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VARIATIONAL ANALYSIS OF DISCRETE DIRICHLET PROBLEMS IN PERIODICALLY PERFORATED DOMAINS

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Abstract. In this paper we study the asymptotic behavior of a family of discrete functionals as the lattice size, $\varepsilon > 0$, tends to zero. We consider pairwise interaction energies satisfying p-growth conditions, $p < d$, d being the dimension of the reference configuration, defined on discrete functions subject to Dirichlet conditions on a δ-periodic array of small squares of side $r_\delta \sim \delta^{d/d-p}$. Our analysis is performed in the framework of Γ-convergence and we prove that, in the regime $\varepsilon = o(r_\delta)$, the discrete energy and their continuum counterpart share the same Γ-limit and the effect of the constraints leads to a capacitary term in the limit energy as in the classical theory of periodically perforated domains for local integral functionals.

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1. Introduction

In the last decades many efforts have been done to investigate the relation between continuum theories and general atomic interaction energies. The reason is related to their relevance for applications in different directions. On one hand the derivation of continuum models from atomistic systems provides a microscopical theoretical justification of continuum theories. On the other hand continuum models, obtained as an approximation of atomic interaction energies usually correspond to a coarse-graining description of the latters, which smooth away fine details but are able to capture the main features of the original problem, bringing a lot of advantages from a numerical view point. In this paper we are interested in studying the asymptotic behaviour of a specific class of atomic interaction energies satisfying suitable p-growth conditions, with $p < d$, d being the dimension of the reference configuration, and subject to a Dirichlet constraint on periodically perforated domains. In the continuous setting, the problem is modelled, in the simple case, by imposing a Dirichlet condition on a periodic array $U_{\delta,R}$ of small balls of radius R and centers on a δ-periodic lattice. For instance, for a suitable choice of the size $R = R_\delta$ of the perforations minimum problems of the form

$$\min\left\{\int_\Omega (|\nabla u|^p - gu)\,\mathrm{d}x : u = 0 \text{ on } U_{\delta,R}\right\}$$

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can be approximated by

$$\min\left\{\int_\Omega (|\nabla u|^p + C_p|u|^p - gu)\,\mathrm{d}x\right\},$$

where C_p is the p-capacity of the unit ball $B_1(0)$ in $\mathbb{R}^d$ and the term $C_p|u|^p$ accounts for the energetic contribution near the perforations. This "extra term" appeared in literature for the first time in the well-known paper by D. Cioranescu and F. Murat [1], where they study the scalar case for $p=2$. For a more general overview of multiscale problems of this type and the techniques used to study them, see for example [2–6]. It must be noted that in the subcritical case $p<d$ the only relevant scaling is

$$R \sim \delta^{\frac{d}{d-p}},$$

since the other cases, *i.e.* $R = o\big(\delta^{\frac{d}{d-p}}\big)$ and $R = O\big(\delta^{\frac{d}{d-p}}\big)$, lead to trivial limit energy. In [7–15] N. Ansini and A. Braides set the problem in the framework of Γ-convergence and extended the analysis to a general class of energies defined on vector valued functions. Under suitable assumptions on the density function f, they prove that, as $\delta\to 0$ and $R_\delta = \delta^{d/d-p}$ functionals of the form

$$\int_\Omega f(\nabla u)\,\mathrm{d}x, \qquad u\in W^{1,p}(\Omega,\mathbb{R}^m),\ u=0 \text{ on } U_{\delta,R}$$

Γ-converge to the functional defined by

$$\int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega \varphi(u)\,\mathrm{d}x, \qquad u\in W^{1,p}(\Omega,\mathbb{R}^m), \tag{1.1}$$

where $\varphi(z)$ is described by the non linear capacitary formula

$$\varphi(z) = \inf\left\{\int_{\mathbb{R}^d} f(\nabla v)\,\mathrm{d}x : v-z\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)\cap L^{p^*}(\mathbb{R}^d;\mathbb{R}^m), v=0 \text{ on } B_1(0)\right\}.$$

In this paper we consider pairwise interaction energies of the form

$$F_\varepsilon(u) = \sum_{\xi\in\mathbb{Z}^d}\ \sum_{\alpha,\alpha+\varepsilon\xi\in\varepsilon\mathbb{Z}^d\cap\Omega} \varepsilon^d f\left(\xi, \frac{u(\alpha+\varepsilon\xi)-u(\alpha)}{\varepsilon|\xi|}\right)$$

defined on functions $u:\alpha\in\varepsilon\mathbb{Z}^d\cap\Omega\mapsto u(\alpha)\in\mathbb{R}^m$, identified with their piecewise constant interpolation on each cell of the lattice $\varepsilon\mathbb{Z}^d$. For any $\xi\in\mathbb{Z}^d$, the energy densities $f(\xi,\cdot)$ are p-homogeneous, locally Lipschitz continuous, and satisfy suitable decay assumptions as $|\xi|\to+\infty$ (see hypotheses (H), (G), and (L) in Sect. 3). Under these assumptions, as a particular case of the asymptotic analysis provided by R. Alicandro and M. Cicalese in [8] (see Thm. 2.3), the Γ-limit of F_ε is given by the local functional

$$F_0(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x, \qquad u\in W^{1,p}(\Omega;\mathbb{R}^m),$$

where $f_{\mathrm{hom}}(M)$ is given by a suitable homogenization formula. We then impose the admissible functions u to satisfy the constraint $u=0$ on a periodic array P_δ of small squares of side $r_\delta\sim\delta^{d/d-p}$ and centers on a δ-periodic lattice. We are interested in the asymptotic behaviour of F_ε as ε and δ go to 0 according to the interplay between the scales ε,δ. The case in which $\varepsilon\sim r_\delta$ has been studied by L. Sigalotti in [9], where she proves that the limit energy is still of the form (1.1) where f is replaced by $f_{\rm hom}$ and φ is given by a non-local capacitary formula. In this paper we focus our attention on the case $\varepsilon = o(r_\delta)$ and we address the question whether or not F_ε and their continuous approximation F_0 share the same asymptotic behaviour under imposed constraints on the admissible functions. We show that this is the case. Indeed, in the main result of this paper, Theorem 2.4, we show that F_ε and F_0 subject to the constraint $u = 0$ on P_δ share the same Γ-limit which is given by

$$\int_\Omega f_{\rm hom}(\nabla u)\,\mathrm{d}x + \int_\Omega \varphi(u)\,\mathrm{d}x$$

where $\varphi(z)$ is described by the capacitary formula induced by F_0 (see (2.13)). It is worth noting that a similar analysis in the case of non-local functionals of convolution type has been performed in a recent paper by R. Alicandro, M.S. Gelli and C. Leone [10]. From a technical point of view we mainly adopt the strategy exploited in [10], which in turn is inspired by [7]. The idea is to use a separation-of-scales argument, formalized in [7] and then deal with the non-locality of our functionals to estimate the contribution near the perforations. A crucial role is played by a discrete version of the Gagliardo-Nirenberg-Sobolev inequality, which is derived from the corresponding non-local variant proved in [10]. This inequality allows us to show the convergence of minimum problems on unbounded domains, defining the approximating capacitary densities, to the limit energy density defined in (2.13).

The paper is organized as follows. In Section 2 we introduce some notation. In Section 3 we present the setting of the problem and state the main result of the paper. In Section 4 we recall some preliminary results. In Section 5 we state and prove the discrete version of the Gagliardo-Nirenberg-Sobolev inequality and some other results which are instrumental for the proof of the main theorem, which is the core of Section 6.

1.1. Notation

In what follows $d, m \in \mathbb{N}$ will be two fixed natural numbers denoting the dimension of the reference and the target spaces of the functions we consider, respectively. The set of vectors $\{e_1, \dots, e_d\}$ will denote the standard orthonormal basis in $\mathbb{R}^d$. Given $t \in \mathbb{R}$, $[t]$ denotes the integer part of t; for $\alpha \in \mathbb{Z}^d, r > 0, Q(\alpha, r) = \alpha + (-r/2, r/2)^d$ (if $\alpha = 0$, simply Q_r) is the open cube in $\mathbb{R}^d$ of center α and side length r. We denote by $\mathbb{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$. If A is a subset of $\mathbb{R}^d$ then $\mathrm{dist}(x, A) = \inf\{|y - x| : y \in A\}$; $\mathcal{A}^{\rm reg}(A)$ is the family of open subsets with Lipschitz boundary. We use standard notation for Γ-convergence [11]. Unless otherwise stated, C will always denote a generic strictly positive constant that may change from line to line.

2. Setting of the problem and the main result

We fix $p \in (1, d)$ and we let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. For fixed $\varepsilon > 0$, we denote by Ω_ε the discrete set $\Omega_\varepsilon := \varepsilon\mathbb{Z}^d \cap \Omega$ and later for a given infinitesimal sequence ε_j we will use the notation $\Omega_j := \Omega_{\varepsilon_j}$. We denote by $\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m)$ the set of functions

$$\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m) := \{u : \Omega_\varepsilon \to \mathbb{R}^m\}.$$

We will identify the functions in $\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m)$ by their piecewise constant interpolation on the cells of the lattice $\varepsilon\mathbb{Z}^d$ that is

$$\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m) = \{u : \mathbb{R}^d \to \mathbb{R}^m : u \text{ constant on } \alpha + [0, \varepsilon)^d \text{ for any } \alpha \in \Omega_\varepsilon\}.$$

Given $\xi \in \mathbb{Z}^d$ and $E \subset \Omega$ we define

$$E_\varepsilon(\xi) := \{\alpha \in E |\ \alpha + \varepsilon\xi \in E\} \cap \varepsilon\mathbb{Z}^d. \tag{2.1}$$

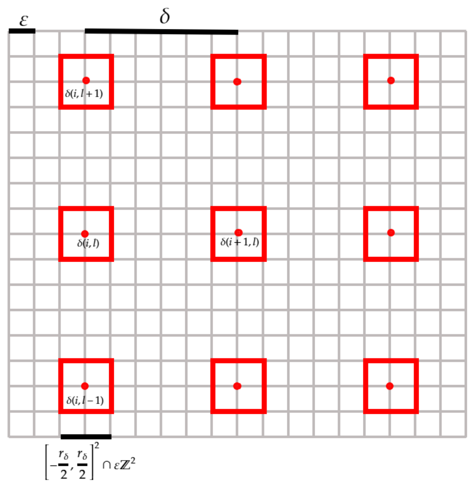

FIGURE 1. Periodically Perforated Domain.

Given a function $v \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m)$, we denote by $D^\xi_\varepsilon v$ the difference quotient along the direction ξ; *i.e.* for $\alpha \in \Omega_\varepsilon(\xi)$

$$D^\xi_\varepsilon v(\alpha) := \frac{v(\alpha + \varepsilon\xi) - v(\alpha)}{\varepsilon|\xi|} \tag{2.2}$$

Let δ, r_δ be given with $\delta > r_\delta > 0$. We assume that

$$\begin{aligned} \delta = \delta_\varepsilon = N_\varepsilon \varepsilon \quad &: N_\varepsilon \in \mathbb{N} \\ r_\delta = 2n_\varepsilon \varepsilon \quad &: n_\varepsilon \in \mathbb{N}. \end{aligned}$$

With this setting we have that $\varepsilon\mathbb{Z}^d \supset \delta\mathbb{Z}^d$. For every $i \in \mathbb{Z}^d$, we define

$$\mathcal{Q}(i\delta, r_\delta) := i\delta + \left(\left[-\frac{r_\delta}{2}, \frac{r_\delta}{2}\right]^d \cap \varepsilon\mathbb{Z}^d\right)$$

and

$$P_\delta := \bigcup_{i \in \mathbb{Z}^d} \mathcal{Q}(i\delta, r_\delta),$$

which, for $d = 2$, is represented in Figure 1. Then, given $f : \mathbb{Z}^d \times \mathbb{R}^m \to [0, +\infty)$ we introduce the family of discrete functionals $F_{\varepsilon,\delta} : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty]$ defined as follows

$$F_{\varepsilon,\delta}(u) := \begin{cases} \displaystyle\sum_{\xi \in \mathbb{Z}^d} \sum_{\alpha \in \Omega_\varepsilon(\xi)} \varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m),\ u \equiv 0 \text{ on } P_\delta \cap \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

We consider the following assumptions on f:

(H) $f(\xi, tz) = |t|^p f(\xi, z), \quad \forall t \in \mathbb{R},\ \forall (\xi, z) \in \mathbb{Z}^d \times \mathbb{R}^m$

(G) For any $i = 1, \dots, d$ and $\xi \in \mathbb{Z}^d$ the functions $m_i = \inf_{\zeta \in \mathbb{S}^{d-1}} f(e_i, \zeta)$ and $M(\xi) = \sup_{\zeta \in \mathbb{S}^{d-1}} f(\xi, \zeta)$ satisfy

  (G0) there exists a constant $\lambda_0 > 0$ such that $\min\{m_i : i = 1, \dots, d\} \geq \lambda_0$

  (G1) $\sum_{\xi \in \mathbb{Z}^d} M(\xi) < +\infty$

(L) $|f(\xi, z_1) - f(\xi, z_2)| \leq CM(\xi)(|z_1|^{p-1} + |z_2|^{p-1})(|z_1 - z_2|), \quad \forall z_1, z_2 \in \mathbb{R}^m,\ \forall \xi \in \mathbb{Z}^d$

Remark 2.1. From (G0) and (H) we get that for every $z \in \mathbb{R}^m$ and $i = 1, \dots, d$, $f(e_i, z) \geq \lambda_0 |z|^p$. Indeed

$$f(e_i, z) = |z|^p f\left(e_i, \frac{z}{|z|}\right) \geq \lambda_0 |z|^p.$$

Analogously for every $(\xi, z) \in \mathbb{Z}^d \times \mathbb{R}^m$, $f(\xi, z) \leq M(\xi)|z|^p$. Moreover, by (G1) we have that for every $\xi \in \mathbb{Z}^d$ and $\eta > 0$ there exists $R_\eta > 0$ such that

$$\sum_{|\xi| > R_\eta} M(\xi) < \eta.$$

Remark 2.2. Hypothesis (L) follows directly if f satisfies (H), (G), and $f(\xi, \cdot)$ is convex for every $\xi \in \mathbb{R}^d$.

We also introduce the 'truncated' discrete functionals $F^T_{\varepsilon,\delta} : L^p(\Omega, \mathbb{R}^m) \to [0, +\infty]$ defined for every $T > 0$ as

$$F^T_{\varepsilon,\delta}(u) := \begin{cases} \displaystyle\sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq T}} \sum_{\alpha \in \Omega_\varepsilon(\xi)} \varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m),\ u \equiv 0 \text{ on } P_\delta \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Let us also consider the unconstrained family of functionals $\mathcal{F}_\varepsilon : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \sum_{\xi \in \mathbb{Z}^d} \sum_{\alpha \in \Omega_\varepsilon(\xi)} \varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise} \end{cases} \tag{2.5}$$

We introduce a localized version of such functionals: given an open set $A \subset \Omega$ we isolate the contribution due to interactions within A by setting, for $u \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^m)$,

$$\mathcal{F}_\varepsilon(u, A) := \sum_{\xi \in \mathbb{Z}^d} \sum_{\alpha \in A_\varepsilon(\xi)} \varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)). \tag{2.6}$$

The truncated version of (2.6) is defined, for any $T > 0$, as

$$\mathcal{F}^T_\varepsilon(u, A) := \sum_{\substack{\xi \in \mathbb{Z}^d \\ |\xi| \leq T}} \sum_{\alpha \in A_\varepsilon(\xi)} \varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)). \tag{2.7}$$

In [8], Theorem 4.1, it is proved the following Γ-convergence result.

Theorem 2.3. *Let $\mathcal{F}_\varepsilon$ be defined by (2.5), with f satisfying assumptions (H) and (G). Then $(\mathcal{F}_\varepsilon)$ Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^m)$-topology to the functional $\mathcal{F}: L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ defined as*

$$\mathcal{F}(u) = \begin{cases} \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.8}$$

where $f_{\mathrm{hom}}: \mathbb{R}^{m\times d} \to [0,+\infty)$ is given by the following homogenization formula:

$$f_{\mathrm{hom}}(M) := \lim_{h\to+\infty} \frac{1}{h^d} \min \Big\{ \sum_{\xi\in\mathbb{Z}^d} \sum_{\alpha\in(Q_h)_1(\xi)} f(\xi, D_1^\xi u(\alpha)), \quad u \in \mathcal{A}_{1,M}(Q_h) \Big\} \tag{2.9}$$

with

$$\mathcal{A}_{1,M}(Q_h) := \{u \in \mathcal{A}_1(\mathbb{R}^d;\mathbb{R}^m) : u(\alpha) = M\alpha \text{ if } (\alpha + [-1,1]^d) \cap Q_h^c \neq \emptyset\}.$$

Moreover, there exist two positive constants $c < C$ such that, for every $M \in \mathbb{R}^{m\times d}$

$$c|M|^p \leq f_{\mathrm{hom}}(M) \leq C|M|^p. \tag{2.10}$$

We now state the main result of the paper.

Theorem 2.4. *Let $F_{\varepsilon,\delta}$ be defined by (2.3), with f satisfying assumptions (H)-(G)-(L) and $1 < p < d$. Assume moreover that*

$$\lim_{\delta\to0} \frac{r_\delta}{\delta^{\frac{d}{d-p}}} = \gamma \tag{2.11}$$

and that $\delta = \delta_\varepsilon$ is such that

$$\lim_{\varepsilon\to0} \frac{\varepsilon}{r_{\delta_\varepsilon}} = 0 \tag{2.12}$$

for some real number $\gamma > 0$. Then

$$\Gamma(L^p) - \lim_{\varepsilon\to0} F_{\varepsilon,\delta_\varepsilon}(u) = \begin{cases} \int_\Omega f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + \gamma^{d-p}\int_\Omega \varphi(u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_{hom}(M)$ is defined by (2.9) and for every $z \in \mathbb{R}^m$

$$\varphi(z) := \inf \Big\{ \int_{\mathbb{R}^d} f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x : u \equiv 0 \text{ in } Q_1, u - z \in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m) \Big\}. \tag{2.13}$$

with $p^* = \frac{pd}{d-p}$.

Remark 2.5. Taking into account the non-degeneracy of the capacitary density (2.13) proved in Proposition 4.8 below, and arguing by comparison, one easily infers that the result stated in Theorem 2.4 can be "continuously" extended to the case $\gamma = 0$ and $\gamma = +\infty$. More in details, if $\gamma = 0$ and (2.12) holds, then the functionals $F_{\varepsilon,\delta_\varepsilon}$

Γ-converge to the energy functional defined in (2.8), while, if $\gamma = +\infty$ and (2.12) holds, the Γ-limit of the functionals $F_{\varepsilon,\delta_\varepsilon}$ is trivially 0 if $u \equiv 0$ and $+\infty$ otherwise.

For later use and reader's convenience we redefine the density functions we have introduced so far in the case $f(\xi, z)$ is replaced by $f^T(\xi, z) = \chi_{B_T}(\xi) f(\xi, z)$. More precisely we set

$$F^T(u) = \int_\Omega f^T_{\rm hom}(\nabla u)\,{\rm d}x + \gamma^{d-p}\int_\Omega \varphi^T(u)\,{\rm d}x$$

with

$$f^T_{\rm hom}(M) = \lim_{h\to+\infty}\frac{1}{h^d}\min\Big\{\sum_{\xi\in\mathbb{Z}^d}\sum_{\alpha\in(Q_h)_1(\xi)} f^T(\xi, D^\xi_1 u(\alpha)), \quad u\in\mathcal{A}_{1,M}(Q_h)\Big\} \tag{2.14}$$

and

$$\varphi^T(z) = \inf\Big\{\int_{\mathbb{R}^d} f^T_{\rm hom}(\nabla u)\,{\rm d}x : u\equiv 0 \text{ in } Q_1, u - z\in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)\cap W^{1,p}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^m)\Big\}. \tag{2.15}$$

3. Preliminary results

For every open set $A\subset\mathbb{R}^d$ and $u\in L^p(A;\mathbb{R}^m)$ let us define

$$G^{1,p}_\varepsilon(u, A) := \sum_{k=1}^d\sum_{\alpha\in A_\varepsilon(e_k)}\varepsilon^d |D^{e_k}_\varepsilon u(\alpha)|^p \tag{3.1}$$

where $A_\varepsilon(e_k)$ and $D^{e_k}_\varepsilon u(\alpha)$ are defined by (2.1) and (2.2), respectively. Concerning these type of energies, the following lemma proved in [8] is instrumental for our analysis.

Lemma 3.1 ([8], Lem. 3.6)**.** *Let $A\in\mathcal{A}(\Omega)$, and set $\tilde A_\varepsilon := \{x\in A : {\rm dist}(x,\partial A) > 2\sqrt{d}\varepsilon\}$. Then there exists a positive constant C such that for any $\xi\in\mathbb{Z}^d$ and $u\in\mathcal{A}_\varepsilon(\Omega;\mathbb{R}^m)$ there holds*

$$\sum_{\alpha\in(\tilde A_\varepsilon)_\varepsilon(\xi)} |D^\xi_\varepsilon u(\alpha)|^p \le C\sum_{k=1}^d\sum_{\alpha\in A_\varepsilon(e_k)} |D^{e_k}_\varepsilon u(\alpha)|^p. \tag{3.2}$$

Remark 3.2. As a direct consequence of the lemma above we can show that long range interactions are negligible for a sequence u_ε such that $G^{1,p}_\varepsilon(u_\varepsilon,\Omega)$ is uniformly bounded. Let us consider $\Omega'\subset\subset\Omega$ and $\eta>0$. Then,

$$\begin{aligned}\mathcal{F}_\varepsilon(u,\Omega') &= \sum_{\xi\in\mathbb{Z}^d}\sum_{\alpha\in(\Omega')_\varepsilon(\xi)}\varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha))\\ &= \sum_{|\xi|\le R_\eta}\sum_{\alpha\in(\Omega')_\varepsilon(\xi)}\varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)) + \sum_{|\xi|> R_\eta}\sum_{\alpha\in(\Omega')_\varepsilon(\xi)}\varepsilon^d f(\xi, D^\xi_\varepsilon u(\alpha)).\end{aligned}$$

Let us call $\tilde{\Omega}_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 2\sqrt{d}\varepsilon\}$. It is clear that for ε small $\Omega' \subset\subset \tilde{\Omega}_\varepsilon$. By (3.2) and Remark 2.1 we get

$$\begin{aligned}\sum_{|\xi|>R_\eta} \sum_{\alpha\in(\Omega')_\varepsilon(\xi)} \varepsilon^d f(\xi, D_\varepsilon^\xi u(\alpha)) &\leq \sum_{|\xi|>R_\eta} \sum_{\alpha\in(\tilde{\Omega}_\varepsilon)_\varepsilon(\xi)} \varepsilon^d f(\xi, D_\varepsilon^\xi u(\alpha)) \\ &\leq \sum_{|\xi|>R_\eta} M(\xi) \sum_{\alpha\in(\tilde{\Omega}_\varepsilon)_\varepsilon(\xi)} \varepsilon^d |D_\varepsilon^\xi u(\alpha)|^p \\ &\leq \sum_{|\xi|>R_\eta} M(\xi) \sum_{k=1}^d \sum_{\alpha\in(\tilde{\Omega}_\varepsilon)_\varepsilon(e_k)} \varepsilon^d |D_\varepsilon^{e_k} u(\alpha)|^p \leq C\eta.\end{aligned}$$

Hence, by the arbitrariness of η, we get the claim.

Remark 3.3. By Remark 2.1 we deduce that, if $T > 1$

$$\mathcal{F}_\varepsilon^T(u, A) \geq \lambda_0 G_\varepsilon^{1,p}(u, A).$$

Indeed,

$$\begin{aligned}\mathcal{F}_\varepsilon^T(u, A) = \sum_{|\xi|\leq T} \sum_{\alpha\in A_\varepsilon(\xi)} \varepsilon^d f(\xi, D_\varepsilon^\xi u(\alpha)) &\geq \sum_{k=1}^d \sum_{\alpha\in A_\varepsilon(e_k)} \varepsilon^d f(e_k, D_\varepsilon^{e_k} u(\alpha)) \\ \geq \sum_{k=1}^d \sum_{\alpha\in A_\varepsilon(e_k)} \varepsilon^d \lambda_0 |D_\varepsilon^{e_k} u(\alpha)|^p &= \lambda_0 G_\varepsilon^{1,p}(u, A).\end{aligned}$$

We recall the discrete version of the Poincarè-Wirtinger inequality and its rescaled version that have been proved in [12], Lemma 5.1, Lemma 5.2.

Lemma 3.4 ([12], Lem. 5.1)**.** *Let $\Omega \subset \mathbb{R}^d$ be a finite union of rectangles, and $p > 1$. There exists ε_0 and a constant $C = C(p, \Omega)$ such that for all $\varepsilon < \varepsilon_0$ and $u : \Omega_\varepsilon \to \mathbb{R}^m$, having set $\tilde{u} = \frac{1}{\sharp\Omega_\varepsilon} \sum_{\alpha\in\Omega_\varepsilon} u(\alpha)$, we have*

$$\sum_{\alpha\in\Omega_\varepsilon} \varepsilon^d |u(\alpha) - \tilde{u}|^p \leq C G_\varepsilon^{1,p}(u, \Omega). \tag{3.3}$$

Lemma 3.5 ([12], Lem. 5.2)**.** *Let Ω, ε_0, p be as in Lemma 3.4 and let C be the constant in (3.3). Let $\delta > 0$ and set $\Omega_\varepsilon^\delta = \delta\Omega \cap \varepsilon\mathbb{Z}^d$. Then, for $\varepsilon < \varepsilon_0\delta$ and for all $u : \Omega_\varepsilon^\delta \to \mathbb{R}^m$ we have*

$$\sum_{\alpha\in\Omega_\varepsilon^\delta} \varepsilon^d |u(\alpha) - \tilde{u}|^p \leq C\delta^p \sum_{k=1}^d \sum_{\alpha\in(\Omega_\varepsilon^\delta)_\varepsilon(e_k)} \varepsilon^d \left|\frac{u(\alpha + e_k\varepsilon) - u(\alpha)}{\varepsilon}\right|^p,$$

where $\tilde{u} = (\sharp\Omega_\varepsilon^\delta)^{-1} \sum_{\alpha\in\Omega_\varepsilon^\delta} u(\alpha)$.

Remark 3.6. The inequality (3.3) is still true if we consider the mean over a sufficiently regular set $E \subset \Omega$ with $|E| > 0$, that is $\tilde{u} = \frac{1}{\sharp E_\varepsilon} \sum_{\alpha\in E_\varepsilon} u(\alpha)$. We note that in this case $\frac{\sharp\Omega_\varepsilon}{\sharp E_\varepsilon} \leq C$, where C that does not depend on ε.

Indeed

$$\begin{aligned}\sum_{\alpha\in\Omega_\varepsilon}\Big|u(\alpha)-\tilde{u}\Big|^p &= \sum_{\alpha\in\Omega_\varepsilon}\Big|u(\alpha)-\frac{1}{\sharp E_\varepsilon}\sum_{\beta\in E_\varepsilon}u(\beta)\Big|^p = \sum_{\alpha\in\Omega_\varepsilon}\Big|\frac{1}{\sharp E_\varepsilon}\sum_{\beta\in E_\varepsilon}u(\alpha)-u(\beta)\Big|^p\\ &\le \sum_{\alpha\in\Omega_\varepsilon}\frac{1}{\sharp E_\varepsilon}\sum_{\beta\in E_\varepsilon}|u(\alpha)-u(\beta)|^p \le \sum_{\alpha\in\Omega_\varepsilon}\frac{1}{\sharp E_\varepsilon}\sum_{\beta\in \Omega_\varepsilon}|u(\alpha)-u(\beta)|^p\\ &= \sum_{\alpha\in\Omega_\varepsilon}\frac{\sharp\Omega_\varepsilon}{\sharp E_\varepsilon}\frac{1}{\sharp\Omega_\varepsilon}\sum_{\beta\in \Omega_\varepsilon}|u(\alpha)-u(\beta)|^p \le C\sum_{\alpha\in\Omega_\varepsilon}\frac{1}{\sharp\Omega_\varepsilon}\sum_{\beta\in \Omega_\varepsilon}|u(\alpha)-u(\beta)|^p.\end{aligned}$$

Then we can proceed as in [12].

4. Supporting results

In this section we state and prove some instrumental results for the proof of Theorem 2.4.

4.1. Gagliardo-Nirenberg-Sobolev type inequality

In this subsection we state a crucial result for our analysis which may be of independent interest and resembles the Gagliardo-Nirenberg-Sobolev inequality in Sobolev spaces. Its proof is a direct consequence of the Gagliardo-Nirenberg-Sobolev type inequality for non-local functionals [10], that we recall in the following.
For $r>0$ and given $p\ge 1$, let $T_\varepsilon : L^p(\mathbb{R}^d;\mathbb{R}^m)\to \mathcal{A}_{\tilde{r}\varepsilon}(\mathbb{R}^d;\mathbb{R}^m)$ be defined by

$$T_\varepsilon u(x) := \frac{1}{(\tilde{r}\varepsilon)^d}\int_{\tilde{r}\varepsilon k+[0,\tilde{r}\varepsilon)^d} u(y)\,dy \quad \text{ on } \tilde{r}\varepsilon k+[0,\tilde{r}\varepsilon)^d,\ k\in\mathbb{Z}^d, \tag{4.1}$$

where $\tilde{r} := \frac{r}{\sqrt{d+3}}$, and let us consider the functionals

$$\mathcal{G}^{r,p}_\varepsilon(u,A) = \int_{B_r}\int_{A_\varepsilon(\xi)}\left|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right|^p \mathrm{d}x\,\mathrm{d}\xi \tag{4.2}$$

defined for every open set $A\subset\mathbb{R}^d$ and $u\in L^p(A;\mathbb{R}^m)$

Theorem 4.1 ([10], Thm. 5.1)**.** *Let $p\in[1,d)$. Then there exists a constant $C=C(p,d,r)>0$ such that for every $u\in L^p(\mathbb{R}^d;\mathbb{R}^m)$*

$$\left(\int_{\mathbb{R}^d}|T_\varepsilon u(x)|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \le C\mathcal{G}^{r,p}_\varepsilon(u,\mathbb{R}^d),$$

where $p^* := \frac{pd}{d-p}$.

Here we present a discrete version of the previous theorem.

Theorem 4.2. *Let $p\in[1,d)$ and $p^* := \frac{pd}{d-p}$. Then there exists a constant $C=C(p,d)>0$ such that for every $u\in\mathcal{A}_\varepsilon(\mathbb{R}^d;\mathbb{R}^m)\cap L^p(\mathbb{R}^d,\mathbb{R}^m)$*

$$\left(\int_{\mathbb{R}^d}|u(x)|^{p^*}\,\mathrm{d}x\right)^{\frac{p}{p^*}} \le CG^{1,p}_\varepsilon(u,\mathbb{R}^d), \tag{4.3}$$

where $G^{1,p}_\varepsilon(u,\mathbb{R}^d)$ is given by (3.1).

Proof. We first observe that when $\tilde{r} = 1$, which means $r = \sqrt{d+3}$, $T_\varepsilon u(x) = u(x) = u(\alpha)$ for $x \in \alpha + [0,\varepsilon)^d$. Thus, by Theorem 4.1 it is sufficient to prove that for $r = \sqrt{d+3}$

$$\mathcal{G}^{r,p}_\varepsilon(u,\mathbb{R}^d) \leq C G^{1,p}_\varepsilon(u,\mathbb{R}^d).$$

As consequence of [13], Lemma 4.1, applied with $\Omega = \mathbb{R}^d$, we have that for every $r > 0$

$$\mathcal{G}^{r,p}_\varepsilon(u,\mathbb{R}^d) \leq C \mathcal{G}^{1,p}_\varepsilon(u,\mathbb{R}^d).$$

Hence, it is sufficient to prove that

$$\mathcal{G}^{1,p}_\varepsilon(u,\mathbb{R}^d) \leq C G^{1,p}_\varepsilon(u,\mathbb{R}^d).$$

The energy $\mathcal{G}^{1,p}_\varepsilon(u,\mathbb{R}^d)$ can be rewritten as

$$\mathcal{G}^{1,p}_\varepsilon(u,\mathbb{R}^d) = \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \int_{B_1} \int_{\alpha\in[0,\varepsilon)^d} \left| \frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon} \right|^p \mathrm{d}x\,\mathrm{d}\xi.$$

Now, we fix $\xi \in B_1$ and $x \in \alpha + [0,\varepsilon)^d$. We set

$$I^\varepsilon_\alpha := \{\beta \in \varepsilon\mathbb{Z}^d : |\beta - \alpha| \leq \sqrt{d}\varepsilon\}.$$

It holds that if $x \in \alpha + [0,\varepsilon)^d$, then $x + \varepsilon\xi \in \beta + [0,\varepsilon)^d$, for some $\beta \in I^\varepsilon_\alpha$. Hence, $u(x+\varepsilon\xi) - u(x) = u(\beta) - u(\alpha)$. Since there exists $\alpha^1, \dots, \alpha^m \in I^\varepsilon_\alpha$, for $m \in \{2, \dots, d+1\}$ such that $\alpha^1 = \alpha, \alpha^m = \beta$ and $\alpha^{i+1} = \alpha^i \pm \varepsilon e_k$ for $k = 1, \dots, d$, it holds

$$|u(\beta) - u(\alpha)|^p = \left| \sum_{i=1}^m u(\alpha^{i+1}) - u(\alpha^i) \right|^p$$

By triangular inquality we get

$$|u(\beta) - u(\alpha)|^p \leq C \sum_{i=1}^{m-1} |u(\alpha^{i+1}) - u(\alpha^i)|^p \tag{4.4}$$

Now, we set

$$S^\varepsilon_\alpha := \bigcup_{\beta,\beta'\in I^\varepsilon_\alpha} (\beta + [0,\varepsilon]^d) \cap (\beta' + [0,\varepsilon]^d)$$

and

$$S^{\varepsilon,\xi}_\alpha := \{x = y + t\xi : \ y \in S^\varepsilon_\alpha, \ -\varepsilon \leq t \leq \varepsilon\} \tag{4.5}$$

We note that $|S_\alpha^{\varepsilon,\xi}| \le C\mathcal{H}^{d-1}(S_\alpha^\varepsilon)\varepsilon|\xi| \le C\varepsilon^d|\xi|$. Moreover, if $x+\varepsilon\xi \in \beta + [0,\varepsilon)^d$ for some $\beta \in I_\alpha^\varepsilon$, such that $\alpha \neq \beta$, then $x \in S_\alpha^{\varepsilon,\xi}$. By (4.4) we have

$$\begin{aligned}
\mathcal{G}_\varepsilon^{1,p}(u,\mathbb{R}^d) &\le C\int_{B_1}\sum_{\alpha\in\varepsilon\mathbb{Z}^d}\int_{S_\alpha^{\varepsilon,\xi}}\left|\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right|^p \mathrm{d}x,\mathrm{d}\xi\\
&\le C\int_{B_1}\sum_{\alpha\in\varepsilon\mathbb{Z}^d}|S_\alpha^{\varepsilon,\xi}|\sum_{\beta\in I_\alpha^\varepsilon}\left|\frac{u(\beta)-u(\alpha)}{\varepsilon}\right|^p \mathrm{d}\xi\\
&\le C\int_{B_1}|\xi|\,\mathrm{d}\xi\sum_{\alpha\in\varepsilon\mathbb{Z}^d}\varepsilon^d\sum_{\beta\in I_\alpha^\varepsilon}\sum_{i=1}^{m-1}\left|\frac{u(\alpha^{i+1})-u(\alpha^i)}{\varepsilon}\right|^p\\
&\le C\int_{B_1}|\xi|\,\mathrm{d}\xi\sum_{k=1}^d\sum_{\alpha\in\varepsilon\mathbb{Z}^d}\varepsilon^d|D_\varepsilon^{e_k}u(\alpha)|^p\\
&= CG_\varepsilon^{1,p}(u,\mathbb{R}^d),
\end{aligned}$$

where the last inequality holds since the same interaction between nearest-neighbours is counted a finite number of times. □

4.2. Two technical lemmas

In this section we recall two technical lemmas which will be used in the proof of Theorem 2.4 and have been proved in [9]. The first one is the so called "joining lemma", which allows to reduce our Γ-convergence to sequences of converging functions that are constants on suitable annuli surrounding the perforations. The second one is the "truncation lemma" that, by the composition with a suitable Lipschitz function, allows us to replace a given sequence u_j with equibounded energies and L^p-norms, by a new sequence uniformly bounded in L^∞, with a small gap in energy.

Lemma 4.3 ([9], Lem. 6.1)**.** *Let $\{u_j\}_j$ be a sequence such that $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ and $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ for some $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Assume that*

$$\sup_j \sum_{\substack{\xi\in\mathbb{Z}^d\\|\xi|\le T}}\sum_{\alpha\in\Omega_\varepsilon(\xi)}\varepsilon_j^d f(\xi, D_{\varepsilon_j}^\xi u_j(\alpha)) < +\infty.$$

Let (ρ_j) be a sequence of the form $\rho_j = \beta\delta_j$ where $\beta < \frac{1}{2}$ with $\varepsilon_j = o(\rho_j)$ as $j \to +\infty$. We denote by $Z_j(\Omega)$ the set of indices

$$Z_j(\Omega) = \left\{ i \in \mathbb{Z}^d \cap \frac{\Omega}{\delta_j} : \mathrm{dist}(i\delta_j,\partial\Omega) > \delta_j \right\}. \tag{4.6}$$

Let $k\in\mathbb{N}$ be fixed. Then for all $i \in Z_j(\Omega)$ there exists $k_i \in \{0,\dots,k-1\}$ such that, having set

$$C_j^i := Q\Big(i\delta_j, \Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j\Big)\Big\backslash Q\Big(i\delta_j, \Big[\frac{\rho_j}{\varepsilon_j}2^{-k_i-1}\Big]\varepsilon_j\Big)$$

$$\rho_j^i := \Big[\frac{3}{4}\frac{\rho_j}{\varepsilon_j}2^{-k_i}\Big]\varepsilon_j,$$

$$u_j^i := \frac{1}{\sharp C_j^i} \sum_{\alpha \in C_j^i} u_j(\alpha),$$

there exists a sequence $w_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ *such that* $w_j \to u$ *in* $L^p(\Omega;\mathbb{R}^m)$ *and*

$$w_j = u_j \text{ on } \Omega_j \setminus \bigcup_{i\in Z_j(\Omega)} C_j^i, \tag{4.7}$$

$$w_j = u_j^i \text{ on } Q(i\delta_j, \rho_j^i + \varepsilon_j T) \setminus Q(i\delta_j, \rho_j^i - \varepsilon_j T), \tag{4.8}$$

$$\left| \sum_{\substack{\xi\in\mathbb{Z}^d\\ |\xi|\le T}} \sum_{\alpha\in\Omega_{\varepsilon_j}(\xi)} \varepsilon_j^d (f(\xi, D^\xi_{\varepsilon_j} u_j(\alpha)) - f(\xi, D^\xi_{\varepsilon_j} w_j(\alpha))) \right| \le \frac{C}{k}. \tag{4.9}$$

for j *large enough.*

Remark 4.4. In [9], Lemma 6.1 condition (4.8) is replaced by $w_j = u_j^i$ on $\partial Q(i\delta_j, \rho_j^i)$, however it can be easly inferred from that proof that w_j can be chosen so that (4.8) is satisfied.

Lemma 4.5 ([9], Lem. 7.2)**.** *Let* $\{u_j\}_j$ *be a sequence such that* $u_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ *with* $\sup_j(\mathcal{F}_\varepsilon(u_j) + \|u_j\|_{L^p(\Omega;\mathbb{R}^m)}) < +\infty$*. Then for every* $\eta > 0$ *and* $L \in \mathbb{N}$ *there exist a subsequence (not relabeled), a constant* $R_L > L$ *and a Lipschitz function* $t_L : \mathbb{R}^m \to \mathbb{R}^m$ *with Lipschitz constant* 1 *such that* $t_L(z) = z$ *if* $|z| < R_L$*,* $t_L(z) = 0$ *if* $|z| > 2R_L$ *and it holds*

$$\liminf_j \mathcal{F}_{\varepsilon_j}(t_L(u_j)) \le \liminf_j \mathcal{F}_{\varepsilon_j}(u_j) + \eta.$$

We recall a compactness result in the strong L^p topology for sequences of functions whose energy defined in (4.2) is uniformly bounded.

Theorem 4.6 ([13], Thm. 4.2)**.** *Let* A *be any open set of* $\mathbb{R}^d$ *with bounded Lipschitz boundary. Let* $\{u_\varepsilon\}_\varepsilon \subset L^p(A;\mathbb{R}^m)$ *be such that for some* $r > 0$

$$\sup_{\varepsilon>0}\{\|u_\varepsilon\|_{L^p(A;\mathbb{R}^m)} + \mathcal{G}^{r,p}_\varepsilon(u_\varepsilon, A)\} < +\infty.$$

Then, given $\varepsilon_j \to 0$*,* $\{u_{\varepsilon_j}\}_j$ *is relatively compact in* $L^p(A;\mathbb{R}^m)$ *and every limit of a converging subsequence is in* $W^{1,p}(A;\mathbb{R}^m)$*.*

4.3. Approximating capacitary energy densities

In this subsection we introduce and investigate the main properties of suitable energy densities defined through minimum problems of capacitary type.
For any $\varepsilon > 0$, $T > 2$, $R > 1 + \varepsilon T$, $z \in \mathbb{R}^m$, set

$$\varphi_{\varepsilon,T,R}(z) := \inf\{\mathcal{F}^T_\varepsilon(v, Q_R) : v \in \mathcal{A}_{\varepsilon,T,z}(Q_R;\mathbb{R}^m)\} \tag{4.10}$$

where $\mathcal{F}^T_\varepsilon$ is defined as in (2.7) and

$$\mathcal{A}_{\varepsilon,T,z}(Q_R;\mathbb{R}^m) := \{v \in \mathcal{A}_\varepsilon(Q_R;\mathbb{R}^m) : v \equiv 0 \text{ in } Q_1, v \equiv z \text{ on } \partial^{\varepsilon T} Q_R\},$$

with the notation $\partial^{\varepsilon T} Q_R = Q_{R+\varepsilon T} \setminus Q_{R-\varepsilon T}$.
Given $v \in \mathcal{A}_{\varepsilon,T,z}(Q_R;\mathbb{R}^m)$, consider the function $u := v - z$. Hence $u \equiv -z$ in Q_1 and $u \equiv 0$ on $\partial^{\varepsilon T} Q_R$. Then we can extend u to the whole $\mathbb{R}^d$ by setting

$$u \equiv 0 \text{ on } \mathbb{R}^d \setminus Q_{R-\varepsilon T}.$$

Identifying u with its extension, it is clear that $\mathcal{F}^T_\varepsilon(u, Q_R) = \mathcal{F}^T_\varepsilon(u, \mathbb{R}^d)$. Hence (4.10) can be written as

$$\varphi_{\varepsilon,T,R}(z) = \inf\{\mathcal{F}^T_\varepsilon(u, \mathbb{R}^d) : u \in \mathcal{A}_\varepsilon(\mathbb{R}^d;\mathbb{R}^m), u \equiv -z \text{ in } Q_1, u \equiv 0 \text{ in } \mathbb{R}^d \setminus Q_{R-\varepsilon T}\}. \tag{4.11}$$

Remark 4.7. If f satisfies (G), (H) and (L), by applying the direct method of Calculus of Variations, the infimum problem defining $\varphi_{\varepsilon,T,R}(z)$ in (4.10) is in fact a minimum.

Next result establishes growth conditions of order p for $\varphi_{\varepsilon,T,R}(\cdot)$.

Proposition 4.8. *Let f satisfy assumptions (H) and (G), and let $T > 2$, $\varepsilon_0 > 0$, and R be fixed such that $R - 2\varepsilon_0 T \geq 2$. Then, for every $0 < \varepsilon \leq \varepsilon_0$ there exists $c_1, c_2 > 0$ such that*

$$c_1|z|^p \leq \varphi_{\varepsilon,T,R}(z) \quad \forall z \in \mathbb{R}^m, \tag{4.12}$$

$$\varphi_{\varepsilon,T,R}(z) \leq c_2|z|^p \quad \forall z \in \mathbb{R}^m. \tag{4.13}$$

Proof. We first prove inequality (4.12). The proof relies on a suitable lower bound of $\mathcal{F}^T_\varepsilon(u, Q_R)$. In order to avoid many technicalities we detail the proof in the case $d = 2$. We will associate to any admissible function u a piecewise affine function obtained by linearly interpolating the values of u on a triangulation of the lattice $\mathbb{Z}^2$. The argument can be generalized to any dimension by linearly interpolating the values of u on a partition of the cells of the lattice $\mathbb{Z}^d$ in d-simplices, known in literature as the Kuhn decomposition (see for example [14]). Set

$$T^- := \{(x_1, x_2) \in [0,1]^2 : x_2 \leq 1 - x_1\},$$

$$T^+ := \{(x_1, x_2) \in [0,1]^2 : x_2 \geq 1 - x_1\},$$

and, given any admissible function u in (4.11), let us define the piecewise-affine interpolation of u on the cells of the lattice as follows

$$\hat{u}(x) := u(\alpha) + \frac{u(\alpha + \varepsilon e_1) - u(\alpha)}{\varepsilon}(x_1 - \alpha_1) + \frac{u(\alpha + \varepsilon e_2) - u(\alpha)}{\varepsilon}(x_2 - \alpha_2) \quad \text{if } x \in \alpha + \varepsilon T^-$$

$$\begin{aligned}\hat{u}(x) = u(\alpha + \varepsilon(e_1 + e_2)) &+ \frac{u(\alpha + \varepsilon(e_1 + e_2)) - u(\alpha + \varepsilon e_2)}{\varepsilon}(x_1 - \alpha_1 - \varepsilon) \\ &+ \frac{u(\alpha + \varepsilon(e_1 + e_2)) - u(\alpha + \varepsilon e_1)}{\varepsilon}(x_2 - \alpha_2 - \varepsilon) \quad \text{if } x \in \alpha + \varepsilon T^+\end{aligned}$$

Since $u \equiv -z$ in Q_1 and $u \equiv 0$ in $\mathbb{R}^d \setminus Q_{R-\varepsilon T}$ we infer that, for ε small enough $\hat{u} \equiv -z$ in $Q_{\frac{1}{2}}$ and $\hat{u} \equiv 0$ on $\mathbb{R}^d \setminus Q_R$. Let us call $P^{\varepsilon,1}_\alpha := \alpha + \varepsilon P_1$ and $P^{\varepsilon,2}_\alpha := \alpha + \varepsilon P_2$ where

$$P_1 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, -x_1 \leq x_2 \leq 1 - x_1\},$$

$$P_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 1, -x_2 \leq x_1 \leq 1 - x_2\}.$$

One can easily show that

$$\partial_{x_k}\hat{u}(x) = D_\varepsilon^{e_k} u(\alpha), \quad k = 1,2 \text{ and } x \in P_\alpha^{\varepsilon,k}.$$

Since $|P_\alpha^{\varepsilon,1}| = |P_\alpha^{\varepsilon,2}| = \varepsilon^2$, we get the following estimate

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla \hat{u}|^p &\le C\Big(\int_{\mathbb{R}^2} |\partial_{x_1}\hat{u}|^p + |\partial_{x_2}\hat{u}|^p\Big) \\
&= C \sum_{\alpha\in\varepsilon\mathbb{Z}^2} \Big(\int_{P_\alpha^{\varepsilon,1}} |\partial_{x_1}\hat{u}|^p + \int_{P_\alpha^{\varepsilon,2}} |\partial_{x_2}\hat{u}|^p\Big) \\
&= C \sum_{\alpha\in\varepsilon\mathbb{Z}^2} \varepsilon^2 (|D_\varepsilon^{e_1} u(\alpha)|^p + |D_\varepsilon^{e_2} u(\alpha)|^p) \\
&= C \sum_{k=1}^{2}\sum_{\alpha\in\varepsilon\mathbb{Z}^2} \varepsilon^2 |D_\varepsilon^{e_k} u(\alpha)|^p = C G_\varepsilon^{1,p}(u,\mathbb{R}^2).
\end{aligned}$$

By Remark 3.3 if follows

$$\mathcal{F}_\varepsilon^T(u,\mathbb{R}^2) \ge \lambda_0 G_\varepsilon^{1,p}(u,\mathbb{R}^2) \ge C\int_{\mathbb{R}^2} |\nabla\hat{u}|^p \ge |z|^p \operatorname{cap}_p(Q_{\frac{1}{2}},\mathbb{R}^2) = c_1|z|^p.$$

We now prove inequality (4.13). Consider a function $u \in C_c^1(Q_{R-2\varepsilon T};\mathbb{R}^m)$ such that $u \equiv -z$ in $Q_{\frac{3}{2}}$, and for every $\alpha\in\varepsilon\mathbb{Z}^d$ let us define the discrete function

$$\hat{u}(\alpha) = \frac{1}{\varepsilon^d}\int_{\alpha+[0,\varepsilon)^d} u(x)\,\mathrm{d}x.$$

Notice that $\hat{u}$ is admissible for the problem defining $\varphi_{\varepsilon,T,R}$ in (4.11). We show that there exists a constant C such that, for $\varepsilon \le \varepsilon_0$

$$G_\varepsilon^{1,p}(\hat{u},\mathbb{R}^d) \le C\int_{\mathbb{R}^d} |\nabla u|^p\,\mathrm{d}x. \tag{4.14}$$

Indeed, fixed $k = 1,\dots,d$ and $x\in\mathbb{R}^d$, we have

$$\frac{u(x+\varepsilon e_k) - u(x)}{\varepsilon} = \int_0^1 \partial_{x_k} u(x+\varepsilon e_k t)\,\mathrm{d}t. \tag{4.15}$$

By Jensen's inequality, Fubini's Theorem and (4.15) we have the following estimate

$$\begin{aligned}
G^{1,p}_\varepsilon(\hat u,\mathbb{R}^d) &= \sum_{k=1}^d \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \varepsilon^d |D^{e_k}_\varepsilon \hat u(\alpha)|^p \\
&= \sum_{k=1}^d \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \varepsilon^d \left|\frac{1}{\varepsilon^d}\left[\int_{\alpha+[0,\varepsilon)^d} \frac{u(x+\varepsilon e_k)-u(x)}{\varepsilon}\,\mathrm{d}x\right]\right|^p \\
&\le \sum_{k=1}^d \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \int_{\alpha+[0,\varepsilon)^d} \left|\frac{u(x+\varepsilon e_k)-u(x)}{\varepsilon}\right|^p \mathrm{d}x \\
&\le \sum_{k=1}^d \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \int_{\alpha+[0,\varepsilon)^d} \int_0^1 |\nabla u(x+\varepsilon e_k t)|^p\,\mathrm{d}t\,\mathrm{d}x \\
&= \sum_{k=1}^d \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \int_0^1 \int_{\alpha+[0,2\varepsilon)^d} |\nabla u(x)|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\int_{\mathbb{R}^d} |\nabla u(x)|^p\,\mathrm{d}x.
\end{aligned}$$

By (G1), Lemma 3.1 and (4.14) we get

$$\begin{aligned}
\mathcal{F}^T_\varepsilon(\hat u,\mathbb{R}^d) &\le \sum_{\substack{\xi\in\mathbb{Z}^d\\ |\xi|\le T}} \sum_{\alpha\in\varepsilon\mathbb{Z}^d} \varepsilon^d M(\xi) |D^\xi_\varepsilon \hat u(\alpha)|^p \\
&\le C\sum_{\xi\in\mathbb{Z}^d} M(\xi) G^{1,p}_\varepsilon(\hat u,\mathbb{R}^d) \\
&\le C\sum_{\xi\in\mathbb{Z}^d} M(\xi)\int_{\mathbb{R}^d} |\nabla u(x)|^p\,\mathrm{d}x.
\end{aligned}$$

Hence, by (4.11) and a density argument

$$\begin{aligned}
\varphi_{\varepsilon,T,R}(z) &\le C\sum_{\xi\in\mathbb{Z}^d} M(\xi)\inf\left\{\int_{Q_{R-2\varepsilon T}} |\nabla u|^p\,\mathrm{d}x : u\equiv -z \text{ in } Q_{\frac32}, u\in C^1_c(Q_{R-2\varepsilon T};\mathbb{R}^m)\right\} \\
&= C\sum_{\xi\in\mathbb{Z}^d} M(\xi)\,\mathrm{cap}_p(Q_{\frac32},Q_{R-2\varepsilon T})|z|^p \le C\sum_{\xi\in\mathbb{Z}^d} M(\xi)\,\mathrm{cap}_p(Q_{\frac32},Q_2)|z|^p = c_2|z|^p.
\end{aligned}$$

□

In the next proposition, we show that the functions $\varphi_{\varepsilon,T,R}$ are uniformly locally Lipschitz continuous.

Proposition 4.9. *Let f satisfy assumptions (H), (G), (L). Then there exists a constant $C>0$ independent of ε, T and R such that for every $z,w\in\mathbb{R}^m$ we have*

$$|\varphi_{\varepsilon,T,R}(w)-\varphi_{\varepsilon,T,R}(z)| \le C(|z|^{p-1}+|w|^{p-1})|w-z|. \tag{4.16}$$

Proof. Note that by (4.13), if $z = 0$ or $w = 0$ the inequality is trivial. Then, we can suppose z and w both not null and consider the map $\phi : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\phi(\zeta) = \frac{|w|}{|z|}\mathcal{R}_z^w(\zeta),$$

where $\mathcal{R}_z^w$ is a rotation that maps z into $\frac{|z|}{|w|}w$. Note that $\phi(0) = 0$, $\phi(z) = w$ and

$$\|\nabla\phi\|_\infty \leq C\frac{|w|}{|z|}, \quad \|\nabla\phi - I\|_\infty \leq C\frac{|w-z|}{|z|}. \tag{4.17}$$

By Remark 4.7, let $v_z \in \mathcal{A}_{\varepsilon,T,z}(Q_R;\mathbb{R}^m)$ be such that

$$\mathcal{F}_\varepsilon^T(v_z, Q_R) = \varphi_{\varepsilon,T,R}(z). \tag{4.18}$$

Now set

$$v_w := \phi \circ v_z,$$

and observe that $v_w \in \mathcal{A}_{\varepsilon,T,w}(Q_R;\mathbb{R}^m)$. Hence, by (4.18)

$$\varphi_{\varepsilon,T,R}(w) \leq \mathcal{F}_\varepsilon^T(v_w, Q_R) = \varphi_{\varepsilon,T,R}(z) + \mathcal{F}_\varepsilon^T(v_w, Q_R) - \mathcal{F}_\varepsilon^T(v_z, Q_R).$$

By hypothesis (L) and (4.17)

$$\begin{aligned}
|f(\xi, D_\varepsilon^\xi v_w(\alpha)) - f(\xi, D_\varepsilon^\xi v_z(\alpha))| &\leq CM(\xi)(|D_\varepsilon^\xi v_z(\alpha)|^{p-1} + |D_\varepsilon^\xi v_w(\alpha)|^{p-1})|D_\varepsilon^\xi v_z(\alpha) - D_\varepsilon^\xi v_w(\alpha)| \\
&= CM(\xi)(|D_\varepsilon^\xi v_z(\alpha)|^{p-1} + |\nabla\phi D_\varepsilon^\xi v_z(\alpha)|^{p-1})|D_\varepsilon^\xi v_z(\alpha)[\nabla\phi - I]| \\
&\leq CM(\xi)(|D_\varepsilon^\xi v_z(\alpha)|^{p-1} + \|\nabla\phi\|_\infty^{p-1}|D_\varepsilon^\xi v_z(\alpha)|^{p-1})\|\nabla\phi - I\|_\infty |D_\varepsilon^\xi v_z(\alpha)| \\
&\leq CM(\xi)|D_\varepsilon^\xi v_z(\alpha)|^{p-1}\left(1 + C\frac{|w|^{p-1}}{|z|^{p-1}}\right)\frac{|w-z|}{|z|}|D_\varepsilon^\xi v_z(\alpha)| \\
&= CM(\xi)\frac{|z|^{p-1} + |w|^{p-1}}{|z|^p}|w-z||D_\varepsilon^\xi v_z(\alpha)|^p
\end{aligned}$$

Then,

$$\varphi_{\varepsilon,T,R}(w) \leq \varphi_{\varepsilon,T,R}(z) + C\frac{|z|^{p-1} + |w|^{p-1}}{|z|^p}|w-z|\sum_{|\xi|\leq T} M(\xi)\sum_{\alpha\in(Q_R)_\varepsilon(\xi)}|D_\varepsilon^\xi v_z(x)|^p. \tag{4.19}$$

By (G0), (4.18) and (4.13)

$$G_\varepsilon^{1,p}(v_z, Q_R) \leq C\mathcal{F}_\varepsilon^T(v_z, Q_R) = C\varphi_{\varepsilon,T,R}(z) \leq C|z|^p. \tag{4.20}$$

Observe that $G_\varepsilon^{1,p}(v_z, Q_R) = G_\varepsilon^{1,p}(v_z, \mathbb{R}^d)$. Then, by Lemma 3.1

$$\sum_{\alpha\in(Q_R)_\varepsilon(\xi)}|D_\varepsilon^\xi v_z(\alpha)|^p \leq CG_\varepsilon^{1,p}(v_z, Q_R).$$

Hence, by (4.19) and (4.20) we get

$$\varphi_{\varepsilon,T,R}(w) \leq \varphi_{\varepsilon,T,R}(z) + C(|z|^{p-1} + |w|^{p-1})|w - z|.$$

Reversing the role of z and w, we get (4.16). □

4.4. Asymptotics of the approximating capacitary energy density

We now show that, if $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ the functions $\varphi_{\varepsilon,T,R_\varepsilon}(z)$ approximate the energy density $\varphi^T(z)$. We recall that they are defined respectively as

$$\varphi^T(z) := \inf\left\{ \int_{\mathbb{R}^d} f^T_{\text{hom}}(\nabla v)\,\mathrm{d}x : v - z \in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m), v \equiv 0 \text{ in } Q_1, v \in W^{1,p}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^m)\right\}, \tag{4.21}$$

$$\varphi_{\varepsilon,T,R_\varepsilon}(z) := \inf\left\{\mathcal{F}^T_\varepsilon(u,\mathbb{R}^d) : u \in \mathcal{A}_\varepsilon(\mathbb{R}^d;\mathbb{R}^m), u \equiv -z \text{ in } Q_1, u \equiv 0 \text{ in } \mathbb{R}^d \setminus Q_{R_\varepsilon - \varepsilon T}\right\}. \tag{4.22}$$

Proposition 4.10. *Let f satisfy assumptions (H), (G) and (L). Let φ^T and $\varphi_{\varepsilon,T,R_\varepsilon}$ be defined by (4.21) and (4.22). Then, if $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ it holds*

$$\lim_{\varepsilon\to 0} \varphi_{\varepsilon,T,R_\varepsilon}(z) = \varphi^T(z) \tag{4.23}$$

uniformly on compact sets.

Proof. By Proposition 4.9 it is sufficient to prove that (4.23) holds pointwise. We will show that (4.23) is a consequence of Theorem 2.3 and Theorem 4.2. Let us fix $z \in \mathbb{R}^m$, $\varepsilon > 0$ and let $v_\varepsilon \in \mathcal{A}_{\varepsilon,T,z}(Q_{R_\varepsilon};\mathbb{R}^m)$ be such that

$$\mathcal{F}^T_\varepsilon(v_\varepsilon, Q_{R_\varepsilon}) = \varphi_{\varepsilon,T,R_\varepsilon}(z).$$

For every fixed $R > 0$ and ε small enough, by (4.13) and by the fact that $\mathcal{F}^T_\varepsilon(v_\varepsilon,\cdot)$ is an increasing set function, we have

$$\mathcal{F}^T_\varepsilon(v_\varepsilon, Q_R) \leq \mathcal{F}^T_\varepsilon(v_\varepsilon, Q_{R_\varepsilon}) \leq c_2|z|^p. \tag{4.24}$$

Now, we show that for ε small enough

$$\mathcal{G}^{1,p}_\varepsilon(v_\varepsilon, Q_R) \leq C G^{1,p}_\varepsilon(v_\varepsilon, Q_{R_\varepsilon}) \leq C|z|^p \tag{4.25}$$

where $\mathcal{G}^{1,p}_\varepsilon$ is defined by (4.2) and the last inequality in (4.25) is a consequence of Remark 3.3. The first inequality in (4.25) follows from the following argument. Let us define $I_\varepsilon(Q_R) := \{\alpha \in \varepsilon\mathbb{Z}^d : (\alpha + [0,\varepsilon)^d) \cap Q_R \neq \emptyset\}$ and $S^{k,\xi}_{\varepsilon,\alpha}$ as in (4.5). It is clear that for ε small enough

$$Q_{R_\varepsilon} \supset \bigcup_{\alpha\in I_\varepsilon(Q_R)} \alpha + [0,\varepsilon)^d \supset Q_R,$$

and then, arguing as in the proof of Theorem 4.2, we get

$$
\begin{aligned}
\mathcal{G}^{1,p}_{\varepsilon}(v_\varepsilon, Q_R) &= \int_{B_1}\int_{(Q_R)_\varepsilon(\xi)} \left|\frac{v_\varepsilon(x+\varepsilon\xi)-v_\varepsilon(x)}{\varepsilon}\right|^p \mathrm{d}x\,\mathrm{d}\xi \\
&\le \int_{B_1}\int_{Q_R} \left|\frac{v_\varepsilon(x+\varepsilon\xi)-v_\varepsilon(x)}{\varepsilon}\right|^p \mathrm{d}x\,\mathrm{d}\xi \\
&\le \int_{B_1}\sum_{\alpha\in I_\varepsilon(Q_R)}\int_{\alpha+[0,\varepsilon)^d} \left|\frac{v_\varepsilon(x+\varepsilon\xi)-v_\varepsilon(x)}{\varepsilon}\right|^p \mathrm{d}x\,\mathrm{d}\xi \\
&\le \int_{B_1}\sum_{\alpha\in I_\varepsilon(Q_R)}\int_{S^{\varepsilon,\xi}_\alpha} \left|\frac{v_\varepsilon(x+\varepsilon\xi)-v_\varepsilon(x)}{\varepsilon}\right|^p \mathrm{d}x\,\mathrm{d}\xi \\
&\le C\int_{B_1}|\xi|\,\mathrm{d}\xi\sum_{k=1}^{d}\sum_{\alpha\in I_\varepsilon(Q_R)}\varepsilon^d|D^{e_k}_\varepsilon v_\varepsilon(\alpha)|^p \\
&\le C G^{1,p}_\varepsilon(v_\varepsilon, Q_{R_\varepsilon}).
\end{aligned}
$$

Moreover, by Lemma 3.4 and Remark 3.6 applied with $E = Q_1$ and $\Omega = Q_R$, we get

$$
\sup_\varepsilon \|v_\varepsilon\|_{L^p(Q_R;\mathbb{R}^m)} < +\infty. \tag{4.26}
$$

By (4.25) and (4.26) we get

$$
\sup_\varepsilon\{\|v_\varepsilon\|_{L^p(Q_R;\mathbb{R}^m)} + \mathcal{G}^{1,p}_\varepsilon(v_\varepsilon, Q_R)\} < +\infty.
$$

and then, by Theorem 4.6, there exists a subsequence (not relabeled) v_ε converging strongly in L^p to $v \in W^{1,p}(Q_R;\mathbb{R}^m)$. By the arbitrariness of $R > 0$ we infer that up to a subsequence v_ε converges strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ to $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ with $v \equiv 0$ in Q_1. As a direct consequence of Theorem 4.2 we infer that $u := v - z \in L^{p^*}(\mathbb{R}^d;\mathbb{R}^m)$. Indeed, let us define

$$
u_\varepsilon = \begin{cases} v_\varepsilon - z & \text{on } Q_{R_\varepsilon} \\ 0 & \text{on } \mathbb{R}^d \setminus Q_{R_\varepsilon}. \end{cases}
$$

By (G0) and (4.24) we get

$$
G^{1,p}_\varepsilon(u_\varepsilon, \mathbb{R}^d) = G^{1,p}_\varepsilon(u_\varepsilon, Q_{R_\varepsilon}) \le C\mathcal{F}^T_\varepsilon(v_\varepsilon, Q_{R_\varepsilon}) \le C|z|^p.
$$

Then by (4.3)

$$
\int_{\mathbb{R}^d}|u_\varepsilon|^{p^*} \le C G^{1,p}_\varepsilon(u_\varepsilon, \mathbb{R}^d) \le C|z|^p,
$$

and then by Fatou's lemma

$$
\int_{\mathbb{R}^d}|u|^{p^*} \le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^d}|u_\varepsilon|^{p^*} < +\infty.
$$

By Theorem 2.3 we deduce that

$$\liminf_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)=\liminf_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,Q_{R_\varepsilon})\geq\liminf_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,Q_R)\geq\int_{Q_R}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x.$$

Letting $R\to+\infty$ we obtain

$$\liminf_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)\geq\int_{\mathbb{R}^d}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x\geq\varphi^T(z).$$

We now prove

$$\limsup_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)\leq\varphi^T(z).$$

In the proof of [10], Proposition 6.7, it is shown that

$$\varphi^T(z)=\inf\left\{\int_{\mathbb{R}^d}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x: u\equiv -z\text{ in }Q_1, u\in W^{1,p}(\mathbb{R}^d;\mathbb{R}^m)\text{ compactly supported}\right\}.$$

Then, using a density argument, given $\eta>0$, there exists $u\in C^\infty_c(\mathbb{R}^d;\mathbb{R}^m)$ such that $u\equiv -z$ on Q_1 and

$$\int_{\mathbb{R}^d}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x\leq\varphi^T(z)+\eta.$$

Fix $R>0$ such that $\operatorname{supp}u\subset Q_R$. Then by [8], Theorem 4.5, applied with $A=Q_R\setminus Q_1$, there exists a family of functions $u_\varepsilon\in L^p(Q_R;\mathbb{R}^m)$, with $u_\varepsilon\equiv -z$ on Q_1 and $u_\varepsilon\equiv 0$ on $\partial^{\varepsilon T}Q_R$ such that

$$\lim_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(u_\varepsilon,\mathbb{R}^d)=\int_{\mathbb{R}^d}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x.$$

Hence,

$$\limsup_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)=\limsup_{\varepsilon\to 0}\mathcal{F}^T_\varepsilon(v_\varepsilon,Q_{R_\varepsilon})\leq\int_{\mathbb{R}^d}f^T_{\rm hom}(\nabla u)\,\mathrm{d}x\leq\varphi^T(z)+\eta.$$

By the arbitrariness of $\eta>0$, we get

$$\limsup_{\varepsilon\to 0}\varphi_{\varepsilon,T,R_\varepsilon}(z)\leq\varphi^T(z),$$

and then the thesis. □

Proposition 4.11. *Let $\varepsilon_j\to 0$ and $R_j\to+\infty$ as $j\to+\infty$ and let $\{u_j\}_j\subset\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$ such that $\sup_j\mathcal{F}^T_{\varepsilon_j}(u_j)<+\infty$. Assume that $u_j\to u$ in $L^p(\Omega;\mathbb{R}^m)$ for some $u\in W^{1,p}(\Omega;\mathbb{R}^m)$. Let $\{\rho_j\}_j$ be a sequence of the form $\rho_j=\beta\delta_j$, with $\beta<\frac{1}{2}$ and for all $i\in Z_j(\Omega)$, where $Z_j(\Omega)$ is defined in (4.6), let u^i_j be defined as in Lemma 4.3. Set $Q^i_j:=i\delta_j+\left(\left[-\frac{\delta_j}{2},\frac{\delta_j}{2}\right)^d\cap\varepsilon_j\mathbb{Z}^d\right)$ and let $\psi^T_j\in\mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ be defined for $\alpha\in\Omega_j$ as*

$$\psi^T_j(\alpha)=\sum_{i\in Z_j(\Omega)}\varphi_{\varepsilon_j,T,R_j}(u^i_j)\chi_{Q^i_j}(\alpha).$$

Then,

$$\lim_{j\to+\infty}\sum_{\alpha\in\Omega_j}\varepsilon_j^d\psi_j^T(\alpha)=\int_\Omega\varphi^T(u)\,\mathrm{d}x.$$

Proof. First of all we observe that

$$\sum_{\alpha\in\Omega_j}\varepsilon_j^d\psi_j^T(\alpha)=\sum_{i\in Z_j(\Omega)}\delta_j^d\varphi_{\varepsilon_j,T,R_j}(u_j^i).$$

Indeed,

$$\begin{aligned}\sum_{\alpha\in\Omega_j}\varepsilon_j^d\psi_j^T(\alpha)&=\sum_{\alpha\in\Omega_j}\sum_{i\in Z_j(\Omega)}\varepsilon_j^d\varphi_{\varepsilon_j,T,R_j}(u_j^i)\chi_{Q_j^i}(\alpha)\\&=\sum_{i\in Z_j(\Omega)}\varepsilon_j^d\left(\frac{\delta_j}{\varepsilon_j}\right)^d\varphi_{\varepsilon_j,T,R_j}(u_j^i)\\&=\sum_{i\in Z_j(\Omega)}\delta_j^d\varphi_{\varepsilon_j,T,R_j}(u_j^i).\end{aligned}$$

Now, we get the following estimate

$$\begin{aligned}&\Big|\sum_{i\in Z_j(\Omega)}\delta_j^d\varphi_{\varepsilon_j,T,R_j}(u_j^i)-\int_\Omega\varphi^T(u(x))\,\mathrm{d}x\Big|\\&\quad=\Big|\sum_{i\in Z_j(\Omega)}\varphi_{\varepsilon_j,T,R_j}(u_j^i)\int_{Q_j^i}\mathrm{d}x-\sum_{i\in Z_j(\Omega)}\int_{Q_j^i}\varphi^T(u(x))\,\mathrm{d}x-\int_{\Omega\setminus\cup_{i\in Z_j(\Omega)}Q_j^i}\varphi^T(u(x))\,\mathrm{d}x\Big|\\&\quad\le\sum_{i\in Z_j(\Omega)}\int_{Q_j^i}\big|\varphi_{\varepsilon_j,T,R_j}(u_j^i)-\varphi^T(u(x))\big|\,\mathrm{d}x+\int_{\Omega\setminus\cup_{i\in Z_j(\Omega)}Q_j^i}|\varphi^T(u(x))|\,\mathrm{d}x\\&\quad\le\sum_{i\in Z_j(\Omega)}\sum_{\alpha\in Q_j^i}\int_{\alpha+[0,\varepsilon_j)^d}\big|\varphi_{\varepsilon_j,T,R_j}(u_j^i)-\varphi_{\varepsilon_j,T,R_j}(u_j(x))\big|\,\mathrm{d}x\\&\qquad+\sum_{i\in Z_j(\Omega)}\sum_{\alpha\in Q_j^i}\int_{\alpha+[0,\varepsilon_j)^d}\big|\varphi_{\varepsilon_j,T,R_j}(u_j(x))-\varphi^T(u(x))\big|\,\mathrm{d}x+\int_{\Omega\setminus\cup_{i\in Z_j(\Omega)}Q_j^i}|\varphi^T(u(x))|\,\mathrm{d}x\\&\quad\le\sum_{i\in Z_j(\Omega)}\sum_{\alpha\in Q_j^i}\varepsilon_j^d\big|\varphi_{\varepsilon_j,T,R_j}(u_j^i)-\varphi_{\varepsilon_j,T,R_j}(u_j(\alpha))\big|\\&\quad+\sum_{i\in Z_j(\Omega)}\sum_{\alpha\in Q_j^i}\varepsilon_j^d\big|\varphi_{\varepsilon_j,T,R_j}(u_j(\alpha))-\varphi^T(u_j(\alpha))\big|\\&\quad+\sum_{i\in Z_j(\Omega)}\sum_{\alpha\in Q_j^i}\int_{\alpha+[0,\varepsilon_j)^d}\big|\varphi^T(u_j(x))-\varphi^T(u(x))\big|\,\mathrm{d}x+\int_{\Omega\setminus\cup_{i\in Z_j(\Omega)}Q_j^i}|\varphi^T(u(x))|\,\mathrm{d}x\\&\quad:=S_j^1+S_j^2+S_j^3+S_j^4.\end{aligned}$$

By hypothesis $u_j\in L^\infty(\Omega;\mathbb{R}^m)$, then by Proposition 4.10 we deduce that $S_j^2\to 0$. Since $\big|\Omega\setminus\cup_{i\in Z_j(\Omega)}Q_j^i\big|\to 0$ and $\varphi^T(z)\le C|z|^p$ we also infer that $S_j^4\to 0$. Moreover, by the fact that φ^T is locally Lipschitz and $u_j\to u$ in

L^p we get that $S_j^3 \to 0$. Finally, by Proposition 4.9, we may estimate S_j^1 as follows:

$$S_j^1 \leq C \sum_{i \in Z_j(\Omega)} \sum_{\alpha \in Q_j^i} \varepsilon_j^d |u_j^i - u_j(\alpha)|.$$

Using a discrete version of Hölder inequality, we get

$$\begin{aligned}\sum_{i \in Z_j(\Omega)} \sum_{\alpha \in Q_j^i} \varepsilon_j^d |u_j^i - u_j(\alpha)| &\leq \varepsilon_j^d \sum_{i \in Z_j(\Omega)} (\sharp Q_j^i)^{1-\frac{1}{p}} \Big(\sum_{\alpha \in Q_j^i} |u_j^i - u_j(\alpha)|^p \Big)^{\frac{1}{p}} \\ &= \delta_j^d \delta_j^{-\frac{d}{p}} \sum_{i \in Z_j(\Omega)} \Big(\sum_{\alpha \in Q_j^i} \varepsilon_j^d |u_j^i - u_j(\alpha)|^p \Big)^{\frac{1}{p}}.\end{aligned}$$

By Lemma 3.5 and Remark 3.6 we deduce that

$$\sum_{\alpha \in Q_j^i} \varepsilon_j^d |u_j^i - u_j(\alpha)|^p \leq C\delta_j^p \sum_{k=1}^d \sum_{\alpha \in (Q_j^i)_{\varepsilon_j}(e_k)} \varepsilon_j^d \Big| \frac{u_j(\alpha + e_k\varepsilon_j) - u_j(\alpha)}{\varepsilon_j} \Big|^p.$$

Then using the fact that the function $x \mapsto x^{\frac{1}{p}}$ is concave, we get

$$\begin{aligned}\sum_{i \in Z_j(\Omega)} \sum_{\alpha \in Q_j^i} \varepsilon_j^d |u_j^i - u_j(\alpha)| &\leq C\delta_j^d \delta_j^{-\frac{d}{p}} \sum_{i \in Z_j(\Omega)} \delta_j \Big(\sum_{k=1}^d \sum_{\alpha \in (Q_j^i)_{\varepsilon_j}(e_k)} \varepsilon_j^d \Big| \frac{u_j(\alpha + e_k\varepsilon_j) - u_j(\alpha)}{\varepsilon_j} \Big|^p \Big)^{\frac{1}{p}} \\ &\leq C\delta_j^d \delta_j^{-\frac{d}{p}} \delta_j (\sharp Z_j(\Omega)) \Big(\frac{1}{(\sharp Z_j(\Omega))} \sum_{i \in Z_j(\Omega)} \sum_{k=1}^d \sum_{\alpha \in (Q_j^i)_{\varepsilon_j}(e_k)} \varepsilon_j^d \Big| \frac{u_j(\alpha + e_k\varepsilon_j) - u_j(\alpha)}{\varepsilon_j} \Big|^p \Big)^{\frac{1}{p}} \\ &= C\delta_j^d \delta_j^{-\frac{d}{p}} \delta_j (\sharp Z_j(\Omega))^{1-\frac{1}{p}} \Big(\sum_{i \in Z_j(\Omega)} \sum_{k=1}^d \sum_{\alpha \in (Q_j^i)_{\varepsilon_j}(e_k)} \varepsilon_j^d \Big| \frac{u_j(\alpha + e_k\varepsilon_j) - u_j(\alpha)}{\varepsilon_j} \Big|^p \Big)^{\frac{1}{p}} \\ &\leq C\delta_j (G^{1,p}_{\varepsilon_j}(u_j, \Omega))^{\frac{1}{p}} \leq C\delta_j (\mathcal{F}^T_{\varepsilon_j}(u_j))^{\frac{1}{p}} \leq C\delta_j (\sup_j \mathcal{F}^T_{\varepsilon_j}(u_j))^{\frac{1}{p}} \leq C\delta_j.\end{aligned}$$

Hence,

$$S_j^1 \leq C\delta_j \to 0.$$

□

5. Proof of Theorem 2.4

We will divide the proof of Theorem 2.4 into three steps. We first show that it suffices to prove the theorem for the truncated functionals $F^T_{\varepsilon,\delta_\varepsilon}$, and then we will deal separately with the Γ-lim inf and the Γ-lim sup inequalities.

Step 1
It is not restrictive to prove the theorem under the additional assumption that there exists $T > 2$ such that

$f(\xi, z) = 0$ if $|\xi| > T$. Indeed, under the hypothesis of Theorem 2.4, assume that for every $T > 2$

$$\Gamma(L^p) - \lim_{\varepsilon} F^T_{\varepsilon,\delta_\varepsilon}(u) = \begin{cases} \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + \gamma^{d-p}\int_\Omega \varphi^T(u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,} \end{cases}$$

where $F^T_{\varepsilon,\delta_\varepsilon}$ is defined by (2.4) and f^T_{hom} and φ^T are defined by (2.14) and (2.15), respectively. If we prove that

$$\Gamma(L^p) - \lim_{\varepsilon} F_{\varepsilon,\delta_\varepsilon}(u) = \lim_{T\to+\infty} \Gamma(L^p) - \lim_{\varepsilon} F^T_{\varepsilon,\delta_\varepsilon}(u). \tag{5.1}$$

then, by the Monotone Convergence Theorem, the statement follows once we prove that for every $M \in \mathbb{R}^{d\times m}$ and $z \in \mathbb{R}^m$

$$\lim_{T\to+\infty} f^T_{\mathrm{hom}}(M) = f_{\mathrm{hom}}(M), \qquad \lim_{T\to+\infty} \varphi^T(z) = \varphi(z).$$

The inequalities above are again a straightforward consequence of Monotone Convergence Theorem. We observe that for every $\varepsilon > 0$ and for every fixed $T > 2$, we have that $F^T_{\varepsilon,\delta_\varepsilon} \le F_{\varepsilon,\delta_\varepsilon}$. Then,

$$\Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u) \le \Gamma(L^p) - \liminf_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u)$$

and passing to the limit as T goes to $+\infty$ we get

$$\lim_{T\to+\infty} \Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u) \le \Gamma(L^p) - \liminf_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u).$$

So, in order to prove (5.1), it suffices to show that

$$\Gamma(L^p) - \limsup_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u) \le \lim_{T\to+\infty} \Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u).$$

For reader's convenience we will specify the dependence of our energies on the open sets $A \subset \mathbb{R}^d$. Setting $F^{''}(u,\Omega) := \Gamma(L^p) - \limsup_{\varepsilon\to 0} F_{\varepsilon,\delta_\varepsilon}(u,\Omega)$ and $F^T(u,\Omega) := \Gamma(L^p) - \lim_{\varepsilon\to 0} F^T_{\varepsilon,\delta_\varepsilon}(u,\Omega)$, let us prove that for every $\eta > 0$ there exists $T_\eta > 0$ such that for every $u \in W^{1,p}(\Omega,\mathbb{R}^m)$

$$F^{''}(u,\Omega) \le F^{T_\eta}(u,\Omega) + C\eta.$$

Since f_{hom} is continuous, a density argument allows us to restrict the proof of the inequality to test functions $u \in C^\infty_c(\mathbb{R}^d,\mathbb{R}^m)$. Let us fix $\eta > 0$, then there exists $T_\eta > 0$ such that $\sum_{|\xi|>T_\eta} M(\xi) < \eta$. By the external regularity of $F^T(u,\Omega)$ there exists $\Omega' \supset\supset \Omega$ such that

$$F^{T_\eta}(u,\Omega') \le F^{T_\eta}(u,\Omega) + \eta. \tag{5.2}$$

Consider $u_\varepsilon \to u$ in $L^p(\Omega';\mathbb{R}^m)$ such that $\mathcal{F}^{T_\eta}_\varepsilon(u_\varepsilon,\Omega')$ converges to $F^{T_\eta}(u,\Omega')$. Observe that, from (2.10), $f^T_{\mathrm{hom}}(M) \le f_{\mathrm{hom}}(M) \le C|M|^p$ for every $M \in \mathbb{R}^{m\times d}$. Moreover by Proposition 4.8 and Proposition 4.10, $\varphi^T(z) \le c_2|z|^p$, for every $z \in \mathbb{R}^m$. Hence

$$\mathcal{F}^{T_\eta}_\varepsilon(u_\varepsilon,\Omega') \le C, \tag{5.3}$$

with C independent of η. We split our energies as follows

$$\mathcal{F}_\varepsilon(u_\varepsilon,\Omega)=\mathcal{F}_\varepsilon^{T_\eta}(u_\varepsilon,\Omega)+\mathcal{F}_\varepsilon^{>T_\eta}(u_\varepsilon,\Omega) \tag{5.4}$$

where

$$\mathcal{F}_\varepsilon^{>T_\eta}(u_\varepsilon,\Omega):=\sum_{|\xi|>T_\eta}\sum_{\alpha\in\Omega_\varepsilon(\xi)}\varepsilon^d f(\xi,D_\varepsilon^\xi u(\alpha)).$$

Let us define $\tilde\Omega'_\varepsilon:=\{x\in\Omega':\mathrm{dist}(x,\partial\Omega')>2\sqrt{d}\varepsilon\}$. If ε is sufficiently small we have $\Omega\subset\subset\tilde\Omega'_\varepsilon$. By Lemma 3.1 applied with $\tilde\Omega'_\varepsilon$, assumption (G) and (5.3) we get

$$\mathcal{F}_\varepsilon^{>T_\eta}(u_\varepsilon,\Omega)\leq\sum_{|\xi|>T_\eta}M(\xi)G_\varepsilon^{1,p}(u_\varepsilon,\Omega')\leq C\eta. \tag{5.5}$$

Hence, by (5.2), (5.4) and (5.5)

$$\begin{aligned}F^{''}(u,\Omega)&\leq\limsup_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,\Omega)\leq\limsup_{\varepsilon\to0}\mathcal{F}_\varepsilon^{T_\eta}(u_\varepsilon,\Omega')+C\eta\\&=F^{T_\eta}(u,\Omega')+C\eta\leq F^{T_\eta}(u,\Omega)+C\eta.\end{aligned}$$

Step 2. Now we prove the validity of the Γ-lim inf inequality for $F^T_{\varepsilon,\delta_\varepsilon}$.
Given $\varepsilon_j\to0$ as $j\to+\infty$, let $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and $u_j\to u$ in $L^p(\Omega;\mathbb{R}^m)$ be such that $\sup_j F^T_{\varepsilon_j,\delta_j}(u_j)<+\infty$. We first assume that $\{u_j\}_j$ is bounded in L^∞ and then we will remove this assumption using Lemma 4.5.
We fix $k\in\mathbb{N}$ and we consider a sequence $\{\rho_j\}_j$ of the form $\rho_j=\beta\delta_j$ with $\beta<\frac12$. Let $Z_j(\Omega)$ and ρ_j^i be defined as in Lemma 4.3. We apply Lemma 4.3 to $\{u_j\}_j$ in order to get a new sequence $\{w_j\}_j$ such that $w_j\to u$ in $L^p(\Omega;\mathbb{R}^m)$ and it satisfies (4.7)–(4.9). In particular the following inequality holds

$$\liminf_{j\to+\infty}F^T_{\varepsilon_j,\delta_j}(u_j)\geq\liminf_{j\to+\infty}F^T_{\varepsilon_j,\delta_j}(w_j)-\frac{C}{k}. \tag{5.6}$$

Set

$$E_j=\bigcup_{i\in Z_j(\Omega)}Q(i\delta_j,\rho_j^i).$$

Note that we have

$$\liminf_{j\to+\infty}F^T_{\varepsilon_j,\delta_j}(w_j)\geq\liminf_{j\to+\infty}F^T_{\varepsilon_j,\delta_j}(w_j,E_j)+\liminf_{j\to+\infty}F^T_{\varepsilon_j,\delta_j}(w_j,\Omega\setminus E_j).$$

We first estimate the energy contribution of $\{u_j\}_j$ on $\Omega\setminus E_j$, *i.e.* far from the perforations. To this end we use the argument exploited in [9], Section 7, Subsection A.1. Let us define $v_j \in \mathcal{A}_{\varepsilon_j}(\Omega;\mathbb{R}^m)$ by modifying w_j as follows

$$v_j(\alpha) = \begin{cases} u_j^i & \alpha \in Q(i\delta_j, \rho_j^i),\ i \in Z_j(\Omega) \\ w_j(\alpha) & \text{otherwise.} \end{cases}$$

Note that since $u_j \in L^\infty(\Omega;\mathbb{R}^m)$, by construction also $v_j, w_j \in L^\infty(\Omega;\mathbb{R}^m)$. Now we prove that $v_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$. By Lemma 3.5 and using the fact that $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$, $w_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$, we have

$$\begin{aligned}
\int_\Omega |v_j - u_j|^p \,\mathrm{d}x &= \sum_{\alpha\in\Omega_j} \varepsilon_j^d |u_j(\alpha) - v_j(\alpha)|^p + o(1) \\
&\le \Bigg[\sum_{\alpha\in\Omega_j\setminus E_j} \varepsilon_j^d |u_j(\alpha) - v_j(\alpha)|^p + \sum_{\alpha\in E_j} \varepsilon_j^d |u_j(\alpha) - v_j(\alpha)|^p \Bigg] + o(1) \\
&\le \Bigg[\sum_{\alpha\in\Omega_j\setminus E_j} \varepsilon_j^d |u_j(\alpha) - w_j(\alpha)|^p + \sum_{\alpha\in E_j} \varepsilon_j^d |u_j(\alpha) - v_j(\alpha)|^p \Bigg] + o(1) \\
&\le \Bigg[\int_\Omega |u_j - w_j|^p \,\mathrm{d}x + \sum_{i\in Z_j(\Omega)} \sum_{\alpha\in Q(i\delta_j,\rho_j^i)} \varepsilon_j^d |u_j(\alpha) - u_j^i|^p \Bigg] + o(1) \\
&\le C\delta_j^p \sum_{i\in Z_j(\Omega)} \sum_{k=1}^d \sum_{\alpha\in (Q(i\delta_j,\rho_j^i))_{\varepsilon_j}(e_k)} \varepsilon_j^d \left| \frac{u_j(\alpha + e_k\varepsilon_j) - u_j(\alpha)}{\varepsilon_j} \right|^p + o(1) \\
&= C\delta_j^p G^{1,p}_{\varepsilon_j}(u_j,\Omega) + o(1) \le C\delta_j^p F^T_{\varepsilon_j,\delta_j}(u_j) + o(1) \le C\delta_j^p + o(1)
\end{aligned}$$

By construction

$$F^T_{\varepsilon_j,\delta_j}(w_j, \Omega_j \setminus E_j) \ge \mathcal{F}^T_{\varepsilon_j}(v_j).$$

Then by Theorem 2.3, we conclude that

$$\liminf_j F^T_{\varepsilon_j,\delta_j}(w_j;\Omega_j\setminus E_j) \ge \liminf_j \mathcal{F}^T_{\varepsilon_j}(v_j) \ge \int_\Omega f^T_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x. \tag{5.7}$$

Now we focus our attention on the contribution of $\{u_j\}_j$ on E_j, *i.e.* close to the perforations. Set $R_j^i := \frac{\rho_j^i}{r_{\delta_j}}$ and $s_j := \frac{\varepsilon_j}{r_{\delta_j}}$. Since $\varepsilon_j = o(r_{\delta_j})$, $s_j \to 0$ when $j \to +\infty$. For $i \in Z_j(\Omega)$ we define the functions $v_{i,j} : s_j\mathbb{Z}^d \to \mathbb{R}^m$ as follows

$$v_{i,j}(\alpha) = \begin{cases} w_j(i\delta_j + r_{\delta_j}\alpha) & \text{if } \alpha \in Q_{R_j^i} \\ u_j^i & \text{if } \alpha \in s_j\mathbb{Z}^d \setminus Q_{R_j^i}, \end{cases}$$

Using hypothesis (H) we get

$$\begin{aligned}
F^T_{\varepsilon_j,\delta_j}(w_j, Q(i\delta_j,\rho^i_j)) &= \sum_{|\xi|\le T}\ \sum_{\alpha\in(Q(i\delta_j,\rho^i_j))_{\varepsilon_j}(\xi)} \varepsilon_j^d f\Big(\xi, \frac{w_j(\alpha+\varepsilon_j\xi)-w_j(\alpha)}{|\xi|\varepsilon_j}\Big)\\
&= \sum_{|\xi|\le T}\ \sum_{\alpha\in\left(Q_{R^i_j}\right)_{s_j}(\xi)} \varepsilon_j^d f\Big(\xi, \frac{v_{i,j}(\alpha+s_j\xi)-v_{i,j}(\alpha)}{r_{\delta_j}s_j|\xi|}\Big)\\
&= \sum_{|\xi|\le T}\ \sum_{\alpha\in\left(Q_{R^i_j}\right)_{s_j}(\xi)} \frac{\varepsilon_j^d}{r^p_{\delta_j}} f\Big(\xi, \frac{v_{i,j}(\alpha+s_j\xi)-v_{i,j}(\alpha)}{s_j|\xi|}\Big)\\
&= \sum_{|\xi|\le T}\ \sum_{\alpha\in\left(Q_{R^i_j}\right)_{s_j}(\xi)} r^{d-p}_{\delta_j} s_j^d f\Big(\xi, \frac{v_{i,j}(\alpha+s_j\xi)-v_{i,j}(\alpha)}{s_j|\xi|}\Big)\\
&= r^{d-p}_{\delta_j}\mathcal{F}^T_{s_j}(v_{i,j}, Q_{R^i_j})\\
&\ge r^{d-p}_{\delta_j}\varphi_{s_j,T,R^i_j}(u^i_j).
\end{aligned}$$

where the last inequality holds since, by construction, $v_{i,j}$ is a good test function for the minimum problem defining φ_{s_j,T,R^i_j}. Summing over $i\in Z_j(\Omega)$,

$$\sum_{i\in Z_j(\Omega)} F^T_{\varepsilon_j,\delta_j}(w_j, Q(i\delta_j,\rho^i_j)) \ge \sum_{i\in Z_j(\Omega)} r^{d-p}_{\delta_j}\varphi_{s_j,T,R^i_j}(u^i_j).$$

Note that the right hand side above, can be written as

$$r^{d-p}_{\delta_j}\sum_{i\in Z_j(\Omega)}\varphi_{s_j,T,R^i_j}(u^i_j) = \frac{r^{d-p}_{\delta_j}}{\left(\frac{\delta_j}{\varepsilon_j}\right)^d}\sum_{\alpha\in\Omega_j}\sum_{i\in Z_j(\Omega)}\varphi_{s_j,T,R^i_j}(u^i_j)\chi_{Q^i_j}(\alpha) = \left(\frac{r_{\delta_j}}{\delta_j^{\frac{d}{d-p}}}\right)^{d-p}\sum_{\alpha\in\Omega_j}\varepsilon_j^d\psi^T_j(\alpha).$$

By Proposition 4.11 we deduce that

$$\liminf_j F^T_{\varepsilon_j,\delta_j}(w_j,E_j) \ge \liminf_j \sum_{i\in Z_j(\Omega)} r^{d-p}_{\delta_j}\varphi_{s_j,T,R^i_j}(u^i_j) \ge \gamma^{d-p}\int_\Omega \varphi^T(u(x))\,\mathrm{d}x. \tag{5.8}$$

By (5.6), (5.7), (5.8) and by the arbitrariness of k, we finally get

$$\liminf_j F^T_{\varepsilon_j,\delta_j}(u_j) \ge \int_\Omega f^T_{\rm hom}(\nabla u) + \gamma^{d-p}\int_\Omega \varphi^T(u)\,\mathrm{d}x.$$

It remains to show that the Γ-liminf inequality holds even if we remove the boundedness assumption on the sequence $\{u_j\}$. For all $L\in\mathbb{N}$ and $\eta>0$ we apply the previous arguments to the truncated sequence $\{t_L(u_j)\}_j$, where t_L is as in the statement of Lemma 4.5 and in particular it is such that

$$\liminf_j F^T_{\varepsilon_j,\delta_j}(t_L(u_j)) \le \liminf_j F^T_{\varepsilon_j,\delta_j}(u_j) + \eta. \tag{5.9}$$

Then, we get

$$\liminf_j F^T_{\varepsilon_j,\delta_{\varepsilon_j}}(t_L(u_j)) \geq \int_\Omega f^T_{\mathrm{hom}}(\nabla t_L(u))\,\mathrm{d}x + \gamma^{d-p}\int_\Omega \varphi^T(t_L(u))\,\mathrm{d}x.$$

Note that $t_L(u) \to u$ as $L \to +\infty$, with respect to the strong convergence of $W^{1,p}(\Omega;\mathbb{R}^m)$. By (5.9) and the arbitrariness of η, we can pass to the limit as $L \to +\infty$ and we get

$$\liminf_j F^T_{\varepsilon_j,\delta_j}(u_j) \geq F^T(u).$$

Step 3. Finally we prove the validity of the Γ-limsup inequality.
By a density argument it is sufficient to prove this inequality for $u \in C^\infty_c(\mathbb{R}^d;\mathbb{R}^m)$. For such u let us consider $\Omega' \in \mathcal{A}^{\mathrm{reg}}(\mathbb{R}^d)$ such that $\Omega' \supset\supset \Omega$. By Theorem 2.3 there exists $\tilde{u}_j \in \mathcal{A}_{\varepsilon_j}(\Omega',\mathbb{R}^m)$, $\tilde{u}_j \to u$ in $L^p(\Omega',\mathbb{R}^m)$ such that

$$\lim_{j\to+\infty} \mathcal{F}^T_{\varepsilon_j}(\tilde{u}_j,\Omega') = \int_{\Omega'} f^T_{\mathrm{hom}}(\nabla u). \tag{5.10}$$

By Lemma 4.5 we may suppose also that $\|\tilde{u}_j\|_{L^\infty(\Omega',\mathbb{R}^m)} < +\infty$. Given $k \in \mathbb{N}$, by Lemma 4.3 there exists $\{w_j\}_j$ satisfying (4.7)–(4.9) such that

$$\mathcal{F}^T_{\varepsilon_j}(w_j,\Omega') \leq \mathcal{F}^T_{\varepsilon_j}(\tilde{u}_j,\Omega') + \frac{C}{k}. \tag{5.11}$$

Now we consider $E_j = \bigcup_{i\in Z_j(\Omega')} Q(i\delta_j,\rho^i_j)$. As in Step 2, let us set $R^i_j := \frac{\rho^i_j}{r_{\delta_j}}$ and $s_j := \frac{\varepsilon_j}{r_{\delta_j}}$. For $i \in Z_j(\Omega')$, let $\tilde{v}_{j,i} \in \mathcal{A}_{s_j,T,\tilde{u}^i_j}(Q_{R^i_j};\mathbb{R}^m)$ such that

$$\mathcal{F}^T_{s_j}(\tilde{v}_{j,i},Q_{R^i_j}) = \varphi_{s_j,T,R^i_j}(\tilde{u}^i_j). \tag{5.12}$$

Set

$$v_{j,i}(\alpha) = \tilde{v}_{j,i}\left(\frac{\alpha - i\delta_j}{r_{\delta_j}}\right), \quad \alpha \in Q(i\delta_j,\rho^i_j),$$

and then

$$u_j(\alpha) = \begin{cases} w_j(\alpha) & \text{if } \alpha \in \Omega' \setminus E_j \\ v_{j,i}(\alpha) & \text{if } \alpha \in Q(i\delta_j,\rho^i_j). \end{cases}$$

By a change of variable we get

$$F^T_{\varepsilon_j,\delta_j}(v_{j,i},Q(i\delta_j,\rho^i_j)) = r^{d-p}_{\delta_j}\mathcal{F}^T_{s_j}(\tilde{v}_{j,i},Q_{R^i_j}) = r^{d-p}_{\delta_j}\varphi_{s_j,T,R^i_j}(\tilde{u}^i_j). \tag{5.13}$$

Since $\tilde v_{j,i}\equiv 0$ on Q_1, then $u_j\equiv 0$ on $Q(i\delta_j, r_{\delta_j})$ for every $i\in Z_j(\Omega')$. We need to prove that $u_j\to u$ in $L^p(\Omega;\mathbb{R}^m)$. By the fact that $\tilde u_j\to u$ in $L^p(\Omega';\mathbb{R}^m)$ and $w_j\to u$ in $L^p(\Omega';\mathbb{R}^m)$, we have

$$\begin{aligned}\int_{\Omega'}|u_j-w_j|^p&=\sum_{\alpha\in\Omega'_j}\varepsilon_j^d|u_j(\alpha)-w_j(\alpha)|^p+o(1)\\&\leq\left[\sum_{\alpha\in E_j}\varepsilon_j^d|u_j(\alpha)-w_j(\alpha)|^p\right]+o(1)\\&\leq C\left[\int_{\Omega'}|w_j-\tilde u_j|^p\,\mathrm{d}x+\sum_{i\in Z_j(\Omega')}\sum_{\alpha\in Q(i\delta_j,\rho_j^i)}\varepsilon_j^d|w_j(\alpha)-v_{j,i}(\alpha)|^p\right]+o(1).\end{aligned}$$

For $\alpha\in Q(i\delta_j,\rho_j^i)$ set

$$g_{j,i}(\alpha):=w_j(\alpha)-v_{j,i}(\alpha).$$

Since $v_{j,i}(\alpha)=\tilde u_j^i$ and $w_j=\tilde u_j^i$ on $\partial^{\varepsilon_j T}Q(i\delta_j,\rho_j^i)$, we get $g_{j,i}=0$ on $\partial^{\varepsilon_j T}Q(i\delta_j,\rho_j^i)$. Then, up to identify $g_{j,i}$ with its piecewise affine interpolation on the cells $\alpha+[0,\varepsilon)^d$ for any $\alpha\in Q(i\delta_j,\rho_j^i)$, we have $g_{j,i}\in W_0^{1,p}(Q(i\delta_j,\rho_j^i);\mathbb{R}^m)$, for which a classical Poincaré inquality holds. Then, since the values of the gradient of the piecewise affine interpolation coincide with the values of the difference quotient along the directions $e_1,\dots,e_d$, by (5.10), (5.12), assumption (G0) and Proposition 4.11, we get

$$\begin{aligned}\sum_{i\in Z_j(\Omega')}\sum_{\alpha\in Q(i\delta_j,\rho_j^i)}\varepsilon_j^d|g_{j,i}(\alpha)|^p&\leq C\delta_j\sum_{i\in Z_j(\Omega')}\sum_{k=1}^d\sum_{\alpha\in(Q(i\delta_j,\rho_j^i))_{\varepsilon_j}(e_k)}\varepsilon_j^d\left|\frac{g_{j,i}(\alpha+\varepsilon_je_k)-g_{j,i}(\alpha)}{\varepsilon_j}\right|^p\\&\leq C\left[\delta_j\sum_{i\in Z_j(\Omega')}G^{1,p}_{\varepsilon_j}(w_j,Q(i\delta_j,\rho_j^i))+\delta_j\sum_{i\in Z_j(\Omega')}G^{1,p}_{\varepsilon_j}(v_{j,i},Q(i\delta_j,\rho_j^i))\right]\\&\leq C\left[\delta_j\mathcal{F}^T_{\varepsilon_j}(w_j,\Omega')+\delta_j\sum_{i\in Z_j(\Omega')}\mathcal{F}^T_{\varepsilon_j}(v_{j,i},Q(i\delta_j,\rho_j^i))\right]\\&\leq C\delta_j\left[\int_{\Omega'}f^T_{\mathrm{hom}}(\nabla u)+\sum_{i\in Z_j(\Omega')}r_{\delta_j}^{d-p}\varphi_{s_j,T,R_j^i}(\tilde u_j^i)+\frac{1}{k}\right]\\&=C\delta_j\left[\int_{\Omega'}f^T_{\mathrm{hom}}(\nabla u)+\gamma^{d-p}\sum_{i\in Z_j(\Omega')}\delta_j^d\varphi_{s_j,T,R_j^i}(\tilde u_j^i)+\frac{1}{k}\right]\\&=o(1)\end{aligned}$$

Finally we pass to the estimate of the energy. By (5.13) we get

$$\begin{aligned}F^T_{\varepsilon_j,\delta_j}(u_j)&\leq\mathcal{F}^T_{\varepsilon_j}(w_j,\Omega')+\sum_{i\in Z_j(\Omega')}F^T_{\varepsilon_j,\delta_j}(v_{j,i},Q(i\delta_j,\rho_j^i))\\&=\mathcal{F}^T_{\varepsilon_j}(w_j,\Omega')+\sum_{i\in Z_j(\Omega')}r_{\delta_j}^{d-p}\varphi_{s_j,T,R_j^i}(\tilde u_j^i).\end{aligned}$$

Applying Proposition 4.11, (5.10) and (5.11) we get

$$\limsup_{j\to+\infty} F^T_{\varepsilon_j,\delta_j}(u_j) \leq \int_{\Omega'} f^T_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + \gamma^{d-p}\int_{\Omega'} \varphi^T(u)\,\mathrm{d}x + \frac{C}{k}.$$

Then, the conclusion follows by the arbitrariness of $k \in \mathbb{N}$ and letting $\Omega' \to \Omega$.

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Data availability statement

All data generated or analysed during this study are included in this published article.

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