})$ with $\tau < 0$, does not embed continuously into $C(\bar{J}; L^2)$ unless $q > 2$. Pointwise-in-time evaluation of $u(\xi)$, however, is required for $\mathcal{J}$ if $\alpha > 0$.

We proceed by setting the control-space to be $L^2(M; \mathcal{H}_l)$ and define the time dependent control operators for a.e. t by

$$B(t) : L^2(M; \mathcal{H}_l) \to H_{\mathfrak{D}}^{-1+\tau} \text{ with } B(t) = \mathcal{I}_t V_t,$$

with

$$V_t : L^2(M; \mathcal{H}_l) \to L^2(M_t; \mathcal{H}_l), \text{ and } \mathcal{I}_t : L^2(M_t; \mathcal{H}_l) \to H_{\mathfrak{D}}^{-1+\tau}.$$

Here $V_t$ is as defined in (5.9) and $\mathcal{I}_t$ as in (5.10). Recall that the $V_t$'s are uniformly (in $t$) *bounded*,

Thus there exists $k_1 > 0$ such that for all $t \in J$:

$$\|V_t\|_{\mathcal{L}(L^2(M;\mathcal{H}_l), L^2(M_t;\mathcal{H}_l))} \leq k_1,$$

and there exist a constant a $k_2$ such that for a.e. $t$

$$\|\mathcal{I}_t\|_{\mathcal{L}(L^2(M_t;\mathcal{H}_l), H_{\mathfrak{D}}^{-1+\tau})} \leq k_2,$$

for $\tau$ as in 5.10, see Theorem 4.8. Consequently $\|B(t)\|_{\mathcal{L}(L^2(M;\mathcal{H}_l), H_{\mathfrak{D}}^{-1+\tau})} \leq k_1 k_2$ for a.a. $t \in J$ and $B(t)$ induces a mapping $B \in \mathcal{L}(L^q(J; L^2(M; \mathcal{H}_l)), L^q(J; H_{\mathfrak{D}}^{-1+\tau}))$, satisfying

$$\|B\|_{\mathcal{L}(L^q(J;L^2(M;\mathcal{H}_l)), L^q(J;H_{\mathfrak{D}}^{-1+\tau}))} \leq k_1 k_2. \tag{5.19}$$

Let $u_0 \in (H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau,})_{1-\frac{1}{q},q}$ be fixed. By Theorem 3.18 and (5.19) for every $\xi \in L^q(J; L^2(M; \mathcal{H}_l))$ the control system (5.15) has a unique solution $u = u(\xi) \in X$ satisfying

$$\|u(\xi)\|_X \leq c(\|\xi\|_{L^q(J;L^2(M;\mathcal{H}_l))} + \|u_0\|_{(H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau})_{1-\frac{1}{q},q}}), \tag{5.20}$$

with $c$ independent of $u_0$ and $\xi$. Here $\epsilon$ is assumed to be sufficiently small so that Theorem 3.18 is applicable for $\tau \in ]0, \epsilon[$, respectively $\tau \in ] - \epsilon, 0[$.

**Theorem 5.8.** *Let Assumption 5.7 hold and let $\epsilon$ be sufficiently small. Then for $l$ and $q$ as in cases (5.16)–(5.18) problem ($\mathcal{P}$) admits a unique solution $\xi^* \in L^2(J; L^2(M; \mathcal{H}_l))$.*

*Proof.* We start by arguing the well-posedness of the cost-functional for each of the three cases. For each of them existence of a solution $u(\xi) \in X$ to (5.15) for all $\xi \in L^2(J; \mathcal{H}_l)$ is guaranteed, see (5.20). For (5.16) moreover $u(\xi) \in C(\bar{J}; L^2)$, see Theorem 3.19, and hence $\mathcal{J}$ is welldefined. Turning to the case $l = d-2$, for (5.17) well-posedness of $\mathcal{J}$ already follows from $u(\xi) \in X$. For (5.18) we utilize Theorem 3.20 with $q \in \mathfrak{I}, q > 2$, and $\tau$ such that $0 < -\tau < \frac{1}{2} - \frac{1}{q}$, then $\theta = \frac{1}{2} - \tau$, to get $\beta = \frac{1}{2} + \tau - \frac{1}{q} > 0$, and $X \subset C^\beta(J; H^{-\tau})$. In particular $u(\xi) \in C(\bar{J}; L^2)$ and $\mathcal{J}$ is also well-defined for the case described by (5.18).

Next, let $\{\xi_n\}$ be a minimizing sequence for ($\mathcal{P}$). Then $\mathcal{J}(\xi_n) \le \mathcal{J}(0) + 1$ for all sufficiently large $n$. Hence $\{\xi_n\}$ is bounded in $L^q(J; L^2(M; \mathcal{H}_l))$. Consequently $\{\xi_n\}$ admits a weakly convergent subsequence, which is denoted by the same symbols, and $\xi^* \in L^q(J; L^2(M; \mathcal{H}_l))$ with $\xi_n \rightharpoonup \xi^*$, see *e.g.* [42], Chapter IV.1 Corollary 2. By the uniform boundedness of $B(t)$ with respect to $t$, it is simple to argue that $B\xi_n \rightharpoonup B\xi^*$ in $L^2(J; H^{-1+\tau}_{\mathfrak{D}})$.

For each $\xi_n$ the solution $u(\xi_n)$ to (5.15) can be decomposed as $u(\xi_n) = u_1(\xi_n) + u_h$ where $u_1(\xi_n)$ is the solution to (5.15) with $\xi = \xi_n, u_0 = 0$, and $u_h$ is the solution to (5.15) with $\xi = 0, u(0) = u_0$. By (5.20) the sequence $u_1(\xi_n)$ is bounded in $X$ and thus there exists a subsequence, again denoted by the same indices, and $u_1^* \in X$ such that $u_1(\xi_n) \rightharpoonup u_1^*$ in $X$. Lemma 3.14 implies that $\mathcal{A}\, u_1^*(\xi_n) \rightharpoonup \mathcal{A} u_1^*$ in $L^q(J; H^{-1+\tau}_{\mathfrak{D}})$. Thus we can take the weak limit in $L^q(J; H^{-1+\tau}_{\mathfrak{D}})$ in the equation

$$\frac{\partial u_1(\xi_n)}{\partial t} - \mathcal{A}\, u_1(\xi_n) = B\xi_n, \quad u(0) = 0,$$

to arrive at

$$\frac{\partial u_1^*}{\partial t} - \mathcal{A}\, u_1^* = B\xi^*, \quad u(0) = 0.$$

Uniqueness of the solution to this equation imply that $u_1^* = u_1(\xi^*)$. It follows that $u(\xi_n) \rightharpoonup u(\xi^*) = u_1(\xi^*) + u_h$ in $X$, with $u(\xi^*)$, the solution to (5.15) setting $\xi = \xi^*$. Consequently $\xi^*$ is an admissible control for ($\mathcal{P}$).

To pass to the limit in the cost functional we use Theorems 3.19 and 3.20 to assert that $\lim_{n\to\infty} u(\xi_n) = u(\xi^*)$ in $C(\bar{J}; L^2)$ for cases (5.16) and (5.18), and we use the compact embedding of $W^{1,2}(J; H^{-1+\tau}_{\mathfrak{D}}) \cap L^2(J; H^{1+\tau}_{\mathfrak{D}})$ into $L^2(J; L^2)$ to obtain $\lim_{n\to\infty} u(\xi_n) = u(\xi^*)$ in $L^2(\bar{J}; L^2)$ for case (5.17), where $\alpha = 0$. These convergence properties, and utilizing that $\alpha = 0$ in case (5.17), justify the following inequalities:

$$\begin{aligned}
&\mathcal{J}(\xi^*) = \tfrac{1}{2} \textstyle\int_0^T \|u(\bar{\xi})(t) - u_d(t)\|^2_{L^2}\, \mathrm{d}t + \tfrac{\alpha}{2} \|u(\xi^*)(T) - u_T\|^2_{L^2} + \tfrac{\beta}{2} \textstyle\int_0^T \|\xi^*\|^2_{L^2(M;\mathcal{H}_l)}\, \mathrm{d}t \\
&\le \lim_{n\to\infty} \tfrac{1}{2} \textstyle\int_0^T \|u(\xi_n)(t) - u_d(t)\|^2_{L^2}\, \mathrm{d}t + \lim_{n\to\infty} \tfrac{\alpha}{2} \|u(\xi_n)(T) - u_T\|^2_{L^2} \ + \liminf_{n\to\infty} \tfrac{\beta}{2} \textstyle\int_0^T \|\xi_n\|^2_{L^2(M;\mathcal{H}_l)}\, \mathrm{d}t \\
&\qquad \le \liminf_{n\to\infty} \mathcal{J}(\xi_n) = \inf_{\xi \in L^2(J; L^2(M;\mathcal{H}_l))} \mathcal{J}(\xi),
\end{aligned}$$

and thus $\xi^*$ is an optimal solution. Uniqueness of this solution follows from the strict convexity of the cost-functional. ☐

### 5.3. Optimality condition

Now we present the optimality condition associated to the unique solution $\xi^*$ of ($\mathcal{P}$). The analysis builds upon the adjoint equation associated to $\xi^*$ which is given by

$$-\frac{\partial \varphi}{\partial t} + \widehat{\mathcal{A}}\, \varphi(t) = -(u(\xi^*)(t) - u_d(t)) \text{ on } J, \quad \varphi(T) = -\alpha(u(\xi^*)(T) - u_T) \tag{5.21}$$

$\widehat{\mathcal{A}}$ defined as in (3.17), with the coefficient function replaced by its adjoint. We have the following regularity results for the adjoint state:

**Lemma 5.9.** *Concerning the regularity of the adjoint state the following properties hold:* $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1}) \cap L^2(J; H_{\mathfrak{D}}^1)$ *for* (5.16), $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1-\tau})$ *for* (5.17), *and* $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1-\tau})$ *for* (5.18) *provided that* $u_T \in H^{-\tau}$.

*Proof.* The solution to (5.21) can be decomposed as $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1$ the solution (5.21) with $\varphi_1(T) = 0$ and $\varphi_2$ the solution to the differential equation in (5.21) with right hand side equal 0.

For case (5.16) we have $\varphi_2 \in W^{1,2}(J; H_{\mathfrak{D}}^{-1}) \cap L^2(J; H_{\mathfrak{D}}^1)$ and $\varphi_1 \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$, and thus $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1}) \cap L^2(J; H_{\mathfrak{D}}^1)$. As a side remark we observe that by Proposition 3.10 we have that $X \subset C(\bar{J}; (H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau})_{\frac{1}{2},2})$, and consequently $u(\xi^*)(T) \in (H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau})_{\frac{1}{2},2}$. Thus if $u_T \in (H_{\mathfrak{D}}^{-1+\tau}, H_{\mathfrak{D}}^{1+\tau})_{\frac{1}{2},2}$, then $\varphi_2 \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$ and subsequently $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau})$.

For case (5.17) we have $\alpha = 0$ and hence $\varphi_2 = 0$, and $u(\xi^*) \in W^{1,2}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1+\tau}) \subset L^2(J; H_{\mathfrak{D}}^{-1-\tau})$ and $\varphi = \varphi_2 \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1-\tau})$.

For case (5.18) we have $u(\xi^*) \in W^{1,q}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^q(J; H_{\mathfrak{D}}^{1+\tau}) \subset C^\beta(\bar{J}; H^{-\tau})$, for some $\beta > 0$, see the proof of Theorem 5.8. Hence $u(\xi^*)(T) \in H_{\mathfrak{D}}^{-\tau} = (H_{\mathfrak{D}}^{-1-\tau}, H_{\mathfrak{D}}^{1-\tau})_{\frac{1}{2},2}$, see Lemma 2.16 below, and consequently $\alpha(u(\xi^*)(T) - u_T) \in H_{\mathfrak{D}}^{-\tau}$. Moreover we have that $u(\xi^*) \in L^q(J; H_{\mathfrak{D}}^{1+\tau}) \subset L^2(J; H_{\mathfrak{D}}^{-1-\tau})$ and consequently $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1-\tau})$. □

The optimality condition is presented next.

**Theorem 5.10.** *Suppose that Assumption 5.7 holds and that* $u_T \in H^{-\tau}$ *in case* (5.18). *Then the necessary and sufficient optimality condition for* $\xi^*$ *to be a minimizer of* ($\mathcal{P}$) *is given by*

$$\beta \|\xi^*(t)\|_{L^2(M;\mathcal{H}_l)}^{q-2} \xi^*(t) = B^*(t)\varphi(t) \text{ for a.a. } t \in J, \tag{5.22}$$

*with* $\varphi$ *given by* (5.21).

*Proof.* Since $\xi \to u(\xi)$ is affine the linearization of this mapping in direction $\delta\xi \in L^q(J; (M, \mathcal{H}_l))$ is given as the solution $u' = u'(\delta\xi)$ to

$$\frac{\partial u'(t)}{\partial t} - \mathcal{A}\, u'(t) = B(t)\delta\xi(t) \;\text{ on } J, \quad u'(0) = 0.$$

*Step 1* We first turn to the cases (5.16), for which $\tau > 0$. In this situation $u' \in \mathrm{MR}_0^2(J; H_{\mathfrak{D}}^{1+\tau}, H_{\mathfrak{D}}^{-1+\tau}) \subset C(\bar{J}; L^2)$, by Theorem 3.16 with $\tau > 0$. This will justify the following identities, where we use (5.21) with $q = 2$:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{J}(\xi^*)\,\delta\xi &= \textstyle\int_0^T (u(\bar{\xi})(t) - u_d(t), u'(\delta\xi)(t))_{L^2}\mathrm{d}t + \alpha(u(\bar{\xi})(T) - u_T, u'(\delta\xi)(T))_{L^2} + \beta \int_0^T (\xi(t), \delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t \\
&= \textstyle\int_0^T \langle \frac{\partial}{\mathrm{d}t}\varphi(t) - \widehat{\mathcal{A}}\varphi(t), u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^{-1}\times H_{\mathfrak{D}}^1}\,\mathrm{d}t - (\varphi(T), u'(\delta\xi)(T))_{L^2} + \beta\int_0^T (\xi^*(t), \delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t \\
&= \textstyle\int_0^T \langle \varphi(t), -\frac{\delta}{\mathrm{d}t}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^1\times H_{\mathfrak{D}}^{-1}}\,\mathrm{d}t + \beta\int_0^T (\xi^*(t), \delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t \\
&= \textstyle\int_0^T \langle \varphi(t), -B(t)\delta\xi(t)\rangle_{H_{\mathfrak{D}}^{1-\tau}\times H_{\mathfrak{D}}^{-1+\tau}}\,\mathrm{d}t + \beta\int_0^T (\xi^*(t), \delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t = 0.
\end{aligned}$$

The above identities hold for all $\delta\xi \in L^2(J; L^2(M; \mathcal{H}_l))$ and for a.a. $t \in J$, and consequently $B^*(t)\varphi(t) = \beta\xi^*(t)$, as desired.

*Step 2* Now we consider case (5.17) where $\tau < 0$ and $\alpha = 0$. In this case $u' \in \mathrm{MR}_0^2(J; H_{\mathfrak{D}}^{1+\tau}, H_{\mathfrak{D}}^{-1+\tau}) \subset C(\bar{J}; H_{\mathfrak{D}}^{\tau}))$, where we use Lemma 2.16, and $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; H_{\mathfrak{D}}^{1-\tau}) \subset C(\bar{J}; H_{\mathfrak{D}}^{-\tau})$ and $\varphi \in W^{1,2}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^2(J; (J; H_{\mathfrak{D}}^{-1-\tau}) \subset C(\bar{J}; H^{-\tau})$. We can now follow the computation of $\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{J}(\xi^*)\,\delta\xi$ as in

Step 1, making appropriate changes in the duality pairings:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{J}(\xi^*)\,\delta\xi &= \textstyle\int_0^T (u(\bar{\xi})(t)-u_d(t),u'(\delta\xi)(t))_{L^2}\mathrm{d}t + \beta\int_0^T(\xi(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T \langle \tfrac{\partial}{\mathrm{d}t}\varphi(t) - \widehat{\mathcal{A}}\varphi(t), u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^{-1-\tau}\times H_{\mathfrak{D}}^{1+\tau}}\,\mathrm{d}t + \beta\int_0^T(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T \langle \varphi(t), -\tfrac{\delta}{\mathrm{d}t}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^{1-\tau}\times H_{\mathfrak{D}}^{-1+\tau}}\,\mathrm{d}t + \beta\int_0^T(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T \langle \varphi(t), -B(t)\delta\xi(t)\rangle_{H_{\mathfrak{D}}^{1-\tau}\times H_{\mathfrak{D}}^{-1+\tau}}\,\mathrm{d}t + \beta\int_0^T(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t = 0,
\end{aligned}$$

where the temporal integration by parts in third equality above can be verified with a density argument. Again we obtain the desired equality (5.22).

*Step 3* In this case we use that $u(\xi^*) \in C(\bar{J}; H_{\mathfrak{D}}^{-\tau})$, (see proof of Thm. 5.8, $\varphi \in W^{1,q}(J; H_{\mathfrak{D}}^{-1-\tau}) \cap L^q(J; H_{\mathfrak{D}}^{1-\tau}) \subset C(\bar{J}; H_{\mathfrak{D}}^{-\tau})$, (see Lem. 5.9 and Prop. 3.10), and $u' \in W^{1,q}(J; H_{\mathfrak{D}}^{-1+\tau}) \cap L^q(J; H_{\mathfrak{D}}^{1+\tau}) \subset C(\bar{J}; H_{\mathfrak{D}}^{-\tau})$, (see Thm. 3.20). We now equate

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{J}(\xi^*)\,\delta\xi &= \textstyle\int_0^T (u(\bar{\xi})(t)-u_d(t),u'(\delta\xi)(t))_{L^2}\mathrm{d}t\\
&\quad +\alpha(u(\bar{\xi})(T)-u_T,u'(\delta\xi)(T))_{L^2} + \beta\textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T \langle \tfrac{\partial}{\mathrm{d}t}\varphi(t) - \widehat{\mathcal{A}}\varphi(t), u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^{-1-\tau}\times H_{\mathfrak{D}}^{1+\tau}}\,\mathrm{d}t - (\varphi(T),u'(\delta\xi)(T))_{L^2}\\
&\qquad +\beta\textstyle\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T \langle \varphi(t), -\tfrac{\delta}{\mathrm{d}t}u'(\delta\xi)(t) - \mathcal{A}\,u'(\delta\xi)(t)\rangle_{H_{\mathfrak{D}}^{1-\tau}\times H_{\mathfrak{D}}^{-1+\tau}}\,\mathrm{d}t + \beta\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t\\
&= \textstyle\int_0^T (\varphi(t), -B(t)\delta\xi(t))_{H_{\mathfrak{D}}^{1-\tau},H_{\mathfrak{D}}^{-1+\tau}}\,\mathrm{d}t + \beta\int_0^T \|\xi^*(t)\|^{q-2}_{L^2(M;\mathcal{H}_l)}(\xi^*(t),\delta\xi(t))_{L^2(M;\mathcal{H}_l)}\mathrm{d}t = 0,
\end{aligned}$$

and again (5.22) follows.

*Step 4* Since the cost-functional $\xi \to \mathcal{J}(\xi)$ is strictly convex, the necessary optimality condition is also sufficient. □

Utilizing the structure of the control operator $B(t) = V_t\mathcal{I}_t$ it will be shown next that the optimal solution $\xi^*$ exhibits extra regularity. This property of increased regularity of the minimizer arises frequently in optimal control, see *e.g.* [43], p. 52.

## 5.4. Extra regularity of the optimal control

It is frequently the case that the optimal control of an optimization problem constrained by a parabolic equation exhibits extra regularity. This is an important property in the context of approximation theory, to obtain error estimates for discretizations of the infinite dimensional problem by finite dimensional ones, or for iterative solution processes when solving ($\mathcal{P}$) numerically. The tool for establishing the extra regularity is the relationship of the optimal control $\xi^*$ to the adjoint state, which is (5.10) in our case. Here this requires us to revisit regularity properties of the adjoint equation, or, equivalently, turning around time, the primal equation. Some additional regularity properties of the problem data will be needed. Throughout this section we restrict ourselves to the case $d \in \{2,3\}$.

**Assumption 5.11.** (a) For some $\alpha > \frac{1}{2}$ the coefficient function satisfies

$$\|\mu(t_1,\cdot) - \mu(t_2,\cdot)\|_{L^\infty(\Omega;\mathbb{C}^{d\times d})} \le c|t_1 - t_2|^\alpha \quad \text{for all } t_1,t_2 \in J. \tag{5.23}$$

(b) In case $d=3$ the operator

$$-\nabla\cdot\mu(t,\cdot)\nabla+1 : W^{1,3}_{\mathfrak{D}} \to W^{-1,3}_{\mathfrak{D}} \tag{5.24}$$

is a topological isomorphism for each $t \in J$, and that the norms of their inverses are uniformly in $t$ bounded.

Moreover for every $t \in J$ all components of the coefficient function $\mu_t = \hat{\mu}(t,\cdot)$ are multipliers on the spaces $W^{s,3}$ at least for small $s \in ]0,\delta_\bullet]$. The corresponding norms as multipliers on these spaces are uniformly in $t$ bounded for $s \in ]0,\delta_\bullet]$ .

Finally, the set $\mathfrak{D} \cap \overline{\partial\Omega \setminus \mathfrak{D}}$ – where the Dirichlet boundary part meets the Neumann part – is a 1-set.

**Remark 5.12.** We are aware that the condition (5.23) restricts the class of admissable coefficients in comparison to Assumption 3.12 considerably. Prototypically, in the latter a (scalar) coefficient function $\hat{\mu}$ is allowed which is identically 1 up to a time point $t_0 \in J$, and from $t_0$ on it is identical 1 on a subdomain $\Omega_\bullet$ and 2 on $\Omega \setminus \Omega_\bullet$. Obviously, such a $\hat{\mu}$ does not satisfy (5.23). But the following is allowed: Let $\Omega_\bullet \subset \Omega$ be a subdomain and $\chi$ its indicator function. If one defines

$$\mu_t = \begin{cases} 1, \text{ if } t \le t_0 \\ 1+(t-t_0)\chi \text{ if } t > t_0, \end{cases}$$

then this coefficient function is admissible.

The following theorem, which is proved at the end of this section, refers to the equation with homogenous initial condition:

$$\frac{\partial u}{\partial t} - \mathcal{A}\,u = f, \quad u(0)=0. \tag{5.25}$$

**Theorem 5.13.** *Let Assumptions 3.12, 3.3, and 5.11 be satisfied.*

(a) *For dimension $d=2$, there exist $\beta>0$ and $r \in ]0,\delta[$ such that the solution $u$ to (5.25) belongs to $C^\beta(J, H^{1+r}_{\mathfrak{D}})$, provided that $f \in L^{\tilde{q}}(J;L^2)$ for $\tilde{q}>2$ sufficiently large.*

(b) *For dimension $d=3$, there exist $\beta>0$ and $r \in ]0,\delta_\bullet[$ such that the solution $u$ to (5.25) belongs to $C^\beta(J, W^{1+r,3}_{\mathfrak{D}})$ provided that $f \in L^{\tilde{q}}(J;L^2)$ for $\tilde{q}>2$ sufficiently large.*

This result allows us to draw conclusions on the regularity of the optimal control $\xi^*$. Henceforth all the Assumptions 2.2, 3.3, 3.12, 5.1, 5.7, and 5.11 are supposed to hold.

**Theorem 5.14.** *Let the assumptions just mentioned hold, let $d \in \{2,3\}$, $\alpha=0$, and $u_d \in L^\infty(J;L^2)$. Then the optimal solution to $(\mathcal{P})$, satisfies*

*(i)* $\xi^* \in L^\infty(J; W^{1-\frac{1}{d+\kappa},d+\kappa}(M))$ *for case (5.16),*
*(ii)* $\xi^* \in L^2(J;W^{-\tau,2}(M))$ *for case (5.17),*
*(iii)* $\xi^*(t) \in W^{1-\frac{2}{3+\kappa},3+\kappa}(M)$ *for a.e. $t \in J$ and $\xi^* \in L^\infty(J;L^2(M;\mathcal{H}_1))$ for case (5.18),*

*for some $\kappa>0$. Recall that $M$ is of Hausdorff dimension $d-1$ in the first case and of Hausdorff dimension 1 in the last two cases.*

*Proof.* Let us recall the adjoint equation (5.21). As established at the beginning of the proof of Theorem 5.8 we have that $u(\xi^*) \in C(J;L^2)$ in the cases (5.16) and (5.18). In view of and the assumption on $u_d$ we have that the right hand side of the adjoint equation satisfies $u(\xi^*)(t) - u_d(t) \in L^\infty(J;L^2)$. Next we observe that after time reversal the adjoint equation is a special case of (5.25), and from Theorem 5.13 we deduce that $\varphi \in C^\beta(J, H^{1+r}_{\mathfrak{D}}) \hookrightarrow C(\overline{J}, W^{1,2+\kappa}_{\mathfrak{D}})$ for $d=2$ and $\varphi \in C^\beta(J, W^{1+r,3}_{\mathfrak{D}}) \hookrightarrow C(\overline{J}, W^{1,3+\kappa}_{\mathfrak{D}})$ for $d=3$, for some $\beta > 0, r>2$ and $\kappa>0$.

Now we recall from Theorem 5.10 that

$$\beta\|\xi^*(t)\|_{L^2(M;\mathcal{H}_l)}^{q-2}\xi^*(t) = B^*(t)\varphi(t) = V_t^*\mathcal{I}_t^*\varphi(t) \text{ for a.a. } t\in J. \tag{5.26}$$

A straight forward computation shows that $V_t^* : L^2(M_t, \mathcal{H}_l) \to L^2(M, \mathcal{H}_l)$ is given by

$$(V_t^*\psi)(x) = \psi(\phi_t(x)),\ x \in M. \tag{5.27}$$

Now we continue with case (5.16) and obtain

$$\beta\,\xi^*(t) = V_t^*\varphi(t)|_{M_t} = \varphi(t,\phi_t(\cdot))|_M \text{ for a.a. t in } J, \tag{5.28}$$

where for the first equality we used (5.26) with $q = 2$, and Theorem 4.6 and (5.27) for the second. Let us now temporarily fix $t$ and denote the function $\varphi(t,\cdot)$ by $\psi$. We point out that the right hand side of (5.28) is to be read as $\big(tr_{M_t}\varphi(t,\cdot)\big)(\phi_t(x)) = \big(tr_{M_t}\psi\big)(\phi_t(x))$. Recall from the discussion in the first paragraph, that $\psi \in W^{1,d+\kappa}(\Omega)$ and consider an extension operator $\mathfrak{E} : W^{1,d+\kappa}(\Omega) \to W^{1,d+\kappa}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. Further, let $\widehat{\phi_t}$ be a Lipschitzian extension of $\phi_t : M \to M_t$, to $\mathbb{R}^d \to \mathbb{R}^d$, having the same Lipschitz constant as $\phi_t$. Each function $\mathfrak{E}\psi$ is continuous on $\mathbb{R}^d$, so, for every $x \in M$, $\big(tr_{M_t}\psi\big)(\phi_t(x)$ is obtained as the *pointwise evaluation* of the function $\mathfrak{E}\psi \circ \widehat{\phi_t}$ in $x$, *i.e.* equals $\big(tr_M(\mathfrak{E}\psi \circ \widehat{\phi_t})\big)(x)$. By construction, it is not hard to see that the family $\mathcal{F} = \{\mathfrak{E}\varphi(t,\widehat{\phi_t}(\cdot))\}_{t\in J}$ is a bounded one in $W^{1,d+\kappa}(\mathbb{R}^d)$. Hence, the family of traces, $\{\beta\xi^*(t)\}_{t\in J} = tr_M\mathcal{F}$ on $M$, is bounded in $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M)$, thanks to Proposition 2.5. It remains to assertain the measurability of $\xi^*$ with values in $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M)$. This follows from the fact that $t \to \xi^*(t) \in L^2(M;\mathcal{H}_l)$ is measurable, that $W^{1-\frac{d-l}{d+\kappa},d+\kappa}(M)$ is reflexive and separable, and the Pettis measurability theorem.

The case (5.18) can be treated with the same techniques as (5.16) except that now (5.26) needs to be considered with $q > 2$. Consequently we obtain $\beta\|\xi^*\|_{L^2(M;\mathcal{H}_1)}^{q-2}\xi^* \in L^\infty(J; W^{1-\frac{2}{3+\kappa},3+\kappa}(M))$ and thus $\xi^*(t) \in W^{1-\frac{2}{3+\kappa},3+\kappa}(M)$ for a.e. $t \in J$. Moreover from (5.26) we deduce that $\xi^* \in L^\infty(J; L^2(M;\mathcal{H}_1))$.

Finally, we turn to case (5.17). In this situation the adjoint variable satisfies $\varphi \in L^2(J; H_{\mathfrak{D}}^{1-\tau})$, see the proof of Lemma 5.9, and recall that $\tau < 0$. We now follow the steps of case (5.16) above. Equation (5.28) is satisfied with $q = 2$. The existence of a continuous extension operator $\mathfrak{E} : H^{1-\tau}(\Omega) \to W^{1-\tau,2}(\mathbb{R}^3)$ is guaranteed by [3], Chapter V.1. The trace mapping $tr_M : W^{1-\tau,2}(\mathbb{R}^3) \to W^{-\tau,2}(M)$ is bounded by Proposition 2.5 in the pointwise $\mathcal{H}_1$ a.a. sense. We thus have $\beta\xi^*(t) = tr_M\mathfrak{E}\varphi(t,\widehat{\phi_t}(\cdot)) \in W^{-\tau,2}(M)$ for a.a. $t \in J$ in case (5.17). The measurability of $J \ni t \to \xi^*(t) \in W^{-\tau,2}(M)$ again follows from the measurability of that mapping with range in $L^2(M;\mathcal{H}_1)$. Finally $\xi^* \in L^2(J; W^{-\tau,2}(M)$ is implied by the boundedness of $\mathfrak{E}$ and the trace operator. □

Now we turn to the proof of Theorem 5.13 and start with two preparatory lemmata.

**Lemma 5.15.** *Assume $d = 2$ and let $s \in ]0, \frac{1}{2}[$. Then $[L^2, H_{\mathfrak{D}}^{1+s}]_\theta$ continuously embeds into $H_{\mathfrak{D}}^{1+r}$ for some $r > 0$ if $\theta$ is sufficiently close to 1.*

*Proof.* The claim follows from Proposition 2.12 ii). □

**Lemma 5.16.** *Assume $d = 3$, and let $s \in ]0, \frac{1}{3}[$. Then $[L^2, W_{\mathfrak{D}}^{1+s,3}]_\theta$ continuously embeds into $W_{\mathfrak{D}}^{1+r,3}$ for some $r > 0$, if $\theta$ is sufficiently close to 1.*

*Proof.* First observe that under the additional Assumption 5.11 (b), one has $[W_{\mathfrak{D}}^{-1,q}, W_{\mathfrak{D}}^{1,q}]_{\frac{1}{2}} = L^q$ ($q \in ]1,\infty[$), see [44]. Then we fix a $\zeta$ such that $L^2$ continuously embeds into $W_{\mathfrak{D}}^{-\zeta,3}$. Clearly, then $[L^2, W_{\mathfrak{D}}^{1+s,3}]_\theta \hookrightarrow [W_{\mathfrak{D}}^{-\zeta,3}, W_{\mathfrak{D}}^{1+s,3}]_\theta$. We may write

$$W_{\mathfrak{D}}^{-\zeta,3} = [L^3, W_{\mathfrak{D}}^{-1,3}]_\zeta = [W_{\mathfrak{D}}^{-1,3}, L^3]_{1-\zeta} = \big[W_{\mathfrak{D}}^{-1,3}, [W_{\mathfrak{D}}^{-1,3}, W_{\mathfrak{D}}^{1,3}]_{\frac{1}{2}}\big]_{1-\zeta} = [W_{\mathfrak{D}}^{-1,3}, W_{\mathfrak{D}}^{1,3}]_{\frac{1-\zeta}{2}} \tag{5.29}$$

Let $\kappa$ be a fixed number from $]\frac{\frac{1}{3}+\zeta}{1+\zeta},\theta[$. Now using (5.29), we may continue

$$[W^{-\zeta,3}_{\mathfrak{D}},W^{1+s,3}_{\mathfrak{D}}]_\theta=\big[[W^{-\zeta,3}_{\mathfrak{D}},W^{1+s,3}_{\mathfrak{D}}]_\kappa,W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}\hookrightarrow\big[[W^{-\zeta,3}_{\mathfrak{D}},W^{1,3}_{\mathfrak{D}}]_\kappa,W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}$$

$$=\Big[\big[[W^{-1,3}_{\mathfrak{D}},W^{1,3}_{\mathfrak{D}}]_{\frac{1-\zeta}{2}},W^{1,3}_{\mathfrak{D}}\big]_\kappa,W^{1+s,3}_{\mathfrak{D}}\Big]_{\frac{\theta-\kappa}{1-\kappa}}=\big[[W^{-1,3}_{\mathfrak{D}},W^{1,3}_{\mathfrak{D}}]_\tau,W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}$$

with $\tau=\frac{1}{2}(1-\zeta)+\frac{\kappa}{2}(1+\zeta)$. Observe that our condition on $\kappa$ implies $\tau>\frac{1}{2}$. So we may continue

$$=\Big[\big[[W^{-1,3}_{\mathfrak{D}},W^{1,3}_{\mathfrak{D}}]_{\frac{1}{2}},W^{1,3}_{\mathfrak{D}}\big]_{2\tau-1},W^{1+s,3}_{\mathfrak{D}}\Big]_{\frac{\theta-\kappa}{1-\kappa}}=\big[[L^3,W^{1,3}_{\mathfrak{D}}\big]_{2\tau-1},W^{1+s,3}_{\mathfrak{D}}\big]_{\frac{\theta-\kappa}{1-\kappa}}=[W^{2\tau-1,3}_{\mathfrak{D}},W^{1+s,3}_{\mathfrak{D}}]_{\frac{\theta-\kappa}{1-\kappa}}. \quad (5.30)$$

Observe that, due to our supposition on $\kappa$, we have $2\tau-1>\frac{1}{3}$. So the results in [44] allow to identify (5.30) with a space $W^{1+r,3}_{\mathfrak{D}}$ , $(r>0)$, if $\theta$ is close to 1. □

We shall also rely on the following result.

**Theorem 5.17.** *(see [45]) Let $V\hookrightarrow H\hookrightarrow V^*$ be a Gelfand triple of Hilbert spaces with dense embeddings. Assume that we are given, for each $t\in J$, a continuous, coercive sesquilinear form $\mathfrak{s}_t$ on $V$ which altogether admit a common coercivity constant. Moreover, suppose that*

$$\sup_{\|\varphi\|_V=\|\psi\|_V=1}\big|\mathfrak{s}_t(\varphi,\psi)-\mathfrak{s}_s(\varphi,\psi)\big|\le c|s-t|^\alpha,\quad s,t\in J \quad (5.31)$$

*for an $\alpha>\frac{1}{2}$.*

*Let $A_t$ be the sectorial operator which is induced by $\mathfrak{s}_t$ on $H$ and $q\in]1,\infty[$.*

*Then, for every $f\in L^q(J,H)\hookrightarrow L^q(J;V^*)$ the solution of the equation*

$$\frac{\partial u}{\partial t}+A(\cdot)u=f,\quad u(0)=0 \quad (5.32)$$

*exists, is unique and satisfies $u\in W^{1,q}(J,H)$, and, consequently,*

$$J\ni t\mapsto A_tu(t)\in L^q(J,H). \quad (5.33)$$

**Theorem 5.18.** *Let $V=H^1_{\mathfrak{D}}$, $\hat{\mu}$ is a – time dependent – coefficient function, bounded and elliptic with a uniform in $t$ ellipticity constant. Additionally, concerning the dependence on $t$, we suppose the condition $\mathfrak{s}_t$ is as in (3.1), there $\mu$ taken as a the coefficient function $\mu(t,\cdot)$. Let $A_t$ denote the operator which is induced by the form $\mathfrak{s}_t$ on $L^2$. Finally, suppose the existence of a reflexive, separable Banach space with dense embedding $X\hookrightarrow L^2$ such that the $X,X^*$ duality extends the $L^2$-self duality and*

$$\|\psi\|_X\le c\|A_t\psi\|_{L^2},\quad \psi\in dom(A_t),\ t\in J \quad (5.34)$$

*$c$ being independt from $t$. Then, for every $L^q(J,L^2)$, the solution $u$ of*

$$\frac{\partial u}{\partial t}+A(\cdot)u=f,\quad u(0)=0 \quad (5.35)$$

*exists and is unique. It belongs to the space $\mathrm{MR}^q_0(J;X,L^2)$.*

*Proof.* First we make use of the fact that condition (5.23) implies condition (5.31) in Theorem 5.17. So existence and uniqueness follow immediately from Theorem 5.17. Moreover, condition (5.34) shows that, for almost every $t$, $u(t)$ indeed belongs to $X$. Let us show that the function $u$ is measurable when considered as $X$-valued. Since it is measurable in $L^2$ also the function $J \ni t \mapsto (u(t), v))_{L^2} = \langle u(t), v\rangle_{X\times X^*}$ is measurable for every $v \in L^2$. But $L^2$ is dense in $X^*$ so $J \ni t \mapsto \langle u(t), v\rangle_{X\times X^*}$ is measurable even for all $v \in X^*$. Hence, $u$ is weakly measurable when considered in $X$ what implies also strong measurability in our case. Knowing this, (5.34) shows in combination with Theorem 5.17 that the assertion is true. □

Having this result at hand, we may apply Proposition 3.10. In combination with the Lemmata 5.15/5.16 this finishes the proof of Theorem 5.13.

**Remark 5.19.** It is not trivial to single out geometries of $\Omega$ and $\mathfrak{D}$ and/or coefficient functions $\mu$ such that (5.24) indeed is a topological isomorphism. Fortunately, a broad zoo of geometries and coefficient functions $\mu$ which implies this isomorphism property is established in [46] and discussed there in great detail.

## 6. Concluding remarks

(a) The assignment

$$C_0(J\times\Omega) \ni f \mapsto \int_J \int_\Omega f(t,x)\mathrm{d}\rho_t(x)\mathrm{d}t \tag{6.1}$$

defines a measure $\varrho$ on $J \times \Omega$ if the mapping $t \mapsto \rho_t \in \mathcal{M}$ is weakly measurable and some integrability condition

$$\int_J \|\rho_t\|^q_{\mathcal{M}}\,\mathrm{d}t < \infty, \quad q \ge 1 \tag{6.2}$$

holds.

Conversely, if $\varrho$ is a measure on $J \times \Omega$, then it always admits a disintegration of type

$$C_0(J\times\Omega) \ni f \mapsto \int_J \int_\Omega f(t,x)\mathrm{d}\rho_t(x)\mathrm{d}\varpi(t), \tag{6.3}$$

with $\rho_t$ a measure on $\Omega$ and $\varpi$ a measure on $\overline{J}$, see [47].

Thus, our result is proved for measures $\varrho$ on $J \times \Omega$ for which the measure $\varpi$ is the Lebesgue measure on $J$ and the measures $\rho_t$ are of the form $\sigma_t \mathcal{H}_l|_{M_t}$, satisfying the integrability condition (6.2).

(b) Condition (6.2) appears to be reasonable for applications, see [15] and [9].

(c) Of course, one can consider right hand sides which are *sums* of – time dependent – measures of the character described above and, say, time dependent $L^2(\Omega)$ functions. By embedding, then the right hand side may be considered as from $L^q(J; H^{-s}_{\mathfrak{D}})$ ($s$ suitably chosen) and the presented parabolic theory still works.

(d) We restricted to the case where the measures live on subsets of *integer* dimension only for technical simplicity. The basis in geometric measure theory on which our results rest is established in [3] for the general case as well. Everything can then be proved quite analogously. Since we are not aware of any applications of this we did not carry out this here but restricted to integral dimensions.

(e) The elliptic result in Proposition 3.5, borrowed from [4], is proved even for systems in that paper. Also in the case of systems the Kato square root problem is solved in the affirmative in an extremely wide range of geometries, see [36]. Moreover, the (elliptic) system operator has a bounded holomorphic calculus on $L^2$, since it is an accretive one. So the above arguments should also work for systems.

(f) As the title of [8] suggests, it can happen that distributional objects are of interest which are *not* necessarily measures. Consider the following situation: Take $\Omega \subset \mathbb{R}^2$ as a Lipschitz domain which contains a subinterval $]-a,a[$ of the $x$-axis. Define the distribution $\Psi$ on $\Omega$ as the PV distribution on $]-a,a[$ as follows:

$$\langle \Psi, v\rangle = \lim_{\epsilon \mapsto 0} \int_{-a}^{-\epsilon} \frac{\overline{v}(x)}{x} \mathrm{d}x + \int_{-a}^{-\epsilon} \frac{\overline{v}(x)}{x} \mathrm{d}x, \quad v \in W^{1,q}_{\mathfrak{D}},\ q > 2. \tag{6.4}$$

It is not hard to see that the mapping in (6.4) is well-defined and continuous on Hölder spaces, hence on $W^{1+\epsilon,2}_{\mathfrak{D}}$ with $\epsilon > 0$ arbitrary. Consequently, the so defined $\Psi$ – *not* being a measure – belongs to any $W^{-1-\epsilon,2}_{\mathfrak{D}}$ and lives on a one-dimensional manifold. We expect that such distributional objects, entering in the parabolic equations as right hand sides, can be treated to a large degree in the same manner as the measures that we have considered above.

We suspect that similar constructions can be found also in higher dimensions, but do not expatiate this further here.

(g) The question arises whether one could have a suitable parabolic theory on the space of bounded Radon measures, $\mathcal{M}(\Omega)$ on $\Omega$, *without* embedding this into suitable Sobolev spaces. We do not know any attempt in this direction, even for autonomous equations. In particular, the domain of second order divergence operators, when considered on $\mathcal{M}(\Omega)$, seems to be unknown. For the action of the $L^2$ semigroup on the pre-dual of $\mathcal{M}(\Omega)$, namely the bounded functions from $C(\Omega)$, some results are known in the pure Dirichlet and in the pure Neumann case, see [48] and [49]. But the generation property of a corresponding analytic semigroup does *not* carry over to $\mathcal{M}$, due to the non-reflexivity. For non-autonomous parabolic operators nothing seems to be known.

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## Data availability statement

No new data/codes were created or analyzed in this study.

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## Appendix A. Appendix

As announced we give some explanations to the proof of **Proposition 4.4**.
The expression in question which one has to estimate is

$$\|G_\alpha \star f\|^2_{L^2(M;\mathcal{H}_l)} = \int_M \left| \int_{\mathbb{R}^d} G_\alpha(x-y) f(y)\,\mathrm{d}y \right|^2 d\mathcal{H}_l(x) \tag{A.1}$$

We follow widely Jonsson/Wallin with the exception to determine the constant $a$ explicitly here – what should allow an easier reading.

We define the number $a$ via

$$\big(d - \frac{d-l}{2}\big)(1-a)2 = \frac{d+l}{2}(1-a)2 = d. \tag{A.2}$$

Re-arranging terms, one obtains

$$\big(d+l\big)a = \big(d - \frac{d-l}{2}\big)a\,2 = l. \tag{A.3}$$

Clearly, this gives $a = \frac{l}{d+l} \in ]0,1[$. Evidently, (A.2) yields

$$\big(d-\alpha\big)(1-a)2 = \big(d - \frac{d-l}{2}\big)(1-a)2 - \big(\alpha - \frac{d-l}{2}\big)(1-a)2 = d - \big(\alpha - \frac{d-l}{2}\big)(1-a)2 < d \tag{A.4}$$

and (A.3) provides

$$(d-\alpha)2a = (d - \frac{d-l}{2})2a + (\frac{d-l}{2} - \alpha)2a = l - (\alpha - \frac{d-l}{2})2a < l, \tag{A.5}$$

thanks to the supposition $\alpha > \frac{d-l}{2}$.

One estimates the r.h.s of (A.1) by

$$\int_M \Big( \int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{1-a} \big|G_\alpha(x-y)\big|^a f(y)\,\mathrm{d}y \Big)^2 \, d\mathcal{H}_l(x).$$

Applying Hölder's inequality, one further estimates

$$\leq \int_M \Big( \int_{\mathbb{R}^d} \big|G_\alpha(x-y)\big|^{2a} |f(y)|^2 \mathrm{d}y \cdot \big( \int_{\mathbb{R}^d} |G_\alpha(x-y)|^{2(1-a)} \mathrm{d}y \big) \Big) \mathrm{d}\mathcal{H}_l(x).$$

The crucial point is to show that the terms

$$\int_{\mathbb{R}^d} |G_\alpha(x-y)|^{2(1-a)} \mathrm{d}y = \int_{\mathbb{R}^d} |G_\alpha(y)|^{2(1-a)} \mathrm{d}y, \tag{A.6}$$

and

$$\int_M |G_\alpha(x-y)|^{2a} \mathrm{d}\mathcal{H}_l(x), \quad y \in \mathbb{R}^d, \tag{A.7}$$

may be estimated uniformly for sets $M$ admitting the same constant $\mathfrak{c}$. Investing the exponential decay of the Bessel kernel at $\infty$ (see [40], Ch. V.3) one can observe that (A.6) makes no difficulties at $\infty$. In a neighborhood of zero (A.6) also converges, thanks to

$$|G_\alpha(z)| \leq \gamma\, |z|^{\alpha - d}, \tag{A.8}$$

(see [40], Ch. V.3) in combination with (A.4).

Finally (A.7) can be written as

$$\int_{M \cap \{x : |x-y| > 1\}} |G_\alpha(x-y)|^{2a} \mathrm{d}\mathcal{H}_l(x) + \int_{M \cap \{x : |x-y| \leq 1\}} |G_\alpha(x-y)|^{2a} \mathrm{d}\mathcal{H}_l(x).$$

According to (A.8), the first integral is not larger than $\gamma^{2a}\ \mathcal{H}_l(M)$, and $\mathcal{H}_l(M)$ is not larger than $\mathfrak{c} \times \mathfrak{n}$ – $\mathfrak{n}$ being the number of( shifted) unit balls $B(z,1)$ required for a covering of $M$. The second integral is estimated by again employing (A.8) in combination with (A.5). This yields $|G_\alpha(x-y)|^{2a} \le \gamma^{2a}|x-y|^{-\sigma}$ with $\sigma < l$. Afterwards one applies [3], Chapter V.1.2, Lemma 1. This shows, first, that (A.7) is indeed finite – and may be estimated uniformly with respect to $y \in \mathbb{R}^d$. But even more, one observes that the constant $\mathfrak{c}$ enters *linearly* in this estimate.