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# A LIOUVILLE THEOREM FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM WITH A POTENTIAL

Stefano Biagi* and Fabio Punzo

**Abstract.** In this paper we are concerned with elliptic equations in divergence form with a potential, posed in a bounded domain $\Omega$. We allow the coefficients of the diffusion matrix $A(x)$ and the potential $Q(x)$ to diverge at the boundary; in addition, we permit that $Q(x)$ vanishes inside $\Omega$, and $A(x)$ loses ellipticity at $\partial\Omega$. The boundary $\partial\Omega$ is assumed to be the (disjoint) union of a finite number $p$ of submanifolds of dimension $\kappa_i \in \{0, \ldots, n-1\}$ $(i = 1, \ldots, p)$. Under suitable assumptions on the behavior of $Q(x)$ and $A(x)$, which also depend on $\kappa_i$, we prove the validity of a Liouville-type theorem. Finally, we show an example for which our hypotheses on $Q$ and $A$ are sharp.

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## 1. Introduction

We study *Schrödinger-type* equations of the following form:

$$\operatorname{div}\{A(x)\nabla u\} - Q(x)u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded and connected open set, $A(x) \equiv (a_{ij}(x))_{ij}$ is an $n \times n$ symmetric matrix whose entries are $C^2$-functions in $\Omega$ and $Q(x)$ is a non-negative (and non-identically vanishing) locally Hölder-continuous function on $\Omega$. We assume that $A(x)$ is elliptic in $\Omega$, whereas ellipticity can be lost when $x \to \partial\Omega$. The case where the degeneracy set of $A$ contains interior points of $\Omega$ is also considered in a second moment. Furthermore, we allow $Q(x)$ and $A(x)$ to become *unbounded* as $x \to \partial\Omega$; in addition, $Q$ can vanish in the interior of $\Omega$.

To simplify the notation, we introduce the *Schrödinger-type* operator

$$\mathcal{L}u := \operatorname{div}\{A(x)\nabla u\} - Q(x)u \qquad (x \in \Omega); \tag{1.2}$$

the homogeneous part of $\mathcal{L}$ will be often indicated by $\mathcal{L}_0$, that is,

$$\mathcal{L}_0 u := \operatorname{div}(A(x)\nabla u).$$

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Dipartimento di Matematica, Politecnico di Milano, Italy.

* Corresponding author: stefano.biagi@polimi.it

Hence

$$\mathcal{L}u = \mathcal{L}_0 u - Q(x)u\,.$$

We investigate a *Liouville-type property* for equation (1.1). More precisely, we provide conditions on the *potential* $Q$ and on the *diffusion* matrix $A$ which imply that $u \equiv 0$ is the unique bounded solution of (1.1), without imposing any boundary condition.

In the special case when $Q > 0$ in $\Omega$, equation (1.1) is clearly equivalent to

$$\frac{1}{Q(x)}\,\mathrm{div}\,\big\{A(x)\nabla u\big\} - u = 0 \quad \text{in } \Omega\,. \tag{1.3}$$

In [1] uniqueness results for equation (1.3) are given, supposing that the matrix $A$ is *uniformly elliptic* in $\overline{\Omega}$. Moreover, uniqueness for (1.3) can be obtained from general results established in [2], which require the existence of a suitable negatively diverging subsolution. In addition, when $\frac{1}{Q}, A \in C^2(\overline{\Omega})$, uniqueness of solutions to (1.3) can be deduced from results obtained in [3] and in [4], under suitable assumptions on the behavior of $Q$ and $A$ on $\partial\Omega$. As is well-known, such uniqueness results are strictly related to stochastic calculus (see *e.g.* [5], [6], [7]). Clearly, all such results cannot be applied in our situation, since our assumptions on $Q$ and $A$ are weaker for various reasons.

The Liouville property for *degenerate-elliptic* operators $L$ in divergence form has been intensively studied through the years, but with particular attention to the case when $\Omega \subseteq \mathbb{R}^n$ is a half-space or the whole of $\mathbb{R}^n$ (see, *e.g.*, [8–12]). We explicitly point out that, due to the global nature of this kind of results, some *global assumptions* on the operator $L$ are needed; the most typical ones in this context are the invariance of $L$ with respect to some group of *translations* and/or *dilations*, or some global assumptions on the intrinsic geometry associated with $L$.

Recently, the Liouville property for equations of the form

$$\Delta u + \langle b(x), \nabla u\rangle - Q(x)u = 0 \quad \text{in } \Omega,$$

has been addressed in [13], taking inspiration from some ideas of [14]. In the present paper, we develop the methods of [13] to deal with equation (1.1). Clearly, now the diffusion matrix $A(x)$ heavily affects the result; furthermore, it causes some difficulties in the proof to be overcome. We always suppose that

**(S)** $\Omega \subseteq \mathbb{R}^n$ is a *bounded and connected* open set such that

$$\partial\Omega = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_p$$

(for some integer $p \geq 1$), where $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are $C^1$ submanifolds of $\mathbb{R}^n$ (compact and without boundary) satisfying the properties listed below:

(i) $\mathcal{M}_i \cap \mathcal{M}_j = \varnothing$ for every $1 \leq i \neq j \leq p$;

(ii) $n - 1 \geq \kappa_1 := \dim(\mathcal{M}_1) \geq \ldots \geq \kappa_p := \dim(\mathcal{M}_p) \geq 0$.

Under suitable assumptions on the behavior of $A(x)$ and $Q(x)$ near the $p$ submanifolds $\mathcal{M}_1, \ldots, \mathcal{M}_p$, which depend also on $\kappa_1, \ldots, \kappa_p$, we show that the Liouville property for equation (1.1) holds.

Observe that, in contrast with [8–12], we do not require any invariance property of the elliptic operator.

We state the main theorem in Section 2 (see Thm. 2.3), where also a couple of examples is presented. In Section 3 an appropriate preliminary *a priori* estimate is obtained (see Prop. 3.2). Then the main result is proved in Section 4. An important point, see Lemma 3.4, is the construction of a function $\zeta(x,t)$ fulfilling

$$Q(x)(\zeta)_t + \langle A(x)\nabla\zeta, \nabla\zeta\rangle \leq 0 \quad \text{for a.e. } x \in \Omega, \text{ for any } t \in (0,\tau),$$

for $\tau > 0$ sufficiently small. Clearly, in order to construct such a function $\zeta$ we need to take into account both the potential $Q$ and the matrix $A$. Finally, in Section 5 we show an example for which the hypotheses made on $Q(x)$ and $A(x)$ in the main result are sharp.

## 2. The main results

For any $E \subseteq \mathbb{R}^n$, we set

$$\mathbf{d}_E(x) := \operatorname{dist}(x, E) \qquad (x \in \mathbb{R}^n).$$

Moreover, for every fixed $r > 0$ we define

$$\mathcal{N}(E,r) := \{x \in \mathbb{R}^n : \mathbf{d}_E(x) < r\}.$$

**Assumptions.** Together with hypothesis (**S**), we make the following structural assumptions on the diffusion matrix $A(x)$ and on the potential $Q(x)$.

**(D)** For every $i,j = 1,\ldots,n$, we have $a_{ij} \in C^2(\Omega)$; moreover, the $n \times n$ (symmetric) matrix $A(x)$ is *positive definite* for every $x \in \Omega$, *i.e.*

$$\langle A(x)\xi, \xi\rangle > 0 \quad \forall\, x \in \Omega, \forall\, \xi \in \mathbb{R}^n \setminus \{0\}\,. \tag{2.1}$$

Finally, we assume that there exist a constant $\rho_0 > 0$ and functions

$$\sigma_1, \ldots, \sigma_p : (0, \rho_0] \to (0, +\infty),$$

continuous in the interval $\mathcal{I} := (0, \rho_0]$, such that (for any $i = 1,\ldots,p$)

$$\langle A(x)\xi, \xi\rangle \leq \sigma_i(\mathbf{d}_{\mathcal{M}_i}(x))|\xi|^2 \quad \forall\, x \in \Omega \cap \mathcal{N}(\mathcal{M}_i, \rho_0), \forall\, \xi \in \mathbb{R}^n\,. \tag{2.2}$$

**(P)** $Q \in C^{\gamma}_{\mathrm{loc}}(\Omega)$ (for some fixed $\gamma \in (0,1)$), and $Q \geq 0$, $Q \not\equiv 0$ in $\Omega$. Furthermore, we assume that there exist *continuous functions* $q_1, \ldots, q_p : \mathcal{I} \to (0,+\infty)$ such that (for any $i = 1,\ldots,p$)

$$Q(x) \geq q_i(\mathbf{d}_{\mathcal{M}_i}(x)) > 0 \qquad \forall\, x \in \Omega \cap \mathcal{N}(\mathcal{M}_i, \rho_0). \tag{2.3}$$

**Remark 2.1.** We explicitly point out that, since the manifolds $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are *(*compact and *) disjoint*, by possibly shrinking $\rho_0 > 0$ we may suppose that

$$\mathcal{N}(\mathcal{M}_1, r) \cap \cdots \cap \mathcal{N}(\mathcal{M}_p, r) = \varnothing \quad \forall\, 0 < r \leq \rho_0. \tag{2.4}$$

Most importantly, since $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are $C^1$, by classical results *(*and again by shrinking $\rho_0 > 0$ if needed*)* we derive that

$$\mathbf{d}_{\mathcal{M}_i} \in C^1(\mathcal{N}(\mathcal{M}_i, \rho_0) \setminus \mathcal{M}_i) \quad \text{and} \quad |\nabla \mathbf{d}_{\mathcal{M}_i}| = 1. \tag{2.5}$$

From now on, we tacitly understand that (2.4) is satisfied.

**Remark 2.2.** It is worth mentioning that the *regularity assumptions* on the matrix $A(x)$ and on the potential $Q(x)$ play a role only in the proof of Proposition 3.1, where they allow us to exploit the classical *local Schauder estimates* for strictly elliptic operators (indeed, it follows from (2.1) and from the regularity of $A(x)$ that $\mathcal{L}$ is strictly elliptic on any open set $D \Subset \Omega$), see, *e.g.*, [15], Corollary 6.3.

We are now in position to state our main result.

**Theorem 2.3.** *Let assumptions* (**S**), (**D**) *and* (**P**) *be satisfied. Moreover, suppose that there exists a constant* $\Lambda > 0$ *such that*

$$\int_0^{\rho_0} \frac{r^{\kappa_i+1-n}}{\sigma_i(r)} \exp\left\{\Lambda\left(\int_r^{\rho_0}\sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t\right)^2\right\}\mathrm{d}r = +\infty \quad (\textit{for any } 1 \le i \le p). \tag{2.6}$$

*Then the* unique solution $u \in C^2(\Omega) \cap L^\infty(\Omega)$ *of equation* (1.1) *is* $u \equiv 0$.

As it can be easily seen from condition (2.6), since the $q_i$'s are related to $Q$ and the $\sigma_i$'s are related to $A$ and the $\kappa_i$'s to $\partial\Omega$, the validity of the Liouville property depends on an interplay between $Q$, $A$ and certain geometric aspects of $\partial\Omega$.

Clearly, Theorem 2.3 implies the following *uniqueness* result.

**Corollary 2.4.** *Let the assumptions of Theorem 2.3 be in force, and let* $f \in C^0(\Omega)$. *Then the equation*

$$\mathcal{L}u = f \quad \textit{in } \Omega$$

*admits at most one bounded solution.*

In what follows, we say that assumption (**D′**) holds, whenever (**D**) is fulfilled with (2.1) replaced by

$$\langle A(x)\xi, \xi\rangle \ge 0 \quad \text{ for any } x \in \Omega, \xi \in \mathbb{R}^n\,. \tag{2.7}$$

As a matter of fact, our techniques allow us to deal also with hypothesis (**D′**). More precisely, we have the following result.

**Theorem 2.5.** *Let assumptions of Theorem 2.3 be satisfied with* (**D**) *replaced by* (**D′**) . *Let* $u \in C^2(\Omega) \cap L^\infty(\Omega)$ *be a* nonnegative *solution of equation* (1.1). *Then*

$$u = 0 \ \ \textit{in} \ \ \mathrm{supp}(Q)\,.$$

*In particular,*

$$u \equiv 0 \ \ \textit{in} \ \ \Omega\,,$$

*provided that* $Q > 0$ *in* $\Omega$ .

The validity of the Liouville-type property for *degenerate-elliptic operators* will be the object of a forthcoming paper [16], where we will consider the case of subelliptic operators modelled on homogeneous Hörmander vector fields.

## 2.1. Examples

Before embarking on the proof of Theorem 2.3, we discuss a couple of examples illustrating the applicability of our main result. The following elementary lemma (see, *e.g.*, [13], Lem. 4.1) will be used.

**Lemma 2.6.** *Let $\rho_0 > 0$ be arbitrarily fixed, and let $\delta_1, \delta_2 \in \mathbb{R}$. We define*

$$I_{\delta_1,\delta_2} := \int_0^{\rho_0} r^{\delta_2} \cdot \exp\Big\{\Big(\int_r^{\rho_0} t^{-\delta_1/2}\,\mathrm{d}t\Big)^2\Big\}\,\mathrm{d}r \in (0,+\infty].$$

*Then, the following facts hold.*

(1) *If $\delta_2 > -1$, then $I_{\delta_1,\delta_2} = +\infty$* if and only if $\delta_1 \geq 2$.
(2) *If $\delta_2 \leq -1$, then $I_{\delta_1,\delta_2} = +\infty$* for every $\delta_1 \in \mathbb{R}$.

**Example 2.7.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set such that the boundary $\partial\Omega$ is a $C^1$-manifold of dimension $n-1$. Moreover, let $\alpha, \beta \in \mathbb{R}$ be such that *either*

$$\text{(i) } \beta < 1 \text{ and } \alpha + \beta \geq 2 \qquad \text{or} \qquad \text{(ii) } \beta \geq 1. \tag{2.8}$$

We then choose two functions $Q_\alpha$, $\psi_\beta$ such that

(a) $Q_\alpha \in C^{\gamma}_{\mathrm{loc}}(\Omega)$ (for some $\gamma \in (0,1)$), $Q \geq 0$ in $\Omega$ and

$$Q \equiv \mathbf{d}_{\partial\Omega}^{-\alpha} \text{ on } \Omega \cap \mathcal{N}(\partial\Omega, \rho_0);$$

(b) $\psi_\beta \in C^2(\Omega)$ and $\psi_\beta \equiv \mathbf{d}_{\partial\Omega}^{\beta}$ on $\Omega \cap \mathcal{N}(\partial\Omega, \rho_0)$

(for a suitable $\rho_0 > 0$), and we consider the Schrödinger-type

$$\mathrm{div}\{\psi_\beta(x)\nabla u\} - Q_\alpha(x)u = 0 \qquad \text{in } \Omega. \tag{2.9}$$

Clearly, the above equation is of the form (1.1), with $A(x) = \psi_\beta(x)\mathrm{Id}_n$. In particular, if $\beta > 0$ the matrix $A(x)$ is degenerate elliptic in $\overline{\Omega}$, and its *degeneracy set* is $\partial\Omega$; if, instead, $\beta < 0$, the matrix $A(x)$ is unbounded in $\Omega$.

We now turn to show that *all the assumptions* of Theorem 2.3 are satisfied, so that equation (2.9) *does not possess* non-trivial bounded solutions.

- Assumption (**S**). Since $\partial\Omega$ is a $C^1$-manifold of dimension $n-1$, we see that assumptions (**S**) is trivially satisfied with the choice

$$p = 1, \quad \mathcal{M}_1 = \partial\Omega, \quad \kappa_1 = n-1.$$

- Assumption (**D**). Owing to the *explicit expression* of the matrix $A$, we immediately recognize that assumption (**D**) is satisfied, with

$$\sigma_1(t) = t^{\beta} \ (0 < t \leq \rho_0). \tag{2.10}$$

- Assumption (**P**). Taking into account the properties of $Q$ listed in (a), and recalling that $\mathcal{M}_1 = \partial\Omega$, we see that assumption (**P**) is satisfied with

$$q_1(t) = t^{-\alpha} \ (0 < t \leq \rho_0). \tag{2.11}$$

Now we have proved the validity of the *structural assumptions* (**S**), (**D**) and (**P**), we turn to prove the validity of assumption (2.6): taking into account (2.10) and (2.11), we need to show that there exists some $\Lambda > 0$ such that

$$\int_0^{\rho_0} r^{-\beta} \exp\Big\{\Lambda\Big(\int_r^{\rho_0} t^{-\frac{\alpha+\beta}{2}}\,\mathrm{d}t\Big)^2\Big\}\,\mathrm{d}r = +\infty,$$

but the validity of the above condition immediately follows by combining (2.8) and Lemma 2.6 (with the choice $\delta_2 = -\beta, \delta_1 = \alpha + \beta$ and $\Lambda = 1$). We will see in Section 5 that the assumption (2.8) on $\alpha, \beta$ is sharp.

We explicitly observe that when $\beta \geq 1$ our Theorem 2.3 can be applied for any $\alpha \in \mathbb{R}$. In particular, the special case $\alpha = 0$ (corresponding to a positive potential $Q$ which is constant near $\partial\Omega$) can be made.

**Example 2.8.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set such that $\partial\Omega = \mathcal{M}_1 \cup \mathcal{M}_2$, where $\mathcal{M}_1$, $\mathcal{M}_2$ are disjoint $C^1$-manifolds of dimension $\kappa_1 = n-1$ and $1 \leq \kappa_2 \leq n-2$, respectively. Moreover, let $\alpha_i, \beta_i \in \mathbb{R}$ (with $i = 1,2$) be such that

(i) $\big[\beta_1 < 1$ and $\alpha_1 + \beta_1 \geq 2\big]$ *or* $\beta_1 \geq 1$;

(ii) $\big[\beta_2 < \kappa_2 - n + 2$ and $\alpha_1 + \beta_1 \geq 2\big]$ *or* $\beta_2 \geq \kappa_2 - n + 2$.

We then choose two functions $Q$, $\psi$ such that

(a) $Q \in C^{\gamma}_{\mathrm{loc}}(\Omega)$ (for some $\gamma \in (0,1)$), $Q \geq 0$ in $\Omega$ and

$$Q \equiv \mathbf{d}_{\mathcal{M}_i}^{-\alpha_i} \text{ on } \Omega \cap \mathcal{N}(\mathcal{M}_i, \rho_0);$$

(b) $\psi \in C^2(\Omega)$ and $\psi \equiv \mathbf{d}_{\mathcal{M}_i}^{\beta_i}$ on $\Omega \cap \mathcal{N}(\mathcal{M}_i, \rho_0)$

(for $i = 1,2$ and a suitable $\rho_0 > 0$), and we consider the Schrödinger-type equation

$$\mathrm{div}\{\psi(x)\nabla u\} - Q(x)u = 0 \qquad \text{in } \Omega. \tag{2.12}$$

Clearly, the above equation is of the form (1.1), with $A(x) = \psi(x)\mathrm{Id}_n$; moreover, we claim that also in this case *all the assumptions* of Theorem 2.3 are satisfied (so that equation (2.12) *does not possess* non-trivial bounded solutions).

- Assumption (**S**). Taking into account the hypotheses on the *structure* of $\partial\Omega$, we see that assumptions (**S**) holds with $p = 2$ and $\mathcal{M}_i$, $\kappa_i$ as above.

- Assumption (**D**). Taking into account the *explicit expression* of $A$ (and the properties of $\psi$ in (b)), we see that assumption (**D**) is satisfied, with

$$\sigma_1(t) = t^{\beta_1} \qquad \text{and} \qquad \sigma_2(t) = t^{\beta_2} \qquad (0 < t \leq \rho_0). \tag{2.13}$$

- Assumption (**P**). Taking into account the properties of $Q$ in (a) (and recalling the structure of $\partial\Omega$), we see that assumption (**P**) is satisfied with the choice

$$q_1(t) = t^{-\alpha_1} \quad \text{and} \quad q_2(t) = t^{-\alpha_2} \qquad (0 < t \leq \rho_0). \tag{2.14}$$

Now we have proved the validity of the *structural assumptions* (**S**), (**D**) and (**P**), we turn to prove the validity of assumption (2.6).

On account of Example 2.7, we know that assumption (2.6) holds for $i = 1$ in view of (a). For $i = 2$, taking into account (2.13) and (2.14), we need to show that there exists some $\Lambda > 0$ such that

$$\int_0^{\rho_0} r^{\kappa_2 + 1 - n - \beta_2} \exp\left\{\Lambda\Big(\int_r^{\rho_0} t^{-\frac{\alpha_2+\beta_2}{2}}\,\mathrm{d}t\Big)^2\right\}\mathrm{d}r = +\infty,$$

but the validity of the above condition (with $\Lambda = 1$) immediately follows by combining the assumption (b) on $\alpha_2, \beta_2$ with Lemma 2.6.

## 3. Preliminary results

In this section we collect some preliminary results which will be used in the proof of Theorem 2.3. Throughout the sequel, given a function $u:\Omega\to\mathbb{R}$ we set

$$u_+ := \max\{u,0\} \quad \text{in } \ \Omega\,.$$

Moreover, we define $S:=\Omega\times[0,+\infty)$ and

$$S_T := \Omega\times[0,T] \quad (\text{for all } T>0).$$

We begin by stating the following proposition, which can be proved (with minor modifications) by arguing essentially as in the proof of [13], Proposition 2.1; here the regularity assumptions on $A(x)$ and on $Q(x)$ play a role.

**Proposition 3.1.** *Let assumptions* (**S**), (**D**) *and* (**P**) *be satisfied. Assume that there exists a nontrivial bounded solution $u$ of equation* (1.1)*. Then there exists a solution $v$ of equation* (1.1) *fulfilling*

$$0<v\leq 1 \quad \text{in } \ \Omega\,. \tag{3.1}$$

We then prove the following proposition.

**Proposition 3.2.** *Let assumptions* (**S**), (**D**) *and* (**P**) *be satisfied, and let $u$ be a non-trivial solution of equation* (1.1) *such that $0\leq u\leq 1$ in $\Omega$. Moreover, let $T>0$ be arbitrarily fixed, and let*

$$v(x,t) := \mathrm{e}^t u(x) - 1, \quad \text{for any } \ (x,t)\in S_T\,. \tag{3.2}$$

*Finally, let $\zeta:S_T\to\mathbb{R}$ and $\varphi:\Omega\to\mathbb{R}$ be such that*

(a) $\varphi\in \mathrm{Lip}_{loc}(\Omega)$ *be with* $\operatorname{supp}\varphi\subset\subset\Omega$.
(b) $\zeta(\cdot,t)\in\mathrm{Lip}_{loc}(\Omega)$ *for any* $t\in[0,T]$*, and*

$$\zeta(x,\cdot)\in C^1([0,T)),\ \mathrm{e}^{\zeta(x,\cdot)}\in C^1([0,T]) \text{ for any } x\in\Omega.$$

*Then we have the following estimate*

$$\begin{aligned}\int_\Omega Q\varphi^2 v_+^2(x,T)\mathrm{e}^{\zeta(x,T)}\mathrm{d}x - \int_\Omega Q\varphi^2 v_+^2(x,0)\mathrm{e}^{\zeta(x,0)}\mathrm{d}x \\ \leq \iint_{S_T}\{Q\zeta_t + \langle A(x)\nabla\zeta,\nabla\zeta\rangle\}\, v_+^2\varphi^2\mathrm{e}^\zeta\,\mathrm{d}x\,\mathrm{d}t \\ + 4\iint_{S_T}\langle A(x)\nabla\varphi,\nabla\varphi\rangle v_+^2\mathrm{e}^\zeta\mathrm{d}x\mathrm{d}t\,.\end{aligned} \tag{3.3}$$

*Proof.* First of all we observe that, since $u$ is a solution of (1.1), by the very definition of $v$ we get

$$Qv_t - \mathcal{L}_0 v = -\mathrm{e}^t\mathcal{L}u = 0 \text{ in } S_T;$$

as a consequence, we have

$$\iint_{S_T} Qv_t\, v_+\,\varphi^2\mathrm{e}^\zeta\mathrm{d}x\mathrm{d}t = \iint_{S_T}(\mathcal{L}_0 v)\, v_+\varphi^2\mathrm{e}^\zeta\mathrm{d}x\mathrm{d}t\,. \tag{3.4}$$

Furthermore, taking into account the properties of $\zeta(x,\cdot)$, we can write

$$\begin{aligned}\iint_{S_T} Qv_t\, v_+\, \varphi^2 \mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t &= \int_\Omega Q\varphi^2\left(\int_0^T v_t\, v_+ \mathrm{e}^{\zeta}\mathrm{d}t\right)\mathrm{d}x \\ &= \int_\Omega Q\varphi^2\left(\int_0^T \left(\frac{1}{2}v_+^2\right)_t \mathrm{e}^{\zeta}\mathrm{d}t\right)\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega Q(x)\varphi^2(x)\left[v_+^2(x,\cdot)\mathrm{e}^{\zeta(x,\cdot)}\right]_{t=0}^{t=T}\mathrm{d}x \\ &\qquad -\frac{1}{2}\iint_{S_T} Q\zeta_t\, v_+^2\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\,.\end{aligned} \tag{3.5}$$

In view of (3.5), to prove the needed (3.3) we now turn to estimate from above the integral in the right-hand side of identity (3.4).

To begin with, owing to assumptions (a)-(b) (and using the divergence structure of $\mathcal{L}_0$), we can perform a simple integration-by-parts argument, giving

$$\begin{aligned}\iint_{S_T}(\mathcal{L}_0 v)\, v_+\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t &= \iint_{S_T}\big(\mathrm{div}\big(A(x)\nabla v\big)\big)v_+\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t \\ &= -\iint_{S_T}\langle A(x)\nabla v, \nabla(v_+\varphi^2\mathrm{e}^{\zeta})\rangle\mathrm{d}x\mathrm{d}t \\ &= -\iint_{S_T}\langle A(x)\nabla v_+, \nabla v_+\rangle\varphi^2\mathrm{e}^{\zeta}\,\mathrm{d}x\,\mathrm{d}t + I_1 + I_2,\end{aligned} \tag{3.6}$$

where we have introduced the notation

$$\begin{aligned}I_1 &:= -2\iint_{S_T}\varphi v_+\mathrm{e}^{\zeta}\langle A(x)\nabla v_+, \nabla\varphi\rangle\mathrm{d}x\mathrm{d}t, \\ I_2 &:= -\iint_{S_T} v_+\varphi^2\mathrm{e}^{\zeta}\langle A(x)\nabla v_+, \nabla\zeta\rangle\mathrm{d}x\mathrm{d}t.\end{aligned}$$

On the other hand, since the matrix $A(x)$ is symmetric and *positive definite* for every $x\in\Omega$, by the Cauchy-Schwarz inequality we have

$$\big|\langle A(x)\xi,\eta\rangle\big| \le \sqrt{\langle A(x)\xi,\xi\rangle}\cdot\sqrt{\langle A(x)\eta,\eta\rangle} \qquad \forall\ \xi,\eta\in\mathbb{R}^n; \tag{3.7}$$

Using (3.7), together with the weighted Young inequality, we then obtain the following estimates, holding true *for every choice of* $\varepsilon_1\,\varepsilon_2>0$:

$$
\begin{aligned}
\bullet\quad |I_1| &\le \iint_{S_T} 2|\varphi| v_+ \mathrm{e}^{\zeta}\sqrt{\langle A(x)\nabla v_+,\nabla v_+\rangle}\cdot\sqrt{\langle A(x)\nabla\varphi,\nabla\varphi\rangle}\mathrm{d}x\mathrm{d}t\\
&= \iint_{S_T} 2\big(|\varphi|\mathrm{e}^{\frac12\zeta}\sqrt{\langle A(x)\nabla v_+,\nabla v_+\rangle}\big)\big(v_+\mathrm{e}^{\frac12\zeta}\sqrt{\langle A(x)\nabla\varphi,\nabla\varphi\rangle}\big)\mathrm{d}x\mathrm{d}t\\
&\le \varepsilon_1\iint_{S_T}\langle A(x)\nabla v_+,\nabla v_+\rangle\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\\
&\qquad+\frac{1}{\varepsilon_1}\iint_{S_T}\langle A(x)\nabla\varphi,\nabla\varphi\rangle v_+^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t;\\
\bullet\quad |I_2| &\le \iint_{S_T} v_+\varphi^2\mathrm{e}^{\zeta}\sqrt{\langle A(x)\nabla v_+,\nabla v_+\rangle}\sqrt{\langle A(x)\nabla\zeta,\nabla\zeta\rangle}\mathrm{d}x\mathrm{d}t\\
&= \iint_{S_T}\Big(|\varphi|\mathrm{e}^{\frac12\zeta}\sqrt{\langle A(x)\nabla v_+,\nabla v_+\rangle}\Big)\Big(v_+|\varphi|\mathrm{e}^{\frac12\zeta}\sqrt{\langle A(x)\nabla\zeta,\nabla\zeta\rangle}\Big)\mathrm{d}x\mathrm{d}t\\
&\le \frac{\varepsilon_2}{2}\iint_{S_T}\langle A(x)\nabla v_+,\nabla v_+\rangle\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\\
&\qquad+\frac{1}{2\varepsilon_2}\iint_{S_T}\langle A(x)\nabla\zeta,\nabla\zeta\rangle v_+^2\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\,.
\end{aligned}
\tag{3.8}
$$

Gathering (3.6)–(3.8), we finally obtain

$$
\begin{aligned}
\iint_{S_T}(\mathcal{L}_0 v)\,v_+\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t &\le \Big(-1+\varepsilon_1+\frac{\varepsilon_2}{2}\Big)\iint_{S_T}\langle A(x)\nabla v_+,\nabla v_+\rangle\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{\varepsilon_1}\iint_{S_T}\langle A(x)\nabla\varphi,\nabla\varphi\rangle v_+^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{2\varepsilon_2}\iint_{S_T}\langle A(x)\nabla\zeta,\nabla\zeta\rangle v_+^2\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.9}
$$

With estimate (3.9) at hand, we can easily complete the proof of (3.3): in fact, choosing $\varepsilon_1=1/2$ and $\varepsilon_2=1$ in (3.9), we get

$$
\begin{aligned}
\iint_{S_T}(\mathcal{L}_0 v)\,v_+\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t &\le \frac12\iint_{S_T}\langle A(x)\nabla\zeta,\nabla\zeta\rangle v_+^2\varphi^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t\\
&\quad+2\iint_{S_T}\langle A(x)\nabla\varphi,\nabla\varphi\rangle v_+^2\mathrm{e}^{\zeta}\mathrm{d}x\mathrm{d}t;
\end{aligned}
\tag{3.10}
$$

from this, recalling (3.4) and (3.5), we derive

$$
\begin{aligned}
&\int_\Omega Q(x)\varphi^2(x)v_+^2(x,T)\mathrm{e}^{\zeta(x,T)}\mathrm{d}x-\int_\Omega Q(x)\varphi^2(x)v_+^2(x,0)\mathrm{e}^{\zeta(x,0)}\mathrm{d}x\\
&\quad\le\iint_{S_T}\{Q(x)\zeta_t(x,t)+\langle A(x)\nabla\zeta(x,t),\nabla\zeta(x,t)\rangle\}\,v_+^2(x,t)\varphi^2(x)\mathrm{e}^{\zeta(x,t)}\mathrm{d}x\\
&\qquad+4\iint_{S_T}\langle A(x)\nabla\varphi,\nabla\varphi\rangle v_+^2(x,t)\mathrm{e}^{\zeta(x,t)}\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

which is precisely the needed (3.3). □

**Remark 3.3.** By carefully scrutinizing the proof of Proposition 3.2, together with the proof of [13], Proposition 2.1, one can plainly recognize that the previous results *do not require the validity of assumptions* (**D**)-(**P**) *in their full strength*: in fact, the explicit bounds from below on $Q(x)$ and from above on $A(x)$ appearing in (**D**) and (**P**), respectively, are not used in the proofs.

We conclude the section with the following key lemma, which provides an explicit *supersolution* of equation $\mathcal{H}u = 0$, where

$$\begin{aligned}\mathcal{H}u &= Q(x)\partial_t u + \sum_{i,j=1}^n a_{ij}(x)\partial_{x_i}u \cdot \partial_{x_j}u \\ &= Q(x)\partial_t u + \langle A(x)\nabla u, \nabla u\rangle\end{aligned}$$

is the differential operator appearing in the right-hand side of (3.3).

**Lemma 3.4.** *Let the assumptions of Theorem 2.3 be fulfilled. Moreover, let $T > 0$ and $0 < \varepsilon < \rho_0/2$ be arbitrarily fixed (here, $\rho_0 > 0$ is as in assumptions (**D**)-(**P**)).*

*We consider the function $\zeta_\varepsilon$ defined on $S_T$ as follows*

$$\zeta_\varepsilon(x,t) := -\frac{1}{T-t}\sum_{i=1}^p f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2 \qquad (\textit{for } (x,t) \in \Omega \times (0,T)),$$

*where, for every $1 \le i \le p$, we have*

$$f_{i,\varepsilon}(r) := \begin{cases} \dfrac{1}{2}\displaystyle\int_r^\varepsilon \sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t & \textit{if } 0 < r \le \varepsilon, \\ 0 & \textit{if } r > \varepsilon. \end{cases}$$

*Then, we have*

$$\begin{gathered}\mathcal{H}\zeta_\varepsilon(x,t) = Q(x)(\zeta_\varepsilon)_t + \langle A(x)\nabla\zeta_\varepsilon, \nabla\zeta_\varepsilon\rangle \le 0 \\ \textit{for a.e. } x \in \Omega \textit{ and every } t \in (0,T).\end{gathered} \tag{3.11}$$

*Proof.* First of all we observe that, since the functions $q_i$'s, $\sigma_i$'s are *continuous and strictly positive on the interval* $\mathcal{I} = [0, \rho_0]$, and since $\varepsilon < \rho_0/2$, from Remark 2.1 we easily recognize that $\zeta_\varepsilon$ is differentiable with respect to $x$ at every point of the *open cylinder* $\mathcal{O}_\varepsilon \times (0,T)$, where

$$\mathcal{O}_\varepsilon := \Omega \setminus \bigcup_{j=1}^p \{x \in \Omega : \mathbf{d}_{\mathcal{M}_j}(x) = \varepsilon\}.$$

As a consequence, since the level sets $\{\mathbf{d}_{\mathcal{M}_j} = \varepsilon\}$ have vanishing Lebesgue measure (so that $|\Omega \setminus \mathcal{O}_\varepsilon| = 0$), to prove the needed (3.11) it suffices to show that

$$\begin{gathered}\mathcal{H}\zeta_\varepsilon(x,t) = Q(x)(\zeta_\varepsilon)_t + \langle A(x)\nabla\zeta_\varepsilon, \nabla\zeta_\varepsilon\rangle \le 0 \\ \textit{for every } x \in \mathcal{O}_\varepsilon \text{ and every } t \in (0,T).\end{gathered} \tag{3.12}$$

In order to establish the above (3.12), we then fix a point $(x_0, t_0) \in \mathcal{O}_\varepsilon \times (0,T)$ and we distinguish two cases (according to the piecewise definition of the $f_{j,\varepsilon}$'s).

CASE I: $\mathbf{d}_{\mathcal{M}_j}(x_0) > \varepsilon$ for all $1 \leq j \leq p$. In this case, since $f_{j,\varepsilon} \equiv 0$ on $(\varepsilon, +\infty)$, we have that $\zeta_\varepsilon \equiv 0$ near $(x_0, t_0)$; as a consequence,

$$\mathcal{H}\zeta_\varepsilon(x_0, t_0) = 0,$$

and (3.12) is trivially satisfied.

CASE II: $\mathbf{d}_{\mathcal{M}_i}(x_0) < \varepsilon$ for some $1 \leq i \leq p$. In this case we first notice that, since the open neighborhoods $\mathcal{N}(\mathcal{M}_j, \varepsilon) = \{\mathbf{d}_{\mathcal{M}_j} < \varepsilon\}$ are pairwise disjoint (recall that $\varepsilon < \rho_0/2$ and see Rem. 2.1), we have

$$x_0 \in \mathcal{N}(\mathcal{M}_i, \varepsilon) \qquad \text{and} \qquad x_0 \notin \overline{\mathcal{N}(\mathcal{M}_j, \varepsilon)} \quad \forall\, j \neq i.$$

On the other hand, since $f_{j,\varepsilon} \equiv 0$ on $(\varepsilon, +\infty)$, by definition we also have

$$\zeta_\varepsilon(x,t) = -\frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2}{T-t} \quad \forall\, x \in \mathcal{N}(\mathcal{M}_i, \varepsilon) \cap \Omega,\, t \in (0,T).$$

As a consequence, since $\mathcal{N}(\mathcal{M}_i, \varepsilon) \cap \Omega$ is an open neighborhood of $x_0$, by using the bounds (2.2)–(2.3) in assumptions (**D**) and (**P**), respectively, we obtain

$$\begin{aligned}
&\mathcal{H}\zeta_\varepsilon(x_0, t_0) = Q(x_0)(\zeta_\varepsilon)_t + \langle A(x_0)\nabla\zeta_\varepsilon, \nabla\zeta_\varepsilon\rangle \\
&\quad = \frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x_0))^2}{(T-t_0)^2}\Big[ - Q(x_0) + \frac{q_i(\mathbf{d}_{\mathcal{M}_i}(x_0))}{\sigma_i(\mathbf{d}_{\mathcal{M}_i}(x_0))} \cdot \langle A(x_0), \nabla\mathbf{d}_{\mathcal{M}_i}, \nabla\mathbf{d}_{\mathcal{M}_i}\rangle\Big] \\
&\quad \leq \frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x_0))^2}{(T-t_0)^2}\Big[ - q_i(\mathbf{d}_{\mathcal{M}_i}(x_0) + \frac{q_i(\mathbf{d}_{\mathcal{M}_i}(x_0))}{\sigma_i(\mathbf{d}_{\mathcal{M}_i}(x_0))} \cdot \sigma_i(\mathbf{d}_{\mathcal{M}_i}(x_0))|\nabla\mathbf{d}_{\mathcal{M}_i}(x_0)|^2\Big] \\
&\quad \text{(since } |\nabla\mathbf{d}_{\mathcal{M}_i}| = 1\text{, see Rem. 2.1)} \\
&\quad \leq \frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x_0))^2}{(T-t_0)^2}\big( - q_i(\mathbf{d}_{\mathcal{M}_i}(x_0)) + q_i(\mathbf{d}_{\mathcal{M}_i}(x_0))\big) = 0,
\end{aligned}$$

which is precisely the needed (3.11). □

## 4. Proofs of theorems 2.3 and 2.5

For further purposes, let us make an observation concerning radial integrals over tubes around a submanifold of $\mathbb{R}^n$, *via* Coarea and Weyl's formula.

**Remark 4.1.** Let $M \subset \mathbb{R}^n$ be a $C^1$ compact $k$–dimensional submanifold, and let

$$d_M(x) := \mathrm{dist}(x, M)$$

be the associated distance function. For a fixed $a > 0$ smaller than the *normal injectivity radius of $M$*, we let $T_a(M)$ be the *tube around $M$ of radius $a$*, that is,

$$T_a(M) := \{x \in \mathbb{R}^n : \, d_M(x) \leq a\}.$$

Let $f(x) = g(d_M(x))$ with $g \in L^1(0,a)$. Then, due to Federer's Coarea formula,

$$\int_{T_a(M)} f(x)\,\mathrm{d}x = \int_0^a g(r)\,\mathcal{H}^{n-1}(\{x : \, d_M(x) = r\})\,\mathrm{d}r.$$

Moreover, by Weyl's tube formula, see [17], there exists a real *polynomial function* $P_M$ on $[0,a]$ such that

(i) $\mathcal{H}^{n-1}(\{d_M = r\}) = r^{\,n-k-1}\,P_M(r)$ for all $r\in(0,a]$;
(ii) $P_M(0) = (n-k)\,\omega_{n-k}\,\mathcal{H}^k(M)$

*(*where $\omega_{n-k}$ is the volume of the unit ball in $\mathbb{R}^{n-k}$*)*. Consequently, we have

$$\int_{T_a(M)} f(x)\,\mathrm{d}x = \int_0^a g(r)\,r^{\,n-k-1}\,P_M(r)\,\mathrm{d}r.$$

In particular, if $g\geq 0$, then

$$\int_{T_a(M)} f(x)\,\mathrm{d}x \leq \Big(\max_{0\leq r\leq a} P_M(r)\Big)\int_0^a g(r)\,r^{\,n-k-1}\,\mathrm{d}r.$$

Thanks to all the preliminary results established so far, we are now ready to provide the proof of our main result, namely Theorem 2.3.

*Proof of Theorem 2.3.* By contradiction, suppose that there exists a (non-trivial) bounded solution to equation (1.1). Then, we know from Proposition 3.1 that there also exists a solution $u$ to to same equation (1.1) such that

$$0 < u \leq 1 \quad \text{pointwise in } \Omega. \tag{4.1}$$

Defining $v(x,t) := \mathrm{e}^t u(x) - 1$ (for $(x,t)\in S = \Omega\times[0,+\infty)$), we claim that

$$v(x,t)\leq 0 \quad \text{for every } x\in\mathrm{supp}(Q) \text{ and } t>0. \tag{4.2}$$

Taking this claim for granted for a moment, we can easily conclude the proof of the theorem: in fact, owing to (4.2) (and exploiting the very definition of $v$), we have

$$0 < u \leq \mathrm{e}^{-t} \quad \text{for every } x\in\mathrm{supp}(Q) \text{ and } t>0;$$

then, by letting $t\to+\infty$, we deduce that $u\leq 0$ on $\mathrm{supp}(Q)\neq\varnothing$, but this is clearly in contradiction with (4.1). Hence, we turn to prove the claimed (4.2).

To begin with, we fix $T>0$ (to be chosen conveniently small later on), and we arbitrarily choose $\varepsilon\in(0,\rho_0/2)$; then, we let $\zeta_\varepsilon$ be as in Lemma 3.4. To proceed further, we choose another parameter $0<\delta<\varepsilon$, and we define

$$\varphi_{\delta,\varepsilon}:\Omega\to\mathbb{R},\quad \varphi_{\delta,\varepsilon}(x) := \phi_1(\mathbf{d}_{\mathcal{M}_1}(x))\cdots\phi_p(\mathbf{d}_{\mathcal{M}_p}(x)),$$

where $\phi_1,\ldots,\phi_p\in\mathrm{Lip}(\mathbb{R})$ satisfy the following properties

(i) $0\leq\phi_i\leq 1$ on $\mathbb{R}$;
(ii) $\phi_i\equiv 0$ on $[-\delta,\delta]$ and $\phi_i\equiv 1$ on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$

We now observe that, by definition, the functions $\varphi_{\delta,\varepsilon}$ and $\zeta_\varepsilon$ clearly satisfy assumptions (a)-(b) in Proposition 3.2; hence, by combining estimate (3.3) (with the *ad-hoc* choice $\varphi=\varphi_{\delta,\varepsilon}$, $\zeta=\zeta_\varepsilon$) with (3.11) in Lemma 3.4, we obtain

$$\begin{aligned}
&\int_\Omega Q\varphi_{\delta,\varepsilon}^2 v_+^2(x,T)\mathrm{e}^{\zeta_\varepsilon(x,T)}\mathrm{d}x - \int_\Omega Q\varphi_{\delta,\varepsilon}^2 v_+^2(x,0)\mathrm{e}^{\zeta_\varepsilon(x,0)}\mathrm{d}x \\
&\quad \le \iint_{S_T} \{Q(x)(\zeta_\varepsilon)_t + \langle A(x)\nabla\zeta_\varepsilon, \nabla\zeta_\varepsilon\rangle\} v_+^2\varphi_{\delta,\varepsilon}^2\mathrm{e}^{\zeta_\varepsilon}\,\mathrm{d}x\mathrm{d}t \\
&\qquad + 4\iint_{S_T} \langle A(x)\nabla\varphi_{\delta,\varepsilon}, \nabla\varphi_{\delta,\varepsilon}\rangle v_+^2\mathrm{e}^{\zeta_\varepsilon}\mathrm{d}x\mathrm{d}t \\
&\quad \le 4\iint_{S_T} \langle A(x)\nabla\varphi_{\delta,\varepsilon}, \nabla\varphi_{\delta,\varepsilon}\rangle v_+^2\mathrm{e}^{\zeta_\varepsilon}\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.3}$$

We then proceed by estimating both sides of (4.3).

- *Estimate of the left-hand side.* First of all we observe that, since $\phi_i \equiv 0$ pointwise on $[-\delta,\delta]$ for every $1\le i\le p$, we have $\varphi_{\delta,\varepsilon}\equiv 1$ pointwise on

$$\Omega_\varepsilon := \{x\in\Omega:\, \mathbf{d}_{\mathcal{M}_i}(x)\ge\varepsilon \text{ for all } 1\le i\le p\} = \{x\in\Omega:\, \mathbf{d}_{\partial\Omega}(x)\ge\varepsilon\}.$$

From this, taking into account that $v_+(x,0) = (u(x)-1)_+ = 0$ in $\Omega$ (see (4.1)) and that $\zeta_\varepsilon(x,\cdot)\equiv 0$ on $(0,T)$ for every $x\in\Omega_\varepsilon$, we obtain

$$\begin{aligned}
&\int_\Omega Q(x)\varphi_{\delta,\varepsilon}^2 v_+^2(x,T)\mathrm{e}^{\zeta_\varepsilon(x,T)}\mathrm{d}x - \int_\Omega Q(x)\varphi_{\delta,\varepsilon}^2 v_+^2(x,0)\mathrm{e}^{\zeta_\varepsilon(x,0)}\mathrm{d}x \\
&\quad = \int_\Omega Q(x)\varphi_{\delta,\varepsilon}^2 v_+^2(x,T)\mathrm{e}^{\zeta_\varepsilon(x,T)}\mathrm{d}x \\
&\quad \ge \int_{\Omega_\varepsilon} Q(x)v_+^2(x,T)\mathrm{e}^{\zeta_\varepsilon(x,T)}\mathrm{d}x = \int_{\Omega_\varepsilon} Q(x)\,v_+^2(x,T)\,\mathrm{d}x.
\end{aligned} \tag{4.4}$$

- *Estimate of the right-hand side.* We first point out that, since the functions $\phi_i$'s are *constant* on $[-\delta,\delta]$ and on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$ (see property (ii)), one has

$$\nabla\varphi_{\delta,\varepsilon}\not\equiv 0 \text{ only on } \Omega_{\delta,\varepsilon},$$

where $\Omega_{\delta,\varepsilon}\Subset\Omega$ is the open set defined as follows:

$$\Omega_{\delta,\varepsilon} := \bigcup_{j=1}^p \mathcal{A}_{\delta,\varepsilon}^j, \quad \text{where } \mathcal{A}_{\delta,\varepsilon}^j := \{x\in\Omega:\, \delta<\mathbf{d}_{\mathcal{M}_j}(x)<\varepsilon\}.$$

Thus, using assumption $(\mathbf{D})$, together with the fact that $0\le v_+(x,t)\le \mathrm{e}^t$ pointwise on the strip $S=\Omega\times[0,+\infty)$ (see (4.1)), we obtain

$$\begin{aligned}
&\iint_{S_T} \langle A(x)\nabla\varphi_{\delta,\varepsilon}, \nabla\varphi_{\delta,\varepsilon}\rangle v_+^2\mathrm{e}^{\zeta_\varepsilon}\mathrm{d}x\mathrm{d}t \\
&\quad = \int_\Omega \langle A(x)\nabla\varphi_{\delta,\varepsilon}, \nabla\varphi_{\delta,\varepsilon}\rangle\Big(\int_0^T v_+^2\,\exp\Big(-\frac{1}{T-t}\sum_{j=1}^p f_{j,\varepsilon}(\mathbf{d}_{\mathcal{M}_j}(x))^2\Big)\mathrm{d}t\Big)\mathrm{d}x \\
&\quad \le T\mathrm{e}^{2T}\int_\Omega \exp\Big(-\frac{1}{T}\sum_{j=1}^p f_{j,\varepsilon}(\mathbf{d}_{\mathcal{M}_j}(x))^2\Big)\langle A(x)\nabla\varphi_{\delta,\varepsilon}, \nabla\varphi_{\delta,\varepsilon}\rangle\,\mathrm{d}x
\end{aligned}$$

$$= T\mathrm{e}^{2T}\int_{\Omega_{\delta,\varepsilon}}\exp\Big(-\frac{1}{T}\sum_{j=1}^{p} f_{j,\varepsilon}(\mathbf{d}_{\mathcal{M}_j}(x))^2\Big)\langle A(x)\nabla\varphi_{\delta,\varepsilon},\nabla\varphi_{\delta,\varepsilon}\rangle\,\mathrm{d}x$$

(since $f_{j,\varepsilon}\equiv 0$ on $[\varepsilon,+\infty)$ and the neighborhoods $\{\mathbf{d}_{\mathcal{M}_j}<\varepsilon\}$ are disjoint)

$$= c_T\sum_{i=1}^{p}\int_{\{\delta<\mathbf{d}_{\mathcal{M}_i}(x)<\varepsilon\}}\mathrm{e}^{-\frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2}{T}}\langle A(x)\nabla\varphi_{\delta,\varepsilon},\nabla\varphi_{\delta,\varepsilon}\rangle\,\mathrm{d}x$$
$$\leq c_T\sum_{i=1}^{p}\int_{\{\delta<\mathbf{d}_{\mathcal{M}_i}(x)<\varepsilon\}}\mathrm{e}^{-\frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2}{T}}\sigma_i(\mathbf{d}_{\mathcal{M}_i}(x))|\nabla\varphi_{\delta,\varepsilon}|^2\,\mathrm{d}x =: (\clubsuit).$$

We now observe that, since $\phi_i\equiv 1$ out of the interval $[-\varepsilon,\varepsilon]$ (and since the neighborhoods $\{\mathbf{d}_{\mathcal{M}_j}<\varepsilon\}$ are disjoint), by definition of $\varphi_{\delta,\varepsilon}$ we have

$$\varphi_{\delta,\varepsilon}(x) = \big(\phi_i(\mathbf{d}_{\mathcal{M}_i}(x))\big)^2 \quad \text{for every } x\in\mathcal{A}^j_{\delta,\varepsilon};$$

hence, by (2.5) we get

$$|\nabla\varphi_{\delta,\varepsilon}(x)|^2 = \phi_i'(\mathbf{d}_{\mathcal{M}_i}(x)) \quad \text{for every } x\in\mathcal{A}^j_{\delta,\varepsilon}.$$

As a consequence, in view of Remark 4.1 (and recalling that $\mathcal{M}_i$ has dimension $\kappa_i$), we obtain the following estimate

$$(\clubsuit) = c_T\sum_{i=1}^{p}\int_{\{\delta<\mathbf{d}_{\mathcal{M}_i}(x)<\varepsilon\}}\mathrm{e}^{-\frac{f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2}{T}}\sigma_i(\mathbf{d}_{\mathcal{M}_i}(x))\big(\phi_i(\mathbf{d}_{\mathcal{M}_i}(x))\big)^2\,\mathrm{d}x$$
$$\leq c_{1,T}\sum_{i=1}^{p}\int_\delta^\varepsilon\exp\Big(-\frac{f_{i,\varepsilon}(r)^2}{T}\Big)\sigma_i(r)(\phi_i')^2 r^{n-1-\kappa_i}\,\mathrm{d}r =: (2\clubsuit).$$

where $c_{1,T}>0$ is a constant depending on $T$ and on the metric structures of the manifolds $\mathcal{M}_1,\ldots,\mathcal{M}_p$. From this, by choosing $\phi_1,\ldots,\phi_p$ in such a way that

$$\phi_i(r) := \Big(\int_\delta^\varepsilon\frac{s^{\kappa_i+1-n}}{\sigma_i(s)}\mathrm{e}^{\frac{f_{i,\varepsilon}(s)^2}{T}}\,\mathrm{d}s\Big)^{-1}\cdot\int_\delta^r\frac{s^{\kappa_i+1-n}}{\sigma_i(s)}\mathrm{e}^{\frac{f_{i,\varepsilon}(s)^2}{T}}\,\mathrm{d}s,$$

for every $\delta\leq r\leq\varepsilon$ (and $1\leq i\leq p$), a direct computation gives

$$(2\clubsuit)\leq c_{1,T}\sum_{i=1}^{p}\Big(\int_\delta^\varepsilon\frac{r^{\kappa_i+1-n}}{\sigma_i(r)}\exp\Big(\frac{f_{i,\varepsilon}(r)^2}{T}\Big)\,\mathrm{d}r\Big)^{-1}.$$

Summing up, we have proved the following estimate

$$\begin{aligned}&\iint_{S_T}\langle A(x)\nabla\varphi_{\delta,\varepsilon},\nabla\varphi_{\delta,\varepsilon}\rangle v_+^2\mathrm{e}^{\zeta_\varepsilon}\,\mathrm{d}x\mathrm{d}t\\&\qquad\leq c_{1,T}\sum_{i=1}^{p}\Big(\int_\delta^\varepsilon\frac{r^{\kappa_i+1-n}}{\sigma_i(r)}\exp\Big(\frac{f_{i,\varepsilon}(r)^2}{T}\Big)\,\mathrm{d}r\Big)^{-1}.\end{aligned}\tag{4.5}$$

Finally, by combining (4.4)–(4.5) with (4.3), and by exploiting the *explicit expression* of the function $f_{i,\varepsilon}(r)$ when $r<\varepsilon$ (see Lem. 3.4), we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} Q(x)\,v_+^2(x,T)\,\mathrm{d}x &\leq c_{2,T}\sum_{i=1}^p\bigg(\int_\delta^\varepsilon \frac{r^{\kappa_i+1-n}}{\sigma_i(r)}\exp\Big(\frac{f_{i,\varepsilon}(r)^2}{T}\Big)\,\mathrm{d}r\bigg)^{-1} \\ &= c_{2,T}\sum_{i=1}^p\bigg(\int_\delta^\varepsilon \frac{r^{\kappa_i+1-n}}{\sigma_i(r)}\exp\bigg\{\frac{1}{4T}\Big(\int_r^\varepsilon\sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t\Big)^2\bigg\}\,\mathrm{d}r\bigg)^{-1},\end{aligned}\tag{4.6}$$

where $c_{2,T}>0$ is another constant only depending on $T$ and on the metric structure of the manifolds $\mathcal{M}_1,\ldots,\mathcal{M}_p$.

With estimate (4.6) at hand, we can now complete the proof of the claimed (4.2). First of all, a standard application of the Young's inequality shows that there exists a *universal number* $T_0>0$, *depending on the constant* $\Lambda$ *in* (2.6) *but independent of fixed* $\delta$ *and* $\varepsilon$, such that

$$\begin{gathered}\frac{1}{4T}\Big(\int_r^\varepsilon\sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t\Big)^2 \geq \Lambda\Big(\int_r^{\rho_0}\sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t\Big)^2 - c_i(\varepsilon) \\ \text{for every } 1\leq i\leq p \text{ and every } 0<r<\varepsilon,\end{gathered}\tag{4.7}$$

provided that $T\leq T_0$ (here, $c_i(\varepsilon)>0$ is a suitable constant depending on $\varepsilon$); as a consequence, by combining (4.6)–(4.7), we have

$$\begin{aligned}&\int_{\Omega_\varepsilon} Q(x)\,v_+^2(x,T)\,\mathrm{d}x \\ &\qquad \leq c_{T,\varepsilon}\sum_{i=1}^p\bigg(\int_\delta^\varepsilon \frac{r^{\kappa_i+1-n}}{\sigma_i(r)}\exp\bigg\{\Lambda\Big(\int_r^{\rho_0}\sqrt{\frac{q_i(t)}{\sigma_i(t)}}\,\mathrm{d}t\Big)^2\bigg\}\,\mathrm{d}r\bigg)^{-1},\end{aligned}\tag{4.8}$$

provided that $T\leq T_0$, where $c_{T,\varepsilon}$ is a suitable constant depending on $T$ and $\varepsilon$.

Now, taking into account that $\delta\in(0,\varepsilon)$ was *arbitrarily chosen*, we can pass to the limit as $\delta\to0^+$ in the above (4.8): using assumption (2.6), we get

$$\int_{\Omega_\varepsilon} Q(x)\,v_+^2(x,T)\,\mathrm{d}x = 0,$$

from which we readily derive that

$$v(x,T)\leq 0 \quad \forall\, x\in\Omega_\varepsilon\cap\mathrm{supp}(Q),\ 0\leq T\leq T_0.$$

From this, recalling that also $\varepsilon\in(0,\rho_0/2)$ was arbitrarily fixed (and taking into account that $\Omega_\varepsilon=\{x\in\Omega:\mathbf{d}_\Omega(x)\geq\varepsilon\}$), we then obtain

$$v(x,T)\leq 0 \quad \forall\, x\in\mathrm{supp}(Q),\ 0\leq T\leq T_0.$$

Now, by repeating the *very same argument* exploited so far, but integrating on the strip $\Omega \times [T_0, T_0 + T]$ and using the function

$$\zeta_\varepsilon(x,t) := -\frac{1}{T - t + T_0} \sum_{i=1}^{p} f_{i,\varepsilon}(\mathbf{d}_{\mathcal{M}_i}(x))^2$$

(which satisfies the analog of (3.11) for a.e. $x \in \Omega$ and every $t \in (T_0, T_0 + T)$), we deduce that $v \leq 0$ on $\mathrm{supp}(Q) \times [T_0, 2T_0]$, and hence

$$v(x,T) \leq 0 \quad \forall\, x \in \mathrm{supp}(Q),\, 0 \leq T \leq 2T_0.$$

By iterating this argument, and by exploiting in a crucial way the fact that $T_0 > 0$ is a *universal number remaining unchanged at any iteration*, we conclude that

$$v(x,T) \leq 0 \quad \forall\, x \in \mathrm{supp}(Q),\, T \geq 0.$$

This is precisely the desired (4.2), and the proof is finally complete. □

Now we prove Theorem 2.5. As a matter of fact, the proof of this result is very similar to that of Theorem 2.3, and so we omit several details. However, the main difference is the following: when (2.7) holds we cannot, in general, obtain and use Proposition 3.1; instead of it, we now exploit the assumption that $u \geq 0$ in $\Omega$.

*Proof of Theorem 2.5.* Let $u$ be a (non-trivial) nonnegative and bounded solution to (1.1) (where assumption **(D)** is replaced by **(D')**). We define

$$\hat{u} = \frac{u}{\sup_\Omega |u|}.$$

By linearity, we see that $\hat{u}$ is a solution of (1.1), further satisfying $0 \lneqq \hat{u} \leq 1$; as a consequence, by using Proposition 3.2, and by arguing exactly as in the proof of Theorem 2.3, we can show that (see, precisely, (4.2))

$$v(x,t) = \mathrm{e}^t u(x) - 1 \leq 0 \quad \text{for every } x \in \mathrm{supp}(Q) \text{ and } t > 0. \tag{4.9}$$

Now, from (4.9) (and recalling that *we are assuming* $u \geq 0$ *in* $\Omega$), we infer that

$$0 \leq u(x) \leq \mathrm{e}^{-t} \text{ for every } x \in \mathrm{supp}(Q) \text{ and every } t > 0;$$

hence, by letting $t \to +\infty$, we immediately obtain that $u \equiv 0$ on $\mathrm{supp}(Q)$. □

## 5. An example of optimality

In this section we construct an example (see Ex. 5.2 below), for which the assumptions on $Q(x)$ and $A(x)$ made in Theorem 2.3 are *optimal*.

We shall use in a crucial way the next result (see [1], Thm. 2.5 or [2], Thm. 2.5).

**Proposition 5.1.** *Let assumptions* (**S**), (**D**) *and* (**P**) *be satisfied. In addition, we assume that*

$$Q(x) > 0 \quad \textit{for any } x \in \Omega,$$

*and we suppose that there exist* $\rho_0 > 0$ *and* $V \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *such that*

$$\text{(i) } \frac{1}{Q(x)}\mathcal{L}_0 V \leq -1 \quad \text{on } \Omega \cap \mathcal{N}(\partial\Omega, \rho_0) = \{x \in \Omega : \mathbf{d}_{\partial\Omega}(x) < \rho_0\}; \tag{5.1}$$

$$\text{(ii) } V > 0 \text{ on } \Omega \cap \mathcal{N}(\partial\Omega, \rho_0) \text{ and } V \equiv 0 \text{ on } \partial\Omega. \tag{5.2}$$

*Then, the equation*

$$\frac{1}{Q(x)}\mathcal{L}_0 u - u = 0 \quad \text{in } \Omega \tag{5.3}$$

*admits infinitely many bounded solutions. In particular, for every* $\gamma \in \mathbb{R}$, *there exists a solution to the Dirichlet problem*

$$\begin{cases} \dfrac{1}{Q(x)}\mathcal{L}_0 u - u = 0 & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

We stress that $u$ is a solution to (5.4) *if and only if* it is a solution of (5.3) and satisfies $u = \gamma$ on $\partial\Omega$.

**Example 5.2.** Let $B_1 \subseteq \mathbb{R}^n$ be the usual Euclidean ball with centre $x_0 = 0$ and radius $r = 1$, and let $\alpha, \beta \in \mathbb{R}$. We consider the equation

$$\operatorname{div}\{\psi_\beta \nabla u\} - (1 - |x|)^{-\alpha} u = 0 \qquad \text{in } B_1, \tag{5.5}$$

where $\psi_\beta \in C^2(B_1)$ is such that $\psi_\beta(x) = (1 - |x|)^\beta$ for $\frac{3}{4} < |x| < 1$. Since $\Omega = B_1$ is a bounded open set with smooth boundary, and since

$$\mathbf{d}_{\partial B_1}(x) = 1 - |x| \text{ for every } x \in B_1,$$

from Example 2.7 we can infer that if

$$\text{(i) } \beta < 1 \text{ and } \alpha + \beta \geq 2 \qquad \text{or} \qquad \text{(ii) } \beta \geq 1,$$

all the assumptions of Theorem 2.3 are satisfied, so that $u = 0$ is the unique bounded solution of (5.5). We now aim to show that these conditions on $\alpha$ and $\beta$ are *optimal*. More precisely, we will prove that if

$$\beta < 1 \text{ and } \alpha + \beta < 2,$$

then there exist *infinitely many bounded solutions* of (5.5).

To this end, since $Q(x) := (1 - |x|)^{-\alpha}$ is strictly positive in $B_1$, we may try to use Proposition 5.1. First of all, given any $C, \gamma > 0$ (to be appropriately fixed in a moment), we take a function $V \in C^2(B_1) \cap C^0(\overline{B}_1)$ satisfying

$$V(x) = C(1 - |x|)^\gamma \quad \text{for } \frac{3}{4} < |x| < 1. \tag{5.6}$$

On account of (5.6), we clearly have that $V > 0$ in $B_1 \cap \mathcal{N}(\partial B_1, 1/4)$ and $V \equiv 0$ on $\partial B_1$. Moreover, a direct computation gives

$$\begin{aligned}\frac{1}{Q(x)}\mathcal{L}_0 V &= C\gamma(1-|x|)^{\beta-2+\alpha+\gamma}\Big[\gamma - 1 - \frac{n-1}{|x|}(1-|x|) + \beta\Big] \\ &\leq C\gamma(\gamma-1+\beta)(1-|x|)^{\beta-2+\alpha+\gamma} \quad \forall\, x \in B_1 \cap \mathcal{N}(\partial B_1, 1/4).\end{aligned}$$

Thus, if we select $0 < \gamma < \min\{1-\beta, 2-\alpha-\beta\}$ (notice that this choice of $\gamma$ is *meaningful*, since we are assuming $\alpha + \beta < 2$ and $\beta < 1$), we obtain

$$\frac{1}{Q(x)}\mathcal{L}_0 V \leq \frac{C\gamma(\gamma+\beta-1)}{4^{-\beta+2-\alpha-\gamma}} = -1 \quad \text{on } B_1 \cap \mathcal{N}(\partial B_1, 1/4),$$

up to properly choosing the constant $C > 0$. In view of this last estimate, we are then entitled to apply Proposition 5.1 (with $\rho_0 = 1/4$), which ensures the existence of infinitely many bounded solutions of (5.5).


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## Data availability statement

My manuscript has no associated data.

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