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TRIVIALISABLE CONTROL-AFFINE SYSTEMS REVISITED

Timothée Schmoderer[1,*] and Witold Respondek[2,3]

Abstract. The purpose of this paper is to explore the concept of trivial control systems, namely systems whose dynamics depends on the controls only. Trivial systems have been introduced and studied by Serres in the context of control-nonlinear systems on the plane with a scalar control. In our work, we begin by proposing an extension of the notion of triviality to control-affine systems with arbitrary number of states and controls. Next, our first result concerns two novel characterisations of trivial control-affine systems, one of them is based on the study of infinitesimal symmetries and is thus geometric. Second, we derive a normal form of trivial control-affine systems whose Lie algebra of infinitesimal symmetries possesses an almost abelian Lie subalgebra. Third, we study and propose a characterisation of trivial control-affine systems on 3-dimensional manifolds with scalar control. In particular, we complete the proof of the previous characterisation obtained by Serres. Our characterisation is based on the properties of two functional feedback invariants: the curvature (introduced by Agrachev) and the centro-affine curvature (used by Wilkens). Finally, we give several normal forms of control-affine systems, for which the curvature and the centro-affine curvature have special properties.

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1. Introduction

In this paper, we consider control-affine systems Σ of the form

$$\Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi) u_i, \quad u_i \in \mathbb{R}, \tag{1.1}$$

where the state ξ belongs to a smooth n-dimensional manifold $\mathcal{M}$ (or an open subset of $\mathbb{R}^n$, since most of our results are local), and f and g_i are smooth vector fields on $\mathcal{M}$, *i.e.* smooth sections of the tangent bundle $T\mathcal{M}$. Throughout the paper, the word "smooth" will always mean C^∞-smooth, and all objects (manifolds, vector fields, differential forms, functions) are assumed to be smooth. We denote a control-affine system by the pair $\Sigma = (f, g)$, where $g = (g_1, \ldots, g_m)$. We will assume throughout that the vector fields g_i are pointwise

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[1] Laboratoire Prisme UR 4229, Université d'Orléans & INSA Centre Val de Loire, F45000 Orléans, France.
[2] Institute of Automatic Control, Łódź University of Technology, 90-537 Łódź, ul. Stefanowskiego 18, Poland.
[3] Emeritus Professor at the Laboratoire de Mathématiques de l'INSA UR 3226 - FR CNRS 3335, INSA Rouen Normandie, Avenue de l'Université, 76800 St Etienne du Rouvray, France.

[*] Corresponding author: timothee.schmoderer@univ-orleans.fr

independent everywhere (see a comment, following Def. 1.1, that justifies this assumption). To any control-affine system $\Sigma = (f, g)$ we attach two distributions:

$$\mathcal{G} = \operatorname{span}\{g_1, \ldots, g_m\} \quad \text{and} \quad \mathcal{G}^1 = \mathcal{G} + [f, \mathcal{G}] = \operatorname{span}\{g_1, \ldots, g_m, [f, g_1], \ldots, [f, g_m]\}. \tag{1.2}$$

We call two control-affine systems $\Sigma = (f, g)$ and $\tilde{\Sigma} = (\tilde{f}, \tilde{g})$ *feedback equivalent*, if there exists a diffeomorphism $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$ and smooth functions $\alpha : \mathcal{M} \to \mathbb{R}^m$ and $\beta : \mathcal{M} \to GL_m(\mathbb{R})$ such that

$$\tilde{f} = \phi_* \left(f + \sum_{i=1}^{m} g_i \alpha_i \right) \quad \text{and} \quad \tilde{g}_i = \phi_* \left(\sum_{j=1}^{m} g_j \beta_i^j \right),$$

where ϕ_* denotes the tangent map of ϕ. If ϕ, α, and β are defined locally around ξ_0, then we say that Σ and $\tilde{\Sigma}$ are locally feedback equivalent at ξ_0 and $\tilde{\xi}_0 = \phi(\xi_0)$, respectively. Feedback equivalence of control-affine systems means, geometrically, equivalence of the affine distributions $\mathcal{A} = f + \mathcal{G}$ and $\tilde{\mathcal{A}} = \tilde{f} + \tilde{\mathcal{G}}$ attached to Σ and $\tilde{\Sigma}$, respectively.

In the thesis [1], Serres proposed the notion of a trivial system of the form

$$(\mathcal{T}) \, : \, \dot{x} = F(w), \quad x \in \mathcal{X}, \quad w \in \mathbb{R}^m,$$

where w is the control that enters nonlinearly and $\mathcal{X}$ is a smooth manifold. The dynamics $F(w)$ of a trivial system does not depend on the state variables x and thus depends on control variables w only. Actually, $(\mathcal{T})$ is called flat in [1] but that name can be misleading because, first, there is a well established notion of flat control systems [2] and, second, the class of trivial control systems does not coincide with control systems of zero-curvature [3], which thus can be considered as geometrically flat, as we will discuss in Section 3. For those reasons, following [4], we call $(\mathcal{T})$ a trivial system and we say that a general control-nonlinear system $\dot{x} = F(x, w)$ is trivialisable if it is equivalent, via a feedback of the form $\tilde{x} = \phi(x)$, $\tilde{w} = \psi(x, w)$, to a trivial system $(\mathcal{T})$, where $(\phi, \psi) : \mathcal{X} \times \mathbb{R}^m \to \tilde{\mathcal{X}} \times \mathbb{R}^m$ is a diffeomorphism. Inspired by the above considerations, we adapt the concept of triviality to control-affine systems as follows.

Definition 1.1 (Trivialisable control-affine systems)**.** We say that a control-affine system $\Sigma = (f, g)$ is (locally) trivialisable if it is (locally) feedback equivalent to a trivial system of the form:

$$(T) \, : \, \begin{cases} \dot{x} &= F(w) \\ \dot{w} &= u \end{cases}, \quad (x, w) \in \mathcal{M} = \mathcal{X} \times \mathbb{R}^m, \quad u \in \mathbb{R}^m,$$

whose $\dot{x}$-dynamics depend on the controlled w-variables only.

Notice that for (T) we have $g_j = \frac{\partial}{\partial w_j}$, for $1 \leq j \leq m$, that justifies our assumption that the vector fields $g_1, \ldots, g_m$ of Σ are pointwise independent everywhere. The notions of trivial and trivialisable general control-nonlinear versus control-affine systems are two sides of the same coin. Indeed, two control-nonlinear systems $\dot{x} = F(x, w)$ and $\dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$ are feedback equivalent if and only if their control-affine extensions $\dot{x} = F(x, w)$, $\dot{w} = u$ and $\dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w})$, $\dot{\tilde{w}} = \tilde{u}$ are equivalent via control-affine feedback transformations, see [5], Proposition 3.4. Therefore a control-nonlinear system $\dot{x} = F(x, w)$ is trivialisable if and only if $\dot{x} = F(x, w)$, $\dot{w} = u$ is trivialisable in the sense of Definition 1.1 and the latter class is the object of our studies in this paper.

Trivial control systems are interesting to study because they model trajectories of dynamical systems under a nonholonomic constraint that does not depend on the point. Indeed, under the additional regularity assumption that $\operatorname{rk} \frac{\partial F}{\partial w}(w) = m$, equivalently, the distribution $\mathcal{G}^1$ of (T) satisfies $\operatorname{rk} \mathcal{G}^1 = 2m$, there exist local coordinates $x = (z, y)$, with $\dim z = n - 2m$ and $y = (y_1, \ldots, y_m)$, and a suitable feedback, such that the equations of (T)

can be rewritten

$$\begin{cases} \dot{z} &= \mathsf{f}(w) \\ \dot{y} &= w \\ \dot{w} &= u \end{cases}$$

and we conclude that the trajectories of (T) satisfy the nonholonomic constraints $\dot{z} = \mathsf{f}(\dot{y})$, whose shape is independent of the point $x = (z, y)$. Denoting by $\mathcal{X}$ the (locally defined) quotient manifold $\mathcal{M}/\mathcal{G}$, we see that a trajectory $x(t) \in \mathcal{X}$ satisfies the nonholonomic constraint $\dot{z} = \mathsf{f}(\dot{y})$ if and only if there exists a smooth control $u(t)$ such that $(x(t), w(t))$ is a trajectory of (T). Connections between equations on the tangent bundle and control systems are explored in [6, 7]. Examples of trivial systems can be found in the literature; *e.g.* in [7] we characterise trivial elliptic, hyperbolic, and parabolic control systems, Dubin's car [8] is a very simple model of system that is trivial, and, finally, trivial control-nonlinear system on surfaces (*i.e.* $n = 2$) and with scalar control have been studied, characterised (and normal forms in particular cases have been given) in [1, 4, 9].

1.1. Outline of the paper

In the next subsection, we develop the main notions of differential geometry and of control theory that we will need in the rest of the paper. Next, in Section 2, we study trivial control-affine systems on manifolds of arbitrary dimension and with an arbitrary number of controls. We propose two characterisations of trivial systems, one of them is based on the Lie algebra of infinitesimal symmetries. Afterwards, in the remaining part of the paper, we focus on the single input case. Using our characterisation of trivial systems via symmetries, we will give a normal form of single-input trivial systems whose Lie algebra of infinitesimal symmetries possesses an almost abelian Lie subalgebra. In particular, in the two- and three-dimensional cases, we give an exhaustive list of trivial systems possessing such a Lie subalgebra of symmetries. Next, in Section 3, we will be interested in revisiting the characterisation of trivial systems discovered by Serres [1] in the context of control-nonlinear systems on surfaces. We propose a characterisation of trivial control-affine systems on three-dimensional manifolds with scalar control. Our characterisation exhibits a discrete invariant, and two fundamental functional invariants: the control curvature introduced by Agrachev [3, 10], and the centro-affine curvature used by Wilkens [11]. Both functional invariants can be computed for any control-affine system. We will fill a gap in the proof of Serres and interpret his results. Finally, in Section 4, we discuss several normal forms (some new and some existing in the literature) of control-affine systems, for which the control curvature and the centro-affine curvature have special properties.

1.2. Preliminaries

In this subsection, we recall the main definitions and notions of differential geometry and of control theory that we need in the paper. The main notations that we use are summarised in Table 1.

Differential Geometry. For a manifold $\mathcal{M}$ we will denote by $T\mathcal{M}$ and $T^*\mathcal{M}$ the tangent and cotangent bundle, respectively. The space of all smooth vector fields (smooth sections of $T\mathcal{M}$) will be denoted $V^\infty(\mathcal{M})$ and the space of all smooth differential p-forms by $\Lambda^p(\mathcal{M})$, except for smooth functions (0-forms) whose space is denoted $C^\infty(\mathcal{M})$. For a diffeomorphism $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$, a vector field $f \in V^\infty(\mathcal{M})$, and a differential p-form $\omega \in \Lambda^p(\tilde{\mathcal{M}})$, we denote by $\phi_* f \in V^\infty(\tilde{\mathcal{M}})$ the push-forward of f, and by $\phi^*\omega \in \Lambda^p(\mathcal{M})$ the pull-back of ω. The (local) flow of a vector field $f \in V^\infty(\mathcal{M})$ is denoted by γ_t^f (for any t for which it is defined). The Lie derivative of a differential p-form ω along a vector field f will be denoted by $\mathrm{L}_f(\omega)$. In particular, for a function $\lambda \in C^\infty(\mathcal{M})$ and its differential $\mathrm{d}\lambda$ (an exact 1-form) we have

$$\mathrm{L}_f(\lambda) = \langle \mathrm{d}\lambda, f \rangle \quad \text{and} \quad \mathrm{L}_f(\mathrm{d}\lambda) = \mathrm{d}\mathrm{L}_f(\lambda).$$

TABLE 1. Main notations for the paper.

$\mathcal{M}$, $T\mathcal{M}$, $\xi = (x, w)$	Smooth n-dimensional manifold, its tangent bundle, and its local coordinates with $\dim w = m$.
ϕ, ϕ_*, ϕ^*	A diffeomorphism, its tangent map, its cotangent map.
$\Sigma = (f, g)$	A control-affine system given by (1.1).
$\mathcal{G}$ and $\mathcal{G}^1$	Distribution spanned by the vector fields $g_1, \dots, g_m$ and the distribution spanned by the vector fields $g_1, \dots, g_m$ and $[f, g_1], \dots, [f, g_m]$; see (1.2).
(T)	Trivial control-affine system; see Definition 1.1.
$\mathfrak{K}$, $\mathfrak{A}$, $\mathfrak{I}$	A real Lie (sub)algebra, a subalgebra, an ideal.
Σ_A, $\Sigma_\lambda^{0,k}$	Normal forms of trivial systems having an almost abelian subalgebra of infinitesimal symmetries; see Theorem 2.4 and Proposition 2.7.
$\Sigma_s = (f_s, g)$	Control-affine system given by a semi-canonical pair; see Definition 3.1.
$\Sigma_c = (f_c, g_c)$	Control-affine system given by the canonical pair; see Definition 3.1.
(k_1, k_2, k_3) and $(\lambda_1, \lambda_2, \lambda_3)$	Structure functions attached to any control-affine system on a 3-dimensional manifold with scalar control; see (3.1).
$(\varepsilon, \kappa, \nu)$	Feedback invariants of control-affine systems; defined for the canonical pair by (3.1') and expressed for any control-affine system by (3.6).

For any smooth functions α, λ, and μ, the Lie derivative possesses the following properties: $\mathrm{L}_{\alpha f}(\lambda) = \alpha \mathrm{L}_f(\lambda)$, and $\mathrm{L}_f(\lambda\mu) = \mathrm{L}_f(\lambda)\mu + \lambda \mathrm{L}_f(\mu)$. Iterative Lie derivatives are defined by $\mathrm{L}_f^k(\lambda) = \mathrm{L}_f\left(\mathrm{L}_f^{k-1}(\lambda)\right)$, for any $k \geq 2$. For any two vector fields $f, g \in V^\infty(\mathcal{M})$, we define their Lie bracket as a new vector field, denoted $[f, g] \in V^\infty(\mathcal{M})$, such that for any smooth function λ we have

$$\mathrm{L}_{[f,g]}(\lambda) = \mathrm{L}_f(\mathrm{L}_g(\lambda)) - \mathrm{L}_g(\mathrm{L}_f(\lambda)).$$

The Lie bracket possesses the following properties: it is bilinear over $\mathbb{R}$, it is skew-commutative, *i.e.* $[f, g] = -[g, f]$, and it satisfies the Jacobi identity:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0, \quad \forall\, f, g, h \in V^\infty(\mathcal{M}).$$

Moreover, for any smooth function α, and any vector fields f, g, and h, we have

$$[f, \alpha g + h] = \alpha [f, g] + \mathrm{L}_f(\alpha) g + [f, h].$$

Two vector fields f and g satisfying $[f, g] = 0$ are said to be commuting; since under diffeomorphisms $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$ the Lie bracket is transformed by $[\phi_* f, \phi_* g] = \phi_* [f, g]$, the commutativity property does not depend on coordinates. The celebrated *Flow-box* theorem (also called the "Straightening-out theorem" or the "Local linearisation lemma") asserts that on a given n-dimensional manifold $\mathcal{M}$ there exists a local coordinate system $(x_1, \dots, x_n)$ such that $f = \frac{\partial}{\partial x_1}$ in a neighbourhood of any point p where $f(p) \neq 0$. This can simultaneously be done for a family of (locally) independent vector fields $(f_1, \dots, f_m)$ if and only if they are mutually commuting. We set $\mathrm{ad}_f^0 g = g$, $\mathrm{ad}_f g = [f, g]$, and the iterated Lie bracket is denoted by $\mathrm{ad}_f^k g = \left[f, \mathrm{ad}_f^{k-1} g\right]$ for $k \geq 1$; see [12], Chapter 1 for a detailed introduction and proofs of the above properties.

Infinitesimal symmetries. We briefly introduce the notion of symmetries of control-affine systems (see [13, 14] for a detailed introduction). For a control-affine system $\Sigma = (f, g)$, given by (1.1), with state $\mathcal{M}$ a smooth n-dimensional manifold, we define the set of admissible velocities $\mathcal{A}$ as

$$\mathcal{A}(\xi) = \{f(\xi) + \sum_{i=1}^{m} g_i(\xi)u_i \,:\, u_i \in \mathbb{R}\} \subset T_\xi\mathcal{M}.$$

Clearly, $\mathcal{A}$ is a collection of affine m-planes, that is, an affine distribution. We say that a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ is a *symmetry* of Σ if it preserves the affine distribution $\mathcal{A} = f + \mathcal{G}$, that is, $\phi_*\mathcal{A} = \mathcal{A}$. We say that a vector field v on $\mathcal{M}$ is an infinitesimal symmetry of $\Sigma = (f, g)$ if the (local) flow γ_t^v of v is a local symmetry, for any t for which it exists, that is, $(\gamma_t^v)_*\mathcal{A} = \mathcal{A}$. Consider the system $\Sigma = (f, g)$ and recall that $\mathcal{G}$ is the distribution spanned by the vector fields $g_1, \dots, g_m$. We have the following characterisation of infinitesimal symmetries.

Proposition 1.2. *A vector field v is an infinitesimal symmetry of the control-affine system $\Sigma = (f, g)$ if and only if*

$$[v, g] = 0 \mod \mathcal{G} \quad \textit{and} \quad [v, f] = 0 \mod \mathcal{G}.$$

By the Jacobi identity, it is easy to see that if v_1 and v_2 are infinitesimal symmetries, then so is $[v_1, v_2]$, hence the set of all infinitesimal symmetries forms a real Lie algebra denoted by $\mathfrak{S}$. Notice that the Lie algebra of infinitesimal symmetries is attached to the affine distribution $\mathcal{A} = f + \mathcal{G}$ and not to a particular pair $(f, g) = (f, g_1, \dots, g_m)$. Different pairs (f, g) related via feedback transformations (α, β), define the same $\mathcal{A}$ and thus have the same Lie algebra of infinitesimal symmetries which, therefore, is a feedback invariant object attached to Σ.

2. Trivial control-affine systems

In this section, we first propose two new characterisations of trivialisable control-affine systems (with the state-space of arbitrary dimension and with an arbitrary number of controls); see Theorem 2.1 below. Second, we give a normal form of trivial systems whose Lie algebra of infinitesimal symmetries possesses an almost abelian Lie subalgebra; see Theorem 2.4 and Proposition 2.7 of this section.

2.1. Characterisations of trivial systems

The following theorem gives two characterisations of trivialisable systems. The first one is technical and shows that triviality is a property that depends on the coordinates (like being a linear control system depends on the choice of coordinates), and the second one is based on infinitesimal symmetries and is thus geometric. Recall that to a control-affine system $\Sigma = (f, g)$ we attach two distributions $\mathcal{G} = \operatorname{span}\{g_1, \dots, g_m\}$ and $\mathcal{G}^1 = \mathcal{G} + [f, \mathcal{G}]$, see (1.2).

Theorem 2.1 (Two characterisations of trivialisable systems)**.** *Consider a control-affine system $\Sigma = (f, g)$ with state on a n-dimensional manifold $\mathcal{M}$ and with $m \geq 1$ controls. The following assertions hold locally around ξ_0:*

(i) Suppose that $\operatorname{rk}\mathcal{G}^1 = m + k$ is constant. The system Σ is locally trivialisable if and only if Σ is locally feedback equivalent to

$$\Sigma_T \,:\, \begin{cases} \dot{x}_i &= h_i(x, w), \quad \textit{for} \quad 1 \leq i \leq n - m, \\ \dot{w}_j &= u_j, \qquad\quad\; \textit{for} \quad 1 \leq j \leq m, \end{cases}$$

where the smooth scalar functions $h_1, \dots, h_{n-m}$ satisfy

$$\operatorname{rk}\operatorname{span}\{\mathrm{d}h_1, \dots, \mathrm{d}h_{n-m}\} = k. \tag{2.1}$$

(ii) *Σ is, locally around ξ_0, trivialisable if and only if the distribution $\mathcal{G}$ is involutive and of constant rank m and, additionally, the Lie algebra of infinitesimal symmetries $\mathfrak{S}$ of Σ possesses an abelian subalgebra $\mathfrak{A}$ such that $\mathfrak{A}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$.*

Observe that the assumption on the rank of the distribution $\mathcal{G}^1$ in statement Theorem *(i)* implies that the dimension n of the manifold $\mathcal{M}$ is greater than or equal to $m + k$. If $n = m + k$, then the trivialisation (T) of Σ_T (and thus of Σ) can be taken (for suitable w and u) as $\dot{x}_i = w_i$, $1 \le i \le k$, $\dot{w}_j = u_j$, $1 \le j \le m$. On the other hand, if $n > m + k$, then $\dot{x}_i = w_i$, $1 \le i \le k$, $\dot{x}_i = F_i(w_1, \ldots, w_k)$, $k + 1 \le i \le n - m$, and $\dot{w}_j = u_j$, $1 \le j \le m$. Notice that $k \le m$, so if $n > 2m$, then there are always nonlinear equations $\dot{x}_i = F_i(w_1, \ldots, w_k)$. In item Theorem *(ii)*, there are no particular relations between the dimension of the state space and the number of controls (other than the obvious $n \ge m$).

Remark 2.2. For the system Σ_T, define $h = (h_1, \ldots, h_{n-m})^t$. Then, under the assumption that $\operatorname{rk} \mathcal{G}^1$ is constant, condition (2.1) can be equivalently reformulated as

$$\operatorname{rk} \frac{\partial h}{\partial w}(x, w) = \operatorname{rk} \frac{\partial h}{\partial (x, w)}(x, w),$$

in a neighbourhood of (x_0, w_0).

Proof.

(i) Suppose that Σ is locally trivialisable, *i.e.* by Definition 1.1, Σ is locally feedback equivalent to (T), which is of the form of Σ_T with $h_i(x, w) = F_i(w)$, for $1 \le i \le n - m$, and we now show that those functions satisfy (2.1). On one hand, the condition $\operatorname{rk} \mathcal{G}^1 = m + k$ implies that the Jacobian matrix $\frac{\partial F}{\partial w}$ is of constant rank k, where $F = (F_1, \ldots, F_{n-m})^T$. On the other hand, we obtain that $\mathrm{d}h_i = \mathrm{d}F_i = \sum_{j=1}^m \frac{\partial F_i}{\partial w_j} \mathrm{d}w_j$. Hence the rank of $\operatorname{span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_{n-m}\}$ is the same as that of $\frac{\partial F}{\partial w}$ and the conclusion follows. Conversely, assume that Σ is feedback equivalent to Σ_T. Using the assumption $\operatorname{rk} \mathcal{G}^1 = m + k$ we can reorder the x-coordinates such that $h = (\hat{h}_1, \ldots, \hat{h}_k, \tilde{h}_{k+1}, \ldots, \tilde{h}_{n-m})$, where $\operatorname{rk} \frac{\partial \hat{h}}{\partial w} = k$. We set $\hat{w}_i = \hat{h}_i(x, w)$, for $1 \le i \le k$, completed by $\hat{w}_{k+1}, \ldots, \hat{w}_m$ (chosen among the w_i's) in such a way that $\hat{w}_1, \ldots, \hat{w}_m$ form a local coordinate system. We conclude, by condition (2.1), that the functions $\tilde{h}_{k+1}, \ldots, \tilde{h}_{n-m}$ depend on the variables $\hat{w}_1, \ldots, \hat{w}_k$ only. Using a feedback transformation that yields $\dot{\hat{w}}_i = \hat{u}_i$, $1 \le i \le k$, we conclude that Σ_T is, indeed, a trivial system in coordinates $(x, \hat{w})$.

(ii) Suppose that $\Sigma = (f, g)$ is locally trivialisable, then for (T) we have $\mathcal{G} = \operatorname{span}\left\{\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_m}\right\}$, which clearly is involutive and of constant rank m. Moreover, the vector fields $v_i = \frac{\partial}{\partial x_i}$, for $1 \le i \le n - m$, are commuting symmetries of (T) that span the abelian Lie algebra $\mathfrak{A}$ satisfying $\mathfrak{A}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$. Conversely, suppose that the Lie algebra of infinitesimal symmetries $\mathfrak{S}$ of $\Sigma = (f, g)$ possesses an abelian subalgebra $\mathfrak{A}$ satisfying $\mathfrak{A}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$. Choose commuting vector fields $v_i \in \mathfrak{A}$, for $1 \le i \le n - m$, that are linearly independent at ξ_0 and local coordinates $\tilde{\xi} = (\tilde{x}, \tilde{w})$ such that $v_i = \frac{\partial}{\partial \tilde{x}_i}$, for $1 \le i \le n - m$. In those coordinates, we have $g_j = A_j(\tilde{x}, \tilde{w}) \frac{\partial}{\partial \tilde{x}} + B_j(\tilde{x}, \tilde{w}) \frac{\partial}{\partial \tilde{w}}$. Since $\mathcal{G}$ is of constant rank m and satisfies $\mathfrak{A}(\tilde{\xi}_0) \oplus \mathcal{G}(\tilde{\xi}_0) = T_{\tilde{\xi}_0}\mathcal{M}$, via a suitable feedback transformation we choose generators of $\mathcal{G}$ as $\tilde{g}_j = \tilde{A}_j(\tilde{x}, \tilde{w}) \frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{w}_j}$ (to simplify notations, we skip the "tildes" and denote $\tilde{g}_j$ by g_j and $\tilde{A}_j$ by A_j). Using that v_i are symmetries of Σ, that is $[v_i, g_j] \in \mathcal{G}$, we deduce that $A_j = A_j(\tilde{w})$, therefore we actually have $[v_i, g_j] = 0$. Moreover, $\mathcal{G}$ is involutive so we deduce that $[g_j, g_k] = 0$. Therefore, all vector fields v_i and g_j commute and thus there exist coordinates $\xi = (x, w)$ such that $v_i = \frac{\partial}{\partial x_i}$, for $1 \le i \le n - m$, and $g_j = \frac{\partial}{\partial w_j}$, for $1 \le j \le m$. The vector fields v_i are symmetries of Σ so $[v_i, f] \in \mathcal{G}$ implying that $f = F \frac{\partial}{\partial x} + \mathrm{F} \frac{\partial}{\partial w}$, where $F = F(w)$ and we achieve $\mathrm{F} = 0$ by a suitable feedback transformation.

□

The previous theorem gives two characterisations of trivialisable systems with arbitrary number of states and controls. In the remaining of the paper, we focus on single-input systems, which reveal to be rich. Namely, we will study trivial single-input systems possessing an almost abelian algebra of symmetries in Section 2.2 and, in Section 3, we give several normal forms of systems with three states and one control for which the curvature (a feedback invariant) has special properties.

2.2. Normal form of trivial systems possessing an almost abelian Lie subalgebra of infinitesimal symmetries

Recall that $\mathfrak{S}$ denotes the Lie algebra of all infinitesimal symmetries of the system $\Sigma = (f, g)$. Theorem 2.1 of the previous subsection asserts that if Σ is trivialisable, then $\mathfrak{S}$ possesses an abelian subalgebra $\mathfrak{A}$ complementary to the distribution $\mathcal{G}$. In many cases, $\mathfrak{A}$ is the Lie algebra of all infinitesimal symmetries $\mathfrak{S}$; for instance, this is the case for the 3-dimensional trivial system $\dot{x} = \mathrm{e}^w$, $\dot{y} = w$, $\dot{w} = u$. To identify specific classes of trivial control systems, it is interesting to study the case of $\mathfrak{A}$ not coinciding with $\mathfrak{S}$, that is, systems that admit additional symmetries (not belonging to $\mathfrak{A}$). The first case to consider is $\mathfrak{A}$ being replaced by an abelian ideal $\mathfrak{J}$ of codimension one in a subalgebra $\mathfrak{K}$ of $\mathfrak{S}$. The picture is thus

$$\mathfrak{J} \underset{1}{\subset} \mathfrak{K} \subset \mathfrak{S},$$

where the subindex 1 below "$\subset$" indicates the codimension. In general, the Lie algebra $\mathfrak{S}$ of all infinitesimal symmetries can be of any dimension (finite or infinite). On the other hand, we will assume that $\dim \mathfrak{K} = n$. Thus, $\dim \mathfrak{J} = n-1$ and if $\mathfrak{J}$ satisifies $\mathfrak{J}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$, then Σ is, first, single-input, and, second, trivialisable due to item Theorem *(ii)* of Theorem 2.1, in which case the required abelian subalgebra $\mathfrak{A}$ is replaced by the abelian ideal $\mathfrak{J}$. Below we provide a normal form, completed by a detailed analysis of all possible casses, of control-affine systems possessing an almost abelian Lie algebras of symmetries.

Definition 2.3 (Almost abelian Lie algebra)**.** Let $\mathfrak{K}$ be a real Lie algebra; following the definition of *[15]*, we call $\mathfrak{K}$ *almost abelian* if it has an abelian ideal $\mathfrak{J}$ of codimension one.

It is a simple application of Lie algebra homology to deduce that an almost abelian Lie algebra (possibly of infinite dimension) is isomorphic to the semi-direct product $\mathfrak{K} \cong \mathfrak{J} \rtimes \mathrm{vect}_{\mathbb{R}}\{v_0\}$ and that its structure is determined by the non-zero action of v_0 on $\mathfrak{J}$, namely by

$$\begin{aligned} \mathrm{ad}_{v_0} \,:\, \mathfrak{J} &\longrightarrow \mathfrak{J} \\ v &\longmapsto [v_0, v]\,. \end{aligned}$$

Moreover, two almost abelian Lie algebras $\mathfrak{K} = \mathfrak{J} \rtimes \mathrm{vect}_{\mathbb{R}}\{v_0\}$ and $\tilde{\mathfrak{K}} = \tilde{\mathfrak{J}} \rtimes \mathrm{vect}_{\mathbb{R}}\{\tilde{v}_0\}$ are isomorphic if and only if there exists a real invertible transformation $P : \mathfrak{J} \to \tilde{\mathfrak{J}}$ and a non-zero constant $\mu \in \mathbb{R}^*$ such that $P\mathrm{ad}_{v_0} = \mu\,\mathrm{ad}_{\tilde{v}_0}P$; see [16], Proposition 11. Therefore, isomorphism classes of almost abelian Lie algebras correspond to similarity classes of the linear operator ad_{v_0} (up to multiplication by a scalar). In particular, if $\mathfrak{K}$ is finite dimensional, then the similarity classes of ad_{v_0} corresponds to the Jordan normal forms. In the following theorem, we consider the case of ad_{v_0} being non-singular (see Rem. 2.6 below for the singular case) and we give a general normal form of control-affine systems, whose Lie algebra of infinitesimal symmetries possesses an almost abelian Lie subalgebra.

Theorem 2.4 (Control systems with almost abelian infinitesimal symmetries)**.** *Consider a control-affine system $\Sigma = (f, g)$ on an n-dimensional state manifold $\mathcal{M}$ and with scalar control. Assume that $f(\xi_0) \notin \mathcal{G}(\xi_0)$ and that the Lie algebra of infinitesimal symmetries $\mathfrak{S}$ possesses an almost abelian Lie subalgebra $\mathfrak{K} = \mathfrak{J} \rtimes \mathrm{vect}_{\mathbb{R}}\{v_0\}$ satisfying: $\mathfrak{J}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$ and the operator ad_{v_0} is non-singular. Then, locally around ξ_0, $\mathfrak{K}$ acts transitively* *on $\mathcal{M}$ and Σ is locally feedback equivalent to a trivial system of the form*

$$\Sigma_A : \begin{cases} \dot{x} &= \exp(Aw)\, F(0) \\ \dot{w} &= u \end{cases}, \quad u \in \mathbb{R},$$

around $(x_0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where A is the matrix representation of ad_{v_0} in the basis $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$, and $F(0) \neq 0 \in \mathbb{R}^{n-1}$ is defined by $f = \sum_{i=1}^{n-1} F_i \frac{\partial}{\partial x_i} \mod \mathcal{G}$.

Remark 2.5. By a linear change of coordinates, the system Σ_A can be brought into a simpler normal form by transforming the matrix A into its real Jordan normal form. Below, I and I' (for complex eigenvalues) denote suitable subsets of the indices $\{1, \ldots, n-1\}$ and the vector $x = (x_1, \ldots, x_{n-1})$ of transformed coordinates consists of blocks x_I and $x_{I'}$ corresponding to the Jordan blocks of A. The matrix exponential of the normalised trivial system decouples over Jordan blocks, which yields:

(1) For a diagonal Jordan block given by a real eigenvalue $\lambda \in \mathbb{R}$, we have:

$$\dot{x}_I = \mathrm{e}^{\lambda w} \eta_I;$$

(2) For a diagonal Jordan block given by a complex eigenvalue $\lambda = a + \mathbf{i}b \in \mathbb{C}$, we have

$$\dot{x}_I = \mathrm{e}^{aw} \cos(bw) \eta_I \quad \text{and} \quad \dot{x}_{I'} = \mathrm{e}^{aw} \sin(bw) \eta_{I'};$$

(3) For a non-diagonal Jordan block given by a real eigenvalue $\lambda \in \mathbb{R}$, we have

$$\dot{x}_I = \mathrm{e}^{\lambda w} N_I, \quad \text{where} \quad N_I = \begin{pmatrix} \eta_1 & \eta_2 w & \cdots & \eta_q w^q \end{pmatrix}^t;$$

(4) For a non-diagonal Jordan block given by a complex eigenvalue $\lambda = a + \mathbf{i}b \in \mathbb{C}$, we have

$$\dot{x}_I = \mathrm{e}^{aw} \cos(bw) N_I, \quad \text{where} \quad N_I = \begin{pmatrix} \eta_1 & \eta_2 w & \cdots & \eta_q w^q \end{pmatrix}^t,$$
$$\dot{x}_{I'} = \mathrm{e}^{aw} \sin(bw) N_{I'}, \quad \text{where} \quad N_{I'} = \begin{pmatrix} \eta'_1 & \eta'_2 w & \cdots & \eta'_q w^q \end{pmatrix}^t;$$

In all of the above cases, η_i and η'_i (present in the vectors η_I, $\eta_{I'}$, N_I and $N_{I'}$) are normalized to either zero or one. See Corollary 2.8 and 2.9 for an illustration of the above cases for two- and three-dimensional systems.

Proof. Consider the control-affine system $\Sigma = (f, g)$ given by vector fields f and g, and let n vector fields $v_1, \ldots, v_{n-1}, v_0$ span the n-dimensional Lie subalgebra $\mathfrak{K} = \mathrm{vect}_{\mathbb{R}} \{v_1, \ldots, v_{n-1}, v_0\}$ of the algebra of infinitesimal symmetries. By assumption, $\mathfrak{K}$ is almost abelian, its abelian ideal is $\mathfrak{I} = \mathrm{vect}_{\mathbb{R}} \{v_1, \ldots, v_{n-1}\}$ and, in the basis $(v_1, \ldots, v_{n-1})$, the linear operator ad_{v_0} is represented by the matrix $A = (a^i_j)_{i,j=1}^{n-1}$, *i.e.*

$$[v_0, v_i] = \sum_{j=1}^{n-1} a_i^j v_j, \quad \forall\, 1 \leq i \leq n-1. \tag{2.2}$$

Since $\mathfrak{I}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$, by statement Theorem *(ii)* of Theorem 2.1, Σ is locally trivialisable and following the proof of that theorem we deduce that there exist local coordinates (x, w) around $(x_0, 0)$ and a feedback transformation such that $v_i = \frac{\partial}{\partial x_i}$, for $1 \leq i \leq n-1$, $g = \frac{\partial}{\partial w}$, and $f = \sum_{i=1}^{n-1} F_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{n-1} F_i v_i$, where $F_i = F_i(w)$. We express the infinitesimal symmetry $v_0 = \sum_{i=1}^{n-1} \gamma_i \frac{\partial}{\partial x_i} + \delta \frac{\partial}{\partial w}$, where $\gamma_i = \gamma_i(x)$, since v_0 is a symmetry of $\mathcal{G} = \mathrm{span}\left\{\frac{\partial}{\partial w}\right\}$, and $\delta = \delta(x, w)$. Using the fact that v_0 is a symmetry of f, *i.e.* $[v_0, f] \in \mathcal{G}$, we deduce the following equations:

$$\sum_{i=1}^{n-1}\left(F_i\,[v_0,v_i]+\delta(w)\frac{\mathrm{d}F_i}{\mathrm{d}w}v_i\right)=0 \quad \text{or, equivalently,}$$
$$\delta(w)\frac{\mathrm{d}F}{\mathrm{d}w}(w)+AF(w)=0, \tag{2.3}$$

where $F=\sum_{i=1}^{n-1}F_i(w)\frac{\partial}{\partial x_i}$. The assumption $f(\xi_0)\notin\mathcal{G}(\xi_0)$ implies that $F(0)\neq 0\in\mathbb{R}^{n-1}$. Moreover, A is non-singular, hence $\delta(0)\neq 0$ and we conclude that $\mathfrak{K}$ acts transitively on $\mathcal{M}$ around ξ_0. Moreover, for all w_1,w_2 we have $\frac{A}{\delta(w_1)}\frac{A}{\delta(w_2)}=\frac{A}{\delta(w_2)}\frac{A}{\delta(w_1)}$. Thus, the Magnus expansion [17] holds and the solution of equation (2.3) is $F(w)=\exp\left(h(w)A\right)F(0)$, with $h(w)=-\int_0^w\frac{1}{\delta(\tau)}\,\mathrm{d}\tau$. Clearly, $h'(0)\neq 0$ and, therefore, $\tilde{w}=h(w)$ is a local diffeomorphism that maps $w_0=0$ into $\tilde{w}_0=0$. Within this new coordinate system (relabelling $\tilde{w}$ as w, for simplicity) we have $F(w)=\exp\left(Aw\right)F(0)$. □

Remark 2.6 (On the singularity of the operator ad_{v_0})**.** The non-singularity of A is crucial to deduce from equation (2.3) that $\mathfrak{K}$ is transitive. Indeed, if this is not the case, then we may have $Af(0)=0$ and we cannot conclude that $\delta(0)\neq 0$. However, under the additional assumption that $\mathfrak{K}$ is transitive on $\mathcal{M}$, *i.e.* $\delta(0)\neq 0$, the same conclusion, as the one of Theorem 2.4, holds even with A being singular.

To show the transitivity of $\mathfrak{K}$, the proof of the previous theorem used $f(\xi_0)\notin\mathcal{G}(\xi_0)$ and, therefore, we gave a normal form around a point which is not an equilibrium. On the other hand, if we drop that condition, then a result similar to Theorem 2.4 holds for an almost abelian Lie subalgebra $\mathfrak{K}$ of symmetries whose rank drops at ξ_0, see Proposition 2.7 below. That case turns out to be much more restrictive on the normal form and on the almost abelian Lie subalgebra. Indeed, we will show that the operator ad_{v_0} has to be necessarily diagonalisable over $\mathbb{R}$ and that all its eigenvalues have to be positive integers. To compare the normal form given in the proposition below and the one of the theorem above, we give the form of Σ_A in the case of A being diagonalisable over $\mathbb{R}$. In that case, item Remark (1) of Remark 2.5 yields $\dot{x}_i=\eta_i\mathrm{e}^{\lambda_i w}$, $1\leq i\leq n-1$, $\dot{w}=u$. Moreover, using the local diffeomorphism $w=\ln(1+\tilde{w})$, the system Σ_A takes the normal form

$$\Sigma_\lambda\,:\,\begin{cases}\dot{x}_i &=\eta_i(w+1)^{\lambda_i}, \quad 1\leq i\leq n-1,\\ \dot{w} &=u\end{cases} \qquad u\in\mathbb{R},$$

around $(x_0,0)\in\mathbb{R}^{n-1}\times\mathbb{R}$, where the λ_i's are the eigenvalues of ad_{v_0} and $\eta=(\eta_1,\ldots,\eta_{n-1})^t$ is a vector whose components are either 0 or 1; notice that, due to $f(\xi_0)\notin\mathcal{G}(\xi_0)$, necessarily $\eta\neq 0$.
Now, we will assume that the system $\Sigma=(f,g)$ satisfies the accessibility rank condition $\dim\mathfrak{L}(\xi_0)=n$, where $\mathfrak{L}=\{f,g\}_{LA}$ is the accessibility Lie algebra of Σ; see [12]. That condition applied to Σ_λ implies that all $\eta_i=1$ and that all eigenvalues λ_i are pair-wise distinct. Therefore, any system Σ satisfying the the accessibility rank condition, the assumptions of Theorem 2.4, and such that the matrix A is diagonalisable over $\mathbb{R}$, is locally feedback equivalent to

$$\Sigma_\lambda^1\,:\,\begin{cases}\dot{x}_i &=(w+1)^{\lambda_i/\lambda_1}, \quad 1\leq i\leq n-1,\\ \dot{w} &=u\end{cases} \qquad u\in\mathbb{R},$$

where $\lambda_1<\lambda_2<\ldots<\lambda_{n-1}$ (recall that $\lambda_i\neq 0$ by the non-singularity of A). Moreover, two such systems Σ_λ^1 and $\Sigma_{\tilde{\lambda}}^1$ are feedback equivalent if and only if $[\lambda_1:\cdots:\lambda_{n-1}]=[\tilde{\lambda}_1:\cdots:\tilde{\lambda}_{n-1}]$ or $[\lambda_1:\cdots:\lambda_{n-1}]=[\tilde{\lambda}_{n-1}:\cdots:\tilde{\lambda}_1]$.
Notice that the lack of accessibility (meaning that either some eigenvalues are equal, or some $\eta_i=0$, or both) implies that the full Lie algebra of infinitesimal symmetries $\mathfrak{S}$ is of infinite dimension; on the other hand for accessible system, $\mathfrak{S}$ is of finite dimension, actually it is the almost abelian algebra $\mathfrak{K}$; we will give details about that analysis elsewhere.

Proposition 2.7. *Consider a control-affine system $\Sigma = (f, g)$ with an n-dimensional state manifold and a scalar control. Assume that Σ satisfies the accessibility rank condition $\dim \mathfrak{L}(\xi_0) = n$, that $f(\xi_0) \in \mathcal{G}(\xi_0)$, and that the Lie algebra of all infinitesimal symmetries $\mathfrak{S}$ possesses an almost abelian Lie subalgebra $\mathfrak{K} \cong \mathfrak{I} \rtimes \mathrm{vect}_{\mathbb{R}} \{v_0\}$ satisfying: $\mathfrak{I}(\xi_0) \oplus \mathcal{G}(\xi_0) = T_{\xi_0}\mathcal{M}$ and the operator ad_{v_0} is non-singular. Then there exists the smallest integer $k \geq 1$ such that $g \wedge \mathrm{ad}_g^k f(\xi_0) \neq 0$, the Lie subalgebra $\mathfrak{K}$ is transitive on $\mathcal{M}\backslash\{\xi_0\}$, the operator ad_{v_0} is diagonalisable over $\mathbb{R}$ and its eigenvalues λ_i are pairwise distinct and such that $k\lambda_i/\lambda_1$, for $2 \leq i \leq n-1$, are positive integers greater than k, where λ_1 is the smallest, in absolute value, eigenvalue of ad_{v_0}. Moreover, the system Σ is locally feedback equivalent to*

$$\Sigma_\lambda^{0,k} : \begin{cases} \dot{x}_i &= w^{k\lambda_i/\lambda_1}, \qquad 1 \leq i \leq n-1 \\ \dot{w} &= u \end{cases} , \quad u \in \mathbb{R},$$

around $(x_0, 0) \in \mathbb{R}^n$.

Recall that the eigenvalues of ad_{v_0} are given up to a multiplication by a non-zero factor and that it is a classical fact that (under the above assumptions) the integer k is an invariant of feedback transformations. Hence two systems $\Sigma_\lambda^{0,k}$ and $\Sigma_{\tilde{\lambda}}^{0,\tilde{k}}$ are locally feedback equivalent around $(x_0, 0)$ if and only if $k = \tilde{k}$ and the ratios λ_i/λ_1 and $\tilde{\lambda}_i/\tilde{\lambda}_1$ are equal for all $2 \leq i \leq n-1$ (after permutations of x_i's and $\tilde{x}_i$'s such that the sequences $(\lambda_i/\lambda_1)_{i=1,n-1}$ and $(\tilde{\lambda}_i/\tilde{\lambda}_1)_{i=1,n-1}$ are both growing).

Observe that the normal form $\Sigma_\lambda^{0,k}$ defines a polynomial system since $k\lambda_i/\lambda_1$ are positive integers. Therefore there are only countably many feedback non-equivalent classes of systems $\Sigma_\lambda^{0,k}$, whereas in the case of Theorem 2.4 there are uncountably many feedback non-equivalent normal forms Σ_λ^1. The difference lies in the assumption that there exists or not an equilibrium at the point ξ_0 under consideration and that the systems have to be smooth around that point. Indeed, both normal forms Σ_λ^1 and $\Sigma_\lambda^{0,k}$ consist of power functions of the variable w, but to be smooth around $w = 0$, the second one needs to actually be given by monomials only.

Proof. The beginning of the proof is the same as that of Theorem 2.4 up to equation (2.3), so we start from there. We have $f = F = \sum_{i=1}^{n-1} F_i(w)\frac{\partial}{\partial x_i}$ and $g = \frac{\partial}{\partial w}$ implying that $\mathfrak{L} = \{f, g\}_{LA} = \mathrm{vect}_{\mathbb{R}}\left\{g,\, \mathrm{ad}_g^j f,\, j \geq 0\right\}$ and by $f(\xi_0) = 0$ we conclude that $\dim \mathfrak{L}(\xi_0) = n$ implies the existence of the smallest k such that $g \wedge \mathrm{ad}_g^k f(\xi_0) \neq 0$; the latter condition being feedback invariant. Differentiating relation (2.3) $k-1$ times and evaluating the result at $w = 0$ yields $\delta(0)F^{(k)}(0) + AF^{(k-1)}(0) = 0$ and thus, by definition of k, we deduce that $\delta(0) = 0$. Moreover, by derivating the same equation once more, we obtain $(A + k\delta'(0)\mathrm{Id})\, F^{(k)}(0) = 0$. Hence, $\left(-k\delta'(0), F^{(k)}(0)\right)$ is an eigenpair of A and, since A is non singular, we conclude that $\delta'(0) = \rho \neq 0$. Thus, the one-dimensional singular vector field $\delta(w)\frac{\partial}{\partial w}$ can be linearized at $w = 0$, see [18], that is, there exists a smooth local diffeomorphism $\tilde{w} = \psi(w)$ that maps $\delta(w)\frac{\partial}{\partial w}$ into $\rho\tilde{w}\frac{\partial}{\partial \tilde{w}}$. Under that transformation, equation (2.3) becomes

$$\rho\tilde{w}\tilde{F}'(\tilde{w}) - A\tilde{F}(\tilde{w}) = 0, \tag{2.4}$$

where $\tilde{F}(\tilde{w}) = F\left(\psi^{-1}(\tilde{w})\right)$. Let $P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be an invertible linear transformation such that $P^{-1}A_\rho P = J_\rho$, where $A_\rho = \frac{1}{\rho}A$ and J_ρ is the real Jordan normal form of A_ρ, whose eigenvalues are λ_i/ρ (recall that λ_i are the eigenvalues of ad_{v_0} represented by A). Set $\bar{x} = Px$, then equation (2.4) in $\bar{x}$-coordinates becomes (we drop the "tildes" for a better readability)

$$wF'(w) - J_\rho F = 0.$$

The system Σ is assumed to satisfy the accessibility rank condition $\dim \mathfrak{L}(\xi_0) = n$, so all components $F_i(w)$ of $F(w)$ are not identically zero. Therefore, by Lemma A.1 of Appendix A, we conclude that all eigenvalues of J_ρ are positive integers ($\lambda_i/\rho \neq 0$ since A is non-singular) and that $F_i(w) = c_i w^{\lambda_i/\rho}$, $\lambda_i/\rho \in \mathbb{N}$ and $c_i \in \mathbb{R}$. Using the accessibility rank condition one more time, we conclude that all eigenvalues λ_i are pair-wise distinct. Permuting the components $\bar{x}_i$'s we can assume that the smallest, among the ratios λ_i/ρ, is λ_1/ρ and, by definition of k, it follows that $k = \lambda_1/\rho$ and thus $\lambda_i/\rho = k\lambda_i/\lambda_1$. Using accessibility once again, we see that $c_i \neq 0$, so replacing $\bar{x}_i$ by $x_i = \frac{\bar{x}_i}{c_i}$ we bring the system Σ into the form $\Sigma^{0,k}_{\lambda}$. □

The previous proposition describes, around an equilibrium, the class of all $\mathcal{C}^\infty$-smooth systems having an almost abelian Lie subalgebra of symmetries and satisfying the accessibility rank condition[1]. If Σ is analytic, then its accessibility at ξ_0 is equivalent to the accessibility rank condition to hold at ξ_0. Thus, both properties imply the existence of the integer k, which is crucial for obtaining the normal form $\Sigma^{0,k}_{\lambda}$. The following example highlights that there are accessible trivial $\mathcal{C}^\infty$-smooth systems possessing an almost abelian Lie algebra of symmetries but failing to satisfy the accessibility rank condition and for which k does not exist. Therefore, such systems are not feedback equivalent to $\Sigma^{0,k}_{\lambda}$. Consider around $\xi_0 = (x_0, 0)$ the system

$$\begin{cases} \dot{x}_i &= \mathfrak{f}(w)^{\lambda_i/\lambda_1}, \quad 1 \leq i \leq n-1 \\ \dot{w} &= u \end{cases}, \quad \text{with} \quad \mathfrak{f}(w) = \exp\left(-\frac{1}{w^2}\right),\ \mathfrak{f}(0) = 0,$$

and the quotients λ_i/λ_1, for $1 \leq i \leq n-1$, are non-zero integers. By a straightforward calculation, one may check that the system possesses an almost abelian Lie subalgebra of infinitesimal symmetries $\mathfrak{K} = \mathfrak{I} \rtimes \mathrm{vect}_{\mathbb{R}}\{v_0\}$, where $\mathfrak{I} = \mathrm{vect}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right\}$ and $v_0 = \sum_{i=1}^{n-1} \frac{\lambda_i}{\lambda_1} x_i \frac{\partial}{\partial x_i} + \frac{1}{2} w^3 \frac{\partial}{\partial w}$, and that $\dim \mathfrak{L}(\xi_0) = 1$.

The above theorem and proposition generalise our previous observations on the Lie algebra of infinitesimal symmetries of null-forms of elliptic, hyperbolic, and parabolic systems for which we have $n = 3$ and, respectively, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; see [19]. Moreover, to extend Theorem 2.4 and Proposition 2.7, we give below the equivalence classes of trivial low dimensional systems possessing an almost abelian Lie subalgebra of symmetries. These results correspond to the different classes of almost abelian Lie algebra of dimension two and three. In dimension two, there is only one class of almost abelian Lie algebra given by $\mathrm{ad}_{v_0} = (1)$. We have:

Corollary 2.8 ($n = 2$)**.** *Let $\Sigma = (f, g)$ be a 2-dimensional trivialisable control-affine system around ξ_0 such that its Lie algebra of infinitesimal symmetries $\mathfrak{S}$ contains an almost abelian subalgebra $\mathfrak{K} = \mathfrak{I} \rtimes \mathrm{vect}_{\mathbb{R}}\{v_0\}$.*

(i) Under the assumptions of Theorem 2.4, the system Σ is locally feedback equivalent to the normal form, given around $(x_0, 0)$, to

$$\begin{cases} \dot{x} &= \eta(w+1) \\ \dot{w} &= u \end{cases} \quad ;\ \text{with} \ \ \mathfrak{I} = \mathrm{vect}_{\mathbb{R}}\left\{\frac{\partial}{\partial x}\right\} \ \text{and} \ \ v_0 = x\frac{\partial}{\partial x} + (w+1)\frac{\partial}{\partial w},$$

where $\eta = 0$ or $\eta = 1$.

(ii) Under the assumptions of Proposition 2.7, there exists the smallest integer $k \geq 1$ such that $g \wedge \mathrm{ad}_g^k f(\xi_0) \neq 0$ and the system Σ is locally feedback equivalent to the normal form given, around $(x_0, 0)$, by

$$\begin{cases} \dot{x} &= w^k \\ \dot{w} &= u \end{cases} \quad ;\ \text{with} \ \ \mathfrak{I} = \mathrm{vect}_{\mathbb{R}}\left\{\frac{\partial}{\partial x}\right\} \ \text{and} \ \ v_0 = x\frac{\partial}{\partial x} + \frac{w}{k}\frac{\partial}{\partial w}.$$

Notice that in both above cases the Lie algebra $\mathfrak{S}$ of all infinitesimal symmetries is of infinite dimension (if $\eta \neq 0$, then that algebra depends on one function of one variable, see [20]) and admits $\mathfrak{K}$ as a proper subalgebra. In dimension three, there are three classes of almost abelian Lie algebra given by: ad_{v_0} is diagonalisable over $\mathbb{R}$, ad_{v_0} has a double real eigenvalue and is not diagonalisable, and ad_{v_0} has a pair of complex conjugate eigenvalues. These three cases yield:

[1] Actually, the proof shows that the key ingredient to obtain the normal form $\Sigma^{0,k}_{\lambda}$ is the existence of the integer k. The proof can be adapted to show that $k \geq 1$ exists as soon as $\dim \mathfrak{L}(\xi_0) \geq 2$. Instead we assume the stronger accessibility rank condition $\dim \mathfrak{L}(\xi_0) = n$ to avoid the case of infinite-dimensional Lie algebra $\mathfrak{S}$ of all infinitesimal symmetries, which we will discuss elsewhere.

Corollary 2.9 ($n = 3$)**.** *Let $\Sigma = (f, g)$ be a 3-dimensional trivialisable control-affine system around ξ_0 such that its Lie algebra of all infinitesimal symmetries $\mathfrak{S}$ contains an almost abelian subalgebra $\mathfrak{K} = \mathfrak{I} \rtimes \mathrm{vect}_{\mathbb{R}} \{v_0\}$.*

(i) *Under the assumptions of Theorem 2.4, the system Σ is locally feedback equivalent to one of the following normal forms, given around $(x_0, y_0, 0)$, and for all of them the ideal $\mathfrak{I}$ is $\mathrm{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$.*

(a) *If ad_{v_0} is diagonalisable over $\mathbb{R}$, then we have the local normal form*

$$\begin{cases} \dot{x} &= \mathrm{e}^{w} \\ \dot{y} &= \eta \mathrm{e}^{\lambda w} \\ \dot{w} &= u \end{cases} \quad ; \textit{ with } \; v_0 = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} + \frac{\partial}{\partial w},$$

where $\lambda \neq 0$ and $\eta = 0$ or $\eta = 1$.

(b) *If ad_{v_0} has two real eigenvalues and is not diagonalisable, then we have the local normal form*

$$\begin{cases} \dot{x} &= \mathrm{e}^{w} (\eta_0 + \eta_1 w) \\ \dot{y} &= \mathrm{e}^{w} \eta_1 \\ \dot{w} &= u \end{cases} \quad ; \textit{ with } \; v_0 = (x + y)\frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{\partial}{\partial w},$$

where $(\eta_0, \eta_1) = (1, 0)$ or $(\eta_0, \eta_1) = (0, 1)$

(c) *If ad_{v_0} has two complex eigenvalues, then we have the local normal form*

$$\begin{cases} \dot{x} &= \mathrm{e}^{\lambda w} \cos(w) \\ \dot{y} &= \mathrm{e}^{\lambda w} \sin(w) \\ \dot{w} &= u \end{cases} \quad ; \textit{ with } \; v_0 = (\lambda x - y)\frac{\partial}{\partial x} + (x + \lambda y) \frac{\partial}{\partial y} + \frac{\partial}{\partial w},$$

where $\lambda \geq 0$.

(ii) *Under the assumptions of Proposition 2.7, there exists the smallest integer $k \geq 1$ such that $g \wedge \mathrm{ad}_g^k f(\xi_0) \neq 0$, the linear operator ad_{v_0} is diagonalisable over $\mathbb{R}$ and Σ is locally feedback equivalent to the following normal form, given around $(x_0, y_0, 0)$, for which $\mathfrak{I} = \mathrm{vect}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$,*

$$\begin{cases} \dot{x} &= w^{k} \\ \dot{y} &= w^{k\lambda} \\ \dot{w} &= u \end{cases} \quad ; \textit{ with } \; v_0 = kx\frac{\partial}{\partial x} + k\lambda y \frac{\partial}{\partial y} + w\frac{\partial}{\partial w},$$

where $k\lambda$ is an integer greater than k.

3. Trivial systems on 3D-manifolds

In this section, we study trivial systems on 3-dimensional manifolds with scalar control. Our aim is to give a new interpretation of the results of [9] and to extend them by giving several normal forms.

Throughout this section, we consider a control-affine system $\Sigma = (f, g)$ of the form

$$\Sigma \, : \, \dot{\xi} = f(\xi) + g(\xi)u, \quad u \in \mathbb{R},$$

where the state ξ belongs to a 3-dimensional manifold $\mathcal{M}$ and the vector fields f and g satisfy, at any $\xi \in \mathcal{M}$, the following regularity assumptions

(A1) $f \wedge g \wedge [g, f] \neq 0$,
(A2) $g \wedge [g, f] \wedge [g, [g, f]] \neq 0$.

To any control-affine system $\Sigma = (f, g)$ we attach 6 structure functions uniquely defined by the following decompositions:

$$\begin{aligned} [f,[f,g]] &= k_1 g + k_2 [g,f] + k_3 [g,[g,f]] , \\ [g,[g,f]] &= \lambda_1 f + \lambda_2 g + \lambda_3 [g,f] . \end{aligned} \tag{3.1}$$

Notice that assumption (A2) implies that $\lambda_1 \neq 0$. We now define two different classes of pairs (f, g).

Definition 3.1 ((Semi)-canonical pairs)**.** We call the pair (f, g) *semi-canonical* if $k_3 \equiv 0$, and we will denote it by (f_s, g). If, additionally $\lambda_1 \equiv \pm 1$, then we call (f, g) a *canonical* pair and we denote it by (f_c, g_c).

Observe that a semi-canonical pair is characterised by the inclusion $[f, [f, g]] \in \operatorname{span}\{g, [g, f]\}$, which is the property of the singular vector field of Σ (justifying the notation (f_s, g) for a semi-canonical pair). Under assumption (A2), that singular vector field is unique and can be computed using the singular control, thus we are not surprised that the following proposition shows that a semi-canonical pair exists (but observe that our proof does not require using the machinery of singular controls).

The following proposition shows that a control-affine system is always feedback equivalent to a system given by a semi-canonical and even by a canonical pair. Moreover, those pairs can be *explicitly* constructed, meaning that the feedback transformations bringing (f, g) into (f_s, g) or into (f_c, g_c) are constructed via differentiation and algebraic operations only (no differential equations to be solved). Furthermore, a canonical pair is *unique* (up to multiplying g_c by -1) hence its structure functions are feedback equivariants (up to a discrete involution, see Rem. 3.4 below).

Proposition 3.2 (Existence of semi-canonical and canonical pairs)**.** *Consider a control-affine system $\Sigma = (f, g)$ satisfying assumptions (A1) and (A2). Then, the following statements hold globally:*

(i) Σ is globally feedback equivalent to $\Sigma_s = (f_s, g)$, where (f_s, g) is a semi-canonical pair;
(ii) Σ is globally feedback equivalent to $\Sigma_c = (f_c, g_c)$, where (f_c, g_c) is a canonical pair.

Moreover (f_s, g) and (f_c, g_c) can be explicitly constructed via the following feedback transformations

$$f_s = f + g k_3, \quad f_c = f_s, \quad \text{and} \quad g_c = |\lambda_1|^{-1/2} g.$$

Furthermore, the canonical pair (f_c, g_c) is unique up to $g_c \mapsto -g_c$.

The proof of the above proposition is based on the following lemma, which gives some relations between the structure functions k_1, k_2, k_3, λ_1, λ_2, λ_3, and shows how they change under feedback transformations. Its proof is a straightforward computation that we detail in Appendix B.

Lemma 3.3. *Consider a control-affine system $\Sigma = (f, g)$ with structure functions (k_1, k_2, k_3) and $(\lambda_1, \lambda_2, \lambda_3)$. Then, the following relations hold:*

$$\mathrm{L}_f(\lambda_1) = -k_2\lambda_1 - \mathrm{L}_g(\lambda_1 k_3), \tag{3.2a}$$

$$\mathrm{L}_f(\lambda_2) - \lambda_3 k_1 + \mathrm{L}_g(k_1) = -k_2\lambda_2 - \mathrm{L}_g(\lambda_2 k_3), \tag{3.2b}$$

$$\mathrm{L}_f(\lambda_3) - \lambda_2 = -k_3\lambda_1 - \mathrm{L}_g(k_2) - \mathrm{L}_g(\lambda_3 k_3). \tag{3.2c}$$

Under the feedback transformation $f \mapsto \tilde{f} = f + g\alpha$ and $g \mapsto \tilde{g} = g\beta$, where α and β are smooth scalar functions satisfying $\beta \neq 0$, the structure functions are transformed by

$$\begin{aligned} \tilde{k}_1 &= k_1 + \mathrm{L}_{[g,f]}(\alpha) + \tfrac{1}{\beta}\left(\mathrm{L}_f(\gamma) + \alpha \mathrm{L}_g(\gamma) - \gamma \mathrm{L}_g(\alpha)\right) + \tilde{k}_2 \tfrac{\gamma}{\beta} \\ &\quad + \tilde{k}_3 \left(\mathrm{L}_{[g,f]}(\beta) + \mathrm{L}_g(\gamma) - \gamma \mathrm{L}_g(\ln|\beta|)\right), \\ \tilde{k}_2 &= k_2 - \mathrm{L}_f(\ln|\beta|) - \tfrac{\gamma}{\beta} - \alpha \mathrm{L}_g(\ln|\beta|) + \tilde{k}_3 \mathrm{L}_g(\beta), \\ \tilde{k}_3 &= \tfrac{1}{\beta}(k_3 - \alpha), \end{aligned} \tag{3.3}$$

$$\tilde{\lambda}_1 = \beta^2 \lambda_1, \quad \tilde{\lambda}_2 = \beta\lambda_2 - \beta\lambda_1\alpha + \gamma\lambda_3 - \mathrm{L}_{[g,f]}(\beta) - \mathrm{L}_g(\gamma) + 2\gamma \mathrm{L}_g(\ln|\beta|), \quad \tilde{\lambda}_3 = \beta\lambda_3 + \mathrm{L}_g(\beta), \tag{3.4}$$

where $\gamma = \mathrm{L}_f(\beta) + \alpha \mathrm{L}_g(\beta) - \beta \mathrm{L}_g(\alpha)$.

Proof of Proposition 3.2. Consider $\Sigma = (f, g)$ whose structure functions are (k_1, k_2, k_3) and $(\lambda_1, \lambda_2, \lambda_3)$. By equations (3.3) and (3.4) we have that under feedback transformations $\beta\tilde{k}_3 = k_3 - \alpha$ and $\tilde{\lambda}_1 = \beta^2\lambda_1$. Hence, choosing $\alpha = k_3$ we obtain that the transformed pair $(\tilde{f}, g)$, where $\tilde{f} = f + gk_3$, is semi-canonical. Moreover, additionally choosing $\beta = |\lambda_1|^{-1/2}$, recall that $\lambda_1 \neq 0$ by assumption (A2), yields a canonical pair $(f_c, g_c) = (\tilde{f}, \tilde{g})$, where $\tilde{g} = \beta g$. Clearly, the singular vector field $f_s = f_c$ is uniquely defined, but the canonical vector field g_c is unique up to $g_c \mapsto \pm g_c$. □

Observe that for the canonical pair (f_c, g_c) we additionally have $k_2 \equiv 0$, due to (3.2a). Thus, the canonical pair (f_c, g_c) satisfies the following decomposition (renaming k_1 to κ, λ_1 to ε, λ_2 to μ, λ_3 to ν)

$$\begin{aligned} [f_c, [f_c, g_c]] &= \kappa g_c, \\ [g_c, [g_c, f_c]] &= \varepsilon f_c + \mu g_c + \nu [g_c, f_c], \end{aligned} \tag{3.1'}$$

where $\varepsilon = \pm 1$. Moreover, using equations (3.2b) and (3.2c) we deduce that κ, μ, and ν are related by

$$\mathrm{L}_{f_c}(\mu) - \nu\kappa + \mathrm{L}_{g_c}(\kappa) = 0 \tag{3.2b'}$$

$$\mathrm{L}_{f_c}(\nu) - \mu = 0, \tag{3.2c'}$$

from which we deduce that the feedback invariants κ and ν are associated via

$$\mathrm{L}^2_{f_c}(\nu) - \nu\kappa + \mathrm{L}_{g_c}(\kappa) = 0. \tag{3.5}$$

Therefore, a canonical pair identifies explicitly a discrete invariant $\varepsilon = \pm 1$ and two constructible feedback invariant functions κ and ν called, respectively, the *curvature* and the *centro-affine curvature* by analogy with Serres' work [4]; see also [11]. Observe that due to (3.2c') above, μ is determined by ν. Moreover, the curvature κ determines the centro-affine curvature ν up to an affine part; *i.e.* if f_c is rectified as $\frac{\partial}{\partial x}$, then ν is determined by κ via (3.5) up to two functions ν_1 and ν_0 satisfying $\mathrm{L}_{f_c}(\nu_i) = \frac{\partial \nu_i}{\partial x} = 0$.

Remark 3.4. A canonical pair is unique up to $g_c \mapsto -g_c$. Hence the centro-affine curvature ν is a feedback equivariant up to the involution $\nu \mapsto -\nu$ (which does not influence our conditions below). We will get back to that subtlety in Proposition 4.1, where we will construct several normal forms. On the other hand, the curvature κ is a true feedback invariant (actually, a feedback equivariant that changes as $\phi^*\kappa$ under a diffeomorphism ϕ).

For a control-affine system $\Sigma_c = (f_c, g_c)$, given by a canonical pair, we will denote by $(\varepsilon, \kappa, \nu)$ the triple of invariants. Although the canonical pair can be constructed without much work, for the sake of completeness, we give the expression of $(\varepsilon, \kappa, \nu)$ for an arbitrary control-affine system. In particular, observe that our formula for the curvature κ generalises the one given in [21], p. 376, where k_3 is already normalised to 0.

Proposition 3.5 (Invariants of control-affine systems)**.** *Consider a control-affine system* $\Sigma = (f, g)$ *on a* 3*-dimensional state-space manifold, and with scalar control, and let* (k_1, k_2, k_3) *and* $(\lambda_1, \lambda_2, \lambda_3)$ *be structure functions defined by (3.1). Then, the invariants* $(\varepsilon, \kappa, \nu)$ *are given by:*

$$\varepsilon = \operatorname{sgn}(\lambda_1), \quad \kappa = k_1 + \frac{1}{2}\mathrm{L}_f(k_2 - \mathrm{L}_g(k_3)) + \frac{1}{4}(k_2 - \mathrm{L}_g(k_3))^2 + \mathrm{L}_{[g,f]}(k_3) + \frac{1}{2}k_3\mathrm{L}_g(k_2 - \mathrm{L}_g(k_3)),$$
$$\text{and} \quad \nu = |\lambda_1|^{-1/2}\left(\lambda_3 - \frac{1}{2}\mathrm{L}_g(\ln|\lambda_1|)\right). \tag{3.6}$$

Our formula for the curvature κ is, indeed, a generalisation of that in [21] because if $k_3 = 0$ (*i.e.* we suppose that f is the singular vector field f_s), then (3.6) reads

$$\kappa = k_1 + \frac{1}{2}\mathrm{L}_f(k_2) + \frac{1}{4}(k_2)^2,$$

which is exactly the formula given by Agrachev [10, 21].

Proof. Consider a pair (f, g) with structure functions (k_1, k_2, k_3) and $(\lambda_1, \lambda_2, \lambda_3)$. To deduce the expression of the invariants $(\varepsilon, \kappa, \nu)$, we apply equations (3.3) and (3.4) with $\alpha = k_3$ and $\beta = |\lambda_1|^{-1/2}$, namely the feedback transformation that constructs the canonical pair. We detail the computation for κ and left the computation for ν to the reader. Recall that, applying a feedback to construct the canonical pair, we obtain as a by-product $\tilde{k}_2 = 0$ (see the proof of Prop. 3.2). First we have,

$$\begin{aligned}\gamma &= -\frac{1}{2}|\lambda_1|^{-1/2}\mathrm{L}_f(\Lambda) - \frac{1}{2}k_3|\lambda_1|^{-1/2}\mathrm{L}_g(\Lambda) - |\lambda_1|^{-1/2}\mathrm{L}_g(k_3)\\ &= |\lambda_1|^{-1/2}\left(-\frac{1}{2}\mathrm{L}_f(\Lambda) - \frac{1}{2}k_3\mathrm{L}_g(\Lambda) - \mathrm{L}_g(k_3)\right),\end{aligned}$$

where $\Lambda = \ln|\lambda_1|$. Second, using $\tilde{k}_2 = 0$ we deduce

$$k_2 = -\frac{1}{2}\mathrm{L}_f(\Lambda) - \frac{1}{2}k_3\mathrm{L}_g(\Lambda) + |\lambda|^{1/2}\gamma = -\mathrm{L}_f(\Lambda) - k_3\mathrm{L}_g(\Lambda) - \mathrm{L}_g(k_3).$$

Therefore, inserting the last expression of k_2 into γ, we get $\gamma = \frac{1}{2}|\lambda_1|^{-1/2}(k_2 - \mathrm{L}_g(k_3))$. Now the curvature reads $\kappa = \tilde{k}_1$, *i.e*

$$\begin{aligned}\kappa &= k_1 + \mathrm{L}_{[g,f]}(k_3) + |\lambda_1|^{1/2}\left(\frac{1}{2}\mathrm{L}_f\left(|\lambda_1|^{-1/2}(k_2 - \mathrm{L}_g(k_3))\right) + \frac{1}{2}k_3\mathrm{L}_g\left(|\lambda_1|^{-1/2}(k_2 - \mathrm{L}_g(k_3))\right)\right.\\ &\qquad\left. - \frac{1}{2}|\lambda_1|^{-1/2}(k_2 - \mathrm{L}_g(k_3))\mathrm{L}_g(k_3)\right)\\ &= k_1 + \mathrm{L}_{[g,f]}(k_3) - \frac{1}{2}(k_2 - \mathrm{L}_g(k_3))\mathrm{L}_g(k_3)\\ &\quad + \frac{1}{2}|\lambda_1|^{1/2}\left(|\lambda_1|^{-1/2}\mathrm{L}_f(k_2 - \mathrm{L}_g(k_3)) - \frac{1}{2}(k_2 - \mathrm{L}_g(k_3))|\lambda_1|^{-3/2}\mathrm{L}_f(|\lambda_1|)\right)\\ &\quad + \frac{1}{2}|\lambda_1|^{1/2}k_3\left(|\lambda_1|^{-1/2}\mathrm{L}_g(k_2 - \mathrm{L}_g(k_3)) - \frac{1}{2}(k_2 - \mathrm{L}_g(k_3))|\lambda_1|^{-3/2}\mathrm{L}_g(|\lambda_1|)\right)\\ &= k_1 + \mathrm{L}_{[g,f]}(k_3) - \frac{1}{2}(k_2 - \mathrm{L}_g(k_3))\mathrm{L}_g(k_3) + \frac{1}{2}\mathrm{L}_f(k_2 - \mathrm{L}_g(k_3))\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}\left(k_2-\mathrm{L}_g\left(k_3\right)\right)\mathrm{L}_f\left(\Lambda\right)+\frac{1}{2}k_3\mathrm{L}_g\left(k_2-\mathrm{L}_g\left(k_3\right)\right)-\frac{1}{4}\left(k_2-\mathrm{L}_g\left(k_3\right)\right)k_3\mathrm{L}_g\left(\Lambda\right)\\
&=k_1+\mathrm{L}_{[g,f]}\left(k_3\right)+\frac{1}{2}\mathrm{L}_f\left(k_2-\mathrm{L}_g\left(k_3\right)\right)+\frac{1}{2}k_3\mathrm{L}_g\left(k_2-\mathrm{L}_g\left(k_3\right)\right)\\
&-\frac{1}{4}\left(k_2-\mathrm{L}_g\left(k_3\right)\right)\left(\mathrm{L}_f\left(\Lambda\right)+\mathrm{L}_g\left(\Lambda\right)+2\mathrm{L}_g\left(k_3\right)\right)\\
&=k_1+\frac{1}{2}\mathrm{L}_f\left(k_2-\mathrm{L}_g\left(k_3\right)\right)+\frac{1}{4}\left(k_2-\mathrm{L}_g\left(k_3\right)\right)^2+\mathrm{L}_{[g,f]}\left(k_3\right)+\frac{1}{2}k_3\mathrm{L}_g\left(k_2-\mathrm{L}_g\left(k_3\right)\right).
\end{aligned}
$$

□

Now consider a trivial system, whose state (x,y,w) belongs to a 3-dimensional manifold $\mathcal{M}$,

$$
(T)\,:\,\left\{\begin{array}{ll}\dot{x} & =F_1(w)\\ \dot{y} & =F_2(w)\\ \dot{w} & =u\end{array}\right.\quad,\quad(x,y,w)\in\mathcal{M},\quad u\in\mathbb{R}.
$$

Notice that (T) is, in general, not given by a canonical pair but is given by a semi-canonical pair since $[f,[f,g]]=0$. Clearly, for trivial systems we have $\kappa=0$, but the converse is not true as discovered in [9] and as we will show in the following theorem.

Theorem 3.6 (Characterisation of trivial systems)**.** *Consider a control-affine system $\Sigma=(f,g)$ together with its structure functions κ and ν. Then, Σ is locally trivialisable if and only if its canonical form $\Sigma_c=(f_c,g_c)$ satisfies*

$$
\kappa=0,\quad \mathrm{L}_{f_c}\left(\nu\right)=0,\quad \textit{and}\quad \mathrm{L}_{[f_c,g_c]}\left(\nu\right)=0. \tag{3.7}
$$

Observe that the conditions of (3.7) can explicitly be tested on the control-affine system $\Sigma=(f,g)$. Indeed, with the help of Proposition 3.2, we explicitly construct the canonical pair (f_c,g_c) of Σ for which the invariants κ and ν can be computed by algebraic operations only. Another way to test condition (3.7) on an arbitrary control-affine system $\Sigma=(f,g)$ is to compute the invariants κ and ν using (3.6) and then to evaluate (3.7) with $f_c=f+gk_3$ and $g_c=|\lambda_1|^{-1/2}g$.

Proof. We begin with necessity and suppose that Σ is trivialisable. Then, (T) is given by $f=F_1(w)\frac{\partial}{\partial x}+F_2(w)\frac{\partial}{\partial y}$ and $g=\frac{\partial}{\partial w}$ (which, *a priori*, is not a canonical pair), whose structure functions are $k_1=k_2=k_3=0$, $\lambda_1=\lambda_1(w)$, $\lambda_2=0$, and $\lambda_3=\lambda_3(w)$. In particular, observe that λ_1 and λ_3 satisfy $\mathrm{L}_f\left(\lambda_1\right)=\mathrm{L}_{[g,f]}\left(\lambda_1\right)=0$ and $\mathrm{L}_f\left(\lambda_3\right)=\mathrm{L}_{[g,f]}\left(\lambda_3\right)=0$. As in the proof of Proposition 3.2, to transform the pair (f,g) of (T) into the canonical pair (f_c,g_c) we use $\alpha=k_3=0$ and $\beta=|\lambda_1|^{-1/2}$, which therefore satisfies $\mathrm{L}_f\left(\beta\right)=0$ and $\mathrm{L}_{[g,f]}\left(\beta\right)=0$. Now, using equations (3.3) and (3.4) of Theorem 3.3, we calculate the structure functions of $(f_c,g_c)=(f,g\beta)$ which are $\tilde{\kappa}=0$, $\tilde{\varepsilon}=\pm1$, $\tilde{\mu}=0$, and $\tilde{\nu}=\tilde{\lambda}_3=\beta\lambda_3+\mathrm{L}_g\left(\beta\right)$. Hence, for the canonical pair (f_c,g_c) of (T) we have

$$
\begin{aligned}
\mathrm{L}_{f_c}\left(\tilde{\nu}\right)&=\beta\mathrm{L}_f\left(\lambda_3\right)+\mathrm{L}_f\left(\mathrm{L}_g\left(\beta\right)\right)=\beta\mathrm{L}_f\left(\lambda_3\right)+\mathrm{L}_g\left(\mathrm{L}_f\left(\beta\right)\right)-\mathrm{L}_{[g,f]}\left(\beta\right)=0,\\
\mathrm{L}_{[g_c,f_c]}\left(\tilde{\nu}\right)&=\beta\mathrm{L}_{[g,f]}\left(\tilde{\nu}\right)=\beta\left(\mathrm{L}_{[g,f]}\left(\lambda_3\right)+\mathrm{L}_{[g,f]}\left(\mathrm{L}_g\left(\beta\right)\right)\right)=\beta\left(\mathrm{L}_g\left(\mathrm{L}_{[g,f]}\left(\beta\right)\right)-\mathrm{L}_{[g,[g,f]]}\left(\beta\right)\right)\\
&=\beta\left(-\lambda_1\mathrm{L}_f\left(\beta\right)-\lambda_3\mathrm{L}_{[g,f]}\left(\beta\right)\right)=0,
\end{aligned}
$$

and the necessity of (3.7) is proved.

Now, conversely, suppose that Σ_c, given by its canonical pair (f_c,g_c), satisfies (3.7). First, due to Lemma C.1 of Appendix C, we apply a local diffeomorphism $(x,y,w)=\phi(\xi)$ that simultaneously rectifies the distribution $\mathcal{F}=\operatorname{span}\left\{f_c,[g_c,f_c]\right\}$ and the vector field g_c, that is $\phi_*\mathcal{F}=\operatorname{span}\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right\}$ and $\phi_*g_c=\frac{\partial}{\partial w}$. In those coordinates, we have $f_c=f_1\frac{\partial}{\partial x}+f_2\frac{\partial}{\partial y}$, with $f_i=f_i(x,y,w)$, and we have $\nu=\nu(w)$ since $\mathrm{L}_{f_c}\left(\nu\right)=\mathrm{L}_{[g_c,f_c]}\left(\nu\right)=0$ and

$f_c \wedge g_c \wedge [g_c, f_c] \neq 0$. Therefore, using relation (3.1') we deduce that f_c satisfies the following two equations (notice that Eq. (3.2c') together with $\mathrm{L}_{f_c}(\nu) = 0$ imply that $\mu = 0$)

$$[f_c, [f_c, g_c]] = 0 \quad \text{and} \quad [g_c, [g_c, f_c]] = \varepsilon f_c + \nu(w)\, [g_c, f_c]\,.$$

The second equation reads

$$\frac{\partial^2 f_c}{\partial w^2} = \varepsilon f_c + \nu(w) \frac{\partial f_c}{\partial w}, \tag{3.8}$$

and, interpreted as a second order linear ODE with respect to w and with parameters (x, y), admits local solutions of the form

$$f_c(x, y, w) = F_1(w) \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} + F_2(w) \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix} = F_1(w)A + F_2(w)B. \tag{3.9}$$

In (3.9), $F_1(w)$ and $F_2(w)$ are smooth fundamental solutions functions of (3.8) (*i.e.* $F_1(w_0) = 1$, $F_1'(w_0) = 0$, $F_2(w_0) = 0$, and $F_2'(w_0) = 1$) and $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$, for $i = 1, 2$, so $A = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}$ and $B = b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y}$ are smooth vector fields on $\mathbb{R}^2$ equipped with coordinates (x, y).

Using the commutativity of f_c and $[g_c, f_c]$ we deduce that

$$[F_1 A + F_2 B, F_1' A + F_2' B] = (F_1 F_2' - F_1' F_2)\, [A, B] = 0.$$

By $F_1 F_2' - F_1' F_2 \neq 0$ (since $f_c \wedge [g_c, f_c] \neq 0$), we conclude that $[A, B] = 0$ and, therefore, there exists a local diffeomorphism $\psi(x, y)$ that simultaneously rectifies A and B (seen as vector fields on $\mathbb{R}^2$). For simplicity, we still denote the new coordinates by (x, y), *i.e.* we have $\psi_* A = \frac{\partial}{\partial x}$ and $\psi_* B = \frac{\partial}{\partial y}$. In the coordinates (x, y, w), the vector fields (f_c, g_c) take the form

$$f_c = F_1(w) \frac{\partial}{\partial x} + F_2(w) \frac{\partial}{\partial y} \quad \text{and} \quad g_c = \frac{\partial}{\partial w}$$

and therefore we conclude that the system $\Sigma_c = (f_c, g_c)$ is trivial. □

Remark that in our proof we start with a canonical pair (f_c, g_c) and we render it trivial by constructing a suitable local coordinate system.

Remark 3.7. The previous theorem was first discovered by Serres in [1]. In the proof of [1], Theorem 4.3.3 (but also in [4], Thm. 4.3 and in [9], Thm. 3.4), he shows, using his notation, that $\alpha_2 = a_2(u, q_2) - q_1$ and $\frac{\partial a_2}{\partial q_2} = b(u)$ and then considers the case $\alpha_2 = a_2(u) - q_1$ and not the general case $\alpha_2 = b(u) q_2 + a_2(u) - q_1$. The proof of [1], Theorem 4.3.3, given for the case $b \equiv 0$ (which, using our notation, is equivalent to $\nu \equiv 0$), still provides an inspiring intuition to treat the general case, as done in our proof.

In the following proposition, we express the structure functions of the trivial system (T) and give two canonical forms of control-affine system that are trivialisable. Both canonical forms are expressed using the canonical pair but in different coordinate systems. For two smooth scalar functions $F(w)$ and $G(w)$, we define their Wronskian as $\mathtt{W}(F, G) = F'G - FG'$. Recall that for any control-affine system $\Sigma = (f, g)$ satisfying (A1) and (A2) we defined, via (3.1), structure functions k_1, k_2, k_3 and $\lambda_1, \lambda_2, \lambda_3$.

Proposition 3.8. *Consider a control-affine system $\Sigma = (f, g)$ and suppose that it satisfies conditions (3.7) of Theorem 3.6, i.e. Σ is trivialisable. Then, locally, the following statements hold:*

(i) Σ *admits the normal form* (T)*, that is,*

$$\Sigma^T : \begin{cases} \dot{x} &= F_1(w) \\ \dot{y} &= F_2(w) \\ \dot{w} &= u \end{cases}$$

whose structure functions are $k_1 = k_2 = k_3 = 0$ *and* $\lambda_1 = -\frac{\mathtt{W}(F_1', F_2')}{\mathtt{W}(F_1, F_2)}$, $\lambda_2 = 0$, $\lambda_3 = \frac{\mathtt{W}'(F_1, F_2)}{\mathtt{W}(F_1, F_2)}$.

(ii) Σ *admits the normal forms* $\Sigma_c^{T,1}$ *and* $\Sigma_c^{T,2}$ *given, respectively, by*

$$\Sigma_c^{T,1} : \begin{cases} \dot{x} &= F_{c,1}(w) \\ \dot{y} &= F_{c,2}(w) \\ \dot{w} &= u \end{cases} \quad \text{and} \quad \Sigma_c^{T,2} : \begin{cases} \dot{x} &= 1 + \varepsilon y u \\ \dot{y} &= (x - \nu(w) y)\, u \\ \dot{w} &= u \end{cases},$$

where $\frac{\mathtt{W}(F_{c,1}', F_{c,2}')}{\mathtt{W}(F_{c,1}, F_{c,2})} \equiv \pm 1$ *and whose invariants are* $(\varepsilon^1, \kappa^1, \nu^1) = \left(-\frac{\mathtt{W}(F_{c,1}', F_{c,2}')}{\mathtt{W}(F_{c,1}, F_{c,2})}, 0, \frac{\mathtt{W}'(F_{c,1}, F_{c,2})}{\mathtt{W}(F_{c,1}, F_{c,2})}\right)$ *and* $(\varepsilon^2, \kappa^2, \nu^2) = (\varepsilon, 0, \nu(w))$*, respectively.*

Remark 3.9. Neither the structure functions k_i nor λ_i are feedback invariant. Item Proposition *(i)* asserts that for the normal form $\Sigma^T = (f, g)$, all $k_i = 0$, so the pair (f, g) is semi-canonical (thus, actually, $f = f_s$) but, in general, it is not canonical since λ_1 is a nontrivial function. Item Proposition *(ii)* assures that, given $\Sigma^T = (f, g)$, we can always choose w_c-coordinate, as $w = \phi(w_c)$, such that $F_{c,i} = \phi^* F_i$ satisfy $\frac{\mathtt{W}(F_{c,1}', F_{c,2}')}{\mathtt{W}(F_{c,1}, F_{c,2})} = \pm 1$ and the corresponding pair (f_c, g_c), where $f_c = F_{c,1}\frac{\partial}{\partial x} + F_{c,2}\frac{\partial}{\partial y}$ and $g_c = \beta g$, with β given by $(\phi^{-1})'\beta = 1$, is canonical.

The two presented canonical forms $\Sigma_c^{T,1} = (f_c^1, g_c^1)$ and $\Sigma_c^{T,2} = (f_c^2, g_c^2)$ are somehow *dual* to each other. Indeed, both are given in terms of the canonical pair (f_c, g_c) of Σ and for $\Sigma_c^{T,1}$ we adopt coordinates for which the vector field g_c^1 is rectified, whereas for $\Sigma_c^{T,2}$ the coordinates are chosen so that f_c^2 is rectified. The two forms carry complementary informations about the control-affine system Σ. The canonical form $\Sigma_c^{T,1}$ exhibits the trivial nature of Σ, but its invariants ε and ν are not immediately visible, and the canonical form $\Sigma_c^{T,2}$ explicitly identifies the invariants $\varepsilon = \pm 1$ and ν but hides the triviality of the system. The two canonical forms show that trivial systems depend on a smooth function of one variable: for $\Sigma_c^{T,2}$ it is clearly $\nu(w)$ and for $\Sigma_c^{T,1}$ it is the function $F_{c,2}(w)$ that determines $F_{c,1}(w)$ (or, equivalently, the other way around) through the ODE $\frac{\mathtt{W}(F_{c,1}', F_{c,2}')}{\mathtt{W}(F_{c,1}, F_{c,2})} = \pm 1$.

Proof. The normal form presented in item Proposition *(i)* is a direct consequence of Theorem 2.1 and it is a straightforward computation to derive the expressions of the structure functions. To obtain the canonical form $\Sigma_c^{T,1}$ of item Proposition *(ii)*, we consider Σ^T and define $g_c = \beta g$, where $\beta = |\lambda_1|^{-1/2}$, see Proposition 3.2. We choose $w = \phi(\hat{w})$ satisfying $(\phi^{-1})'\beta = 1$. Then in the coordinates $(x, y, \hat{w})$, the system Σ^T takes the form $\Sigma_c^{T,1}$, where $F_{c,i} = \phi^* F_i$ and whose third equation reads $\dot{\hat{w}} = \hat{u}$. Finally, the canonical form $\Sigma_c^{T,2}$ is a special case of item Proposition *(i)* of Proposition 4.1 presented in the next section. □

Let us shed a new light on the result of Theorem 3.6. Recall that any system Σ satisfying (A1) and (A2) admits a canonical pair (f_c, g_c). On the other hand, Σ is trivialisable if and only if it can be brought to the normal form $\Sigma_c^{T,2}$ for which $(f_c^{T,2}, g_c^{T,2})$ forms also a canonical pair. Therefore, Σ is trivialisable if and only if its canonical pair (f_c, g_c) is equivalent, via a diffeomorphism ϕ, to $(f_c^{T,2}, g_c^{T,2})$, that is, $\phi_* f_c = f_c^{T,2}$ and $\phi_* g_c = g_c^{T,2}$. The equivalence of control systems via diffeomorphisms was solved in the C^∞-category in [5], Theorem 2.4 and we will use this result to give another proof of a slightly weaker version of Theorem 3.6.

Proposition 3.10. *The system* $\Sigma_c = (f_c, g_c)$ *is locally trivialisable, with its normal form* $\Sigma_c^{T,2}$ *satisfying* $\nu^{(j)}(0) = 0$*, for* $1 \le j \le \rho - 1$*, and* $\nu^{(\rho)}(0) \neq 0$*, if and only if* Σ_c *satisfies (3.7) and, moreover,*

$$\mathrm{L}^j_{g_c}(\nu)(0) = 0, \ \ \textit{for } 1 \le j \le \rho - 1, \ \ \textit{and } \mathrm{L}^\rho_{g_c}(\nu)(0) \neq 0. \tag{3.10}$$

Remark, that the above proposition, based on the general result of [5], is slighly weaker than Theorem 3.6. Indeed, it assumes that the structure function ν has a finite order, whereas in Theorem 3.6 we allow ν to be any smooth function; in particular, ν may be a smooth function flat at 0, *i.e.* $\nu^{(j)}(0) = 0$, for all $j \ge 0$.

Proof. Necessity of (3.7) is shown in the proof of Theorem 3.6, while necessity of conditions (3.10) on the derivatives of ν is obvious.

To prove sufficiency we will use [5], Theorem 2.4. For $\Sigma_c = (f_c, g_c)$ satisfying (3.7) and (3.10) we choose, due to Lemma C.1 of Appendix C, coordinates (x, y, w) such that $g_c = \frac{\partial}{\partial w}$ and $\mathrm{span}\{f_c, [f_c, g_c]\} = \mathrm{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$. To apply the result of [5], we set $f_0 = f_c$, $f_1 = g$, $f_2 = [f_0, f_1]$ and define structure functions c^k_{ij} via

$$[f_i, f_j] = \sum_{k=0}^{2} c^k_{ij} f_k, \quad 1 \le i, j \le 2.$$

We denote the successive derivatives of c^k_{ij} by $(c^k_{ij})^{i_1,\ldots,i_q} = \mathrm{L}_{f_{i_q}}\left(\cdots \mathrm{L}_{f_{i_1}}\left(c^k_{ij}\right)\cdots\right)$, where $q \ge 0$ and $0 \le i_j \le 2$. For Σ_c satisfying (3.7) we have $c^1_{02} = 1$, $c^0_{12} = -\varepsilon$, $c^2_{12} = \nu(w)$, and all the other structure functions vanish. Hence, $(c^2_{12})^{i_1,\ldots,i_q} = \mathrm{L}^q_{f_1}\left(c^2_{12}\right) = \frac{\mathrm{d}^q \nu}{\mathrm{d}w^q}$ and thus the rank of the family $\mathcal{C} = \left\{(c^k_{ij})^{i_1,\ldots,i_q},\ 0 \le i, j \le 2,\ q \ge 0\right\}$ is $r = 1$ and its order is ρ (see [5] for the definition of the rank and order of a family of smooth functions).

On the other hand, consider the normal form $\tilde{\Sigma} = \Sigma_c^{T,2}$ given by $\tilde{f}_0 = f_c^{T,2} = \frac{\partial}{\partial \tilde{x}}$ and $\tilde{f}_1 = g_c^{T,2} = \tilde{\varepsilon}\tilde{y}\frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{\nu}(\tilde{w})\tilde{y})\frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{w}}$, where we define $\tilde{\varepsilon} = \varepsilon$ and $\tilde{\nu}(\tilde{w}) = \nu(\tilde{w})$, that is, the function ν of Σ_c applied to the argument $\tilde{w}$. By a straightforward calculation, we conclude that the family $\tilde{\mathcal{C}} = \left\{(\tilde{c}^k_{ij})^{i_1,\ldots,i_q},\ 0 \le i, j \le 2,\ q \ge 0\right\}$, where the structure functions $\tilde{c}^k_{ij}$ are defined as above, is of rank $\tilde{r} = 1$ and order ρ (notice that $\mathrm{L}^q_{\tilde{f}_1}(\tilde{\nu}) = \frac{\mathrm{d}^q \tilde{\nu}}{\mathrm{d}\tilde{w}^q}$ since $\tilde{\nu}$ depends on $\tilde{w}$ only and the component of $\tilde{f}_1$ along $\frac{\partial}{\partial \tilde{w}}$ is one).

Now observe that Σ_c and $\tilde{\Sigma}$ satisfy $r = \tilde{r} = 1$, $\rho = \tilde{\rho}$, and the identity diffeomorphism $(\tilde{x}, \tilde{y}, \tilde{w}) = \psi(x, y, w) = (x, y, w)$ maps c^k_{ij} into $\tilde{c}^k_{ij}$ and $(c^k_{ij})^{i_1,\ldots,i_q}$ into $(\tilde{c}^k_{ij})^{i_1,\ldots,i_q}$, for $1 \le q \le \rho + 1$; the latter is obvious because $\tilde{w} = w$ and, according to our definition of $\tilde{\Sigma}$, $\tilde{\nu}(\tilde{w}) = \nu(w)$. Therefore, it follows by [5], Theorem 2.4 that there exists a local diffeomorphism ϕ such that $\phi_* f_c = f_c^{T,2}$ and $\phi_* g_c = g_c^{T,2}$, thus proving that Σ is locally trivialisable. ☐

Remark 3.11. Notice that the third components of ϕ (that conjugates the systems) and ψ (that conjugates the structure functions) coincide and are $w = \tilde{w}$ while the two first components of ϕ establish the equivalence of (f_c, g_c) with $(f_c^{T,2}, g_c^{T,2})$ and have nothing to do with the first two components of ψ, which have been chosen as the identity but can be taken arbitrarily since $\nu = \nu(w)$. It is a consequence of $r = \tilde{r} = 1$, see [5], Proposition 2.4 and [22], Remark 3.

4. Normal forms of flat and centro-flat control-affine systems on 3D-manifolds

We have shown that the curvature κ and the centro-affine curvature ν are two functional feedback equivariants of control-affine systems, hence, their properties define non-equivalent classes of systems. In this section, we propose a normal form for each class of control-systems that is presented in Table 2 below. The presented classes describe all the cases for which the curvature κ and the centro-affine curvature ν satisfy $\kappa\nu \equiv 0$ together with the particular sub-cases for which, additionally, either κ or ν is constant.

TABLE 2. Nomenclature of subclasses of flat and centro-flat control-affine systems.

Notation	Name	Properties
$\Sigma^{\varepsilon,\kappa=0,\nu}$	Flat	Curvature κ vanishes
$\Sigma^{\varepsilon,\kappa,\nu=0}$	Centro-flat	Centro-affine curvature ν vanishes
$\Sigma^{\varepsilon,\kappa'=0,\nu=0}$	Flat-constant	Curvature κ is constant and the centro-affine curvature ν vanishes
$\Sigma^{\varepsilon,\kappa=0,\nu'=0}$	Centro-flat-constant	Curvature κ vanishes, and the centro-affine curvature ν is constant
$\Sigma^{\varepsilon,\kappa=0,\nu=0}$	Completely flat	Curvatures κ and ν vanish

Each class of control system presented in the above table is denoted by an upper index I given by three components and defined as follows. The first element is always $\varepsilon = \pm 1$ to emphasize the dependence of the normal forms on the invariant ε; the second element is either $\kappa = 0$ to say that the curvature vanishes or $\kappa' = 0$ to express that the curvature is constant (this notation is a bit abusive because κ is not a function of one variable in general); finally, the third index is either $\nu = 0$ or $\nu' = 0$ with the same interpretation as previously. The following proposition provides a normal form Σ_c^I for each class of control-affine systems Σ^I, where the upper multi-index I is one of the five given in Table 2. The lower index c indicates that all normal forms Σ_c^I are expressed using their canonical pairs. Recall that the structure function ν is unique up to its sign, *i.e.* changing $g_c \mapsto -g_c$ yields $\nu \mapsto -\nu$, hence in normal forms below we suppose that $\nu \geq 0$. All normal forms below are given around $0 \in \mathbb{R}^3$ for better readability, but the same result holds around arbitrary point $(x_0, y_0, w_0) \in \mathbb{R}^3$.

Proposition 4.1 (Normal forms of flat control-affine systems)**.** *Consider a control-affine system $\Sigma = (f, g)$ together with its invariants ε, κ, and ν. Then, the following statements hold locally (all normal forms below are represented by a canonical pair (f_c, g_c) and considered around $0 \in \mathbb{R}^3$).*

(i) If $\kappa = 0$, then Σ is locally feedback equivalent to

$$\Sigma_c^{\varepsilon,\kappa=0,\nu} : \begin{cases} \dot{x} &= 1 + \quad a(y,w)u \\ \dot{y} &= \quad (x + b(y,w))\, u, \\ \dot{w} &= \quad c(y,w)u \end{cases}$$

whose invariants are ε, $\kappa = 0$, and $\nu = \nu_1(y,w)x + \nu_0(y,w)$, and the functions a, b and c satisfy the following differential equations

$$\frac{\partial a}{\partial y} = \varepsilon + \nu_1(y,w)a(y,w), \qquad a(0,w) = 0,$$
$$\frac{\partial b}{\partial y} = \nu_1(y,w)b(y,w) - \nu_0(y,w), \qquad b(0,w) = 0,$$
$$\frac{\partial c}{\partial y} = \nu_1(y,w)c(y,w), \qquad c(0,w) = 1,$$

and thus are given by

$$a(y,w) = \left[\varepsilon \int_0^y \exp\left(-\int_0^\tau \nu_1(t,w)\mathrm{d}t\right)\mathrm{d}\tau\right] \exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right),$$
$$b(y,w) = \left[-\int_0^y \nu_0(\tau,w)\exp\left(-\int_0^\tau \nu_1(t,w)\mathrm{d}t\right)\mathrm{d}\tau\right] \exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right),$$
$$c(y,w) = \exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right).$$

(ii) If $\nu = 0$, then Σ is locally feedback equivalent to

$$\Sigma_c^{\varepsilon,\kappa,\nu=0} : \begin{cases} \dot{x} &= r(x,y)\, c_\varepsilon(w) \\ \dot{y} &= r(x,y)\, s_\varepsilon(w) \\ \dot{w} &= \varepsilon \frac{\partial r}{\partial y}\, c_\varepsilon(w) + \frac{\partial r}{\partial x}\, s_\varepsilon(w) + u \end{cases},$$

where

$$c_\varepsilon(w) = \frac{\mathrm{e}^{w\sqrt{\varepsilon}} + \mathrm{e}^{-w\sqrt{\varepsilon}}}{2} \quad \text{and} \quad s_\varepsilon(w) = \frac{\mathrm{e}^{w\sqrt{\varepsilon}} - \mathrm{e}^{-w\sqrt{\varepsilon}}}{2\sqrt{\varepsilon}},$$

whose invariants are ε, $\kappa = \kappa(x,y)$, $\nu = 0$, and the function $r(x,y)$ satisfies $r > 0$ and the following non-linear partial differential equation

$$-r(x,y)^2 \left(\frac{\partial^2}{\partial x^2} - \varepsilon \frac{\partial^2}{\partial y^2} \right) (\ln r(x,y)) = \kappa(x,y). \tag{4.1}$$

(iii) If κ and ν are constant, then

$$\kappa\nu = 0, \tag{4.2}$$

i.e. at least one of them vanishes.

(iv) If $\kappa = 0$ and ν is constant, then Σ is locally feedback equivalent to

(a) If $\varepsilon = 1$, then

$$\Sigma_c^{+,\kappa=0,\nu'=0} : \begin{cases} \dot{x} &= \mathrm{e}^{\nu w}\mathrm{e}^{w\sqrt{\nu^2+4}} \\ \dot{y} &= \mathrm{e}^{\nu w}\mathrm{e}^{-w\sqrt{\nu^2+4}} \\ \dot{w} &= \frac{1}{2}u \end{cases}, \quad \text{where} \quad \nu \geq 0.$$

(b) If $\varepsilon = -1$ and $\nu > 2$, then

$$\Sigma_c^{-,\kappa=0,\nu'=0,+} : \begin{cases} \dot{x} &= \mathrm{e}^{\nu w}\mathrm{e}^{w\sqrt{\nu^2-4}} \\ \dot{y} &= \mathrm{e}^{\nu w}\mathrm{e}^{-w\sqrt{\nu^2-4}} \\ \dot{w} &= \frac{1}{2}u \end{cases}.$$

(c) If $\varepsilon = -1$ and $\nu = 2$, then

$$\Sigma_c^{-,\kappa=0,\nu'=0,0} : \begin{cases} \dot{x} &= \mathrm{e}^{w} \\ \dot{y} &= w\mathrm{e}^{w} \\ \dot{w} &= u \end{cases}.$$

(d) If $\varepsilon = -1$ and $0 \leq \nu < 2$, then

$$\Sigma_c^{-,\kappa=0,\nu'=0,-} : \begin{cases} \dot{x} &= \mathrm{e}^{\nu w}\cos\left(w\sqrt{4-\nu^2}\right) \\ \dot{y} &= \mathrm{e}^{\nu w}\sin\left(w\sqrt{4-\nu^2}\right) \\ \dot{w} &= \frac{1}{2}u \end{cases}.$$

Moreover, for the four normal forms above, ε, $\kappa = 0$, and ν are complete invariants.

(v) If $\nu = 0$ and κ is constant, then Σ is locally feedback equivalent to

$$\Sigma_c^{\varepsilon,\kappa'=0,\nu=0} : \begin{cases} \dot{x} &= \left(1 - \frac{\kappa}{4}\left(x^2 - \varepsilon y^2\right)\right) c_\varepsilon(w) \\ \dot{y} &= \left(1 - \frac{\kappa}{4}\left(x^2 - \varepsilon y^2\right)\right) s_\varepsilon(w) \\ \dot{w} &= \frac{-\kappa}{2}\left(y\, c_\varepsilon(w) - x\, s_\varepsilon(w)\right) + u \end{cases},$$

whose complete invariants are ε, κ, and $\nu = 0$.

(vi) If $\kappa = 0$ and $\nu = 0$, then Σ is locally feedback equivalent to

$$\Sigma_c^{\varepsilon,\kappa=0,\nu=0} : \begin{cases} \dot{x} &= c_\varepsilon(w) \\ \dot{y} &= s_\varepsilon(w) \\ \dot{w} &= u \end{cases}.$$

Before presenting a proof for those normal forms, we give some remarks about them. For item Proposition *(i)* we adopt coordinates, where the vector field f_c is rectified, whereas for the other normal forms we choose coordinates in which the vector field g_c is rectified. The first normal form $\Sigma_c^{\varepsilon,\kappa=0,\nu}$ of flat control-affine systems describes the most general form of a system for which the curvature κ vanishes. On the other hand, the normal form $\Sigma_c^{\varepsilon,\kappa,\nu=0}$ of item Proposition *(ii)*, describes systems for which the centro-affine curvature ν vanishes. All other items are then special cases of those two general normal forms.

Recall that ν is unique up to its sign, that is why in item Proposition *(iv)* we have $\nu \geq 0$ for Proposition *(iv)*-Proposition *(a)* to Proposition *(iv)*-Proposition *(d)*. It is remarkable that if κ and ν are constant (hence true invariants) then at least one of them is zero as asserted in item Proposition *(iii)*. Moreover, relation (4.2) already appeared in [11], where the four families of normal forms given by $\kappa = 0$ and ν constant were listed (but the non-invariance of the sign of ν was not discussed there). The two normal forms of Proposition *(vi)* with $\varepsilon = \pm 1$ and $\kappa = \nu = 0$ are , respectively, given by

$$\Sigma_c^{+,\kappa=0,\nu=0} : \begin{cases} \dot{x} &= \cosh(w) \\ \dot{y} &= \sinh(w) \\ \dot{w} &= u \end{cases} \quad \text{and} \quad \Sigma_c^{-,\kappa=0,\nu=0} : \begin{cases} \dot{x} &= \cos(w) \\ \dot{y} &= \sin(w) \\ \dot{w} &= u \end{cases}$$

correspond to hyperbolic and elliptic systems without parameters and have been extensively analysed and differently characterised in [7, 19].

Proof. For each item, we consider a control-affine system $\Sigma_c = (f_c, g_c)$ given by the canonical pair and with $\varepsilon = \pm 1$ and structure functions κ and ν.

(i) Since $\kappa = 0$, using relation (3.1'), we conclude that the vector fields f_c and $[f_c, g_c]$ are commuting. Therefore, we can rectify them simultaneously to get $f_c = \frac{\partial}{\partial x}$ and $[f_c, g_c] = \frac{\partial}{\partial y}$. Afterwards, we determine the form of the vector field g_c. First, it satisfies $\left[\frac{\partial}{\partial x}, g_c\right] = \frac{\partial}{\partial y}$ and thus we immediately conclude

$$g_c = a(y,w)\frac{\partial}{\partial x} + (x + b(y,w))\frac{\partial}{\partial y} + c(y,w)\frac{\partial}{\partial w}. \tag{4.3}$$

Moreover, assumption (A1) implies that $c \neq 0$. Second, for g_c we have $\left[g_c, -\frac{\partial}{\partial y}\right] = \varepsilon\frac{\partial}{\partial x} + \mu g_c - \nu\frac{\partial}{\partial y}$, where the functions μ and ν satisfy (3.2b') and (3.2c') and therefore $\nu = \nu_1(y,w)x + \nu_0(y,w)$ and $\mu = \nu_1(y,w)$. Hence, the functions a, b, and c of g_c satisfy

$$\frac{\partial a}{\partial y}(y,w) = \varepsilon + \nu_1(y,w)a(y,w),$$

$$\frac{\partial b}{\partial y}(y,w) = \nu_1(y,w)b(y,w) - \nu_0(y,w),$$
$$\frac{\partial c}{\partial y}(y,w) = \nu_1(y,w)c(y,w).$$

Solutions of those equations are, respectively,

$$a(y,w) = \left[\varepsilon \int_0^y \exp\left(-\int_0^\tau \nu_1(t,w)\mathrm{d}t\right)\mathrm{d}\tau + A(w)\right]\exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right), \tag{4.4a}$$
$$b(y,w) = \left[-\int_0^y \nu_0(\tau,w)\exp\left(-\int_0^\tau \nu_1(t,w)\mathrm{d}t\right)\mathrm{d}\tau + B(w)\right]\exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right), \tag{4.4b}$$
$$c(y,w) = C(w)\exp\left(\int_0^y \nu_1(\tau,w)\mathrm{d}\tau\right). \tag{4.4c}$$

Updating the coordinates, we can set $C(w) = 1$, and $A(w) = B(w) = 0$; see Appendix D. In those coordinates we obtain the normal form $\Sigma_c^{\varepsilon,\kappa=0,\nu}$.

(ii) Suppose that $\nu = 0$ and choose coordinates $(\bar{x},\bar{y},\bar{w})$ such that $g_c = \frac{\partial}{\partial \bar{w}}$. Then, by relation (3.1') we conclude that

$$f_c = \bar{A}(\bar{x},\bar{y})c_\varepsilon(\bar{w}) + \bar{B}(\bar{x},\bar{y})s_\varepsilon(\bar{w}), \quad c_\varepsilon(\bar{w}) = \frac{\mathrm{e}^{\bar{w}\sqrt{\varepsilon}} + \mathrm{e}^{-\bar{w}\sqrt{\varepsilon}}}{2}, \quad s_\varepsilon(\bar{w}) = \frac{\mathrm{e}^{\bar{w}\sqrt{\varepsilon}} - \mathrm{e}^{-\bar{w}\sqrt{\varepsilon}}}{2\sqrt{\varepsilon}},$$

where $\bar{A} = a_1\frac{\partial}{\partial \bar{x}} + a_2\frac{\partial}{\partial \bar{y}} + a_3\frac{\partial}{\partial \bar{w}}$, with $a_i = a_i(\bar{x},\bar{y})$, and $\bar{B} = b_1\frac{\partial}{\partial \bar{x}} + b_2\frac{\partial}{\partial \bar{y}} + b_3\frac{\partial}{\partial \bar{w}}$, with $b_i = b_i(\bar{x},\bar{y})$, being smooth vector fields. By assumption (A1), we conclude that $a_1b_2 - a_2b_1 \neq 0$, hence $\bar{\bar{A}} = a_1\frac{\partial}{\partial \bar{x}} + a_2\frac{\partial}{\partial \bar{y}}$ and $\bar{\bar{B}} = b_1\frac{\partial}{\partial \bar{x}} + b_2\frac{\partial}{\partial \bar{y}}$ form a moving frame of the tangent bundle of $\mathcal{X} = \mathcal{O}/\mathcal{G}$, where $\mathcal{G} = \operatorname{span}\left\{\frac{\partial}{\partial \bar{w}}\right\}$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^3$, in which the rectifying coordinates $(\bar{x},\bar{y},\bar{w})$ are defined. We define a metric $\bar{\bar{\mathsf{g}}}$ on $\mathcal{X}$ by declaring $(\bar{\bar{A}},\bar{\bar{B}})$ orthonormal, *i.e.*

$$\bar{\bar{\mathsf{g}}}(\bar{\bar{A}},\bar{\bar{A}}) = 1, \quad \bar{\bar{\mathsf{g}}}(\bar{\bar{A}},\bar{\bar{B}}) = 0, \quad \text{and} \quad \bar{\bar{\mathsf{g}}}(\bar{\bar{B}},\bar{\bar{B}}) = -\varepsilon.$$

Notice that the signature of $\bar{\bar{\mathsf{g}}}$ is $(+,-\mathrm{sgn}\,(\varepsilon))$, hence $\bar{\bar{\mathsf{g}}}$ is definite for $\varepsilon = -1$ and indefinite for $\varepsilon = 1$. Since all metrics on 2-dimensional manifolds are locally conformally flat, we conclude that there exists an isometry $(x,y) = \psi(\bar{x},\bar{y})$ such that $\bar{\bar{\mathsf{g}}} = \psi^*\mathsf{g}$, where $\mathsf{g} = \varrho(x,y)(\mathrm{d}x^2 - \varepsilon\mathrm{d}y^2)$, with $\varrho > 0$. In the coordinates (x,y), both the pair $(\tilde{A},\tilde{B})$, with $\tilde{A} = \psi_*\bar{\bar{A}}$ and $\tilde{B} = \psi_*\bar{\bar{B}}$, and $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$ form an orthornormal frame for g so we have

$$(\tilde{A},\tilde{B}) = r(x,y)I(x,y)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right),$$

where $r = \frac{1}{\sqrt{\varrho}}$ and $I(x,y)$ is a a linear isometry, *i.e.* it belongs to the (pseudo)-orthonormal group $O(1,-\varepsilon)$. Using a suitable change of the variable $w = \bar{w} + h(x,y)$ we can get rid of $I(x,y)$. Finally, in the coordinates (x,y,w), the vector field f_c of the control system takes the form

$$f_c = r(x,y)c_\varepsilon(w)\frac{\partial}{\partial x} + r(x,y)s_\varepsilon(w)\frac{\partial}{\partial y} + \left(a(x,y)c_\varepsilon(w) + b(x,y)s_\varepsilon(w)\right)\frac{\partial}{\partial w}.$$

We now use the structure equations (3.1') and deduce that necessarily

$$a(x,y) = \varepsilon \frac{\partial r}{\partial y} \quad \text{and} \quad b(x,y) = \frac{\partial r}{\partial x}$$

and that r satisfies

$$\varepsilon r \frac{\partial^2 r}{\partial y^2} - \varepsilon \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial x}\right)^2 - r\frac{\partial^2 r}{\partial x^2} = \kappa(x,y).$$

which can be expressed in the form of (4.1).

(iii) If κ and ν are constants, then due to relation (3.5), we immediately conclude (4.2).

(iv) Suppose that $\kappa = 0$ and ν is constant, then Σ satisfies condition (3.7) of Theorem 3.6 and thus Σ is locally trivialisable. Using the results of item Proposition *(ii)* of Proposition 3.8, we bring Σ in the form of $\Sigma_c^{T,1}$ for which $f_c = F_{c,1}(w)\frac{\partial}{\partial x} + F_{c,2}(w)\frac{\partial}{\partial y}$ and $g_c = \frac{\partial}{\partial w}$ form a canonical pair. Using (3.1'), we conclude that the functions $F_{c,i}$, for $i = 1, 2$, satisfy the following second order linear ordinary differential equation

$$F''_{c,i}(w) = \varepsilon F_{c,i}(w) + \nu F'_{c,i}(w). \tag{4.5}$$

Solutions are dictated by the sign of the discriminant $\Delta = \nu^2 + 4\varepsilon$ of the characteristic polynomial of the ODE. Moreover, the roots of the characteristic polynomial are $r_{1/2} = \frac{\nu \pm \sqrt{\Delta}}{2}$. Recall that the sign of ν is not invariant and thus by choosing w suitably we can always get $\nu \geq 0$. Moreover, it is a trivial calculation to check that the solutions given below are fundamental solutions of (4.5), *i.e.* we just need to compute the Wronskian at w_0.

(a) If $\varepsilon = +1$, then $\Delta > 0$ for all $\nu \geq 0$. Solutions of (4.5) are given by (after normalising w with $\frac{1}{2}$)

$$F_{c,1}(w) = \mathrm{e}^{\nu w}\mathrm{e}^{w\sqrt{\nu^2+4}} \quad \text{and} \quad F_{c,2}(w) = \mathrm{e}^{\nu w}\mathrm{e}^{-w\sqrt{\nu^2+4}},$$

and we obtain the normal form $\Sigma_c^{+,\kappa=0,\nu'=0}$.

(b) If $\varepsilon = -1$ and $\nu > 2$, then $\Delta > 0$ and solving (4.5) gives

$$F_{c,1}(w) = \mathrm{e}^{\nu w}\mathrm{e}^{w\sqrt{\nu^2-4}} \quad \text{and} \quad F_{c,2}(w) = \mathrm{e}^{\nu w}\mathrm{e}^{-w\sqrt{\nu^2-4}},$$

and we obtain the normal form $\Sigma_c^{-,\kappa=0,\nu'=0,+}$.

(c) If $\varepsilon = -1$ and $\nu = 2$, then $\Delta = 0$ and the solutions of (4.5) are

$$F_{c,1}(w) = w\mathrm{e}^{w} \quad \text{and} \quad F_{c,2}(w) = \mathrm{e}^{w},$$

which gives $\Sigma_c^{-,\kappa=0,\nu'=0,0}$.

(d) If $\varepsilon = -1$ and $0 \leq \nu < 2$, then $\Delta < 0$, and the solutions of (4.5) are

$$F_{c,1}(w) = \mathrm{e}^{\nu w}\cos\left(w\sqrt{4-\nu^2}\right) \quad \text{and} \quad F_{c,2}(w) = \mathrm{e}^{\nu w}\sin\left(w\sqrt{4-\nu^2}\right),$$

which gives $\Sigma_c^{-,\kappa=0,\nu'=0,-}$.

(v) Assume that κ is constant and $\nu = 0$, then we refine the normal form $\Sigma_c^{\varepsilon,\kappa,\nu=0}$ of item Proposition *(ii)*. We recognize that equation (4.1) satisfied by $r(x,y)$ describes the curvature (in the usual differential geometry sense) of the metric $\mathsf{g} = \frac{1}{r^2}(\mathrm{d}x^2 - \varepsilon\mathrm{d}y^2)$. By assumption, the curvature of g is constant (equal to $-\kappa$) and

by Minding's theorem, surfaces with the same constant curvature are locally isometric. Therefore, there exists an isometry $(\tilde{x},\tilde{y})=\psi(x,y)$ such that $\mathsf{g}=\psi^*\tilde{\mathsf{g}}$ with

$$\tilde{\mathsf{g}}=\left(\frac{1}{1-\frac{\kappa}{4}\left(\tilde{x}^2-\varepsilon\tilde{y}^2\right)}\right)^2\left(\mathrm{d}\tilde{x}^2-\varepsilon\mathrm{d}\tilde{y}^2\right),$$

which is also of curvature $-\kappa$. The action of the isometry on $(\dot{x},\dot{y})$ can be compensated by applying $w\mapsto w+h(x,y)$, for a suitable function h, thus we obtain that the system takes the form of $\Sigma_c^{\varepsilon,\kappa,\nu=0}$ with $r(x,y)=1-\frac{\kappa}{4}\left(x^2-\varepsilon y^2\right)$, *i.e.* we get $\Sigma_c^{\varepsilon,\kappa'=0,\nu=0}$.

(vi) The normal form $\Sigma_c^{\varepsilon,\kappa=0,\nu=0}$ is a special case of item *(v)* with $\kappa=0$.

□

5. Conclusions and perspectives

In this paper, we have analysed in detail the notion of triviality adapted to the context of control-affine systems. We proposed two new characterisations of trivial control-affine system, one of them is based on the existence of an abelian subalgebra of the Lie algebra of infinitesimal symmetries. In particular, we gave a normal form of trivial control-affine systems for which the Lie algebra of infinitesimal symmetries has a transitive almost abelian Lie subalgebra. In the future, we will be interested in extending our result to the case of multi-input systems and we will try to propose other characterisation of triviality that are purely geometric. In the second part of the paper, we have revisited results due to Serres [1] and we give a novel proof of his characterisation of trivial systems on 3-dimensional manifolds with scalar inputs. In particular, our characterisation uses a discrete invariant $\varepsilon=\pm1$ and two well-defined functional invariants of feedback transformations: the curvature κ (introduced by Agrachev and Gamkrelidze [3]) and the centro-affine curvature ν (studied by Wilkens [11]). We show that those invariants can explicitly be computed for any control-affine system and that a canonical pair of vector fields (f_c,g_c), for which κ and ν appear explicitly, can also be constructed with a purely algebraically defined feedback transformation. Then, we extended the results of Serres and Wilkens by giving several normal forms of control-affine systems. In the future, our goal is two-folds: first we will be interested in the question of how to use the triple (ε,κ,ν) in order to express a set of *complete invariants* of control-affine systems (on 3D manifolds with scalar control). Identifying a set of complete invariants would be helpful in obtaining normal forms of control-affine system in dimension three. Second, we will be interested in generalising our characterisation of trivial control-affine systems to the multi-input case, in particular the notion of curvature of dynamics pairs, as proposed in [23], seems promising.

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Data availability statement

No new data and code were created or analyzed in this study

References

[1] U. Serres, *Geometry and Feedback Classification of Low-Dimensional Non-Linear Control Systems.* PhD thesis, Université de Bourgogne (2006).

[2] M. Fliess, J. Levine, P. Martin and P. Rouchon, A Lie–Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automatic Control* **44** (1999) 922–937.

[3] A.A. Agrachev and R.V. Gamkrelidze, Feedback-invariant optimal control theory and differential geometry—I. Regular extremals. *J. Dyn. Control Systems* **3** (1997) 343–389.

[4] U. Serres, On curvature and feedback classification of two-dimensional optimal control systems. *J. Math. Sci.* **144** (2007) 3841–3847.

[5] B. Jakubczyk, Equivalence and invariants of nonlinear control systems, in *Nonlinear Controllability and Optimal Control*, vol. 133, edited by H.J. Sussmann, editor. Marcel Dekker, New York (1990) 177–218.

[6] T. Schmoderer, *Study of Control Systems under Quadratic Nonholonomic Constraints. Motion Planning, Introduction to the Regularised Continuation Method.* PhD thesis, INSA Rouen Normandie (2022).

[7] T. Schmoderer and W. Respondek, Conic nonholonomic constraints on surfaces and control systems. *J. Dyn. Control Syst.* **29** (2023) 1981–2022.

[8] L.E. Dubins, On Curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *Am. J. Math.* **79** (1957) 497.

[9] U. Serres, Control systems of zero curvature are not necessarily trivializable. arXiv:0902.2332 [math] (2009).

[10] A.A. Agrachev, Feedback-invariant optimal control theory and differentiaL geometry, II. Jacobi curves for singular extremals. *J. Dyn. Control Syst.* **4** (1998) 583–604.

[11] G.R. Wilkens, Centro-affine geometry in the plane and feedback invariants of two-state scalar control systems, in *Proceedings of Symposia in Pure Mathematics*, vol. 64, edited by G. Ferreyra, R. Gardner, H. Hermes and H. Sussmann. American Mathematical Society, Providence, Rhode Island (1998) 319–333.

[12] A. Isidori, Nonlinear Control Systems. Communications and Control Engineering, Nonlinear Control Systems, 3rd edn. Springer-Verlag, London (1995).

[13] J. Grizzle and S. Marcus, The structure of nonlinear control systems possessing symmetries. *IEEE Trans. Automatic Control* **30** (1985) 248–258.

[14] W. Respondek and I.A. Tall, Nonlinearizable single-input control systems do not admit stationary symmetries. *Syst. Control Lett.* **46** (2002) 1–16.

[15] D. Burde and M. Ceballos, Abelian ideals of maximal dimension for solvable Lie algebras *J. Lie Theory* **22** (2012) 741–756.

[16] Z. Avetisyan, The structure of almost Abelian Lie algebras. *Int. J. Math.* **33** (2022) 2250057.

[17] W. Magnus, On the exponential solution of differential equations for a linear operator. *Commun. Pure Appl. Math.* **7** (1954) 649–673.

[18] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, 2nd edn. Grundlehren der mathematischen Wissenschaften. Springer (1988).

[19] T. Schmoderer and W. Respondek, Null-forms of conic systems in $\mathbb{R}^3$ are determined by their symmetries. *Syst. Control Lett.* **170** (2022) 105397.

[20] W. Respondek, Symmetries and minimal flat outputs of nonlinear control systems, in *New Trends in Nonlinear Dynamics and Control and their Applications.* Springer (2004) 65–86.

[21] A.A. Agrachev and Y. Sachkov, *Control Theory from the Geometric Viewpoint*, vol. 87. Springer Science & Business Media (2013).

[22] W. Respondek and S. Ricardo. Equivariants of mechanical control systems. *SIAM J. Control Optim.* **51** (2013) 3027–3055.

[23] B. Jakubczyk and W. Kryński, Vector fields with distributions and invariants of ODEs. *J. Geom. Mech.* **5** (2013) 85–129.

Appendix A. Smooth solutions of Cauchy–Euler system

Let J be the real Jordan normal form of an $N \times N$ matrix, *i.e.*

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_\ell \end{pmatrix}, \quad \text{where either} \quad J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix} \quad \text{or} \quad J_j = \begin{pmatrix} \Lambda_j & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \Lambda_j \end{pmatrix},$$

with $\lambda_j \in \mathbb{R}$, $\Lambda_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$, where $a_j, b_j \in \mathbb{R}$, $b_j \neq 0$, and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma A.1. *If the Cauchy–Euler system*

$$wF'(w) - JF(w) = 0, \quad F(w) \in \mathbb{R}^N \tag{A.1}$$

possesses a smooth solution $F(w)$, around $w = 0$, such that all components F are not identically zero, then the matrix J is diagonal with non-negative integer eigenvalues $\lambda_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Moreover, the solution of (A.1) is given by $F_j(w) = c_j w^{\lambda_j}$, for $1 \leq j \leq N$, with $c_j \neq 0$.

Proof. Four cases are to be discussed. First, the Jordan block is of the form $J_j = (\lambda_j)$, with $\lambda_j \in \mathbb{R}$. For simplicity of notation, denote $\lambda = \lambda_j$ and the corresponding scalar component of F by F_1, so we have the scalar equation $wF_1' - \lambda F_1 = 0$. The solution of the latter is $F_1(w) = \theta|w|^\lambda$, where $\theta = \theta^+$ for $w > 0$ and $\theta = \theta^-$ for $w < 0$, with θ^+ and θ^- being real constants. The function $F_1(w)$ is C^∞-smooth if and only if $\lambda \in \mathbb{N}_0$, $\theta^+ = (-1)^\lambda \theta^-$, and $\theta \neq 0$ (since all F_j are assumed not identicallly zero).

Second, the Jordan block is of the form $J_j = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}$. Denote $\lambda = \lambda_j$ and the corresponding components of F by $(F_1, \ldots, F_{q_j})^t$, where J_j is a $(q_j \times q_j)$-matrix. The last row of J_j gives $wF_q' - \lambda F_q = 0$, where for simplicity we denoted $q = q_j$, and by the above analysis of the first case, we conclude that $F_q(w) = \theta_q|w|^\lambda$, with $\lambda \in \mathbb{N}_0$ and $\theta_q^+ = (-1)^\lambda \theta_q^-$. Now, one before the last row of J_j gives $wF_{q-1}' - \lambda F_{q-1} - F_q = 0$ and hence $F_{q-1}(w) = |w|^\lambda (\theta_q \ln|w| + \theta_{q-1})$, where θ_{q-1} is a constant possibly depending on the sign of w. Clearly, F_{q-1} being smooth at $w = 0$ implies that $\theta_q = 0$, which contradicts the fact that $F_q \neq 0$. Therefore all Jordan blocks with $q \geq 2$ are excluded.

Third, the Jordan block of the form $J_j = \Lambda_j$. For simplicity of notation, denote $\lambda = \lambda_j = a + b\mathbf{i}$, $\Lambda = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, and the corresponding components of F by $(F_1, F_2)^t$. Solving $(wF_1', wF_2')^t - \Lambda(F_1, F_2)^t = 0$ we conclude that $F_1(w)$ and $F_2(w)$ are $\mathbb{R}$-linear combinations of $|w|^a \cos(b \ln|w|)$ and $|w|^a \sin(b \ln|w|)$ but none of them is smooth at $w = 0$, which contradicts the assumption. So Jordan blocks of the form $J_j = \Lambda_j$ are excluded.

Fourth, the Jordan blocks of the form $J_j = \begin{pmatrix} \Lambda_j & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & \Lambda_j \end{pmatrix}$ produce along the last two rows a system of equations similar to that of the third case and, as previously, we conclude that there are no smooth non-zero solutions; hence, the Jordan blocks with complex eigenvalues are excluded as well.

To summarize, the only smooth non-zero solutions exist in the case of the diagonal matrix J with eigenvalues $\lambda_j \in \mathbb{N}_0$ and, moreover, are of the for $F_j(w) = \theta_j|w|^{\lambda_j}$, where $\theta_j^+ = (-1)^{\lambda_j}\theta_j^-$ and $\theta_j \neq 0$. Finally, removing the absolute value, we obtain that $F_j(w) = c_j w^{\lambda_j}$, where $c_j = \theta_j^+ \neq 0$. □

Appendix B. Detailed computations for Theorem 3.3

In this appendix, we detail the computation to obtain relations (3.2a)–(3.2c) between structure functions and we prove transformation rules (3.3) and (3.4) that show how the structure functions are changed under a feedback transformation. Consider a control-affine system $\Sigma = (f, g)$ with structure functions (k_1, k_2, k_3) and $(\lambda_1, \lambda_2, \lambda_3)$.

First, by applying the Jacobi identity to $[f,[g,[g,f]]]$ we deduce that $[f,[g,[g,f]]] = -[g,[f,[f,g]]]$. We compute the left-hand-side and the right-hand-side separately:

$$\begin{aligned}
[f,[g,[g,f]]] &= \mathrm{L}_f(\lambda_1) f + \mathrm{L}_f(\lambda_2) g - \lambda_2 [g,f] + \mathrm{L}_f(\lambda_3) - \lambda_3 (k_1 g + k_2 [g,f] + k_3 [g,[g,f]]), \\
&= \mathrm{L}_f(\lambda_1) f + (\mathrm{L}_f(\lambda_2) - \lambda_3 k_1) g + (\mathrm{L}_f(\lambda_3) - \lambda_2 - \lambda_3 k_2)[g,f] \\
&\quad - \lambda_3 k_3 (\lambda_1 f + \lambda_2 g + \lambda_3 [g,f]), \\
&= (\mathrm{L}_f(\lambda_1) - \lambda_3 k_3 \lambda_1) f + (\mathrm{L}_f(\lambda_2) - \lambda_3 k_1 - \lambda_3 k_3 \lambda_2) g \\
&\quad + \left(\mathrm{L}_f(\lambda_3) - \lambda_2 - \lambda_3 k_2 - \lambda_3^2 k_3\right)[g,f].
\end{aligned}$$

And on the other hand we have

$$\begin{aligned}
[g,[f,[f,g]]] &= \mathrm{L}_g(k_1) g + \mathrm{L}_g(k_2)[g,f] + k_2(\lambda_1 f + \lambda_2 g + \lambda_3 [g,f]) \\
&\quad + \mathrm{L}_g(k_3)(\lambda_1 f + \lambda_2 g + \lambda_3 [g,f]) + k_3 [g, \lambda_1 f + \lambda_2 g + \lambda_3 [g,f]], \\
&= (k_2\lambda_1 + \lambda_1 \mathrm{L}_g(k_3)) f + (\mathrm{L}_g(k_1) + k_2\lambda_2 + \mathrm{L}_g(k_3)\lambda_2) g \\
&\quad + (\mathrm{L}_g(k_2) + k_2\lambda_3 + \mathrm{L}_g(k_3)\lambda_3)[g,f] \\
&\quad + k_3 (\mathrm{L}_g(\lambda_1) f + \lambda_1 [g,f] + \mathrm{L}_g(\lambda_2) g + \mathrm{L}_g(\lambda_3)[g,f] + \lambda_3(\lambda_1 f + \lambda_3 g + \lambda_3 [g,f])), \\
&= (k_2\lambda_1 + \lambda_1 \mathrm{L}_g(k_3) + k_3 \mathrm{L}_g(\lambda_1) + k_3\lambda_3\lambda_1) f \\
&\quad + (\mathrm{L}_g(k_1) + k_2\lambda_2 + \mathrm{L}_g(k_3)\lambda_2 + k_3\mathrm{L}_g(\lambda_2) + k_3\lambda_3\lambda_2) g \\
&\quad + \left(\mathrm{L}_g(k_2) + k_2\lambda_3 + \mathrm{L}_g(k_3)\lambda_3 + k_3\lambda_1 + k_3\mathrm{L}_g(\lambda_3) + k_3\lambda_3^2\right)[g,f].
\end{aligned}$$

Identifying the terms in front of f, g, and $[g,f]$ we obtain equations (3.2a) to (3.2c).

Now, we apply a feedback transformation of the form $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$ and we get first $\left[\tilde{f},\tilde{g}\right] = \beta [f,g] + \gamma g$, where $\gamma = \mathrm{L}_f(\beta) + \alpha \mathrm{L}_g(\beta) - \beta \mathrm{L}_g(\alpha)$. Second,

$$\begin{aligned}
\left[\tilde{g},\left[\tilde{g},\tilde{f}\right]\right] &= \beta^2 [g,[g,f]] + \beta \mathrm{L}_g(\beta)[g,f] + \left(-\beta \mathrm{L}_{[g,f]}(\beta) - \beta\mathrm{L}_g(\gamma) + \gamma \mathrm{L}_g(\beta)\right) g \\
&= \beta^2\lambda_1 f + \left(\beta^2\lambda_2 - \beta\mathrm{L}_{[g,f]}(\beta) - \beta\mathrm{L}_g(\gamma) + \gamma\mathrm{L}_g(\beta)\right) g + \left(\beta^2\lambda_3 + \beta\mathrm{L}_g(\beta)\right)[g,f] \\
&= \beta^2\lambda_1 \tilde{f} + \left(-\beta^2\lambda_1\alpha + \beta^2\lambda_2 - \beta\mathrm{L}_{[g,f]}(\beta) - \beta\mathrm{L}_g(\gamma) + \gamma\mathrm{L}_g(\beta) + \gamma(\beta\lambda_3 + \mathrm{L}_g(\beta))\right) g \\
&\quad + (\beta\lambda_3 + \mathrm{L}_g(\beta))\left[\tilde{g},\tilde{f}\right],
\end{aligned}$$

implying that $\tilde{\lambda}_1 = \beta^2\lambda_1$, $\tilde{\lambda}_2 = \beta\lambda_2 - \beta\lambda_1\alpha + \gamma\lambda_3 - \mathrm{L}_{[g,f]}(\beta) - \mathrm{L}_g(\gamma) + 2\gamma\mathrm{L}_g(\ln|\beta|)$, and $\tilde{\lambda}_3 = \beta\lambda_3 + \mathrm{L}_g(\beta)$. Third, we have

$$\begin{aligned}
\left[\tilde{f},\left[\tilde{f},\tilde{g}\right]\right] &= [f + g\alpha, \beta[f,g] + \gamma g] \\
&= \beta[f,[f,g]] + \mathrm{L}_f(\beta)[f,g] + \mathrm{L}_f(\gamma) g + \gamma[f,g] \\
&\quad + \alpha\beta[g,[f,g]] + \alpha\mathrm{L}_g(\beta)[f,g] - \beta\mathrm{L}_{[f,g]}(\alpha) g + \alpha\mathrm{L}_g(\gamma) g - \gamma\mathrm{L}_g(\alpha) g, \\
&= \left(\beta k_1 + \mathrm{L}_f(\gamma) + \beta\mathrm{L}_{[g,f]}(\alpha) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha)\right) g + (\beta k_2 - \mathrm{L}_f(\beta) - \gamma - \alpha\mathrm{L}_g(\beta))[g,f] \\
&\quad + (\beta k_3 - \alpha\beta)[g,[g,f]], \\
&= \left(\beta k_1 + \mathrm{L}_f(\gamma) + \beta\mathrm{L}_{[g,f]}(\alpha) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha)\right) g + (\beta k_2 - \mathrm{L}_f(\beta) - \gamma - \alpha\mathrm{L}_g(\beta))[g,f] \\
&\quad + \frac{1}{\beta^2}(\beta k_3 - \alpha\beta)\left\{\left[\tilde{g},\left[\tilde{g},\tilde{f}\right]\right] - \beta\mathrm{L}_g(\beta)[g,f] + \beta\mathrm{L}_{[g,f]}(\beta) g + \beta\mathrm{L}_g(\gamma) g - \gamma\mathrm{L}_g(\beta) g\right\},
\end{aligned}$$

implying $\tilde{k}_3 = \frac{1}{\beta}(k_3 - \alpha)$. Next, continuing the computation (denoting $\tilde{h} = \tilde{k}_3\left[\tilde{g},\left[\tilde{g},\tilde{f}\right]\right]$):

$$\left[\tilde{f},\left[\tilde{f},\tilde{g}\right]\right] = \Big(\beta k_1 + \mathrm{L}_f(\gamma) + \beta\mathrm{L}_{[g,f]}(\alpha) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha) + \tilde{k}_3\left(\beta\mathrm{L}_{[g,f]}(\beta) + \beta\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\beta)\right)\Big) g + \tilde{h}$$

$$+\left(\beta k_2 - \mathrm{L}_f(\beta) - \gamma - \alpha \mathrm{L}_g(\beta) - \tilde{k}_3\beta\mathrm{L}_g(\beta)\right)[g,f] + \tilde{h}$$
$$= \left(\beta k_1 + \mathrm{L}_f(\gamma) + \beta\mathrm{L}_{[g,f]}(\alpha) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha) + \tilde{k}_3\left(\beta\mathrm{L}_{[g,f]}(\beta) + \beta\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\beta)\right)\right) g$$
$$\frac{1}{\beta}\left(\beta k_2 - \mathrm{L}_f(\beta) - \gamma - \alpha\mathrm{L}_g(\beta) - \tilde{k}_3\beta\mathrm{L}_g(\beta)\right)\left\{\left[\tilde{g},\tilde{f}\right] + \gamma g\right\} + \tilde{h},$$

implying $\tilde{k}_2 = k_2 - \mathrm{L}_f(\ln|\beta|) - \frac{\gamma}{\beta} - \alpha\mathrm{L}_g(\ln|\beta|) - \tilde{k}_3\mathrm{L}_g(\beta)$ and finally

$$\tilde{k}_1 = \frac{1}{\beta}\left(\beta k_1 + \mathrm{L}_f(\gamma) + \beta\mathrm{L}_{[g,f]}(\alpha) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha) + \tilde{k}_3\left(\beta\mathrm{L}_{[g,f]}(\beta) + \beta\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\beta)\right) + \tilde{k}_2\gamma\right)$$
$$= k_1 + \mathrm{L}_{[g,f]}(\alpha) + \frac{1}{\beta}\left(\mathrm{L}_f(\gamma) + \alpha\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\alpha) + \tilde{k}_3\left(\beta\mathrm{L}_{[g,f]}(\beta) + \beta\mathrm{L}_g(\gamma) - \gamma\mathrm{L}_g(\beta)\right) + \tilde{k}_2\gamma\right).$$

Appendix C. Technical lemma for the proof of Theorem 3.6

The sufficiency part of the proof of Theorem 3.6 relies on the existence of a diffeomorphism that simultaneously rectifies the distribution $\operatorname{span}\{f_c, [g_c, f_c]\}$ and the vector field g_c as proven by the following lemma.

Lemma C.1. *Consider a control-affine system $\Sigma_c = (f_c, g_c)$ given by its canonical pair and set $\mathcal{F} = \operatorname{span}\{f_c, [g_c, f_c]\}$. If the structure functions of Σ_c satisfy condition (3.7) of Theorem 3.6, then there exists a local diffeomorphism $(x, y, w) = \phi(\xi)$ such that $\phi_*\mathcal{F} = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ and $\phi_* g_c = \frac{\partial}{\partial w}$.*

Proof. First, we prove that there exists smooth solutions h for the system

$$\mathrm{L}_{f_c}(h) = 0, \quad \mathrm{L}_{[g_c,f_c]}(h) = 0, \quad \text{and} \quad \mathrm{L}_{g_c}(h) = 1.$$

We need to check three integrability conditions:

1. $\mathrm{L}_{[f_c,[g_c,f_c]]}(h) = \mathrm{L}_{f_c}\left(\mathrm{L}_{[g_c,f_c]}(h)\right) - \mathrm{L}_{[g_c,f_c]}\left(\mathrm{L}_{f_c}(h)\right) = 0$ and $[f_c, [g_c, f_c]] = 0$, so $0 = 0$ and the first condition holds.
2. $\mathrm{L}_{[f_c,g_c]}(h) = \mathrm{L}_{f_c}\left(\mathrm{L}_{g_c}(h)\right) - \mathrm{L}_{g_c}\left(\mathrm{L}_{f_c}(h)\right) = 0$ and $\mathrm{L}_{[f_c,g_c]}(h) = 0$ so $0 = 0$ and the second integrability condition holds.
3. $\mathrm{L}_{[[g_c,f_c],g_c]}(h) = \mathrm{L}_{[g_c,f_c]}\left(\mathrm{L}_{g_c}(h)\right) - \mathrm{L}_{g_c}\left(\mathrm{L}_{[g_c,f_c]}(h)\right) = 0$ and $[[g_c, f_c], g_c] = -\varepsilon f_c - \nu[g_c, f_c]$. Therefore $\mathrm{L}_{[[g_c,f_c],g_c]}(h) = -\varepsilon\mathrm{L}_{f_c}(h) - \nu\mathrm{L}_{[g_c,f_c]}(h) = 0$ and $0 = 0$ so the third condition holds.

Take a smooth solution h of the above system, rename it $\phi_3 = h$, and choose ϕ_1, ϕ_2 such that $\mathrm{d}\phi_1$ and $\mathrm{d}\phi_2$ annihilate g_c and are independent (they exist since $g_c \neq 0$). The local diffeomorphism $\phi = (\phi_1, \phi_2, \phi_3) = (x, y, w)$ is such that $\phi_*\mathcal{F} = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ and $\phi_* g_c = \frac{\partial}{\partial w}$. □

Appendix D. Technical result for the proof of Proposition 4.1

We show that there exists a local diffeomorphism $(\tilde{x}, \tilde{y}, \tilde{w}) = \phi(x, y, w)$, around $0 \in \mathbb{R}^3$, bringing f_c into $\phi_* f_c = \frac{\partial}{\partial \tilde{x}}$ and g_c, given by (4.3), into

$$\phi_* g_c = \tilde{a}(\tilde{y}, \tilde{w})\frac{\partial}{\partial \tilde{x}} + \left(\tilde{x} + \tilde{b}(\tilde{y}, \tilde{w})\right)\frac{\partial}{\partial \tilde{y}} + \tilde{c}(\tilde{y}, \tilde{w})\frac{\partial}{\partial \tilde{w}} \tag{D.1}$$

such that $\tilde{a}(0, \tilde{w}) = 0$, $\tilde{b}(0, \tilde{w}) = 0$, and $\tilde{c}(0, \tilde{w}) = 1$. To this end, define $(\tilde{x}, \tilde{y}, \tilde{w}) = (x - \alpha(w), y - \beta(w), w)$, where $\alpha(w)$ and $\beta(w)$ are smooth functions defined below satisfying $\alpha(0) = 0$ and $\beta(0) = 0$. Then, $\tilde{a}(\tilde{y}, w) = a(\tilde{y} + \beta(w), w) - \alpha'(w)c(\tilde{y} + \beta(w), w)$ and $\tilde{a}(0, w) = 0$ if and only if

$$a(0 + \beta(w), w) - \alpha'(w)c(0 + \beta(w), w) = 0. \tag{D.2}$$

Using, $x = \tilde{x} + \alpha(w)$, we conclude that $\tilde{b}(\tilde{y}, w) = \alpha(w) + b(\tilde{y} + \beta(w), w) - \beta'(w)c(\tilde{y} + \beta(w), w)$ and thus $\tilde{b}(0, w) = 0$ if and only if

$$\alpha(w) + b(\beta(w), w) - \beta'(w)c(\beta(w), w) = 0. \tag{D.3}$$

Now, consider the second order ordinary differential equation (notice that $c(\beta(w), w) \neq 0$ since $c(0,0) \neq 0$ and $\beta(0) = 0$)

$$\begin{aligned}\frac{\mathrm{d}^2\beta}{\mathrm{d}w^2} = \frac{1}{c(\beta(w), w)}\frac{\mathrm{d}\beta}{\mathrm{d}w}(w)\left(\frac{\partial b}{\partial y}(\beta(w), w) - \frac{\mathrm{d}\beta}{\mathrm{d}w}(w)\frac{\partial c}{\partial y}(\beta(w), w) - \frac{\partial c}{\partial w}(\beta(w), w)\right)\\ + \frac{1}{c(\beta(w), w)^2}\left(a(w) + c(\beta(w), w)\frac{\partial b}{\partial w}(\beta(w), w)\right),\end{aligned}$$

obtained by differentiating (D.3) with respect to w and using that $\alpha'(w) = \frac{a(\beta(w),w)}{c(\beta(w),w)}$. Take the solution $\beta(w)$ satisfying $\beta(0) = 0$, $\beta'(0) = \frac{b(0,0)}{c(0,0)}$ and define $\alpha(w) = \beta'(w)c(\beta(w), w) - b(\beta(w), w)$ and thus $\alpha(0) = 0$. By construction, the functions $\alpha(w)$ and $\beta(w)$ satisfy equations (D.2) and (D.3). Therefore, the local diffeomorphism $(\tilde{x}, \tilde{y}, \tilde{w}) = \phi(x, y, w) = (x - \alpha(w), y - \beta(w), w)$ brings g_c, given by (4.3), into $\phi_* g_c$, given by (D.1), where $\tilde{a}(\tilde{y}, \tilde{w})$ and $\tilde{b}(\tilde{y}, \tilde{w})$ satisfy, respectively, $\tilde{a}(0, \tilde{w}) = 0$ and $\tilde{b}(0, \tilde{w}) = 0$. Now it remains to normalise the function $\tilde{c}$ such that $\tilde{c}(0, 0) = 1$. To this end, apply the diffeomorphism $\bar{w} = \gamma(\tilde{w})$ such that $\gamma(0) = 0$ and $\gamma'(\tilde{w})\tilde{c}(0, \tilde{w}) = 1$. This local diffeomorphism leaves conditions (D.2) and (D.3) unchanged, thus the calculation is complete.