ESAIM: COCV 31 (2025) 100
https://doi.org/10.1051/cocv/2025075

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

SECOND-ORDER OPTIMALITY CONDITIONS FOR TIME-OPTIMAL CONTROL PROBLEMS GOVERNED BY SEMILINEAR PARABOLIC EQUATIONS

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Abstract. A class of time-optimal control problems governed by semilinear parabolic equations with mixed pointwise constraints and final point constraints is considered. By introducing the locally optimal solution to time-optimal control problems, we derive first- and second-order necessary optimality conditions of KKT-type and second-order sufficient conditions for locally optimal solutions to the problem and establish regularity of Lagrange multipliers.

Mathematics Subject Classification. 49K20, 35K20.

Received January 9, 2025. Accepted September 9, 2025.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N = 2$ or 3 and its boundary $\Gamma = \partial\Omega$ is of class C^2. Let $D = H^2(\Omega) \cap H_0^1(\Omega)$, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $Q_T := \Omega \times (0,T)$. We consider the problem of finding $T > 0$, control function $u \in L^\infty(Q_T)$ and the corresponding state $y \in C(\overline{Q}_T) \cap W_2^{1,1}(0,T;D,H)$ which solve

$$J(T,y,u) := \psi_0(T,y(T)) + \int_0^T \int_\Omega L(x,t,y(x,t),u(x,t))\mathrm{d}x\mathrm{d}t \to \inf \tag{1.1}$$

s.t.

$$(P) \qquad \frac{\partial y}{\partial t} + Ay + \psi(x,t,y) = u \quad \text{in} \quad Q_T,\ y(x,t) = 0 \text{ on } \Sigma_T = \Gamma \times [0,T], \tag{1.2}$$

$$y(0) = y_0 \quad \text{in} \quad \Omega, \tag{1.3}$$

$$\psi_i(T,y(T)) \le 0, \quad i = 1,2,...,m, \tag{1.4}$$

$$a \le g(x,t,y(x,t),u(x,t)) \le b \quad \text{a.a.} \quad (x,t) \in Q_T, \tag{1.5}$$

Keywords and phrases: Time-optimal control, necessary first- and second-order optimality conditions, second-order sufficient condition, semilinear parabolic control, mixed pointwise constraints, final point constraints.

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where $y_0 \in H_0^1(\Omega) \cap C(\overline{\Omega})$, $a < b$, A denotes a second-order elliptic operator of the form

$$Ay = -\sum_{i,j=1}^{N} D_j\left(a_{ij}(x) D_i y\right),$$

$L, g : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and

$$\psi_i(T, y(T)) := \int_\Omega L_i(x, T, y(x, T))\mathrm{d}x, \ \ i = 0, 1, 2, ..., m$$

with $L_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions.

Throughout of the paper, we denote by Φ the feasible set of problem (P), that is, Φ consists of triples $(T, y, u) \in (0, +\infty) \times \left(C(\overline{Q}_T) \cap W_2^{1,1}(0, T; D, H)\right) \times L^\infty(Q_T)$ which satisfies conditions (1.2)–(1.5). The goal of problem (P) is to find a control function u which plays a role as heat source to control the temperature of the system in the shortest time $T > 0$.

The class of time-optimal control problems is important for modern technology. They have several applications in different areas. In particular, there are many problems in the field of Aerospace and Rocketry can be solved by finding solutions of time-optimal control problems such as: Soft landing on the Moon (see [1]), Optimal ascent trajectories and Orbital transfer problems (see [2]), Launch vehicle guidance and trajectory optimization (see [3] and [4]). Recently, scientists of SpaceX have used time-optimal control for guiding reusable launch vehicles, such as the SpaceX Falcon 9, back to Earth for a soft landing. Here, the goal is to minimize the time for descent and landing while conserving fuel and ensuring a safe touchdown.

In order to find solution to time-optimal control problems, we often derive necessary optimality conditions and solve the system of optimality conditions by computer to obtain approximate solutions. Then we prove that the approximate solutions converge to the exact solution and give error estimate betwen the exact solution and approximate solutions. This can be done by using second-order sufficient optimality conditions. Besides, we can use the second-order sufficient conditions to check whether or not an extremal point is an optimal solution to the problem. Therefore, optimality conditions for optimal control problems are significant. In this paper we aim at deriving first and second-order necessary optimality conditions and second-order sufficient optimality conditions for problem (P).

Although, there have been many papers dealing with first and second-order optimality conditions for optimal control problems with fixed end-time (see for instance [5]–[7], [8], [9]–[12], [13], [14], [15], [16] and [17]), few of papers study second-order optimality conditions for time-optimal control problems governed by PDEs or ODEs. This is because the structure of a time-optimal control problem is quite complicated. It is never convex even though the constraints are of linear forms; the function spaces vary in time variable T. This causes difficulties for the existence of an optimal solution and how to define the so-called locally optimal solution to the problem. In [18], H. Maurer and H.J. Oberle gave sufficient optimality conditions for optimal control problems governed by ODE with mixed constraints, while H. Maurer and N.P. Osmolovskii [19] studied second-order optimal conditions for time-optimal control which is linear in control variable. In [20] J.-P. Raymond and H. Zidani established a Pontryagin's principle (first-order optimality conditions) for time-optimal control problems governed by semilinear parabolic equations with pointwise state constraints and unbounded controls. Also, N. Arada and J.-P. Raymond [21] derived first-order optimality conditions for time- optimal problems with Dirichlet boundary controls. Recently, in [22], L. Bonifacius *et al.* have considered a time-optimal control problem governed by a linear parabolic equation with a quadratic cost function and pure control constraint. They established necessary and sufficient second-order optimality conditions and gave error estimates between the exact solution of the original problem and approximate solutions of discretization problems which are discretized by the space-time finite element method.

It is noted that, to derive optimality conditions for optimal solution (T_*, y_*, u_*) to time-optimal control problems, in the previous articles the authors assumed that it was a strongly local solution, that is, (T_*, y_*, u_*)

is an optimal solution to the problem in a neighborhood of (T_*, y_*) (see Def. 3.3 in [8]). To our best knowledge, so far there have been no paper dealing with first and second-order optimality conditions for a locally optimal solutions to time-optimal control problems governed by seminlinear parabolic equations with mixed constraints. Therefore, our obtained results in this paper are novel.

In the present setting of problem (P), we will face with some challenges: the regularity of solution to the state equation, how to define a locally optimal solution; how to derive optimality conditions in both state variable and control variable and how to transform the problem to a new problem with fixed end-time. To overcome these challenges, we first establish the existence and regularity of solution to the state equation. Then we give a suitable definition for locally optimal solution by extending the sate functions and scaling control functions. Since the implicit function theorem is fail to apply for (P), we need to derive optimality conditions in both variable y and u. For this, we will derive optimality conditions for a specific mathematical programming problem in Banach spaces and apply the obtained result for (P). In order to transform (P) into an optimal control problem with fixed end-time, we use the simplest way by changing the variable $t = Ts$ with $s \in [0, 1]$ to reduce (P) to a fixed end-time problem (P_1), where (T, v) plays a role as a new control variable. Note that although the appearance of the variable T in (P_1) causes some complications in establishing sufficient second-order conditions, the space containing T is $\mathbb{R}$, where the weak convergence and strong convergence of sequences are coincide. Therefore, we only need to deal with the weak convergence of sequences in the space of control variable v.

The remainder of the paper is organized as follows. In Section 2, we establish some related results on the regularity of solution to the state equation and derive optimality conditions for a specific mathematical programming in Banach spaces. The statement of main results are given in Section 3. In section 4, we derive first and second-order optimality conditions for a reduced problem with fixed final time. Section 5 is destined for proofs of main results.

2. Related results

2.1. State equation

In the sequel, we assume that $\partial\Omega$ is of class C^2. This condition guarantees the $W_p^{2,1}$-regularity of the state y (see Lem. 2.1). Let $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ and $D := H^2(\Omega) \cap H_0^1(\Omega)$. The norm and the scalar product in H are denoted by $|\cdot|$ and $(\cdot,\cdot)_H$, respectively. It is known that

$$D \hookrightarrow\hookrightarrow V \hookrightarrow\hookrightarrow H$$

and each space is dense in the following one. Hereafter, $\hookrightarrow\hookrightarrow$ denotes a compact embedding.

$W^{m,p}(\Omega)$ for m integer and $p \geq 1$ is a Banach space of elements in $v \in L^p(\Omega)$ such that their generalized derivatives $D^\alpha v \in L^p(\Omega)$ for $|\alpha| \leq m$. The norm of a element $v \in W^{m,p}(\Omega)$ is given by

$$\|u\|_{m,p} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_p,$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$. When $p = 2$, we write $H^m(\Omega)$ and $H_0^m(\Omega)$ for $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$, respectively.
$W^{s,p}(\Omega)$ for real number $s \geq 0$ consists of elements v so that

$$\|v\|_{s,p} = \Big[\|v\|_{m,p}^p + \sum_{|\alpha|=m} \int_\Omega\int_\Omega \frac{|D^\alpha v(x) - D^\alpha v(x')|^p}{|x-x'|^{N+\sigma p}} dxdx'\Big]^{1/p} < +\infty,$$

where $s = m + \sigma$ with $m = [s]$ and $\sigma \in (0, 1)$. It is known that

$$W^{s,p}(\Omega) = \big(W^{k_0,p}(\Omega), W^{k_1,p}(\Omega)\big)_{\theta,1}, \tag{2.1}$$

where $s = (1-\theta)k_0 + \theta k_1$, $0 < \theta < 1$, $k_0 < k_1$ and $1 \le p \le \infty$. Here $\big(W^{k_0,p}(\Omega), W^{k_1,p}(\Omega)\big)_{\theta,1}$ is a interpolation space (see [23], Chapter 22, p. 109 and Chapter 34, p. 164 and [24]).
$W_p^{2l,l}(Q_T)$ for l integer and $p \ge 1$ is a Banach space of elements in $v \in L^p(Q_T)$ such that their generalized derivatives $D_t^r D_x^s v \in L^p(\Omega)$ with $2r + s \le 2l$. The norm of a element $v \in W_p^{2l,l}(\Omega)$ is given by

$$\|v\|_{p,Q_T}^{(2l)} = \sum_{j=0}^{2l} \langle\langle v \rangle\rangle_{p,Q_T}^{(j)},$$

where

$$\langle\langle v \rangle\rangle_{p,Q_T}^{(j)} := \sum_{2l+s=j} \|D_t^r D_x^s v\|_{L^p(Q_T)}.$$

Given $\alpha \in (0,1)$ and $T > 0$, we denote by $C^{0,\alpha}(\Omega)$ and $C(\overline{Q}_T)$ the space of Hölder continuous functions on Ω and the space of continuous functions on $\overline{Q}_T$, respectively.

Let $H^{-1}(\Omega)$ be the dual of $H_0^1(\Omega)$. We define the following function spaces

$$\begin{aligned}
&H^1(Q_T) = W_2^{1,1}(Q_T) = \{y \in L^2(Q_T) : \frac{\partial y}{\partial x_i}, \frac{\partial y}{\partial t} \in L^2(Q_T)\},\\
&V_2(Q_T) = L^\infty(0,T;H) \cap L^2(0,T;V),\\
&W(0,T) = \{y \in L^2(0,T;H_0^1(\Omega)) : y_t \in L^2(0,T;H^{-1}(\Omega))\},\\
&W_2^{1,1}(0,T;V,H) = \{y \in L^2([0,T],V) : \frac{\partial y}{\partial t} \in L^2([0,T],H)\},\\
&W_2^{1,1}(0,T;D,H) = \{y \in L^2([0,T],D) : \frac{\partial y}{\partial t} \in L^2([0,T],H)\},\\
&U_T = L^\infty(Q_T),\\
&Y_T = \big\{y \in W_2^{1,1}(0,T;D,H) \cap W_p^{2,1}(Q_T) | Ay \in L^p(Q_T)\big\}.
\end{aligned}$$

Hereafter, we assume that

$$1 + \frac{N}{2} < p < N + 2 \tag{2.2}$$

and q is the conjugate number of p. Then Y_T is a Banach space under the graph norm

$$\|y\|_{Y_T} := \|y\|_{W_2^{1,1}(0,T;D,H)} + \|y\|_{W_p^{2,1}(Q_T)} + \|Ay\|_{L^p(Q_T)}.$$

Note that if $y \in W_p^{2,1}(Q_T)$, then $y_t \in L^p(0,T;L^p(\Omega))$. By the proof of Lemma 2.1, we have

$$W_p^{2,1}(Q_T) \hookrightarrow\hookrightarrow C(\overline{Q}_T). \tag{2.3}$$

Hence

$$Y_T \hookrightarrow\hookrightarrow C(\overline{Q}_T). \tag{2.4}$$

Besides, we have the following continuous embeddings:

$$W_2^{1,1}(0,T;D,H) \hookrightarrow C([0,T],V), \quad W(0,T) \hookrightarrow C([0,T],H), \tag{2.5}$$

where $C([0,T],X)$ stands for the space of continuous mappings $v : [0,1] \to X$ with X is a Banach space.

Recall that given $y_0 \in H$ and $u \in L^2(0,T;H)$, a function $y \in W(0,T)$ is said to be a weak solution of the semilinear parabolic equation (1.2)–(1.3) if

$$\begin{cases} \langle y_t, v\rangle + \sum_{i,j=1}^n \int_\Omega a_{ij} D_i y D_j v \mathrm{d}x + (\psi(\cdot,\cdot,y), v)_H = (u,v)_H \quad \forall v \in H_0^1(\Omega), \quad \text{a.a. } t \in [0,T] \\ y(0) = y_0. \end{cases} \tag{2.6}$$

If a weak solution y such that $y \in W_2^{1,1}(0,T;D,H)$ and $\psi(\cdot,\cdot,y) \in L^2(0,T;H)$ then we have $y_t + Ay + \psi(\cdot,\cdot,y) \in L^2(0,T;H)$ and

$$\langle y_t, v\rangle + (Ay, v) + (\psi(\cdot,\cdot,y), v) = (u,v)$$

for each $v \in H_0^1(\Omega)$ and a.a. $t \in [0,T]$. Since $H_0^1(\Omega)$ is dense in $H = L^2(\Omega)$, we have

$$(y_t, v) + (Ay, v) + (\psi(\cdot,\cdot,y), v) = (u,v)$$

for each $v \in H$ and a.a. $t \in [0,T]$. Hence

$$y_t + Ay + \psi(\cdot,\cdot,y) = u \quad \text{a.a. } t \in [0,T], \ y(0) = y_0.$$

In this case we say y is a *strong solution* of (1.2)–(1.3). From now on a solution to (1.2)–(1.3) is understood a strong solution.

Let us make the following assumptions which are related to the state equation.

$(H1)$ Coefficients $a_{ij} = a_{ji} \in C(\overline{\Omega})$ for every $1 \le i,j \le N$, satisfy the uniform ellipticity conditions, *i.e.*, there exists a number $\gamma > 0$ such that

$$\gamma |\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^N \text{ for all } x \in \Omega. \tag{2.7}$$

$(H2)$ The mapping $\psi : \overline{\Omega} \times [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and for each $x \in \overline{\Omega}$, $\psi(x,\cdot,\cdot)$ is of class C^2 and satisfies the property that

$$\begin{aligned} &\exists C_\psi \in \mathbb{R} : \psi_y(x,t,y) \ge C_\psi \quad \forall (x,t,y) \in \overline{\Omega} \times [0,+\infty) \times \mathbb{R}, \\ &\psi(\cdot,\cdot,0) \in L^p(Q_T) \quad \forall T > 0. \end{aligned}$$

Furthermore, ψ and its derivative are local bounded and local Lipschitz in the sense that for each $M > 0$, there exists a number $k_{\psi,M} > 0$ such that

$$\begin{aligned} &|\psi_t(x,t,y)| + |\psi_y(x,t,y)| + |\psi_{yy}(x,t,y)| \le k_{\psi,M}, \\ &|\psi(x,t_1,y_1) - \psi(x,t_2,y_2)| + |\psi_t(x,t_1,y_1) - \psi_t(x,t_2,y_2)| + |\psi_y(x,t_1,y_1) - \psi_y(x,t_2,y_2)| \\ &+ |\psi_{tt}(x,t_1,y_1) - \psi_{tt}(x,t_2,y_2)| + |\psi_{ty}(x,t_1,y_1) - \psi_{ty}(x,t_2,y_2)| + |\psi_{yy}(x,t_1,y_1) - \psi_{yy}(x,t_2,y_2)| \\ &\le k_{\psi,M}(|t_1 - t_2| + |y_1 - y_2|), \end{aligned}$$

for all t, y, t_i, y_i satisfying $|t|, |y|, |t_i|, |y_i| \le M$ with $i = 1,2$.

$(H3)$ $y_0 \in H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega)$, where $p > 1$ satisfying (2.2).

Note that $(H3)$ makes sure that $y_0 \in C(\overline{\Omega})$. Since $2 > 2 - \frac{2}{p} > 1$, Theorem 1.4.3.2 in [25] implies that

$$W^{2,p}(\Omega) \hookrightarrow \hookrightarrow W^{2-\frac{2}{p},p}(\Omega) \hookrightarrow \hookrightarrow W^{1,p}(\Omega).$$

Hence the condition $y_0 \in W^{2-\frac{2}{p},p}(\Omega)$ relaxes the condition $y_0 \in W^{2,p}(\Omega)$.

Lemma 2.1. *Suppose that* $(H1), (H2)$ *and* $(H3)$ *are valid. Then for each* $T > 0$ *and* $u \in L^p(Q_T)$ *with* p *satisfying* (2.2), *the state equation* (1.2)–(1.3) *has a unique solution* $y \in Y_T$ *and there exist positive constants* $C_1 > 0$ *and* $C_2 > 0$ *such that*

$$\|y\|_{C(\overline{Q}_T)} + \|\psi(\cdot,\cdot,y)\|_{L^2(0,T;H)} \le C_1(\|u\|_{L^p(Q_T)} + \|y_0\|_{L^\infty(\Omega)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)}) \tag{2.8}$$

and

$$\|y_t\|_{L^p(Q_T)} + \|y\|_{L^p(Q_T)} + \|Dy\|_{L^p(Q_T)} + \|D^2y\|_{L^p(Q_T)} \le C_2(\|y_0\|_{C(\overline{\Omega})} + \|u\|_{L^p(Q_T)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)}), \tag{2.9}$$

where C_2 *depends on* $T, \Omega, \|y_0\|_{C(\overline{\Omega})}$, $\|u\|_{L^p(Q_T)}$ *and* $\|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)}$.

Proof. We first claim that $y_0 \in C(\overline{\Omega})$. In fact, by (2.2), $(2 - \frac{2}{p}) - \frac{N}{p} \in (0,1)$. By [25], Theorems 1.4.3.1 and 1.4.4.1, we have $W^{2-\frac{2}{p},p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ with $\alpha = (2 - \frac{2}{p}) - \frac{N}{p}$. This implies that $y_0 \in C(\overline{\Omega})$. Hence $y_0(x) = 0$ for $x \in \Gamma$ and the claim is justified.

By [11],Theorem 2.1 (see also proofs of [12], Thm. 2.1 and [12], Lem. 2.2.III), for each $u \in L^p(0,T;L^p(\Omega)) = L^p(Q_T)$ with $p > \frac{N+2}{2}$, the state equation has a unique solution $y \in L^\infty(Q) \cap W_2^{1,1}(0,T;V,H)$ such that $\psi(\cdot,\cdot,y) \in L^2(0,T;H)$ and inequalities (2.8) is fulfilled.

By $(H2)$, we have $\psi(\cdot,\cdot,y) \in L^\infty(Q_T)$. Let us show that $y \in W_p^{2,1}(Q_T)$. For this, we consider equation

$$z_t + Az = u - \psi(x,t,y),\ z(0) = y_0. \tag{2.10}$$

Since $u - \psi(\cdot,\cdot,y) \in L^p(Q_T)$, [26],Theorem 9.1, p. 341 implies that the equation (2.10) has a unique solution $z \in W_p^{2,1}(Q_T)$ such that $z(0) = y_0$ and $z(x,t) = 0$ for $(x,t) \in \Gamma \times [0,T]$. Besides, there exists a constant $C > 0$ such that

$$\|z\|^{(2)}_{p,Q_T} \le C(\|y_0\|_{2-\frac{2}{p},p} + \|u - \psi(\cdot,\cdot,y)\|_{L^p(Q_T)}). \tag{2.11}$$

By definition of norm $\|\cdot\|^{(2)}_{p,Q_T}$, we have

$$\|z_t\|_{L^p(Q_T)} + \|z\|_{L^p(Q_T)} + \|Dz\|_{L^p(Q_T)} + \|D^2z\|_{L^p(Q_T)} \le C(\|y_0\|_{2-\frac{2}{p},p} + \|u - \psi(\cdot,\cdot,y)\|_{L^p(Q_T)}).$$

We now prove that (2.3) is valid and so $z \in C(\overline{Q}_T)$. In fact, consider the space $W^{s,p}(\Omega)$, where $s \in (\frac{N}{p}, 2 - \frac{2}{p})$. By [25], Theorem 1.4.3.2, the embeddings

$$W^{2,p}(\Omega) \hookrightarrow \hookrightarrow W^{s,p}(\Omega) \hookrightarrow \hookrightarrow L^p(\Omega). \tag{2.12}$$

By (2.2), $0 < s - N/p < 1$. Using [25],Theorem 1.4.3.1 and Theorem 1.4.4.1 again, we have

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega}). \tag{2.13}$$

Here we used the fact that if Ω is bounded domain, then $C^{0,\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega})$. By the generalized Gagliardo-Nirenberg inequality in [27], Theorem 1, part A, we have

$$\|v\|_{s,p} \le C\|v\|_{2,p}^{\theta}\|v\|_{0,p}^{1-\theta} \qquad \forall v \in W^{2,p}(\Omega),$$

where $\theta = s/2 \in (\frac{N}{2p}, 1-\frac{1}{p}) \subset (0,1)$. By [24], Section 1.10.1, Lemma (a), p. 61, we have

$$\big(L^p(\Omega), W^{2,p}(\Omega)\big)_{\theta,1} \hookrightarrow W^{s,p}(\Omega) \hookrightarrow L^p(\Omega). \tag{2.14}$$

Using [28],Theorem 7.4.3, p. 268 for the case $p_\theta = p_0 = p_1 = p$ and $0 < 1-\theta-\frac{1}{p}$, we obtain

$$W_p^{2,1}(Q_T) \hookrightarrow\hookrightarrow C([0,T], W_p^s(\Omega)).$$

From this and (2.13), we get

$$W_p^{2,1}(Q_T) \hookrightarrow\hookrightarrow C([0,T], C(\overline{\Omega})) \hookrightarrow C(\overline{Q}_T)$$

and (2.3) is justified.

Since $z(x,t) = 0$ on $\Gamma \times [0,T]$, we get $z \in C([0,T], W_0^{1,p}(\Omega)) \subset L^2(0,T;H_0^1(\Omega))$. By the uniqueness, we have $z = y$.

Let $M > C_1(\|u\|_{L^p([0,T],H)} + \|y_0\|_{L^\infty(\Omega)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)})$. By $(H2)$ and (2.8), there exists $k_M > 0$ such that

$$|\psi(x,t,y)| \le |\psi(x,t,y) - \psi(x,t,0)| + |\psi(x,t,0)| \le k_M|y| + |\psi(x,t,0)|.$$

Combining this with (2.11) and (2.8), we have

$$\begin{aligned}
\|y\|_{p,Q_T}^{(2)} &\le C\big(\|y_0\|_{2-\frac{2}{p},p} + \|u\|_{L^p(Q_T)} + k_M\|y\|_{L^p(Q_T)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)}\big)\\
&\le C(\|y_0\|_{2-\frac{2}{p},p} + \|u\|_{L^p(Q_T)} + k_M'\|y\|_{C(Q_T)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)})\\
&\le C_2(\|y_0\|_{C(\overline{\Omega})} + \|u\|_{L^p(Q_T)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)})
\end{aligned}$$

for some constant $C_2 > 0$ which depends on $T, \Omega, \|y_0\|_{C(\overline{\Omega})}$, $\|u\|_{L^p(Q_T)}$ and $\|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)}$. By definition of norm $\|y\|_{p,Q_T}^{(2)}$, we have

$$\|y_t\|_{L^p(Q_T)} + \|y\|_{L^p(Q_T)} + \|Dy\|_{L^p(Q_T)} + \|D^2y\|_{L^p(Q_T)} \le C_2(\|y_0\|_{C(\overline{\Omega})} + \|u\|_{L^p(Q_T)} + \|\psi(\cdot,\cdot,0)\|_{L^p(Q_T)})$$

which is inequality (2.9). Since $Ay = u - \psi(\cdot,\cdot,y) - y_t \in L^p(Q_T)$, $y \in Y_T$. The lemma is proved. □

Let us consider the linearized equation

$$y_t + Ay + c(x,t)y = u, \quad y(0) = \phi_0. \tag{2.15}$$

Lemma 2.2. *Suppose $u \in L^p(Q_T)$, $c \in L^\infty(Q_T)$ and $\phi_0 \in W^{2-\frac{2}{p},p}(\Omega) \cap H_0^1(\Omega)$ with p satisfying (2.2). Then equation (2.15) has a unique solution $y \in Y_T$ and there exist positive constant C_1 and C_2 which depend on T such that*

$$\|y_t\|_{L^p(Q_T)} + \|y\|_{L^p(Q_T)} + \|Dy\|_{L^p(Q_T)} + \|D^2y\|_{L^p(Q_T)} \le C_1\big(\|u\|_p + \|\phi_0\|_{W^{2-\frac{2}{p},p}(\Omega)}\big), \tag{2.16}$$

$$\|y\|_{C(\overline{Q}_T)} \leq C_2\big(\|u\|_p + \|\phi_0\|_{W^{2-\frac{2}{p},p}(\Omega)}\big). \tag{2.17}$$

Proof. The conclusion of the lemma follows directly from [26],Theorem 9.1, p. 341. By embedding (2.3), we have estimate (2.17). □

2.2. Mathematical programming problem

Let E, W and Z be Banach spaces with the dual spaces E^*, W^* and Z^*, respectively. We denote by $\tau(\mathbb{R}^m)$ the strong topology in $\mathbb{R}^m$, by $\sigma(E^*, E)$ the weak* topology in E^* and by $\sigma(W^*, W)$ the weak* topology in W^*. Given a Banach space X, $B_X(x_0, r)$ stands for the open ball with center x_0 and radius r. Given a subset M in X. We shall denote by Int$M$ and $\overline{M}$ the interior and the closure of M, respectively.

Let $f : Z \to \mathbb{R}$, $F : Z \to E$, $G : Z \to W$ and $H_i : Z \to \mathbb{R}$ with $i = 1, 2, ..., m$ be mappings and K be a nonempty convex set in E. We consider the following optimization problem

$$\text{(MP)} \qquad \begin{cases} f(z) \to \min \\ \text{s.t.} \\ F(z) = 0,\ H_i(z) \leq 0,\ G(z) \in K,\ i = 1, 2, ..., m. \end{cases}$$

Denote Σ by the feasible set of (MP), that is,

$$\Sigma := \{z \in Z \mid F(z) = 0, H_i(z) \leq 0, G(z) \in K, i = 1, 2, ..., m\}.$$

Let us recall some concepts of variational analysis. Given a nonempty subset M of X and $\overline{x} \in \overline{M}$, the set

$$T(M, \overline{x}) := \left\{h \in X \mid \exists t_n \to 0^+, \exists h_n \to h, \overline{x} + t_n h_n \in M \quad \forall n \in \mathbb{N}\right\},$$

is called *the contingent cone* to M at $\overline{x}$. It is well known that if M is convex then

$$T(M, \overline{x}) = \overline{\text{cone}(M - \overline{x})},$$

where $\text{cone}\,(M - \overline{x}) = \{\lambda(v - \overline{x}) \mid v \in M, \lambda > 0\}$. When M is a convex set, *the normal cone* to M at $\overline{x}$ is defined by

$$N(M, \overline{x}) := \{x^* \in X^* \mid \langle x^*, x - \overline{x}\rangle \leq 0 \quad \forall x \in M\} = \{x^* \in X^* \mid \langle x^*, h\rangle \leq 0 \quad \forall h \in T(M, \overline{x})\}.$$

Given a feasible point $z_0 \in \Sigma$, we denote by $f'(z_0)$ or $Df(z_0)$ the first-order derivative of f at z_0 and by $f''(z_0)$ or $D^2f(z_0)$ the second-order derivative of f at z_0. Let us impose the following assumptions.

$(A1)$ int$K \neq \emptyset$;
$(A2)$ The mappings f, F and G are of class C^2 around z_0;
$(A3)$ The range of $DF(z_0)$ is closed in W;
$(A4)$ $DF(z_0)$ is onto and there exists $\widetilde{z} \in Z$ such that

$$\begin{cases} DF(z_0)\widetilde{z} = 0, \\ H_i(z_0) + DH_i(z_0)\widetilde{z} < 0\ \forall i = 1, 2, ..., m \\ G(z_0) + DG(z_0)\widetilde{z} \in \text{int}(K). \end{cases} \tag{2.18}$$

It is noted that $(A4)$ is a type of the Mangasarian–Fromowitz condition. Obviously, $(A4)$ implies $(A3)$.

To deal with second-order optimality conditions, we need the so-called critical cone which is defined as follows. Let $\mathcal{C}_0[z_0]$ be a set of vectors $d \in Z$ satisfying conditions:

(a_1) $Df(z_0)d \le 0$,
(a_2) $DF(z_0)d = 0$,
(a_3) $DH_i(z_0)d \le 0$ for $i \in I_0$, where $I_0 := \{i \in \{1, 2, ..., m\} : H_i(z_0) = 0\}$,
(a_4) $DG(z_0)d \in \text{cone}(K - G(z_0))$.

Then the closure of $\mathcal{C}_0[z_0]$ in Z is called the critical cone at z_0 and denoted by $\mathcal{C}[z_0]$. Each vector $d \in \mathcal{C}[z_0]$ is called a critical direction.

The generalized Lagrange function which is associated with the problem (MP) is defined as follows

$$\mathcal{L} : Z \times \mathbb{R} \times E^* \times \mathbb{R}^m \times W^* \to \mathbb{R},$$
$$\mathcal{L}(z, \lambda, e^*, l, w^*) = \lambda f(z) + \langle e^*, F(z)\rangle + \sum_{i=1}^{m} l_i H_i(z) + \langle w^*, G(z)\rangle.$$

We say that vector $(\lambda, e^*, l, w^*) \in \mathbb{R} \times E^* \times \mathbb{R}^m \times W^*$ is a generalized Lagrange multipliers of (MP) at z_0 if the following conditions are fulfilled:

$$D_z\mathcal{L}\,(z_0, \lambda, e^*, l, w^*) = \lambda Df(z_0) + DF(z_0)^*e^* + \sum_{i=1}^{m} l_i DH_i(z_0) + DG(z_0)^*w^* = 0, \tag{2.19}$$

$$|\lambda|^2 + \|e^*\|^2 + \sum_{i=1}^{m} |l_i|^2 + \|w^*\|^2 \neq 0, \tag{2.20}$$

$$\lambda \ge 0,\ \ l_i \ge 0,\ \ l_i H_i(z_0) = 0, i = 1, 2, .., m,\ \ w^* \in N(K, G(z_0)). \tag{2.21}$$

We denote by $\Lambda[z_0]$ the set of generalized Lagrange multipliers at z_0. In addition, if $\lambda = 1$, then we say $(1, e^*, l, w^*)$ are normal. We denote by $\Lambda_*[z_0]$ the set of Lagrange multipliers (e^*, l, w^*) such that $(1, e^*, l, w^*) \in \Lambda[z_0]$.

We now have the following important lemma.

Lemma 2.3. *Suppose that $z_0 \in \Sigma$ is a locally optimal solution of (MP) under which $(A1)$, $(A2)$ and $(A3)$ are valid, and $DF(z_0)$ is surjective. Then for each $d \in \mathcal{C}_0[z_0]$, the system*

$$\begin{cases} f'(z_0)z + \frac{1}{2}f''(z_0)d^2 < 0, \\ H_i'(z_0)z + \frac{1}{2}H_i''(z_0)d^2 < 0, \quad \forall i \in I_0, \\ DF(z_0)z + \frac{1}{2}D^2F(z_0)(d, d) = 0, \\ DG(z_0)z + \frac{1}{2}D^2G(z_0))(d, d) \in \text{cone}(\text{int}K - G(z_0)) \end{cases} \tag{2.22}$$

has no solution $z \in Z$.

Proof. The proof is performed analogously to the proof of [29], Lemma 3.2. For this, we need to use a Taylor expansion and the Ljusternik theorem. □

The following result gives first- and second-order necessary optimality conditions for (MP).

Proposition 2.4. *Suppose that z_0 is a locally optimal solution to (MP). Then the following assertions are fulfilled:*
(a) If $(A1), (A2)$ and $(A3)$ are satisfied, then $\Lambda[z_0] \neq \emptyset$ and for each $d \in \mathcal{C}_0[z_0]$, there exist $(\lambda, e^*, l, w^*) \in \Lambda[z_0]$ such that

$$D_z^2\mathcal{L}(z_0, \lambda, e^*, l, w^*)[d, d] \ge 0. \tag{2.23}$$

As a consequence,

$$\sup_{(\lambda,e^*,l,w^*)\in\Lambda[z_0]} D_z^2\mathcal{L}(z_0,\lambda,e^*,l,w^*)[d,d]\geq 0 \quad \forall d\in\mathcal{C}_0[z_0].$$

(b) If (A1), (A2) and (A4) are satisfied, then $\Lambda_*[z_0]$ *is nonempty and compact in the topology* $\sigma(E^*,E)\times\sigma(\mathbb{R}^k,\mathbb{R}^k)\times\sigma(W^*,W)$ *and for each* $d\in\mathcal{C}[z_0]$*, there exist* $(e^*,l,w^*)\in\Lambda_*[z_0]$ *such that*

$$D_z^2\mathcal{L}(z_0,1,e^*,l,w^*)[d,d]\geq 0. \tag{2.24}$$

As a consequence,

$$\max_{(e^*,l,w^*)\in\Lambda_*[z_0]} D_z^2\mathcal{L}(z_0,1,e^*,l,w^*)[d,d]\geq 0 \quad \forall d\in\mathcal{C}[z_0]. \tag{2.25}$$

Proof. (a). Fixing any $d\in\mathcal{C}_0[z_0]$, we consider two cases.
Case 1. $DF(z_0)Z\neq E$.

Then there is a point $e\in E$ such that $e\notin DF(z_0)Z$. Since $DF(z_0)Z$ is a closed subspace, the separation theorem (see [30], Thm. 3.4) implies that there exists a non-zero functional $e^*\in E^*$ which separates e and $DF(z_0)Z$, that is, $\langle e^*,e\rangle\geq\langle e^*,DF(z_0)z\rangle$ for all $z\in Z$. This implies $DF(z_0)^*e^*=0$. Putting $\lambda=0, l=0, w^*=0$, we get $(0,e^*,0,0)\in\Lambda(z_0)$. If $e^*DF(z_0)d^2\leq 0$, then we replace e^* by $-e^*$. Then $(0,-e^*,0,0)$ satisfies the conclusion of (a).
Case 2. $DF(z_0)Z=E$.

Let $m_0=|I_0|$. We define a set S which consists of vectors $(\mu,\gamma,e,w)\in\mathbb{R}\times\mathbb{R}^{m_0}\times E\times W$ such that there exists $z\in Z$ satisfying

$$\begin{cases} f'(z_0)z+\frac{1}{2}f''(z_0)d^2<\mu, \\ H_j'(z_0)z+\frac{1}{2}H_j''(z_0)d^2<\gamma_j, j\in I_0, \\ DF(z_0)z+\frac{1}{2}D^2F(z_0)d^2=e \\ DG(z_0)z+\frac{1}{2}D^2G(z_0)d^2-w\in\text{cone}(\text{int}K-G(z_0)). \end{cases} \tag{2.26}$$

It is clear that S is convex. We now claim that S is open. In fact, take $(\hat{\mu},\hat{\gamma},\hat{e},\hat{w})\in S$ corresponding to $\hat{z}$. Choose $\epsilon>0$ such that

$$f'(z_0)\hat{z}+\frac{1}{2}f''(z_0)d^2<\hat{\mu}-\epsilon, \quad H_j'(z_0)\hat{z}+\frac{1}{2}H_j''(z_0)d^2<\hat{\gamma}_j-\epsilon, \quad \forall j\in I_0.$$

Then there is $\delta>0$ such that for all $z\in B_Z(\hat{z},\delta)$ one has

$$f'(z_0)z+\frac{1}{2}f_i''(z_0)d^2<\hat{\mu}-\epsilon/2 \quad H_j'(z_0)z+\frac{1}{2}H_j''(z_0)d^2<\hat{\gamma}_j-\epsilon/2 \quad \forall j\in I_0.$$

It follows that

$$\begin{aligned} &f'(z_0)z+\frac{1}{2}f''(z_0)d^2<\mu, \quad \forall z\in B(\hat{z},\delta) \text{ and } |\mu-\hat{\mu}|<\epsilon/2, \\ &H_j'(z_0)z+\frac{1}{2}H_j''(z_0)d^2<\gamma_j, \quad \forall z\in B(\hat{z},\delta) \text{ and } |\gamma_i-\hat{\gamma}_i|<\epsilon/2, j\in I_0. \end{aligned}$$

Since cone(int$K - G(z_0)$) is an open convex cone and

$$DG(z_0)\hat{z} + \frac{1}{2}D^2G(z_0)d^2 - \hat{w} \in \text{cone}(\text{int}(K) - G(z_0)),$$

the continuity of $DG(z_0)$ implies that there exit balls $B_Z(\hat{z}, \hat{\delta})$ and $B_W(\hat{w}, r)$ with $\hat{\delta} < \delta$ such that

$$DG(z_0)z + \frac{1}{2}D^2G(z_0)d^2 - w \in \text{cone}(\text{int}(K) - G(z_0)) \quad \forall (z, w) \in B_Z(\hat{z}, \hat{\delta}) \times B_W(\hat{w}, r).$$

Since $DF(z_0)$ is surjective, the open mapping theorem implies that $DF(z_0)[B_Z(\hat{z}, \hat{\delta})] + \frac{1}{2}D^2F(z_0)d^2$ is open. Hence, there exists a number $\alpha > 0$ such that

$$B_E(\hat{e}, \alpha) \subset DF(z_0)[B_Z(\hat{z}, \hat{\delta})] + \frac{1}{2}D^2G(z_0)d^2.$$

Thus, for any $(\mu, \gamma, e, w) \in B_{\mathbb{R}}(\hat{\mu}, \epsilon/2) \times B_{\mathbb{R}^{m_0}}(\hat{\gamma}, \epsilon/2) \times B_E(\hat{e}, \alpha) \times B_W(\hat{w}, r)$, there exists $z \in B_Z(\hat{z}, \hat{\delta})$ such that

$$f'(z_0)z + \frac{1}{2}f_i''(z_0)d^2 < \mu, \quad H_j'(z_0)z + \frac{1}{2}H_j''(z_0)d^2 < \gamma_j \quad \forall j \in I_0.$$
$$e = DF(z_0)z + \frac{1}{2}D^2F(z_0)d^2, \quad DG(\overline{z})z + \frac{1}{2}D^2G(z_0)d^2 - w \in \text{cone}(\text{int}(K) - G(\overline{z})).$$

This means

$$B_{\mathbb{R}}(\hat{\mu}, \epsilon/2) \times B_{\mathbb{R}^{m_0}}(\hat{\gamma}, \epsilon/2) \times B_E(\hat{e}, \alpha) \times B_W(\hat{w}, r) \subset S.$$

Hence S is open. The claim is justified.

By Lemma 2.3, we have $(0, 0, 0, 0) \notin S$. By the separation theorem (see [31], Thm. 1, p. 163, there exists a nonzero vector $(\lambda, l, e^*, w^*) \in \mathbb{R} \times \mathbb{R}^{m_0} \times E^* \times \times W^*$ such that

$$\lambda\mu + l^T\gamma + \langle e^*, e\rangle + \langle w^*, w\rangle \geq 0 \quad \forall (\mu, \gamma, e, w) \in S. \tag{2.27}$$

Fix any $z \in Z$, $w' \in$ cone(int$K - G(z_0)$), $r > 0$ and $r_j > 0$. Put

$$\mu = r + f'(z_0)z + \frac{1}{2}f''(z_0)d^2, \;\; \gamma_j = r_j + H_j'(z_0)z + \frac{1}{2}H_j''(z_0)d^2,$$
$$e = DF(z_0)z + \frac{1}{2}D^2F(z_0)d^2, \;\; w = DG(z_0)z + \frac{1}{2}D^2G(z_0)d^2 - w'.$$

Then $(\mu, \gamma, e, w) \in S$. From this and (2.27), we get

$$\lambda(r + f'(z_0)z + \frac{1}{2}f''(z_0)d^2) + \sum_{j \in I_0} l_j(r_j + H_j'(z_0)z + \frac{1}{2}H_j''(z_0)d^2)$$
$$+ \langle e^*, DF(z_0)z + \frac{1}{2}D^2F(z_0)d^2\rangle + \langle w^*, DG(z_0)z + \frac{1}{2}D^2G(z_0)d^2\rangle - \langle w^*, w'\rangle \geq 0.$$

If $\lambda < 0$ then by letting $r \to +\infty$, the term on the left hand side approach to $-\infty$ which is impossible. Hence we must have $\lambda \geq 0$. Similarly, we have $l_j \geq 0$ for all $j \in I_0$.

By letting $r \to 0$ and $r_j \to 0$, we get

$$\lambda f'(z_0) + \sum_{j \in I_0} l_j H_j'(z_0) + DF(z_0)^* e^* + DG(z_0)^* w^* = 0 \tag{2.28}$$

and

$$\frac{1}{2}\big(\lambda f''(z_0)d^2 + \sum_{j \in I_0} l_i H_i''(z_0)d^2 + \langle e^*, D^2F(z_0)d^2\rangle + \langle w^*, D^2G(z_0)d^2\rangle\big) \geq \langle w^*, w'\rangle$$

for all $w' \in \text{cone}(\text{int}K - G(z_0))$. It follows that

$$\langle w^*, w'\rangle \leq 0, \forall w' \in \text{cone}(\text{int}K - G(z_0)).$$

Since $K \subseteq \overline{K} = \overline{\text{int}K}$ (see [31], Prop. 4, p. 163), we obtain

$$\langle w^*, w'\rangle \leq 0, \forall w' \in \text{cone}(K - G(z_0)).$$

This implies that $w^* \in N(K, G(z_0))$. By letting $w' \to 0$, we get

$$\lambda f''(z_0)d^2 + \sum_{j \in I_0} l_j H_i''(z_0)d^2 + \langle e^*, D^2F(z_0)d^2\rangle + \langle w^*, D^2G(z_0)d^2\rangle \geq 0. \tag{2.29}$$

We now take $l_j = 0$ for $j \in \{1, 2, ..., m\} \setminus I_0$. Then we have $l_j \geq 0$ and $l_j H_j(z_0) = 0$ for all $j = 1, 2, .., m$ and (2.28) and (2.29) become

$$\lambda f'(z_0) + \sum_{j=1}^{n} l_j H_j'(z_0) + DF(z_0)^* e^* + DG(z_0)^* w^* = 0$$

and

$$\lambda f''(z_0)d^2 + \sum_{j=1}^{n} l_j H_i''(z_0)d^2 + \langle e^*.D^2F(z_0)d^2\rangle + \langle w^*, D^2G(z_0)d^2\rangle \geq 0.$$

This implies that

$$\sup_{(\lambda, e^*, l, w^*) \in \Lambda[z_0]} \Big[\lambda f''(z_0)d^2 + \sum_{j=1}^{n} l_j H_i''(z_0)d^2 + \langle e^*.D^2F(z_0)d^2\rangle + \langle w^*, D^2G(z_0)d^2\rangle\Big] \geq 0.$$

If we take $d = 0$, then we obtain the first-order optimality conditions, that is, $\Lambda[z_0] \neq \emptyset$. Hence the assertion (a) is proved.
(b). We claim that $\lambda \neq 0$. In fact, if $\lambda = 0$, then

$$DH(z_0)^* l^T + DF(z_0)^* e^* + DG(z_0)^* w^* = 0.$$

Here $H := (H_1, H_2, ..., H_m)$. Let $\tilde{z}$ be a vector satisfying $(A4)$. Then we have

$$l^T DH(z_0)\tilde{z} + \langle e^*, DF(z_0)\tilde{z}\rangle + \langle w^*, DG(z_0)^*\tilde{z}\rangle = 0.$$

This implies that

$$l^T DH(z_0)\tilde{z} + \langle w^*, DG(z_0)\tilde{z}\rangle = 0,$$

where

$$DH(z_0)\tilde{z} \in (-\infty, 0)^m - H(z_0) = \mathrm{int}((-\infty, 0]^m - H(z_0))$$

and

$$DG(z_0)\tilde{z} \in \mathrm{int}K - G(z_0) = \mathrm{int}(K - G(z_0)).$$

Then for any $\xi \in \mathbb{R}^m$ and $w \in W$, there exist $\gamma > 0$ and $s > 0$ small enough such that

$$\gamma\xi + DH(z_0)\tilde{z} \in (-\infty, 0]^m - H(z_0)$$

and

$$sw + DG(z_0)\tilde{z} \in K - G(z_0).$$

Since $l \geq 0$, $l^T H(z_0) = 0$ and $w^* \in N(K, G(z_0))$, we have

$$\gamma l^T \xi + s\langle w^*, w\rangle = l^T(\tau\xi + DH(z_0)\tilde{z}) + \langle w^*, sw + DG(z_0)\tilde{z}\rangle \leq 0.$$

Hence $l^T\xi \leq 0$ for all $\xi \in \mathbb{R}^n$ and $\langle w^*, w\rangle \leq 0$ for all $w \in W$. This implies that $l = 0$ and $w^* = 0$. Consequently, $DF(z_0)^* e^* = 0$. Since $DF(z_0)$ is surjective, we get $e^* = 0$. Hence $(\lambda, e^*, l, w^*) = (0, 0, 0, 0)$ which is absurd. The claim is justified.

Let us define $\widehat{K} = \{0\} \times (-\infty, 0]^m \times K$ and $\widehat{G} = (F, H, G)$. Then the constraint of (MP) is equivalent to the constraint $\widehat{G}(z) \in \widehat{K}$. According to [32], Corollary 2.101, p. 70, $(A4)$ is equivalent to the Robinson constraint qualification:

$$0 \in \mathrm{int}\{\tilde{G}(z_0) + D\tilde{G}(z_0)Z - \tilde{K}\}.$$

By [32], Theorem 3.9, p. 151, $\Lambda_*[z_0]$ is nonempty convex bounded and compact in the topology $\sigma(E^*, E) \times \tau(\mathbb{R}^m) \times \sigma(W^*, W)$. Then by the same argument as the proof of Proposition 3.53 in [32], we can show that

$$\sup_{(e^*, l, w^*) \in \Lambda_*[z_0]} D_z^2 \mathcal{L}(z_0, 1, e^*, l, w^*)(d, d) \geq 0 \ \forall d \in \mathcal{C}[z_0].$$

Moreover, since $\Lambda_*[z_0]$ is nonempty convex bounded and compact in the topology $\sigma(E^*, E) \times \tau(\mathbb{R}^m) \times \sigma(W^*, W)$, we get

$$\max_{(e^*, l, w^*) \in \Lambda_*[z_0]} D_z^2 \mathcal{L}(z_0, 1, e^*, l, w^*)(d, d) \geq 0 \ \forall d \in \mathcal{C}[z_0].$$

The proof of the position is complete. □

Recently, optimality conditions for problem (MP) have been studied by some authors. Under the constraint qualification condition, Bonnans [32], Bonnans and Zidani [33], and Cominetti [34] derived first-and second-order optiality conditions of KKT-type. Meanwhile, Zowe and Kurcyusz [35] gave first-order optimality conditions when K is a cone. In [36] Ben-tal and Zowe gave first-and second-order optimality conditions for (MP) of Fritz-John type and KKT-type for the case where f is a vector function and K is a cone and $H_j = 0$.

Proposition 2.4 gives first-and second-order optimality conditions of Fritz-John type and KKT-type for (MP), where K may not be a cone. Assertion (b) of Proposition 2.4 is a shaper version of [32], Proposition 3.53 and [33], Theorem 2.7, where we do not require the extended polyhedricity condition and supremum is replaced by maximum. However, Proposition 2.4 is established under the key assumption $(A1)$ which is a price we have to pay.

It is noted that our technique for proving Proposition 2.4 is different from those in [36], [32], [33], [34] and [35]. We only use the separation theorem and Taylor expansions to establish the result.

Remark 2.5. When K is a cone and $H_j = 0$ for all $j = 1, 2, ..., m$, we get a similar result with [36], Theorem 8.2. However, in this case, the obtained result is failed to apply for problem (P). In fact, the constraint (1.4) corresponds to the constraints $H_j \leq 0$ while the constraint (1.5) corresponds to constraint $g(\cdot, y, u) \in K$ with $K := \{w \in L^\infty(Q_T) | a \leq w(x,t) \leq b\}$ which is not a cone.

3. Main results

Let $L = L(x,t,y,u) : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. Given a triple $(T_*, y_*, u_*) \in \Phi$, the symbols $L[x,t]$, $L_t[x,t], L_y[x,t]$, $L_u[x,t]$, $L[\cdot,\cdot]$, *etc.*, stand for $L(x,t,y_*(x,t),u_*(x,t))$, $L_t(x,t,y_*(x,t),u_*(x,t))$, $L_y(x,t,y_*(x,t),u_*(x,t))$, $L_u(x,t,y_*(x,t),u_*(x,t))$, $L(\cdot,\cdot,y_*(\cdot,\cdot),u_*(\cdot,\cdot))$, *etc.*, respectively. Let $L_i = L_i(x,T,\zeta) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The symbols $L_i[T_1,T_2], L_{iT}[T_1,T_2], L_{i\zeta}[T_1,T_2]$, *etc.*, stand for $L_i(x,T_1,y_*(T_2))$, $L_{iT}(x,T_1,y_*(T_2))$, $L_{i\zeta}(x,T_1,y_*(T_2))$, *etc.*, respectively. Also, the symbols $\psi_i[T_1,T_2]$, $\psi_{iT}[T_1,T_2]$, $\psi_{i\zeta}[T_1,T_2]$, etc stand for $\psi_i(T_1,y_*(T_2))$, $\psi_{iT}(T_1,y_*(T_2))$, $\psi_{i\zeta}(T_1,y_*(T_2))$, *etc.*, respectively. Put

$$K_{T_*} := \{v \in L^\infty(Q_{T_*}) : a \leq v(x,t) \leq b \text{ a.a. } (x,t) \in Q_{T_*}\}.$$

Let $\ell : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a mapping which stands for L and g. We impose the following hypotheses.

$(H4)$ (i) ℓ is a Carathéodory function and for each $x \in \Omega$, $\ell(x,\cdot,\cdot,\cdot)$ is of class C^2 and satisfies the following property: for each $M > 0$, there exists $k_{\ell,M} > 0$ such that

$$\begin{aligned}
&|\ell(x,t_1,y_1,u_1) - \ell(x,t_2,y_2,u_2)| + |\ell_t(x,t_1,y_1,u_1) - \ell_t(x,t_2,y_2,u_2)| \\
&+ |\ell_y(x,t_1,y_1,u_1) - \ell_y(x,t_2,y_2,u_2)| + |\ell_u(x,t_1,y_1,u_1) - \ell_u(x,t_2,y_2,u_2)| \\
&+ |\ell_{tt}(x,t_1,y_1,u_1) - \ell_{tt}(x,t_2,y_2,u_2)| + |\ell_{yy}(x,t_1,y_1,u_1) - \ell_{yy}(x,t_2,y_2,u_2)| \\
&+ |\ell_{uu}(x,t_1,y_1,u_1) - \ell_{uu}(x,t_2,y_2,u_2)| + |\ell_{yu}(x,t_1,y_1,u_1) - \ell_{yu}(x,t_2,y_2,u_2)| \\
&+ |\ell_{ty}(x,t_1,y_1,u_1) - \ell_{ty}(x,t_2,y_2,u_2)| + |\ell_{tu}(x,t_1,y_1,u_1) - \ell_{tu}(x,t_2,y_2,u_2)| \\
&\leq k_{\ell,M}(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|),
\end{aligned}$$

for a.a. $x \in \Omega$ and for all $t_i, y_i, u_i \in \mathbb{R}$ satisfying $|t_i|, |y_i|, |u_i| \leq M$ with $i = 1, 2$. Furthermore, we require that the functions $\ell(\cdot,0,0,0)$, $\ell_t(\cdot,0,0,0)$, $\ell_y(\cdot,0,0,0)$, $\ell_u(\cdot,0,0,0)$, $\ell_{tt}(\cdot,0,0,0)$, $\ell_{yy}(\cdot,0,0,0)$, $\ell_{uu}(\cdot,0,0,0)$, $\ell_{ty}(\cdot,0,0,0)$, $\ell_{tu}(\cdot,0,0,0)$, $\ell_{yu}(\cdot,0,0,0)$, belong to $L^\infty(\Omega)$.

(ii) For $i = 0, 1, 2, ..., m$, L_i is a Carathéodory function and for each $x \in \Omega$, $L_i(x,\cdot,\cdot)$ is of class C^2 and satisfies the following property: for each $M > 0$, there exists $k_{L_i,M} > 0$ such that

$$\begin{aligned}
&|L_i(x,t_1,\zeta_1) - L_i(x,t_2,\zeta_2)| + |L_{iT}(x,t_1,\zeta_1) - L_{iT}(x,t_2,\zeta_2)| \\
&+ |L_{i\zeta}(x,t_1,\zeta_1) - L_{i\zeta}(x,t_2,\zeta_2)| + |L_{iTT}(x,t_1,\zeta_1) - L_{iTT}(x,t_2,\zeta_2)| \\
&+ |L_{i\zeta\zeta}(x,t_1,\zeta_1) - L_{i\zeta\zeta}(x,t_2,\zeta_2)| + |L_{iT\zeta}(x,t_1,\zeta_1) - L_{iT\zeta}(x,t_2,\zeta_2)| \leq k_{L_i,M}(|t_1 - t_2| + |\zeta_1 - \zeta_2|),
\end{aligned}$$

for a.a. $x \in \Omega$ and for all $t_j, \zeta_j \in \mathbb{R}$ satisfying $|t_j|, |\zeta_j| \le M$ with $j = 1, 2$. We also require that the functions $L_{iT}(\cdot, 0, 0)$, $L_{i\zeta}(\cdot, 0, 0)$, $L_{iTT}(\cdot, 0, 0)$, $L_{iT\zeta}(\cdot, 0, 0)$, $L_{i\zeta\zeta}(\cdot, 0, 0)$ belong to $L^\infty(\Omega)$ for $i = 0, 1, 2, ..., m$.

$(H5)$ The function $\frac{1}{g_u(\cdot,\cdot,y,u)}$ belongs to $L^\infty(Q_T)$ for all $(T, y, u) \in \Phi$.

$(H6)$ There exists $(\widehat{T}, \widehat{y}, \widehat{u}) \in \mathbb{R} \times Y_{T_*} \times U_{T_*}$ such that the following conditions are fulfilled:

(i) $\dfrac{\partial \widehat{y}}{\partial t} + A\widehat{y} + \psi_t[x, t]\dfrac{\widehat{T}t}{T_*} + \psi_y[x, t]\widehat{y} - \widehat{u} + \dfrac{\widehat{T}}{T_*}(Ay_* + \psi[x, t] - u_*) = 0,\ \widehat{y}(0) = 0;$

(ii) $\psi_i[T_*, T_*] + \psi_{iT}[T_*, T_*]\widehat{T} + \psi_{i\zeta}[T_*, T_*]\widehat{y}(T_*) < 0,\ i = 1, 2, ..., m;$

(iii) $g[x, t] + g_t[x, t]\dfrac{\widehat{T}t}{T_*} + g_y[x, t]\widehat{y} + g_u[x, t]\widehat{u} \in \mathrm{int}(K_{T_*}).$

Note that hypothesis $(H4)$ makes sure that J and g are of class C^2 on $Y_T \times U_T$ for each fixed $T > 0$. Meanwhile, $(H5)$ guarantees that the Lagrange multipliers belong to $L^1(Q_{T_*})$ and $(H6)$ ensures that the Lagrange multipliers are normal. Note that since the control variable belongs to $L^\infty(Q_T)$, the Lagrange multipliers belong to the dual space $(L^\infty(Q_T))^*$, which are signed additive measures rather than functions in $L^1(Q_T)$. This causes difficulties in using optimality conditions for computing numerical solutions. Therefore, condition $(H5)$ is important. Let us give an example showing that $(H5)$ and $(H6)$ are fulfilled.

Example 3.1. Let $g(x, t, y, u) = -u^3 - u(y^2 + 1)$. It is easy to see that $\frac{1}{g_u(\cdot,\cdot,y,u)} = \frac{1}{-3u^2-(y^2+1)}$ belongs to $L^\infty(Q_T)$ for all $(T, y, u) \in \Phi$ and so $(H5)$ is valid. Let $b = 0$ and $\psi(x, t, y) = y^3 + y$. Then we have $\psi_y(x, t, y) = 3y^2 + 1 \ge 1 > 0$. Since $g[x, t] \le b = 0$, we have $0 \le u_* \in L^\infty(Q_{T_*})$. The maximum principle implies that the solution y_* of the state equation corresponding to u_* satisfies the property that $y_* \ge 0$ on $\overline{Q}_{T_*}$ and so $y_*(x, T_*) \ge 0$ a.a. $x \in \Omega$. Let $\delta > 0$ and take $\widehat{u} \in L^\infty(Q_{T_*})$ such that $\widehat{u}(x, t) \ge \delta > 0$ a.a. $(x, t) \in Q_{T_*}$. By Lemma 2.2, equation

$$\frac{\partial \widehat{y}}{\partial t} + A\widehat{y} + \psi_y[x, t]\widehat{y} - \widehat{u} = 0, \quad \widehat{y}(0) = 0$$

has unique solution $\widehat{y} \in Y_{T_*}$. Also, the maximum principle implies that $\widehat{y}(x, t) \ge 0$ on Q_{T_*}. Let ψ_i defined by

$$\psi_i(T, y(T)) = -\int_\Omega T^2[y^{2i+1}(x, T) + 1]\mathrm{d}x, \quad i = 1, 2, ..., m.$$

Take

$$\begin{aligned} a := -\Big[&\|u_*\|^3_{L^\infty(Q_{T_*})} + \|u_*\|_{L^\infty(Q_{T_*})}(\|y_*\|^2_{L^\infty(Q_{T_*})} + 1) + 2\|y_*\|_{L^\infty(Q_{T_*})}\|u_*\|_{L^\infty(Q_{T_*})}\|\widehat{y}\|_{L^\infty(Q_{T_*})} \\ &+ 3\|u_*\|^2_{L^\infty(Q_{T_*})}\|\widehat{u}\|_{L^\infty(Q_{T_*})} + \|\widehat{u}\|_{L^\infty(Q_{T_*})}(\|y_*\|^2_{L^\infty(Q_{T_*})} + 1)\Big] - \delta \end{aligned}$$

(more general, $a < 0$ and $|a|$ is large enough). With $\widehat{y}$ and $\widehat{u}$ as above and $\widehat{T} = 0$, it's easy to see that condition (i) of $(H6)$ is satisfied. Moreover, we have

$$\psi_i[T_*, T_*] + \psi_{i\zeta}[T_*, T_*]\widehat{y}(T_*) = -\int_\Omega T_*^2[y_*^{2i+1}(T_*) + 1]\mathrm{d}x - (2i+1)\int_\Omega T_*^2 y_*^{2i}(T_*)\widehat{y}(T_*)\mathrm{d}x \le -T_*^2|\Omega| < 0,$$

for $i = 1, 2, ..., m$. Hence (ii) is satisfied with $(0, \widehat{y}, \widehat{u})$. Since $y_*, u_*, \widehat{y} \ge 0$, $\widehat{u} \ge \delta$ on $\overline{Q}_{T_*}$ and definition of number a, we have

$$a < a + \delta \le g[x, t] + g_y[x, t]\widehat{y} + g_u[x, t]\widehat{u} \le -\delta < 0 = b \quad \text{a.a. on } Q_{T_*}.$$

Hence $g[x, t] + g_y[x, t]\widehat{y} + g_u[x, t]\widehat{u} \in \mathrm{int}(K_{T_*})$ and so (iii) is satisfied with $(0, \widehat{y}, \widehat{u})$. Thus $(H6)$ is valid.

Given a triple $(T, y, u) \in \Phi$, we define the extension of y on the right by setting

$$y_e(x,t) = \begin{cases} y(x,t) & \text{if} \quad (x,t) \in \overline{\Omega} \times [0,T] \\ y(x,T) & \text{if} \quad (x,t) \in \overline{\Omega} \times (T,+\infty). \end{cases}$$

Then y_e is continuous on $\overline{\Omega} \times [0,+\infty)$. Moreover, for each $T_0 \in (T,+\infty)$, y_e is uniform continuous on the compact set $\overline{\Omega} \times [0,T_0]$.

Definition 3.2. A triple $(T_*, y_*, u_*) \in \Phi$ is said to be a locally optimal solution of (P) if there exists some $\varepsilon > 0$ such that

$$\begin{aligned} &J(T_*, y_*, u_*) \le J(T,y,u) \quad \forall (T,y,u) \in \Phi \text{ satisfying} \\ &|T - T_*| + \max_{(x,t)\in\overline{\Omega}\times[0,T_*\vee T]} |y_e(x,t) - y_{*e}(x,t)| + \text{esssup}_{(x,t)\in\Omega\times[0,T_*]} |u(x, \frac{Tt}{T_*}) - u_*(x,t)| < \varepsilon. \end{aligned} \tag{3.1}$$

Hereafter, $T_* \vee T = \max(T_*, T)$.

It is noted that when $T = T_*$, Definition 3.2 becomes a definition for locally optimal solutions to the problem with fixed end-time T_*.

Let us denote by $\mathcal{C}_0[(T_*, y_*, u_*)]$ the set of vectors $(T, y, u) \in \mathbb{R} \times Y_{T_*} \times U_{T_*}$ which satisfy the following conditions:

(c_1) $\int_\Omega \big(L_{0T}[T_*, T_*]T + L_{0\zeta}[T_*, T_*]y(T_*)\big)\mathrm{d}x + \int_{Q_{T_*}} \Big(\frac{T}{T_*}L[x,t] + L_t[x,t]\frac{Tt}{T_*} + L_y[x,t]y + L_u[x,t]u\Big)\mathrm{d}x\mathrm{d}t \le 0;$

(c_2) $\frac{\partial y}{\partial t} + Ay + \psi_t[x,t]\frac{Tt}{T_*} + \psi_y[x,t]y - u + \frac{T}{T_*}(Ay_* + \psi[x,t] - u_*) = 0, \quad y(0) = 0;$

(c_3) $\int_\Omega \big(L_{iT}[T_*, T_*]T + L_{i\zeta}[T_*, T_*]y(T_*)\big)\mathrm{d}x \le 0$ for $i \in \{1, 2, .., m | \psi_i[T_*, T_*] = 0\};$

(c_4) $g_t[\cdot,\cdot]\frac{Tt}{T_*} + g_y[\cdot,\cdot]y + g_u[\cdot,\cdot]u \in \text{cone}\big(K_{T_*} - g[\cdot,\cdot]\big).$

Define $\mathcal{C}[(T_*, y_*, u_*)] = \overline{\mathcal{C}_0[(T_*, y_*, u_*)]}$, which is the closure of $\mathcal{C}_0[(T_*, y_*, u_*)]$ in $\mathbb{R} \times Y_{T_*} \times U_{T_*}$. The set $\mathcal{C}[(T_*, y_*, u_*)]$ is called a critical cone to the problem (P). Each vector $d \in \mathcal{C}[(T_*, y_*, u_*)]$ is called a critical direction to the (P) at (T_*, y_*, u_*). It is clear that $\mathcal{C}[(T_*, y_*, u_*)]$ is a closed convex cone containing 0.

We have the following main result on necessary optimality conditions.

Theorem 3.3. *Suppose that assumptions $(H1) - (H5)$ are satisfied and $(T_*, y_*, u_*) \in \Phi$ is a locally optimal solution to (P). Then there exist Lagrange multipliers $\lambda \in \mathbb{R}_+$, $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, $\widetilde{\varphi} \in L^\infty(Q_{T_*}) \cap L^2(0, T_*; H_0^1(\Omega))$, $\widetilde{e} \in L^\infty(Q_{T_*})$ and an absolutely continuous function $\widetilde{\phi} : [0, T_*] \to \mathbb{R}$ which are not all zeros and satisfy the following conditions:*
(i) (the adjoint equations)

$$\begin{cases} -\frac{\partial \widetilde{\varphi}}{\partial t} + A^*\widetilde{\varphi} + \psi_y[\cdot,\cdot]\widetilde{\varphi} = -\lambda L_y[\cdot,\cdot] - \widetilde{e} g_y[\cdot,\cdot] \quad \text{in } Q_{T_*}, \\ \widetilde{\varphi} = 0 \quad \text{on } \Sigma_{T_*}, \\ \widetilde{\varphi}(T_*) = -\lambda L_{0\zeta}[T_*, T_*] - \sum_{i=1}^m \mu_i L_{i\zeta}[T_*, T_*] \quad \text{in} \quad \Omega \end{cases} \tag{3.2}$$

and

$$\begin{cases} \widetilde{\phi}'(t) &= -\int_\Omega \big(\lambda L_t[x,t] + \widetilde{\varphi}\psi_t[x,t] + \widetilde{e} g_t[x,t]\big)\mathrm{d}x \quad \text{in } (0, T_*), \\ \widetilde{\phi}(T_*) &= 0 \end{cases} \tag{3.3}$$

where A^* is the adjoint operator of A, which is defined by $A^*\widetilde{\varphi} = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_j\widetilde{\varphi})$;
(ii) (optimality condition for u_*)

$$\lambda L_u[x,t] - \widetilde{\varphi}(x,t) + \widetilde{e}(x,t)g_u[x,t] = 0 \quad \text{a.a. } (x,t) \in Q_{T_*}; \tag{3.4}$$

(iii) (optimality condition for T_*)

$$\int_0^{T_*} \widetilde{\phi}(t)\mathrm{d}t + \int_{Q_{T_*}} \big(\lambda L[x,t] + \widetilde{\varphi}\big(Ay_* + \psi[x,t] - u_*\big)\mathrm{d}x\mathrm{d}t + T_*\int_\Omega \big(\lambda L_{0\zeta}[T_*,T_*] + \sum_{i=1}^m \mu_i L_{i\zeta}[T_*,T_*]\big)\mathrm{d}x = 0; \tag{3.5}$$

(iv) (the complementary conditions)

$$\mu_i \geq 0 \text{ and } \mu_i \int_\Omega L_i(x,T_*,y_*(T_*))\mathrm{d}x = 0,\ i = 1,2,...,m, \tag{3.6}$$

$$\widetilde{e}(x,t) \in N([a,b], g[x,t]) \ \text{a.a. } (x,t) \in Q_{T_*}. \tag{3.7}$$

Moreover, one has
(v) (the nonnegative second-order condition)

$$\begin{aligned}
&\sup_{(\lambda,\widetilde{\varphi},\mu,\widetilde{\phi})\in\Lambda[(T_*,y_*,u_*)]} D^2_{(T,y,u)}\mathcal{L}^P(T_*,y_*,u_*,\lambda,\widetilde{\varphi},\mu,\widetilde{\phi},\widetilde{e})[d,d] = \\
&\sup_{(\lambda,\widetilde{\varphi},\mu,\widetilde{\phi})\in\Lambda[(T_*,y_*,u_*)]} \Big[\lambda\int_\Omega \Big(L_{0TT}[T_*,T_*]T^2 + 2L_{0T\zeta}[T_*,T_*]Ty(T_*) + L_{0\zeta\zeta}[T_*,T_*]y(T_*)^2\Big)\mathrm{d}x \\
&+ \lambda\int_{Q_{T_*}} \Big(L_{tt}[x,t](\frac{Tt}{T_*})^2 + 2L_{ty}[x,s]\frac{Tt}{T_*}y(x,t) + 2L_{tu}[x,t]\frac{Tt}{T_*}u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ \lambda\int_{Q_{T_*}} \Big(L_{yy}[x,t]y(x,t)^2 + L_{uu}[x,t]u(x,t)^2 + 2L_{yu}[x,t]y(x,t)u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ 2\lambda\int_{Q_{T_*}} \frac{T}{T_*}\Big(L_t[x,t]\frac{Tt}{T_*} + L_y[x,t]y(x,t) + L_u[x,t]u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} \widetilde{\varphi}(x,t)\Big(\psi_{tt}[x,t](\frac{Tt}{T_*})^2 + 2\psi_{ty}[x,s]\frac{Tt}{T_*}y(x,t) + \psi_{yy}[x,t]y(x,t)^2\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} 2\frac{T}{T_*}\widetilde{\varphi}(x,t)\Big(Ay + \psi_t[x,t]\frac{Tt}{T_*} + \psi_y[x,t]y - u\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} \widetilde{e}(x,s)\big(g_{tt}[x,t](\frac{Tt}{T_*})^2 + g_{yy}[x,t]y^2 + g_{uu}[x,t]u^2 + 2g_{ty}[x,t]\frac{Tt}{T_*}y + 2g_{tu}[x,t]\frac{Tt}{T_*}u + 2g_{yu}[x,t]yu\big)\mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^m \mu_i\int_\Omega \big(L_{iTT}[T_*,T_*]T^2 + 2L_{iT\zeta}[T_*,T_*]Ty(T_*) + L_{i\zeta\zeta}[T_*,T_*]y(T_*)^2\big)\Big] \geq 0
\end{aligned} \tag{3.8}$$

for all $d = (T,y,u) \in \mathcal{C}_0[(T_*,y_*,u_*)]$. Here $\Lambda[(T_*,y_*,u_*)]$ denotes the set of Lagrange multipliers $(\lambda,\widetilde{\varphi},\mu,\widetilde{\phi})$ satisfying conditions $(i)-(iv)$. In addition, if $(H6)$ is satisfied then multipliers $(\lambda,\widetilde{\varphi},\mu,\widetilde{\phi}) \in \Lambda[(T_*,y_*,u_*)]$ are normal, that is, $\lambda = 1$ and one has

$$\max_{(1,\widetilde{\varphi},\mu,\widetilde{\phi})\in\Lambda[(T_*,y_*,u_*)]} D^2_{(T,y,u)}\mathcal{L}^P(T_*,y_*,u_*,1,\widetilde{\varphi},\mu,\widetilde{\phi},\widetilde{e})[d,d] \geq 0 \quad \forall d \in \mathcal{C}[(T_*,y_*,u_*)]. \tag{3.9}$$

To deal with second-order sufficient conditions, we need to enlarge the critical cone $\mathcal{C}[(T_*, y_*, u_*)]$. For this we define $\mathcal{C}'[(T_*, y_*, u_*)]$ being a set of vectors $(T, y, u) \in \mathbb{R} \times W_2^{1,1}(0, T_*, H, D) \times L^2(Q_{T_*})$ satisfying conditions (c_1), (c_2), (c_3) and

(c_4') $g_t[x,t]\dfrac{Tt}{T_*} + g_y[x,t]y(x,t) + g_u[x,t]u(x,t) \in T\big([a,b], g[x,t]\big)$ for a.a. $(x,t) \in Q_{T_*}$.

The following theorem gives second-order sufficient conditions for locally optimal solution to (P).

Theorem 3.4. *Suppose assumptions $(H1)-(H5)$, $T_* > 0$ and $(T_*, y_*, u_*) \in \Phi$. Assume that there exist Lagrange multipliers $\lambda = 1$, $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, $\widetilde{\varphi} \in L^\infty(Q_{T_*}) \cap L^2(0, T_*; H_0^1(\Omega))$, $\widetilde{e} \in L^\infty(Q_{T_*})$ and an absolutely continuous function $\widetilde{\phi} : [0, T_*] \to \mathbb{R}$ such that conditions (i)–(iv) of Theorem 3.3 are fulfilled. Furthermore, assume that the following conditions are fulfilled:*
(v) (the strictly second-order condition)

$$D^2_{(T,y,u)}\mathcal{L}^P(T_*, y_*, u_*, 1, \widetilde{\varphi}, \mu, \widetilde{\phi}, \widetilde{e})[(\widehat{T}, \widehat{y}, \widehat{u}), (\widehat{T}, \widehat{y}, \widehat{u})] > 0 \quad \forall (\widehat{T}, \widehat{y}, \widehat{u}) \in \mathcal{C}'[(T_*, y_*, u_*)] \setminus \{(0,0,0)\}; \tag{3.10}$$

(vi) (the Legendre–Clebsch condition) there is a number $\Lambda_0 > 0$ such that

$$L_{uu}[x,t] + \widetilde{e}(x,t)g_{uu}[x,t] \geq \Lambda_0 \quad \text{a.a. } (x,t) \in Q_{T_*}. \tag{3.11}$$

Then there exist numbers $\varepsilon_0 > 0$ and $\kappa_0 > 0$ such that

$$J(T, y, u) \geq J(T_*, y_*, u_*) + \kappa_0\Big((T - T_*)^2 + \frac{1}{T_*}\int_{Q_{T_*}} |u(x, \frac{Tt}{T_*}) - u_*(x,t)|^2 \mathrm{d}x\mathrm{d}t\Big) \tag{3.12}$$

for all $(T, y, u) \in \Phi \cap N\big((T_*, y_*, u_*), \varepsilon_0\big)$. In particular, (T_*, y_*, u_*) is a locally optimal solution to (P). Here

$$\Phi \cap N\big((T_*, y_*, u_*), \varepsilon_0\big) := \Big\{(T, y, u) \in \Phi : \mathrm{dist}\big[(T, y, u), (T_*, y_*, u_*)\big] < \epsilon_0\Big\},$$

with

$$\mathrm{dist}[(T, y, u), (T_*, y_*, u_*)] := |T - T_*| + \max_{(x,t)\in\overline{\Omega}\times[0, T_*\vee T]} |y_e(x,t) - y_{*e}(x,t)| + \operatorname*{esssup}_{(x,t)\in\Omega\times[0,T_*]} |u(x, \frac{Tt}{T_*}) - u_*(x,t)|.$$

Remark 3.5. 1) By using similar arguments as in the proof of [37], Proposition 2.6, we can show that if strictly second-order condition (3.10) and the Legendre-Clebsch condition (3.11) are satisfied, then the following strongly second-order condition holds:

$$\exists \gamma_0 > 0 : \ D^2_{(T,y,u)}\mathcal{L}^P(T_*, y_*, u_*, 1, \widetilde{\varphi}, \mu, \widetilde{\phi}, \widetilde{e})[(\widehat{T}, \widehat{y}, \widehat{u}), (\widehat{T}, \widehat{y}, \widehat{u})] \geq \gamma_0\big(\widehat{T}^2 + \|\widehat{u}\|_2^2\big) \ \forall (\widehat{T}, \widehat{y}, \widehat{u}) \in \mathcal{C}'[(T_*, y_*, u_*)].$$

Conversely, by a modification of the proof of [38], Lemma 5.1, we can prove that if the above strongly second-order condition is satisfied and $L_{i\zeta\zeta}[\cdot,\cdot] = 0$, then the strictly second-order condition and the Legendre-Clebsch condition are fulfilled.
2) When $T = T_*$, (P) becomes a problem with fixed end-time. In this case the conclusion of Theorem 3.3 is similar to the first conclusion of [16], Theorem 5.2 except that the critical cone in Theorem 3.3 is somewhat different from those in [16], Theorem 5.2. Meanwhile, Theorem 3.4 implies [16], Theorem 5.8.

4. Problem with fixed end-time

Recall that $Q_1 = \Omega \times (0,1)$. We put

$$U_1 = L^\infty(Q_1), \quad Y_1 = \Big\{\zeta \in W_2^{1,1}(0,1;D,H) \cap W_p^{2,1}(Q_1) |\ A\zeta \in L^p(Q_1)\Big\}.$$

Then Y_1 is a Banach space under the graph norm

$$\|\zeta\|_{Y_1} := \|\zeta\|_{W_2^{1,1}(0,1;D,H)} + \|\zeta\|_{W_p^{2,1}(Q_1)} + \big\|A\zeta\big\|_{L^p(Q_1)}.$$

To establish optimality conditions for time-optimal problem (P), we need to change variables to transform (P) into a problem with a fixed terminal time. Taking any $T > 0$, we define a function $\xi : [0,1] \to [0,T]$ by setting $\xi(s) = Ts$.

Let

$$\zeta(x,s) := y(x,\xi(s)), \quad v(x,s) := u(x,\xi(s)).$$

By changing variable $t = \xi(s) = Ts$ in (P), we obtain the following optimal control problem with fixed final time:

$$(P_1) \qquad \begin{cases} \widehat{J}(\xi,T,\zeta,v) := \psi_0(T,\zeta(1)) + \displaystyle\int_0^1 \int_\Omega TL(x,\xi(s),\zeta(x,s),v(x,s))\mathrm{d}x\mathrm{d}s \to \inf \\ \text{s.t.} \\ \dfrac{\partial \zeta}{\partial s} + TA\zeta + T\psi(x,\xi,\zeta) = Tv \quad \text{in } Q_1, \quad \zeta(x,s) = 0 \quad \text{on } \Sigma_1 = \Gamma \times [0,1], \\ \zeta(0) = y_0 \quad \text{in } \Omega, \\ \xi = \displaystyle\int_0^{(\cdot)} T d\tau, \\ \psi_i(T,\zeta(1)) \le 0, \quad i = 1,2,...,m, \\ a \le g(x,\xi(s),\zeta(x,s),v(x,s)) \le b \quad \text{a.a. } x \in \Omega,\ \forall s \in [0,1]. \end{cases}$$

We shall denote by Φ_1 the feasible set of (P_1) and put

$$Z = C([0,1],\mathbb{R}) \times \mathbb{R} \times Y_1 \times U_1.$$

Given a vector $z = (\xi,T,\zeta,v) \in Z$, we define

$$\|z\|_* := \|\xi\|_{C([0,1],\mathbb{R})} + |T| + \|\zeta\|_{C(\overline{Q}_1)} + \|v\|_{U_1}, \quad \|z\|_Z := \|\xi\|_{C([0,1],\mathbb{R})} + |T| + \|\zeta\|_{Y_1} + \|v\|_{U_1}.$$

Given $z_0 \in Z$ and $r > 0$, we denote by $B_{*Z}(z_0,r)$ and $B_Z(z_0,r)$ the balls center at z_0 and radius r in norm $\|\cdot\|_*$ and $\|\cdot\|_Z$, respectively.

Definition 4.1. Vector $z_* = (\xi_*,T_*,\zeta_*,v_*) \in \Phi_1$ is a locally optimal solution to (P_1) if there exists $\delta > 0$ such that

$$\widehat{J}(z_*) \le \widehat{J}(z) \quad \forall z \in \Phi_1 \text{ satisfying } \|z - z_*\|_* \le \delta. \tag{4.1}$$

Since $Y_1 \hookrightarrow C(\overline{Q}_1)$, there exists $\gamma > 0$ such that $\|z\|_* \le \gamma\|z\|_Z$ for all $z \in Z$. Therefore, if $\|z - z_*\|_Z \le \frac{\delta}{\gamma}$, then $\|z - z_*\|_* \le \delta$. Consequently, if z_* is a locally optimal solution to (P_1) in norm $\|\cdot\|_*$, then it is also locally optimal solution to (P_1) in norm $\|\cdot\|_Z$.

The following propositions give relations between optimal solutions of (P) and (P_1).

Proposition 4.2. *Suppose that* $(T_*, y_*, u_*) \in \Phi$ *is a locally optimal solution to* (P). *Let* $\xi_*(s) = T_* s$ *for* $s \in [0,1]$ *and*

$$\zeta_*(x,s) := y_*(x, \xi_*(s)),\ \ v_*(x,s) := u_*(x, \xi_*(s)).$$

Then the vector $(\xi_*, T_*, \zeta_*, v_*)$ *is a locally optimal solution to* (P_1) *and* $\widehat{J}(\xi_*, T_*, \zeta_*, v_*) = J(T_*, y_*, u_*)$.

Proof. By definition, $T_* > 0$ and there exists $\epsilon > 0$ such that (3.1) is valid. Fix a number $T_0 > T_*$. By the uniform continuity of y_{*e} on $\overline{\Omega} \times [0, T_0]$, there exists $\delta \in (0, \frac{\varepsilon}{4})$ such that

$$|y_{*e}(x,t_1) - y_{*e}(x,t_2)| \le \frac{\epsilon}{4} \quad \forall t_1, t_2 \in [0, T_0] \ \text{ satisfying } \ |t_1 - t_2| < \delta. \tag{4.2}$$

We can choose $\delta > 0$ small enough such that $T > 0$ whenever $|T - T_*| < \delta$. Clearly, $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$. Let $(\xi, T, \zeta, v) \in \Phi_1$ such that

$$\|\xi - \xi_*\|_{C([0,1],\mathbb{R})} + |T - T_*| + \|\zeta - \zeta_*\|_{C(\overline{Q}_1)} + \|v - v_*\|_{L^\infty(\overline{Q}_1)} < \min(\delta, T_0 - T_*). \tag{4.3}$$

We define $y(x,t) = \zeta(x, \xi^{-1}(t)),\ u(x,t) = v(x, \xi^{-1}(t))$, where $\xi^{-1}(t) = \frac{t}{T}$. It is easy to check that $(T, y, u) \in \Phi$. By (4.3) we have

$$|T - T_*| + \|\xi - \xi_*\|_{C([0,1],\mathbb{R})} \le \delta \le \frac{\varepsilon}{4}.$$

Without loss of generality, we may assume that $T > T_*$. Then from condition $T - T_* < T_0 - T_*$, we have $T < T_0$ and $[0, T \vee T_*] = [0, T]$. Therefore, for $(x,t) \in \Omega \times [0, T]$, we have

$$\begin{aligned}
|y_e(x,t) - y_{*e}(x,t)| &= \Big| y(x,t) - y_{*e}(x,t) \Big| \\
&= \Big| \zeta\big(x, \xi^{-1}(t)\big) - y_{*e}(x, \xi(\xi^{-1}(t))\Big| \\
&\le \Big| \zeta\big(x, \xi^{-1}(t)\big) - y_*\big(x, \xi_*\big(\xi^{-1}(t)\big)\big)\Big| + \Big| y_*\big(x, \xi_*\big(\xi^{-1}(t)\big)\big) - y_{*e}\big(x, \xi\big(\xi^{-1}(t)\big)\big)\Big| \\
&= \Big| \zeta\Big(x, \xi^{-1}(t)\Big) - \zeta_*\Big(x, \xi^{-1}(t)\Big)\Big| + \Big| y_{*e}\big(x, \xi_*\big(\xi^{-1}(t)\big)\big) - y_{*e}\big(x, \xi\big(\xi^{-1}(t)\big)\big)\Big|.
\end{aligned} \tag{4.4}$$

Note that $\xi^{-1}(t) \in [0,1]$ for all $t \in [0, T]$. By (4.3) we deduce that

$$\Big| \zeta\Big(x, \xi^{-1}(t)\Big) - \zeta_*\Big(x, \xi^{-1}(t)\Big)\Big| \le \|\zeta - \zeta_*\|_{C(\overline{Q}_1)} \le \delta \le \frac{\varepsilon}{4} \tag{4.5}$$

for all $(x,t) \in \overline{\Omega} \times [0, T]$. Moreover, for all $t \in [0, T]$ we have $t_1 = \xi_*\big(\xi^{-1}(t)\big) \in [0, T_*] \subset [0, T]$, $t_2 = \xi\big(\xi^{-1}(t)\big) = t \in [0, T]$ and

$$|t_1 - t_2| \le \|\xi - \xi_*\|_{C([0,1],\mathbb{R})} \le \delta.$$

Therefore, we have from (4.2) that

$$\Big| y_{*e}\Big(x, \xi_*\big(\xi^{-1}(t)\big)\Big) - y_{*e}\Big(x, \xi\big(\xi^{-1}(t)\big)\Big)\Big| \le \frac{\varepsilon}{4} \quad \forall (x,t) \in \overline{\Omega} \times [0, T].$$

Combining this with (4.4) and (4.5), we get

$$|y_e(x,t) - y_{*e}(x,t)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \quad \forall (x,t) \in \overline{\Omega} \times [0,T].$$

This implies that $\max\limits_{(x,t)\in\overline{\Omega}\times[0,T\vee T_*]} |y_e(x,t) - y_{*e}(x,t)| \le \frac{\varepsilon}{2}$. Hence

$$|T - T_*| + \max_{(x,t)\in\overline{\Omega}\times[0,T\vee T_*]} |y_e(x,t) - y_{*e}(x,t)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \frac{3\varepsilon}{4}. \tag{4.6}$$

On the other hand

$$\text{esssup}_{t\in[0,T_*]}|u(\frac{Tt}{T_*}) - u_*(t)| = \text{essup}_{t\in[0,T_*]}|v(x,\frac{t}{T_*}) - v_*(\frac{t}{T_*})| \le \|v - v_*\|_{L^\infty(Q_1)} \le \delta < \frac{\epsilon}{4}.$$

Combining this with (4.6), yields

$$|T - T_*| + \max_{(x,t)\in\overline{\Omega}\times[0,T\vee T_*]} |y_e(x,t) - y_{*e}(x,t)| + \text{esssup}_{t\in[0,T_*]}|u(\frac{Tt}{T_*}) - u_*(t)| < \epsilon.$$

From this and (3.1), we have $J(T_*, y_*, u_*) \le J(T, y, u)$. Hence

$$\widehat{J}(\xi_*, T_*, \zeta_*, v_*) = J(T_*, y_*, u_*) \le J(T, y, u) = \widehat{J}(\xi, T, \zeta, v).$$

The proof of the proposition is complete. □

Proposition 4.3. *Suppose that $T_* > 0$ and $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$ is a locally optimal solution to (P_1). Let*

$$y_*(x,t) := \zeta_*(x, \xi_*^{-1}(t)), \;\; u_*(x,t) := v_*(x, \xi_*^{-1}(t)).$$

Then the vector (T_*, y_*, u_*) is a locally optimal solution to (P) and $J(T_*, y_*, u_*) = \widehat{J}(\xi_*, T_*, \zeta_*, v_*)$.

Proof. Let $\delta > 0$ such that (4.1) is valid. Fix a number $T_0 > T_*$. Since y_{*e} is uniform continuous on $\overline{\Omega} \times [0, T_0]$, there exists $\epsilon \in (0, \delta)$ such that $|y_{*e}(x,t) - y_{*e}(x,t')| < \frac{\delta}{3}$ for all $t, t' \in [0, T_0]$ satisfying $|t - t'| < \epsilon$. We can choose $\epsilon > 0$ small enough so that $T > 0$ whenever $|T - T_*| < \epsilon$. We now take $(T, y, u) \in \Phi$ satisfying

$$|T - T_*| + \max_{(x,t)\in\overline{\Omega}\times[0,T\vee T_*]} |y_e(x,t) - y_{*e}(x,t)| + \text{esssup}_{(x,t)\in\Omega\times[0,T_*]}|u(x,\frac{Tt}{T_*}) - u_*(x,t)| < \min(\frac{\epsilon}{6}, T_0 - T_*).$$

Without loss of generality, we can assume that $T > T_*$. Then we have $T < T_0$. Let $\xi(s) = Ts$ and $\zeta(x,s) = y(x, \xi(s)), v(x,s) = u(x, \xi(s))$. Then $(\xi, T, \zeta, v) \in \Phi_1$. Note that

$$\|\xi - \xi_*\|_{C([0,1],\mathbb{R})} = \max_{s\in[0,1]} |Ts - T_*s| = |T - T_*| < \frac{\epsilon}{6}. \tag{4.7}$$

From this and uniform continuity of y_{*e}, we have for all $(x,s) \in \overline{\Omega} \times [0,1]$ that

$$\begin{aligned}
|\zeta(x,s) - \zeta_*(x,s)| &= |y(x,\xi(s)) - y_*(x,\xi_*(s))| \\
&\le |y(x,\xi(s)) - y_{*e}(x,\xi(s))| + |y_{*e}(x,\xi(s)) - y_{*e}(x,\xi_*(s))| \\
&\le \frac{\epsilon}{6} + \frac{\delta}{3} \le \frac{\delta}{2}.
\end{aligned}$$

This implies that

$$\|\zeta - \zeta_*\|_{C(\overline{Q}_1)} \leq \frac{\delta}{2}. \tag{4.8}$$

Also, we have

$$\begin{aligned}\text{esssup}_{(x,s)\in\Omega\times[0,1]}|v(x,s) - v_*(x,s)| &= \text{esssup}_{(x,s)\in\Omega\times[0,1]}|u(x,Ts) - u_*(x,T_*s)| \\ &= \text{esssup}_{(x,t)\in\Omega\times[0,T_*]}|u(x,\frac{Tt}{T_*}) - u_*(x,t))| \leq \frac{\epsilon}{6} < \frac{\delta}{6}.\end{aligned}$$

Hence $\|v - v_*\|_{L^\infty(Q_1)} \leq \frac{\delta}{6}$. Combining this with (4.7) and (4.8), we obtain

$$\|\xi - \xi_*\|_{C([0,1],\mathbb{R})} + \|\zeta - \zeta_*\|_{C(\overline{Q}_1)} + \|v - v_*\|_{L^\infty(Q_1)} + |T - T_*| < \frac{\delta}{6} + \frac{\delta}{2} + \frac{\delta}{6} + \frac{\delta}{6} = \delta.$$

Since $(\xi_*, T_*, \zeta_*, v_*)$ is a locally optimal solution to (P_1), we have

$$J(T,y,u) = \widehat{J}(\xi,T,\zeta,v) \geq \widehat{J}(\xi_*,T_*,\zeta_*,v_*) = J(T_*,y_*,u_*).$$

The proof of the proposition is complete. □

To reduce (P_1) to a mathematical programming problem, we put

$$E = L^p(Q_1) \times \Big(H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega)\Big) \times C([0,1],\mathbb{R}), \quad W = L^\infty(Q_1).$$

Define mappings $F : Z \to E$, $H : Z \to \mathbb{R}^m$ and $G : Z \to W$ by setting

$$\begin{aligned}F(\xi,T,\zeta,v) &= \Big(F_1(\xi,T,\zeta,v),\ F_2(\xi,T,\zeta,v),\ F_3(\xi,T,\zeta,v)\Big) \\ &= \Big(\frac{\partial\zeta}{\partial s} + TA\zeta + T\psi(\cdot,\xi,\zeta) - Tv,\quad \zeta(0) - y_0,\quad \xi - \int_0^{(\cdot)} Td\tau\Big),\end{aligned}$$

$$\begin{aligned}H(\xi,T,\zeta,v) &= \Big(H_1(\xi,T,\zeta,v), ..., H_m(\xi,T,\zeta,v)\Big) \\ &= \Big(\psi_1(T,\zeta(1)), ..., \psi_m(T,\zeta(1))\Big) = \Big(\int_\Omega L_1(x,T,\zeta(x,1))\mathrm{d}x, ..., \int_\Omega L_m(x,T,\zeta(x,1))\mathrm{d}x\Big),\end{aligned}$$

and

$$G(\xi,T,\zeta,v) = g(\cdot,\xi,\zeta,v).$$

By definition of space Y_1, if $(\xi,T,\zeta,v) \in Z$ then $\frac{\partial\zeta}{\partial s}, A\zeta \in L^p(Q_1)$, $\psi(\cdot,\xi,\zeta) \in L^\infty(Q_1) \subset L^p(Q_1)$ (since $\xi \in C([0,1],\mathbb{R})$, $\zeta \in C(\overline{Q}_1)$) and $v \in L^\infty(Q_1) \subset L^p(Q_1)$. Hence

$$\frac{\partial\zeta}{\partial s} + TA\zeta + T\psi(\cdot,\xi,\zeta) - Tv \in L^p(Q_1)$$

and so F_1 is well defined. F_2 is also well defined. Indeed, from $(H3)$ we have $y_0 \in H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega)$. Since $\zeta \in Y_1$, $\zeta \in W_2^{1,1}(0,1;D,H) \cap W_p^{2,1}(Q_1)$. Since $W_2^{1,1}(0,1;D,H) \hookrightarrow C([0,1],V)$, $\zeta(0) \in H_0^1(\Omega)$. Since $\zeta \in W_p^{2,1}(Q_1)$, [26], Lemma 3.4, p. 82 implies that $\zeta(0) \in W^{2-\frac{2}{p},p}(\Omega)$. So $\zeta(0) \in H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega)$. Consequently, $\zeta(0) - y_0 \in H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega)$. Therefore, F is well defined. Similarly, H and G are well defined. We now rewrite problem (P_1) in the form of (MP):

$$
\text{(MP1)} \qquad \begin{cases} \widehat{J}(\xi, T, \zeta, v) \to \inf \\ \text{s.t.} \\ F(\xi, T, \zeta, v) = 0, \\ H(\xi, T, \zeta, v) \le 0, \\ G(\xi, T, \zeta, v) \in K_1, \end{cases}
$$

where

$$
K_1 := \{u \in L^\infty(Q_1) | a \le u(x,s) \le b \text{ a.a. } (x,s) \in Q_1\}.
$$

Next, we will apply Proposition 2.4 for the problem (MP1) to derive necessary optimality conditions.

As before, given a vector $z_* = (\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$, the symbols $L[x,s]$, $L_t[x,s]$, $L_y[x,s]$, *etc.*, stand for $L(x, \xi_*(s), \zeta_*(x,s), v_*(x,s))$, $L_t(x, \xi_*(s), \zeta_*(x,s), v_*(x,s))$, $L_y(x, \xi_*(s), \zeta_*(x,s), v_*(x,s))$, *etc.*, respectively.

The following lemmas shows that (P_1) satisfies all conditions of Proposition 2.4.

Lemma 4.4. *Suppose that assumptions $(H2)$ and $(H4)$ are satisfied. Then the mappings $\widehat{J}, F, H$ and G are of class C^2 around z_*.*

Proof. Let $B_Z(z_*, \epsilon)$ be a neighborhood of z_* in Z. By assumptions $(H2)$, $(H4)$ and some standard arguments, we can show that $\widehat{J}, F, H$ and G are of class C^2 in $B_Z(z_*, \epsilon)$ (see a detail proof in the appendix). Namely, given a vector $\widehat{z} = (\widehat{\xi}, \widehat{T}, \widehat{\zeta}, \widehat{v}) \in B_Z(z_*, \epsilon)$, we have

$$
\begin{aligned}
D\widehat{J}(\widehat{z})[(\xi, T, \zeta, v)] &= \int_\Omega \big(L_{0T}(x, \widehat{T}, \widehat{\zeta}(1))T + L_{0\zeta}(x, \widehat{T}, \widehat{\zeta}(1))\zeta(1)\big)\mathrm{d}x \\
&\quad + \int_{Q_1} \Big(TL(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) + \widehat{T}[L_t(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})\xi + L_y(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})\zeta + L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})v]\Big)\mathrm{d}x\mathrm{d}s, \qquad (4.9)
\end{aligned}
$$

$$
DF(\widehat{z}) = \Big(DF_1(\widehat{z}),\ DF_2(\widehat{z}),\ DF_3(\widehat{z})\Big), \tag{4.10}
$$

$$
DF_1(\widehat{z})[(\xi, T, \zeta, v)] = \frac{\partial \zeta}{\partial s} + \widehat{T}A\zeta + \widehat{T}\psi_y(\cdot, \widehat{\xi}, \widehat{\zeta})\zeta + \widehat{T}\psi_t(\cdot, \widehat{\xi}, \widehat{\zeta})\xi - \widehat{T}v + T(A\widehat{\zeta} + \psi(\cdot, \widehat{\xi}, \widehat{\zeta}) - \widehat{v}), \tag{4.11}
$$

$$
DF_2(\widehat{z})[(\xi, T, \zeta, v)] = \zeta(0), \tag{4.12}
$$

$$
DF_3(\widehat{z})[(\xi, T, \zeta, v)] = \xi - \int_0^{(\cdot)} T d\tau, \tag{4.13}
$$

$$
DH(\widehat{z}) = \Big(DH_1(\widehat{z}),\ DH_2(\widehat{z}), ..., DH_m(\widehat{z})\Big), \tag{4.14}
$$

$$
DH_i(\widehat{z})[(\xi, T, \zeta, v)] = \int_\Omega \big(L_{iT}(x, \widehat{T}, \widehat{\zeta}(1))T + L_{i\zeta}(x, \widehat{T}, \widehat{\zeta}(1))\zeta(1)\big)\mathrm{d}x,\ i = 1, 2, ..., m, \tag{4.15}
$$

$$
DG(\widehat{z})[(\xi, T, \zeta, v)] = \langle g_t(\cdot, \widehat{\xi}, \widehat{\zeta}, \widehat{v}), \xi\rangle + \langle g_y(\cdot, \widehat{\xi}, \widehat{\zeta}, \widehat{v}), \zeta\rangle + \langle g_u(\cdot, \widehat{\xi}, \widehat{\zeta}, \widehat{v}), v\rangle \tag{4.16}
$$

and

$$
\begin{aligned}
&D^2\widehat{J}(\widehat{z})(\xi,T,\zeta,v)^2 = \int_\Omega \Big(L_{0TT}(x,\widehat{T},\widehat{\zeta}(1))T^2 + L_{0\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1))\zeta^2(1) + 2L_{0T\zeta}(x,\widehat{T},\widehat{\zeta}(1))T\zeta(1)\Big)\mathrm{d}x \\
&+ \int_{Q_1} \widehat{T}\Big[L_{tt}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\xi^2 + L_{yy}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\zeta^2 + L_{uu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})v^2\Big]\mathrm{d}x\mathrm{d}s \\
&+ \int_{Q_1} 2\widehat{T}\Big[L_{ty}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\xi\zeta + L_{tu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\xi v + L_{yu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\zeta v\Big]\mathrm{d}x\mathrm{d}s, \\
&+ \int_{Q_1} 2T\big[L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\xi + L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\zeta + L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})v\big]\mathrm{d}x\mathrm{d}s, \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
&D^2F(\widehat{z})(\xi,T,\zeta,v)^2 \\
&= \Big(\widehat{T}\psi_{tt}(\cdot,\widehat{\xi},\widehat{\zeta})\xi^2 + \widehat{T}\psi_{yy}(\cdot,\widehat{\xi},\widehat{\zeta})\zeta^2 + 2\widehat{T}\psi_{ty}(\cdot,\widehat{\xi},\widehat{\zeta})\xi\zeta + 2T\big(A\zeta + \psi_t(\cdot,\widehat{\xi},\widehat{\zeta})\xi + \psi_y(\cdot,\widehat{\xi},\widehat{\zeta})\zeta - v\big),\ 0,\ 0\Big), \qquad (4.18)
\end{aligned}
$$

$$
D^2H(\widehat{z})(\xi,T,\zeta,v)^2 = \big(D^2H_1(\widehat{z}),\ D^2H_2(\widehat{z}),...,D^2H_m(\widehat{z})\big)(\xi,T,\zeta,v)^2, \qquad (4.19)
$$

$$
D^2H_i(\widehat{z})(\xi,T,\zeta,v)^2 = \int_\Omega \big(L_{iTT}(x,\widehat{T},\widehat{\zeta}(1))T^2 + 2L_{iT\zeta}(x,\widehat{T},\widehat{\zeta}(1))T\zeta(1) + L_{i\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1))\zeta^2(1)\big)\mathrm{d}x, \qquad (4.20)
$$

$$
\begin{aligned}
D^2G(\widehat{z})(\xi,T,\zeta,v)^2 &= g_{tt}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[\xi,\xi] + g_{yy}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[\zeta,\zeta] + g_{uu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[v,v] \\
&\quad + 2g_{ty}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[\xi,\zeta] + 2g_{tu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[\xi,v] + 2g_{yu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v})[\zeta,v] \qquad (4.21)
\end{aligned}
$$

for all $(\xi,T,\zeta,v) \in Z$. The lemma is proved. □

Lemma 4.5. *The operator $DF(\xi_*,T_*,\zeta_*,v_*) : Z \to E$ is onto. In particular, $DF(\xi_*,T_*\zeta_*,v_*)Z$ is closed.*

Proof. Taking any $(\phi_0,\zeta_0,\xi_0) \in E$, we consider the following equation on Z

$$DF(\xi_*,T_*,\zeta_*,v_*)[(\xi,T,\zeta,v)] = (\phi_0,\zeta_0,\xi_0).$$

By (4.10)–(4.13), it is equivalent to the system of equations:

$$
\begin{aligned}
&\frac{\partial\zeta}{\partial s} + T_*A\zeta + T_*\psi_y(\cdot,\xi_*,\zeta_*)\zeta + T_*\psi_t(\cdot,\xi_*,\zeta_*)\xi - T_*v + T(A\zeta_* + \psi(\cdot,\xi_*,\zeta_*) - v_*) = \phi_0, \\
&\zeta(0) = \zeta_0, \\
&\xi(s) - Ts = \xi_0(s).
\end{aligned}
$$

Taking $T = T_*$ and $v = 0$, we get $\xi = \xi_0 + T_*s$. Consider parabolic equation

$$
\frac{\partial\zeta}{\partial s} + T_*A\zeta + T_*\psi_y(\cdot,\xi_*,\zeta_*)\zeta = \phi_0 - T_*\psi_t(\cdot,\xi_*,\zeta_*)\xi - T_*(A\zeta_* + \psi(\cdot,\xi_*,\zeta_*) - v_*), \qquad (4.22)
$$

$$
\zeta(0) = \zeta_0. \qquad (4.23)
$$

By $(H2)$, $T_*\psi_t(\cdot,\xi_*,\zeta_*)\xi \in L^\infty(Q_1)$. Also we have $T_*(A\zeta_* + \psi(\cdot,\xi_*,\zeta_*) - v_*) = -\frac{\mathrm{d}}{\mathrm{d}s}\zeta_* \in L^p(Q_1)$. Hence $\phi_0 - T_*\psi_t(\cdot,\xi_*,\zeta_*)\xi - T_*(A\zeta_* + \psi(\cdot,\xi_*,\zeta_*) - v_*) \in L^p(Q_1)$. By Lemma 2.2, equations (4.22)–(4.23) has a solution $\zeta \in Y_1$. Therefore, $DF(\xi_*,T_*,\zeta_*,v_*)$ is onto. The lemma is proved. □

The following lemma shows that the Mangasarian-Fromowitz condition is valid.

Lemma 4.6. *If $(H6)$ is satisfied, then $(A4)$ is also fulfilled at $z_* = (\xi_*,T_*,\zeta_*,v_*)$.*

Proof. Recall that $\xi_*(s) = T_*s,\ \zeta_*(x,s) = y_*(x,T_*s),\ v_*(x,s) = u_*(x,T_*s)$. By Lemma 4.5, the operator $DF(z_*) : Z \to E$ is onto. Let $(\widetilde{T}, \widetilde{y}, \widetilde{u})$ satisfy $(H6)$. By defining

$$\widetilde{\xi}(s) = \widetilde{T}s,\ \widetilde{\zeta}(x,s) = \widetilde{y}(x,T_*s),\ \widetilde{v}(x,s) = \widetilde{u}(x,T_*s),$$

we see that $(H6)$ is equivalent to:

$$(H6)'\begin{cases} \dfrac{\partial\widetilde{\zeta}}{\partial t} + T_*\Big(A\widetilde{\zeta} + \psi_t(\cdot,\xi_*,\zeta_*)\widetilde{\xi} + \psi_y(\cdot,\xi_*,\zeta_*)\widetilde{\zeta} - \widetilde{u}\Big) + \widetilde{T}\Big(A\zeta_* + \psi(\cdot,\xi_*,\zeta_*) - v_*\Big) = 0,\ \widetilde{\zeta}(0) = 0 \\ \psi_i(T_*,\zeta_*(1)) + \psi_{iT}(T_*,\zeta_*(1))\widetilde{T} + \psi_{i\zeta}(T_*,\zeta_*(1))\widetilde{\zeta}(1) < 0, \quad i = 1,2,...,m \\ g(\cdot,\xi_*,\zeta_*,v_*) + g_t(\cdot,\xi_*,\zeta_*,v_*)\widetilde{\xi} + g_y(\cdot,\xi_*,\zeta_*,v_*)\widetilde{\zeta} + g_u(\cdot,\xi_*,\zeta_*,v_*)\widetilde{v} \in \mathrm{int}K_1. \end{cases}$$

This means that

$$\begin{cases} DF(z_*)(\widetilde{\xi},\widetilde{T},\widetilde{\zeta},\widetilde{v}) = 0, \\ H_i(z_*) + DH_i(z_*)(\widetilde{\xi},\widetilde{T},\widetilde{\zeta},\widetilde{v}) < 0, \quad i = 1,2,...,m, \\ G(z_*) + DG(z_*)(\widetilde{\xi},\widetilde{T},\widetilde{\zeta},\widetilde{v},) \in \mathrm{int}K_1. \end{cases}$$

Hence $(A4)$ is satisfied. The lemma is proved. □

Let us denote by $\mathcal{C}_{1,0}[z_*]$ the critical cone of vectors $(\xi,T,\zeta,v) \in Z$ satisfying conditions:

(b_1) $\displaystyle\int_\Omega \big(L_{0T}[T_*,1]T + L_{0\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x + \int_{Q_1}\Big(TL[x,s] + T_*\big(L_t[x,s]\xi + L_y[x,s]\zeta + L_u[x,s]v\big)\Big)\mathrm{d}x\mathrm{d}s \le 0;$

(b_2) $\dfrac{\partial\zeta}{\partial s} + T_*\big(A\zeta + \psi_t[\cdot,\cdot]\xi + \psi_y[\cdot,\cdot]\zeta - v\big) + T(A\zeta_* + \psi[\cdot,\cdot] - v_*) = 0, \quad \zeta(0) = 0;$

(b_3) $\xi(s) = Ts$ for all $s \in [0,1]$;

(b_4) $\displaystyle\int_\Omega \big(L_{iT}[T_*,1])\xi(1) + L_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x \le 0$ for $i \in \{1,2,..,m|\psi_i[T_*,1] = 0\}$;

(b_5) $g_t[\cdot,\cdot]\xi + g_y[\cdot,\cdot]\zeta + g_u[\cdot,\cdot]v \in \mathrm{cone}\big(K_1 - g[\cdot,\cdot]\big).$

The closure of $\mathcal{C}_{1,0}[z_*]$ in Z is denoted by $\mathcal{C}_1[z_*]$. It is also called a critical cone to (P_1) at z_*.

Let us define a Lagrange function associated to (P_1):

$$\mathcal{L} : Z \times \mathbb{R}_+ \times L^q(Q_1) \times (H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega))^* \times C([0,1],\mathbb{R})^* \times \mathbb{R}^m \times L^\infty(Q_1)^* \to \mathbb{R}$$

by setting

$$\begin{aligned}&\mathcal{L}(z,\lambda,\phi_1,\phi_2^*,\nu_1^*,\mu,w^*) \\ &= \lambda\int_\Omega L_0(x,T,\zeta(1))\mathrm{d}x + \lambda\int_{Q_1} TL(x,\xi,\zeta,v)\mathrm{d}x\mathrm{d}s + \int_{Q_1}\phi_1\Big(\frac{\partial\zeta}{\partial s} + TA\zeta + T\psi(x,\xi,\zeta) - Tv\Big)\mathrm{d}x\mathrm{d}s \\ &+ \langle\phi_2^*,\zeta(0) - y_0\rangle + \langle\nu_1^*,\xi - \int_0^{(\cdot)} Td\tau\rangle + \sum_{i=1}^m \mu_i\int_\Omega L_i(x,T,\zeta(1))\mathrm{d}x + \langle w^*, g(\cdot,\xi,\zeta,v)\rangle,\ z = (\xi,T,\zeta,v) \in Z.\end{aligned}$$

The following result gives necessary optimality conditions for (P_1).

Proposition 4.7. *Suppose assumptions $(H1)-(H5)$, $T_* > 0$ and $z_* = (\xi_*,T_*,\zeta_*,v_*) \in \Phi_1$ is a locally optimal solution to (P_1). Then there exist Lagrange multipliers $\lambda \in \mathbb{R}_+$, $\mu = (\mu_1,\mu_2,...,\mu_m) \in \mathbb{R}^m$, $\varphi \in L^\infty(Q_1) \cap L^2(0,1;H_0^1(\Omega))$, $e \in L^\infty(Q_1)$ and an absolutely continuous function $\phi : [0,1] \to \mathbb{R}$ not all are zero and satisfy the following conditions:*

(*i*) *(the adjoint equations)*

$$\begin{cases} -\dfrac{\partial\varphi}{\partial s} + T_*A^*\varphi + T_*\psi_y[\cdot,\cdot]\varphi = -\lambda T_* L_y[\cdot,\cdot] - eg_y[\cdot,\cdot] \quad \text{in } Q_1, \\ \varphi = 0 \quad \text{on } \Sigma_1, \\ \varphi(1) = -\lambda L_{0\zeta}[T_*,1] - \sum_{i=1}^m \mu_i L_{i\zeta}[T_*,1] \end{cases} \tag{4.24}$$

and

$$\begin{cases} \phi'(s) &= -\displaystyle\int_\Omega \big(\lambda T_* L_t[x,s] + \varphi T_*\psi_t[x,s] + eg_t[x,s]\big)\mathrm{d}x \quad \text{in } (0,1), \\ \phi(1) &= 0 \end{cases} \tag{4.25}$$

where A^* *is the adjoint operator of* A*, which is defined by* $A^*\varphi = -\sum_{i,j=1}^N D_i(a_{ij}(x)D_j\varphi)$*;*
(*ii*) *(optimality condition for* v_**)*

$$\lambda T_* L_u[x,s] - T_*\varphi(x,s) + e(x,s)g_u[x,s] = 0 \quad \text{a.a. } (x,s) \in Q_1; \tag{4.26}$$

(*iii*) *(optimality condition for* T_**)*

$$\int_0^1 \phi(s)\mathrm{d}s + \int_{Q_1} \Big(\lambda L[x,s] + \varphi\big(A\zeta_* + \psi[x,s] - v_*\big)\Big)\mathrm{d}x\mathrm{d}s + \int_\Omega \big(\lambda L_{0T}[T_*,1] + \sum_{i=1}^m \mu_i L_{iT}[T_*,1]\big)\mathrm{d}x = 0; \tag{4.27}$$

(*iv*) *(the complementary conditions)*

$$\begin{cases} \quad \mu_i \geq 0 \text{ and } \mu_i\psi_i(T_*,\zeta_*(1)) = \mu_i \displaystyle\int_\Omega L_i(x,T_*,\zeta_*(1))\mathrm{d}x = 0, \ i = 1,2,...,m, \\ e(x,s) \in N([a,b], g[x,s]) \ \text{a.a. } (x,s) \in Q_1. \end{cases} \tag{4.28}$$

Moreover, one has
(*v*) *(the nonnegative second-order condition)*

$$\begin{aligned} &\sup_{(\lambda,\varphi,\mu,\phi,e)\in\Lambda[z_*]} D^2_{(\xi,T,\zeta,v)}\mathcal{L}\big(\xi_*,T_*,\zeta_*,v_*,\lambda,\varphi,\mu,\phi,e\big)[z,z] \\ &= \sup_{(\lambda,\varphi,\mu,\phi,e)\in\Lambda[z_*]} \Big[\lambda\int_\Omega \Big(L_{0TT}[T_*,1]T^2 + 2L_{0T\zeta}[T_*,1]T\zeta(1) + L_{0\zeta\zeta}[T_*,1]\zeta^2(1)\Big)\mathrm{d}x \\ &+ \lambda\int_{Q_1} T_*\Big(L_{tt}[x,s]\xi^2 + 2L_{ty}[x,s]\xi\zeta + 2L_{tu}[x,s]\xi v + L_{yy}[x,s]\zeta^2 + L_{uu}[x,s]v^2 + 2L_{yu}[x,s]\zeta v\Big)\mathrm{d}x\mathrm{d}s \\ &+ \lambda\int_{Q_1} 2T\Big(L_t[x,s]\xi + L_y[x,s]\zeta + L_u[x,s]v\Big)\mathrm{d}x\mathrm{d}s \\ &+ \int_{Q_1}\varphi\Big[T_*\Big(\psi_{tt}[x,s]\xi^2 + \psi_{yy}[x,s]\zeta^2 + 2\psi_{ty}[x,s]\xi\zeta\Big) + 2T\Big(A\zeta + \psi_t[x,s]\xi + \psi_y[x,s]\zeta - v\Big)\Big]\mathrm{d}x\mathrm{d}s \\ &+ \int_{Q_1} e(x,s)\Big(g_{tt}[x,s]\xi^2 + g_{yy}[x,s]\zeta^2 + g_{uu}[x,s]v^2 + 2g_{ty}[x,s]\xi\zeta + 2g_{tu}[x,s]\xi v + 2g_{yu}[x,s]\zeta v\Big)\mathrm{d}x\mathrm{d}s \\ &+ \sum_{i=1}^m \mu_i\int_\Omega \Big(L_{iTT}[T_*,1]T^2 + 2L_{iT\zeta}[T_*,1]T\zeta(1) + L_{i\zeta\zeta}[T_*,1]\zeta^2(1)\Big)\mathrm{d}x\Big] \geq 0 \end{aligned} \tag{4.29}$$

for all $z = (\xi, T, \zeta, v) \in \mathcal{C}_{1,0}[z_*]$. *Here* $\Lambda[z_*]$ *denotes the set of multipliers* $(\lambda, \varphi, \mu, \phi, e)$ *satisfying conditions (i)–(iv). In addition, if the assumption* $(H6)'$ *is satisfied then multipliers* $(\lambda, \varphi, \mu, \phi, e)$ *are normal, that is,* $\lambda = 1$ *and one has*

$$\max_{(1,\varphi,\mu,\phi,e)\in\Lambda[z_*]} D^2_{(\xi,T,\zeta,v)}\mathcal{L}\big(\xi_*, T_*, \zeta_*, v_*, \lambda, \varphi, \mu, \phi, e\big)[z, z] \geq 0 \quad \forall z \in \mathcal{C}_1[z_*]. \tag{4.30}$$

Proof. It is easy to see that

$$\text{int}K_1 = \{v \in L^\infty(Q_1) : a < \text{essinf} v \leq \text{esssup} v < b\}.$$

Hence $\text{int}K_1$ is nonempty and so $(A1)$ is valid. By Lemma 4.4 and Lemma 4.5, $(A2)$ and $(A3)$ are valid. Thus all conditions of Proposition 2.4 (part (a)) are fulfilled. Accordingly, there exist Lagrange multipliers $\lambda \in \mathbb{R}_+$, $\phi_1 \in L^q(Q_1)$, $\phi_2^* \in (H_0^1(\Omega) \cap W^{2-\frac{2}{p},p}(\Omega))^*$, $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, $\nu_1^* \in C([0,1], \mathbb{R})^*$ and $w^* \in L^\infty(Q_1)^*$ such that the following conditions hold:

$$\begin{aligned}
&D_z\mathcal{L}(z_*, \lambda, \phi_1, \phi_2^*, \nu_1^*, \mu, w^*) \\
&= \lambda D\widehat{J}(z_*) + \big\langle(\phi_1, \phi_2^*, \nu_1^*), DF(z_*)\big\rangle + \big\langle\mu, DH(z_*)\big\rangle + \Big\langle w^*, DG(z_*)\Big\rangle = 0, \qquad (4.31)\\
&\lambda \geq 0,\ \mu_i \geq 0,\ \mu_i H_i(z_*) = 0,\ i = 1, 2, ..., m, \qquad (4.32)\\
&w^* \in N(K_1, g[\cdot, \cdot]), \qquad (4.33)\\
&\sup_{(\lambda,\phi_1,\phi_2^*,\nu_1^*,\mu,w^*)\in\Lambda[z_*]} D_z^2\mathcal{L}(z_*, \lambda, \phi_1, \phi_2^*, \nu_1^*, \mu, w^*)[z, z] \geq 0 \quad \forall z = (\xi, T, \zeta, v) \in \mathcal{C}_{1,0}[z_*]. \qquad (4.34)
\end{aligned}$$

Note that ν_1^* is a signed Radon measure which is absolutely continuous w.r.t the Lebesgue measure $|\cdot|$ on $[0, 1]$. By Riesz's Representation (see [31], Chapter 01, p. 19 and [39], Thm. 3.8, p. 73), there exists a function of bounded variation ν_1, which is continuous from the right and vanishes at zero such that

$$\langle \nu_1^*, \vartheta\rangle = \int_0^1 \vartheta(s)d\nu_1(s) \qquad \forall\vartheta \in C([0, 1], \mathbb{R}),$$

where the integral stands for the Riemann-Stieltjes integral. We define function $\phi : [0, 1] \to \mathbb{R}$ by setting

$$\phi(s) := -\nu_1((s, 1]) = \nu_1(s) - \nu_1(1). \tag{4.35}$$

Then the function ϕ is of bounded variation and $\phi(1) = 0$. Then for all $(\xi, T) \in C([0, 1], \mathbb{R}) \times \mathbb{R}$, we have

$$\langle \nu_1^*, \xi - \int_0^{(\cdot)} T d\tau\rangle = \int_0^1 (\xi(s) - Ts)d\nu_1(s).$$

Hence (4.31) becomes

$$\begin{aligned}
&\lambda\int_\Omega \Big(L_{0T}(x, T_*, \zeta_*(1))T + L_{0\zeta}(x, T_*, \zeta_*(1))\zeta(1)\Big)\text{d}x \\
&+ \lambda\int_{Q_1} \Big(TL(x, \xi_*, \zeta_*, v_*) + T_*[L_t(x, \xi_*, \zeta_*, v_*)\xi + L_y(x, \xi_*, \zeta_*, v_*)\zeta + L_u(x, \xi_*, \zeta_*, v_*)v]\Big)\text{d}x\text{d}s \\
&+ \int_{Q_1} \phi_1\Big(\frac{\partial\zeta}{\partial s} + T_*A\zeta + T_*\psi_t(x, \xi_*, \zeta_*)\xi + T_*\psi_y(x, \xi_*, \zeta_*)\zeta - T_*v\Big)\text{d}x\text{d}s
\end{aligned}$$

$$+\int_{Q_1}\phi_1\Big(A\zeta_*+\psi(x,\xi_*,\zeta_*)-v_*\Big)T\mathrm{d}x\mathrm{d}s$$
$$+\langle\phi_2^*,\zeta(0)\rangle+\int_0^1(\xi(s)-Ts)d\nu_1(s)+\sum_{i=1}^m\mu_i\int_\Omega\Big(L_{iT}(x,T_*,\zeta_*(1))T+L_{i\zeta}(x,T_*,\zeta_*(1))\zeta(1)\Big)\mathrm{d}x$$
$$+\langle w^*,g_t(\cdot,\xi_*,\zeta_*,v_*)\xi+g_y(\cdot,\xi_*,\zeta_*,v_*)\zeta+g_u(\cdot,\xi_*,\zeta_*,v_*)v\rangle=0,\quad\forall(\xi,T,\zeta,v)\in Z.\tag{4.36}$$

Step 1. Deriving optimality conditions for v_*.

By inserting $(\xi,T,\zeta)=(0,0,0)$ into the above equality to get

$$\lambda\int_{Q_1}T_*L_u[x,s]v(x,s)\mathrm{d}x\mathrm{d}s-\int_{Q_1}\phi_1T_*v\mathrm{d}x\mathrm{d}s+\langle w^*,g_u[\cdot,\cdot]v\rangle=0,\qquad\forall v\in U_1.\tag{4.37}$$

It follows from (4.37) that

$$g_u[\cdot,\cdot]w^*=T_*\phi_1-\lambda T_*L_u[\cdot,\cdot].$$

Hence

$$w^*=\frac{T_*\phi_1-\lambda T_*L_u[\cdot,\cdot]}{g_u[\cdot,\cdot]}\quad\text{on }L^\infty(Q_1).\tag{4.38}$$

By $(H5)$, $\frac{1}{g_u[\cdot,\cdot]}\in L^\infty(Q_1)$. Since $T_*\phi_1-\lambda T_*L_u[\cdot,\cdot]\in L^q(Q_1)$, we have $\frac{T_*\phi_1-\lambda T_*L_u[\cdot,\cdot]}{g_u[\cdot,\cdot]}\in L^q(Q_1)$. Consequently, w^* can be represented by a function $e\in L^q(Q_1)$. From this and (4.38), we obtain

$$e(x,s)=\frac{T_*\phi_1(x,s)-\lambda T_*L_u[x,s]}{g_u[x,s]}\quad\text{a.a }(x,s)\in Q_1.$$

This implies that

$$\lambda T_*L_u[x,s]-T_*\phi_1(x,s)+e(x,s)g_u[x,s]=0\quad\text{a.a. }(x,s)\in Q_1\tag{4.39}$$

which is in the form of assertion (ii) of the proposition. In Step 2, we will find ϕ_1. By (4.33) and Corollary 4 in [40], we have $e(x,s)\in N([a,b],g[x,s])$ for a.a. $(x,s)\in Q_1$.

Step 2. Deriving the first adjoint equation.

Inserting $(\xi,T,v)=(0,0,0)$ into equality (4.36), we get

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\psi_y[x,s]\zeta\Big)\phi_1\mathrm{d}x\mathrm{d}s+\langle\phi_2^*,\zeta(0)\rangle=-\int_\Omega\big(\lambda L_{0\zeta}[T_*,1]\zeta(1)+\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x$$
$$-\lambda\int_{Q_1}T_*L_y[x,s]\zeta\mathrm{d}x\mathrm{d}s-\int_{Q_1}eg_y[x,s]\zeta\mathrm{d}x\mathrm{d}s,\quad\forall\zeta\in Y_1.\tag{4.40}$$

Inserting $v=\frac{g_y[x,s]}{g_u[x,s]}\zeta$ into (4.37), we get

$$\lambda\int_{Q_1}T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\mathrm{d}x\mathrm{d}s-\int_{Q_1}\phi_1T_*\frac{g_y[x,s]}{g_u[x,s]}\zeta\mathrm{d}x\mathrm{d}s+\int_{Q_1}eg_y[x,s]\zeta\mathrm{d}x\mathrm{d}s=0,\quad\forall\zeta\in Y_1.\tag{4.41}$$

Combining this with (4.40), yields

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\psi_y[x,s]\zeta\Big)\phi_1\mathrm{d}x\mathrm{d}s+\langle\phi_2^*,\zeta(0)\rangle=-\int_\Omega\big(\lambda L_{0\zeta}[T_*,1]\zeta(1)+\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x$$
$$-\lambda\int_{Q_1}T_*L_y[x,s]\zeta\mathrm{d}x\mathrm{d}s+\lambda\int_{Q_1}T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\mathrm{d}x\mathrm{d}s-\int_{Q_1}\phi_1T_*\frac{g_y[x,s]}{g_u[x,s]}\zeta\mathrm{d}x\mathrm{d}s,\quad\forall\zeta\in Y_1.$$

This is equivalent to

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\psi_y[x,s]\zeta+T_*\frac{g_y[x,s]}{g_u[x,s]}\zeta\Big)\phi_1\mathrm{d}x\mathrm{d}s+\langle\phi_2^*,\zeta(0)\rangle=$$
$$-\int_\Omega\big(\lambda L_{0\zeta}[T_*,1]\zeta(1)+\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x-\lambda\int_{Q_1}T_*L_y[x,s]\zeta\mathrm{d}x\mathrm{d}s+\lambda\int_{Q_1}T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\mathrm{d}x\mathrm{d}s \tag{4.42}$$

for all $\zeta\in Y_1$. Let us consider the equation

$$\begin{cases}-\dfrac{\partial\varphi}{\partial s}+T_*A^*\varphi+T_*\big(\psi_y[x,s]+\dfrac{g_y[x,s]}{g_u[x,s]}\big)\varphi=-\lambda T_*L_y[x,s]+\lambda T_*L_u[x,s]\dfrac{g_y[x,s]}{g_u[x,s]}\quad\text{in } Q_1,\\ \varphi=0\quad\text{on }\Sigma_1,\\ \varphi(1)=-\lambda L_{0\zeta}[T_*,1]-\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\quad\text{in }\Omega.\end{cases}\tag{4.43}$$

By $(H4)$ and $(H5)$, $-\lambda L_{0\zeta}[T_*,1]-\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\in L^\infty(\Omega)$, $T_*\big(\psi_y[\cdot,\cdot]+\frac{g_y[\cdot,\cdot]}{g_u[\cdot,\cdot]}\big)\in L^\infty(Q_1)$ and $-\lambda T_*L_y[\cdot,\cdot]+\lambda T_*L_u[\cdot,\cdot]\frac{g_y[\cdot,\cdot]}{g_u[\cdot,\cdot]}\in L^\infty(Q_1)$. By changing variable $\widetilde{\varphi}(x,s')=\varphi(x,1-s')$ and using Theorem 2.1 in [11], we see that the above equation has a unique solution $\varphi(x,s)=\widetilde{\varphi}(x,1-s)$ which belongs to $L^\infty(Q_1)\cap L^2(0,1;H_0^1(\Omega))$. Multiplying both sides of (4.43) with $\zeta\in Y_1$ and integrating on Q_1, we have

$$\int_{Q_1}\Big(-\frac{\partial\varphi}{\partial s}+T_*A^*\varphi+T_*\big(\psi_y[x,s]+\frac{g_y[x,s]}{g_u[x,s]}\big)\varphi\Big)\zeta\mathrm{d}x\mathrm{d}s=\int_{Q_1}\Big(-\lambda T_*L_y[\cdot,\cdot]\zeta+\lambda T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\Big)\mathrm{d}x\mathrm{d}s$$

Using integration by part formula, we get

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\big(\psi_y[x,s]+\frac{g_y[x,s]}{g_u[x,s]}\big)\zeta\Big)\varphi\mathrm{d}x\mathrm{d}s+(\varphi(0),\zeta(0))_H-(\varphi(1),\zeta(1))_H$$
$$=\int_{Q_1}\Big(-\lambda T_*L_y[\cdot,\cdot]\zeta+\lambda T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\Big)\mathrm{d}x\mathrm{d}s.$$

Since $\varphi(1)=-\lambda L_{0\zeta}[T_*,1]-\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]$, we obtain

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\big(\psi_y[x,s]+\frac{g_y[x,s]}{g_u[x,s]}\big)\zeta\Big)\varphi\mathrm{d}x\mathrm{d}s+\int_\Omega\varphi(0)\zeta(0)\mathrm{d}x$$
$$=-\int_\Omega\big(\lambda L_{0\zeta}[T_*,1]\zeta(1)+\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x+\int_{Q_1}\Big(-\lambda T_*L_y[\cdot,\cdot]\zeta+\lambda T_*L_u[x,s]\frac{g_y[x,s]}{g_u[x,s]}\zeta\Big)\mathrm{d}x\mathrm{d}s.$$

Subtracting this equality from (4.42), we get

$$\int_{Q_1}\Big(\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\big(\psi_y[x,s]+\frac{g_y[x,s]}{g_u[x,s]}\big)\zeta\Big)\Big(\phi_1-\varphi\Big)\mathrm{d}x\mathrm{d}s+\langle\phi_2^*,\zeta(0)\rangle-\int_\Omega\varphi(0)\zeta(0)\mathrm{d}x=0\quad\forall\zeta\in Y_1.\quad(4.44)$$

By Lemma 2.2, we see that for each $\vartheta\in L^p(Q_1)$ and $\zeta_0\in H_0^1(\Omega)\cap W^{2-\frac{2}{p},p}(\Omega)$ the equation

$$\frac{\partial\zeta}{\partial s}+T_*A\zeta+T_*\big(\psi_y[x,s]+\frac{g_y[x,s]}{g_u[x,s]}\big)\zeta=\vartheta\quad\text{in } Q_1,\qquad\zeta=0\quad\text{on }\Sigma_1,\qquad\zeta(0)=\zeta_0\quad\text{in }\Omega$$

has a unique solution $\zeta\in Y_1$. Inserting such a solution into (4.44), we obtain

$$\begin{aligned}&\int_{Q_1}\big(\phi_1-\varphi\big)\vartheta\mathrm{d}x\mathrm{d}s=0\qquad\forall\vartheta\in L^p(Q_1),\\&\langle\phi_2^*,\zeta_0\rangle-\int_\Omega\varphi(0)\zeta_0\mathrm{d}x=0\quad\forall\zeta_0\in H_0^1(\Omega)\cap W^{2-\frac{2}{p},p}(\Omega).\end{aligned}$$

This implies that $\phi_2^*=\varphi(0)$ and $\phi_1=\varphi\in L^\infty(Q_1)\cap L^2(0,1;H_0^1(\Omega))$. Hence (4.39) becomes

$$\lambda T_*L_u[x,s]-T_*\varphi(x,s)+e(x,s)g_u[x,s]=0\quad\text{a.a. }(x,s)\in Q_1.$$

We obtain assertion (ii) of the proposition. Besides, we have

$$e=\frac{-\lambda T_*L_u[\cdot,\cdot]+T_*\varphi}{g_u[\cdot,\cdot]}\in L^\infty(Q_1).$$

We now rewrite equation (4.43) in the form:

$$\begin{cases}-\dfrac{\partial\varphi}{\partial s}+T_*A^*\varphi+T_*\psi_y[x,s]\varphi=-\lambda T_*L_y[x,s]+\lambda T_*L_u[x,s]\dfrac{g_y[x,s]}{g_u[x,s]}-T_*\dfrac{g_y[x,s]}{g_u[x,s]}\varphi\quad\text{in } Q_1,\\ \varphi=0\quad\text{on }\Sigma_1,\\ \varphi(1)=-\lambda L[T_*,1]-\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\quad\text{in }\Omega.\end{cases}$$

This equivalent to

$$\begin{cases}-\dfrac{\partial\varphi}{\partial s}+T_*A^*\varphi+T_*\psi_y[x,s]\varphi=-\lambda T_*L_y[x,s]-eg_y[x,s]\quad\text{in } Q_1,\\ \varphi=0\quad\text{on }\Sigma_1,\\ \varphi(1)=-\lambda L[T_*,1]-\sum_{i=1}^m\mu_iL_{i\zeta}[T_*,1]\quad\text{in }\Omega.\end{cases}$$

We obtain the first adjoint equation of assertion (i).

Step 3. Deriving the second adjoint equation (optimality condition for ξ_*).

Inserting $(T,\zeta,v)=(0,0,0)$ into equality (4.36), we get

$$-\int_0^1\xi(s)d\nu_1(s)=\int_0^1\int_\Omega\big(\lambda T_*L_t[x,s]+\varphi T_*\psi_t[x,s]+eg_t[x,s]\big)\xi\mathrm{d}x\mathrm{d}s\quad\forall\xi\in C([0,1],\mathbb{R})\qquad(4.45)$$

By Lemma 5.1 in [41], the above equality is valid for $\xi(s) = \xi_0\chi_{(\tau,1]}(s)$ with $0 \le \tau < 1$, where $\chi_{(\tau,1]}$ is the indicator function of the set $(\tau, 1] \subset [0, 1]$. Inserting $\xi(s)$ into (4.45), we get

$$-\int_\tau^1 \xi_0 d\nu_1(s) = \int_\tau^1 \xi_0 \int_\Omega \big(\lambda T_* L_t[x,s] + \varphi T_*\psi_t[x,s] + eg_t[x,s]\big)\mathrm{d}x\mathrm{d}s \quad \forall \xi_0 \in \mathbb{R},\ \forall \tau \in [0,1).$$

From this and definition of function ϕ in (4.35), we obtain

$$\phi(\tau) = \int_\tau^1 \int_\Omega \big(\lambda T_* L_t[x,s] + \varphi T_*\psi_t[x,s] + eg_t[x,s]\big)\mathrm{d}x\mathrm{d}s, \quad \forall \tau \in [0,1).$$

This implies that $\lim_{\tau\to 1}\phi(\tau) = 0 = \phi(1)$. We obtain equation (4.25).

Step 4. Establishing optimality conditions for T_*.

Inserting $(\xi, \zeta, v) = (0, 0, 0)$ into (4.36) and notice that $-\int_0^1 Tsd\nu_1(s) = \int_0^1 T\phi(s)\mathrm{d}s$, we have

$$\int_0^1 T\phi(s)\mathrm{d}s + \int_0^1\int_\Omega \lambda L[x,s]T\mathrm{d}x\mathrm{d}s + \int_0^1\int_\Omega \varphi\Big(A\zeta_* + \psi[x,s] - v_*\Big)T\mathrm{d}x\mathrm{d}s$$
$$+ \lambda\int_\Omega L_{0T}[T_*,1]T\mathrm{d}x + \sum_{i=1}^m \mu_i \int_\Omega L_{iT}[T_*,1]Tdx = 0, \quad \forall T \in \mathbb{R}.$$

Hence we obtain

$$\int_0^1 \phi(s)\mathrm{d}s + \int_{Q_1}\Big(\lambda L[x,s] + \varphi\big(A\zeta_* + \psi[x,s] - v_*\big)\Big)\mathrm{d}x\mathrm{d}s + \int_\Omega\Big(\lambda L_{0T}[T_*,1] + \sum_{i=1}^m \mu_i L_{iT}[T_*,1]\Big)\mathrm{d}x = 0$$

which is equation (4.27).

Step 5. Deriving the non-negative second-order condition.

Assertion (v) of the theorem follows from condition (4.34) and formulas (4.17)–(4.21).

Finally, if $(H6)'$ is satisfied, then we have from Lemma 4.6 that $(A4)$ is valid. By the part (b) of Proposition 2.4, we get $\lambda = 1$. The proof of the proposition is complete. □

In the rest of this section, we focus on second-order sufficient optimality conditions for problem (P_1). For this we need to enlarge the critical cone $\mathcal{C}_1[z_*]$ by the cone $\mathcal{C}_1'[z_*]$ which consists of vectors $(\xi, T, \zeta, v) \in C([0,1],\mathbb{R}) \times \mathbb{R} \times W_2^{1,1}(0,1;D,H) \times L^2(Q_1)$ satisfying the following conditions:

(b_1') $\int_\Omega \big(L_{0T}[T_*,1]T + L_{0\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x + \int_{Q_1}\Big(TL[x,s] + T_*(L_t[x,s]\xi + L_y[x,s]\zeta + L_u[x,s]v)\Big)\mathrm{d}x\mathrm{d}s \le 0;$

(b_2') $\dfrac{\partial\zeta}{\partial s} + T_*\big(A\zeta + \psi_t[\cdot,\cdot]\xi + \psi_y[\cdot,\cdot]\zeta - v\big) + T(A\zeta_* + \psi[\cdot,\cdot] - v_*) = 0, \quad \zeta(0) = 0;$

(b_3') $\xi(s) = Ts \quad \forall s \in [0,1];$

(b_4') $\int_\Omega \big(L_{iT}[T_*,1]T + L_{i\zeta}[T_*,1]\zeta(1)\big)\mathrm{d}x \le 0$ for $i \in \big\{1, 2, .., m | \psi_i[T_*,1] = 0\big\};$

(b_5') $g_t[x,s]\xi(s) + g_y[x,s]\zeta(x,s) + g_u[x,s]v(x,s) \in T([a,b], g[x,s])$ for a.a. $(x,s) \in Q_1$.

The following proposition gives second-order sufficient conditions for locally optimal solutions to (P_1).

Proposition 4.8. *Suppose $z_* = (\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$ with $T_* > 0$, assumptions $(H1) - (H5)$, and there exist multipliers $(\lambda, \mu, \varphi, \phi, e)$ with $\lambda = 1$ satisfying conditions (i)–(iv) of Proposition 4.7. Furthermore, assume that the following conditions are fulfilled:*

(*v*) *(the strictly second-order condition)*

$$\begin{aligned}
&\int_\Omega \Big(L_{0TT}[T_*,1]T^2 + 2L_{0T\zeta}[T_*,1]T\zeta(1) + L_{0\zeta\zeta}[T_*,1]\zeta^2(1)\Big)\mathrm{d}x \\
&+ \int_{Q_1} T_*\Big(L_{tt}[x,s]\xi^2 + 2L_{ty}[x,s]\xi\zeta + 2L_{tu}[x,s]\xi v + L_{yy}[x,s]\zeta^2 + L_{uu}[x,s]v^2 + 2L_{yu}[x,s]\zeta v\Big)\mathrm{d}x\mathrm{d}s \\
&+ \int_{Q_1} 2T\Big(L_t[x,s]\xi + L_y[x,s]\zeta + L_u[x,s]v\Big)\mathrm{d}x\mathrm{d}s \\
&+ \int_{Q_1} \varphi\Big[T_*\Big(\psi_{tt}[x,s]\xi^2 + \psi_{yy}[x,s]\zeta^2 + 2\psi_{ty}[x,s]\xi\zeta\Big) + 2T\Big(A\zeta + \psi_t[x,s]\xi + \psi_y[x,s]\zeta - v\Big)\Big]\mathrm{d}x\mathrm{d}s \\
&+ \int_{Q_1} e(x,s)\Big(g_{tt}[x,s]\xi^2 + g_{yy}[x,s]\zeta^2 + g_{uu}[x,s]v^2 + 2g_{ty}[x,s]\xi\zeta + 2g_{tu}[x,s]\xi v + 2g_{yu}[x,s]\zeta v\Big)\mathrm{d}x\mathrm{d}s \\
&+ \sum_{i=1}^m \mu_i \int_\Omega \Big(L_{iTT}[T_*,1]T^2 + 2L_{iT\zeta}[T_*,1]T\zeta(1) + L_{i\zeta\zeta}[T_*,1]\zeta^2(1)\Big)\mathrm{d}x > 0
\end{aligned} \tag{4.46}$$

for all $z = (\xi, T, \zeta, v) \in \mathcal{C}_1'[z_*] \setminus \{(0,0,0,0)\}$.
(*vi*) *(the Legendre–Clebsch condition) there is a number* $\Lambda > 0$ *such that*

$$T_*L_{uu}[x,s] + e(x,s)g_{uu}[x,s] \geq \Lambda \quad \text{a.a. } (x,s) \in Q_1. \tag{4.47}$$

Then there exist numbers $\delta > 0$ *and* $\kappa > 0$ *such that*

$$\widehat{J}(z) \geq \widehat{J}(z_*) + \kappa\Big((T - T_*)^2 + \|v - v_*\|^2_{L^2(Q_1)}\Big) \tag{4.48}$$

for all $z = (\xi, T, \zeta, v) \in \Phi_1 \cap B_{*Z}(z_*, \delta)$. *In particular,* z_* *is a locally optimal solution to* (P_1).

Proof. Let us define the so-called Lagrange function for (P_1):

$$\mathcal{L}\big(\xi, T, \zeta, v, \varphi, \phi, \mu, e\big) = \widehat{J}(\xi, T, \zeta, v) + \langle(\varphi, \phi), F(\xi, T, \zeta, v)\rangle + \langle\mu, H(\xi, T, \zeta, v)\rangle + \langle e, G(\xi, T, \zeta, v)\rangle, \tag{4.49}$$

where

$$\begin{aligned}
&\langle(\varphi,\phi), F(\xi,T,\zeta,v)\rangle = \int_{Q_1}\Big[\zeta\big(-\frac{\partial\varphi}{\partial s} + TA^*\varphi\big) + \big(\psi(x,\xi,\zeta) - v\big)T\varphi\Big]\mathrm{d}x\mathrm{d}s + \langle\varphi(1), \zeta(1)\rangle \\
&\quad + \int_0^1 \xi(s)d\phi(s) + \int_0^1 T\phi(s)\mathrm{d}s + \phi(1)\xi(1), \\
&\langle\mu, H(\xi,T,\zeta,v)\rangle = \sum_{i=1}^m \mu_i\psi_i(T,\zeta(1)) = \sum_{i=1}^m \mu_i \int_\Omega L_i(x,T,\zeta(x,1))\mathrm{d}x, \\
&\langle e, G(\xi,T,\zeta,v)\rangle = \int_{Q_1} eg(x,\xi,\zeta,v)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{4.50}$$

Then from conditions (*i*)–(*iii*) of Proposition 4.7 we can show that

$$D_z\mathcal{L}(z_*, \varphi, \phi, \mu, e) = 0. \tag{4.51}$$

Besides, for each $d = (\xi, T, \zeta, v) \in \mathcal{C}_1'[z_*]$, $D^2_{zz}\mathcal{L}(z_*, \varphi, \phi, \mu, e)[d, d]$ equals to the left hand side of (4.46). Hence

$$D^2_{zz}\mathcal{L}(z_*, \varphi, \phi, \mu, e)[d, d] > 0 \quad \forall d \in \mathcal{C}_1'[z_*] \setminus \{(0, 0, 0, 0)\}. \tag{4.52}$$

We now suppose to the contrary that the conclusion of the proposition were false. Then we could find sequences $\{z_n = (\xi_n, T_n, \zeta_n, v_n)\} \subset \Phi_1$ and $\{\gamma_n\} \subset \mathbb{R}_+$ such that $\xi_n \to \xi_*$ in $C([0, 1], \mathbb{R})$, $\zeta_n \to \zeta_*$ in Y_1, $v_n \to v_*$ in $L^\infty(Q_1)$, $T_n \to T_*$ in $\mathbb{R}$, $\gamma_n \to 0^+$ and

$$\widehat{J}(z_n) < \widehat{J}(z_*) + \gamma_n\Big(\|v_n - v_*\|^2_{L^2(Q_1)} + |T_n - T_*|^2\Big). \tag{4.53}$$

If $v_n = v_*$ and $T_n = T_*$ then by the uniqueness we have $\zeta_n = \zeta_*$ and $\xi_n = \xi_*$. This leads to $\widehat{J}(z_*) < \widehat{J}(z_*)$ which is absurd. Therefore, we can assume that $\|v_n - v_*\|^2_{L^2(Q_1)} + |T_n - T_*|^2 \neq 0$ for all $n > 0$. Define

$$t_n = \Big(\|v_n - v_*\|^2_{L^2(Q_1)} + |T_n - T_*|^2\Big)^{1/2}.$$

Then $t_n \to 0^+$ as $n \to +\infty$ and we get from (4.53) that

$$\widehat{J}(z_n) - \widehat{J}(z_*) \le o(t_n^2). \tag{4.54}$$

Put $\widehat{\xi}_n = \frac{\xi_n - \xi_*}{t_n}$, $\widehat{\zeta}_n = \frac{\zeta_n - \zeta_*}{t_n}$, $\widehat{v}_n = \frac{v_n - v_*}{t_n}$ and $\widehat{T}_n = \frac{T_n - T_*}{t_n}$. Then we have

$$|\widehat{T}_n|^2 + \|\widehat{v}_n\|^2_{L^2(Q_1)} = 1.$$

Therefore, we can assume that $\widehat{T}_n \to \widehat{T}$ in $\mathbb{R}$ and $\widehat{v}_n \rightharpoonup \widehat{v}$ in $L^2(Q_1)$. On the other hand we have

$$\begin{aligned}
&\|\widehat{\xi}_n\|_{C([0,1],\mathbb{R})} = |\widehat{T}_n| \le 1,\\
&|\widehat{\xi}_n(s_1) - \widehat{\xi}_n(s_2)| \le |s_1 - s_2||\widehat{T}_n| \le |s_1 - s_2| \quad \forall s_1, s_2 \in [0, 1].
\end{aligned}$$

Hence $\{\widehat{\xi}_n\}$ is equicontinuous. By the Arzelá–Ascoli theorem, we can assume that $\widehat{\xi}_n \to \widehat{\xi}$ in $C([0, 1], \mathbb{R})$.

Let us claim that $\widehat{\zeta}_n \rightharpoonup \widehat{\zeta}$ for some in $\widehat{\zeta} \in W^{1,1}_2(0, 1; D, H)$ and $\widehat{\zeta}_n(1) \to \widehat{\zeta}(1)$ in $L^2(\Omega)$. In fact, since $(\xi_n, T_n, \zeta_n, v_n) \in \Phi_1$ and $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$ we have

$$\begin{aligned}
&\frac{\partial \zeta_n}{\partial s} + T_n A\zeta_n + T_n\psi(x, \xi_n, \zeta_n) = T_n v_n, \quad \zeta_n(0) = y_0,\\
&\frac{\partial \zeta_*}{\partial s} + T_* A\zeta_* + T_*\psi(x, \xi_*, \zeta_*) = T_* v_*, \quad \zeta_*(0) = y_0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{\partial(\zeta_n - \zeta_*)}{\partial s} + T_n A(\zeta_n - \zeta_*) + T_n[\psi(x, \xi_*, \zeta_n) - \psi(x, \xi_*, \zeta_*)]\\
&= T_n\Big[(v_n - v_*) - (\psi(x, \xi_n, \zeta_n) - \psi(x, \xi_*, \zeta_n))\Big] + (T_n - T_*)[v_* - A\zeta_* - \psi(x, \xi_*, \zeta_*)],\\
&(\zeta_n - \zeta_*)(0) = 0.
\end{aligned} \tag{4.55}$$

By Taylor's expansions, there exist measurable functions $\theta_{1,n}, \theta_{2,n}$ such that

$$\psi(x,\xi_n,\zeta_n) - \psi(x,\xi_*,\zeta_n) = \psi_t(x,\xi_* + \theta_{1,n}(\xi_n - \xi_*),\zeta_n)(\xi_n - \xi_*), \quad 0 \le \theta_{1,n}(x,t) \le 1,$$
$$\psi(x,\xi_*,\zeta_n) - \psi(x,\xi_*,\zeta_*) = \psi_y(x,\xi_*,\zeta_* + \theta_{2,n}(\zeta_n - \zeta_*))(\zeta_n - \zeta_*), \quad 0 \le \theta_{2,n}(x,t) \le 1.$$

Note that $T_n \to T_*$ in $\mathbb{R}$, $\xi_n \to \xi_*$ in $C([0,1],\mathbb{R})$ and $\zeta_n \to \zeta_*$ in $L^\infty(Q_1)$ so there exists $M > 0$ such that $|T_n| + \|\xi_n\|_{C([0,1],\mathbb{R})} + \|\zeta_n\|_{L^\infty(Q_1)} + \|\xi_* + \theta_{1,n}(\xi_n - \xi_*)\|_{C([0,1],\mathbb{R})} + \|\zeta_* + \theta_n(\zeta_{2,n} - \zeta_*)\|_{L^\infty(Q_1)} \le M$. By $(H2)$, there exists $k_M > 0$ such that

$$|\psi_t(x,\xi_* + \theta_{1,n}(\xi_n - \xi_*),\zeta_n)| + |\psi_y(x,\xi_*,\zeta_* + \theta_{2,n}(\zeta_n - \zeta_*))| \le k_M.$$

Hence $T_n\psi_y(x,\xi_*,\zeta_* + \theta_{2,n}(\zeta_n - \zeta_*))$ is bounded in $L^\infty(Q_1)$. From (4.55) we get

$$\begin{aligned} &\frac{\partial \widehat{\zeta}_n}{\partial s} + T_n A\widehat{\zeta}_n + T_n\psi_y(x,\xi_*,\zeta_* + \theta_{2,n}(\zeta_n - \zeta_*))\widehat{\zeta}_n \\ &= T_n[\widehat{v}_n - \psi_t(x,\xi_* + \theta_{1,n}(\xi_n - \xi_*),\zeta_n)\widehat{\xi}_n] + [v_* - A\zeta_* - \psi(x,\xi_*,\zeta_*)]\widehat{T}_n, \qquad \widehat{\zeta}_n(0) = 0. \end{aligned} \tag{4.56}$$

By [42], Theorem 5, p. 360, there exists a constant $c > 0$ such that

$$\begin{aligned} \|\widehat{\zeta}_n\|_{W_2^{1,1}(0,1;D,H)} &\le c\Big\|T_n[\widehat{v}_n - \psi_t(x,\xi_* + \theta_{1,n}(\xi_n - \xi_*),\zeta_n)\widehat{\xi}_n] + [v_* - A\zeta_* - \psi(x,\xi_*,\zeta_*)]\widehat{T}_n\Big\|_{L^2(Q_1)} \\ &\le cM + cMk_M|Q_1|^{\frac{1}{2}} + c\|v_* - A\zeta_* - \psi(x,\xi_*,\zeta_*)\|_{L^2(Q_1)}. \end{aligned}$$

Hence $\{\widehat{\zeta}_n\}$ is bounded in $W_2^{1,1}(0,1;D,H)$. Without loss of generality, we may assume that $\widehat{\zeta}_n \rightharpoonup \widehat{\zeta}$ in $W_2^{1,1}(0,1,D,H)$. By the Aubin theorem, the imbedding $W_2^{1,1}(0,1,D,H) \hookrightarrow L^2(0,1;H)$ is compact. Hence $\widehat{\zeta}_n \to \widehat{\zeta}$ in norm of $L^2(0,1;H)$. This implies that $\widehat{\zeta}_n(t) \to \widehat{\zeta}(t)$ a.a. $t \in [0,1]$. On the other hand $W_2^{1,1}(0,1,D,H) \hookrightarrow C([0,1],H)$. Hence $\widehat{\zeta}_n(1) \to \widehat{\zeta}(1)$ in H. The claim is justified.

Let us divide the remainder of the proof into some steps.

Step 1. Showing that $(\widehat{\xi},\widehat{T},\widehat{\zeta},\widehat{v}) \in \mathcal{C}_1'[(\xi_*,T_*,\zeta_*,v_*)]$.

We now use the procedure in the proof of [42], Theorem 3, p. 356. By passing to the limit, we obtain from (4.56) that

$$\frac{\partial \widehat{\zeta}}{\partial s} + T_* A\widehat{\zeta} + T_*\psi_y[\cdot,\cdot]\widehat{\zeta} = T_*(\widehat{v} - \psi_t[\cdot,\cdot]\widehat{\xi}) + (v_* - A\zeta_* - \psi[\cdot,\cdot])\widehat{T}, \quad \widehat{\zeta}(0) = 0$$

which implies that condition (b_2') is satisfied. Also, we have

$$|\widehat{T}_n s - \widehat{\xi}(s)| = |\widehat{\xi}_n(s) - \widehat{\xi}(s)| \le \max_{s\in[0,1]}|\widehat{\xi}_n(s) - \widehat{\xi}(s)| = \|\widehat{\xi}_n - \widehat{\xi}\|_{C([0,1],\mathbb{R})} \quad \forall s \in [0,1]. \tag{4.57}$$

Letting $n \to \infty$ in (4.57) and notice that $\widehat{\xi}_n \to \widehat{\xi}$ in $C([0,1],\mathbb{R})$, we obtain

$$\widehat{\xi}(s) = \widehat{T}s \quad \forall s \in [0,1].$$

This implies that (b_3') is valid.

By the mean value theorem, we have from (4.54) that

$$\int_\Omega \big(L_{0T}(x,T_* + \rho_1(T_n - T_*),\zeta_n(1))\widehat{T}_n + L_{0\zeta}(x,T_*,\zeta_* + \rho_2(\zeta_n(1) - \zeta_*(1)))\widehat{\zeta}_n(1)\big)\mathrm{d}x$$

$$+\int_{Q_1}\widehat{T}_n L(x,\xi_n,\zeta_n,v_n)\mathrm{d}x\mathrm{d}s+\int_{Q_1}T_*L_t(x,\xi_*+\eta_1(\xi_n-\xi_*),\zeta_n,v_n)\widehat{\xi}_n\mathrm{d}x\mathrm{d}s$$
$$+\int_{Q_1}T_*L_y(x,\xi_*,\zeta_*+\eta_2(\zeta_n-\zeta_*),v_n)\widehat{\zeta}_n\mathrm{d}x\mathrm{d}s+\int_{Q_1}T_*L_u(x,\xi_*,\zeta_*,v_*+\eta_3(v_n-v_*))\widehat{v}_n\mathrm{d}x\mathrm{d}s\le\frac{o(t_n^2)}{t_n},\quad (4.58)$$

where $0\le\rho_i,\eta_j\le 1$. Since $\xi_n\to\xi_*$ in $C([0,1],\mathbb{R})$, $T_n\to T_*$ in $\mathbb{R}$, $\zeta_n\to\zeta_*$ in Y_1, $v_n\to v_*$ in $L^\infty(Q_1)$ and $(H4)$, we have

$$\begin{aligned}
&\|L_{0T}(\cdot,T_*+\rho_1(T_n-T_*),\zeta_n(1))-L_{0T}(\cdot,T_*,\zeta_*(1))\|_{L^\infty(\Omega)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L_{0\zeta}(\cdot,T_*,\zeta_*+\rho_2(\zeta_n(1)-\zeta_*(1)))-L_{0\zeta}(\cdot,T_*,\zeta_*(1))\|_{L^\infty(\Omega)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L(\cdot,\xi_n,\zeta_n,v_n)-L(\cdot,\xi_*,\zeta_*,v_*)\|_{L^\infty(Q_1)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L_t(\cdot,\xi_*+\eta_1(\xi_n-\xi_*),\zeta_n,v_n)-L_t(\cdot,\xi_*,\zeta_*,v_*)\|_{L^\infty(Q_1)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L_y(\cdot,\xi_*,\zeta_*+\eta_2(\zeta_n-\zeta_*),v_n)-L_y(\cdot,\xi_*,\zeta_*,v_*)\|_{L^\infty(Q_1)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L_u(\cdot,\xi_*,\zeta_*,v_*+\eta_3(v_n-v_*))-L_u(\cdot,\xi_*,\zeta_*,v_*)\|_{L^\infty(Q_1)}\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Combining this with the fact $\widehat{\zeta}_n(1)\to\widehat{\zeta}(1)$ in $L^2(\Omega)$, we deduce from (4.58) when $n\to\infty$ that

$$\int_\Omega\big(L_{0T}[T_*,1]\widehat{T}+L_{0\zeta}[T_*,1]\widehat{\zeta}(1)\big)\mathrm{d}x+\int_{Q_1}\Big(\widehat{T}L[x,s]+T_*[L_t[x,s]\widehat{\xi}+L_y[x,s]\widehat{\zeta}+L_u[x,s]\widehat{v}]\Big)\mathrm{d}x\mathrm{d}s\le 0.$$

Hence condition (b_1') of $\mathcal{C}_1'[z_*]$ is valid. It remains to verify conditions (b_4') and (b_5') of $\mathcal{C}_1'[z_*]$. For each $i\in\{1,2,...,m\}$, we have

$$\begin{aligned}
&\int_\Omega L_i(x,T_n,\zeta_n(x,1))\mathrm{d}x-\int_\Omega L_i(x,T_*,\zeta_*(x,1))\mathrm{d}x\\
&\qquad=\psi_i(T_n,\zeta_n(1))-\psi_i(T_*,\zeta_*(1))\in(-\infty,0]-\psi_i(T_*,\zeta_*(1)).
\end{aligned}\quad (4.59)$$

Using the mean value theorem for the left hand side of (4.59) and dividing both sides by t_n, we get

$$\begin{aligned}
&\int_\Omega\Big(L_{iT}(x,T_*+\eta_4(T_n-T_*)),\zeta_n(1))\widehat{T}_n+L_{i\zeta}(x,T_*,\zeta_*(1)+\eta_5(\zeta_n(1)-\zeta_*(1)))\widehat{\zeta}_n(1)\Big)\mathrm{d}x\\
&\in\frac{1}{t_n}\Big((-\infty,0]-\psi_i(T_*,\zeta_*(1))\Big)\subseteq T\big((-\infty,0];\psi_i(T_*,\zeta_*(1))\big),
\end{aligned}\quad (4.60)$$

where $0\le\eta_4,\eta_5\le 1$. Notice that $\widehat{T}_n\to\widehat{T}$ in $\mathbb{R}$, $\widehat{\zeta}_n(1)\to\widehat{\zeta}(1)$ in $L^2(\Omega)$ and

$$\begin{aligned}
&\|L_{iT}(\cdot,T_*+\eta_4(T_n-T_*),\zeta_n(1))-L_{iT}(\cdot,T_*,\zeta_*(1))\|_{L^\infty(\Omega)}\to 0\quad\text{as}\quad n\to\infty,\\
&\|L_{i\zeta}(\cdot,T_*,\zeta_*(1)+\eta_5(\zeta_n(1)-\zeta_*(1)))-L_{i\zeta}(\cdot,T_*,\zeta_*(1))\|_{L^\infty(\Omega)}\to 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Letting $n\to\infty$ in (4.60) and using $(H4)$, we get

$$\int_\Omega\big(L_{iT}(x,T_*,\zeta_*(1))\widehat{T}+L_{i\zeta}(x,T_*,\zeta_*(1))\widehat{\zeta}(1)\big)\mathrm{d}x\in T((-\infty,0];\psi_i(T_*,\zeta_*(1))),\quad i=1,2,..,m.$$

This implies that

$$\int_\Omega\big(L_{iT}(x,T_*,\zeta_*(1))\widehat{T}+L_{i\zeta}(x,T_*,\zeta_*(1))\widehat{\zeta}(1)\big)\mathrm{d}x\le 0\quad\text{for}\quad i\in\big\{1,2,..,m:\psi_i(T_*,\zeta_*(1))=0\big\}.$$

Hence condition (b_4') is valid. Also, we have

$$G(\xi_n, T_n, \zeta_n, v_n) - G(\xi_*, T_*, \zeta_*, v_*) \in (K_1 - g[\cdot,\cdot]).$$

By a Taylor expansion and the definitions of $\widehat{\xi}_n, \widehat{\zeta}_n, \widehat{v}_n$, we have

$$g_t[\cdot,\cdot]\widehat{\xi}_n + g_y[\cdot,\cdot]\widehat{\zeta}_n + g_u[\cdot,\cdot]\widehat{v}_n + \frac{o(t_n)}{t_n} \in \frac{1}{t_n}\big(K_1 - g[\cdot,\cdot]\big) \subseteq \text{cone}(K_1 - g[\cdot,\cdot]) \subseteq T_{L^2}(K_1; g[\cdot,\cdot]) \tag{4.61}$$

Since $T_{L^2}(K_1; g[\cdot,\cdot])$ is a closed convex subset of $L^2(Q_1)$, it is a weakly closed set of $L^2(Q_1)$ and so $T_{L^2}(K_1; g[\cdot,\cdot])$ is sequentially weakly closed. Let us claim that

$$g_t[\cdot,\cdot]\widehat{\xi}_n + g_y[\cdot,\cdot]\widehat{\zeta}_n + g_u[\cdot,\cdot]\widehat{v}_n \rightharpoonup g_t[\cdot,\cdot]\widehat{\xi} + g_y[\cdot,\cdot]\widehat{\zeta} + g_u[\cdot,\cdot]\widehat{v} \quad \text{in } L^2(Q_1). \tag{4.62}$$

Set $M' := \|\xi_*\|_{C([0,1],\mathbb{R})} + \|\zeta_*\|_{C(\overline{Q}_1)} + \|v_*\|_{L^\infty(Q_1)}$. By the assumption $(H4)$-(i), there exists $k_{g,M'} > 0$ such that

$$\begin{aligned}
|g_t[x,s]| &\le |g_t(x,\xi_*,\zeta_*,v_*) - g_t(x,0,0,0)| + |g_t(x,0,0,0)| \le k_{g,M'}M' + \|g_t(\cdot,0,0,0)\|_{L^\infty(\Omega)},\\
|g_y[x,s]| &\le |g_y(x,\xi_*,\zeta_*,v_*) - g_y(x,0,0,0)| + |g_y(x,0,0,0)| \le k_{g,M'}M' + \|g_y(\cdot,0,0,0)\|_{L^\infty(\Omega)},\\
|g_u[x,s]| &\le |g_u(x,\xi_*,\zeta_*,v_*) - g_u(x,0,0,0)| + |g_u(x,0,0,0)| \le k_{g,M'}M' + \|g_u(\cdot,0,0,0)\|_{L^\infty(\Omega)},
\end{aligned}$$

for a.a. $(x,s) \in Q_1$. From this, for each $\eta \in L^2(Q_1)$, we have

$$\begin{aligned}
&\Big|\int_{Q1} \big(g_t[x,s]\widehat{\xi}_n + g_y[x,s]\widehat{\zeta}_n + g_u[x,s]\widehat{v}_n\big)\eta \mathrm{d}x\mathrm{d}s - \int_{Q1} \big(g_t[x,s]\widehat{\xi} + g_y[x,s]\widehat{\zeta} + g_u[x,s]\widehat{v}\big)\eta \mathrm{d}x\mathrm{d}s\Big|\\
&\le \int_{Q_1} |g_t[x,s]||\widehat{\xi}_n - \widehat{\xi}||\eta|\mathrm{d}x\mathrm{d}s + \int_{Q_1} |g_y[x,s]||\widehat{\zeta}_n - \widehat{\zeta}||\eta|\mathrm{d}x\mathrm{d}s + \Big|\int_{Q_1} g_u[x,s]\eta(\widehat{v}_n - \widehat{v})\mathrm{d}x\mathrm{d}s\Big|\\
&\le (k_{g,M'}M' + \|g_t(\cdot,0,0,0)\|_{L^\infty(\Omega)}) \int_{Q_1} |\widehat{\xi}_n - \widehat{\xi}||\eta|\mathrm{d}x\mathrm{d}s\\
&+ (k_{g,M'}M' + \|g_y(\cdot,0,0,0)\|_{L^\infty(\Omega)}) \int_{Q_1} |\widehat{\zeta}_n - \widehat{\zeta}||\eta|\mathrm{d}x\mathrm{d}s + \Big|\int_{Q_1} g_u[x,s]\eta(\widehat{v}_n - \widehat{v})\mathrm{d}x\mathrm{d}s\Big|\\
&\le (k_{g,M'}M' + \|g_t(\cdot,0,0,0)\|_{L^\infty(\Omega)})|Q_1|^{\frac{1}{2}}\|\eta\|_{L^2(Q_1)}\|\widehat{\xi}_n - \widehat{\xi}\|_{C([0,1],\mathbb{R})}\\
&+ (k_{g,M'}M' + \|g_y(\cdot,0,0,0)\|_{L^\infty(\Omega)})\|\eta\|_{L^2(Q_1)}\|\widehat{\zeta}_n - \widehat{\zeta}\|_{L^2(Q_1)} + \Big|\int_{Q_1} g_u[x,s]\eta(\widehat{v}_n - \widehat{v})\mathrm{d}x\mathrm{d}s\Big|
\end{aligned} \tag{4.63}$$

Since $g_u[\cdot,\cdot] \in L^\infty(Q_1)$ and $\eta \in L^2(Q_1)$, we have $g_u[\cdot,\cdot]\eta \in L^2(Q_1)$. Combining this with $\widehat{v}_n \rightharpoonup \widehat{v}$ in $L^2(Q_1)$, we deduce that

$$\int_{Q_1} g_u[x,s]\eta(\widehat{v}_n - \widehat{v})\mathrm{d}x\mathrm{d}s \to 0 \quad \text{as } n \to +\infty. \tag{4.64}$$

It follows from (4.63), (4.64) and facts $\widehat{\xi}_n \to \widehat{\xi}$ in $C([0,1],\mathbb{R})$, $\widehat{\zeta}_n \to \widehat{\zeta}$ in $L^2(Q_1)$ that

$$\int_{Q1} \big(g_t[x,s]\widehat{\xi}_n + g_y[x,s]\widehat{\zeta}_n + g_u[x,s]\widehat{v}_n\big)\eta \mathrm{d}x\mathrm{d}s \to \int_{Q1} \big(g_t[x,s]\widehat{\xi} + g_y[x,s]\widehat{\zeta} + g_u[x,s]\widehat{v}\big)\eta \mathrm{d}x\mathrm{d}s,\ \forall \eta \in L^2(Q_1), \tag{4.65}$$

which implies that (4.62) is valid. The claim is justified.
Letting $n \to \infty$ in (4.61), we obtain

$$g_t[\cdot,\cdot]\widehat{\xi} + g_y[\cdot,\cdot]\widehat{\zeta} + g_u[\cdot,\cdot]\widehat{v} \in T_{L^2}(K_1; g[\cdot,\cdot]). \tag{4.66}$$

By Lemma 2.4 in [43], we have

$$T_{L^2}(K_1; g[\cdot,\cdot]) = \{v \in L^2(Q_1) : v(x,s) \in T([a,b]; g[x,s]) \quad \text{a.a. } (x,s) \in Q_1\}.$$

Combining this with (4.66), yields

$$g_t[x,s]\xi(s) + g_y[x,s]\zeta(x,s) + g_u[x,s)v(x,s) \in T([a,b], g[x,s]) \quad \text{a.a. } (x,s) \in Q_1.$$

Hence condition (b_5') is valid. Consequently, $(\widehat{\xi}, \widehat{T}, \widehat{\zeta}, \widehat{v}) \in \mathcal{C}_1'[(\xi_*, T_*, \zeta_*, v_*)]$.

Step 2. Proving that $(\widehat{\xi}, \widehat{T}, \widehat{\zeta}, \widehat{v}) = (0,0,0,0)$.

Since $F(\xi_n, T_n, \zeta_n, v_n) = F(\xi_*, T_*, \zeta_*, v_*) = 0$ and integration by part, we have from (4.50) that

$$\langle (\varphi, \phi), F(\xi_n, T_n, \zeta_n, v_n) - F(\xi_*, T_*, \zeta_*, v_*)\rangle = 0.$$

Since $\langle \mu, H(\xi_*, T_*, \zeta_*, v_*)\rangle = 0$ and $\mu_i \geq 0$, $H_i(\xi_n, T_n, \zeta_n, v_n) \leq 0$, $i = 1, 2, ..., m$, we have

$$\langle \mu, H(\xi_n, T_n, \zeta_n, v_n) - H(\xi_*, T_*, \zeta_*, v_*)\rangle \leq 0.$$

Since $e \in N(K_1, g[\cdot,\cdot])$, we get

$$\langle e, G(\xi_n, T_n, \zeta_n, v_n) - G(\xi_*, T_*, \zeta_*, v_*)\rangle \leq 0.$$

From these facts and definition of $\mathcal{L}$ in (4.49), we obtain

$$\begin{aligned}
\mathcal{L}\big((\xi_n, T_n, \zeta_n, v_n), \varphi, \phi, \mu, e\big) &- \mathcal{L}\big((\xi_*, T_*, \zeta_*, v_*), \varphi, \phi, \mu, e\big) \\
&= \widehat{J}(\xi_n, T_n, \zeta_n, v_n) - \widehat{J}(\xi_*, T_*, \zeta_*, v_*) \\
&+ \langle (\varphi, \phi), F(\xi_n, T_n, \zeta_n, v_n) - F(\xi_*, T_*, \zeta_*, v_*)\rangle \\
&+ \langle \mu, H(\xi_n, T_n, \zeta_n, v_n) - H(\xi_*, T_*, \zeta_*, v_*)\rangle \\
&+ \langle e, G(\xi_n, T_n, \zeta_n, v_n) - G(\xi_*, T_*, \zeta_*, v_*)\rangle \\
&\leq \widehat{J}(\xi_n, T_n, \zeta_n, v_n) - \widehat{J}(\xi_*, T_*, \zeta_*, v_*) \leq o(t_n^2).
\end{aligned} \tag{4.67}$$

Here the last estimate follows from (4.54). Using (4.67), (4.51) and Taylor expansions for four terms in (4.67) (see also Lem. A in the appendix for similar techniques), and collecting terms of $\widehat{v}_n^2$ into a group, we get

$$\begin{aligned}
\frac{o(t_n^2)}{t_n^2} &\geq \frac{2}{t_n^2}\Big\{\mathcal{L}\Big((\xi_n, \zeta_n, v_n, T_n), \varphi, \phi, \mu, e\Big) - \mathcal{L}\Big((\xi_*, \zeta_*, v_*, T_*), \varphi, \phi, \mu, e\Big)\Big\} \\
&= \int_\Omega \Big(L_{0TT}(x, T_* + \beta_{01}(T_n - T_*), \zeta_n(1))\widehat{T}_n^2 + 2L_{0T\zeta}(x, T_*, \zeta_*(1) + \beta_{02}(\zeta_n(1) - \zeta_*(1)))\widehat{T}_n\widehat{\zeta}_n(1) \\
&+ L_{0\zeta\zeta}(x, T_*, \zeta_*(1) + \beta_{03}(\zeta_n(1) - \zeta_*(1)))\widehat{\zeta}_n^2(1)\Big)\mathrm{d}x \\
&+ \int_{Q_1} T_n\Big[L_{tt}(x, \xi_* + \alpha_1(\xi_n - \xi_*), \zeta_n, v_n)\widehat{\xi}_n^2 + L_{yy}(x, \xi_*, \zeta_* + \alpha_2(\zeta_n - \zeta_*), v_n)\widehat{\zeta}_n^2\Big]\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_1} T_n\Big[L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*))\widehat{v}_n^2+2L_{ty}(x,\xi_*,\zeta_*+\alpha_4(\zeta_n-\zeta_*),v_n)\widehat{\xi}_n\widehat{\zeta}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} T_n\Big[2L_{tu}(x,\xi_*,\zeta_*,v_*+\alpha_5(v_n-v_*))\widehat{\xi}_n\widehat{v}_n+2L_{yu}(x,\xi_*,\zeta_*,v_*+\alpha_6(v_n-v_*))\widehat{\zeta}_n\widehat{v}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} 2\widehat{T}_n\Big[L_t[x,s]\widehat{\xi}_n+L_y[x,s]\widehat{\zeta}_n+L_u[x,s]\widehat{v}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} \varphi T_n\Big[\psi_{tt}(x,\xi_*+\alpha_7(\xi_n-\xi_*),\zeta_n)\widehat{\xi}_n^2+\psi_{yy}(x,\xi_*,\zeta_*+\alpha_8(\zeta_n-\zeta_*))\widehat{\zeta}_n^2\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} \Big[2\varphi T_n\psi_{ty}(x,\xi_*,\zeta_*+\alpha_9(\zeta_n-\zeta_*))\widehat{\xi}_n\widehat{\zeta}_n+2\varphi\widehat{T}_n\Big(A\widehat{\zeta}_n+\psi_t[x,s]\widehat{\xi}_n+\psi_y[x,s]\widehat{\zeta}_n-\widehat{v}_n\Big)\Big]\mathrm{d}x\mathrm{d}s\\
&+\sum_{i=1}^m\mu_i\int_\Omega\Big(L_{iTT}(x,T_*+\beta_{i1}(T_n-T_*),\zeta_n(1))\widehat{T}_n^2+L_{iT\zeta}(x,T_*,\zeta_*(1)+\beta_{i2}(\zeta_n(1)-\zeta_*(1)))\widehat{T}_n\widehat{\zeta}_n(1)\\
&+L_{i\zeta\zeta}(x,T_*,\zeta_*(1)+\beta_{i3}(\zeta_n(1)-\zeta_*(1)))\widehat{\zeta}_n^2(1)\Big)\mathrm{d}x\\
&+\int_{Q_1} e(x,s)\Big(g_{tt}(x,\xi_*+\alpha_{10}(\xi_n-\xi_*),\zeta_n,v_n)\widehat{\xi}_n^2+g_{yy}(x,\xi_*,\zeta_*+\alpha_{11}(\zeta_n-\zeta_*),v_n)\widehat{\zeta}_n^2\Big)\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} e(x,s)\Big(g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))\widehat{v}_n^2+2g_{ty}(x,\xi_*,\zeta_*+\alpha_{13}(\zeta_n-\zeta_*),v_n)\widehat{\xi}_n\widehat{\zeta}_n\Big)\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} e(x,s)\Big(2g_{tu}(x,\xi_*,\zeta_*,v_*+\alpha_{14}(v_n-v_*))\widehat{\xi}_n\widehat{v}_n+2g_{yu}(x,\xi_*,\zeta_*,v_*+\alpha_{15}(v_n-v_*))\widehat{\zeta}_n\widehat{v}_n\Big)\mathrm{d}x\mathrm{d}s\\
&=:\Sigma_n+\int_{Q_1}\big[T_nL_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*))\widehat{v}_n^2+e(x,s)g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))\widehat{v}_n^2\big]\mathrm{d}x\mathrm{d}s,
\end{aligned}
\tag{4.68}
$$

where $0\le\alpha_i,\beta_{kj}\le 1$, $i=1,2,...,15$, $k=0,1,2,...,m$, $j=1,2,3$, and

$$
\begin{aligned}
\Sigma_n=&\int_\Omega\Big(L_{0TT}(x,T_*+\beta_{01}(T_n-T_*),\zeta_n(1))\widehat{T}_n^2+2L_{0T\zeta}(x,T_*,\zeta_*(1)+\beta_{02}(\zeta_n(1)-\zeta_*(1)))\widehat{T}_n\widehat{\zeta}_n(1)\\
&+L_{0\zeta\zeta}(x,T_*,\zeta_*(1)+\beta_{03}(\zeta_n(1)-\zeta_*(1)))\widehat{\zeta}_n^2(1)\Big)\mathrm{d}x\\
&+\int_{Q_1} T_n\Big[L_{tt}(x,\xi_*+\alpha_1(\xi_n-\xi_*),\zeta_n,v_n)\widehat{\xi}_n^2+L_{yy}(x,\xi_*,\zeta_*+\alpha_2(\zeta_n-\zeta_*),v_n)\widehat{\zeta}_n^2\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} T_n\Big[2L_{ty}(x,\xi_*,\zeta_*+\alpha_4(\zeta_n-\zeta_*),v_n)\widehat{\xi}_n\widehat{\zeta}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} T_n\Big[2L_{tu}(x,\xi_*,\zeta_*,v_*+\alpha_5(v_n-v_*))\widehat{\xi}_n\widehat{v}_n+2L_{yu}(x,\xi_*,\zeta_*,v_*+\alpha_6(v_n-v_*))\widehat{\zeta}_n\widehat{v}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} 2\widehat{T}_n\Big[L_t[x,s]\widehat{\xi}_n+L_y[x,s]\widehat{\zeta}_n+L_u[x,s]\widehat{v}_n\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} \varphi T_n\Big[\psi_{tt}(x,\xi_*+\alpha_7(\xi_n-\xi_*),\zeta_n)\widehat{\xi}_n^2+\psi_{yy}(x,\xi_*,\zeta_*+\alpha_8(\zeta_n-\zeta_*))\widehat{\zeta}_n^2\Big]\mathrm{d}x\mathrm{d}s\\
&+\int_{Q_1} \Big[2\varphi T_n\psi_{ty}(x,\xi_*,\zeta_*+\alpha_9(\zeta_n-\zeta_*))\widehat{\xi}_n\widehat{\zeta}_n+2\varphi\widehat{T}_n\Big(A\widehat{\zeta}_n+\psi_t[x,s]\widehat{\xi}_n+\psi_y[x,s]\widehat{\zeta}_n-\widehat{v}_n\Big)\Big]\mathrm{d}x\mathrm{d}s\\
&+\sum_{i=1}^m\mu_i\int_\Omega\Big(L_{iTT}(x,T_*+\beta_{i1}(T_n-T_*),\zeta_n(1))\widehat{T}_n^2+2L_{iT\zeta}(x,T_*,\zeta_*(1)+\beta_{i2}(\zeta_n(1)-\zeta_*(1)))\widehat{T}_n\widehat{\zeta}_n(1)
\end{aligned}
$$

$$+ L_{i\zeta\zeta}(x, T_*, \zeta_*(1) + \beta_{i3}(\zeta_n(1) - \zeta_*(1))\widehat{\zeta}_n^2(1)\Big)\mathrm{d}x$$
$$+ \int_{Q_1} e(x,s)\Big(g_{tt}(x, \xi_* + \alpha_{10}(\xi_n - \xi_*), \zeta_n, v_n)\widehat{\xi}_n^2 + g_{yy}(x, \xi_*, \zeta_* + \alpha_{11}(\zeta_n - \zeta_*), v_n)\widehat{\zeta}_n^2\Big)\mathrm{d}x\mathrm{d}s$$
$$+ \int_{Q_1} e(x,s)\Big(2g_{ty}(x, \xi_*, \zeta_* + \alpha_{13}(\zeta_n - \zeta_*), v_n)\widehat{\xi}_n\widehat{\zeta}_n\Big)\mathrm{d}x\mathrm{d}s$$
$$+ \int_{Q_1} e(x,s)\Big(2g_{tu}(x, \xi_*, \zeta_*, v_* + \alpha_{14}(v_n - v_*))\widehat{\xi}_n\widehat{v}_n + 2g_{yu}(x, \xi_*, \zeta_*, v_* + \alpha_{15}(v_n - v_*))\widehat{\zeta}_n\widehat{v}_n\Big)\mathrm{d}x\mathrm{d}s.$$

It follows from (4.68) that

$$0 \geq \lim_{n\to\infty}\Big[\Sigma_n + \int_{Q_1}\big[T_n L_{uu}(x, \xi_*, \zeta_*, v_* + \alpha_3(v_n - v_*))\widehat{v}_n^2 + e(x,s)g_{uu}(x, \xi_*, \zeta_*, v_* + \alpha_{12}(v_n - v_*))\widehat{v}_n^2\big]\mathrm{d}x\mathrm{d}s\Big]. \tag{4.69}$$

Recall that $\|\xi_n - \xi_*\|_{C([0,1],\mathbb{R})} \to 0$, $\|\zeta_n - \zeta_*\|_{L^\infty(Q_1)} \to 0$, $\|v_n - v_*\|_{L^\infty(Q_1)} \to 0$, $|T_n - T_*| \to 0$, $\|\widehat{\zeta}_n - \widehat{\zeta}\|_{L^2(Q_1)} \to 0$, $|\widehat{T}_n - \widehat{T}| \to 0$, $\widehat{v}_n \rightharpoonup \widehat{v}$ in $L^2(Q_1)$ and $\|\widehat{\zeta}_n(1) - \widehat{\zeta}(1)\|_{L^2(\Omega)} \to 0$. Combining these facts with $(H4)$, we get

$$\begin{aligned}
\lim_{n\to\infty}\Sigma_n = &\int_\Omega\Big(L_{0TT}[T_*,1]\widehat{T}^2 + 2L_{0T\zeta}[T_*,1]\widehat{T}\widehat{\zeta}(1) + L_{0\zeta\zeta}[T_*,1]\widehat{\zeta}^2(1)\Big)\mathrm{d}x\\
&+ \int_{Q_1} T_*\Big[L_{tt}[x,s]\widehat{\xi}^2 + L_{yy}[x,s]\widehat{\zeta}^2\Big]\mathrm{d}x\mathrm{d}s + \int_{Q_1} T_*\Big[2L_{ty}[x,s]\widehat{\xi}\widehat{\zeta}\Big]\mathrm{d}x\mathrm{d}s\\
&+ \int_{Q_1} T_*\Big[2L_{tu}[x,s]\widehat{\xi}\widehat{v} + 2L_{yu}[x,s]\widehat{\zeta}\widehat{v}\Big]\mathrm{d}x\mathrm{d}s + \int_{Q_1} 2\widehat{T}\Big[L_t[x,s]\widehat{\xi} + L_y[x,s]\widehat{\zeta} + L_u[x,s]\widehat{v}\Big]\mathrm{d}x\mathrm{d}s\\
&+ \int_{Q_1}\varphi\Big[T_*\Big(\psi_{tt}[x,s]\xi^2 + \psi_{yy}[x,s]\zeta^2 + 2\psi_{ty}[x,s]\xi\zeta\Big) + 2T\Big(A\zeta + \psi_t[x,s]\xi + \psi_y[x,s]\zeta - v\Big)\Big]\mathrm{d}x\mathrm{d}s\\
&+ \sum_{i=1}^m \mu_i\int_\Omega\Big(L_{iTT}[T_*,1]\widehat{T}^2 + 2L_{iT\zeta}[T_*,1]\widehat{T}\widehat{\zeta}(1) + L_{i\zeta\zeta}[T_*,1]\widehat{\zeta}^2(1)\Big)\mathrm{d}x\\
&+ \int_{Q_1} e(x,s)\big(g_{tt}[x,s]\widehat{\xi}^2 + g_{yy}[x,s]\widehat{\zeta}^2 + 2g_{ty}[x,s]\widehat{\xi}\widehat{\zeta} + 2g_{tu}[x,s]\widehat{\xi}\widehat{v} + 2g_{yu}[x,s]\widehat{\zeta}\widehat{v}\big)\mathrm{d}x\mathrm{d}s\\
&=: \Sigma.
\end{aligned} \tag{4.70}$$

We now deal with the second term in (4.69). By using assumption (vi), we see that the function

$$L^2(Q_1) \ni v \mapsto \int_{Q_1}\big[T_* L_{uu}[x,s] + e(x,s)g_{uu}[x,s]\big]v^2(x,s)\mathrm{d}x\mathrm{d}s$$

is convex and so it is sequentially lower semicontinuous. Hence

$$\liminf_{n\to\infty}\int_{Q_1}\big[T_* L_{uu}[x,s] + e(x,s)g_{uu}[x,s]\big]\widehat{v}_n^2(x,s)\mathrm{d}x\mathrm{d}s \geq \int_{Q_1}\big[T_* L_{uu}[x,s] + e(x,s)g_{uu}[x,s]\big]\widehat{v}^2(x,s)\mathrm{d}x\mathrm{d}s.$$

This implies that

$$\liminf_{n\to\infty}\int_{Q_1}\Big(T_n L_{uu}(x, \xi_*, \zeta_*, v_* + \alpha_3(v_n - v_*)) + e g_{uu}(x, \xi_*, \zeta_*, v_* + \alpha_{12}(v_n - v_*))\Big)\widehat{v}_n^2\mathrm{d}x\mathrm{d}s$$

$$\geq \liminf_{n\to\infty} \int_{Q_1} \Big(T_n L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*)) + e g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))$$
$$- T_* L_{uu}[x,s] - e g_{uu}[x,s]\Big)\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s + \liminf_{n\to\infty}\int_{Q_1}\Big(T_* L_{uu}[x,s] + e g_{uu}[x,s]\Big)\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$\geq \int_{Q_1}\Big(T_* L_{uu}[x,s] + e g_{uu}[x,s]\Big)\widehat{v}^2 \mathrm{d}x\mathrm{d}s.$$

Combining this with (4.70) and (4.69) we have

$$0 \geq \liminf_{n\to\infty} \Sigma_n$$
$$+ \liminf_{n\to\infty}\Big[\int_{Q_1}\big[T_n L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*))\widehat{v}_n^2 + e(x,s) g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))\widehat{v}_n^2\big]\mathrm{d}x\mathrm{d}s\Big]$$
$$\geq \Sigma + \int_{Q_1}\Big(T_* L_{uu}[x,s] + e g_{uu}[x,s]\Big)\widehat{v}^2 \mathrm{d}x\mathrm{d}s$$
$$= D^2_{zz}\mathcal{L}(z_*,\varphi,\mu,\phi,e)\big[(\widehat{\xi},\widehat{T},\widehat{\zeta},\widehat{v}),(\widehat{\xi},,\widehat{T},\widehat{\zeta},\widehat{v})\big].$$

From this and strictly second-order condition (4.52), we conclude that $(\widehat{\xi},\widehat{T},\widehat{\zeta},\widehat{v}) = (0,0,0,0)$.

Step 3. Showing a contradiction.

From Step 2, we have $T_n \to T_*$, $|\widehat{T}_n|^2 \to 0$, $\widehat{\xi}_n \to 0$ in $C([0,1],\mathbb{R})$ and $\widehat{\zeta}_n \to 0$ in $L^2(Q_1)$. Hence $\lim_{n\to\infty}\Sigma_n = \Sigma = 0$. Note that $|\widehat{T}_n|^2 + \|\widehat{v}_n\|^2_{L^2(Q_1)} = 1$. Using these facts and (4.47), we have from (4.69) that

$$0 \geq \liminf_{n\to\infty} \Sigma_n$$
$$+ \liminf_{n\to\infty}\int_{Q_1}\big[T_n L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*)) + e(x,s) g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))\big]\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$\geq 0 + \liminf_{n\to\infty}\int_{Q_1}\big[T_n L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*)) + e(x,s) g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))\big]\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$\geq \liminf_{n\to\infty}\int_{Q_1}\Big[T_n L_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_3(v_n-v_*)) + e(x,s) g_{uu}(x,\xi_*,\zeta_*,v_*+\alpha_{12}(v_n-v_*))$$
$$- T_* L_{uu}[x,s] - e(x,s) g_{uu}[x,s]\Big]\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$+ \liminf_{n\to\infty}\int_{Q_1}\big[T_* L_{uu}[x,s] - e(x,s) g_{uu}[x,s]\big]\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$= 0 + \liminf_{n\to\infty}\int_{Q_1}\big[T_* L_{uu}[x,s] - e(x,s) g_{uu}[x,s]\big]\widehat{v}_n^2 \mathrm{d}x\mathrm{d}s$$
$$\geq \liminf_{n\to\infty}\int_{Q_1}\Lambda v_n^2 \mathrm{d}x\mathrm{d}s$$
$$= \liminf_{n\to\infty}\Big(\int_{Q_1}\Lambda v_n^2 \mathrm{d}x\mathrm{d}s + \Lambda|\widehat{T}_n|^2\Big) = \liminf_{n\to\infty}\Lambda\big(\|\widehat{v}_n\|^2_{L^2(Q_1)} + |\widehat{T}_n|^2\big) \geq \Lambda$$

which is absurd. The proof of proposition is complete. □

5. Proof of main results

We first prove the following lemmas.

Lemma 5.1. *If* $(T, y, u) \in \mathcal{C}[(T_*, y_*, u_*)]$, *then* $(\xi, T, \zeta, v) \in \mathcal{C}_1[(\xi_*, T_*, \zeta_*, v_*)]$, *where* $\xi(s) = Ts$, $\zeta(x, s) = y(x, T_*s)$, $v(x, s) = u(x, T_*s)$. *In particular, if* $(T, y, u) \in \mathcal{C}_0[(T_*, y_*, u_*)]$, *then* $(\xi, T, \zeta, v) \in \mathcal{C}_{1,0}[(\xi_*, T_*, \zeta_*, v_*)]$.

Proof. Assume that $(T, y, u) \in \mathcal{C}[(T_*, y_*, u_*)]$. By definition of $\mathcal{C}[(T_*, y_*, u_*)]$, there exists a sequence $\{(T_k, y_k, u_k)\} \subset \mathcal{C}_0[(T_*, y_*, u_*)]$ converging to (T, y, u). Since $\{(T_k, y_k, u_k)\} \subset \mathcal{C}_0[(T_*, y_*, u_*)]$, it satisfies the following conditions:

(c_1) $\int_\Omega \big(L_{0T}[T_*, T_*]T_k + L_{0\zeta}[T_*, T_*]y_k(T_*)\big)\mathrm{d}x + \int_{Q_{T_*}} \Big(\frac{T_k}{T_*}L[x, t] + L_t[x, t]\frac{T_k t}{T_*} + L_y[x, t]y_k + L_u[x, t]u_k\Big)\mathrm{d}x\mathrm{d}t \leq 0$;

(c_2) $\frac{\partial y_k}{\partial t} + Ay_k + \psi_t[x, t]\frac{T_k t}{T_*} + \psi_y[x, t]y_k - u_k + \frac{T_k}{T_*}(Ay_* + \psi[x, t] - u_*) = 0, \quad y(0) = 0$;

(c_3) $\int_\Omega \big(L_{iT}[T_*, T_*]T_k + L_{i\zeta}[T_*, T_*]y_k(T_*)\big)\mathrm{d}x \leq 0$ for $i \in \{1, 2, .., m : \psi_i[T_*, T_*] = 0\}$;

(c_4) $g_t[\cdot, \cdot]\frac{T_k t}{T_*} + g_y[\cdot, \cdot]y_k + g_u[\cdot, \cdot]u_k \in \mathrm{cone}\big(K_{T_*} - g[\cdot, \cdot]\big)$.

We define $\xi_k = T_k s$, $\zeta_k(x, s) = y_k(x, \xi_*(s))$ and $v_k(x, s) = u_k(x, \xi_*(s))$. By changing variable $t = \xi_*(s) = T_*s$, we obtain

(b_1) $\int_\Omega \big(L_{0T}[T_*, 1]T_k + L_{0\zeta}[T_*, 1]\zeta_k(1)\big)\mathrm{d}x + \int_{Q_1} \Big(T_k L[x, s] + T_*\big(L_t[x, s]\xi_k + L_y[x, s]\zeta_k + L_u[x, s]v_k\big)\Big)\mathrm{d}x\mathrm{d}s \leq 0$;

(b_2) $\frac{\partial \zeta_k}{\partial s} + T_*\big(A\zeta_k + \psi_t[\cdot, \cdot]\xi_k + \psi_y[\cdot, \cdot]\zeta_k - v_k\big) + T_k(A\zeta_* + \psi[\cdot, \cdot] - v_*) = 0, \quad \zeta(0) = 0$;

(b_3) $\xi_k(s) = T_k s$ for all $s \in [0, 1]$;

(b_4) $\int_\Omega \big(L_{iT}[T_*, 1]T_k + L_{i\zeta}[T_*, 1]\zeta_k(1)\big)\mathrm{d}x \leq 0 \quad$ for $i \in \{1, 2, .., m : \psi_i[T_*, 1] = 0\}$;

(b_5) $g_t[\cdot, \cdot]\xi_k + g_y[\cdot, \cdot]\zeta_k + g_u[\cdot, \cdot]v_k \in \mathrm{cone}\big(K_1 - g[\cdot, \cdot]\big)$.

This implies that $\{(\xi_k, T_k, \zeta_k, v_k)\} \subset \mathcal{C}_{1,0}[(\xi_*, T_*, \zeta_*, v_*)]$. It is clear that $(\xi_k, T_k, \zeta_k, v_k) \to (\xi, T, \zeta, v)$. Hence, $(\xi, T, \zeta, v) \in \mathcal{C}_1[(\xi_*, T_*, \zeta_*, v_*)]$. The lemma is proved. □

Lemma 5.2. *Suppose* $T_* > 0$, $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$ *and* $(\xi, T, \zeta, v) \in \mathcal{C}_1'[(\xi_*, T_*, \zeta_*, v_*)]$. *Then* $(T, y, u) \in \mathcal{C}'[(T_*, y_*, u_*)]$, *where* $y(x, t) := \zeta(x, \frac{t}{T_*}), u(x, t) := v(x, \frac{t}{T_*}), y_*(x, t) := \zeta_*(x, \frac{t}{T_*})$ *and* $u_* := v_*(x, \frac{t}{T_*})$.

Proof. By definition of Φ_1, $(\xi_*, T_*, \zeta_*, v_*) \in C([0, 1], \mathbb{R}) \times \mathbb{R} \times Y_1 \times U_1$ and satisfies the following constraints:

$$
\begin{aligned}
&\frac{\partial \zeta_*}{\partial s} + T_*A\zeta_* + T_*\psi(x, \xi_*, \zeta_*) = T_*v_* \quad \text{in } Q_1, \quad \zeta_*(x, s) = 0 \quad \text{on } \Sigma_1 = \Gamma \times [0, 1],\\
&\zeta_*(0) = y_0 \quad \text{in } \Omega,\\
&\xi_*(s) = T_*s \text{ for all } s \in [0, 1],\\
&\psi_i(T_*, \zeta_*(1)) \leq 0, \quad i = 1, 2, ..., m,\\
&a \leq g(x, \xi_*(s), \zeta_*(x, s), v_*(x, s)) \leq b \quad \text{a.a. } x \in \Omega,\ \forall s \in [0, 1].
\end{aligned}
$$

By definition of y_*, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}y_*(x, t) = \frac{\mathrm{d}}{\mathrm{d}s}\zeta_*(x, \frac{t}{T_*})\frac{1}{T_*}.$$

Changing variable $s = \frac{t}{T_*}$ with $t \in [0, T_*]$ in the above constraints, we obtain

$$\frac{\partial y_*}{\partial t} + Ay_* + \psi(x, t, y_*) = u_* \quad \text{in } Q_{T_*}, \quad y_*(x, t) = 0 \quad \text{on } \Sigma_{T_*} = \Gamma \times [0, T_*],$$

$$y_*(0) = y_0 \quad \text{in } \Omega,$$
$$\psi_i(T_*, y_*(T_*)) \le 0, \quad i = 1, 2, ..., m,$$
$$a \le g(x, t, y_*(x,t), u_*(x,t)) \le b \quad \text{a.a. } x \in \Omega,\ t \in [0, T_*].$$

This implies that $(T_*, y_*, u_*) \in \Phi$. We now assume that $(\xi, T, \zeta, v) \in \mathcal{C}_1'[(\xi_*, T_*, \zeta_*, v_*)]$. By definition, $(\xi, T, \zeta, v) \in C([0,1], \mathbb{R}) \times \mathbb{R} \times W_2^{1,1}(0,1; D, H) \times L^2(Q_1)$ and satisfies conditions $(b_1') - (b_5')$. Namely, we have

$$(b_1)' \int_\Omega \big(L_{0T}[T_*, 1]T + L_{0\zeta}[T_*, 1]\zeta(1)\big)\mathrm{d}x + \int_{Q_1} \Big(TL[x,s] + T_*[L_t[x,s]\xi(s) + L_y[x,s]\zeta + L_u[x,s]v]\Big)\mathrm{d}x\mathrm{d}s \le 0;$$
$$(b_2)' \frac{\partial \zeta}{\partial s} + T_*\big(A\zeta + \psi_t[\cdot,\cdot]\xi + \psi_y[\cdot,\cdot]\zeta - v\big) + T(A\zeta_* + \psi[\cdot,\cdot] - v_*) = 0, \quad \zeta(0) = 0;$$
$$(b_3)' \ \xi(s) = Ts \quad \forall s \in [0,1];$$
$$(b_4)' \int_\Omega \big(L_{iT}[T_*, 1]T + L_{i\zeta}[T_*, 1]\zeta(1)\big)\mathrm{d}x \le 0 \text{ for } i \in \big\{1, 2, .., m : \psi_i[T_*, 1] = 0\big\};$$
$$(b_5)' \ g_t[x,s]\xi(s) + g_y[x,s]\zeta(x,s) + g_u[x,s)v(x,s) \in T([a,b], g[x,s]) \text{ for a.a. } (x,s) \in Q_1.$$

Changing variable $s = \frac{t}{T_*}$ with $t \in [0, T_*]$ and putting $y(x,t) = \zeta(x, \frac{t}{T_*})$ and $u = v(x, \frac{t}{T_*})$, we obtain

$$(c_1) \int_\Omega \big(L_{0T}[T_*, T_*]T + L_{0y}[T_*, T_*]y(T_*)\big)\mathrm{d}x + \int_{Q_{T_*}} \big(\frac{T}{T_*}L[x,t] + [L_t[x,t]\frac{Tt}{T_*} + L_y[x,t]y + L_u[x,t]u]\big)\mathrm{d}x\mathrm{d}t \le 0;$$
$$(c_2) \ \frac{\partial y}{\partial t} + Ay + \psi_t[x,t]\frac{Tt}{T_*} + \psi_y[x,t]y - u + \frac{T}{T_*}(Ay_* + \psi[x,t] - u_*) = 0, \quad y(0) = 0;$$
$$(c_3) \int_\Omega \big(L_{iT}[T_*, T_*]T + L_{i\zeta}[T_*, T_*]y(T_*)\big)\mathrm{d}x \le 0 \text{ for } i \in \big\{1, 2, .., m : \psi_i[T_*, T_*] = 0\big\};$$
$$(c_4') \ g_t[x,t]\frac{Tt}{T_*} + g_y[x,t]y(x,t) + g_u[x,t)u(x,t) \in T([a,b], g[x,t]) \text{ for a.a. } (x,t) \in Q_{T_*}.$$

Hence $(T, y, u) \in \mathcal{C}'[(T_*, y_*, u_*)]$. The lemma is proved. □

5.1. Proof of Theorem 3.1

Let $(T_*, y_*, u_*) \in \Phi$ be a locally optimal solution to (P). As before, we define

$$\xi_*(s) = T_* s, \quad \zeta_*(x,s) = y_*(x, \xi_*(s)), \quad v_*(x,s) = u_*(x, \xi_*(s)).$$

By Proposition 4.2, vector $(\xi_*, T_*, \zeta_*, v_*)$ is a locally optimal solution to (P_1) and $\widehat{J}(\xi_*, T_*, \zeta_*, v_*) = J(T_*, y_*, u_*)$. According to Proposition 4.7, there exist Lagrange multipliers $\lambda \in \mathbb{R}_+$, $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, $\varphi \in L^\infty(Q_1) \cap L^2(0,1; H_0^1(\Omega))$, $e \in L^\infty(Q_1)$ and an absolutely continuous function $\phi : [0,1] \to \mathbb{R}$ satisfying the conditions (4.24)–(4.29). Define

$$\widetilde{\varphi}(x,t) = \varphi(x, \frac{t}{T_*}), \quad \widetilde{\phi}(t) = \phi(\frac{t}{T_*}), \quad \widetilde{e}(x,t) = \frac{1}{T_*}e(x, \frac{t}{T_*}).$$

By replacing $s = \xi_*^{-1}(t) = \frac{t}{T_*}$ into (4.24)–(4.28), we can show that $(\lambda, \mu, \widetilde{\varphi}, \widetilde{\phi}, \widetilde{e})$ satisfies conditions (i)–(iv) of Theorem 3.3.

Let $d = (T, y, u) \in \mathcal{C}_0[(T_*, y_*, u_*)]$. By Lemma 5.1, vector $z := (\xi, T, \zeta, v) \in \mathcal{C}_{1,0}[(\xi_*, T_*, \zeta_*, v_*)]$, where $\xi(s) = Ts$, $\zeta(x,s) = y(x, \xi_*(s))$, $v(x,s) = u(x, \xi_*(s))$. Then condition (v) of Proposition 4.7 is valid for $z = (\xi, T, \zeta, v)$. By changing variable $s = \frac{t}{T_*}$ in (4.29), we obtain condition (v) of Theorem 3.3.

The last assertion of Theorem 3.3 follows from the last assertion of Proposition 4.7 and Lemma 4.6. The proof of Theorem 3.3 is complete. □

5.2. Proof of Theorem 3.2

Let $(T_*, y_*, u_*) \in \Phi$. We define

$$\xi_*(s) = T_*s, \quad \zeta_*(x,s) = y_*(x, T_*s), \quad v_*(x,s) = u_*(x, T_*s) \tag{5.1}$$

for all $s \in [0,1]$. By a simple argument, we can show that $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$. Besides, we also have $y_*(x,t) = \zeta_*(x, \frac{t}{T_*})$ and $u_*(x,t) = v_*(x, \frac{t}{T_*})$ for all $t \in [0, T_*]$. Let $(1, \mu, \widetilde{\varphi}, \widetilde{\phi}, \widetilde{e})$ be multipliers which satisfy conditions (i)–(iv) of Theorem 3.3 and define

$$\varphi(x,s) = \widetilde{\varphi}(x, T_*s), \quad \phi(s) = \widetilde{\phi}(T_*s), \quad e(x,s) = T_*\widetilde{e}(x, T_*s) \tag{5.2}$$

for all $s \in [0,1]$. Then by changing variable $t = \xi_*(s) = T_*s$ in (i)–(iv) of Theorem 3.3, it is easy to check that $(1, \mu, \varphi, \phi, e)$ satisfies conclusions (i)–(iv) of Proposition 4.7. Now we take any $(\xi, T, \zeta, v) \in \mathcal{C}_1'[(\xi_*, T_*, \zeta_*, v_*)] \setminus \{(0,0,0,0)\}$. Define

$$y(x,t) = \zeta(x, \frac{t}{T_*}),\ u(x,t) = v(x, \frac{t}{T_*}) \tag{5.3}$$

for $t \in [0, T_*]$. By Lemma 5.2, we have $(T, y, u) \in \mathcal{C}'[(T_*, y_*, u_*)] \setminus \{0,0,0\}$. It follows from this and (3.10) that

$$\begin{aligned}
&\int_\Omega \Big(L_{0TT}[T_*, T_*]T^2 + 2L_{0Ty}[T_*, T_*]Ty(T_*) + L_{0yy}[T_*, T_*]y(T_*)^2\Big)\mathrm{d}x \\
&+ \int_{Q_{T_*}} \Big(L_{tt}[x,t](\frac{Tt}{T_*})^2 + 2L_{ty}[x,s]\frac{Tt}{T_*}y(x,t) + 2L_{tu}[x,t]\frac{Tt}{T_*}u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} \Big(L_{yy}[x,t]y(x,t)^2 + L_{uu}[x,t]u(x,t)^2 + 2L_{yu}[x,t]y(x,t)u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ 2\int_{Q_{T_*}} \frac{T}{T_*}\Big(L_t[x,t]\frac{Tt}{T_*} + L_y[x,t]y(x,t) + L_u[x,t]u(x,t)\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} \widetilde{\varphi}(x,t)\Big(\psi_{tt}[x,t](\frac{Tt}{T_*})^2 + 2\psi_{ty}[x,s]\frac{Tt}{T_*}y(x,t) + \psi_{yy}[x,t]y(x,t)^2\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} 2\frac{T}{T_*}\widetilde{\varphi}(x,t)\Big(Ay + \psi_t[x,t]\frac{Tt}{T_*} + \psi_y[x,t]y - u\Big)\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_{T_*}} \widetilde{e}(x,s)\big(g_{tt}[x,t](\frac{Tt}{T_*})^2 + g_{yy}[x,t]y^2 + g_{uu}[x,t]u^2 + 2g_{ty}[x,t]\frac{Tt}{T_*}y + 2g_{tu}[x,t]\frac{Tt}{T_*}u + 2g_{yu}[x,t]yu\big)\mathrm{d}x\mathrm{d}t \\
&+ \sum_{i=1}^m \mu_i \int_\Omega \Big(L_{iTT}[T_*, T_*]T^2 + 2L_{iT\zeta}[T_*, T_*]Ty(T_*) + L_{i\zeta\zeta}[T_*, T_*]y(T_*)^2\Big)\mathrm{d}x \geq 0.
\end{aligned}$$

By changing the variable $t = T_*s$ with $s \in [0,1]$ and noting that $\xi(s) = Ts$, (5.1), (5.2) and (5.3), we obtain that the condition (v) of Proposition 4.8 is satisfied. On the other hand, changing variable $t = T_*s$ in (3.11) and noting that $e(x,s) = T_*\widetilde{e}(x, T_*s)$, we obtain

$$L_{uu}[x,s] + \frac{1}{T_*}e(x,s)g_{uu}[x,s] \geq \Lambda_0 \quad \text{a.a. } (x,s) \in Q_1.$$

This implies that the condition (vi) of Proposition 4.8 is satisfied with $\Lambda = T_*\Lambda_0$. Thus $(\xi_*, T_*, \zeta_*, v_*) \in \Phi_1$ and $(1, \mu, \varphi, \phi, e) \in \Lambda[(\xi_*, T_*, \zeta_*, v_*)]$ which satisfy all sufficient conditions of Proposition 4.8. Accordingly,

$(\xi_*, T_*, \zeta_*, v_*)$ is a locally optimal solution to (P_1). Since $y_*(x,t) = \zeta_*(x, \frac{t}{T_*})$ and $u_*(x,t) = v_*(x, \frac{t}{T_*})$ for $t \in [0, T_*]$, Proposition 4.3 implies that (T_*, y_*, u_*) is a locally optimal solution to (P). Moreover, according to the conclusion of Proposition 4.8, there exist numbers $\delta > 0$ and $\kappa > 0$ such that

$$\widehat{J}(z) \geq \widehat{J}(z_*) + \kappa\Big((T - T_*)^2 + \|v - v_*\|^2_{L^2(Q_1)}\Big) \tag{5.4}$$

for all $z = (\xi, T, \zeta, v) \in \Phi_1 \cap B_Z(z_*, \delta)$. According to the proof of Proposition 4.3, there exists $\epsilon_0 \in (0, \frac{\delta}{6})$ such that, if $(T, y, u) \in \Phi$ and

$$\mathrm{dist}[(T, y, u), (T_*, y_*, u_*)] < \epsilon_0,$$

then $(\xi, T, \zeta, v) \in \Phi_1 \cap B_Z(z_*, \delta)$ with $\xi(s) = Ts$, $\zeta(x, s) = y(x, Ts)$ and $v(x, s) = u(x, Ts)$ for $s \in [0, 1]$. Therefore, for all $(T, y, u) \in \Phi \cap N\big((T_*, y_*, u_*), \epsilon_0\big)$, we have from (5.4) that

$$\begin{aligned} J(T, y, u) = \widehat{J}(z) &\geq \widehat{J}(z_*) + \kappa\Big((T - T_*)^2 + \|v - v_*\|^2_{L^2(Q_1)}\Big) \\ &= J(T_*, y_*, u_*) + \kappa\Big((T - T_*)^2 + \int_{Q_1} |u(x, Ts) - u_*(x, T_* s)|^2 \mathrm{d}x\mathrm{d}s\Big) \\ &= J(T_*, y_*, u_*) + \kappa\Big((T - T_*)^2 + \frac{1}{T_*}\int_{Q_{T_*}} |u(x, \frac{Tt}{T_*}) - u_*(x, t)|^2 \mathrm{d}x\mathrm{d}t\Big) \end{aligned}$$

which is the conclusion of Theorem 3.4. The proof of Theorem 3.4 is complete. □

Acknowledgments

The second author would like to thank the Alexander Von Humboldt Foundation for their support during his visit to the Faculty of Mathematics of the University of Duisburg-Essen.

Funding

This research was supported by Vietnam Academy of Science and Technology under grant number CTTH.00.03/25-26.

Data availability statement

The research data associated with this article are included in the article.

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Appendix A.

Proof of Fréchet differentiability of $\widehat{J}, F, H$ and G in Lemma 4.4.

Let $B_Z(z_*,\varepsilon)$ be a neighborhood of z_* in $Z=C([0,1],\mathbb{R})\times\mathbb{R}\times Y_1\times U_1$. Fix $\widehat{z}=(\widehat{\xi},\widehat{T},\widehat{\zeta},\widehat{v})\in B_Z(z_*,\varepsilon)$.

- Firstly, we prove that $\widehat{J}$ is Fréchet differentiable at $\widehat{z}$. For this, consider mapping $B:Z\to\mathbb{R}$, $h=(h_1,h_2,h_3,h_4)\mapsto Bh$,

$$\begin{aligned}Bh :=& \int_\Omega \big(L_{0T}(x,\widehat{T},\widehat{\zeta}(1))h_2 + L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))h_3(1)\big)\mathrm{d}x\\ &+\int_{Q_1}\Big(h_2L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})+\widehat{T}[L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_1+L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_3+L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_4]\Big)\mathrm{d}x\mathrm{d}s. \end{aligned}\tag{A.1}$$

Then B is linear and bounded. Taking any $\widehat{h}=(\widehat{h}_1,\widehat{h}_2,\widehat{h}_3,\widehat{h}_4)\in Z$ with $\|\widehat{h}\|_Z\le\varepsilon$. We have

$$\begin{aligned}&|\widehat{J}(\widehat{z}+\widehat{h})-\widehat{J}(\widehat{z})-B\widehat{h}|\\
&\le\left|\int_\Omega\Big[L_0(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))-L_0(x,\widehat{T},\widehat{\zeta}(1))-L_{0T}(x,\widehat{T},\widehat{\zeta}(1))\widehat{h}_2-L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))\widehat{h}_3(1)\Big]\mathrm{d}x\right|\\
&+\bigg|\int_{Q_1}\Big[(\widehat{T}+\widehat{h}_2)L(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-\widehat{T}L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\\
&-\Big(\widehat{h}_2L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})+\widehat{T}[L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_1+L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_3+L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_4]\Big)\Big]\mathrm{d}x\mathrm{d}s\bigg|\\
&\le\int_\Omega\Big|L_0(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))-L_0(x,\widehat{T},\widehat{\zeta}(1))-L_{0T}(x,\widehat{T},\widehat{\zeta}(1))\widehat{h}_2-L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))\widehat{h}_3(1)\Big|\mathrm{d}x\\
&+\int_{Q_1}\Big|(\widehat{T}+\widehat{h}_2)L(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-\widehat{T}L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\\
&-\Big(\widehat{h}_2L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})+\widehat{T}[L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_1+L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_3+L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\widehat{h}_4]\Big)\Big|\mathrm{d}x\mathrm{d}s\\
&=:\int_\Omega P_1(x)\mathrm{d}x+\int_{Q_1}P_2(x,s)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{A.2}$$

Also,

$$\begin{aligned}P_1(x) &= \Big|L_0(x,\widehat T+\widehat h_2,\widehat\zeta(1)+\widehat h_3(1)) - L_0(x,\widehat T,\widehat\zeta(1)) - L_{0T}(x,\widehat T,\widehat\zeta(1))\widehat h_2 - L_{0\zeta}(x,\widehat T,\widehat\zeta(1))\widehat h_3(1)\Big|\\ &= \Big|\Big[L_0(x,\widehat T+\widehat h_2,\widehat\zeta(1)+\widehat h_3(1)) - L_0(x,\widehat T,\widehat\zeta(1)+\widehat h_3(1))\Big] + \Big[L_0(x,\widehat T,\widehat\zeta(1)+\widehat h_3(1)) - L_0(x,\widehat T,\widehat\zeta(1))\Big]\\ &\quad - L_{0T}(x,\widehat T,\widehat\zeta(1))\widehat h_2 - L_{0\zeta}(x,\widehat T,\widehat\zeta(1))\widehat h_3(1)\Big|, \qquad (A.3)\end{aligned}$$

for a.a. $x\in\Omega$. By Taylor's expansions of $T\mapsto L_0(x,T,\widehat\zeta(1)+\widehat h_3(1))$ and $\zeta\mapsto L_0(x,\widehat T,\zeta)$, there exist measurable functions $0\le\eta_1,\eta_2(x)\le 1$ such that

$$L_0(x,\widehat T+\widehat h_2,\widehat\zeta(1)+\widehat h_3(1)) - L_0(x,\widehat T,\widehat\zeta(1)+\widehat h_3(1)) = L_{0T}(x,\widehat T+\eta_1\widehat h_2,\widehat\zeta(1)+\widehat h_3(1))\widehat h_2, \qquad (A.4)$$

$$L_0(x,\widehat T,\widehat\zeta(1)+\widehat h_3(1)) - L_0(x,\widehat T,\widehat\zeta(1)) = L_{0\zeta}(x,\widehat T,\widehat\zeta(1)+\eta_2\widehat h_3(1))\widehat h_3(1) \qquad (A.5)$$

for a.a. $x\in\Omega$. From (A.3), (A.4) and (A.5), we obtain

$$\begin{aligned}P_1(x) &\le \Big|L_{0T}(x,\widehat T+\eta_1\widehat h_2,\widehat\zeta(1)+\widehat h_3(1)) - L_{0T}(x,\widehat T,\widehat\zeta(1))\Big||\widehat h_2|\\ &\quad + \Big|L_{0\zeta}(x,\widehat T,\widehat\zeta(1)+\eta_2\widehat h_3(1)) - L_{0\zeta}(x,\widehat T,\widehat\zeta(1))\Big||\widehat h_3(1)|, \qquad (A.6)\end{aligned}$$

for a.a. $x\in\Omega$. We have

$$\begin{aligned}&\|\widehat z\|_Z \le \|\widehat z - z_*|_Z + \|z_*\|_Z \le \|z_*\|_Z+\varepsilon, \quad \|\widehat h\|_Z\le\varepsilon,\\ &|\widehat T+\eta_1\widehat h_2| \le |\widehat T| + |\widehat h_2| \le \|\widehat z\|_Z + \|\widehat h\|_Z \le \|z_*\|_Z + 2\varepsilon,\\ &|\widehat\zeta(1)+\eta_2\widehat h_3(1)| \le |\widehat\zeta(1)| + |\widehat h_3(1)| \le \|\widehat\zeta\|_{C(\overline Q_1)} + \|\widehat h_3\|_{C(\overline Q_1)} \le \|\widehat z\|_Z + \|\widehat h\|_Z \le \|z_*\|_Z + 2\varepsilon,\end{aligned}$$

for a.a. $(x,s)\in Q_1$. Set $M := \|z_*\|_Z + 2\varepsilon$. By $(H4)$-(ii), there exists $k_{L_0,M}$ such that

$$P_1(x) \le k_{L_0,M}(|\widehat h_2|^2 + |\widehat h_2||\widehat h_3(1)| + |\widehat h_3(1)|^2) \le k_{L_0,M}(|\widehat h_2|^2 + |\widehat h_2|\|\widehat h_3\|_{C(\overline Q_1)} + \|\widehat h_3\|^2_{C(\overline Q_1)}) \le 3k_{L_0,M}\|\widehat h\|^2_Z, \qquad (A.7)$$

for a.a. $x\in\Omega$. We now evaluate P_2. By $(H4)$-(i), we have for a.a. $(x,s)\in Q_1$,

$$\begin{aligned}&|L(x,\widehat\xi+\widehat h_1,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta,\widehat v)|\\ &\le k_{L,M}(|\widehat h_1| + |\widehat h_3| + |\widehat h_4|) \le k_{L,M}(\|\widehat h_1\|_{C([0,1],\mathbb R)} + \|\widehat h_3\|_{C(\overline Q_1)} + \|\widehat h_4\|_{L^\infty(Q_1)}) \le 3k_{L,M}\|\widehat h\|_Z. \qquad (A.8)\end{aligned}$$

It follows that

$$\begin{aligned}&P_2(x,s) \le 3k_{L,M}\|\widehat h\|_Z|\widehat h_2|\\ &+ |\widehat T||L(x,\widehat\xi+\widehat h_1,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta,\widehat v) - L_t(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_1 - L_y(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_3 - L_u(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_4|\\ &\le 3k_{L,M}\|\widehat h\|^2_Z + MP_3(x,s), \qquad (A.9)\end{aligned}$$

where $P_3(x,s) := |L(x,\widehat\xi+\widehat h_1,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta,\widehat v) - L_t(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_1 - L_y(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_3 - L_u(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_4|$ a.a. $(x,s)\in Q_1$. Moreover, we have

$$\begin{aligned}P_3(x,s) &= \Big|\Big(L(x,\widehat\xi+\widehat h_1,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4)\Big)\\ &\quad + \Big(L(x,\widehat\xi,\widehat\zeta+\widehat h_3,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta,\widehat v+\widehat h_4)\Big)\\ &\quad + \Big(L(x,\widehat\xi,\widehat\zeta,\widehat v+\widehat h_4) - L(x,\widehat\xi,\widehat\zeta,\widehat v)\Big)\\ &\quad - L_t(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_1 - L_y(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_3 - L_u(x,\widehat\xi,\widehat\zeta,\widehat v)\widehat h_4\Big| \qquad (A.10)\end{aligned}$$

for a.a. $(x,s)\in Q_1$. By Taylor's expansions of $\xi\mapsto L(x,\xi,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)$, $\zeta\mapsto L(x,\widehat{\xi},\zeta,\widehat{v}+\widehat{h}_4)$ and $v\mapsto L(x,\widehat{\xi},\widehat{\zeta},v)$, there exist measurable functions $0\le\eta_3(s),\eta_4(x,s),\eta_5(x,s)\le 1$ such that

$$L(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L(x,\widehat{\xi},\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)=L_t(x,\widehat{\xi}+\eta_3\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_1, \tag{A.11}$$

$$L(x,\widehat{\xi},\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{h}_4)=L_y(x,\widehat{\xi},\widehat{\zeta}+\eta_4\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_3, \tag{A.12}$$

$$L(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{h}_4)-L(x,\widehat{\xi},\widehat{\zeta},\widehat{v})=L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\eta_5\widehat{h}_4)\widehat{h}_4. \tag{A.13}$$

Combining these facts with (A.10), we obtain

$$\begin{aligned}P_3(x,s)&\le|L_t(x,\widehat{\xi}+\eta_3\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})||\widehat{h}_1|\\&\le|L_y(x,\widehat{\xi},\widehat{\zeta}+\eta_4\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})||\widehat{h}_3|\\&\le|L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\eta_5\widehat{h}_4)-L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})||\widehat{h}_4|\end{aligned} \tag{A.14}$$

for a.a. $(x,s)\in Q_1$. We have

$$\begin{aligned}&|(\widehat{\xi}+\widehat{h}_1)(s)|,\ |(\widehat{\zeta}+\widehat{h}_3)(x,s)|,\ |(\widehat{v}+\widehat{h}_4)(x,s)|\le\|\widehat{z}\|_Z+\|\widehat{h}\|_Z\le\|z_*\|_Z+2\varepsilon,\\&|(\widehat{\xi}+\eta_3\widehat{h}_1)(s)|\le|\widehat{\xi}(s)|+|\widehat{h}_1(s)|\le\|z_*\|_Z+2\varepsilon,\\&|\widehat{\zeta}+\eta_4\widehat{h}_3|\le|\widehat{\zeta}(x,s)|+|\widehat{h}_3(x,s)|\le\|z_*\|_Z+2\varepsilon,\\&|(\widehat{v}+\eta_5\widehat{h}_4)(x,s)|\le|\widehat{v}(x,s)|+|\widehat{h}_4(x,s)|\le\|z_*\|_Z+2\varepsilon,\end{aligned}$$

for a.a. $(x,s)\in Q_1$. Recall $M=\|z_*\|_Z+2\varepsilon$. From this and $(H4)$-(i), we deduce that

$$P_3(x,s)\le k_{L,M}\Big((|\widehat{h}_1|+|\widehat{h}_3|+|\widehat{h}_4|)|\widehat{h}_1|+(|\widehat{h}_3|+|\widehat{h}_4|)|\widehat{h}_3|+|\widehat{h}_4|^2\Big)\le 6k_{L,M}\|\widehat{h}\|_Z^2, \tag{A.15}$$

for a.a. $(x,s)\in Q_1$. From (A.9) and (A.15), we have

$$P_2(x,s)\le 3k_{L,M}(1+2M)\|\widehat{h}\|_Z^2 \tag{A.16}$$

for a.a. $(x,s)\in Q_1$. Combining (A.2), (A.7) and (A.16), we obtain

$$|\widehat{J}(\widehat{z}+\widehat{h})-\widehat{J}(\widehat{z})-B\widehat{h}|\le 3|\Omega|k_{L_0,M}\|\widehat{h}\|_Z^2+3|Q_1|k_{L,M}(1+2M)\|\widehat{h}\|_Z^2. \tag{A.17}$$

Hence,

$$0\le\lim_{\|\widehat{h}\|_Z\to 0}\frac{|\widehat{J}(\widehat{z}+\widehat{h})-\widehat{J}(\widehat{z})-B\widehat{h}|}{\|\widehat{h}\|_Z}\le\lim_{\|\widehat{h}\|_Z\to 0}\Big(3|\Omega|k_{L_0,M}+3|Q_1|k_{L,M}(1+2M)\Big)\|\widehat{h}\|_Z=0, \tag{A.18}$$

which implies that $\widehat{J}$ is Fréchet differentiable at $\widehat{z}$ and $D\widehat{J}(\widehat{z})=B$.

- We now show that $\widehat{J}$ is second-order Fréchet differentiable at $\widehat{z}$. For this, consider bounded bilinear operator $\widehat{B}: Z\times Z\to\mathbb{R}$, $(\widehat{h},h)=\Big((\widehat{h}_1,\widehat{h}_2,\widehat{h}_3,\widehat{h}_4),(h_1,h_2,h_3,h_4)\Big)\mapsto\widehat{B}(\widehat{h},h)$ with

$$\widehat{B}(\widehat{h},h):=\int_\Omega[\widehat{h}_2\quad\widehat{h}_3(1)]\Big(a_{ij}\Big)_{1\le i,j\le 2}[h_2\quad h_3(1)]^T\mathrm{d}x+\int_{Q_1}\widehat{h}^T\Big(b_{ij}\Big)_{1\le i,j\le 4}h\mathrm{d}x\mathrm{d}s, \tag{A.19}$$

where the matrices $\big(a_{ij}\big)_{1\le i,j\le 2}$ and $\big(b_{ij}\big)_{1\le i,j\le 4}$ are given by

$$a_{11}=L_{0TT}(\cdot,\widehat{T},\widehat{\zeta}(1)),\quad a_{12}=a_{21}=L_{0T\zeta}(\cdot,\widehat{T},\widehat{\zeta}(1)),\quad a_{22}=L_{0\zeta\zeta}(\cdot,\widehat{T},\widehat{\zeta}(1)),$$

$$b_{11} = \widehat{T}L_{tt}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{12} = b_{21} = L_t(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{13} = b_{31} = \widehat{T}L_{ty}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}),$$
$$b_{14} = b_{41} = \widehat{T}L_{tu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{22} = 0, \quad b_{23} = b_{32} = L_y(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{24} = b_{42} = L_u(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}),$$
$$b_{33} = \widehat{T}L_{yy}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{34} = b_{43} = \widehat{T}L_{yu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}), \quad b_{44} = \widehat{T}L_{uu}(\cdot,\widehat{\xi},\widehat{\zeta},\widehat{v}).$$

Taking any $\widehat{h} = (\widehat{h}_1,\widehat{h}_2,\widehat{h}_3,\widehat{h}_4) \in Z$ with $\|\widehat{h}\|_Z \le \varepsilon$. We need to prove that

$$\lim_{\|\widehat{h}\|_Z\to 0}\frac{\|D\widehat{J}(\widehat{z}+\widehat{h}) - D\widehat{J}(\widehat{z}) - \widehat{B}(\widehat{h},\cdot)\|_{Z^*}}{\|\widehat{h}\|_Z} = 0. \tag{A.20}$$

For this, take $h = (h_1,h_2,h_3,h_4) \in Z$ such that $\|h\|_Z \le 1$. Then we have

$$\begin{aligned}
&|D\widehat{J}(\widehat{z}+\widehat{h})h - D\widehat{J}(\widehat{z})h - \widehat{B}(\widehat{h},h)|\\
&\le \int_\Omega \Big| L_{0T}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_2 + L_{0\zeta}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_3(1)\\
&\quad - L_{0T}(x,\widehat{T},\widehat{\zeta}(1))h_2 - L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))h_3(1)\\
&\quad - L_{0TT}(x,\widehat{T},\widehat{\zeta}(1))h_2\widehat{h}_2 - L_{0\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1))h_3(1)\widehat{h}_3(1) - L_{0T\zeta}(x,\widehat{T},\widehat{\zeta}(1))[h_2\widehat{h}_3(1)+\widehat{h}_2h_3(1)]\Big|\mathrm{d}x\\
&+ \int_{Q_1}\Big| h_2L(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4) + (\widehat{T}+\widehat{h}_2)L_t(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)h_1\\
&\quad + (\widehat{T}+\widehat{h}_2)L_y(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)h_3 + (\widehat{T}+\widehat{h}_2)L_u(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)h_4\\
&\quad - h_2L(x,\widehat{\xi},\widehat{\zeta},\widehat{v}) - \widehat{T}L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_1 - \widehat{T}L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_3 - \widehat{T}L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_4\\
&\quad - \widehat{T}L_{tt}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_1\widehat{h}_1 - \widehat{T}L_{yy}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_3\widehat{h}_3 - \widehat{T}L_{uu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})h_4\widehat{h}_4\\
&\quad - \widehat{T}L_{ty}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_1\widehat{h}_3+\widehat{h}_1h_3] - \widehat{T}L_{tu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_1\widehat{h}_4+\widehat{h}_1h_4] - \widehat{T}L_{yu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_3\widehat{h}_4+\widehat{h}_3h_4]\\
&\quad - L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_2\widehat{h}_1+\widehat{h}_2h_1] - L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_2\widehat{h}_3+\widehat{h}_2h_3] - L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})[h_2\widehat{h}_4+\widehat{h}_2h_4]\Big|\mathrm{d}x\mathrm{d}s\\
&=: \int_\Omega \widehat{P}_1(x)\mathrm{d}x + \int_{Q_1}\widehat{P}_2(x,s)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{A.21}$$

By Taylor's expansions, there exist $0 \le \widehat{\eta}_1,\widehat{\eta}_2(x),\widehat{\eta}_3,\widehat{\eta}_4(x) \le 1$ such that

$$\begin{aligned}
&L_{0T}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_2 - L_{0T}(x,\widehat{T},\widehat{\zeta}(1))h_2\\
&= [L_{0T}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1)) - L_{0T}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{h}_3(1))]h_2 + [L_{0T}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{h}_3(1)) - L_{0T}(x,\widehat{T},\widehat{\zeta}(1))]h_2\\
&= L_{0TT}(x,\widehat{T}+\widehat{\eta}_1\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_2\widehat{h}_2 + L_{0T\zeta}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{\eta}_2\widehat{h}_3(1))h_2\widehat{h}_3(1)
\end{aligned}$$

and

$$\begin{aligned}
&L_{0\zeta}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_3(1) - L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))h_3(1)\\
&= [L_{0\zeta}(x,\widehat{T}+\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1)) - L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{h}_3(1))]h_3(1)\\
&+ [L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{h}_3(1)) - L_{0\zeta}(x,\widehat{T},\widehat{\zeta}(1))]h_3(1)\\
&= L_{0\zeta T}(x,\widehat{T}+\widehat{\eta}_3\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1))h_3(1)\widehat{h}_2 + L_{0\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{\eta}_4\widehat{h}_3(1))h_3(1)\widehat{h}_3(1).
\end{aligned}$$

for a.a. $x \in \Omega$. From these facts, the assumption $(H4)$-(ii) and $\|h\|_Z \le 1$, we obtain

$$\begin{aligned}
\widehat{P}_1(x) \le{}& \Big|L_{0TT}(x,\widehat{T}+\widehat{\eta}_1\widehat{h}_2,\widehat{\zeta}(1)+\widehat{h}_3(1)) - L_{0TT}(x,\widehat{T},\widehat{\zeta}(1))\Big||h_2\widehat{h}_2|\\
&+ \Big|L_{0\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1)+\widehat{\eta}_4\widehat{h}_3(1)) - L_{0\zeta\zeta}(x,\widehat{T},\widehat{\zeta}(1))\Big||h_3(1)\widehat{h}_3(1)|
\end{aligned}$$

$$
\begin{aligned}
&+ \Big| L_{0T\zeta}(x, \widehat{T}, \widehat{\zeta}(1) + \widehat{\eta}_2 \widehat{h}_3(1)) - L_{0T\zeta}(x, \widehat{T}, \widehat{\zeta}(1)) \Big| |h_2 \widehat{h}_3(1)| \\
&+ \Big| L_{0\zeta T}(x, \widehat{T} + \widehat{\eta}_3 \widehat{h}_2, \widehat{\zeta}(1) + \widehat{h}_3(1)) - L_{0\zeta T}(x, \widehat{T}, \widehat{\zeta}(1)) \Big| |h_3(1) \widehat{h}_2| \\
&\le k_{L_0,M} \Big((|\widehat{h}_2| + |\widehat{h}_3(1)|)|\widehat{h}_2| + |\widehat{h}_3(1)|^2 + |\widehat{h}_3(1)|^2 + (|\widehat{h}_2| + |\widehat{h}_3(1)|)|\widehat{h}_2| \Big) \\
&\le 2k_{L_0,M} \Big((|\widehat{h}_2| + \|\widehat{h}_3\|_{C(\overline{Q}_1)})|\widehat{h}_2| + \|\widehat{h}_3\|^2_{C(\overline{Q}_1)} \Big) \le 6k_{L_0,M} \|\widehat{h}\|_Z^2.
\end{aligned} \tag{A.22}
$$

for a.a. $x \in \Omega$ and $M = \|z_*\|_Z + 2\varepsilon$. We now evaluate $\widehat{P}_2$. By Taylor's expansions, there exist $0 \le \widehat{\eta}_i(x,s) \le 1$, $i = 5, 6, 7$ such that

$$
\begin{aligned}
&L(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) - L(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) \\
&= (L(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) - L(x, \widehat{\xi}, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4)) \\
&+ (L(x, \widehat{\xi}, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) - L(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v} + \widehat{h}_4)) \\
&+ (L(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v} + \widehat{h}_4) - L(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})) \\
&= L_t(x, \widehat{\xi} + \widehat{\eta}_5 \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4)\widehat{h}_1 + L_y(x, \widehat{\xi}, \widehat{\zeta} + \widehat{\eta}_6 \widehat{h}_3, \widehat{v} + \widehat{h}_4)\widehat{h}_3 + L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v} + \widehat{\eta}_7 \widehat{h}_4)\widehat{h}_4.
\end{aligned} \tag{A.23}
$$

From this, the assumption $(H4)$-(i) and $\|h\|_Z \le 1$, we have

$$
\begin{aligned}
&\Big| h_2 L(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) - h_2 L(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) - L_t(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_2 \widehat{h}_1 - L_y(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_2 \widehat{h}_3 - L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_2 \widehat{h}_4 \Big| \\
&\le |L_t(x, \widehat{\xi} + \widehat{\eta}_5 \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) - L_t(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})||\widehat{h}_1| + |L_y(x, \widehat{\xi}, \widehat{\zeta} + \widehat{\eta}_6 \widehat{h}_3, \widehat{v} + \widehat{h}_4) - L_y(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})||\widehat{h}_3| \\
&+ |L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v} + \widehat{\eta}_7 \widehat{h}_4) - L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})||\widehat{h}_4| \\
&\le k_{L,M} \Big((|\widehat{h}_1| + |\widehat{h}_3| + |\widehat{h}_4|)|\widehat{h}_1| + (|\widehat{h}_3| + |\widehat{h}_4|)|\widehat{h}_3| + |\widehat{h}_4|^2 \Big) \le 6k_{L,M} \|\widehat{h}\|_Z^2,
\end{aligned} \tag{A.24}
$$

and

$$
\begin{aligned}
&\Big| \widehat{h}_2 L_t(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_1 + \widehat{h}_2 L_y(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_3 + \widehat{h}_2 L_u(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_4 \\
&- L_t(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) \widehat{h}_2 h_1 - L_y(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) \widehat{h}_2 h_3 - L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) \widehat{h}_2 h_4 \Big| \\
&\le 3k_{L,M} (|\widehat{h}_1| + |\widehat{h}_3| + |\widehat{h}_4|)|\widehat{h}_2| \le 9k_{L,M} \|\widehat{h}\|_Z^2,
\end{aligned} \tag{A.25}
$$

for a.a. $(x,s) \in Q_1$. From (A.21), (A.24) and (A.25), we deduce that

$$
\widehat{P}_2(x,s) \le 15k_{L,M} \|\widehat{h}\|_Z^2 + M\widehat{P}_3(x,s), \tag{A.26}
$$

for a.a. $(x,s) \in Q_1$, $M = \|z_*\|_Z + 2\varepsilon$ and

$$
\begin{aligned}
&\widehat{P}_3(x,s) \\
&:= \Big| L_t(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_1 + L_y(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_3 + L_u(x, \widehat{\xi} + \widehat{h}_1, \widehat{\zeta} + \widehat{h}_3, \widehat{v} + \widehat{h}_4) h_4 \\
&- L_t(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_1 - L_y(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_3 - L_u(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_4 \\
&- L_{tt}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_1 \widehat{h}_1 - L_{yy}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_3 \widehat{h}_3 - L_{uu}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v}) h_4 \widehat{h}_4 \\
&- L_{ty}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})[h_1 \widehat{h}_3 + \widehat{h}_1 h_3] - L_{tu}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})[h_1 \widehat{h}_4 + \widehat{h}_1 h_4] - L_{yu}(x, \widehat{\xi}, \widehat{\zeta}, \widehat{v})[h_3 \widehat{h}_4 + \widehat{h}_3 h_4] \Big|,
\end{aligned} \tag{A.27}
$$

for a.a. $(x,s)\in Q_1$. By similar arguments as in proof of (A.23) above, there exist $0\le\widehat{\eta}_i(x,s)\le 1$, $i=8,...,16$ such that

$$\begin{aligned}
&L_t(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_t(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\\
&=L_{tt}(x,\widehat{\xi}+\widehat{\eta}_8\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_1+L_{ty}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_9\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_3+L_{tu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{10}\widehat{h}_4)\widehat{h}_4,\\
&L_y(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_y(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\\
&=L_{yt}(x,\widehat{\xi}+\widehat{\eta}_{11}\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_1+L_{yy}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_{12}\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_3+L_{yu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{13}\widehat{h}_4)\widehat{h}_4,\\
&L_u(x,\widehat{\xi}+\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_u(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\\
&=L_{ut}(x,\widehat{\xi}+\widehat{\eta}_{14}\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_1+L_{uy}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_{15}\widehat{h}_3,\widehat{v}+\widehat{h}_4)\widehat{h}_3+L_{uu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{16}\widehat{h}_4)\widehat{h}_4
\end{aligned}\tag{A.28}$$

for a.a. $(x,s)\in Q_1$. From this, (A.27), the assumption $(H4)$-(i) and $\|h\|_Z\le 1$, we deduce that

$$\begin{aligned}
\widehat{P}_3(x,s)&\le\Big|L_{tt}(x,\widehat{\xi}+\widehat{\eta}_8\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{tt}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_1|\\
&+\Big|L_{yy}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_{12}\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{yy}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_3|+\Big|L_{uu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{16}\widehat{h}_4)-L_{uu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_4|\\
&+\Big|L_{ty}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_9\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{ty}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_3|+\Big|L_{tu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{10}\widehat{h}_4)-L_{tu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_4|\\
&+\Big|L_{yt}(x,\widehat{\xi}+\widehat{\eta}_{11}\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{yt}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_1|+\Big|L_{yu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v}+\widehat{\eta}_{13}\widehat{h}_4)-L_{yu}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_4|\\
&+\Big|L_{ut}(x,\widehat{\xi}+\widehat{\eta}_{14}\widehat{h}_1,\widehat{\zeta}+\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{ut}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_1|+\Big|L_{uy}(x,\widehat{\xi},\widehat{\zeta}+\widehat{\eta}_{15}\widehat{h}_3,\widehat{v}+\widehat{h}_4)-L_{uy}(x,\widehat{\xi},\widehat{\zeta},\widehat{v})\Big||\widehat{h}_3|\\
&\le k_{L,M}\Big(3(|\widehat{h}_1|+|\widehat{h}_3|+|\widehat{h}_4|)|\widehat{h}_1|+3(|\widehat{h}_3|+|\widehat{h}_4|)|\widehat{h}_3|+3|\widehat{h}_4|^2\Big)\le 18k_{L,M}\|\widehat{h}\|_Z^2,
\end{aligned}\tag{A.29}$$

for a.a. $(x,s)\in Q_1$. Combining (A.26) and (A.29), we get

$$\widehat{P}_2(x,s)\le(15+18M)k_{L,M}\|\widehat{h}\|_Z^2,\tag{A.30}$$

for a.a. $(x,s)\in Q_1$. From (A.21), (A.22) and (A.30), we obtain

$$|D\widehat{J}(\widehat{z}+\widehat{h})h-D\widehat{J}(\widehat{z})h-\widehat{B}(\widehat{h},h)|\le(6|\Omega|k_{L_0,M}+(15+18M)k_{L,M}|Q_1|)\|\widehat{h}\|_Z^2,\tag{A.31}$$

for all $h\in Z$ with $\|h\|_Z\le 1$. It follows that

$$\|D\widehat{J}(\widehat{z}+\widehat{h})-D\widehat{J}(\widehat{z})-\widehat{B}(\widehat{h},\cdot)\|_{Z^*}\le(6|\Omega|k_{L_0,M}+(15+18M)k_{L,M}|Q_1|)\|\widehat{h}\|_Z^2.\tag{A.32}$$

Hence, we have

$$\lim_{\|\widehat{h}\|_Z\to 0}\frac{\|D\widehat{J}(\widehat{z}+\widehat{h})-D\widehat{J}(\widehat{z})-\widehat{B}(\widehat{h},\cdot)\|_{Z^*}}{\|\widehat{h}\|_Z}\le\lim_{\|\widehat{h}\|_Z\to 0}(6|\Omega|k_{L_0,M}+(15+18M)k_{L,M}|Q_1|)\|\widehat{h}\|_Z=0.\tag{A.33}$$

Thus, the mapping $\widehat{J}$ is second-order Fréchet differentiable at $\widehat{z}$ and $D^2\widehat{J}(\widehat{z})=\widehat{B}$.
The Fréchet differentiability of F,H and G are proved analogously. □

Lemma A Let $f:\mathbb{R}^3\to\mathbb{R}$, $(a,b,c)\mapsto f(a,b,c)$, be a function. Suppose that f is of class C^2 and $(a_0,b_0,c_0)\in\mathbb{R}^3$. Then, for each $(a,b,c)\in\mathbb{R}^3$, there exist $\alpha_i\in[0,1], i=1,...,6$, such that

$$\begin{aligned}
&f(a,b,c)-f(a_0,b_0,c_0)\\
&=f'_a(a_0,b_0,c_0)(a-a_0)+f'_b(a_0,b_0,c_0)(b-b_0)+f'_c(a_0,b_0,c_0)(b-b_0)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}f''_{aa}(a_0+\alpha_1(a-a_0),b,c)(a-a_0)^2+\frac{1}{2}f''_{bb}(a_0,b_0+\alpha_2(b-b_0),c)(b-b_0)^2\\
&+\frac{1}{2}f''_{cc}(a_0,b_0,c_0+\alpha_3(c-c_0))(c-c_0)^2+f''_{ab}(a_0,b_0+\alpha_4(b-b_0),c)(a-a_0)(b-b_0)\\
&+f''_{ac}(a_0,b_0,c_0+\alpha_5(c-c_0))(a-a_0)(c-c_0)+f''_{bc}(a_0,b_0,c_0+\alpha_6(c-c_0))(b-b_0)(c-c_0). \qquad \text{(A.34)}
\end{aligned}
$$

Proof. For each $(a,b,c)\in\mathbb{R}^3$, we have

$$
\begin{aligned}
f(a,b,c)-f(a_0,b_0,c_0)&=[f(a,b,c)-f(a_0,b,c)]+[f(a_0,b,c)-f(a_0,b_0,c)]+[f(a_0,b_0,c)-f(a_0,b_0,c_0)]\\
&=: I_1+I_2+I_2. \qquad \text{(A.35)}
\end{aligned}
$$

By Taylor's expansions of functions $f(\cdot,b,c)$, $f(a_0,\cdot,c)$ and $f(a_0,b_0,\cdot)$, there exist $\alpha_1,\alpha_2,\alpha_3\in[0,1]$ such that

$$I_1=f'_a(a_0,b,c)(a-a_0)+\frac{1}{2}f''_{aa}(a_0+\alpha_1(a-a_0),b,c)(a-a_0)^2, \qquad \text{(A.36)}$$

$$I_2=f'_b(a_0,b_0,c)(b-b_0)+\frac{1}{2}f''_{bb}(a_0,b_0+\alpha_2(b-b_0),c)(b-b_0)^2, \qquad \text{(A.37)}$$

$$I_3=f'_c(a_0,b_0,c_0)(c-c_0)+\frac{1}{2}f''_{cc}(a_0,b_0,c_0+\alpha_3(c-c_0))(c-c_0)^2. \qquad \text{(A.38)}$$

By Taylor's expansions of functions $f'_a(a_0,\cdot,c)$, $f'_a(a_0,b_0,\cdot)$ and $f'_b(a_0,b_0,\cdot)$, there exist $\alpha_4,\alpha_5,\alpha_6\in[0,1]$ such that

$$f'_a(a_0,b,c)=f'_a(a_0,b_0,c)+f''_{ab}(a_0,b_0+\alpha_4(b-b_0),c)(b-b_0), \qquad \text{(A.39)}$$

$$f'_a(a_0,b_0,c)=f'_a(a_0,b_0,c_0)+f''_{ac}(a_0,b_0,c_0+\alpha_5(c-c_0))(c-c_0), \qquad \text{(A.40)}$$

$$f'_b(a_0,b_0,c)=f'_b(a_0,b_0,c_0)+f''_{bc}(a_0,b_0,c_0+\alpha_6(c-c_0))(c-c_0). \qquad \text{(A.41)}$$

From (A.35), (A.36), (A.37), (A.38), (A.39), (A.40) and (A.41), we obtain (A.34). The lemma is proved. □