

AN ANISOTROPIC POINCARÉ INEQUALITY IN $GSBV^p$ AND THE LIMIT OF STRONGLY ANISOTROPIC MUMFORD–SHAH FUNCTIONALS

JANUSZ GINSTER^{1,*}  AND PETER GLADBACH²

Abstract. We show that functions in $GSBV^p$ in three-dimensional space with small variation in 2 of 3 directions are close to a function of one variable outside an exceptional set. Bounds on the volume and the perimeter in these two directions of the exceptional sets are provided. As a key tool we prove an approximation result for such functions by functions in $W^{1,p}$. For this we present a two-dimensional countable ball construction that allows to carefully remove the jumps of the function. As a direct application, we show Γ -convergence of an anisotropic three-dimensional Mumford–Shah model to a one-dimensional model.

Mathematics Subject Classification. 26D10; 49J45.

Received January 2, 2023. Accepted January 2, 2025.

1. INTRODUCTION

Poincaré inequalities in their various forms are a standard tool in the study of PDEs or variational problems. In particular, in the Sobolev space $W^{1,p}(\Omega)$ of a connected open bounded Lipschitz domain it holds

$$\min_{a \in \mathbb{R}} \int_{\Omega} |u - a|^p \, dx \leq C \int_{\Omega} |\nabla u|^p \, dx.$$

By density this can be extended to the space of functions of bounded variation, $BV(\Omega)$, where the right hand side is replaced by $|Du|(\Omega)$, see [1]. Clearly, in BV one cannot expect that the left hand side can be bounded in terms of the absolutely continuous part of Du , ∇u , alone. However, in [2] De Giorgi *et al.* show that for functions $u \in SBV^p(\Omega)$ whose jump set is not too large there exists an exceptional set $\omega \subset \Omega$ such that for $1 \leq p < n$

$$\left(\min_{a \in \mathbb{R}} \int_{\Omega \setminus \omega} |u - a|^{p^*} \, dx \right)^{1/p^*} \leq C \left(\int_{\Omega} |\nabla u|^p \, dx \right)^{1/p} \quad (1.1)$$

Keywords and phrases: Functional inequalities, Sobolev spaces, functions of bounded variation, anisotropic Mumford–Shah.

¹ Weierstrass Institute, Anton-Wilhelm-Amo-Strasse 39, 10117 Berlin, Germany.

² Peter Gladbach Institut für Angewandte Mathematik Rheinische Friedrich-Wilhelms-Universität Bonn Endenicher Allee 60 53115 Bonn, Germany.

* Corresponding author: ginster@wias-berlin.de.

and $\mathcal{L}^n(\omega) \leq C\mathcal{H}^{n-1}(J_u)^{n/(n-1)}$. In [3], Theorem 2.3 (see [4] for an application to a compactness result in GSBV^p) a variant for functions in SBV whose jump set is not necessarily small is shown: In this setting there exists a Cacciopoli partition whose total boundary can be estimated by the length of the jump set of u and on each set in the Cacciopoli partition the difference between u and a constant can be estimated by ∇u .

In the same spirit as (1.1), in [5], Theorem 1.2 Cagnetti *et al.* show the more sophisticated Korn–Poincaré inequality in the space GSBD^p in arbitrary dimension (see also [6, 7] for the two-dimensional versions) and even provide bounds on the perimeter of the exceptional set ω . The strategy of the proof is to find for $u \in \text{GSBD}^p$ a function $w \in W^{1,p}$ which agrees with u outside an exceptional set $\omega \subseteq \Omega$ and satisfies appropriate bounds. Then one can apply the ordinary Korn–Poincaré inequality to the more regular function w to obtain an estimate on u outside ω . In addition, the same article provides not only a bound on $\mathcal{L}^n(\omega)$ but also on $\text{Per}(\omega)$ in terms of $\mathcal{H}^{n-1}(J_u)$. A simple modification of this argument yields also a version of (1.1) with an additional bound on the perimeter of the exceptional set, see Section 2.1.

In this paper we are interested in functions in GSBV^p whose jump set is approximately parallel to one of the three directions. For such functions it is possible to estimate the L^p -difference to a function only depending on this one variable outside an exceptional set satisfying appropriate bounds. More precisely, we prove the following.

Proposition 1.1. *Let $S \subseteq \mathbb{R}^2$ be open, bounded, connected, with Lipschitz boundary and $I \subseteq \mathbb{R}$ an open, bounded interval. Then there exists for $\Omega = I \times S$ a number $\eta(\Omega) > 0$ such that the following holds:*

For all $p \in [1, \infty)$ there exists a constant $C(p, \Omega) > 0$ such that for all functions $u \in \text{GSBV}^p(\Omega)$ with $\int_{J_u} |\nu'| d\mathcal{H}^2 \leq \eta(\Omega)$ (here $\nu = (\nu_1, \nu') \in S^2$ denotes the measure theoretic normal to J_u) there exists a set $\omega \subseteq \mathbb{R}^3$ and a measurable function $a : I \rightarrow \mathbb{R}$ satisfying

$$\int_{\Omega \setminus \omega} |u(x) - a(x_1)|^p dx \leq C(p, \Omega) \int_{\Omega} |\nabla' u|^p dx,$$

where $\nabla' = (\partial_2, \partial_3)$. Moreover,

$$\mathcal{L}^3(\omega) \leq C(p, \Omega) \int_{J_u} |\nu'| d\mathcal{H}^2 \tag{1.2}$$

and

$$\sup_{\varphi \in C_c^0(I \times S), |\varphi| \leq 1} \int_{\omega} \partial_i \varphi dx \leq C(p, \Omega) \int_{J_u} |\nu'| d\mathcal{H}^2 \quad \text{for } i = 2, 3. \tag{1.3}$$

At first glance, it appears as if an application of inequality (1.1) in each slice $\{x_1\} \times S$ can be used to show the above result (without the bounds on the exceptional set). The exceptional set ω would then be given by the union of the corresponding exceptional sets in each slice. However, note that in order to guarantee \mathcal{L}^3 -measurability of a set $\omega \subseteq \mathbb{R}^3$ it is not enough to ensure \mathcal{L}^2 -measurability of every slice $(\{x_1\} \times S) \cap \omega$. Hence, a simple application of inequality (1.1) or similar results in every slice cannot guarantee the measurability of the exceptional set ω .

Instead we present a simple two-dimensional argument (see Thm. 2.2) that allows us to remove the jumps of u in a small neighborhood around the jump set. This two-dimensional construction is heavily inspired by the ball construction which was developed in the context of the Ginzburg–Landau functional, see [8–10] (*cf.* [11–14] for an application in the context of dislocations). We cover the jump set of u by countably many balls and grow them according to the well-known ball construction until u has on the boundaries of the grown balls trace values that can be extended without jumps into the balls. This yields again an approximation outside a small set $\omega \subset \Omega$ from which the inequality (1.1) can be easily derived. Moreover, this technique can be transferred to

three dimensions by applying it simultaneously in each slice while at the same time guaranteeing measurability of the exceptional set.

In three dimensions we prove the following approximation result.

Theorem 1.2. *Let $S \subseteq \mathbb{R}^2$ be open, bounded, connected with Lipschitz boundary and $I \subseteq \mathbb{R}$ an open, bounded interval. Then there exists for $\Omega = I \times S$ a number $\eta(\Omega) > 0$ such that the following hold:*

For all $p \in [1, \infty)$ there exists a constant $C(p, \Omega) > 0$ such that for all functions $u \in \text{GSBV}^p(\Omega)$ with $\int_{J_u} |\nu'| d\mathcal{H}^2 \leq \eta(\Omega)$ (as before $\nu = (\nu_1, \nu') \in S^2$ is the measure theoretic normal to J_u) there exists a set $\omega \subseteq \mathbb{R}^3$ with

$$\mathcal{L}^3(\omega) \leq C(p, \Omega) \int_{J_u} |\nu'| d\mathcal{H}^2 \quad (1.4)$$

and

$$\sup_{\varphi \in C_c^1(I \times S), |\varphi| \leq 1} \int_{\Omega} \partial_i \varphi dx \leq C(p, \Omega) \int_{J_u} |\nu'| d\mathcal{H}^2 \quad \text{for } i = 1, 2, \quad (1.5)$$

and a function $w \in L^p(\Omega)$ with $w = u$ on $\Omega \setminus \omega$ and

$$\int_{\Omega} |\nabla' w|^p dx \leq C(p, S) \int_{\Omega} |\nabla' u|^p dx, \quad (1.6)$$

where $\nabla' = (\partial_2, \partial_3)$.

Using the usual Poincaré inequality for w on each slice for the function $a(x_1) = \int_S w(x_1, x') dx'$ this implies Proposition 1.1.

Another consequence of Theorem 1.2 is the following two-dimensional statement (which in turn implies inequality 1.1), which follows directly by considering functions $u(x_1, x') = u(x')$.

Theorem 1.3. *Let $S \subseteq \mathbb{R}^2$ be open, bounded, connected with Lipschitz boundary. Then there exists a number $\eta(S) > 0$ and a constant $C(S)$ such that the following holds:*

For all $p \in [1, \infty)$ there exists a constant $C(p, S) > 0$ such that for all functions $u \in \text{GSBV}^p(S)$ with $\mathcal{H}^1(J_u) \leq \eta(S)$ there exists a bounded set $\omega \subseteq \mathbb{R}^2$ with $\text{Per}(\omega) \leq C(S) \mathcal{H}^1(J_u)$ and a function $w \in W^{1,p}(S)$ with $w = u$ in $S \setminus \omega$ and

$$\int_S |\nabla w|^p dx \leq C(p, S) \int_S |\nabla u|^p dx.$$

We remark that the statement in Theorem 1.3 is already well-known to the community in arbitrary dimensions, as it can also be proved by following the complex argument in [5], Theorem 1.2 (see also [7] for the two-dimensional result) for the space GSBD^p carefully. Additionally, in Section 2.1, we show how to obtain Theorem 3.1 directly from [5], Theorem 1.2.

We note that the conditions $\mathcal{H}^1(J_u) \leq \eta(S)$ here and $\int_{J_u} |\nu'| d\mathcal{H}^2 \leq \eta(\Omega)$ in Theorem 1.2 are necessary, as can be seen in Figure 1.

As an immediate application of Proposition 1.1, we investigate an anisotropic version of the Mumford–Shah functional (see e.g. [1, 15]) in a three dimensional domain $I \times S$. Here, one may consider $x_1 \in I$ as a time variable and $x' = (x_2, x_3)$ as a space variable in a grayscale video file $u \in \text{GSBV}^p(I \times S)$. We study in Section 4 the functional

$$F_\varepsilon(u) := \int_{I \times S} \frac{1}{\varepsilon} |\nabla' u|^p + |\partial_1 u|^p dx + \int_{J_u} \frac{1}{\varepsilon} |\nu'| + |\nu_1| d\mathcal{H}^2.$$

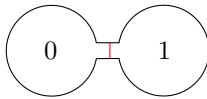


FIGURE 1. Functions with jump set larger than $\eta(S)$ cannot be approximated with $W^{1,p}(S)$ functions in most of Ω . The above function is piecewise constant but not close to any constant function.

Here $\nu \in S^2$ is the measure theoretic normal to the jump set J_u and $\varepsilon > 0$ is a small parameter, so that variations in x' are penalized much more harshly than variations in x_1 . As ε tends to zero, we expect compactness of finite-energy sequences, with limit functions depending only on x_1 and Γ -convergence of F_ε to the one-dimensional Mumford–Shah functional $F_0 : L^1(I \times S) \rightarrow [0, \infty]$,

$$F_0(u) = \begin{cases} \mathcal{L}^2(S) \left(\int_I |u'|^p dx_1 + \#J_u \right) & \text{if } u(x) = u(x_1) \in \text{SBV}^p(I) \\ \infty & \text{otherwise.} \end{cases}$$

We do indeed prove this in Theorem 4.1. While Γ -convergence is obvious, the compactness statement is not. In fact, the main difficulty lies in showing that a function with finite F_ε is close in measure to a function only of x_1 . Here we use Proposition 1.1.

The rest of the article is structured as follows: First, we introduce the necessary notation. Then we present the two-dimensional countable ball construction. For better readability we first use this construction to prove the two-dimensional result Theorem 1.3 in Section 2. In Section 3 we present the transfer of this construction to three dimensions and show Theorem 1.2. Then we showcase how this can be used to prove a Γ -convergence result for the anisotropic Mumford–Shah functional in Section 4.

1.1. Notation

We will write C or c for generic constants that may change from line to line but do not depend on the problem parameters. For set of skew-symmetric matrices in $\mathbb{R}^{n \times n}$ we write $Skew(n)$. For the ease of notation, we always identify vectors in \mathbb{R}^n with their transposes. For vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we write $x' = (x_2, x_3)$. Moreover, we write $\nabla' = (\partial_2, \partial_3)$. For a measurable set $B \subseteq \mathbb{R}^n$ we use the notation $|B|$ or $\mathcal{L}^n(B)$ to denote its n -dimensional Lebesgue measure. Similarly, we denote by $\mathcal{H}^k(B)$ its k -dimensional Hausdorff measure. Moreover, we use standard notation for L^p -spaces and Sobolev spaces $W^{1,p}$. In addition, we denote by BV the space of functions with bounded variation, by SBV^p the space of special functions of bounded variation (with p -integrability of the absolutely continuous part), and by GSBV^p the space of generalized special functions of bounded variation as introduced in [1]. In particular, we use for a function $u \in \text{SBV}^p$ the decomposition

$$Du = \nabla u d\mathcal{L}^n + [u] \otimes \nu d\mathcal{H}_{|J_u}^{n-1},$$

where J_u is the jump set of u , $\nu \in S^{n-1}$ is the measure theoretic normal to J_u and $[u] = u^+ - u^-$ for the approximate limits u^+ and u^- on J_u . In the specific case of a characteristic function $u = \mathbb{1}_\omega$ for a measurable $\omega \subseteq \mathbb{R}^n$ we write $\text{Per}(\omega) := |Du|(\mathbb{R}^n)$ for its perimeter. Eventually, we write GSBD^p for the space of specialized special functions of bounded deformation as introduced in [16]. In particular, for $u \in \text{GSBD}^p$ we denote by $e(u)$ the density of the absolutely continuous part of the symmetrized derivative Eu . In the specific case that $u \in W^{1,p}$, this means $e(u) = \frac{\nabla u + \nabla u^T}{2}$.

2. TWO-DIMENSIONAL RESULT: PROOF OF THEOREM 1.3

2.1. A Proof using Korn's Inequality in SBD^p

One way to prove Theorem 1.3 is to use the existing (and more complicated) approximation results in SBD^p which were used to prove a Korn inequality in SBD^p.

Proof of Theorem 1.3. Let $u \in \text{GSBV}^p(S)$. We then define the vector-valued function $v \in \text{GSBD}^p(S)$ by $v = (u, 0)$. By [5], Theorem 1.2 there exists $\omega \subseteq S$ with $\mathcal{L}^2(\omega)^{\frac{1}{2}} + \mathcal{H}^1(\partial^*\omega) \leq C\mathcal{H}^1(J_v) = C\mathcal{H}^1(J_u)$ and a function $w \in W^{1,p}(S; \mathbb{R}^2)$ such that $v = w$ on $S \setminus \omega$ and

$$\int_S |e(w)|^p dx \leq C \int_S |e(v)|^p dx.$$

By Korn's inequality applied to w there exists $W \in \text{Skew}(2)$ such that

$$\int_S |\nabla w - W|^p dx \leq C \int_S |e(v)|^p dx.$$

In particular, we find

$$\mathcal{L}^2(S \setminus \omega) |W_{21}|^p \leq \int_{S \setminus \omega} \left| \begin{pmatrix} \partial_1 u & \partial_2 u \\ 0 & 0 \end{pmatrix} - W \right|^p dx \leq C \int_S |e(v)|^p dx. \quad (2.1)$$

If $\mathcal{L}^2(\omega) \leq \frac{1}{2}\mathcal{L}^2(S)$, i.e. $\mathcal{H}^1(J_u)$ is small enough, we obtain $|W_{21}|^p \leq \frac{2C}{\mathcal{L}^2(S)} \int_S |e(v)|^p dx$. As W is skew-symmetric this already implies $|W|^p \leq \frac{2C}{\mathcal{L}^2(S)} \int_S |e(v)|^p dx$. Hence, by the triangle inequality we find

$$\int_S |\nabla w|^2 dx \leq 2^p \int_S |\nabla w - W|^p dx + 2^p \int_S |W|^p dx \quad (2.2)$$

$$\leq C \int_S |e(v)|^p dx \quad (2.3)$$

$$\leq C \int_S |\nabla u|^p dx, \quad (2.4)$$

which concludes the proof. \square

We close this section with the following remark.

Remark 2.1. Later we would like to apply this result in each slice of the form $\{x_1\} \times S$ to obtain exceptional sets $\omega_{x_1} \subseteq S$ and functions $w_{x_1} \in W^{1,p}(S)$ satisfying the conclusion of Theorem 1.3. However, it is not immediate from this proof that one can define a measurable set $\omega = \bigcup_{x_1} \{x_1\} \times S$ and a measurable function $w : I \times S \rightarrow \mathbb{R}$ through $w(x_1, x_2, x_3) = w_{x_1}(x_2, x_3)$. As the construction in [5] is rather involved, we present a simpler method in this paper to construct the desired set and function simultaneously in all slices using the ball construction technique. From our construction it is easy to see that all defined quantities are measurable. On the other hand, the construction presented in this manuscript, cannot easily be used to prove the result from [5] or [7] as in this setting the symmetric part of w needs to be controlled by $\int_S |e(u)|^p dx$. This is typically not the case in our simple construction as w essentially emerges from u by extending good trace values of u into balls. Then gradient estimates for w follow from gradient bounds for u on the boundary of these balls. Even if the symmetric part of the gradient would be controlled on the boundary of these balls, the naively extended w would generally not have this property.

2.2. The countable ball construction and the proof of Theorem 2.2

In this section, we prove the following more general version of Theorem 1.3:

Theorem 2.2. *Let $S \subseteq \mathbb{R}^2$ be open, bounded, connected with Lipschitz boundary. Then there exists a number $\eta(S) > 0$ and a constant $C(S)$ such that the following holds:*

For all $p \in [1, \infty)$ there exists a constant $C(p, S)$ such that for all $T > 0$ and all functions $u \in \text{GSBV}^p(S)$ with $\mathcal{H}^1(J_u) \leq e^{-T}\eta(S)$ there exists a bounded set $\omega \subseteq \mathbb{R}^2$ with $\text{Per}(\omega) \leq e^T C(S)\mathcal{H}^1(J_u)$ and a function $w \in W^{1,p}(S)$ with $w = u$ in $S \setminus \omega$ and

$$\int_S |\nabla w|^p dx \leq \left(1 + \frac{C(p, S)}{T}\right) \int_S |\nabla u|^p dx.$$

Theorem 1.3 follows from Theorem 2.2 by taking $T = 1$. In fact, Theorem 2.2 is a direct consequence of [5], Remark 3.5. The more precise version above shows in a quantitative way that the smaller the jump set of u is, the closer is the constant in the estimate for $\int_S |\nabla w|^p dx$ to 1. This is of course consistent with the fact that if $\mathcal{H}^1(J_u) = 0$ then w can be chosen to be u .

Before we present our proof of Theorem 2.2, we briefly introduce the ball-construction technique from the seminal paper [9]:

Lemma 2.3. *Given a finite set of balls $B_i := B(x_i, r_i) \subset \mathbb{R}^d$, $i = 1, \dots, N$, there is a family of finite sets of balls $B_i^t := B(x_i^t, r_i^t)$, $i = 1, \dots, N$, $t \in [0, t_i]$ with collapse times $t_i \in [0, \infty]$ such that for $I_t := \{i = 1, \dots, N; t < t_i\}$ we have*

- (i) $\bigcup_{i=1}^N B_i \subseteq \bigcup_{i \in I_s} B_i^s \subseteq \bigcup_{i \in I_t} B_i^t$ whenever $0 \leq s \leq t < \infty$.
- (ii) $\sum_{i \in I_t} r_i^t = e^t \sum_{i=1}^N r_i$.
- (iii) $(\overline{B_i^t})_{i \in I_t}$ is pairwise disjoint.
- (iv) $B(x_i^s, e^{t-s}r_i^s) \subseteq B_i^t$ whenever $0 \leq s \leq t < t_i$.

Proof. We first replace the balls B_i with balls B_i^0 such that (iii) holds. If $\overline{B_i} \cap \overline{B_j} \neq \emptyset$, $i < j$, then $|x_i - x_j| \leq r_i + r_j$. But then

$$B_i \cup B_j \subset B(x_i^0, r_i^0), \text{ where } x_i^0 := \frac{r_i}{r_i + r_j}x_i + \frac{r_j}{r_i + r_j}x_j \text{ and } r_i^0 := r_i + r_j. \quad (2.5)$$

Also we set $t_j := 0$.

We may repeat this replacement at most $N - 1$ times until (iii) is satisfied, defining I_0 and $(B_i^0)_{i \in I_0}$. We define $I_t := I_0$ and $B_i^t := B(x_i^0, e^t r_i^0)$ for $t < \varepsilon$, where

$$\varepsilon := \inf\{t > 0 : \overline{B_i^t} \cap \overline{B_j^t} \neq \emptyset \text{ for some } i \neq j \in I_0\}.$$

If $\varepsilon = \infty$, we are done. Otherwise, at time ε , we repeat first the replacement and then the growing scheme to some larger time. Since at every such time the number of balls is strictly decreased, there are at most $N - 1$ such collapsing times, at which point a single ball remains.

This defines I_t and the balls $(B_i^t)_{i \in I_t}$ for all $t \in [0, \infty)$.

It is straightforward to check that (i)–(iv) are satisfied for all times. □

Remark 2.4. We see that as N is replaced by $N + 1$, all collapsing times t_i decrease and all radii increase. This is important in the following:

We wish to apply a version of Lemma 2.3 to a covering of the jump set J_u . However, as the jump set of a function $u \in \text{GSBV}^p(S)$ is in general not compact, countably many balls are required to cover it. We can however extend the ball construction to the case of countably many balls with $\sum_{i \in \mathbb{N}} r_i < \infty$, cf. also [14].

Before we prove the countable version of Lemma 2.3, we first show an auxiliary lemma that ensures we can apply the one-dimensional SBV-compactness result to the sequence of constructed radii and centers, cf. [1], Exercise 4.4.

Lemma 2.5. *Let $(a_n)_n$ be a sequence of positive numbers such that $\sum_n a_n < \infty$. Then there exists $h : [0, \infty) \rightarrow [0, \infty)$ such that $h(0) = 0$, h is non-decreasing, subadditive, $\lim_{t \searrow 0} \frac{h(t)}{t} = \infty$ and $\sum_n h(a_n) < \infty$.*

Proof. Assume that $\phi \in W_{\text{loc}}^{1,1}([0, \infty))$ with $\sum_n \phi(a_n) < \infty$. Then

$$\sum_{n \in \mathbb{N}} \phi(a_n) = \sum_{n \in \mathbb{N}} \int_0^{a_n} \phi'(t) dt = \int_0^\infty N(t) \phi'(t) dt, \quad (2.6)$$

where $N(t) = \#\{n \in \mathbb{N} : a_n \geq t\}$. In particular for $\phi(t) = t$ we have $\int_0^\infty N(t) dt = \sum_n a_n < \infty$.

Determine for $k \geq k_0$ the points $t_k \in (0, \infty)$ such that $\int_0^{t_k} N(t) dt = 4^{-k}$ and set $h' = 2^k$ in (t_{k+1}, t_k) , $h(0) = 0$, $h' = 0$ on $[k_0, \infty)$. In particular for this choice of h we have

$$\int_0^\infty N(t) h'(t) dt \leq \sum_{k=k_0}^\infty 4^{-k} 2^k < \infty. \quad (2.7)$$

Since $N(t) \geq 1$ on $[0, k_0]$, we have that $h \in W_{\text{loc}}^{1,1}([0, \infty))$, and since h' is non-increasing and non-negative, h is indeed subadditive and non-decreasing, with $\lim_{t \searrow 0} \frac{h(t)}{t} \geq \lim_{t \searrow 0} h'(t) = \infty$, and $\sum_{n \in \mathbb{N}} h(a_n) < \infty$. \square

We now prove the countable ball construction:

Lemma 2.6. *Given an at most countable set of balls $B_i := B(x_i, r_i) \subset \mathbb{R}^d$, $i \in \mathbb{N}$ with*

$$\sum_{i \in \mathbb{N}} r_i < \infty,$$

there is a family of at most countable sets of balls $B_i^t := B(x_i^t, r_i^t)$, $i \in \mathbb{N}$, $t \in [0, t_i)$ with collapse times $t_i \in [0, \infty]$ such that for $I_t := \{i \in \mathbb{N} ; t < t_i\}$ we have

- (i) $\bigcup_{i \leq N} B_i \subseteq \bigcup_{i \leq N, i \in I_s} B_i^s \subseteq \bigcup_{i \leq N, i \in I_t} B_i^t$ whenever $0 \leq s \leq t < \infty$, for all $N \in \mathbb{N}$.
- (ii) $\sum_{i \in I_t} r_i^t \leq e^t \sum_{i \in \mathbb{N}} r_i$ for all $t > 0$.
- (iii) For all $t > 0$ the family $(\overline{B_i^t})_{i \in I_t}$ is pairwise disjoint.
- (iv) $B(x_i^s, e^{t-s} r_i^s) \subseteq B_i^t$ whenever $0 \leq s \leq t < t_i$.
- (v) The balls B_i^t are monotone in the following sense: If $B_i \subseteq B_i'$ for every $i \in \mathbb{N}$, then $t_i' \leq t_i$ and $B_i^t \subseteq (B_i')^t$ for every $i \in \mathbb{N}$, $t < t_i'$. Also

$$\bigcup_{i \leq N, i \in I_t} B_i^t \subseteq \bigcup_{i \leq N, i \in I_t'} (B_i^t)' \text{ for all } N \in \mathbb{N}, t \geq 0.$$

- (vi) The functions r_i^t as functions of t are non-decreasing. As functions of t it holds $r_i^t \in \text{SBV}((0, t_i))$ and $x_i^t \in \text{SBV}((0, t_i); \mathbb{R}^2)$ satisfying $J_{r_i^t} \cup J_{x_i^t} \subseteq \{t_j : j \in \mathbb{N}\}$, $(r_i^t)' = r_i^t$ and $(x_i^t)' = 0$.

Note that the first condition implies in particular that $t \mapsto \bigcup_{i \in I_t} B_i^t$ is monotone. In addition, the index $i(t)$ of the ball covering B_i^s decreases with t .

Proof. We start with the situation where $r_i = 0$ for all $i > N$ for some $N \in \mathbb{N}$. We show (i)-(v) in this case by induction over N . Consider first the case $N = 0$, where we simply set $t_0 := \infty, x_0^t = x_0, r_0^t := e^t r_0$ for all $t \geq 0$. In particular we note that if $B_0 \subseteq \tilde{B}_0$, then $B_0^t \subseteq \tilde{B}_0^t$.

Now assume (i)-(v) hold for $N - 1$. Let $\tilde{t}_i \in [0, \infty], \tilde{B}_i^t \subset \mathbb{R}^d$ for $i = 0, \dots, N - 1, t \in [0, \tilde{t}_i]$, be a family of sets of balls for B_1, \dots, B_{N-1} . Consider $B_N^t := B(x_N, e^t r_N)$ for $t \geq 0$. Let

$$t_N := \inf\{t \geq 0 : \overline{B_N^t} \cap \bigcup_{i=0}^{N-1} \overline{\tilde{B}_i^t} \neq \emptyset\}.$$

By our construction, there is $i_0 \in \{0, \dots, N - 1\}$ such that $\tilde{t}_{i_0} > t_N$ and $\overline{B_N^{t_N}} \cap \overline{\tilde{B}_{i_0}^{t_N}} \neq \emptyset$.

We now set $B_i^t := \tilde{B}_i^t$ for $i = 0, \dots, N - 1, t < \min(\tilde{t}_i, t_N)$.

At time t_N , we replace any two balls $B = B(x, r), B' = B(x', r')$ among $\{B_i^{t_N} : i = 0, \dots, N - 1, t_i > t_N\} \cup \{B_N^{t_N}\}$ with $\overline{B} \cap \overline{B'} \neq \emptyset$ with the new ball $B'' := B(x'', r'')$, where

$$x'' := \frac{r}{r+r'}x + \frac{r'}{r+r'}x', \quad r'' := r + r'. \quad (2.8)$$

We note that if $\overline{B} \cap \overline{B'} \neq \emptyset$, then $|x - x'| \leq r + r'$ and thus $B \cup B' \subseteq B''$. Also, if $B \subseteq \tilde{B}$ and $B' \subseteq \tilde{B}'$ then $B'' \subseteq \tilde{B}''$.

We repeat the replacement of two balls as above until $\overline{B} \cap \overline{B'} = \emptyset$ for all balls in the set. As the replacement happens at least once, namely $\overline{\tilde{B}_{i_0}^{t_N}} \cap \overline{B_N^{t_N}} \neq \emptyset$, there are at most $N - 1$ balls remaining after replacements. The replacements are monotone in the sense of (v). From time t_N onwards, we restart the growing of balls and see that (i)-(v) remain satisfied.

This finishes the proof in the case of finitely many balls.

For the case of countably many balls, we perform the finite construction above for every $N \in \mathbb{N}$, yielding nonincreasing sequences of collapse times $t_{i,N} \in [0, \infty], N \geq i$ and nondecreasing sequences of families of balls $B_{i,N}^t \subseteq \mathbb{R}^d, N \geq i, t < t_{i,N}$. Because of the monotonicities there exist limits for the collapse times $t_i := \lim_{N \rightarrow \infty} t_{i,N}$ and limit balls $\tilde{B}_i^t := \bigcup_{N \rightarrow \infty} B_{i,N}^t$ for $t < t_i$. This is indeed a ball because it is the countable union of a nondecreasing sequence of balls, where $r_{i,N}^t \leq e^t \sum_{j \in \mathbb{N}} r_j$ is bounded, so that the union is in fact an open ball.

The family thus created unfortunately does not satisfy (i) or (v). For example, if the ball $B_{2,N}^{t_2}$ is absorbed by $B_{1,N}^{t_1}$ at time $t_{2,N} = 1 + 1/N$, then $t_2 = 1$ but $B_{1,N}^{t_2}$ will not include the absorbed ball, so that (i),(v) are not satisfied.

In order to satisfy (i)-(vi), we define the right-continuous version

$$B_i^t := \left(\bigcap_{t < r < t_i} \tilde{B}_i^r \right)^\circ \quad \text{for } t < t_i.$$

We now go through the properties (i)-(v).

For (i), we fix $N \in \mathbb{N}$. Then

$$\bigcup_{i \leq N, i \in I_t} B_i^t = \bigcup_{i \leq N, i \in I_t} \left(\bigcap_{t < r < t_N} \tilde{B}_i^r \right)^\circ = \left(\bigcap_{t < r < \min\{t_i : i \leq N, t_i > t\}} \bigcup_{i \leq N, r < t_i} \tilde{B}_i^r \right)^\circ.$$

This shows (i).

For (ii) by Fatou's Lemma

$$\sum_{i \in I_t} r_i^t \leq \liminf_{r \searrow t} \sum_{i \in I_r} \tilde{r}_i^r \leq \liminf_{r \searrow t} \limsup_{N \rightarrow \infty} \sum_{i \in I_{r,N}} r_{i,N}^r \leq e^t \sum_{i \in \mathbb{N}} r_i.$$

For (iv), since (iv) holds for $N \in \mathbb{N}$, we have $\tilde{B}(x_i^{s+\varepsilon}, e^{t-s}\tilde{r}_i^{s+\varepsilon}) \subseteq \tilde{B}_i^{t+\varepsilon}$ for every $\varepsilon > 0$. Taking the limit $\varepsilon \rightarrow 0$, it follows that $B(x_i^s, e^{t-s}r_i^s) \subseteq B_i^t$.

For (iii), take two balls B_i^t, B_j^t with $i \neq j \in I_t$ and assume that $\overline{B_i^t} \cap \overline{B_j^t} \neq \emptyset$. Since (iv) holds, we also have $\tilde{B}_i^{t+\varepsilon} \cap \tilde{B}_j^{t+\varepsilon} \neq \emptyset$ for $\varepsilon > 0$ small enough. But then already $B_{i,N}^{t+\varepsilon} \cap B_{j,N}^{t+\varepsilon} \neq \emptyset$ for N large, a contradiction.

Next, (v) is clear from the construction.

Finally, we show (vi). First let $h : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying the assertions in Lemma 2.5 for the sequence of radii r_i . Recall that $r_{i,N}^t$ is the radius of the ball $B_{i,N}^t$ in the finite construction for the first N initial balls. It follows that $r_{i,N}^t$ is a nondecreasing sequence of functions. Moreover, $r_{i,N}^t \rightarrow r_i^t$ for all $t \notin \{t_j : j \in \mathbb{N}\}$. In particular, by (ii) the sequence $r_{i,N}^t$ is uniformly bounded on $(0, t_i)$. Moreover, it holds by construction that $r_{i,N}^t \in \text{SBV}(0, t_i)$ as a function of t , $(r_{i,N}^t)' = r_{i,N}^t$, $J_{r_{i,N}^t} \subseteq \{t_{1,N}, \dots, t_{N,N}\}$ and (cf. (2.8))

$$\begin{aligned} \sum_{s \in J_{r_{i,N}^t}^t} h(|[r_{i,N}^s]|) &\leq \sum_{s \in J_{r_{i,N}^t}^t} h\left(\sum_{1 \leq j \leq N: s=t_{j,N}, B_j \subseteq B_{i,N}^t} r_{j,N}^s\right) \\ &\leq \sum_{s \in J_{r_{i,N}^t}^t} \sum_{1 \leq j \leq N: s=t_{j,N}, B_j \subseteq B_{i,N}^t} h(r_{j,N}^s) \\ &\leq \sum_{s \in J_{r_{i,N}^t}^t} \sum_{1 \leq j \leq N: s=t_{j,N}, B_j \subseteq B_{i,N}^t} h\left(\sum_{B_k \subseteq B_{j,N}^s} e^s r_k\right) \\ &\leq \sum_k h(e^{t_i} r_k) \leq \lceil e^{t_i} \rceil \sum_k h(r_k), \end{aligned}$$

where we used that h is nondecreasing and subadditive. Hence, we may apply the compactness result in $\text{SBV}(0, t_i)$ to obtain that $r_i^t \in \text{SBV}(0, t_i)$. Moreover, $(r_{i,N}^t)' \rightarrow (r_i^t)'$ in $L^p(0, t_i)$ for all $1 < p < \infty$ and $r_{i,N}^t \rightarrow r_i^t$ in $L^1(0, t_i)$. In particular, $(r_i^t)' = r_i^t$. Consider the center points $x_{i,N}^t$ of the balls $B_{i,N}^t$. By construction it holds that $x_{i,N}^t \in \text{SBV}(0, t_i)$ with $(x_{i,N}^t)' = 0$, $J_{x_{i,N}^t} \subseteq \{t_{1,N}, \dots, t_{N,N}\}$ and for $t \in J_{x_{i,N}^t}$ it holds (cf. (2.8))

$$|[x_{i,N}^t]| \leq \sum_{1 \leq j \leq N: t=t_{j,N}, B_j \subseteq B_{i,N}^t} r_{j,N}^t.$$

By a similar argument as for the functions $r_{i,N}^t$ we can then using Lemma 2.5 invoke the compactness result in SBV to obtain that $x_i^t \in \text{SBV}(0, t_i)$ and the claimed properties. \square

With this, we can prove the following:

Lemma 2.7. *Given a family of balls $B_i^t = B(x_i^t, r_i^t)$ satisfying (i), (iii), (iv) from Lemma 2.6 and a nonnegative function $f \in L^1(\mathbb{R}^d)$, we have*

$$\int_0^\infty \sum_{i \in I_t} r_i^t \int_{\partial B_i^t} f \, d\mathcal{H}^{d-1} \, dt \leq \int_{\mathbb{R}^d} f(x) \, dx. \quad (2.9)$$

In particular, the surface integral is well-defined for almost every $t \in [0, \infty)$.

Proof. We define for $N \in \mathbb{N}$ the function $g_N : [0, \infty) \rightarrow [0, \infty)$,

$$g_N(t) := \sum_{i \in I_t, i \leq N} \int_{B_i^t} f(x) \, dx.$$

Then by (i) every g_N is nondecreasing and therefore differentiable almost everywhere. Then we have by (iii)

$$\int_{\mathbb{R}^d} f(x) \, dx \geq \lim_{T \rightarrow \infty} g_N(T) \geq \int_0^\infty g'_N(t) \, dt.$$

If $g'_N(t)$ exists and $t \neq t_i$ for any $i \leq N$, then by (iv)

$$\begin{aligned} g'_N(t) &= \lim_{h \searrow 0} \sum_{i \in I_t, i \leq N} \frac{1}{h} \int_{B_i^{t+h} \setminus B_i^t} f(x) \, dx \geq \lim_{h \searrow 0} \sum_{i \in I_t, i \leq N} \frac{1}{h} \int_{B(x_i^t, e^h r_i^t) \setminus B_i^t} f(x) \, dx \\ &= \sum_{i \in I_t, i \leq N} r_i^t \int_{\partial B_i^t} f \, d\mathcal{H}^{d-1}. \end{aligned}$$

By Fatou's Lemma we obtain

$$\int_0^\infty \sum_{i \in I_t} r_i^t \int_{\partial B_i^t} f \, d\mathcal{H}^{d-1} \leq \liminf_{N \rightarrow \infty} \int_0^\infty \sum_{i \in I_t, i \leq N} r_i^t \int_{\partial B_i^t} f \, d\mathcal{H}^{d-1} \leq \liminf_{N \rightarrow \infty} \int_0^\infty g'_N(t) \, dt \leq \int_{\mathbb{R}^d} f(x) \, dx, \quad (2.10)$$

where for every $i \in \mathbb{N}$ we have

$$\int_{\partial B_i^t} f \, d\mathcal{H}^{d-1} = \lim_{h \searrow 0} \frac{1}{h} \int_{B(x_i^t, e^h r_i^t) \setminus B_i^t} f(x) \, dx$$

for almost every $t \in [0, t_i)$, which completes the proof. \square

Before we prove the main result of this section, we show that the traces of a SBV-function on most circles in the ball construction are in the Sobolev space $W^{1,p}(\partial B_i^t)$.

Lemma 2.8. *Let $\Omega \subseteq \mathbb{R}^2$ and $u \in \text{SBV}^p(\Omega)$. Let $B_i = B(x_i, r_i) \subseteq \Omega$ such that $J_u \subseteq \bigcup_i B(x_i, r_i)$ and $\sum_i r_i < \infty$. Let $(B_i^t)_i = (B(x_i^t, r_i^t))_i$ be the sequence of balls as constructed in Lemma 2.6. Then for every i the following is true for almost every $t \in [0, t_i)$: Let $Tu \in L^1(\partial B_i^t)$ be the outer trace of u on ∂B_i^t . Then it holds that $Tu \in W^{1,p}(\partial B_i^t)$, i.e. Tu has a weak tangential derivative on ∂B_i^t . In addition,*

$$r_i^t \int_{\partial B_i^t} |\partial_\tau Tu|^p \, dx \leq \lim_{h \searrow 0} \frac{C}{h} \int_{B(x_i^t, e^h r_i^t) \setminus B_i^t} |\nabla u|^p \, dx. \quad (2.11)$$

Proof. Define $\tilde{u} : \Omega \rightarrow \mathbb{R}$ by $\tilde{u} = u \mathbf{1}_{\Omega \setminus \bigcup_i B_i}$. Then $\tilde{u} \in \text{SBV}^p(\Omega)$ with

$$J_{\tilde{u}} \subseteq \bigcup_i \partial B_i \quad \text{and} \quad |\nabla \tilde{u}|^p \leq |\nabla u|^p \text{ almost everywhere.}$$

Next, note that since the function $t \mapsto r_i^t \in SBV([0, t_i])$ constructed in Lemma 2.6 is non-decreasing, it is differentiable at a.e. $t \in (0, t_i)$ with derivative $(r_i^t)' = r_i^t$ (cf. Lem. 2.6 (vi)). We choose such a $t \in [0, t_i)$, so that

$$r_i^t = \lim_{h \searrow 0} \frac{r_i^{t+h} - r_i^t}{h} = \lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} (r_i^s)' ds + \lim_{h \searrow 0} \frac{1}{h} \sum_{t < s < t+h, s \in J_{r_i^t}} |[r_i^s]|.$$

In particular, for every such $t \in [0, t_i)$

$$\lim_{h \searrow 0} \frac{1}{h} \sum_{t < s < t+h, s \in J_{r_i^t}} |[r_i^s]| = 0.$$

The fact that the radius grows as slowly as possible at t means that the radii of the balls that are absorbed between times t and $t+h$ are small. More precisely, if $(r_i^t)' = r_i^t$ and $(r_i^{t+h})' = r_i^{t+h}$, then ∂B_i^t and ∂B_i^{t+h} intersect no initial ball B_j and

$$\lim_{h \searrow 0} \frac{1}{h} \mathcal{H}^1(J_{\bar{u}} \cap B_i^{t+h} \setminus \overline{B_i^t}) \leq \lim_{h \searrow 0} \frac{1}{h} \sum_{j: B_j \subseteq (B_i^{t+h} \setminus \overline{B_i^t})} 2\pi r_j \leq \lim_{h \searrow 0} \frac{1}{h} \sum_{t < s < t+h, s \in J_{r_i^t}} 2\pi |[r_i^s]| = 0.$$

Since $B(x_i^t, e^h r_i^t) \setminus B_i^t \subseteq B_i^{t+h} \setminus B_i^t$ by construction (cf. Lem. 2.6 (iv)), it follows that

$$\lim_{h \searrow 0} \frac{1}{h} \mathcal{H}^1(J_{\bar{u}} \cap B(x_i^t, e^h r_i^t) \setminus B_i^t) = 0. \quad (2.12)$$

We now finalize our choice of t : We assume $(r_i^t)' = r_i^t$ and

$$\lim_{h \searrow 0} \frac{1}{h} \int_{B(x_i^t, e^h r_i^t) \setminus B_i^t} |\nabla \tilde{u}|^p dx < \infty, \quad (2.13)$$

which holds for almost every $t \in [0, t_i)$ by Lemma 2.7.

We conclude by invoking the BV trace theorem, see *e.g.* [1], Theorem 3.87. To this end, consider the rescaled functions $w_h \in SBV(B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)})$, $w_h(y) = \tilde{u}\left(x_i^t + e^{h(|y|-1)} \frac{y}{|y|}\right)$. By the trace theorem, $w_h(y)$ converges in $L^1(B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)})$ to the trace $T\tilde{u}(x_i^t + r_i^t \frac{y}{|y|})$.

On the other hand, by the chain rule and a change of variables (2.13) implies

$$\limsup_{h \searrow 0} \int_{B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)}} |\nabla w_h|^p dx \leq C \lim_{h \searrow 0} \frac{1}{h} \int_{B(x_i^t, e^h r_i^t) \setminus B_i^t} |\nabla \tilde{u}|^p dx$$

and (2.12) implies

$$\lim_{h \searrow 0} \mathcal{H}^1(J_{w_h}) = 0.$$

Since w_h is bounded in L^1 by the trace theorem, the SBV^p compactness theorem [1], Theorem 4.8 yields that the limit (which is the trace $w(y) = T\tilde{u}(x_i^t + r_i^t \frac{y}{|y|})$) is in $W^{1,p}(B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)})$, with the bound

$$r_i^t \int_{\partial B_i^t} |\partial_\tau T\tilde{u}|^p dx = C \int_{B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)}} |\nabla w|^p dx \leq \liminf_{h \searrow 0} C \int_{B(0, 2r_i^t) \setminus \overline{B(0, r_i^t)}} |\nabla w_h|^p dx,$$

which implies (2.11). □

We can now finally prove Theorem 2.2:

Proof of Theorem 2.2. Step 1: Extension and covering

We begin by noting that there exists an open bounded set $S' \subset \mathbb{R}^2$ with Lipschitz boundary such that $S \Subset S'$ and for any $u \in \text{GSBV}^p(S)$ there exists an extension $U \in \text{GSBV}^p(S')$ with $U = u$ in S and

$$\int_{S'} |\nabla U|^p dx \leq C(S) \int_S |\nabla u|^p dx, \quad \mathcal{H}^1(J_U) \leq C(S) \mathcal{H}^1(J_u).$$

The construction of the extension is the same as used for $W^{1,p}(S)$ in [17], Theorem 12.15.

We wish to cover the singular set

$$S_U := \{x \in S' : U \text{ is not approximately continuous at } x\},$$

where we recall that U is approximately continuous at x if for any $\varepsilon > 0$, we have

$$\lim_{r \searrow 0} \frac{|\{y \in B(x, r) : |u(y) - u(x)| > \varepsilon\}|}{r^2} = 0.$$

Recall that $\mathcal{H}^1(S_U \setminus J_U) = 0$ and thus $\mathcal{H}^1(S_U) = \mathcal{H}^1(J_U) \leq C(S) \mathcal{H}^1(J_u)$. By the definition of the Hausdorff measure, we can cover S_U with countably many balls $B_i = B(x_i, r_i)$, $i \in \mathbb{N}$ such that $\sum_{i \in \mathbb{N}} r_i \leq C(S) \mathcal{H}^1(J_u)$.

Step 2: Application of the countable ball construction

We apply Lemma 2.6 to this family, yielding B_i^t satisfying (i)–(vi). Let us now fix $R > 0$ and consider the truncated functions $U^R = \max\{\min\{U, R\}, -R\}$ and $u^R = \max\{\min\{u, R\}, -R\}$. Note that $U^R \in \text{SBV}^p(S')$ with $\int_{S'} |\nabla U^R|^p dx \leq \int_{S'} |\nabla U|^p dx \leq C(S) \int_S |\nabla u|^p dx$ and $S_{U^R} \subseteq S_U$. We then apply Lemma 2.7 to this family with the functions $|\nabla U^R|^p \mathbf{1}_{S'} \in L^1(\mathbb{R}^2)$. This yields

$$\int_0^\infty \sum_{i \in I_t} r_i^t \int_{\partial B_i^t \cap S'} |\nabla U^R|^p d\mathcal{H}^1 dt \leq C(S) \int_S |\nabla u|^p dx. \quad (2.14)$$

By Lemma 2.8 and the integral mean value theorem applied to (2.14), we find $t_0 \in (0, T)$ such that

$$\sum_{i \in I_{t_0}} r_i^{t_0} \int_{\partial B_i^{t_0} \cap S'} |\nabla U^R|^p d\mathcal{H}^1 \leq \frac{C(S)}{T} \int_S |\nabla u|^p dx \quad (2.15)$$

and for all $i \in I_{t_0}$ the trace of U^R is in $W^{1,p}(\partial B_i^{t_0})$ in the sense of Lemma 2.8 so that estimate (2.11) hold. Note that it can be seen from the proof of Lemma 2.8 that we may use the same t_0 for all R .

In addition, by Lemma 2.6 (ii) we have

$$\sum_{i \in I_{t_0}} r_i^{t_0} \leq C(S) e^T \mathcal{H}^1(J_u). \quad (2.16)$$

In particular, setting $\omega := \bigcup_{i \in I_{t_0}} B_i^{t_0}$ we find that $\text{Per}(\omega) \leq 2\pi C(S) e^T \mathcal{H}^1(J_u)$.

We further define $I := \{i \in I_{t_0} : B_i^{t_0} \cap S \neq \emptyset\}$. Because by (2.16) any $\text{diam}(B_i^{t_0}) = 2r_i^{t_0} \leq 2C(S) e^T \mathcal{H}^1(J_u) \leq \text{dist}(S, \partial S')$ if $e^T \mathcal{H}^1(J_u) \leq \frac{\text{dist}(S, \partial S')}{2C(S)} =: \eta(S)$, for all $i \in I$ we have $\partial B_i^{t_0} \subset S'$.

Step 3: Extension of the truncated function

Since for every $i \in I$, we have $B_i^{t_0} \subset S'$, we have access to the outer trace of U^R on every circle. Define $w^R := u^R = U^R$ in $S \setminus \bigcup_{i \in I} B_i^{t_0}$, and use the following extension into every $B_i^{t_0}$, $i \in I$:

Using polar coordinates to write any point $y \in B_i^{t_0}$ as $y = (1 - \theta)x_i^{t_0} + \theta z$ for $z \in \partial B_i^{t_0}$ and $\theta \in [0, 1)$ we define w^R in $B_i^{t_0}$ as

$$w^R((1 - \theta)x_i^{t_0} + \theta z) := (1 - \theta) \int_{\partial B_i^{t_0}} U^R d\mathcal{H}^1 + \theta U^R(z). \quad (2.17)$$

Then $\|w^R\|_{L^\infty} \leq R$ and $w^R = U^R$ in $S \setminus \omega$. Moreover, it follows by a straightforward computation that $w^R \in W^{1,p}(B_i^{t_0})$ and

$$\int_{B_i^{t_0}} |\nabla w^R|^p dx \leq C(p)r_i^{t_0} \int_{\partial B_i^{t_0}} |\nabla U^R|^p d\mathcal{H}^1, \quad (2.18)$$

where the surface integral is understood in the sense of Lemma 2.8. In particular, summing up over all $i \in I$ we obtain

$$\int_{\omega \cap S} |\nabla w^R|^p \leq \frac{C(p,S)}{T} \int_S |\nabla u|^p dx. \quad (2.19)$$

Finally we have to show that $w^R \in W^{1,p}(S)$. Clearly w is the pointwise almost everywhere limit of the functions

$$w_N^R(x) := \begin{cases} w^R(x) & x \in B_i^{t_0}, i \in I, i \leq N \\ u^R(x) & \text{otherwise.} \end{cases}$$

We see that it holds for the singular set of w_N^R that $S_{w_N^R} \subseteq S_u \cap \bigcup_{i \in I, i > N} B_i^{t_0}$. By (i) however,

$$\mathcal{H}^1(S_u \cap \bigcup_{i \in I, i > N} B_i^{t_0}) \leq \mathcal{H}^1(S_u \cap \bigcup_{i > N} B_i) \rightarrow_{N \rightarrow \infty} 0. \quad (2.20)$$

By the SBV compactness theorem, we have that $\mathcal{H}^1(S_{w^R}) = 0$ and thus $w^R \in W^{1,p}(S)$ with

$$\int_S |\nabla w^R|^p dx \leq \left(1 + \frac{C(p,S)}{T}\right) \int_S |\nabla u|^p dx.$$

Step 4: Reversal of truncation

Finally, we need to pass to the limit in $R \rightarrow \infty$. Choosing $\eta(S)$ smaller if necessary, we can ensure by the isoperimetric inequality that $\mathcal{L}^2(\omega) \leq C\eta(S)^2 \leq \frac{1}{3}\mathcal{L}^2(S)$. On the other hand, since U is measurable, there is $R_0 > 0$ such that $\mathcal{L}^2(\{x \in S : U(x) = U^{R_0}(x)\}) \geq \frac{2}{3}\mathcal{L}^2(S)$. For all $R > R_0$ we then have $\mathcal{L}^2(\{x \in S : w^R(x) = U^{R_0}(x) = U(x)\}) \geq \frac{1}{3}\mathcal{L}^2(S)$. By the Poincaré–Wirtinger inequality Proposition 2.9 below, it follows that

$$\int_S |w^R(x)|^p dx \leq C \left(\mathcal{L}^2(S)|R_0|^p + \int_S |\nabla w^R|^p dx \right),$$

and the right-hand side is bounded independently of R . Letting $R \rightarrow \infty$, we use weak compactness in $W^{1,p}(S)$ to pass to the limit $w \in W^{1,p}(S)$, completing the proof. \square

Proposition 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, connected with Lipschitz boundary and $p \in (1, \infty)$. Then for every $\alpha \in (0, \mathcal{L}^n(\Omega))$ there exists $C > 0$ such that for all $\Omega' \subseteq \Omega$ with $\mathcal{L}^n(\Omega') \geq \alpha$ and $u \in W^{1,p}(\Omega)$ it holds $\|u - \int_{\Omega'} u \, dx\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$.*

The more general Korn–Poincaré–Wirtinger version of this result can be found in [5], Lemma 4.4. Proposition 2.9 follows immediately from that lemma by considering the vector-valued function $w = ue_1 \in W^{1,p}(\Omega; \mathbb{R}^n)$.

3. THREE-DIMENSIONAL RESULT: PROOF OF THEOREM 1.2

Theorem 1.2 follows immediately from Theorem 3.1 below by setting $T = 1$.

Theorem 3.1. *Let $S \subseteq \mathbb{R}^2$ be open, bounded, connected, with Lipschitz boundary and $I \subseteq \mathbb{R}$ an open, bounded interval. Then there exists for $\Omega = I \times S$ a number $\eta(\Omega) > 0$ such that the following hold:*

For all $p \in [1, \infty)$ there exists a constant $C(p, \Omega) > 0$ such that for all $T > 0$ and all functions $u \in \text{GSBV}^p(\Omega)$ with $\int_{J_u} |\nu'| \, d\mathcal{H}^2 \leq e^{-T} \eta(\Omega)$ there exists a set $\omega \subseteq \mathbb{R}^3$ with

$$\mathcal{L}^3(\omega) \leq C(p, \Omega) e^T \int_{J_u} |\nu'| \, d\mathcal{H}^2 \quad (3.1)$$

and

$$\sup_{\varphi \in C_c^1(I \times S), |\varphi| \leq 1} \int_{\omega} \partial_j \varphi \, dx \leq C(p, \Omega) e^T \int_{J_u} |\nu'| \, d\mathcal{H}^2 \quad \text{for } j = 2, 3, \quad (3.2)$$

and a function $w \in L^p(\Omega)$ with $w = u$ on $\Omega \setminus \omega$, with distributional derivatives $\partial_2 w, \partial_3 w \in L^p(\Omega)$ and

$$\int_{\Omega} |\nabla' w|^p \, dx \leq \left(1 + \frac{C(p, S)}{T}\right) \int_{\Omega} |\nabla' u|^p \, dx. \quad (3.3)$$

Remark 3.2. We note that the estimates (3.1) and (3.2) are optimal in the sense that one cannot bound $\mathcal{L}^3(\omega)$ or $\sup_{\varphi \in C_c^1(I \times S), |\varphi| \leq 1} \int_{\omega} \partial_i \varphi \, dx$ by $\left(\int_{J_u} |\nu'| \, d\mathcal{H}^2\right)^\alpha$ for some $\alpha > 1$. Moreover, a bound on the full perimeter of ω instead of (3.2) cannot be expected. Indeed: Let $I = (-1, 1)$, $S = B(0, 1) \subseteq \mathbb{R}^2$ and $h > 0$.

(a) Consider the function $u \in \text{SBV}^p(I \times S)$ given by

$$u(x) = \begin{cases} 0 & \text{if } x \in (-h/2, h/2) \times B(0, 1/2) \\ 1 & \text{else.} \end{cases} \quad (3.4)$$

Then $\nabla u = 0$ and $\int_{J_u} |\nu'| \, d\mathcal{H}^2 \leq Ch$. It follows that any function w satisfying (1.6) is constant on each slice $\{x_1\} \times S$ for almost every x_1 . Hence, the exceptional set ω satisfies for almost every $x_1 \in (-h/2, h/2)$ that $\{x_1\} \times B(0, 1/2) \subseteq \omega$ or $\{x_1\} \times (B(0, 1) \setminus B(0, 1/2)) \subseteq \omega$. In particular, by Fubini's theorem $\mathcal{L}^3(\omega) \geq ch$. This shows that (3.1) is optimal. Moreover, by (3.1) it holds $\mathcal{L}^3(\omega) \leq Ch$. Hence, by Fubini's theorem it holds $\mathcal{L}^1(\{(x_2, x_3) \in S : \mathcal{L}^1(\{x_1 \in (-1, 1) : (x_1, x_2, x_3) \in \omega\}) = 2\}) \leq Ch$. This implies together with the observation above that for $h > 0$ small enough it holds $\mathcal{L}^2(\{(x_2, x_3) \in S : x_1 \rightarrow \chi_\omega(x_1, x_2, x_3) \text{ is not constant a.e.}\}) \geq c > 0$, where $c > 0$ does not depend on h . Hence, if ω is a set of finite perimeter then it holds by slicing that $\mathcal{H}^2(\partial^* \omega) \geq c$ but $\int_{J_u} |\nu'| \, d\mathcal{H}^2 \leq Ch$. Sending $h \rightarrow 0$ shows that in (3.2) the left hand side cannot be replaced by the full perimeter of ω .

(b) Now, consider the function $u \in \text{SBV}^p(I \times S)$ defined by

$$\begin{cases} 0 & \text{if } x \in (-1, 1) \times B(0, h) \\ 1 & \text{else.} \end{cases}$$

Again, $\nabla u = 0$, $\int_{J_u} |\nu'| \, d\mathcal{H}^2 \leq Ch$ and any w satisfying (1.6) is constant on almost every slice. If the exceptional set ω satisfies (3.1) then we have for small h necessarily that $w = 1$. Hence, we have on almost every slice $\{x_1\} \times B(0, h) \subseteq \omega$. On the other hand, on at least half of the slices we have $\mathcal{L}^2(\{x' \in S : (x_1, x') \in \omega\}) \leq Ch$. This implies that $\max_{i=2,3} \sup_{\varphi \in C_c^0(I \times S), |\varphi| \leq 1} \int_{\omega} \partial_i \varphi \, dx \geq ch$, which shows that (3.2) is optimal.

Remark 3.3. Note that the 1 + 1-dimensional version of Theorem 1.2 is much simpler. For $\Omega \subseteq \mathbb{R}^2$ and $u \in \text{SBV}^p(\Omega)$ we define $\omega := \{(x_1, x_2) \in \Omega : (\{x_1\} \times \mathbb{R}) \cap J_u \neq \emptyset\}$. Then u is absolutely continuous in the x_2 direction on $\Omega \setminus \omega$. Moreover, $\mathcal{L}^2(\omega) \leq \int_{J_u} |\nu_2| \, d\mathcal{H}^1$ and $\sup_{\varphi \in C_c^0(\Omega), |\varphi| \leq 1} \int_{\omega} \partial_2 \varphi \, dx = 0$.

Proof of Theorem 3.1. Step 1: Slicing and definitions of ω and w .

Ideally, we would use Theorem 2.2 in almost every slice $\{x_1\} \times S$ and thus obtain the pair (ω, w) . Unfortunately, the measurable selection theorem does not guarantee measurability in (x_1, x') . Instead, we have to look into the proof of Theorem 2.2 and pick a global stopping time $t_0 \in (0, T)$, which does not depend on x_1 , in order to ensure measurability in both variables.

We claim that a careful modification of the proof of Theorem 2.2 yields a slightly reduced Borel subset $I_{\text{ad}} \subseteq I$ satisfying first

$$\mathcal{L}^1(I \setminus I_{\text{ad}}) \leq \frac{e^T}{\eta(\Omega)} \int_{J_u} |\nu'| \, d\mathcal{H}^2 \quad (3.5)$$

and a Lebesgue measurable pair $\omega \subset I_{\text{ad}} \times \mathbb{R}^2$ bounded, $w : I_{\text{ad}} \times S \rightarrow \mathbb{R}$ such that second

$$\text{Per}(\omega_{x_1}) \leq C(\Omega) e^T \mathcal{H}^1(J_{u_{x_1}}) \leq \eta(\Omega), \text{ for almost every } x_1 \in I_{\text{ad}}, \quad (3.6)$$

where $\eta(\Omega) = \eta(S)$ from the proof of Theorem 2.2. Third, we claim that $w_{x_1} = w(x_1, \cdot)$ is in $W^{1,p}(S)$ for almost every $x_1 \in I_{\text{ad}}$ with

$$\int_{I_{\text{ad}}} \int_S |\nabla' w_{x_1}(x')|^p \, dx' \, dx_1 \leq \left(1 + \frac{C(p, \Omega)}{T}\right) \int_{I \times S} |\nabla' u(x)|^p \, dx. \quad (3.7)$$

By extending ω as $\omega = (I \setminus I_{\text{ad}} \times S) \cup \omega$ and extending $w : I \times S \rightarrow \mathbb{R}$ by zero, we claim third that $w = u$ on $(I \times S) \setminus \omega$.

The construction of the measurable pair (ω, w) satisfying (3.6) and (3.7) is postponed to Steps 4 and 5.

Step 2: Quantitative estimates on ω .

We show (3.2): Pick any test function $\varphi \in C_c^1(I \times S)$ with $|\varphi| \leq 1$ everywhere. Then by Fubini's theorem we have for $j = 2, 3$

$$\int_{\omega} \partial_j \varphi \, dx = \int_{\omega \cap (I_{\text{ad}} \times S)} \partial_j \varphi \, dx \leq \int_{I_{\text{ad}}} C(\Omega) e^T \mathcal{H}^1(J_{u_{x_1}}) \, dx_1 \leq C(\Omega) e^T \int_{J_u} |\nu'| \, d\mathcal{H}^2,$$

where we used (3.6).

To show (3.1), first estimate

$$\mathcal{L}^3(\omega \setminus (I_{\text{ad}} \times \mathbb{R}^2)) = \mathcal{L}^1(I \setminus I_{\text{ad}}) \mathcal{L}^2(S) \leq C(\Omega) e^T \int_{J_u} |\nu'| \, d\mathcal{H}^2.$$

On I_{ad} , use the isoperimetric inequality to obtain

$$\mathcal{L}^3(\omega \cap (I_{\text{ad}} \times \mathbb{R}^2)) = \int_{I_{\text{ad}}} \mathcal{L}^2(\omega_{x_1}) \, dx_1 \leq \int_{I_{\text{ad}}} C \text{Per}(\omega_{x_1})^2 \, dx_1 \leq C(\Omega) e^T \int_{J_u} |\nu'| \, d\mathcal{H}^2.$$

Step 3: Quantitative estimates on w .

We show that $w \in L^p(\Omega)$. (3.3) will then follow immediately from (3.7) by Fubini's theorem.

Define for $M \in \mathbb{N}$ the measurable set $S_M = \{x' \in S : \|u(\cdot, x')\|_{L^\infty(I)} \leq M\}$. We know by slicing that $u(\cdot, x') \in \text{GSBV}^p(I)$ for almost every $x' \in S$, and since for a bounded interval $\text{GSBV}^p(I) \subseteq L^\infty(I)$, we have $S = \bigcup_{M \in \mathbb{N}} S_M$. We choose $M_0 \in \mathbb{N}$ large enough that $\mathcal{L}^2(S_{M_0}) \geq \frac{2}{3} \mathcal{L}^2(S)$. We now specify our choice of $\eta(\Omega)$. In particular, we demand by the isoperimetric inequality that $\mathcal{L}^2(\omega_{x_1}) \leq C(S) \text{Per}(\omega_{x_1})^2 \leq \frac{1}{3} \mathcal{L}^2(S)$.

In particular, for almost every $x_1 \in I_{\text{ad}}$, we have $\mathcal{L}^2(S_{M_0} \setminus \omega_{x_1}) \geq \frac{1}{3} \mathcal{L}^2(S)$. Almost everywhere on $S_{M_0} \setminus \omega_{x_1}$ we have $|w(x_1, x')| = |u(x_1, x')| \leq M_0$. The Poincaré-Wirtinger inequality in Proposition 2.9 then yields

$$\int_{I_{\text{ad}}} \int_S |w(x_1, x')|^p \, dx' \, dx_1 \leq C(p) \int_{I_{\text{ad}}} \left(M_0^p + C(p, S) \int_S |\nabla' w_{x_1}(x')|^p \, dx' \right) \, dx_1 < \infty,$$

showing that $w \in L^p(\Omega)$.

Step 4: Covering of the jump set with cylinders

We mirror the proof of Theorem 2.2, with an additional x_1 variable. First, u is extended to $I \times S'$, yielding $U \in \text{GSBV}^p(I \times S')$. We cover the singular set S_U with countably many cylinders $C_i = [a_i, b_i] \times B_i \subset I \times \mathbb{R}^2$, $i \in \mathbb{N}$, such that

$$\sum_{i \in \mathbb{N}} (b_i - a_i) r_i \leq C(S) \int_{J_u} |\nu'| \, d\mathcal{H}^2. \quad (3.8)$$

Note that we only estimate the x_2 - x_3 component of the surface of the cylinders. In order to find the cylinders, first consider the case of no x_2 - x_3 jumps. If $\int_{J_u} |\nu'| \, d\mathcal{H}^2 = 0$, simply take $\omega = \emptyset$, $w = u$. Otherwise, set $\delta = \frac{1}{\mathcal{H}^2(J_U)} \int_{J_U} |\nu'| \, d\mathcal{H}^2 > 0$. Utilize the change of variables $\phi_\delta : I \times S' \rightarrow I \times \delta S'$, $\phi(x_1, x') = (x_1, \delta x')$. Define $U_\delta = U \circ \phi_\delta^{-1} \in \text{GSBV}^p(I \times \delta S')$. Then the change-of-variables formula yields

$$\begin{aligned} \mathcal{H}^2(S_{U_\delta}) &= \mathcal{H}^2(\phi_\delta(J_U)) = \int_{J_U} |\text{cof } D\phi_\delta \nu| \, d\mathcal{H}^2 &= \int_{J_U} |(\delta^2 \nu_1, \delta \nu')| \, d\mathcal{H}^2 \\ &\leq \delta^2 \mathcal{H}^2(J_U) + \delta \int_{J_U} |\nu'| \, d\mathcal{H}^2 &= 2\delta \int_{J_U} |\nu'| \, d\mathcal{H}^2. \end{aligned} \quad (3.9)$$

By the definition of the Hausdorff measure, we can cover the singular set S_{U_δ} with cylinders (which are balls of a specific norm on \mathbb{R}^3) $\tilde{C}_i = [a_i, b_i] \times \tilde{B}_i \subset I \times \mathbb{R}^2$ such that $\sum_{i \in \mathbb{N}} (b_i - a_i) \tilde{r}_i \leq C\delta \int_{J_U} |\nu'| \, d\mathcal{H}^2$. Rescaling the cylinders by taking $C_i = \phi_\delta^{-1}(\tilde{C}_i)$ yields a cover of S_U satisfying (3.8).

Step 5: Measurable simultaneous ball construction in almost every slice $\{x_1\} \times \mathbb{R}^2$.

Define the reduced set $I_{\text{ad}} \subseteq I$ by

$$I_{\text{ad}} = \{x_1 \in I : \sum_{i: x_1 \in [a_i, b_i]} r_i \leq e^{-T} \eta(\Omega)\}.$$

By Markov's inequality

$$\mathcal{L}^1(I \setminus I_{\text{ad}}) \leq \frac{e^T}{\eta(\Omega)} \sum_{i \in \mathbb{N}} (b_i - a_i) r_i \leq C(\Omega) e^T \int_{J_u} |\nu'| d\mathcal{H}^2,$$

which shows (3.5).

Perform the countable ball construction from Lemma 2.6 for every $x_1 \in I_{\text{ad}}$ simultaneously, yielding three Borel measurable functions: The extinction time $t_i : I_{\text{ad}} \rightarrow [0, \infty)$, the radius $r_i : \{(x_1, t) \in I_{\text{ad}} \times [0, \infty) : t \in [0, t_i(x_1))\} \rightarrow [0, \infty)$ and the center $x'_i : \{(x_1, t) \in I_{\text{ad}} \times [0, \infty) : t \in [0, t_i(x_1))\} \rightarrow \mathbb{R}^2$.

A careful observation yields that all three functions are Borel measurable, first for finitely many balls, and in the passage to countably many through a monotone limit and taking the right-continuous version of a monotone function.

We specify our choice of the stopping time $t_0 \in (0, T)$. We demand the following two properties:

$$\begin{cases} \int_{I_{\text{ad}}} \sum_{i: t_0 < t_i(x_1)} r_i(t_0, x_1) \int_{\{x_1\} \times \partial B_i(t_0, x_1)} |\nabla U|^p d\mathcal{H}^1 dx_1 \leq \frac{1}{T} \int_{I_{\text{ad}} \times S'} |\nabla U|^p dx, \\ \partial_t r_i(x_1, t_0) = r_i(x_1, t_0), \text{ for almost every } x_1 \in I_{\text{ad}} \text{ with } t_0 < t_i(x_1). \end{cases} \quad (3.10)$$

The first property holds by Markov's inequality for a subset of $(0, T)$ of positive measure by Lemma 2.7. The second property holds for almost every $t \in (0, T)$ by the following argument: Consider the measurable set $N_i = \{(t, x_1) : x_1 \in I_{\text{ad}}, t < t_i(x_1), \limsup_{h \searrow 0} \frac{r_i(t+h, x_1) - r_i(t, x_1)}{h} > r_i(t, x_1)\}$ and note that on the complement of N_i intersected with $I_{\text{ad}} \times S'$ it holds $\partial_t r_i(x_1, t) = r_i(x_1, t)$. By Fubini's theorem and property (vi) from Lemma 2.6, we have

$$\mathcal{L}^2(N_i) = \int_{I_{\text{ad}}} \mathcal{L}^1(N_i^{x_1}) dx_1 = 0,$$

so that a second application of Fubini's theorem implies

$$\int_{(0, T)} \mathcal{L}^1(N_i^t) dt = 0,$$

i.e. that $\mathcal{L}^1(N_i^t) = 0$ for almost every $t \in (0, T)$.

We can thus pick $t_0 \in (0, T)$ satisfying (3.10). The resulting pair is

$$\omega = \bigcup_{i \in \mathbb{N}} \{(x_1, y') : t_0 < t_i(x_1), |y' - x'_i(t_0, x_1)| < r_i(t_0, x_1)\},$$

which is a Borel set by construction, and w^R as defined in (3.11) in every slice, which becomes on ω

$$w^R(x_1, x') := (1 - \theta_i(x_1, x')) \int_{\partial\{x_1\} \times B_i(t_0, x_1)} U^R d\mathcal{H}^1 + \theta_i(x_1, x') U^R(x_1, z'_i(x_1, x')), \quad (3.11)$$

where $U^R = \max\{\min\{u, R\}, -R\}$ as in the proof of Theorem 2.2. Note that by modifying U on a null set, we may assume that U is in fact Borel measurable. As a composition of Borel measurable functions, w^R is Borel measurable. Again, as in the proof of Theorem 2.2 we may take the limit as $R \rightarrow \infty$ in almost every slice. Again, the limit satisfies (3.7) by Lemma 2.8. \square

4. APPLICATION TO AN ANISOTROPIC MUMFORD–SHAH FUNCTIONAL

We now provide an application of Theorem 1.2 to the following Mumford–Shah type functionals $F_\varepsilon : \text{GSBV}^p(I \times S) \rightarrow \mathbb{R}$,

$$F_\varepsilon(u) := \int_{I \times S} \frac{1}{\varepsilon} |\nabla' u|^p + |\partial_1 u|^p \, dx + \int_{J_u} \frac{1}{\varepsilon} |\nu'| + |\nu_1| \, d\mathcal{H}^2, \quad (4.1)$$

where $\nu \in S^2$ is the measure theoretic normal to J_u and $S \subseteq \mathbb{R}^2$ is open, bounded, connected and with Lipschitz boundary.

Here u can be interpreted as a grayscale movie on a film of cross section $S \subset \mathbb{R}^2$, with $x_1 \in I$, interpreted as time, and $I \subseteq \mathbb{R}$ is a bounded interval. In this setting, it appears natural to penalize changes in time and changes in space differently.

We show the following result concerning the asymptotics of F_ε as $\varepsilon \rightarrow 0$:

Theorem 4.1. *As $\varepsilon \rightarrow 0$ the functionals F_ε Γ -converge compactly under convergence in measure to the functional $F_0 : L^1(I \times S) \rightarrow [0, \infty]$,*

$$F_0(u) = \begin{cases} \mathcal{L}^2(S) \left(\int_I |u'|^p \, dx_1 + \#J_u \right) & \text{if } u(x) = u(x_1) \in \text{SBV}^p(I) \\ \infty & \text{otherwise.} \end{cases} \quad (4.2)$$

More precisely:

- (i) *Let $u_\varepsilon \in \text{GSBV}^p(I \times S)$ with $\sup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) < \infty$. Then there exist functions $w_\varepsilon \in \text{SBV}_{\text{loc}}^p(I)$ such that $u_\varepsilon - w_\varepsilon \rightarrow 0$ in measure, with*

$$\limsup_{\varepsilon \rightarrow 0} \#J_{w_\varepsilon} \leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\mathcal{L}^2(S)} \int_{J_{u_\varepsilon}} |\nu_1| \, d\mathcal{H}^2 \quad (4.3)$$

and

$$\liminf_{\varepsilon \rightarrow 0} F_0(w_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (4.4)$$

In particular, there exists a sequence of piecewise constant functions $c_\varepsilon \in \text{SBV}_{\text{loc}}^p(I)$, functions $w \in \text{SBV}_{\text{loc}}^p(I)$ and a (not relabelled) subsequence such that $u_\varepsilon - c_\varepsilon \rightarrow w$ in measure and

$$F_0(w) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(w_\varepsilon - c_\varepsilon) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon).$$

- (ii) *For any $u \in \text{SBV}^p(I)$ and any sequence $u_\varepsilon \in \text{SBV}^p(I \times S)$ such that $u_\varepsilon(x) \rightarrow u(x_1)$ in measure we have $F_0(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$.*
 (iii) *For any $u \in \text{SBV}^p(I)$ we have $F_\varepsilon(u) = F_0(u)$.*

Note that a function $u \in \text{GSBV}^p(I \times S)$ with $\nabla' u = 0$ and $\int_{J_u} |\nu'| \, d\mathcal{H}^2 = 0$ satisfies $D_2 u = D_3 u = 0$ in a distributional sense. Since S is connected this implies that u agrees almost everywhere with a function that is a

function of x_1 only. Indeed, for a smooth function this is true. The general case can be shown by truncation and mollification. Together with the lower-semicontinuity of all F_ε this guarantees (ii). Indeed, it holds for all $M > 0$ and $\varepsilon_0 > 0$ that the sequence $v_\varepsilon = \max\{\min\{u_\varepsilon, M\}u_\varepsilon, -M\}$ converges in L^1 to $v = \max\{\min\{u, M\}, -M\}$ and that

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} F_{\varepsilon_0}(v_\varepsilon) \geq F_{\varepsilon_0}(v).$$

Sending $M \rightarrow \infty$ yields $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F_{\varepsilon_0}(u)$. Then (ii) follows by sending $\varepsilon_0 \rightarrow 0$. Moreover, (iii) follows immediately.

Hence, the only difficulty lies in the compactness statement (i), which does not yet exist in the literature. Indeed, the related dimension reduction results for brittle materials in [18–21] all assume either a uniform L^∞ -bound on the sequence $(u_\varepsilon)_\varepsilon$ or use the compactness result from [22] to obtain compactness for u_ε outside a set on which $|u_\varepsilon| \rightarrow \infty$. In the latter case, no further information is kept on this set. In particular, the energy created by ∇u on this set is ignored. The compactness result presented here is more in the spirit of the compactness results in [4], Theorem 3.1 and [23], Theorem 4.1. It identifies the different sets on which u_ε needs to be renormalized in order to obtain compactness. The corresponding limit function w of the renormalized functions $u_\varepsilon - c_\varepsilon$ inherits the information about the behavior of ∇u_ε everywhere. To show (i), we will have to quantify exactly where a function with bounded F_ε differs from a function of only x_1 .

For the proof, the following lemma states that we may remove jumps from a SBV^p function on an interval if it is close (outside of a small set A) to a different SBV^p function with fewer jumps.

Lemma 4.2. *Let $I \subseteq \mathbb{R}$ be an interval, $p \in (1, \infty)$. Let $u, v \in \text{SBV}^p(I)$ with $\#J_u = N \leq M = \#J_v$. Let $\delta > 0$ and $A \subseteq I$ be a Borel set with $\mathcal{L}^1(A) \leq \delta$. Then there is a function $w \in \text{SBV}^p(I)$ and a set \tilde{A} with $\#J_w \leq N$,*

$$\int_I |w'|^p dx \leq \int_I |v'|^p dx, \quad (4.5)$$

$$\begin{aligned} \|w - v\|_{L^\infty(I \setminus \tilde{A})} \leq C(M) & \left(\|u - v\|_{L^p(I \setminus A)}^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p})^{1/p} \right. \\ & \left. + \mathcal{L}^1(A)^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p}) \right), \end{aligned} \quad (4.6)$$

and

$$\mathcal{L}^1(\tilde{A}) \leq C(M) \left(\mathcal{L}^1(A) + \frac{\|u - v\|_{L^p(I \setminus A)}}{\|u'\|_{L^p} + \|v'\|_{L^p}} \right). \quad (4.7)$$

Remark 4.3. Note that if u and v are piecewise constant but $u \neq v$ then $\tilde{A} = I$ is consistent with the estimates of the lemma.

Proof. Assume first that $N = 0$, by restricting to a smaller interval if necessary. Then $u \in W_{\text{loc}}^{1,p}(I)$.

Let $J_v = \{t_1, \dots, t_M\}$ with $t_i \leq t_{i+1}$. Define $t_0 := \inf I$, $t_{M+1} := \sup I$. Define the subintervals $I_i := (t_i, t_{i+1})$.

Let $l \geq 2\mathcal{L}^1(A)$. Let \tilde{A} be the union of all subintervals of length less than l . Then $\mathcal{L}^1(\tilde{A}) \leq (M+1)l$. On all other subintervals I_i we also have $v \in W_{\text{loc}}^{1,p}(I_i)$.

If $x \in I_i$ and $\mathcal{L}^1(I_i) \geq l$, there exists a point $y \in B(x, l) \cap I_i \setminus A$ such that

$$|u(y) - v(y)|^p \leq \frac{2}{l} \int_{I_i \setminus A} |u - v|^p dz. \quad (4.8)$$

Also by the Sobolev embedding theorem both u and v are in $C^{0,1/p'}(I_i)$ and

$$|u(x) - u(y)|^p + |v(x) - v(y)|^p \leq l^{p/p'} \int_{I_i} |u'|^p + |v'|^p \, dz. \quad (4.9)$$

Combining (4.8) and (4.9) yields

$$\|u - v\|_{L^\infty(I \setminus \tilde{A})} \lesssim l^{-1/p} \|u - v\|_{L^p(I \setminus A)} + l^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p}). \quad (4.10)$$

Optimizing l over $[2\mathcal{L}^1(A), \infty)$ yields

$$l := \max \left\{ \frac{\|u - v\|_{L^p(I \setminus A)}}{\|u'\|_{L^p} + \|v'\|_{L^p}}, 2\mathcal{L}^1(A) \right\}, \quad (4.11)$$

whereupon we obtain (4.7) and

$$\|u - v\|_{L^\infty(I \setminus \tilde{A})} \lesssim \max \left\{ \|u - v\|_{L^p(I \setminus A)}^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p})^{1/p}, \mathcal{L}^1(A)^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p}) \right\}. \quad (4.12)$$

We now define the function w in $I \setminus \tilde{A}$ as follows: Pick a point $x_0 \in I \setminus \tilde{A}$ and define

$$w(x) := v(x_0) + \int_{x_0}^x v'(z) \, dz. \quad (4.13)$$

It is clear that $w \in W_{\text{loc}}^{1,p}(I)$ with $\|w'\|_{L^p} = \|v'\|_{L^p}$. To see that w is close to v in $L^\infty(I \setminus \tilde{A})$, we bound the jumps of v between intervals. It is clear that on the interval $I_{i_0} \subset I \setminus \tilde{A}$ containing x_0 , we have $w = v$ on I_{i_0} . Let $I_i = (t_i, t_{i+1})$ and $I_{i+k} = (t_{i+k}, t_{i+k+1})$ be two consecutive intervals contained in $I \setminus \tilde{A}$. Since $1 \leq k \leq M$ and each interval I_{i+j} , $1 \leq j \leq k-1$, has length at most l , we have $|t_{i+k} - t_{i+1}| \leq Ml$. By the triangle inequality and the Hölder-continuity of u we have

$$|v(t_{i+k}) - v(t_{i+1})| \leq 2\|u - v\|_{L^\infty(I \setminus \tilde{A})} + \|u'\|_{L^p}(Ml)^{1/p'}. \quad (4.14)$$

As $I \setminus \tilde{A}$ consists of at most M intervals, we obtain

$$\|w - v\|_{L^\infty(I \setminus \tilde{A})} \leq C(M)\|u - v\|_{L^\infty(I \setminus \tilde{A})} + C(M)\|u'\|_{L^p} l^{1/p'}. \quad (4.15)$$

Then (4.6) follows from (4.12) and the definition of l .

Let us now briefly discuss the case $N > 0$. Let $J_u = \{s_1, \dots, s_N\}$ with $s_k < s_{k+1}$ and set $s_0 = \inf I$ and $s_{N+1} = \sup I$. Then argue on the intervals (s_k, s_{k+1}) for $k = 0, \dots, N$ as before up to inequality (4.10) to obtain for $l \geq 2\mathcal{L}^1(A)$ the analogue of (4.10) for the sets $\tilde{A}_k \subseteq (s_k, s_{k+1})$ as above

$$\|u - v\|_{L^\infty((s_k, s_{k+1}) \setminus \tilde{A}_k)} \lesssim l^{-1/p} \|u - v\|_{L^p((s_k, s_{k+1}) \setminus A)} + l^{1/p'} (\|u'\|_{L^p(s_k, s_{k+1})} + \|v'\|_{L^p(s_k, s_{k+1})}).$$

We set $\tilde{A} = \bigcup_{k=0}^N \tilde{A}_k$. Summation over k on the right hand side yields

$$\|u - v\|_{L^\infty(I \setminus \tilde{A})} \lesssim l^{-1/p} \|u - v\|_{L^p(I \setminus A)} + l^{1/p'} (\|u'\|_{L^p(I)} + \|v'\|_{L^p(I)}) \quad (4.16)$$

Optimizing over $l \in [2\mathcal{L}^1(A), \infty)$ yields again

$$\|u - v\|_{L^\infty(I \setminus \tilde{A})} \lesssim \max \left\{ \|u - v\|_{L^p(I \setminus A)}^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p})^{1/p}, \mathcal{L}^1(A)^{1/p'} (\|u'\|_{L^p} + \|v'\|_{L^p}) \right\} \quad (4.17)$$

and

$$\mathcal{L}^1(\tilde{A}) \leq (M + N + 1)l \leq (2M + 1) \max \left\{ \frac{\|u - v\|_{L^p(I \setminus A)}}{\|u'\|_{L^p} + \|v'\|_{L^p}}, 2\mathcal{L}^1(A) \right\}.$$

Then one constructs w as before in every interval (s_k, s_{k+1}) separately. Hence, $\#J_w \leq N$. The remaining estimates for w follow as before. \square

We now turn to the proof of Theorem 4.1 (i). Note that it is impossible to replace the lim sup by a lim inf, since different subsequences may minimize different parts of the energy.

Proof of Theorem 4.1 (i). For this proof, we wish to apply Lemma 4.2 to u_ε restricted to two slices $x' = \text{const} \in S$.

Before we continue, let

$$M := \sup_{\varepsilon > 0} \frac{1}{\mathcal{L}^2(S)} F_\varepsilon(u_\varepsilon), \quad (4.18)$$

$$N := \left[\limsup_{\varepsilon \rightarrow 0} \frac{1}{\mathcal{L}^2(S)} \int_{J_{u_\varepsilon}} |\nu_1| \, d\mathcal{H}^2 \right]. \quad (4.19)$$

Extract a subsequence $\varepsilon \rightarrow 0$ (not relabeled) such that

$$\lim_{\varepsilon \rightarrow 0} \int_{I \times S} |\partial_1 u_\varepsilon|^p \, dx = \liminf_{\varepsilon \rightarrow 0} \int_{I \times S} |\partial_1 u_\varepsilon|^p \, dx. \quad (4.20)$$

Then apply Proposition 1.1 to each function u_ε , yielding sets $\omega_\varepsilon \subset I \times S$ and measurable functions $a_\varepsilon : I \rightarrow \mathbb{R}$ such that

$$\mathcal{L}^3(\omega_\varepsilon) \leq C(p, S, I)M\varepsilon \quad (4.21)$$

and

$$\int_{(I \times S) \setminus \omega_\varepsilon} |u_\varepsilon - a_\varepsilon|^p \, dx \leq C(p, S, I)M\varepsilon. \quad (4.22)$$

We do not use the boundary estimate (1.5) from Theorem 1.2.

Next, we define for $\lambda \geq 1$ four subsets of S :

$$A_\varepsilon^\lambda := \left\{ x' \in S : \int_I |\partial_1 u_\varepsilon(x_1, x')|^p \, dx_1 \leq \frac{\lambda}{\mathcal{L}^2(S)} \int_{I \times S} |\partial_1 u_\varepsilon(x)|^p \, dx \right\}, \quad (4.23)$$

$$B_\varepsilon^\lambda := \left\{ x' \in S : u_\varepsilon^{x'} = u_\varepsilon(\cdot, x') \in \text{SBV}^p(I) \text{ and } \#(J_{u_\varepsilon^{x'}}) \leq \frac{\lambda}{\mathcal{L}^2(S)} \int_{J_{u_\varepsilon}} |\nu_1| \, d\mathcal{H}^2 \right\}, \quad (4.24)$$

$$C_\varepsilon^\lambda := \left\{ x' \in S : \int_{I \setminus \omega_\varepsilon^{x'}} |u_\varepsilon(x_1, x') - a_\varepsilon(x_1)|^p dx_1 \leq \frac{\lambda}{\mathcal{L}^2(S)} \int_{(I \times S) \setminus \omega_\varepsilon} |u_\varepsilon(x) - a_\varepsilon(x_1)|^p dx \right\}, \quad (4.25)$$

$$D_\varepsilon^\lambda := \left\{ x' \in S : \mathcal{L}^1(\omega_\varepsilon^{x'}) \leq \frac{\lambda}{\mathcal{L}^2(S)} \mathcal{L}^3(\omega_\varepsilon) \right\}, \quad (4.26)$$

where, for $x' \in S$, $\omega_\varepsilon^{x'} \subset I$ is the slice

$$\omega_\varepsilon^{x'} := \{x_1 \in I : (x_1, x') \in \omega_\varepsilon\}. \quad (4.27)$$

By Markov's inequality and standard slicing properties of BV-functions we have

$$\min \{ \mathcal{L}^2(A_\varepsilon^\lambda), \mathcal{L}^2(B_\varepsilon^\lambda), \mathcal{L}^2(C_\varepsilon^\lambda), \mathcal{L}^2(D_\varepsilon^\lambda) \} \geq \frac{\lambda - 1}{\lambda} \mathcal{L}^2(S). \quad (4.28)$$

Thus for every $\delta \in (0, 1)$, we have

$$\mathcal{L}^2(A_\varepsilon^{1+\delta} \cap B_\varepsilon^{8/\delta} \cap C_\varepsilon^{8/\delta} \cap D_\varepsilon^{8/\delta}) \geq \frac{\delta}{8} \mathcal{L}^2(S). \quad (4.29)$$

Consequently, we may find $x' \in A_\varepsilon^{1+\delta} \cap B_\varepsilon^{8/\delta} \cap C_\varepsilon^{8/\delta} \cap D_\varepsilon^{8/\delta}$ and define $b_\varepsilon(x_1) := u_\varepsilon(x_1, x') \in \text{SBV}^p(I)$, noting that from the properties of the four sets we have

$$\begin{aligned} \int_I |b'_\varepsilon(x_1)|^p dx_1 &\leq \frac{1+\delta}{\mathcal{L}^2(S)} \int_{I \times S} |\partial_1 u_\varepsilon(x)|^p dx, \\ \#J_{b_\varepsilon} &\leq \frac{8M}{\delta}, \\ \int_{I \setminus \omega_\varepsilon^{x'}} |b_\varepsilon(x_1) - a_\varepsilon(x_1)|^p dx_1 &\leq \frac{C(p, S, I)M}{\delta} \varepsilon, \\ \mathcal{L}^1(\omega_\varepsilon^{x'}) &\leq \frac{C(p, S, I)M}{\delta} \varepsilon. \end{aligned} \quad (4.30)$$

Similarly, we can find $y' \in A_\varepsilon^{8/\delta} \cap B_\varepsilon^{1+\delta} \cap C_\varepsilon^{8/\delta} \cap D_\varepsilon^{8/\delta}$ and define $d_\varepsilon(x_1) := u_\varepsilon(x_1, y') \in \text{SBV}_{\text{loc}}^p(I)$, such that

$$\begin{aligned} \int_I |d'_\varepsilon(x_1)|^p dx_1 &\leq \frac{C(M, S)}{\delta}, \\ \#J_{d_\varepsilon} &\leq \frac{1+\delta}{\mathcal{L}^2(S)} \int_{J_{u_\varepsilon}} |\nu_1| d\mathcal{H}^2, \\ \int_{I \setminus \omega_\varepsilon^{y'}} |d_\varepsilon(x_1) - a_\varepsilon(x_1)|^p dx_1 &\leq \frac{C(p, S, I)M}{\delta} \varepsilon, \\ \mathcal{L}^1(\omega_\varepsilon^{y'}) &\leq \frac{C(p, S, I)M}{\delta} \varepsilon. \end{aligned} \quad (4.31)$$

We note that by our choice of subsequence we have for $\delta \leq \delta_0(N, S)$ and $\varepsilon \leq \varepsilon_0(N, S)$ that $\#J_{d_\varepsilon} \leq N$ uniformly since $\#J_{d_\varepsilon}$ is an integer that is strictly less than $N + 1$. If $\#J_{b_\varepsilon} \leq \#J_{d_\varepsilon}$ then set $w_\varepsilon = b_\varepsilon$. It follows immediately that $F_\varepsilon(w_\varepsilon) \leq (1+\delta)F_\varepsilon(u_\varepsilon)$ and by the estimates (4.21), (4.22) and the definition of x' $w_\varepsilon - u_\varepsilon \rightarrow 0$ in measure. In addition, we may assume that $\liminf_{\varepsilon \rightarrow 0} \int_I |d'_\varepsilon|^p dx_1 > 0$. Otherwise we may assume up to a further (not relabeled) subsequence that $\lim_{\varepsilon \rightarrow 0} \int_I |d'_\varepsilon|^p dx_1 = 0$. In this scenario, set $w_\varepsilon = d_\varepsilon$. Then we

obtain from (4.21), (4.22) and the definition of y' that $u_\varepsilon - w_\varepsilon \rightarrow 0$ in measure and $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(w_\varepsilon) \leq (1 + \delta) \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon)$.

Consequently, from now on, we assume that for $\varepsilon > 0$ small enough it holds $\int_I |d'_\varepsilon|^p dx_1 > c > 0$ and $\#J_{d_\varepsilon} \leq \#J_{b_\varepsilon}$. Then apply Lemma 4.2 to the pair of functions $b_\varepsilon, d_\varepsilon$, with exceptional set $A_\varepsilon := \omega_\varepsilon^{x'} \cup \omega_\varepsilon^{y'}$, yielding a new exceptional set $\tilde{A}_\varepsilon \subset I$ with

$$\mathcal{L}^1(\tilde{A}_\varepsilon) \leq C(M, p, S, c) \frac{\varepsilon^{1/p}}{\delta} \quad (4.32)$$

and a function $w_\varepsilon \in \text{SBV}^p(I)$ such that

$$\begin{aligned} \int_I |w'_\varepsilon(x_1)|^p dx_1 &\leq \frac{1 + \delta}{\mathcal{L}^2(S)} \int_{I \times S} |\partial_1 u_\varepsilon(x)|^p dx, \\ \#J_{w_\varepsilon} &\leq \frac{1 + \delta}{\mathcal{L}^2(S)} \int_{J_{u_\varepsilon}} |\nu_1| d\mathcal{H}^2 \leq N, \\ \|w_\varepsilon(x_1) - b_\varepsilon(x_1)\|_{L^\infty(I \setminus \tilde{A}_\varepsilon)} &\leq \frac{C(M, p, S)}{\delta} \varepsilon^{1/(pp')}. \end{aligned} \quad (4.33)$$

The corresponding piecewise constant function $c_\varepsilon : I \rightarrow \mathbb{R}$ is then simply given as the average of w_ε in between two jumps of w_ε . In particular, by the Poincaré inequality, we obtain L^p -boundedness of $w_\varepsilon - c_\varepsilon$. Hence, there exists $w \in L^p(I)$ such that $w_\varepsilon - c_\varepsilon \rightarrow w$ in L^p . Even more, we obtain by the GSBV^p compactness statement that $w \in \text{SBV}^p$ and $F_0(w) \leq \liminf_{\varepsilon \rightarrow 0} F_0(w_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(w_\varepsilon - c_\varepsilon)$. Moreover, by (4.33) we have

$$F_\varepsilon(w_\varepsilon - c_\varepsilon) \leq (1 + \delta) F_\varepsilon(u_\varepsilon). \quad (4.34)$$

Combining (4.21), (4.22) and (4.33) implies for every $\delta_0(N, S) > \delta > 0$ that $w_\varepsilon - u_\varepsilon \rightarrow 0$ in measure and hence $u_\varepsilon - c_\varepsilon \rightarrow w$ in measure. By a diagonal argument, we may ignore the multiplicative error term $1 + \delta$ in (4.33) and (4.34) asymptotically without losing the convergence in measure. \square

ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers for their valuable suggestions to improve the quality of this manuscript.

DATA AVAILABILITY STATEMENT

No new data/codes were created or analyzed in this study.

REFERENCES

- [1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2000).
- [2] E. De Giorgi, M. Carriero and A. Leaci, Existence theorem for a minimum problem with free discontinuity set. *Arch. Ration. Mech. Anal.* **108** (1989) 195–218.
- [3] M. Friedrich, A piecewise Korn inequality in *SBD* and applications to embedding and density results. *SIAM J. Math. Anal.* **50** (2018) 3842–3918.
- [4] M. Friedrich, A compactness result in *GSBV^p* and applications to Γ -convergence for free discontinuity problems. *Calc. Var. PDE* **58** (2019) 1–31.
- [5] F. Cagnetti, A. Chambolle and L. Scardia, Korn and Poincaré–Korn inequalities for functions with a small jump set. *Math. Ann.* **383** (2022) 1179–1216.
- [6] A. Chambolle, S. Conti and G. Francfort, Korn–Poincaré inequalities for functions with a small jump set. *Indiana Univ. Math. J.* **65** (2016) 1373–1399.

- [7] S. Conti, M. Focardi and F. Iurlano, Integral representation for functionals defined on SBD_p in dimension two. *Arch. Ration. Mech. Anal.* **223** (2017) 1337–1374.
- [8] R.L. Jerrard, Lower bounds for generalized Ginzburg–Landau functionals. *SIAM J. Math. Anal.* **30** (1999) 721–746.
- [9] E. Sandier, Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.* **152** (1998) 379–403.
- [10] E. Sandier and S. Serfaty, Vortices in the magnetic Ginzburg–Landau model. Vol. 70 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA (2007).
- [11] R. Alicandro, M. Cicalese and L. De Luca, Screw dislocations in periodic media: variational coarse graining of the discrete elastic energy (2021). [cvgmt preprint](#).
- [12] L. De Luca, A. Garroni and M. Ponsiglione, Γ -convergence analysis of systems of edge dislocations: the self energy regime. *Arch. Ration. Mech. Anal.* **206** (2012) 885–910.
- [13] J. Ginster, Plasticity as the Γ -limit of a two-dimensional dislocation energy: the critical regime without the assumption of well-separateness. *Arch. Ration. Mech. Anal.* **233** (2019) 1253–1288.
- [14] P. Gladbach, *A phase-field model of dislocations on several slip-planes*. PhD thesis, Universität Bonn (2016).
- [15] D.B. Mumford and J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems. *Commun. Pure Appl. Math.* **42** (1989) 577–685.
- [16] G. Dal Maso, Generalised functions of bounded deformation. *J. Eur. Math. Soc.* **15** (2013) 1943–1997.
- [17] G. Leoni, *A First Course in Sobolev Spaces*. American Mathematical Society (2017).
- [18] S. Almi and E. Tasso, Brittle fracture in linearly elastic plates. *Proc. Roy. Soc. Edinb. Sect. A: Math.* **153** (2023) 68–103.
- [19] J.-F. Babadjian and D. Henao, Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination. *Interfaces Free Bound.* **18** (2016) 545–578.
- [20] A.A. León Baldelli, J.-F. Babadjian, B. Bourdin, D. Henao and C. Maurini, A variational model for fracture and debonding of thin films under in-plane loadings. *J. Mech. Phys. Solids* **70** (2014) 320–348.
- [21] B. Schmidt, A Griffith–Euler–Bernoulli theory for thin brittle beams derived from nonlinear models in variational fracture mechanics. *Math. Models Methods Appl. Sci.* **27** (2017) 1685–1726.
- [22] A. Chambolle and V. Crismale, Compactness and lower semicontinuity in *GSBD*. *J. Eur. Math. Soc.* **23** (2021) 701–719.
- [23] J. Ginster and P. Gladbach, The Euler–Bernoulli limit of thin brittle linearized elastic beams. *J. Elasticity* (2023) 1–31.



Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.