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DIFFERENTIALLY OBSERVABLE SYSTEMS AND NORMAL FORM

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Abstract. By normal form of observability (or canonical form), we mean a controlled dynamical system with a certain triangular structure that respects the input-output map of the system. The problem of transforming a single output controlled dynamical system using a local diffeomorphism (local coordinate change) was solved in the 1980s and 1990s. Under the assumption of uniform observability, it was shown that the system can be locally transformed almost everywhere into the so-called normal form. The global aspect consists of finding an injective transformation that sends the initial system into normal form, while preserving the input-output map. To get around the problem of the singularities involved in constructing this transformation, we have proposed a purely analytic condition which, combined with an observability condition (differential observability), solves this problem.

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1. Introduction

An observer is a dynamical system through which the state of a process can be estimated using its measurable variables (inputs and outputs). Observer theory for linear systems is well-understood and many practical observers have been designed for both single and multi-output cases. The most famous linear observers are those of D.-G. Luenberger [1] for deterministic linear systems and R.-E. Kalman [2] in the stochastic case.

Observers for nonlinear systems however are much more challenging to design, and nonlinear observer research has received considerable attention since early 1980s. Among these approaches, geometric methods occupy an important place in the literature. These methods consist in transforming a nonlinear system in another for which the synthesis of the observer is easier to carry out: i) the error linearization problem consists of transforming a nonlinear observable system in a linear system plus an output injection [3–10]. For this class of systems, a Luenberger or Kalman observer can be designed. ii): the high-gain observer concerns the systems which are observable independently of inputs (uniformly observable systems). Many normal forms characterizing these systems exist in the literature, and all these normal forms have a triangular structure. The high gain observer construction has been initiated by the authors in [11]. Since then, several works have been established on this subject [12], [13] [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]) for different algorithms. In order to design a high gain observer for a uniformly observable system, the fundamental problem consists of showing how the system can be transformed in a normal form. For the single output uniformly observable systems which are

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control affine, the authors in [25] (and for a different proof see [11]) shown that the system can be transformed by a diffeomorphism locally almost everywhere if and only if it is uniformly observable. In the same spirit the authors in [26] have extended the above result to non control affine uniformly observable nonlinear systems. For the multi-output uniformly observable systems, the problem is far from being solved.
Although a single output uniformly observable system can be locally transformed by a diffeomorphism in a normal form (see [25], [11]), this diffeomorphism cannot be generally extended to a global one. This is mainly due to some topological obstructions and to the singularity phenomenon. In [15] (see also [27]), a geometric formulation of observability is introduced and used for obtaining a global transformation permitting to transform a nonlinear system in a normal form of observability. This transformation, that generally depends on the control and its time derivatives has been used in [28] to solve the problem of the output feedback stabilization. However, from engineering point of view, (due to the noised inputs), this normal form is not desirable for the high gain observers algorithm for open loop systems. More recently, the authors of [29] proposed a transformation which does not depend on the inputs. However this construction does not really exploit the singularities of the problem, moreover the complete triangularization of the system is not achieved.

In this paper we give a sufficient condition under which the singularities can be taken into account in the process of triangularization permitting to obtain the normal form of observability.

For the sake of simplicity, the work presented here concerns single-control systems. Generalization to multi-control systems is straightforward.

This paper is organized as follows: in Section 2, first we present the class of control systems to which our results apply, and we recall some observability concepts and some existing results concerning the immersion of a class of uniformly observable systems into a normal form of observability. In section 3, we give a sufficient condition permitting to solve the problem of immersion of uniformly observable systems into a normal form. This condition takes into account a large class of singularities, and makes it possible to extend the work presented in [29].

2. Observability concepts and normal form

Consider the following controlled systems:

$$\begin{cases} \dot{x} = f(x) + ug(x), \quad x \in \mathcal{M}, \quad u \in \mathbb{R} \\ y = h(x), \quad y \in \mathbb{R} \end{cases} \tag{2.1}$$

where $\mathcal{M}$ is an m-dimensional manifold, f, g are smooth vector fields on $\mathcal{M}$, and h is a smooth function which represents the output map which models the measured data of the system. The smoothness of a morphism means that this one is of class $\mathcal{C}^p$, $p \geq 1$. In this paper p may be chosen sufficiently large. The control $u(.)$ is assumed to be locally integrable in the Lebesgue sense.

In the sequel, we will sometimes use the notation (f, g, h) instead of Equation (2.1). It should also be noted that we have deliberately chosen to take a scalar input so as not to complicate the notations, and that all the main results in this paper extend to the multi-input case, without any additional assumptions.

2.1. Observability concepts

From practical point of view, the observability concept stands for the possibility of reconstructing the unknown trajectory $x(.)$ from the observed data issued from the input control $u(.)$, and the output measurement $y(.) = h(x(.))$. Several notions of observability exist in the literature (see for instance [27, 30]). Let us recall some of them.

Let $u(.)$ be a locally integrable function, let $x \in \mathcal{M}$ and let $x_u(.)$ be the trajectory of system (2.1) such that $x_u(0) = x$, and set $y(x, .) = h(x_u(.))$.

- An input u renders system (2.1) **observable** on an interval $[0,T]$ if, for every different initial states x, x', the associated outputs $y(x,.)$, $y(x',.)$ are not identically equal on the maximal interval $[0,T_{xx'}[\subset [0,T]$ on which $x_u(t)$ and $x'_u(t)$ are well defined.
- System (2.1) is **uniformly observable** (observable for any input), if for every $T>0$; for every $u \in L^\infty([0,T],\mathbb{R})$, u renders system (2.1) observable on $[0,T]$.

A sufficient condition permitting to check if a sufficiently smooth input $u(.)$ renders the system observable, is to find an integer k such that the following map:
$x \to \left(y(x,0), \dfrac{\mathrm{d}y(x,t)}{\mathrm{d}t}|_{t=0}, \ldots, \dfrac{\mathrm{d}^k y(x,t)}{\mathrm{d}t^k}|_{t=0} \right)$ is an injective map from $\mathcal{M}$ into $\mathbb{R}^{k+1}$. This sufficient condition leads us to the following formulation which is introduced by the authors in [27]:

In the sequel, $\underline{u}_k = (u_1,\ldots,u_k)$ is an element of $\mathbb{R}^k$. Let $N \geq m$ be a fixed integer ($m = \dim \mathcal{M}$), and consider the vector field F defined on $\mathcal{M}\times\mathbb{R}^{N-1}$ by $F(x,\underline{u}_{N-1}) = f(x) + u_1 g(x) + \sum_{i=2}^{N-1} u_i \frac{\partial}{\partial u_{i-1}}$ (for $N=2$, $\underline{u}_{N-1} = u_1$ and $F(x,u_1) = f(x)+u_1 g(x)$). In the sequel $\mathcal{M}$ is viewed as sub-manifold of $\mathcal{M}\times\mathbb{R}^{N-1}$ by using the natural injection (inclusion). Now consider a class $\mathcal{C}^N$-function $u(.)$ defined on some $[0,T]$ and $x\in\mathcal{M}$, and let $x_u(.)$ be the trajectory of system (2.1) which is associated to the input $u(.)$ and such that $x_u(0)=x$, and finally set $y(x,.) = h(x_u(.))$. We will use the following notations: $\dfrac{\mathrm{d}^k u}{\mathrm{d}t^k}(.) = u_{k+1}(.)$, $\underline{u}_k(.) = (u_1(.),\ldots,\underline{u}_k(.))$ and $y^{(k)}(x,.) = \dfrac{\mathrm{d}^k y(x,.)}{\mathrm{d}t^k}$, for $k=0,\ldots,N-1$, then we can observe that:

$$y^{(k)}(x,t) = L_F^k(h)(x(t),\underline{u}_{N-1}(t)) = L_F^k(h)(x(t),\underline{u}_k(t)) \tag{2.2}$$

where $L_F^i(h)$ is the ith Lie derivative of h along the vector field F. This leads to the following definition stated in [27]:

Definition 2.1. System (2.1) is said to be **differentially observable** of order N, if the following input-output map defined from $\mathcal{M}\times\mathbb{R}^{N-1}$ into $\mathbb{R}^N\times\mathbb{R}^{N-1}$ by:

$$\Sigma_N : (x,\underline{u}_{N-1}) \to (h(x), L_F(h)(x,\underline{u}_{N-1}),\ldots,L_F^{N-1}(h)(x,\underline{u}_{N-1}),\underline{u}_{N-1}) \tag{2.3}$$

is injective.

It is important to note that the differential observability concept appears as an original concept permitting to answer the output feedback stabilization based on the high gain technics (see [15, 26, 27]). Also, we can note that if a system (2.1) is uniformly observable, then it is locally differentially observable of order $N = \dim\mathcal{M}$ almost everywhere (this is a consequence of the fact that a uniformly observable system can transform into the normal form of observability, locally almost everywhere, see section 2.4.1 of [27] for similar argument).

2.2. Normal form of observability and some existing results

The normal form of observability that we consider in this paper takes the following triangular structure:

$$\begin{cases} \dot{z}_1 = z_2 + uG_1(z_1) \\ \vdots \\ \dot{z}_{k-1} = z_k + uG_{k-1}(z_1,\ldots,z_{k-1}) \\ \vdots \\ \dot{z}_N = \widetilde{G}_N(z) + uG_N(z) \\ y = z_1 \end{cases} \tag{2.4}$$

where $u(.)$ is a control function which is generally locally integrable in the Lebesgue sense. $y(.)$ is the output of the system.

The interest of these normal forms lies in the fact that one can estimate the unknown trajectories of system (2.4), using only the information contained in the input and output signals. In the case when the functions $\widetilde{G}_N$ and G_i's are global Lipschitz w.r.t. z, the algorithm (observer) permitting to estimate the unknown trajectories of (2.4) (see for instance [11], [14]) is given by the following dynamical system:

$$\begin{cases} \dot{\widehat{z}}_1 = \widehat{z}_2 + uG_1(\widehat{z}_1) + K_1(\widehat{z}_1 - y) \\ \vdots \\ \dot{\widehat{z}}_{N-1} = \widehat{z}_N + uG_{N-1}(\widehat{z}_1, \dots, \widehat{z}_{N-1}) + K_{n-1}(\widehat{z}_1 - y) \\ \dot{\widehat{z}}_N = \widetilde{G}_N(\widehat{z}) + uG_N(\widehat{z}) + K_n(\widehat{z}_1 - y) \end{cases} \tag{2.5}$$

where the K_i's are constants which must be judicially chosen, and $y(.) = z_1(.)$ is the output of system (2.4). Moreover the estimation is an exponential one: $\|\widehat{z}(t) - z(t)\| \leq \mu e^{-at}\|\widehat{z}(0) - z(0)\|$, $\mu > 0$, $a > 0$ are constants. Motivated by this observer construction, the first step is to analyze the possibility to transform a nonlinear system (2.1) in the normal form (2.4) by using an injective differential transformation $z = \Phi(x)$ satisfying $h(\Phi^{-1}(z)) = z_1$. If this transformation exists, then the second step is to find an inversion method to estimate $x(t) = \Phi^{-1}(z(t))$.

In the sequel an injective transformation $z = \Phi(x)$ which sends any trajectory of system (2.1) to a trajectory of system (2.4) and which preserves the input-output map will be called an **immersion** between the two systems, we say again that Φ **immerses** system (2.1) in system (2.4). Note that this immersion between the two systems is an immersion in the geometric sense, if in addition the tangent map $T_x\Phi$ is everywhere of rank $m = \dim \mathcal{M}$.

In this paper we are not concerned with the estimation techniques allowing to estimate the unknown state of system (2.1) based on the observer (2.5). The question we are going to answer in this article is therefore: what are the key conditions allowing to transform a nonlinear system (2.1) in a normal form (2.4)? In other words, what are the key conditions permitting to solve the following problem:

$$\begin{cases} \text{For } 1 \leq k \leq N, \text{ there exists a function } G_k : \mathbb{R}^k \to \mathbb{R} \text{ and } \widetilde{G}_N : \mathbb{R}^N \to \mathbb{R} \text{ such :} \\ L_g(L_f^{k-1}(h)) = G_k \circ \varphi_k, \quad \text{and} \quad L_f^N(h) = \widetilde{G}_N \circ \varphi_N; \quad \text{where} \quad \varphi_k = (h, \dots, L_f^{k-1}(h)) \end{cases} \tag{2.6}$$

This is the same as finding the key condition for the solution of the following problem:

$$\begin{cases} \text{For } 1 \leq k \leq N-1, \text{ if } (\underline{x}, \underline{x}') \in \mathcal{M} \times \mathcal{M} \text{ is such } \varphi_k(\underline{x}) = \varphi_k(\underline{x}'), \text{ then,} \\ L_g(L_f^{k-1}(h))(\underline{x}) = L_g(L_f^{k-1}(h))(\underline{x}'), \text{ where } \varphi_k = (h, \dots, L_f^{k-1}(h)) \end{cases} \tag{2.7}$$

Moreover we will show that if the φ_k's are semi-proper (see Def. 3.2 of Sect. 3), then the terms G_k's (resp. $\widetilde{G}_N$) become continuous on $\varphi_k(\mathcal{M})$ (resp. on $\varphi_N(\mathcal{M})$). Moreover, in the analytic case, these terms can be extended to smooth functions on an open set containing $\varphi_k(\mathcal{M}) \setminus \varphi_k(S_k)$, where S_k is the set of singular points of φ_k.

Let us start by the following example:

Example 2.2. Consider the following system defined on the cylinder $\mathfrak{C}^+ = \{(x_1, x_2, x_3) :\ x_1^2 + x_2^2 = 1,\ x_3 > 0\}$ by:

$$\begin{cases} \dot{x} = f(x) + ug(x) \\ y = h(x) \end{cases} \tag{2.8}$$

where $f = x_3[-x_2\dfrac{\partial}{\partial x_1} + x_1\dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_3}]$; $g = x_3\dfrac{\partial}{\partial x_3}$ and $h(x) = x_1x_3$.

In this example, we show that system (2.8) satisfies the following properties:

i) the map $\varphi_2 = (h, L_f(h))$ is everywhere a local diffeomorphism which locally transforms the system in a normal form of observability, however φ_2 is not injective, ii) $\varphi_3 = (h, L_f(h), L_f^2(h))$ is injective and immerses the system in a normal form.

The following calculations are easy to obtain:

$L_f(h)(x) = -x_2x_3^2 + x_1x_3$, $L_f^2(h)(x) = x_1x_3 - 3x_2x_3^2 - x_1x_3^3$, $L_g(h) = h$, $L_gL_f(h) = 2L_f(h) - h$, $L_fL_g(h) = L_f(h)$.

Set $\varphi_2 = (h, L_f(h))$, the jacobian $J(\varphi_2) = \begin{pmatrix} x_3 & 0 & x_1 \\ x_3 & -x_3^2 & x_1 - 2x_2x_3 \end{pmatrix}$

is of rank 2 since $x_3 > 0$. Hence for every fixed point $\underline{x} \in \mathfrak{C}^+$, $(z_1, z_2) = (h(x), L_f(h)(x))$ forms a system of coordinates around $\underline{x}$ in which system (2.8) takes the following normal form:

$$\begin{cases} \dot{z}_1 = z_2 + uz_1 \\ \dot{z}_2 = \lambda(z_1, z_2) + u(2z_2 - z_1) \\ y = z_1 \end{cases} \tag{2.9}$$

for some function λ.

φ_2 is not injective: indeed, consider $x = (1, 0, 1)$, $x' = (-1, 0, -1)$, clearly x, x' are in $\mathfrak{C}^+$ and $\varphi_2(x) = \varphi_2(x')$. We have then answered to point i).

Concerning ii), it is not difficult to check that $\varphi_3 = (h, L_f(h), L_f^2(h))$ is injective, and that φ_3 immerses system (2.8) in the following normal form:

$$\begin{cases} \dot{z}_1 = z_2 + uz_1 \\ \dot{z}_2 = z_3 + u(2z_2 - z_1) \\ \dot{z}_2 = \mu(z_1, z_2, z_3) + u\widetilde{\mu}(z_1, z_2, z_3)) \\ y = z_1 \end{cases} \tag{2.10}$$

for some functions μ, $\widetilde{\mu}$.

This example leads us to the following generalization:

Theorem 2.3. *Assume that system (2.1) is uniformly observable, and satisfying the following hypotheses:*

h_1) $\varphi_m = (h, \ldots, L_f^{m-1}(h))$ is everywhere a local diffeomorphism.

h_2) $\varphi_{m+2} = (h, \ldots, L_f^{m+1}(h))$ is injective.

Then φ_{m+2} immerses system (2.1) in a normal form of observability (2.4) on $\mathbb{R}^{m+2}$.

This theorem can be obtained by following the same approach as that used in [11].

However, how can this result be extended to the case where φ_m admits singularities? The answer to this question is the subject of the paper [29], where the authors propose a well-designed hypothesis to weaken hypothesis h_1). This hypothesis is formulated as follows:

For every k, $2 \leq k \leq m+1$, the following property holds:

The hypothesis $\mathcal{B}(k)$: for every $(\underline{x}, \underline{x}')$ such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x}')$, $\mathcal{B}(k, \underline{x}, \underline{x}')$ is satisfied, and where, $\mathcal{B}(k, \underline{x}, \underline{x}')$ says:

$\mathcal{B}(k, \underline{x}, \underline{x}')$: there exists a sequence $(\xi(j), \xi'(j))$ which converges to $(\underline{x}, \underline{x}')$ such that $\varphi_k(\xi(j)) = \varphi_k(\xi'(j))$, and that either one of the following tangent map $T_{\xi(j)}\varphi_{k-1}$, $T_{\xi'(j)}\varphi_{k-1}$ is of full rank.

Theorem 2.4 ([29])**.** *Assume that system (2.1) is uniformly observable and that φ_{m+2} is injective, and $\mathcal{B}(k)$ is satisfied for $2 \leq k \leq m+1$, then φ_{m+2} immerses system (2.1) in a normal form of observability, namely, for $1 \leq k \leq m+1$, $L_gL_f^{k-1}(h) = G_k \circ \varphi_k$, for some function G_k.*

In the following, we will emphasize the importance of property $\mathcal{B}(k)$ in the proof of theorem 2.4.

First, we give some local properties of a submersion, which we will refer to in a number of passages in this paper.

Remark 2.5. Let $\mathcal{M}$, $\mathcal{N}$ be two manifolds. Let $\varphi : \mathcal{M} \to \mathcal{N}$, $\psi : \mathcal{M} \to \mathbb{R}$ be two $\mathcal{C}^1$ functions such that φ is a submersion at some ξ, and that $\psi = \chi \circ \varphi$ for some function χ. Then we can find two open neighbourhoods W_ξ, W_z of the respective points ξ and $z = \varphi(\xi)$, and a class $\mathcal{C}^1$-map $s : W_z \to W_\xi$ such that:

i) $\varphi \circ s = id|_{W_z}$ (the identity map of W_z).
ii) the function χ is of class $\mathcal{C}^1$ on W_z.

Le us set $m = \dim \mathcal{M}$, $n = \dim \mathcal{N}$, a one way to check i), ii) of this remark, is to observe that locally φ is equivalent to the natural projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$. Indeed, locally around ξ and z, we can find two charts $\sigma_1 : W_\xi \to \mathbb{R}^m$, $\sigma_2 : W_z \to \mathbb{R}^n$, with $\sigma_1(\xi) = 0$, $\sigma_2(z) = 0$, such that $\sigma_2 \circ \varphi \circ \sigma_1^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection, and hence i) follows immediately.

Concerning ii), it suffices to check that $\chi = \psi \circ s$.

Brief proof of theorem 2.4

The proof of $L_g L_f^{k-1}(h) = G_k \circ \varphi_k$, for $1 \leq k \leq m+1$, will be obtained by induction on k.

For $k = 1$, only the uniform observability is required to show that $L_g(h) = G_1 \circ \varphi_1$, where $\varphi_1 = h$ (see [29] for more details).

The next step is to show that if $L_g L_f^{i-1}(h) = G_i \circ \varphi_i$, for $1 \leq i \leq k$, and $\mathcal{B}(k+1)$ is satisfied, then $L_g L_f^k(h) = G_{k+1} \circ \varphi_{k+1}$, or equivalently:

$$\text{For every } (\underline{x}, \underline{x}'),\ \varphi_{k+1}(\underline{x}) = \varphi_{k+1}(\underline{x}') \text{ implies } L_g L_f^k(h)(\underline{x}) = L_g L_f^k(h)(\underline{x}') \tag{2.11}$$

Using the assumption that $\mathcal{B}(k+1, \underline{x}, \underline{x}')$ is satisfied, we can find a sequence $(\xi(j), \xi'(j))$ which converges to $(\underline{x}, \underline{x}')$ such that $\varphi_{k+1}(\xi(j)) = \varphi_{k+1}(\xi'(j))$, and for instance $T_{\xi(j)}\varphi_k$ is of full rank. To show (2.11), using an argument of continuity, it suffices to check the following:

$$\text{For every } j,\ L_g L_f^k(h)(\xi(j)) = L_g L_f^k(h)(\xi'(j)) \tag{2.12}$$

Using the induction hypothesis and the fact that $T_{\xi(j)}\varphi_k$ is of full rank, remark 2.5 implies that $G_1, \ldots, G_k$ are respectively smooths on a some open neighbourhoods $W_1, \ldots, W_k$ of the respective points $\varphi_1(\xi(j)), \ldots, \varphi_k(\xi(j))$, and the set $\bigcap_{1 \leq i \leq k} \varphi_i^{-1}(W_i)$ is an open set containing $\xi(j)$ and $\xi'(j)$ (since $\varphi_{k+1}(\xi(j)) = \varphi_{k+1}(\xi'(j))$).

Now assume that (2.12) is not satisfied, then there exists j such that $L_g L_f^k(h)(\xi(j)) \neq L_g L_f^k(h)(\xi'(j))$, hence we can find open neighbourhoods V, V' of the respective points $\xi(j)$, $\xi'(j)$ such that $V \subset \bigcap_{1 \leq i \leq k} \varphi_i^{-1}(W_i)$, $V' \subset \bigcap_{1 \leq i \leq k} \varphi_i^{-1}(W_i)$ and that:

$$L_g L_f^k(h)(x)) \neq L_g L_f^k(h)(x'), \quad \text{for every } (x, x') \text{ in } V \times V' \tag{2.13}$$

Set $u_k(x, x') = -\dfrac{L_f^{k+1}(h)(x) - L_f^{k+1}(h)(x')}{L_g L_f^k(h)(x)) - L_g L_f^k(h)(x')}$, the following closed system is then will defined on $V \times V'$:

$$\begin{cases} \dot{x} = f(x) + u_k(x, x')g(x) \\ \dot{x}' = f(x') + u_k(x, x')g(x') \end{cases} \tag{2.14}$$

Now let us use the following notations: $(\xi(j,t),\xi'(j,t))$ denotes the trajectory of system (2.14) which is issued from $(\xi(j),\xi'(j))$ at $t=0$, $u_k(t)=u_k(\xi(j,t),\xi'(j,t))$, $\underline{e}(t)=\varphi_{k+1}(\xi(j,t))-\varphi_{k+1}(\xi'(j,t))$, $v(t)=\dfrac{\mathrm{d}(\xi_{k+1}(j,t))}{\mathrm{d}t}$, then $(\underline{e}(t),\varphi_{k+1}(\xi(j,t)))$ is a solution of the following smooth system:

$$\begin{cases} \text{for } 1\leq i\leq k,\ \dot{e}_i=z_{i+1}+u_k(.)(G_i(e_1,\ldots,e_i))-G_i(z_1-e_1,\ldots,z_i-e_i)),\ \text{and } \dot{e}_{k+1}=0 \\ \text{for } 1\leq i\leq k,\ \dot{z}_i=z_{i+1}+u_k(.)G_i(z),\ \text{and } \dot{z}_{k+1}=v(t) \end{cases} \tag{2.15}$$

Hence $(0,\varphi_{k+1}(\xi(j,t)))$ is the unique solution of the smooth system (2.15) which is issued from $(0,\varphi_{k+1}(\xi(j))$ at $t=0$. Consequently $\underline{e}(.)\equiv 0$, and hence $h(\xi(.))\equiv h(\xi'(.))$, which contradicts the uniform observability of the system. Thus necessarily $L_gL_f^k(h)(\xi(j))=L_gL_f^k(h)(\xi'(j))$.

We end this section by some remarks.

Let E, E' be two subsets of a topological space $\mathcal{A}$ such that $E\cap E'\neq\emptyset$. Let $x\in E\cap E'$, E, E' are said to be equivalent at x, if there exists an open neighbourhood U_0 of x such that for every open neighbourhood $U\subset U_0$ of x, we have $U\cap E=U\cap E'$. The class of equivalence is called the germ of E (or E') at x, and we will denote it by $E(x)$. A property $(\mathcal{P})$ is satisfied for $E(x)$, if there exists an open neighbourhood U_0 of x such that for every open neighbourhood U of x satisfying $U\subset U_0$, $(\mathcal{P})$ is verified for $E\cap U$. In practice, $E(x)$ can be identified with $E\cap U$, where U is an open neighbourhood of x.

Let $\underline{z}\in\mathbb{R}^k$ be such that the set $D_{\varphi_k}^{\underline{z}}=\{(x,x'):\ \varphi(x)=\varphi(x')=\underline{z}\}$ is not empty. Let $(\underline{x},\underline{x}')\in D_{\varphi_k}^{\underline{z}}$, in the sequel, we will use the notation $D_{\varphi_k}(\underline{x},\underline{x}')$ instead $D_{\varphi_k}^{\underline{z}}(\underline{x},\underline{x}')$ (the germ of $D_{\varphi_k}^{\underline{z}}$ at $(\underline{x},\underline{x}')$).

Let S_{k-1} be the set of singular points of φ_{k-1} and set $S_{k-1}(x)$ to be the germ of S_{k-1} at x. The property $\mathcal{B}(k)$ can be rewritten as follows:

$$\mathcal{B}(k,\underline{x},\underline{x}'):\quad \varphi_k(\underline{x})=\varphi_k(\underline{x})\text{ implies } D_{\varphi_k}(\underline{x},\underline{x}')\nsubseteq S_{k-1}(\underline{x})\times S_{k-1}(\underline{x}') \tag{2.16}$$

$$\mathcal{B}(k):\qquad \text{for every } (\underline{x},\underline{x}')\ s.t.\ \varphi_k(\underline{x})=\varphi_k(\underline{x}),\ \mathcal{B}(k,\underline{x},\underline{x}')\text{ is satisfied} \tag{2.17}$$

Definition 2.6. Let $\varphi:\mathcal{A}_1\to\mathcal{A}_2$ be a continuous map between two topological spaces, and set $D_\varphi=\{(x,x')\in\mathcal{A}_1\times\mathcal{A}_1:\ \varphi(x)=\varphi(x')\}$ (the φ-diagonal set of $\mathcal{A}_1\times\mathcal{A}_1$).

φ is bi-semi-open at an element (x,x') of D_φ, if for every open neighbourhoods U_x, $U_{x'}$ of x, x', the interior of $\varphi(U)\cap\varphi(U')$ is not empty.

φ is bi-semi-open on a subset C of D_φ, if it is bi-semi-open at each (x,x') of C.

Remark 2.7. 1) If φ is open, then it is be-semi-open.
2) If $\mathcal{B}(k)$ holds for $k\geq 2$, then φ_{k-1} is bi-semi-open on D_{φ_k}.
3) Imposing condition $\mathcal{B}(k)$ leads to exclude non controlled systems that do not meet property $\mathcal{B}(k)$, whereas the injectivity of φ_{m+2} suffices to transform a non controlled system under the normal form of observability (see for instance example 2.8 bellow).
4) Property $B(k)$ excludes the singular points which satisfy $D_{\varphi_{k+1}}(\underline{x},\underline{x}')\subset S_k(\underline{x})\times S_k(\underline{x}')$.
5) In theorem 2.4, the problem of immersion in normal forms in higher dimensions ($N>m+2$) is not addressed.

Example 2.8. In this example, we will give a system in dimension 2, which can be transformed in a normal form of observability in dimension 3, and which not satisfies $\mathcal{B}(2)$.

Consider the following autonomous system:

$$\begin{cases} \dot{x}=f(x) \\ y=h(x) \end{cases} \tag{2.18}$$

where $f = \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}$ and $h(x) = x_1 x_2^2$. $L_f(h)(x) = x_2^2 + 2x_1x_2$; $L_f^2(h)(x) = 2(x_1 + 2x_2)$ and $L_f^3(h) = 6$.

Set $\varphi_1 = h$, $\varphi_2 = (h, L_f(h))$, $\varphi_3 = (h, L_f(h), L_f^2(h))$, in this example, we will show the following:

i) Property $\mathcal{B}(2)$ is not satisfied.
ii) φ_3 is an injective map which immerses system (2.18) in the normal form:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = 6 \\ y = z_1 \end{cases} \tag{2.19}$$

Concerning i), set $\underline{x} = (1, 0)$, $\underline{x}' = (-1, 0)$, clearly $\varphi_2(\underline{x}) = \varphi_2(\underline{x}')$. It is not difficult to see that for very $\epsilon > 0$ sufficiently small, and for every x, x' s.t. $\|x - \underline{x}\| < \epsilon$, $\|x' - \underline{x}'\| < \epsilon$, $\varphi_1(x) = \varphi_1(x')$ implies $x_2 = x_2' = 0$, and hence $\varphi_1(x) = \varphi_1(x') = 0$. Thus φ_1 is not bi-semi-open on $(\underline{x}, \underline{x}')$, and from 2) of remark 2.7, $\mathcal{B}(2)$ is not satisfied.

ii) The injectivity of $\varphi_3 = (h, L_f(h), L_f^2(h))$:

For a fixed $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$, let us show that the set of solution of the equation $\varphi_3(x) = \varphi_3(\widetilde{x})$ is reduced to the singleton $\{\widetilde{x}\}$. The equation $\varphi_3(x) = \varphi_3(\widetilde{x})$ can be rewritten as follows:

$$\begin{cases} x_1 = \widetilde{x}_1 + 2\widetilde{x}_2 - 2x_2 \\ x_2^2 + 2(\widetilde{x}_1 + 2\widetilde{x}_2 - 2x_2)x_2 = \widetilde{x}_2^2 + 2\widetilde{x}_1\widetilde{x}_2 \\ (\widetilde{x}_1 + 2\widetilde{x}_2 - 2x_2)x_2^2 = \widetilde{x}_1\widetilde{x}_2^2 \end{cases} \tag{2.20}$$

The two last equations of system (2.20) take the following form:

$$\begin{cases} x_2^2 - 2(\widetilde{x}_1 + 2\widetilde{x}_2)x_2 + (\widetilde{x}_2^2 + 2\widetilde{x}_1\widetilde{x}_2) = 0 \\ 2x_2^3 - (\widetilde{x}_1 + 2\widetilde{x}_2)x_2^2 + \widetilde{x}_1\widetilde{x}_2^2 = 0 \end{cases} \tag{2.21}$$

In order to show the injectivity of φ_3, it suffices to check that the unique solution of system (2.21) is $x_2 = \widetilde{x}_2$.

Note that the first equation of (2.21) admits two solutions: $\widetilde{x}_2$ and $s = 2\widetilde{x}_1 + 3\widetilde{x}_2$. To check that $\widetilde{x}_2$ is the unique solution of (2.21), it suffices to show that if $s \neq \widetilde{x}_2$, then s is not a solution of the second equation of (2.21).

Assume the contrary, then $\widetilde{x}_2$ and s are both solutions of the second equation of (2.21), hence for every fixed $\widetilde{x}_1$, the second equation of (2.21) admits three solutions: $\widetilde{x}_2$, s and $\widetilde{s}$, with $\widetilde{x}_2 \neq s$, and hence:

$$2x_2^3 - (\widetilde{x}_1 + 2\widetilde{x}_2)x_2^2 + \widetilde{x}_1\widetilde{x}_2^2 = 2(x_2 - \widetilde{x}_2)(x_2 - s)(x_2 - \widetilde{s}) \tag{2.22}$$

Let's identify the two polynomials on either side of equality, we get:

$$\begin{cases} 2(s + \widetilde{s} + \widetilde{x}_2) = \widetilde{x}_1 + 2\widetilde{x}_2 \\ (s + \widetilde{s})\widetilde{x}_2 + s\widetilde{s} = 0 \\ 2\widetilde{x}_2 s\widetilde{s} = -\widetilde{x}_1\widetilde{x}_2^2 \end{cases} \tag{2.23}$$

Using the expression $s = 2\widetilde{x}_1 + 3\widetilde{x}_2$, the first equality of (2.22) gives $2\widetilde{s} = -(3\widetilde{x}_1 + 6\widetilde{x}_2)$. Now replace these expressions in the second equality of (2.22), we deduce:

$$P(\widetilde{x}_2) = 18\widetilde{x}_2^2 + 10\widetilde{x}_1\widetilde{x}_2 + 3\widetilde{x}_1^2 = 0 \tag{2.24}$$

For a fixed $\widetilde{x}_1$, the polynomial P admits real zeros if its discriminant is greater than or equal to 0. This last condition is satisfied iff $\widetilde{x}_1 = 0$. Now replace $\widetilde{x}_1 = 0$ in (2.24), we get $\widetilde{x}_2 = 0$, and this implies $s = \widetilde{x}_2 = 0$, which contradicts the fact that $\widetilde{x}_2 \neq s$. Hence φ_3 is injective.

Finally, a simple computation shows that if $x(t)$ is a trajectory of (2.18), then $\varphi_3(x(t))$ is a trajectory of (2.19).

In this paper, we propose a sufficient condition allowing on the one hand to take into account a large class of singularities, on the other hand, to extend immersion to normal forms in arbitrary dimension.

3. The main theorem and some preliminary results

In the first subsection, we present the main result. In the second one, we illustrate the main theorem with examples, finally, in the third subsection, we establish some technical results relating to the problem of extending functions to continuous functions, or even to smooth functions outside singularities.

3.1. The main theorem

Let $\mathcal{M}$, $\mathcal{N}$ be two smooths manifolds, $T\mathcal{M}$ and $T\mathcal{N}$ denote their tangent space. Let φ be a smooth map from $\mathcal{M}$ into $\mathcal{N}$, and $T\varphi$ denotes the tangent map.

Definition 3.1. 1) Let A be a subset of $\mathcal{N}$. An element z of A is said to be a regular point of A, if there exists an open neighbourhood V_z of z in $\mathcal{N}$, such that $N_z = V_z \cap A$ is a smooth sub-manifold of $\mathcal{N}$. The set of regular points of A will be denoted by A_{Reg}.

2) Set $A = \varphi(\mathcal{M})$ equipped with the topology induced by that of $\mathcal{N}$. An element x of $\mathcal{M}$ is said to be a regular point of φ if: i) $z = \varphi(x)$ is an element of A_{Reg}, ii) there exist an open neighbourhood U_x of x in $\mathcal{M}$ and an open neighbourhood V_z of z in $\mathcal{N}$ such that φ is a submersion from U_x onto the manifold $N_z = V_z \cap A$ stated in 1). When $\dim \mathcal{N} \leq \dim \mathcal{M}$, this definition coincides with the condition that the rank of $T_x\varphi$ is equal to $n = \dim \mathcal{N}$.

3) x is a singular point of φ if it is not a regular point of φ.

Note that in the above definition $\varphi(\mathcal{M})$ is equipped with the topology induced by that of $\mathcal{N}$. Generally $\varphi(\mathcal{M})$ has no interesting structure, and the set of regular point of $\varphi(\mathcal{M})$ may be empty (for instance, it suffices to consider the famous immersion from $\mathbb{R}$ in the torus, whose image is dense). An interesting class concerns analytic semi-proper morphisms $\varphi : \mathcal{M} \to \mathcal{N}$ (for the semi-property, see Def. 3.2 bellow). For this class of morphisms (see [31, 32]), $\varphi(\mathcal{M})$ becomes a sub-analytic set, moreover, $\varphi(\mathcal{M})_{Reg}$ is an open dense subset of $\varphi(\mathcal{M})$.

Definition 3.2. A map $\varphi : \mathcal{M} \to \mathcal{N}$ between two manifolds is said to be semi-proper if, for every compact set L of $\mathcal{N}$; there exists a compact subset K of $\mathcal{M}$ such that $\varphi(K) = L \cap \varphi(\mathcal{M})$.

For example, periodic scalar maps on $\mathbb{R}$ are semi-proper but not proper.

Remark 3.3. Unlike proper functions, the restriction of a semi-proper function to a closed subset is not necessarily semi-proper.

Indeed, set $\varphi(x) = \sin(x)$, and consider the closed discrete set $E = \{2k\pi + \dfrac{1}{k} :\ k = 1, 2 \ldots, \}$. We can verify that restriction of φ to E is not semi-proper.

Property 3.4. *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a semi-proper and continuous map between two manifolds, then $\varphi(\mathcal{M})$ is a closed subset of $\mathcal{N}$.*

This property can be obtained using a compactness argument.

In the sequel, φ_k denote the map $(h, \ldots, L_f^{k-1}(h))$, for $k \geq 1$.

Definition 3.5. Let $\mathfrak{F}$ be a family of smooths scalar functions on $\mathcal{M}$, let X be a vector field on $\mathcal{M}$, and let k be an integer, $k \geq 2$. We say that $\mathfrak{F}$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property, if for every ψ in $\mathfrak{F}$ such that $\psi = \chi \circ \varphi_{k-1}$; there exists a scalar function χ' such that $L_X(\psi) = \chi' \circ \varphi_k$.

Remark 3.6. If the function χ in the above definition is smooth, then we can always find χ' such that $L_X(\psi) = \chi' \circ \varphi_k$.

In all the following developments, we only consider differentially observable systems of order $N \geq 3$. This latter fact is justified by the following remark:

Remark 3.7. If system (f, g, h) is differentially observable of order 2, then it can be immersed in a normal form of observability defined on $\mathbb{R}^2$, without any additional hypothesis. However for $N \geq 3$, additional assumptions will be required in order to guarantee the immersion of the system in a normal form of observability.

Indeed, if the system is differentially observable of order 2, then a simple computation shows that the system is necessarily uniformly observable and $\varphi_2 = (h, L_f(h))$ is injective. As in [11], [29], we deduce that $L_g(h) = G_1 \circ h$, for some function G_1, which implies that φ_2 immerses system (2.1) in the normal form of observability in dimension 2.

In order to state the mean theorem of this paper, the following notations will be required.

Let k be an integer $1 \leq k \leq N-1$, let $\underline{X}_k = (X_1, \ldots, X_k)$ be a k-tuple of vector fields X_i which belong to $\{f, g\}$. Let $\nu_g(\underline{X}_k)$ denote the cardinal of the set of components of $\underline{X}_k$ which are equal to g, and set $\mathbb{X}(k) = \{\underline{X}_k = (X_1, \ldots, X_k)$, where $X_j \in \{f, g\}$, and that $\nu_g(\underline{X}_k) \geq 1\}$. Set $L_{\underline{X}_k}(h) = L_{X_k} \ldots L_{X_1}(h)$ where $L_{X_k} \ldots L_{X_1}$ is the composition of the Lie derivatives L_{X_i}, and $\mathcal{X}(k) = \{L_{\underline{X}_k}(h),\ \underline{X}_k \in \mathbb{X}(k)\}$.

Theorem 3.8 (the proof is given in Sect. 4). *Assume that system (2.1) is differentially observable of order $N \geq 3$, and for every integer k, $2 \leq k \leq N-1$; for every X in $\{f, g\}$, $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property, then:*

1) *φ_N immerses system (2.1) in the normal form (2.4).*
2) *If φ_k is semi-proper, for $k = 1, \ldots, N$, then the nonlinear term G_k (respectively $\widetilde{G}_N$) in the normal form is continuous on $\varphi_k(\mathcal{M})$ (resp. on $\varphi_N(\mathcal{M})$), and where $\varphi_k(\mathcal{M})$ is equipped with the topology induced by that of $\mathcal{N}$.*
3) *For $k = 1, \ldots, N$, let S_k be the set of singular points of φ_k, and assume that $S_k \subsetneq \mathcal{M}$. Then G_k (resp. $\widetilde{G}_N$) can be extended to a smooth function on some open set of $\mathbb{R}^k$ (resp. of $\mathbb{R}^N$) which contains $\varphi_k(\mathcal{M} \setminus S_k)$ (resp. $\varphi_N(\mathcal{M} \setminus S_N)$).*

The following remark concerns the part 3) of the theorem:

Remark 3.9. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth map, then the set of singular points of φ may be the whole space. Indeed, it suffices to consider the famous immersion from $\mathbb{R}$ in the torus.

An illustration of the proof of the theorem will be given for differentially observable systems of order 3 (see Ex. 3.15).

As we will see in Section 4, the proof of part 1) of this theorem is completely different from that given in [11] and [29].

In the rest of this section, we analyse the $(\varphi_{k-1}, \varphi_k, X)$- composite function property, then give some examples to illustrate theorem 3.8.

Let $\varphi : \mathcal{M} \to \mathbb{R}^k$ be a smooth map, and X a vector field on $\mathcal{M}$. Let $(\xi, \xi') \in \mathcal{M} \times \mathcal{M}$ such that $\varphi(\xi) = \varphi(\xi') = z$ and set $D_\varphi(\xi, \xi')$ to be the germ at (ξ, ξ') of the φ-diagonal set $\{(x, x') : \varphi(x) = \varphi(x') = z\}$.

Let S_k be the set of singular points of φ_k, let ξ be an element of S_k, and denote by $S_k(\xi)$ the germ of S_k at ξ.

The property we will establish in the following definition extends the property $\mathcal{B}(k)$ given in the section above.

Definition 3.10. Let $(\underline{x}, \underline{x}') \in \mathcal{M} \times \mathcal{M}$ be such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x}') = \underline{z}$. We say that **property** $\mathfrak{C}(k)(\underline{x}, \underline{x}')$ is satisfied if there exists a sequence $\{\xi_1, \dots, \xi_p\}$ such that: i) $\xi_1 = \underline{x}$, $\xi_p = \underline{x}'$, and for all j, $\varphi_k(\xi_j) = \underline{z}$, ii) $D_{\varphi_k}(\xi_j, \xi_{j+1}) \nsubseteq S_{k-1}(\xi_j) \times S_{k-1}(\xi_{j+1})$, it means that $\mathcal{B}(k, \xi_j, \xi_{j+1})$ (see (2.16)) is satisfied.

Property $\mathfrak{C}(k)$ is satisfied if, for every $(\underline{x}, \underline{x}') \in \mathcal{M} \times \mathcal{M}$ such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x})$, $\mathfrak{C}(k)(\underline{x}, \underline{x}')$ is satisfied.

Remark 3.11. Property $\mathcal{B}(k)$ is a particular case of $\mathfrak{C}(k)$.

This remark follows directly from the definitions of $\mathcal{B}(k)$ and $\mathfrak{C}(k)$.

Proposition 3.12 (the proof is given in the appendix)**.** *Assume that system (2.1) is differentially observable of order N, and that property $\mathfrak{C}(k)$ is satisfied for $2 \le k \le N-1$. Then $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property for $2 \le k \le N-1$, and from theorem 3.8, φ_N immerses system (2.1) in a normal form of observability.*

The following proposition states that systems stated in theorem 2.4 are necessarily differentially observable.

Proposition 3.13 (the proof is given in Sect. 4)**.** *If system (2.1) satisfies the hypotheses of theorem 2.4, then it is differentially observable of order $m+2$, where $m = \dim \mathcal{M}$.*

Corollary 3.14. *Theorem 2.4 is a special case of theorem 3.8.*

Indeed, since $\mathcal{B}(k)$ is a special case of property $\mathfrak{C}(k)$, the corollary follows trivially from propositions 3.12 and 3.13.

3.2. Some examples

In what follows, we will present three examples. In the first example, we illustrate theorem 3.8 by giving its proof for differentially observable systems of order 3. In the second example, we give a numerical application of a 2-dimensional systems that take the normal form in 3 dimensions. For these systems, the assumptions of theorem 3.8 are satisfied, but they do not satisfy the $\mathfrak{C}(k)$'s properties. Finally, in the last example, we present a two-dimensional system that transforms into a normal form, and which satisfies the $\mathfrak{C}(k)$'s properties, but not the $\mathcal{B}(k)$'s properties.

Example 3.15. In this example, we illustrate the proof of the theorem 3.8 using differentially observable systems of order 3.

Let $f = f_1 \dfrac{\partial}{\partial x_1} + f_2 \dfrac{\partial}{\partial x_2}$, $g = g_1 \dfrac{\partial}{\partial x_1} + g_2 \dfrac{\partial}{\partial x_2}$ be smooths vector fields on $\mathbb{R}^2$, and h is a smooth scalar function (for instance f, g and h are of class $\mathcal{C}^2$), $\varphi_1 = h$ and $\varphi_2 = (h, L_f(h))$.

Assume that the system defined by (f, g, h) satisfies the assumptions of theorem 3.8 with a differentiability order N=3. This assumption can therefore be summarized as follows:

- $H_1)$ System (f, g, h) is differentially observable of order $N = 3$.
- $H_2)$ For X in $\{f, g\}$, $\mathcal{X}(1) = \{L_g(h)\}$ satisfies the $(\varphi_1, \varphi_2, X)$-composite function property, for $X = f, g$. Namely, if there exists a function χ such that $L_g(h) = \chi \circ h$, then there exist functions χ_1, χ_2 such that $L_f L_g(h) = \chi_1 \circ \varphi_2$ and $L_g^2(h) = \chi_2 \circ \varphi_2$, where $\varphi_2 = (h, L_f(h))$.

Note that if $L_g(h)$ is of class $\mathcal{C}^1$, then $H_2)$ is always satisfied.

The purpose of this example is to show that under hypotheses $H_1)$ and $H_2)$, we obtain:

$$L_g(h) = G_1 \circ h, \quad L_g L_f(h) = G_2 \circ \varphi_2 \tag{3.1}$$

Let F and Σ_3 be the vector field and the observability map which are stated in Definition 2.1:

$F(x, \underline{u}_2) = f + u_1 g + u_2 \dfrac{\partial}{\partial u_1}$ where $\underline{u}_2 = (u_1, u_2)$, and $\Sigma_3(x, \underline{u}_2) = ((h(x), L_F(h)(x, \underline{u}_2), L_F^2(h)(x, \underline{u}_2)), \underline{u}_2)$.

Denote by $\Sigma_3(.,\underline{u}_2)$ the map from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by $\Sigma_3(.,\underline{u}_2)(x) = (h(x), L_F(h)(x,\underline{u}_2), L_F^2(h)(x,\underline{u}_2))$. A simple computation yields:

$$\begin{aligned}\Sigma_3(.,\underline{u}_2)(x) = (h(x), L_f(h)(x) + u_1 L_g(h)(x), L_f^2(h)(x) + u_1 L_f L_g(h)(x) + u_1 L_g L_f(h)(x) \\ + u_1^2 L_g^2(h)(x) + u_2 L_g(h)(x))\end{aligned} \tag{3.2}$$

The differential observability of order 3 (see Def. 2.1) means that for every fixed $\underline{u}_2$, the map $\Sigma_3(.,\underline{u}_2)$ is injective, and hence the hypotheses $H_1)$, $H_2)$ are then equivalent to:

- $\widetilde{H}_1)$ $\Sigma_3(.,\underline{u}_2)$ is injective for every fixed $\underline{u}_2$.
- $\widetilde{H}_2)$ If $L_g(h) = G_1 \circ h$ for some function G_1, then there exist functions χ_1, χ_2 such that $L_f L_g(h) = \chi_1 \circ \varphi_2$ and $L_g^2(h) = \chi_2 \circ \varphi_2$ (the $(\varphi_1, \varphi_2, X)$-composite function property of $\mathcal{X}(1)$).

In the following we will give the proof of (3.1). In other words, we will show the following:

$$\begin{cases} \text{if } h(x) = h(x') \text{ then } \ L_g(h)(x) = L_g(h)(x'); \\ \text{if } \varphi_2(x) = \varphi_2(x') \text{ then } \ L_g L_f(h)(x) = L_g L_f(h)(x') \end{cases} \tag{3.3}$$

- Proof of the first property of (3.3)
  Assume that the first property of (3.3) is not true, then we can find $\underline{x}$, $\underline{x}'$ satisfying $Lg(h)(\underline{x}) \neq L_g(h)(\underline{x}')$ with $h(\underline{x}) = h(\underline{x}')$. In this case set $u_1^* = -\dfrac{L_f(h)(\underline{x}) - L_f(h)(\underline{x}')}{L_g(h)(\underline{x}) - L_g(h)(\underline{x}')}$ and $u_2^* = -\dfrac{\lambda(\underline{x},\underline{u}_1^*) - \lambda(\underline{x}',\underline{u}_1^*)}{L_g(h)(\underline{x}) - L_g(h)(\underline{x}')}$, where $\lambda(x,u_1) = L_f^2(h)(x) + u_1 L_f L_g(h)(x) + u_1 L_g L_f(h)(x) + u_1^2 L_g^2(h)(x)$, and using the definition the expression of Σ_3 given in (3.2), we deduce that $\Sigma_3(\underline{x},\underline{u}_2^*) = \Sigma_3(\underline{x}',\underline{u}_2^*)$. But $\underline{x} \neq \underline{x}'$ (since $Lg(h)(\underline{x}) \neq L_g(h)(\underline{x}')$), this contradicts the injectivity of $\Sigma_3(.,\underline{u}_2^*)$.
- Let us show the second property of (3.3).
  From the first property of (3.3), we know that there exists a function G_1, such that $L_g(h) = G_1 \circ h$, and from hypothesis $\widetilde{H}_2)$, there exist functions χ_1, χ_2 such that $L_f L_g(h) = \chi_1 \circ \varphi_2$ and $L_g^2(h) = \chi_2 \circ \varphi_2$. Combining this last fact with expression (3.2), we obtain:

$$\Sigma_3(.,\underline{u}_2) = (h(x), L_f(h)(x) + u_1 G_1 \circ h(x), L_f^2(h)(x) + u_1 \chi_1 \circ \varphi_2 + u_1^2 \chi_2 \circ \varphi_2 + u_2 G_1 \circ h(x) + u_1 L_g L_f(h)(x)) \tag{3.4}$$

  Assume that the second property of (3.3) is not true, then there exists $\widetilde{x}$, $\widetilde{x}'$, $\widetilde{x} \neq \widetilde{x}'$ such that $\varphi_2(\widetilde{x}) = \varphi_2(\widetilde{x}')$ and that $L_g L_f(h)(\widetilde{x}) \neq L_g L_f(h)(\widetilde{x}')$. Set $\widetilde{u}_1 = -\dfrac{L_f^2(h)(\widetilde{x}) - L_f^2(h)(\widetilde{x}')}{L_g L_f(h)(\widetilde{x}) - L_g L_f(h)(\widetilde{x}')}$ and $\underline{\widetilde{u}}_2 = (\widetilde{u}_1, \widetilde{u}_2)$, where $\widetilde{u}_2$ is arbitrary constant, a simple computation gives $\Sigma_3(\widetilde{x},\underline{\widetilde{u}}_2) = \Sigma_3(\widetilde{x}',\underline{\widetilde{u}}_2)$, which contradicts the injectivity of $\Sigma_3(.,\underline{\widetilde{u}}_2)$. Therefore the second property of property of (3.3) is then satisfied.

Remark 3.16. Note that in this proof we only used algebraic tools and no dynamical properties related to the system (f,g,h) were deployed.

Example 3.17. Consider the input-output system (f,g,h) defined on $\mathbb{R}^2$ by $f = x_2 \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}$, $g = g_1 \dfrac{\partial}{\partial x_1} + g_2 \dfrac{\partial}{\partial x_2}$, $h(x_1,x_2) = x_1^3$, where g_1, g_2 are smooths functions (of class $\mathcal{C}^p$, $p \geq 3$) and as above, $\varphi_k = (h, \ldots, L_f^{k-1}(h))$.

The class of systems (f,g,h) that we consider are those that can be transformed by the map $\varphi_4 = (h, \ldots, L_f^3(h))$ into a normal form of observability. This means that there exists functions G_i, $1 \leq i \leq 3$ (which are not necessarily smooths) such that:

$$\begin{cases} L_g(h) = G_1 \circ \varphi_1 = G_1 \circ h \\ L_g L_f(h) = G_2 \circ \varphi_2 = G_2 \circ (h, L_f(h)) \\ L_g L_f^2(h) = G_2 \circ \varphi_3 = G_2 \circ (h, L_f(h), L_f^2(h)) \end{cases} \tag{3.5}$$

For these systems, we will show that the assumptions of theorem 3.8 are satisfied, but they do not satisfy the $\mathfrak{C}(k)$'s properties.

More precisely, we will check the following properties:

1) The class of systems (f, g, h) satisfying property (3.5) is not empty.
2) If system (f, g, h) satisfies property (3.5), then the set $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property, for $X \in \{f, g\}$ and $2 \leq k \leq N-1$ (here $N = 4$).
3) Systems (f, g, h) which satisfy property (3.5) are differentially observable of order $N = 4$.
4) Property $\mathfrak{C}(2)$ is not satisfied.

1) First let us show φ_4 is injective.
Using the expression of f, g, h, we get:

$$h = x_1^3,\ L_f(h) = 3x_1^2 x_2,\ L_f^2(h) = 3(2x_1 x_2^2 + x_1^2),\ L_f^3(h) = 6(x_2^3 + 3x_1 x_2) \tag{3.6}$$

Let $x = (x_1, x_2)$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ such that $\varphi_4(x) = \varphi_4(\widetilde{x})$, and let us show that $x = \widetilde{x}$. Note that if $x_1 \neq 0$, then $(h(x), L_f(h)(x)) = (h(\widetilde{x}), L_f(h)(\widetilde{x}))$ implies $x = \widetilde{x}$, and if $x_1 = 0$, then $\widetilde{x}_1 = 0$, and hence $L_f^3(h)(x) = L_f^3(h)(\widetilde{x})$ implies $x_2 = \widetilde{x}_2$.
Clearly, if we set $g = 0$, then φ_4 transforms system (f, g, h) into a non controlled normal form. Hence the set of systems (f, g, h) satisfying property (3.5) is not empty.

2) Assume that system (f, g, h) satisfies property (3.5), and let us check that:

$$\text{for } 2 \leq k \leq N-1 = 3 \text{ and } X \in \{f, g\},\ \text{the set } \mathcal{X}(k-1) \text{ satisfies the } (\varphi_{k-1}\varphi_k, X) - \textit{property} \tag{3.7}$$

In other words, we will check:

$$\begin{cases} \text{for } 2 \leq k \leq 3, \\ \text{if } \psi \in \mathcal{X}(k-1) \text{ is such that } \psi = \chi \circ \varphi_{k-1},\ \text{for some function } \chi \text{ then:} \\ \text{there exist functions } \chi', \widetilde{\chi}' \text{ such that } L_f(\psi) = \chi' \circ \varphi_k \text{ and } L_g(\psi) = \widetilde{\chi}' \circ \varphi_k. \end{cases} \tag{3.8}$$

The first equality of (3.5) implies that the smooth function g_1 only depends on x_1: $g_1 = g_1(x_1)$.
Now consider $x = (0, x_2)$ and using the last equality of (3.5) , we deduce that $6x_2^2 g_1(0) = G_3(0)$ for every x_2, thus $g_1(0) = 0$. Now using the fact that g_1 is of class $\mathcal{C}^p$, $p \geq 3$, we deduce that:

$$g_1 = x_1 \widetilde{g}_1(x_1),\ \text{for some smooth function } \widetilde{g}_1 \tag{3.9}$$

and hence (3.5) takes the following expression:

$$\begin{cases} L_g(h) = 3x_1^3 \widetilde{g}_1(x_1) = G_1 \circ \varphi_1 \\ L_g L_f(h) = 6x_1^2 x_2 \widetilde{g}_1(x_1) + 3x_1^2 g_2 = G_2 \circ \varphi_2 = G_2(x_1^3, 3x_1^2 x_2) \\ L_g L_f^2(h) = 6x_1(x_2^2 + x_1)\widetilde{g}_1(x_1) + 12x_1 x_2 g_2 = G_3 \circ \varphi_3 = G_3(x_1^3, 3x_1^2 x_2, 6x_1 x_2^2 + 3x_1^2) \end{cases} \tag{3.10}$$

Now let us check (3.8).

i) For $k=2$, $\mathcal{X}(k-1)=\mathcal{X}(1)=\{L_g(h)\}$, $\varphi_1=h=x_1^3$, $\varphi_2=(h,L_f(h))=(x_1^3,3x_1^2x_2)$.
In this case, $\psi\in\mathcal{X}(1)$ means that $\psi=L_g(h)$, and from the first equality of (3.10), $\psi=3x_1^3\widetilde{g}_1(x_1)$, and hence there exists a function χ such that $\psi=\chi\circ\varphi_1$. In order to check (3.8) for $k=2$ we have to show that there exist functions χ', $\widetilde{\chi}'$ such that:

$$\begin{cases} L_f(\psi)=L_fL_g(h)=\chi'\circ\varphi_2 \\ L_g(\psi)=L_g^2(h)=\widetilde{\chi}'\circ\varphi_2 \end{cases} \tag{3.11}$$

a) $L_f(\psi)=L_fL_g(h)=3x_1^2x_2(3\widetilde{g}_1(x_1)+x_1\widetilde{g}'(x_1))$, and hence $L_f(\psi)=\chi'\circ\varphi_2$, for some function χ'.
b) $L_g(\psi))=L_g(3x_1^3\widetilde{g}_1(x_1))=x_1\widetilde{g}_1(x_1)(9x_1^2\widetilde{g}_1(x_1)+3x_1^3\widetilde{g}_1'(x_1))$, thus we can find a function χ' such that $L_g(\psi)=\widetilde{\chi}'\circ\varphi_1$.

ii) For $k=3$, $\mathcal{X}(k-1)=\mathcal{X}(2)=\{L_g^2(h),L_fL_g(h),L_gL_f(h)\}$, and from (3.10), we know that $L_gL_f(h)=G_2\circ\varphi_2$. Combining this last fact with (3.11), we obtain:

$$\text{For every } \psi \text{ in } \mathcal{X}(2);\ \text{there exists a function } \chi \text{ such that } \psi=\chi\circ\varphi_2 \tag{3.12}$$

In order to show (3.8) for $k=3$, it suffices to check the following property:

$$\text{For every } \psi \text{ in } \mathcal{X}(2),\ \text{for every } X \text{ in } \{f,g\},\ \text{if } \varphi_3(x)=\varphi_3(x') \text{ then } L_X(\psi)(x)=L_X(\psi)(x') \tag{3.13}$$

which implies that $L_X(\psi)=\widetilde{\chi}\circ\varphi_3$ for some function $\widetilde{\chi}$.
To do so, let x, x' be such that $\varphi_3(x)=\varphi_3(x')$.
- If $x_1\neq 0$, then $x=x'$, and hence (3.13) is trivially satisfied
- If $x_1=0$, then $x_1'=0$. In this case, using the two first expression of (3.10) and a) above, we deduce that every $\psi\in\mathcal{X}(2)$ takes the form: $\psi=x_1^2\widetilde{\psi}$ for some differentiable function $\widetilde{\psi}$, and hence $L_X(\psi)(x)=L_X(\psi)(x')=0$.

This ends the proof of (3.8).
We end this part by the following remark:
- By definition $\mathcal{X}(3)=\{L_gL_f^2(h)\}\cup L_f(\mathcal{X}(2))\cup L_g\mathcal{X}(2))$, and form (3.12), we have: for every $\psi\in L_f(\mathcal{X}(2))\cup L_g\mathcal{X}(2))$; there exists χ such that $\psi=\chi\circ\varphi_2$.
- From the last equality of (3.10), we know that $L_gL_f^2(h)=G_3\circ\varphi_3$.

Combining these facts, we get:

$$\text{For every } \psi \text{ in } \mathcal{X}(3);\ \text{there exists a function } \chi \text{ such that } \psi=\chi\circ\varphi_3 \tag{3.14}$$

3) In this part, we will show that system (f,g,h) is differentially observable of order $N=4$.
Let F and Σ_4 be the vector field and the observability map which are stated in Definition 2.3:
$F(x,\underline{u}_3)=f+u_1g+u_2\dfrac{\partial}{\partial u_1}+u_3\dfrac{\partial}{\partial u_2}$ where $\underline{u}_2=(u_1,u_2)$, $\underline{u}_3=(u_1,u_2,u_3)$, and $\Sigma_4(x,\underline{u}_3)=(h(x),L_F(h)(x,u_1),L_F^2(h)(x,\underline{u}_2),L_F^3(h)(x,\underline{u}_3),\underline{u}_3)$.
For every fixed $\underline{u}_3$, let $\Sigma_4(.,\underline{u}_3):\mathbb{R}^2\to\mathbb{R}^4$ be the map defined by: $\Sigma_4(.,\underline{u}_3)(x)=(h(x),L_F(h)(x,u_1),L_F^2(h)(x,\underline{u}_2),L_F^3(h)(x,\underline{u}_3))$.
In the sequel, we will show that for every $\underline{u}_3$, $\Sigma_4(.,\underline{u}_3)$ is injective, which means that system (f,g,h) is differentially observale.
Recall that $\mathcal{X}(1)=\{L_g(h)\}$, $\mathcal{X}(2)=\{L_gL_f(h),L_fL_g(h),L_g^2(h)\}$ and $\mathcal{X}(3)=\{L_gL_f^2(h)\}\cup L_f(\mathcal{X}(2))\cup L_g(\mathcal{X}(2))$, and set $\widehat{\mathcal{X}}(1)=\mathcal{X}(1)$, $\widehat{\mathcal{X}}(2)=\mathcal{X}(1)\cup\mathcal{X}(2)$ and $\widehat{\mathcal{X}}(3)=\mathcal{X}(1)\cup\mathcal{X}(2)\cup\mathcal{X}(3)$. In order to compute the Lie derivatives $L_F^k(h)$, we will use the following formula. Let $\sigma(x,\underline{u}_2)$ be a smooth function with respect

to $(x, \underline{u}_2)$, then

$$L_F(\sigma) = L_{f+u_1 g}(\sigma(., \underline{u}_2)) + \frac{\partial \sigma(x,.)}{\partial \underline{u}_2}\underline{u}_3 \tag{3.15}$$

Let us compute the expression of the Lie derivatives $L_F(h)$, $L_F^2(h)$, $L_F^3(h)$:

$$L_F(h) = L_f(h) + u_1 L_g(h) \tag{3.16}$$

Applying (3.15) to formula (3.16), we obtain:

$$L_F^2(h) = L_{f+u_1 g}^2(h) + u_2 L_g(h) = L_f^2(h) + \sum_{\psi \in \widehat{\mathcal{X}}(2)} P_\psi(\underline{u}_2)\psi \tag{3.17}$$

where $P_\psi(\underline{u}_2)$ is a polynomial function of the variable $\underline{u}_2$.
Applying again (3.15) to the function $L_F^2(h)$, we get:
$L_F^3(h) = L_{f+u_1 g} L_f^2(h) + \sum_{\psi \in \widehat{\mathcal{X}}(2)} P_\psi(\underline{u}_2) L_{f+u_1 g}(\psi) + \sum_{\psi \in \widehat{\mathcal{X}}(2)} [\frac{\partial P_\psi(\underline{u}_2)}{\partial \underline{u}_2}\underline{u}_3]\psi$, and hence,

$$L_F^3(h) = L_f^3(h) + \sum_{\psi \in \widehat{\mathcal{X}}(3)} P_\psi(\underline{u}_3)\psi \tag{3.18}$$

where $P_\psi(\underline{u}_3)$ is a polynomial function of the variable $\underline{u}_3$.
From (3.12) and (3.14), we deduce that:

$$\begin{cases} \sum_{\psi \in \widehat{\mathcal{X}}(2)} P_\psi(\underline{u}_2)\psi(x) = \sigma_2(\varphi_2(x), \underline{u}_2) \\ \sum_{\psi \in \widehat{\mathcal{X}}(3)} P_\psi(\underline{u}_3)\psi(x) = \sigma_3(\varphi_3(x), \underline{u}_3) \end{cases} \tag{3.19}$$

where $\sigma_k(z_1, \ldots, z_k, \underline{u}_k)$ is a function which not necessarily smooth w.r.t. $(z_1, \ldots, z_k)$.
Now combining (3.16), (3.17), (3.18) and (3.19) with the fact that $L_g(h) = G_1 \circ h$ (see (3.5)), we obtain:

$$\begin{cases} L_F(h) = L_f(h) + u_1 \sigma_1 \circ \varphi_1 \\ L_F^2(h) = L_f^2(h) + \sigma_2(\varphi_2(x), \underline{u}_2) \\ L_F^3(h) = L_f^3(h) + \sigma_3(\varphi_3(x), \underline{u}_3) \end{cases} \tag{3.20}$$

where $\sigma_1 = G_1$.
Finally combining (3.20) with the fact that $\varphi_4 = (h, L_f(h), L_f^2(h), L_f^3(h))$ is injective (see 1) above), we deduce the injectivity of $\Sigma_4(., \underline{u}_3)$ for every fixed $\underline{u}_3$.

4) In this part, we will show that $\mathfrak{C}(2)$ is not satisfied.
Recall that $\varphi_1(x) = h(x) = x_1^3$, $\varphi_2(x) = (h(x), L_f(h)(x)) = (x_1^3, 3x_1^2 x_2)$.
The set of singularities of φ_1 is $S_1 = (x_1 = 0)$. Also, note that if $\underline{x} = (0,0)$ and $\underline{x}' = (0, \underline{x}_2)$, $\underline{x}_2 \neq 0$, then for every ϵ in the interval $]0, \frac{|\underline{x}'|}{2}[$, $\varphi_2(B_\epsilon(\underline{x})) \cap \varphi_2(B_\epsilon(\underline{x}')) = \{(0,0)\}$ (where $B_\epsilon(x)$ is the ball centered at x and of radius ϵ). Thus, if $(x, x') \in B_\epsilon(\underline{x}) \times B_\epsilon(\underline{x}')$ is such that $\varphi_2(x) = \varphi_2(x')$, then $x_1 = x_1' = 0$, and hence x, x' are singular points of φ_1. Consequently, we cannot approximate $(\underline{x}, \underline{x}')$ by (x, x') such that $\varphi_2(x) = \varphi_2(x')$ and that x or x' is a regular point of φ_1. Hence $\mathcal{B}(2, \underline{x}, \underline{x}')$ cannot be satisfied. Using this last fact, we can directly deduce that we cannot construct a finite sequence $\xi_1, \ldots, \xi_p$ such that $\xi_1 = \underline{x}$,

$\xi_p = \underline{x}'$, $\varphi_2(\xi_i) = \varphi_2(\xi_{i+1})$ and such that $\mathcal{B}(2, \xi_i, \xi_{i+1})$ is satisfied for $i = 1, \dots, p$. Consequently, $\mathfrak{C}(2)$ is not fulfilled.

Example 3.18. In this example, we will consider a class of differentially observable systems of order 3, and satisfying the property $\mathfrak{C}(2)$, but which on the other hand do not satisfy the property $\mathcal{B}(2)$. We also note that according to the proposition 3.12, these systems transform into a normal form of observability.

Set $f = \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}$, $h(x) = x_1 x_2^2$, and consider vector fields $g = g_1 \dfrac{\partial}{\partial x_1} + g_2 \dfrac{\partial}{\partial x_2}$, we have:

$$\begin{cases} \varphi_1(x) = h(x) = x_1 x_2^2;\ \varphi_2(x) = (h(x), L_f(h)(x)) = (x_1 x_2^2, x_2^2 + 2x_1 x_2); \\ \varphi_3(x) = (h(x), L_f(h)(x), f^2(h)(x)) = (x_1 x_2^2, x_2^2 + 2x_1 x_2, 2(x_1 + 2x_2)) \end{cases} \tag{3.21}$$

We claim that the set of systems (f, g, h) which are differentially observable and for which $\mathcal{B}(2)$ is not satisfied is a nonempty set. Indeed, according to example 2.8, property $\mathcal{B}(2)$ is not satisfied, and $\varphi_3 = (h, L_f(h), L_f^2(h))$ is injective. Hence for $g = 0$, the system (f, g, h) is differentially observable and $\mathcal{B}(2)$ is not satisfied.

Now let (f, g, h) be as above, and assume that (f, g, h) is a differentially observable system. In what follows, we will show that $\mathfrak{C}(2)$ is satisfied, and hence from proposition 3.12, φ_3 transforms system (f, g, h) in a normal form of observability.

In order to show $\mathfrak{C}(2)$, it suffices to check $\mathfrak{C}(2)(\underline{x}, \underline{x}')$ for every singular points $\underline{x}$, $\underline{x}'$ of $\varphi_1 = h$ for which $\varphi_2(\underline{x}) = \varphi_2(\underline{x})$, namely for $\underline{x} = (\underline{x}_1, 0)$, $\underline{x}' = (\underline{x}_1', 0)$ (since the set of singular points of $\varphi_1 = h$ is $\mathbb{R} \times \{0\}$). This problem can be solved if we can show that $\mathcal{B}(2, \underline{x}, \underline{x}')$ is satisfied for $\underline{x} = (0, 0)$ and $\underline{x}' = (\underline{x}_1', 0)$, where $\underline{x}_1'$ is arbitrary. Indeed, if this is the case, then for any two points $\underline{x} = (\underline{x}_1, 0)$, $\underline{x}' = (\underline{x}_1', 0)$, the elements $\xi_1 = \underline{x}$, $\xi_2 = (0, 0)$ and $\xi_3 = \underline{x}'$ satisfy properties i)-ii) of Definition 3.10 (namely, $\mathcal{B}(2, \xi_1, \xi_2)$, $\mathcal{B}(2, \xi_1, \xi_2)$ are fulfilled), and hence $\mathfrak{C}(2)(\underline{x}, \underline{x}')$ holds for every singular points $\underline{x}, \underline{x}'$ of φ_1.

Without loss of generality, consider $\underline{x} = (0, 0)$, $\underline{x}' = (\underline{x}_1', 0)$, with $\underline{x}_1' > 0$, and let us check $\mathcal{B}(2, \underline{x}, \underline{x}')$. For this propose, we will show that for every $\epsilon > 0$ sufficiently small, we can find $x = (x_1, x_2)$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ such that $\|x\| < \epsilon$, $\|\widetilde{x} - \underline{x}'\| < \epsilon$, $x_2 \neq 0$ and $\varphi_2(x) = \varphi_2(\widetilde{x})$: which means that x is a regular point of $\varphi_1 = h$, and hence $\mathcal{B}(2, \underline{x}, \underline{x}')$ is satisfied. To do this, we will characterize all $(x, \widetilde{x})$ which are close to $(\underline{x}, \underline{x}')$ and satisfying $\varphi_2(x) = \varphi_2(\widetilde{x})$, and $x_1 \neq 0$, $x_2 \neq 0$, it means:

$$\begin{cases} x_1 \neq 0, \quad x_2 \neq 0, \text{ and,} \\ x_1 x_2^2 = \widetilde{x}_1 \widetilde{x}_2^2 \\ x_2^2 + 2x_1 x_2 = \widetilde{x}_2^2 + 2\widetilde{x}_1 \widetilde{x}_2 \end{cases} \tag{3.22}$$

Take $\epsilon \in]0, \frac{\underline{x}_1'}{2}[$ sufficiently small, and le $\widetilde{x}_1$ be any element of $]\underline{x}_1' - \epsilon, \underline{x}_1' + \epsilon[$ (by construction $\widetilde{x}_1 > \frac{\underline{x}_1'}{2} > 0$). Let x_1 be any sufficiently small and positive number. A solution (x_1, x_2), $(\widetilde{x}_1, \widetilde{x}_2)$ of (3.22) can be computed as follows:

$$\begin{cases} \widetilde{x}_2^2 = \dfrac{x_1}{\widetilde{x}_1} x_2^2, \text{ (or } \ \widetilde{x}_2 = \pm \sqrt{\dfrac{x_1}{\widetilde{x}_1}} x_2) \\ x_2 = \dfrac{-2x_1 \pm 2\sqrt{x_1 \widetilde{x}_1}}{(1 - \frac{x_1}{\widetilde{x}_1})} \end{cases} \tag{3.23}$$

Thus for every $\epsilon > 0$, we can solve $\varphi_2(x) = \varphi_2(\widetilde{x})$, with $x_2 \neq 0$ and $\|x\| < \epsilon$, $\|\widetilde{x} - \underline{x}'\| < \epsilon$. Thus $\mathcal{B}(2, \underline{x}, \underline{x}')$ is satisfied for $\underline{x} = (0, 0)$, $\underline{x}' = (\underline{x}_1', 0)$.

3.3. Some preliminary results

In this subsection we give some results that allow us to extend some functions to continuous or smooth functions.

Lemma 3.19. *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a continuous map between two topological spaces, and let $\psi : \mathcal{M} \to \mathbb{R}$ be a continuous map satisfying $\psi = \theta \circ \varphi$, for some scalar function θ defined on $\varphi(\mathcal{M})$. Then for every compact subset K of $\mathcal{M}$, the restriction of θ to the compact set $\varphi(K)$ is continuous (of course, $\varphi(K)$ is equipped with the relative topology induced by that of $\mathcal{N}$).*

The proof of this lemma can easily be obtained using a compactness argument.

Lemma 3.20. *Let φ, ψ and θ be as in lemma 3.19, and assume that φ is semi-proper. Then θ is continuous on $\varphi(\mathcal{M})$ equipped with the relative topology induced by that of $\mathcal{N}$. Moreover, using Tietz extension theorem, θ can be extended to a continuous function on $\mathcal{N}$.*

Proof. of lemma 3.20

It suffices to check that for every z in $\varphi(\mathcal{M})$; there exists an open neighbourhood V_z of z in $\varphi(\mathcal{M})$ such that the restriction of θ to V_z is continuous. To do so, let W_z be a relatively compact open neighbourhood of z in $\mathcal{N}$, and consider the open neighbourhood V_z of $\varphi(\mathcal{M})$ defined by $V_z = W_z \cap \varphi(\mathcal{M})$. Since φ is a semi-proper continuous map, from property 3.4, $\varphi(\mathcal{M})$ is then a closed subset of $\mathcal{N}$. Hence the closure $\overline{V_z}$ is a compact subset of $\varphi(\mathcal{M})$. Using again the fact that φ is a semi-proper continuous map, it follows that $\overline{V_z} = \varphi(K)$ for some compact K of $\mathcal{M}$ (see Def. 3.2). Finally, according to lemma 3.19, it follows that the restriction of θ to $\overline{V_z}$ is continuous. This ends the proof of the lemma. □

If we omit the semi-proper property of φ, the continuity of θ is no longer guaranteed even if $\varphi(M)$ is a closed subset of $\mathcal{N}$, as the following example shows.

Example 3.21. Here $\mathcal{M} = \mathbb{R}$, $\mathcal{N} = \mathbb{R}^2$, φ, ψ and θ are defined as follows:

- $\varphi(x) = (x+1, 0)$, if $x \leq 0$, and $\varphi(x) = (\mathrm{e}^{-x}, x\mathrm{e}^{-x})$, if $x > 0$.
- $\psi(x) = x$
- $\theta(z_1, 0) = z_1 - 1$; $\theta(z_1, z_2) = \dfrac{z_2}{z_1}$ if $z_1 \neq 0$ and $z_2 \neq 0$; $\theta(0, z_2) = \theta_0(z_2)$ if $z_2 \neq 0$, and where θ_0 is any function.

Clearly,

i) φ is continuous on $\mathbb{R}$ and $\varphi(\mathbb{R})$ is a closed subset of $\mathbb{R}^2$, and $\theta \circ \varphi = \psi$.
ii) θ is not continuous on $\varphi(\mathbb{R})$ at $(0, 0)$.
iii) φ is not semi-proper. Indeed, there is no compact L of $\mathbb{R}$ such that $\varphi(L) = \overline{B(0,1)} \cap \varphi(\mathbb{R})$, where $\overline{B(0,1)}$ is the closed ball centered at 0 and of radius 1.

Note that in the above example, we can choose another φ which is sufficiently smooth and such that properties i), ii), iii) are satisfied.

In order to prove the part 3) of theorem 3.8, the following lemma will be required.

Lemma 3.22. *Let $\varphi : \mathcal{M} \to \mathbb{R}^k$ be a smooth map and assume that $\mathcal{M} \setminus S \neq \emptyset$, where S is the set of singular points of φ. Let χ be a scalar map on $\varphi(\mathcal{M})$, such that $\chi \circ \varphi$ is smooth on $\mathcal{M}$. Then χ can be extended to a smooth function on an open set of $\mathbb{R}^k$ which contains $\varphi(\mathcal{M} \setminus S)$.*

Proof. Set $\mathcal{Z} = \varphi(\mathcal{M} \setminus S)$, by construction $\mathcal{Z} \subset \varphi(\mathcal{M})_{Reg}$ (the set of regular points of $\varphi(\mathcal{M})$). Now according to 2) of Definition 3.1, for every z in $\mathcal{Z}$; for every x in $\varphi^{-1}(z) \cap (\mathcal{M} \setminus S)$ (a regular point of φ), we can find an open neighbourhood U_x of x in $\mathcal{M}$ and an open neighbourhood V_z of z in $\mathcal{N}$ such that: i) $N_z = V_z \cap \varphi(\mathcal{M})$ is a sub-manifold of $\mathbb{R}^k$, ii) the restriction map $\varphi|_{U_x}$ is a submersion from U_x onto the manifold N_z. By construction, it is obvious to check that:

$$N_z = V_z \cap \varphi(\mathcal{M}) \subset \mathcal{Z} \tag{3.24}$$

Now since $\psi = \chi \circ \varphi$ is smooth on $\mathcal{M}$, using property ii) of remark 2.5, we deduce that χ is smooth on N_z. Let W_z be any fixed neighbourhood of z in $\mathcal{N}$ such that $\overline{W}_z$ (the closure of W_z) is contained in V_z. Note that

$\overline{W}_z \cap \varphi(\mathcal{M}) = \overline{W}_z \cap N_z$ is a closed subset of the manifold N_z, hence it is also a closed subset of $\mathcal{N}$ (because N_z is a submanifold of $\mathcal{N}$). Moreover χ is smooth on $\overline{W}_z \cap \varphi(\mathcal{M})$, hence we can extend $\chi|_{\overline{W}_z \cap \varphi(\mathcal{M})}$ to a smooth function $\widehat{\chi}_z$ on V_z.

Now set $\Omega = \bigsqcup_{z\in\mathcal{Z}}$, since Ω is an open set (in particular a paracompact set), we can find a partition of unity of Ω denoted by $\Lambda = (\lambda_z)_{z\in\mathcal{Z}}$ which is subordinate to the open cover $(W_z)|_{z\in\mathcal{Z}}$ of Ω, namely:

- For every ζ in Ω; there is a neighbourhood of ζ where all but a finite of the family Λ are 0.
- The λ_z's are smooths.
- $\sum_{z\in\mathcal{Z}} \lambda_z(\zeta) = 1$, for every ζ in Ω.

From (3.24), we deduce that $\Omega \cap (\varphi(\mathcal{M}) \setminus \mathcal{Z}) = \emptyset$. Now set $\widehat{\chi}(\zeta) = \sum_{z\in\mathcal{Z}} \lambda_z(\zeta)\widehat{\chi}_z(\zeta)$ if $\zeta \in \Omega$ and $\widehat{\chi}(\zeta) = \chi(\zeta)$ if $\zeta \in \varphi(\mathcal{M}) \setminus \mathcal{Z}$. By construction $\widehat{\chi}$ is smooth on the open set Ω which contains $\varphi(\mathcal{M}) \setminus S$, and its restriction to $\varphi(\mathcal{M})$ coincides with χ, which ends the proof of the lemma. $\square$

4. Proof of Theorem 3.8

We recall some notations introduced in Section 2.1:

- $\underline{u}_k = (u_1, \ldots, u_k)$ is a k-tuple of $\mathbb{R}^k$,
- F is the vector field defined on $\mathcal{M} \times \mathbb{R}^{N-1}$ by $F = F(x, \underline{u}_{N-1}) = f(x) + u_1 g(x) + \sum_{j=2}^{N-1} u_j \dfrac{\partial}{\partial u_{j-1}}$,
- $y^{(k)} = L_F^k(h)(x, \underline{u}_{N-1})$. Note that $y^{(k)}$ depends at most on $(x, \underline{u}_k)$.
- $\varphi_k = (h, \ldots, L_f^{k-1}(h))$, for $k \geq 1$.

Consider the map $\Sigma_N : \mathcal{M} \times \mathbb{R}^{N-1} \to \mathbb{R}^N \times \mathbb{R}^{N-1}$ defined by:

$\Sigma_N(x, \underline{u}_{N-1}) = (h(x), L_F(h)(x, \underline{u}_{N-1}), \ldots, L_F^{N-1}(h)(x, \underline{u}_{N-1}), \underline{u}_{N-1})$, and recall that the input-output dynamical system defined by (f, g, h) is differentially observable if Σ_N is injective.

Recall also some notations used to state theorem 3.8:

Let k be an integer such that $1 \leq k \leq N-1$ and set $\underline{X}_k = (X_1, \ldots, X_k)$, where $X_j \in \{f, g\}$ and $L_{\underline{X}_k}(h) = L_{X_k} \ldots L_{X_1}(h)$. Let $\nu_g(\underline{X}_k)$ denote the cardinal of the set of components of $\underline{X}_k$ which are equal to g. Finally, set $\mathbb{X}(k) = \{\underline{X}_k = (X_1, \ldots, X_k) : \ X_j \in \{f, g\},$ and that $\nu_g(\underline{X}_k \geq 1\}$, and $\mathcal{X}(k) = \{L_{\underline{X}_k}(h) : \ \underline{X}_k \in \mathbb{X}(k)\}$.

The proof of theorem 3.8 is based on the following proposition:

Proposition 4.1. *(the proof is presented in Sect. 5).*
The following properties hold:

1)

$$\begin{cases} y^{(1)} = L_f(h)(x) + u_1 L_g(h)(x) \\ \vdots \\ y^{(k)} = \lambda_k(\underline{u}_{k-1}, x) + u_k L_g(h)(x) \\ \vdots \\ y^{(N-1)} = \lambda_{N-1}(\underline{u}_{N-2}, x) + u_{N-1} L_g(h)(x) \end{cases} \tag{4.1}$$

where the $\lambda_k(\underline{u}_{k-1}, x)$'s are smooth w.r.t. x and polynomial w.r.t. $\underline{u}_{k-1}$.

2) In this part, $N \geq 3$.
Assume that the following property holds for $2 \leq k \leq N-1$:
***property** $\mathcal{P}(k)$:*

i) For every j, $2 \leq j \leq k$; for every X in $\{f, g\}$, $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1}, \varphi_j, X)$-composite function property.

ii) For every j, $1 \leq j \leq k-1$; there exists a function G_j such that $L_g L_f^{j-1}(h) = G_j \circ \varphi_j$.

Then the following formulas are satisfied:

$$\begin{cases} \text{for } 1 \leq j \leq k-1, \ y^{(j)} = \Lambda_j(\varphi_{j+1}(x), \underline{u}_j) \\ y^{(k)} = L_f^k(h)(x) + u_1 L_g L_f^{k-1}(h)(x) + \omega_2^k(\underline{u}_k, \varphi_k(x)) \\ y^{(k+1)} = L_f^{k+1}(h)(x) + u_2 L_g L_f^{k-1}(h)(x) + \omega_1^{k+1}(u_1, x) + \omega_2^{k+1}(\underline{u}_{k+1}, \varphi_k(x)) \\ \vdots \\ y^{(k+i)} = L_f^{k+i}(h)(x) + u_{i+1} L_g L_f^{k-1}(h)(x) + \omega_1^{k+i}(\underline{u}_i, x) + \omega_2^{k+i}(\underline{u}_{k+i}, \varphi_k(x)) \\ \vdots \\ y^{(N-1)} = L_f^{N-1}(h)(x) + u_{N-k} L_g L_f^{k-1}(h)(x) + \omega_1^{N-1}(\underline{u}_{N-k-1}, x) + \omega_2^{N-1}(\underline{u}_{N-1}, \varphi_k(x)) \end{cases} \tag{4.2}$$

where the $\omega_1^{k+i}(\underline{u}_i, x)$'s (resp. the $\omega_2^{k+i}(\underline{u}_j, z_1, \ldots, z_k)$)'s) are smooth w.r.t. x and polynomial w.r.t. $\underline{u}_i$ (resp. smooth on $(z_1, \ldots, z_k)$ and polynomial on $\underline{u}_j$).

4.1. Proof of Theorem 3.8

Proof. Part 1) of Theorem 3.8.

It suffices to show the following:

$$\begin{cases} \text{for } 1 \leq k \leq N, \ L_g L_f^{k-1}(h) = G_k \circ \varphi_k \\ L_f^N(h) = \widetilde{G}_N \circ \varphi_N \end{cases} \tag{4.3}$$

The last property of (4.3) is obvious and follows from the injectivity of φ_N (since Σ_N is injective). Indeed, let $z \in \varphi_N(\mathcal{M})$ and let $x \in \mathcal{M}$ be such that $z = \varphi_N(x)$, then $L_f^N(h)(x) = L_f^N(h)(\varphi_N^{-1}(z))$. Hence $\widetilde{G}_N$ is defined on $\varphi_N(\mathcal{M})$ by $\widetilde{G}_N = L_f^N(h) \circ \varphi_N^{-1}$.

Now let us show the first property of (4.3). To do so, it suffices to check the following property:

$$\text{For every } x, x', \ \varphi_k(x) = \varphi_k(x') \text{ implies } L_g L_f^{k-1}(h)(h)(x) = L_g L_f^{k-1}(h)(h)(x') \tag{4.4}$$

Property (4.4) will be obtained by induction on k.

A) If property (4.4) is not true for $k = 1$, then we can find $\underline{x} \neq \underline{x}'$ such that $h(\underline{x}) = h(\underline{x}')$ and $L_g(h)(\underline{x}) \neq L_g(h)(\underline{x}')$.

Now using formula (4.1), we deduce that:

$$L_F^k(h) = y^{(k)} = \lambda_k(\underline{u}_{k-1}, x) + u_k L_g(h)(x), \text{ for } 1 \leq k \leq N-1 \tag{4.5}$$

Thus $\underline{u}_i^* = (u_1^*, \ldots, u_{N-1}^*)$ given by the following triangular formula is well defined:

$$\begin{cases} u_1^* = \dfrac{L_f(h)(\underline{x}') - L_f(h)(\underline{x})}{L_g(h)(\underline{x}) - L_g(h)(\underline{x})} \\ \vdots \\ u_k^* = \dfrac{\lambda_i(\underline{u}_{k-1}^*, \underline{x}') - \lambda_k(\underline{u}_{k-1}^*, \underline{x})}{L_g(h)(\underline{x}) - L_g(h)(\underline{x})} \\ \vdots \\ u_{N-1}^* = \dfrac{\lambda_{N-1}(\underline{u}_{N-2}^*, \underline{x}') - \lambda_{N-1}(\underline{u}_{N-2}^*, \underline{x})}{L_g(h)(\underline{x}) - L_g(h)(\underline{x})} \end{cases} \tag{4.6}$$

and by construction we have $\Sigma_N(\underline{x}, \underline{u}^*_{N-1}) = \Sigma_N(\underline{x}', \underline{u}^*_{N-1})$, which contradicts the fact that system (f, g, h) is differentially observable. Hence (4.4) is true for $k = 1$, which means that $L_g(\varphi_1) = G_1 \circ \varphi_1$, here $\varphi_1 = h$.

B) Let $k \geq 2$, and assume that for $1 \leq i \leq k-1$; there exist functions G_i such that $L_g L_f^{i-1}(h) = G_i \circ \varphi_i$, and let us show property (4.4).
Assume that (4.4) is not true, then we can find $\underline{x} \neq \underline{x}'$ such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x}')$ and that $L_g L_f^{k-1}(h)(\underline{x}) \neq L_g L_f^{k-1}(h)(\underline{x}')$. As for $k = 1$, we will construct a sequence $u^*_{N-1} = (u^*_1, \ldots, u^*_{N-1})$ such that $\Sigma_N(\underline{x}, u^*_{N-1}) = \Sigma_N(\underline{x}', u^*_{N-1})$, which contradicts the differential observability of order N of system (f, g, h).
From the hypothesis of theorem 3.8, we know that for every X in $\{f, g\}$, $\mathcal{X}(i-1)$ satisfies the $(\varphi_{i-1}, \varphi_i, X)$-composite function property. Combining this property with the fact that for $1 \leq i \leq k-1$; there exist functions G_i such that $L_g L_f^{i-1}(h) = G_i \circ \varphi_i$, we deduce that property $\mathcal{P}(k)$ of proposition 4.1 is satisfied.
Now consider the $\omega_1^{k+i}(\underline{u}_i, x))$'s established in (4.2) of proposition 4.1, and let $(u^*_1, \ldots, u^*_{N-k})$ be the sequence given by the following triangular formula:

$$\begin{cases} u^*_1 = \dfrac{L_f^k(h)(\underline{x}') - L_f^k(h)(\underline{x})}{L_g L_f^{k-1}(h)(\underline{x}) - L_g L_f^{k-1}(h)(\underline{x}')} \\ u^*_2 = \dfrac{(L_f^{k+1}(h)(\underline{x}') + \omega_1^{k+1}(u^*_1, \underline{x}')) - (L_f^{k+1}(h)(\underline{x}) + \omega_1^{k+1}(u^*_1, \underline{x}))}{L_g L_f^{k-1}(h)(\underline{x}) - L_g L_f^{k-1}(h)(\underline{x}')} \\ \vdots \\ u^*_{i+1} = \dfrac{(L_f^{k+i}(h)(\underline{x}') + \omega_1^{k+i}(\underline{u}^*_i, \underline{x}')) - (L_f^{k+i}(h)(\underline{x}) + \omega_1^{k+i}(\underline{u}^*_i, \underline{x}))}{L_g L_f^{k-1}(h)(\underline{x}) - L_g L_f^{k-1}(h)(\underline{x}')} \\ \vdots \\ u^*_{N-k} = \dfrac{(L_f^{N-1}(h)(\underline{x}') + \omega_1^{N-1}(\underline{u}^*_{N-k-1}, \underline{x}')) - (L_f^{N-1}(h)(\underline{x}) + \omega_1^{N-1}(\underline{u}^*_{N-k-1}, \underline{x}))}{L_g L_f^{k-1}(h)(\underline{x}) - L_g L_f^{k-1}(h)(\underline{x}')} \end{cases} \tag{4.7}$$

Set $\underline{u}^*_{N-1} = (u^*_1, \ldots, u^*_{N-k}, u_{N-k+1}, \ldots, u_{N-1})$ where $u_{N-k+1}, \ldots, u_{N-1}$ are arbitrary constants.
Now put $\underline{u}^*_{N-1}$ in formula (4.2), we deduce that $L_F^i(h)(\underline{x}, \underline{u}^*_{N-1}) = L_F^i(h)(\underline{x}', \underline{u}^*_{N-1})$, for $1 \leq i \leq N-1$. Hence $\Sigma_N(\underline{x}, \underline{u}^*_{N-1}) = \Sigma_N(\underline{x}', \underline{u}^*_{N-1})$, which contradicts the differential observability of system (f, g, h). This ends the proof of part 1) of theorem 3.8.

□

We conclude this part with the following remark which summarizes the proof of part 1) of the theorem.

Remark 4.2. The proof of the part 1) of theorem 3.8 is based on the following:

i) The differential observability of the system defined by (f, g, h) implies that $L_g(h) = G_1 \circ \varphi_1$ for some function G_1, and where $\varphi_1 = h$.
ii) Let k, $2 \leq k \leq N-1$ and assume that for $1 \leq i \leq k-1$, there exist functions G_i such that $L_g L_f^{i-1}(h) = G_i \circ \varphi_i$, then property $\mathcal{P}(k)$ of proposition 4.1 is satisfied.
iii) The differential observability implies that: if property $\mathcal{P}(k)$ of proposition 4.1 is satisfied, then $L_g L_f^{k-1}(h) = G_k \circ \varphi_k$.

– i) is proved in A) above.
– ii) follows from the definition of property $\mathcal{P}(k)$ and from the hypothesis of theorem 3.8, which says that for every $X \in \{f, g\}$; for every i, $2 \leq i \leq N-1$, the set $\mathcal{X}(i-1)$ satisfies the $(\varphi_{i-1}, \varphi_i, X)$-composite function property.
– Property iii) follows from the proof given in B) above.

Proof. of 2) and 3) of theorem 3.8.

- Proof of 2).
  From part 1) of theorem 3.8, we know that $L_g L_f^{k-1}(h) = G_k \circ \varphi_k$. Now using the fact that φ_k is semi-proper and applying lemma 3.20, we deduce that G_k is continuous on $\varphi_k(\mathcal{M})$.
- Proof of 3).
  This part is a direct consequence of lemma 3.22.

□

We end this paragraph with the proof of proposition 3.13.

Proof. of proposition 3.13
We will prove that if system (2.1) is uniformly observable, and φ_{m+2} ($m = \dim \mathcal{M}$) is injective, and that properties $\mathcal{B}(k)$ are satisfied for $2 \leq k \leq m+1$, then system (2.1) is differentially observable. In other words, we will show that the map Σ_{m+2} is injective, or equivalently, for every fixed $\underline{u}_{m+1}$, the map $x \to (h(x), L_F(h)(x, u_1), \ldots, L_F^{m+1}(h)(x, \underline{u}_{m+1})) = (y^{(0)}, y^{(1)}, \ldots, y^{(m+1)})$ is injective.
From theorem 2.4, we know that:

$$\text{for } \ 1 \leq k \leq m+1,\ L_g L_f^{k-1}(h) = G_k \circ \varphi_k \tag{4.8}$$

where $\varphi_k = (h, \ldots, L_f^{k-1}(h))$.
Now, using the fact that $\mathcal{B}(k)$ is a special case of the property $\mathfrak{C}(k)$ (see Rem. 3.11), and taking into account (4.8), we deduce from lemma 5.4 (see the appendix) that $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property for all X in $\{f, g\}$. Therefore the property $\mathcal{P}(k)$ of proposition 4.1 is satisfied. Then we can apply (4.2):

$$\text{for } 2 \leq k \leq m+1, \quad y^{(k)} = L_f^k(h)(x) + u_1 L_g L_f^{k-1}(h)(x) + \omega_2^k(\underline{u}_k, \varphi_k(x)) \tag{4.9}$$

Combining (4.9) with the fact that $y^{(1)} = L_f(h) + u_1 L_g(h)$, we get:

$$\text{for } 1 \leq k \leq m+1, \quad y^{(k)} = L_f^k(h)(x) + u_1 L_g L_f^{k-1}(h)(x) + \omega_2^k(\underline{u}_k, \varphi_k(x)) \tag{4.10}$$

where $\omega_2^1(\underline{u}_1, \varphi_1(x)) = 0$.
Let x, x' be such that $\Sigma_{m+2}(x, u_{m+1}) = \Sigma_{m+2}(x', u_{m+1})$ and let us check that $x = x'$.
Combining (4.8) and (4.10), we obtain:

$$\begin{cases} h(x) = h(x') \\ L_f(h)(x) + u_1 G_1 \circ h(x) = L_f(h)(x') + u_1 G_1 \circ h(x') \\ \vdots \\ L_f^k(h)(x) + u_1 G_k \circ \varphi_k(x) + \omega_2^k(\underline{u}_k, \varphi_k(x)) = L_f^k(h)(x') + u_1 G_k \circ \varphi_k(x') + \omega_2^k(\underline{u}_k, \varphi_k(x')) \\ \vdots \\ L_f^{m+1}(h)(x) + u_1 G_{m+1} \circ \varphi_{m+1}(x) + \omega_2^{m+1}(\underline{u}_{m+1}, \varphi_{m+1}(x)) = L_f^{m+1}(h)(x') + u_1 G_{m+1} \circ \varphi_{m+1}(x') \\ \quad + \omega_2^{m+1}(\underline{u}_{m+1}, \varphi_{m+1}(x')) \end{cases} \tag{4.11}$$

which implies that $\varphi_{m+2}(x) = \varphi_{m+2}(x')$, and by hypothesis φ_{m+2} is injective, thus $x = x'$. □

5. Appendix

In this section, we begin by giving the proof of proposition 4.1, then we will give the proof of we will give the proof of proposition 3.12.

5.1. Proof of Proposition 4.1

First, we give some preliminary results for the preparation of the proof of proposition 4.1, next we give its proof.

We recall some notations:

- $\underline{u}_k = (u_1, \dots, u_k) \in \mathbb{R}^k$, for $k = 1, \dots, N-1$, and for $k = 0$, $\underline{u}_0 = 0$.
- $F = F(x, \underline{u}_{N-1}) = f(x) + u_1 g(x) + \sum_{j=2}^{N-1} u_j \dfrac{\partial}{\partial u_{j-1}}$ (for $N = 2$, $F = f + u_1 g$).
- $y^{(k)} = L_F^k(h) = L_F^k(h)(x, \underline{u}_k)$.
- $\varphi_k = (h, \dots, L_f^{k-1}(h))$.
- $\underline{X}_k = (X_1, \dots, X_k)$ is a k-tuple of vector fields such that $X_i \in \{f, g\}$, and $\nu_g(\underline{X}_k)$ is the cardinal of the set $\{j :\ X_j = g\}$, and set $\mathbb{X}(k) = \{\underline{X}_k = (X_1, \dots, X_k) :\ X_j \in \{f, g\},$ and such that $\nu_g(\underline{X}_k) \geq 1\}$, in particular $\mathbb{X}(1) = \{g\}$.
- Set $L_{\underline{X}_k}(h) = L_{X_k} \dots L_{X_1}(h)$, and $\mathcal{X}(k) = \{L_{\underline{X}_k}(h) :\ \underline{X}_k \in \mathbb{X}(k)\}$, in particular $\mathcal{X}(1) = \{L_g(h)\}$.
- Set $\underline{X}_1^f = g$, and for $k \geq 2$, $\underline{X}_k^f$ is the k-tuple of vector fields such that $X_1 = \dots = X_{k-1} = f$ and $X_k = g$. Finally set $\mathbb{X}^f(k) = \mathbb{X}(k) \setminus \{\underline{X}_k^f\}$ and $\mathcal{X}^f(k) = \{L_{\underline{X}_k}(h) :\ \underline{X}_k \in \mathbb{X}^f(k)\}$. In particular, we have $\mathbb{X}^f(1) = \emptyset$ and $\mathcal{X}^f(1) = \emptyset$, and for $k \geq 2$, $\mathcal{X}(k) = \mathcal{X}^f(k) \cup \{L_g L_f^{k-1}(h)\}$.
- For $j \geq 2$, and for X in $\{f, g\}$, $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1}, \varphi_j, X)$-composite function property if for every ψ in $\mathcal{X}(j-1)$ for which there exists a function χ such that $\psi = \chi \circ \varphi_{j-1}$; there exists a function χ' such that $L_X(\psi) = \chi' \circ \varphi_j$.
- **Property** $\mathcal{P}(k)$ of proposition 4.1:
  - i) For every j, $2 \leq j \leq k$; for every X in $\{f, g\}$, $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1}, \varphi_j, X)$-composite function property.
  - ii) For every j, $1 \leq j \leq k-1$; there exists a function G_j such that $L_g L_f^{j-1}(h) = G_j \circ \varphi_j$.

The following result is useful in the proof of the proposition 4.1:

Claim 5.1. *Assume that property $\mathcal{P}(k)$ holds, then for every j, $1 \leq j \leq k$, we have:*

$$\text{For every } \psi \text{ in } \mathcal{X}(j);\ \text{there exists a scalar map } \chi \text{ such that } \psi = \chi \circ \varphi_j \tag{5.1}$$

Proof. The proof will be obtained by induction on j.

For $j = 1$, $\mathcal{X}(1) = \{L_g(h)\}$ and from ii) of property $\mathcal{P}(k)$, $L_g(h) = G_1 \circ \varphi_1$ (here $\varphi_1 = h$). Hence (5.1) holds for $j = 1$.

Let j, $2 \leq j \leq k-1$ and assume that (5.1) holds for $\mathcal{X}(j-1)$, and let us show it for $\mathcal{X}(j)$. Note that $\mathcal{X}(j) = \{L_g L_f^{j-1}(h)\} \cup L_f(\mathcal{X}(j-1)) \cup L_g(\mathcal{X}(j))$, thus $\psi \in \mathcal{X}(j)$ takes one of the following form: $\psi = L_g L_f^{j-1}(h)$ or $\psi = L_X(\widetilde{\psi})$, $\widetilde{\psi} \in \mathcal{X}(j-1)$ and $X \in \{f, g\}$.

a) If $\psi = L_g L_f^{j-1}(h)$, then ii) of property $\mathcal{P}(k)$ implies that $\psi = L_g L_f^{j-1}(h) = G_j \circ \varphi_j$.

b) If $\psi = L_X(\widetilde{\psi})$ for some $\widetilde{\psi} \in \mathcal{X}(j-1)$ and $X \in \{f, g\}$, using the expression $\widetilde{\psi} = \chi \circ \varphi_{j-1}$ (the induction hypothesis), and the fact that $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1}, \varphi_j, X)$-composite function property (property i) of property $\mathcal{P}(k)$), it follows that $\psi = L_X(\widetilde{\psi}) = \chi' \circ \varphi_j$ for some function χ'.

$\square$

Let $\mathfrak{V}$ be a vector subspace of $\mathcal{C}^p(\mathcal{M} \times \mathbb{R}^{N-1})$: the space of $\mathcal{C}^p$ scalar functions on $\mathcal{M} \times \mathbb{R}^{N-1}$, where $N \geq 2$. The argument of elements of $\mathcal{C}^p(\mathcal{M} \times \mathbb{R}^{N-1})$ is denoted by $(x, \underline{u}_{N-1})$. Let $H_1(x, \underline{u}_{N-1})$, $H_2(x, \underline{u}_{N-1})$ be any two element of $\mathcal{C}^p(\mathcal{M} \times \mathbb{R}^{N-1})$, we say that H_1, H_2 are equivalent modulo $\mathfrak{V}$, if $H_1 - H_2$ belongs to $\mathfrak{V}$. This relation defines an equivalence relation on $\mathcal{C}^p(\mathcal{M} \times \mathbb{R}^{N-1})$, and the class of equivalence of H_1 is the same as that of H_2, and it is equal to the orbit $H_2 + \mathfrak{V}$. If no confusion is to be feared, we identify H_1 and H_2 with

$H_2 + \mathfrak{V}$, and we can set:

$$H_1 = H_2 + \mathfrak{V} \tag{5.2}$$

which means: $H_1 = H_2 + H$ for some H in $\mathfrak{V}$.

Let $\mathfrak{F}$ be a subset of $\mathcal{C}^p(\mathcal{M})$ the space of $\mathcal{C}^p$ scalar functions of $\mathcal{M}$. Let j, $1 \leq j \leq N-1$, in the sequel $\mathbb{P}[\underline{u}_j]$ will denote the ring of polynomial functions of $\underline{u}_j = (u_1, \ldots, u_j)$, and $\mathbb{P}[\underline{u}_j][\mathfrak{F}]$ will denote the $\mathbb{P}[\underline{u}_j]$-module spanned by $\mathfrak{F}$.

Let $P(\underline{u}_i)$ be an element of $\mathbb{P}[\underline{u}_i]$, and using the expression of F, we obtain:

$$\begin{cases} \text{for } 1 \leq i \leq N-2, \\ L_F(P(\underline{u}_i)L_{\underline{X}_l}(h)) = \dfrac{\partial P(\underline{u}_i)}{\partial \underline{u}_i}\underline{u}_{i+1}L_{\underline{X}_l}(h) + P(\underline{u}_i)[L_f L_{\underline{X}_l}(h) + u_1 L_g L_{\underline{X}_l}(h)] \\ L_f(\mathcal{X}(l)) \subset \mathcal{X}^f(l+1); \quad L_g(\mathcal{X}(l)) \subset \mathcal{X}^f(l+1) \end{cases} \tag{5.3}$$

and hence:

$$\begin{cases} L_F(\mathbb{P}[\underline{u}_i][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_i][\mathcal{X}^f(l+1)] + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(l)] \subset \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(l+1)] \\ L_F(\mathbb{P}[\underline{u}_i][\mathcal{X}^f(l)]) \subset \mathbb{P}[\underline{u}_i][\mathcal{X}^f(l+1)] + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(l)] \subset \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(l+1)] \end{cases} \tag{5.4}$$

Lemma 5.2. *The following algebraic expressions are satisfied:*

$$\begin{cases} \textit{for } 2 \leq j \leq N-1, \\ y^{(j)} = L_f^j(h) + u_1 L_g L_f^{j-1}(h) + \mathbb{P}[u_1][\mathcal{X}^f(j)] + \sum_{l=1}^{j-1} \mathbb{P}[\underline{u}_{j+1-l}][\mathcal{X}(l)] \end{cases} \tag{5.5}$$

$$\begin{cases} \textit{for } 1 \leq k \leq N-2, \ \textit{for } 1 \leq i \leq N-k-1: \\ y^{(k+i)} = L_f^{k+i}(h) + u_{i+1} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum\limits_{\substack{l=1 \\ l \neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \end{cases} \tag{5.6}$$

Remark 5.3. Note that the terms $\mathbb{P}[\underline{u}_d][\mathcal{X}(l)]$ in the sum symbol of the formula (5.5) (resp. in the sum symbol of (5.6)) is characterized by the graduation $d + l = j + 1$ (resp. $d + l = k + i + 1$ for $l \neq k$). In the following, we will use this rule to check whether a formal expression verifies the formula (5.5) (resp. (5.6)).

Proof. of lemma 5.2.

- Proof of (5.5). The proof will be given by induction on j.
  For $j = 2$, $y^{(2)} = L_F(y^{(1)}) = L_F(L_f(h) + u_1 L_g(h))$, thus

$$y^{(2)} = L_f^2(h) + u_1 L_g L_f(h) + (u_1 L_f L_g(h) + u_1^2 L_g^2(h)) + u_2 L_g(h) \tag{5.7}$$

  Here $\mathcal{X}^f(2) = \{L_g L_f(h), L_f L_g(h)\}$ and $\mathcal{X}(1) = \{L_g(h)\}$, obviously $(u_1 L_f L_g(h) + u_1^2 L_g^2(h)) \in \mathbb{P}[u_1][\mathcal{X}^f(2)]$, and $u_2 L_g(h) \in \mathbb{P}[\underline{u}_2][\mathcal{X}(1)]$. Thus (5.5) is satisfied for $j = 2$.
  Now assume that:

$$y^{(j-1)} = L_f^{j-1}(h) + u_1 L_g L_f^{j-2}(h) + \mathbb{P}[u_1][\mathcal{X}^f(j-1)] + \sum_{l=1}^{j-2} \mathbb{P}[\underline{u}_{j-l}][\mathcal{X}(l)] \tag{5.8}$$

and let us show:

$$y^{(j)} = L_f^j(h) + u_1 L_g L_f^{j-1}(h) + \mathbb{P}[u_1][\mathcal{X}^f(j)] + \sum_{l=1}^{j-1} \mathbb{P}[\underline{u}_{j+1-l}][\mathcal{X}(l)] \tag{5.9}$$

By construction we have:

$$\begin{aligned} y^{(j)} = L_F(y^{(j-1)}) = L_F[L_f^{j-1}(h) + u_1 L_g L_f^{j-2}(h)] + L_F(\mathbb{P}[u_1][\mathcal{X}^f(j-1)]) \\ + \textstyle\sum_{l=1}^{j-2} L_F(\mathbb{P}[\underline{u}_{j-l}][\mathcal{X}(l)]) \end{aligned} \tag{5.10}$$

On the one hand, using (5.4), we get:

$$\begin{aligned} &L_F(\mathbb{P}[u_1][\mathcal{X}^f(j-1)]) \subset \mathbb{P}[u_1][\mathcal{X}^f(j)] + \mathbb{P}[\underline{u}_2][\mathcal{X}(j-1)] \\ &\text{for } 1 \leq l \leq j-2,\ L_F(\mathbb{P}[\underline{u}_{j-l}][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_{j-l}][\mathcal{X}(l+1)] + \mathbb{P}[\underline{u}_{j+1-l}][\mathcal{X}(l)] \end{aligned} \tag{5.11}$$

On the other hand,

$$\begin{aligned} &L_F[L_f^{j-1}(h) + u_1 L_g L_f^{j-2}(h)] = L_f^j(h) + u_1 L_g L_f^{j-1}(h) + \\ &u_1(L_f L_g L_f^{j-2}(h) + u_1 L_g^2 L_f^{j-2}(h)) + u_2 L_g L_f^{j-2}(h); \\ &\text{where, } u_1(L_f L_g L_f^{j-2}(h) + u_1 L_g^2 L_f^{j-2}(h)) \in \mathbb{P}[u_1][\mathcal{X}^f(j)]; \\ &\text{and, } u_2 L_g L_f^{j-2}(h) \in \mathbb{P}[\underline{u}_2][\mathcal{X}(j-1)] \end{aligned} \tag{5.12}$$

Combining (5.10), (5.11), (5.12) and remark 5.3, we deduce formula (5.5) for j, namely:

$$y^{(j)} = L_f^j(h) + u_1 L_g L_f^{j-1}(h) + \mathbb{P}[u_1][\mathcal{X}^f(j)] + \sum_{l=1}^{j-1} \mathbb{P}[\underline{u}_{j+1-l}][\mathcal{X}(l)] \tag{5.13}$$

- Proof of formula (5.6).
  In what follows, we will prove formula (5.6) by induction on i. Namely, for $i \geq 1$, we have:

$$y^{(k+i)} = L_f^{k+i}(h) + u_{i+1} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l \neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \tag{5.14}$$

i) For $i = 1$, let us show:

$$y^{(k+1)} = L_f^{k+1}(h) + u_2 L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l \neq k}}^{k+1} \mathbb{P}[\underline{u}_{k+2-l}][\mathcal{X}(l)] \tag{5.15}$$

From formula (5.5), we know that:

$$y^{(k)} = L_f^k(h) + u_1 L_g L_f^{k-1}(h) + \mathbb{P}[u_1][\mathcal{X}^f(k)] + \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)] \tag{5.16}$$

Using the fact that $y^{(k+1)} = L_F(y^{(k)})$, we get:

$$y^{(k+1)} = L_F(L_f^k(h) + u_1 L_g L_f^{k-1}(h)) + L_F(\mathbb{P}[u_1][\mathcal{X}^f(k)]) + \sum_{l=1}^{k-1} L_F(\mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)]) \tag{5.17}$$

On the one hand, according to (5.4), we know that:

$$\begin{aligned} &L_F(\mathbb{P}[u_1][\mathcal{X}^f(k)]) \subset \mathbb{P}[u_1][\mathcal{X}^f(k+1)] + \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] \\ &\subset \mathbb{P}[u_1][\mathcal{X}(k+1)] + \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] \end{aligned} \tag{5.18}$$

and,

$$\begin{aligned} &\text{for } 1 \leq l \leq k-1, \\ &L_F(\mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_{k+2-l}][\mathcal{X}(l)] + \mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}^f(l+1)] \end{aligned} \tag{5.19}$$

Note that in the particular case when $l = k-1$:

$$L_F(\mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)]) = L_F(\mathbb{P}[\underline{u}_2][\mathcal{X}(k-1)]) \subset \mathbb{P}[\underline{u}_3][\mathcal{X}(k-1)] + \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] \tag{5.20}$$

Combining (5.18), (5.19) and (5.20), we deduce:

$$L_F(\mathbb{P}[u_1][\mathcal{X}^f(k)]) + \sum_{l=1}^{k-1} L_F(\mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l \neq k}}^{k+1} \mathbb{P}[\underline{u}_{k+2-l}][\mathcal{X}(l)] \tag{5.21}$$

On the other hand, $L_F(L_f^k(h) + u_1 L_g L_f^{k-1}(h)) = L_{f+u_1 g}(L_f^k(h) + u_1 L_g L_f^{k-1}(h)) + \frac{\partial(L_f^k(h) + u_1 L_g L_f^{k-1}(h))}{\partial u_1} u_2$ $= L_f^{k+1}(h) + u_2 L_g L_f^{k-1}(h) + u_1[L_g L_f^k(h) + L_f L_g L_f^{k-1}(h) + u_1 L_g^2 L_f^{k-1}(h)]$, thus,

$$L_F(L_f^k(h) + u_1 L_g L_f^{k-1}(h)) = L_f^{k+1}(h) + u_2 L_g L_f^{k-1}(h) + \mathbb{P}[u_1][\mathcal{X}(k+1)] \tag{5.22}$$

Finally, combining (5.17), (5.21) and (5.22), we deduce (5.15).

ii) Let $i \geq 2$, and assume that formula (5.14) is true for $i-1$, namely:

$$y^{(k+i-1)} = L_f^{k+i-1}(h) + u_i L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_i][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l \neq k}}^{k+i-1} \mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}(l)] \tag{5.23}$$

and let us check the following:

$$y^{(k+i)} = L_f^{k+i}(h) + u_{i+1} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l \neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \tag{5.24}$$

Since $y^{(k+i)} = L_F(y^{(k+i-1)})$, we obtain:

$$y^{(k+i)} = L_F(L_f^{k+i-1}(h) + u_i L_g L_f^{k-1}(h)) + L_F(\mathbb{P}[\underline{u}_i][\mathcal{X}^f(k)]) + \sum_{\substack{l=1 \\ l \neq k}}^{k+i-1} L_F(\mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}(l)]) \tag{5.25}$$

Now we are going to develop the Lie derivatives $L_F(.)$ in (5.25):
On the one hand, using similar argument as for $i=1$ (see also expression (5.4)), we get:

$$L_F(\mathbb{P}[\underline{u}_i][\mathcal{X}^f(k)]) \subset \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \mathbb{P}[\underline{u}_i][\mathcal{X}(k+1)] \tag{5.26}$$

and for $1 \le l \le k+i-1,\ l \neq k$,

$$L_F(\mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] + \mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}^f(l+1)] \tag{5.27}$$

In the particular case when $l=k-1$, we have:

$$L_F(\mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}(l)]) = L_F(\mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(k-1)]) \subset \mathbb{P}[\underline{u}_{i+2}][\mathcal{X}(k-1)] + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] \tag{5.28}$$

Now combining (5.26), (5.27) and (5.28), we obtain:

$$L_F(\mathbb{P}[\underline{u}_i][\mathcal{X}^f(k)]) + \sum_{\substack{l=1\\ l\neq k}}^{k+i-1} L_F(\mathbb{P}[\underline{u}_{k+i-l}][\mathcal{X}(l)]) \subset \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum_{\substack{l=1\\ l\neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \tag{5.29}$$

On the other hand, let us develop the first Lie derivative of expression (5.25):
$L_F(L_f^{k+i-1}(h) + u_iL_gL_f^{k-1}(h)) = L_{f+u_1g}(L_f^{k+i-1}(h) + u_iL_gL_f^{k-1}(h)) + \frac{\partial(L_f^{k+i-1}(h)+u_iL_gL_f^{k-1}(h))}{\partial u_i}u_{i+1} = L_f^{k+i}(h) + u_{i+1}L_gL_f^{k-1}(h) + u_1L_gL_f^{k+i-1}(h) + u_i(L_fL_gL_f^{k-1}(h) + u_1L_g^2L_f^{k-1}(h))$.
Now using the fact that:

$$\begin{aligned} &u_1L_gL_f^{k+i-1}(h) \in \mathbb{P}[\underline{u}_1][\mathcal{X}(k+i)] \\ &u_i(L_fL_gL_f^{k-1}(h) + u_1L_g^2L_f^{k-1}(h)) \in \mathbb{P}[\underline{u}_i][\mathcal{X}(k+1)] \subset \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(k+1)] \end{aligned} \tag{5.30}$$

we deduce,

$$\begin{aligned} L_F(L_f^{k+i-1}(h) + u_iL_gL_f^{k-1}(h)) = {} & L_f^{k+i}(h) + u_{i+1}L_gL_f^{k-1}(h) + \mathbb{P}[\underline{u}_1][\mathcal{X}(k+i)] \\ & + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}(k+1)] \end{aligned} \tag{5.31}$$

Finally the expression (5.24) follows from (5.24), (5.29) and (5.31). Which ends the proof of lemma 5.2. □

Proof. of proposition 4.1.

1) Let us prove 1) of the proposition, namely:

$$\begin{cases} y^{(1)} = L_f(h)(x) + u_1L_g(h)(x) \\ \vdots \\ y^{(k)} = \lambda_k(\underline{u}_{k-1}, x) + u_kL_g(h)(x) \\ \vdots \\ y^{(N-1)} = \lambda_{N-1}(\underline{u}_{N-2}, x) + u_{N-1}L_g(h)(x) \end{cases} \tag{5.32}$$

Set $\underline{u}_0 = 0$ and $\lambda_1(\underline{u}_0, x) = L_f(h)(x)$, and let us prove by induction on k the following formula:

$$y^{(k)} = \lambda_k(\underline{u}_{k-1}, x) + u_kL_g(h)(x) \tag{5.33}$$

Recall that $F = F(x, \underline{u}_{N-1}) = f(x) + u_1g(x) + \sum_{j=2}^{N-1} u_j \dfrac{\partial}{\partial u_{j-1}}$, and $y^{(k)} = L_F^k(h)$.

For $k=1$, $y^{(1)}=L_F(h)=L_f(h)(x)+u_1L_g(h)(x)$ and hence (5.33) is satisfied for $k=1$.
Now let $k\geq 2$ and assume that:

$$y^{(k-1)}=\lambda_{k-1}(\underline{u}_{k-2},x)+u_{k-1}L_g(h)(x) \tag{5.34}$$

where λ_{k-1} is a smooth function w.r.t. x and polynomial w.r.t. $\underline{u}_{k-2}$, and let show :

$$y^{(k)}=\lambda_k(\underline{u}_{k-1},x)+u_kL_g(h)(x) \tag{5.35}$$

By definition $y^{(k)}=L_F(y^{(k-1)})$ and hence $y^{(k)}=L_F(\lambda_{k-1}(\underline{u}_{k-2},x))+L_F(u_{k-1}L_g(h)(x))$.
On the one hand we have $L_F(\lambda_{k-1}(\underline{u}_{k-2},x))=L_{f+u_1g}(\lambda_{k-1}(\underline{u}_{k-2},x))+\dfrac{\partial\lambda_{k-1}(\underline{u}_{k-2},x)}{\partial\underline{u}_{k-2}}\underline{u}_{k-1}$, which is a smooth function on x, and polynomial on $\underline{u}_{k-1}$.
On the other hand, $L_F(u_{k-1}L_g(h))=L_{f+u_1g}(u_{k-1}L_g(h))+\dfrac{\partial(u_{k-1}L_g(h))}{\partial u_{k-1}}u_k=u_{k-1}L_{f+u_1g}L_g(h)(x)+u_kL_g(h)(x)$,
thus $y^{(k)}=\lambda_k(\underline{u}_{k-1},x)+u_kL_g(h)(x)$, where $\lambda_k(\underline{u}_{k-1},x)=L_F(\lambda_{k-1}(\underline{u}_{k-2},x))+u_{k-1}L_{f+u_1g}L_g(h)(x)$.

2) Proof of 2) of proposition 4.1.
Let k, $2\leq k\leq N-1$ and assume that the following property holds.
property $\mathcal{P}(k)$:
 i) For every j, $2\leq j\leq k$; for every X in $\{f,g\}$, $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1},\varphi_j,X)$-composite function property.
 ii) For every j, $1\leq j\leq k-1$; there exists a function G_j such that $L_gL_f^{j-1}(h)=G_j\circ\varphi_j$.

Let us show the following formulas:

$$\begin{cases} \text{for } 1\leq j\leq k-1,\ y^{(j)}=\Lambda_j(\varphi_{j+1}(x),\underline{u}_j);\\ y^{(k)}=L_f^k(h)(x)+u_1L_gL_f^{k-1}(h)(x)+\omega_2^k(\underline{u}_k,\varphi_k(x))\\ y^{(k+1)}=L_f^{k+1}(h)(x)+u_2L_gL_f^{k-1}(h)(x)+\omega_1^{k+1}(u_1,x)+\omega_2^{k+1}(\underline{u}_{k+1},\varphi_k(x))\\ \vdots\\ y^{(k+i)}=L_f^{k+i}(h)(x)+u_{i+1}L_gL_f^{k-1}(h)(x)+\omega_1^{k+i}(\underline{u}_i,x)+\omega_2^{k+i}(\underline{u}_{k+i},\varphi_k(x))\\ \vdots\\ y^{(N-1)}=L_f^{N-1}(h)(x)+u_{N-k}L_gL_f^{k-1}(h)(x)+\omega_1^{N-1}(\underline{u}_{N-k-1},x)+\omega_2^{N-1}(\underline{u}_{N-1},\varphi_k(x)) \end{cases} \tag{5.36}$$

We start by showing the first formula of (5.36), namely:

$$\text{for } 1\leq j\leq k-1,\quad y^{(j)}=\Lambda_j(\varphi_{j+1}(x),\underline{u}_j) \tag{5.37}$$

For $j=1$, $y^{(1)}=L_f(h)(x)+u_1L_g(h)(x)$, and by hypothesis $L_g(h)(x)=G_1\circ\varphi_1$, and $\varphi_2=(h,L_f(h))$, thus $y^{(1)}=\Lambda_1(\varphi_2(x),\underline{u}_1)$, and hence (5.37) is satisfied for $j=1$.
Now let us show formula (5.37), for $2\leq j\leq k-1$.
From lemma 5.2 (see expression (5.5)), we know that:

$$\begin{cases} \text{for } 2\leq j\leq k-1,\\ y^{(j)}=L_f^j(h)+u_1L_gL_f^{j-1}(h)+\mathbb{P}[u_1][\mathcal{X}^f(j)]+\sum_{l=1}^{j-1}\mathbb{P}[\underline{u}_{j+1-l}][\mathcal{X}(l)] \end{cases} \tag{5.38}$$

Since **property** $\mathcal{P}(k)$ is satisfied, claim 5.1 implies that for every l, $1\leq l\leq k$; for every ψ in $\mathcal{X}(l)$; there exists χ such that $\psi=\chi\circ\varphi_l$. Combining these facts with the fact that $\varphi_{j+1}=(h,\dots,L_f^j(h))$, we deduce that $y^{(j)}=\Lambda_j(\varphi_{j+1}(x),\underline{u}_j)$, for some function Λ_j.

To end the proof of the proposition, it only remains to show the following:

$$\begin{cases} \text{for } \ 2 \leq k \leq N-1, \\ y^{(k)} = L_f^k(h)(x) + u_1 L_g L_f^{k-1}(h)(x) + \omega_2^k(\underline{u}_k, \varphi_k(x)) \\ y^{(k+1)} = L_f^{k+1}(h)(x) + u_2 L_g L_f^{k-1}(h)(x) + \omega_1^{k+1}(u_1, x) + \omega_2^{k+1}(\underline{u}_{k+1}, \varphi_k(x)) \\ \vdots \\ y^{(k+i)} = L_f^{k+i}(h)(x) + u_{i+1} L_g L_f^{k-1}(h)(x) + \omega_1^{k+i}(\underline{u}_i, x) + \omega_2^{k+i}(\underline{u}_{k+i}, \varphi_k(x)) \\ \vdots \\ y^{(N-1)} = L_f^{N-1}(h)(x) + u_{N-k} L_g L_f^{k-1}(h)(x) + \omega_1^{N-1}(\underline{u}_{N-k-1}, x) + \omega_2^{N-1}(\underline{u}_{N-1}, \varphi_k(x)) \end{cases} \tag{5.39}$$

From expressions (5.14) and (5.16) we know that:

$$\begin{cases} y^{(k)} = L_f^k(h) + u_1 L_g L_f^{k-1}(h) + \mathbb{P}[u_1][\mathcal{X}^f(k)] + \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)] \\ y^{(k+1)} = L_f^{k+1}(h) + u_2 L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_2][\mathcal{X}^f(k)] + \sum\limits_{\substack{l=1 \\ l\neq k}}^{k+1} \mathbb{P}[\underline{u}_{k+2-l}][\mathcal{X}(l)] \\ \vdots \\ y^{(k+i)} = L_f^{k+i}(h) + u_{i+1} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum\limits_{\substack{l=1 \\ l\neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \\ \vdots \\ y^{(N-1)} = L_f^{N-1}(h) + u_{N-k} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{N-k}][\mathcal{X}^f(k)] + \sum\limits_{\substack{l=1 \\ l\neq k}}^{N-1} \mathbb{P}[\underline{u}_{N-l}][\mathcal{X}(l)] \end{cases} \tag{5.40}$$

Let us show the first equality of (5.39).
As above, using the fact that **property** $\mathcal{P}(k)$ is satisfied, claim 5.1 implies that for every l, $1 \leq l \leq k$; for every ψ in $\mathcal{X}(l)$; there exists χ such that $\psi = \chi \circ \varphi_l$, and thus element of $\mathbb{P}[u_1][\mathcal{X}^f(k)] + \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+1-l}][\mathcal{X}(l)]$ takes the form $\omega(\underline{u}_k, \varphi_k(x))$. Consequently $y^{(k)} = L_f^k(h)(x) + u_1 L_g L_f^{k-1}(h)(x) + \omega^k(\underline{u}_k, \varphi_k(x))$, which ends the proof of the first formula of (5.39).
To end the proof of (5.39), it only remains to show that for $1 \leq i \leq N-k-1$:

$$y^{(k+i)} = L_f^{k+i}(h)(x) + u_{i+1} L_g L_f^{k-1}(h)(x) + \omega_1^{k+i}(\underline{u}_i, x) + \omega_2^{k+i}(\underline{u}_{k+i}, \varphi_k(x)) \tag{5.41}$$

Using (5.40), the following expression holds for $1 \leq i \leq N-k-1$:

$$y^{(k+i)} = L_f^{k+i}(h) + u_{i+1} L_g L_f^{k-1}(h) + \mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum_{\substack{l=1 \\ l\neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \tag{5.42}$$

We now separate the sum of the second term of the above equality into two parts:

$$\sum_{\substack{l=1 \\ l\neq k}}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] = \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] + \sum_{l=k+1}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \tag{5.43}$$

We can also note that:

$$\begin{cases} \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \subset \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+i}][\mathcal{X}(l)] \\ \sum_{l=k+1}^{k+i} \mathbb{P}[\underline{u}_{k+i+1-l}][\mathcal{X}(l)] \subset \sum_{l=k+1}^{k+i} \mathbb{P}[\underline{u}_{i}][\mathcal{X}(l)] \end{cases} \tag{5.44}$$

Using again **property** $\mathcal{P}(k)$, claim 5.1 implies that elements of $\mathbb{P}[\underline{u}_{i+1}][\mathcal{X}^f(k)] + \sum_{l=1}^{k-1} \mathbb{P}[\underline{u}_{k+i}][\mathcal{X}(l)]$ take the form $\omega_2^{k+i}(\underline{u}_{k+i}, \varphi_k(x))$. As for the elements of $\sum_{l=k+1}^{k+i} \mathbb{P}[\underline{u}_i][\mathcal{X}(l)]$, they take the form $\omega_1^{k+i}(\underline{u}_i, x)$. Which ends the proof of the proposition.

□

5.2. Proof of Proposition 3.12

First recall that $\varphi_j = (h, \ldots, L_f^{j-1}(h))$, and that if ζ, ζ' are such that $\varphi_j(\zeta) = \varphi_j(\zeta') = z$, then $D_{\varphi_j}(\zeta, \zeta') = \{(x, x') :\ \varphi_j(x) = \varphi_j(x') = z\}$. In the sequel, we will identify $D_{\varphi_j}(\zeta, \zeta')$ with its germ at (ζ, ζ'), and we denote by $S_j(\zeta)$ the germ at ζ of the set of the set of singular points of φ_j.

Let $(\underline{x}, \underline{x}')$ be such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x}')$ and recall that **property** $\mathfrak{C}(k)(\underline{x}, \underline{x}')$ is satisfied if there exists a sequence $\{\xi_1, \ldots, \xi_p\}$ such that: i) $\xi_1 = \underline{x}$, $\xi_p = \underline{x}'$, and $\varphi_k(\xi_i) = \varphi_k(\xi_{i+1})$, ii) $D_{\varphi_k}(\xi_i, \xi_{i+1}) \nsubseteq S_{k-1}(\xi_i) \times S_{k-1}(\xi_{i+1})$, and property $\mathfrak{C}(k)$ is satisfied if, for every $(\underline{x}, \underline{x}') \in \mathcal{M} \times \mathcal{M}$ such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x})$, $\mathfrak{C}(k)(\underline{x}, \underline{x}')$ holds.

The proof of proposition 3.12 is based on the following lemma:

Lemma 5.4. *Assume that property $\mathfrak{C}(k)$ is satisfied for a given k, $2 \leq k \leq N-1$, and that $L_g L_f^{i-1}(h) = G_i \circ \varphi_i$, for $1 \leq i \leq k-1$. Then $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property, for $X = f, g$. Namely, for every ψ in $\mathcal{X}(k-1)$ such that $\psi = \chi \circ \varphi_{k-1}$ for some scalar function χ; there exists χ' such that $L_X(\psi) = \chi' \circ \varphi_k$.*

Proof. of the lemma.

In order to prove the lemma, it suffices to check the following property:

$$\begin{cases} \text{if } \psi = \chi \circ \varphi_{k-1} \text{ for some scalar function } \chi, \text{ then :} \\ \text{for every } \underline{x},\ \underline{x}',\ \varphi_k(\underline{x}) = \varphi_k(\underline{x}') \text{ implies } L_X(\psi)(\underline{x}) = L_X(\psi)(\underline{x}') \end{cases} \tag{5.45}$$

Let $(\underline{x},\ \underline{x}')$ such that $\varphi_k(\underline{x}) = \varphi_k(\underline{x}')$, since $\mathfrak{C}(k)(\underline{x}, \underline{x}')$ is satisfied, then there exists a sequence $(\xi_i)_{1 \leq i \leq p}$ such that the following properties hold:

i) $\xi_1 = \underline{x}$, $\xi_p = \underline{x}'$, and $\varphi_k(\xi_i) = \underline{z}$, for $1 \leq i \leq p$
ii) $D_{\varphi_k}(\xi_i, \xi_{i+1}) \nsubseteq S_{k-1}(\xi_i) \times S_{k-1}(\xi_{i+1})$, for $1 \leq i \leq p$.

To show (5.45), it suffices to check that:

$$L_X(\psi)(\xi_i) = L_X(\psi)(\xi_{i+1}), \quad \text{for } 1 \leq i \leq p \tag{5.46}$$

Using property ii), it follows that for every open neighbourhoods O_i, O_{i+1} of the respective elements ξ_i, ξ_{i+1}, we can find (ζ_i, ζ_{i+1}) in $O_i \times O_{i+1}$ such that $\varphi_k(\zeta_i) = \varphi_k(\zeta_{i+1})$ and that either ζ_i or ζ_{i+1} is a regular point of φ_{k-1}. In order to check (5.46), it suffices to show that $L_X(\psi)(\zeta_i) = L_X(\psi)(\zeta_{i+1})$, and hence by continuity, we deduce that $L_X(\psi)(\xi_i) = L_X(\psi)(\xi_{i+1})$.

Assume for instance that ζ_i is a regular point of φ_{k-1}, and set $\eta_i = \varphi_{k-1}(\zeta_i) = \varphi_{k-1}(\zeta_{i+1})$. From 2) of Definition 3.1, we can find an open neighbourhood U_i of ζ_i contained in O_i, and an open neighbourhood V_i of η_i in $\mathbb{R}^{k-1}$, such that $N_i = V_i \cap \varphi(\mathcal{M})$ is a sub-manifold of $\mathbb{R}^{k-1}$, and that φ_{k-1} is a submersion from U_i onto the sub-manifold N_i. Hence, we obtain the following diagram:

$$\begin{tikzcd} U_i \arrow[rr, "\varphi_{k-1}"] \arrow[dr, "\psi"'] & & N_i \arrow[dl, "\chi"] \\ & \mathbb{R} & \end{tikzcd} \qquad \text{(Diag 1)}$$

Applying remark 2.5, we deduce that χ is smooth in some neighbourhood of η_i.

On the other hand, $\varphi_{k-1}(\mathcal{M})$ is a topological space which is equipped with the topology induced by that of $\mathbb{R}^{k-1}$, thus N_i is an open subset of $\varphi_{k-1}(\mathcal{M})$, and hence $\varphi_{k-1}^{-1}(N_i)$ is an open subset of $\mathcal{M}$ and contains ζ_{i+1} (since $\varphi_{k-1}(\zeta_i) = \varphi_{k-1}(\zeta_{i+1}) = \eta_i$). Set $U_{i+1} = O_{i+1} \cap \varphi_{k-1}^{-1}(N_i)$, U_{i+1} is then an open neighbourhood which contains ζ_{i+1}, and the following diagram holds:

$$\begin{tikzcd} U_{i+1} \arrow[rr, "\varphi_{k-1}"] \arrow[dr, "\psi"'] & & N_i \arrow[dl, "\chi"] \\ & \mathbb{R} & \end{tikzcd} \qquad \text{(Diag 2)}$$

Let X in $\{f, g\}$, and using the two above diagrams, we deduce that:

$$L_X\psi(\zeta_i) = T_{\eta_i}\chi[T_{\zeta_i}\varphi_{k-1}(X(\zeta_i))] \tag{5.47}$$

$$L_X\psi(\zeta_{i+1}) = T_{\eta_i}\chi[T_{\zeta_{i+1}}\varphi_{k-1}(X(\zeta_{i+1}))] \tag{5.48}$$

Recall that $\varphi_{k-1} = (h, \ldots, L_f^{k-2}(h))$, and let us show that $L_X\psi(\zeta_i) = L_X\psi(\zeta_{i+1})$:

- If $X = f$, then $T_x\varphi_{k-1}(f(x))$ can be identified with $(L_f(h)(x), \ldots, L_f^{k-1}(h)(x))$ in the canonical system of coordinates of $\mathbb{R}^k$. Then since $\varphi_k(\zeta_i) = \varphi_k(\zeta_{i+1})$, it follows that $T_{\zeta_i}\varphi_{k-1}(f(\zeta_i)) = T_{\zeta_{i+1}}\varphi_{k-1}(f(\zeta_{i+1}))$, and according to (5.47)-(5.48), we deduce that $L_f\psi(\zeta_i) = L_f\psi(\zeta_{i+1})$.
- If $X = g$, as above, $T_x\varphi_{k-1}(g(x))$ can be identified with $(L_g(h)(x), \ldots, L_gL_f^{k-2}(h)(x))$ in the canonical system of coordinates of $\mathbb{R}^k$. From the hypothesis of the lemma, we know that $L_gL_f^{l-1}(h) = G_l \circ \varphi_l$, for $1 \le l \le k-1$. Moreover, $\varphi_l(\zeta_i) = \varphi_l(\zeta_{i+1})$, for $1 \le l \le k-1$ (since $\varphi_k(\zeta_i) = \varphi_k(\zeta_{i+1})$), and hence $L_gL_f^{l-1}(h)(\zeta_i) = L_gL_f^{l-1}(h)(\zeta_{i+1})$ for $1 \le l \le k-1$, which implies that $T_{\zeta_i}\varphi_{k-1}(g(\zeta_i)) = T_{\zeta_{i+1}}\varphi_{k-1}(g(\zeta_{i+1}))$. Combining this last fact with (5.47)-(5.48), it follows that $L_g(\psi)(\zeta_i) = L_g(\psi)(\zeta_{i+1})$.

This ends the proof of the lemma. □

Proof. of proposition 3.12.

In what follows, we will show that if system (2.1) is differentially observable of order N, and that property $\mathfrak{C}(k)$ is satisfied for $2 \le k \le N-1$, then $\mathcal{X}(k-1)$ satisfies the $(\varphi_{k-1}, \varphi_k, X)$-composite function property.

In fact, we will show a more stronger property, namely: if system (2.1) is differentially observable of order N, and property $\mathfrak{C}(k)$ is satisfied for $2 \le k \le N-1$, then property $\mathcal{P}(k)$ of proposition 4.1 holds.

First let us recall property $\mathcal{P}(k)$ of proposition 4.1 which is used in the proof of theorem 3.8.

Property $\mathcal{P}(k)$:

i) For every j, $2 \le j \le k$; for every X in $\{f, g\}$, $\mathcal{X}(j-1)$ satisfies the $(\varphi_{j-1}, \varphi_j, X)$-composite function property.

ii) For every j, $1 \le j \le k-1$; there exists a function G_j such that $L_gL_f^{j-1}(h) = G_j \circ \varphi_j$.

The proof of proposition 3.12 is based on lemma 5.4 and remark 4.2, and it will be given by induction:

1) For $k = 2$, let us check property $\mathcal{P}(2)$.
By definition, $\mathcal{X}(1) = \{L_g(h)\}$, and from i) of remark 4.2, we know that the differential observability implies that $L_g(h) = G_1 \circ \varphi_1$ for some function G_1, and where $\varphi_1 = h$. Now, using the hypothesis of

proposition 3.12, we know that property $\mathfrak{C}(2)$ is satisfied. Hence from lemma 5.4, it follows that $\mathcal{X}(1)$ satisfies the $(\varphi_1, \varphi_2, X)$-composite function property, for $X = f, g$.
As a consequence, property $\mathcal{P}(2)$ is satisfied.

2) Now assume that $\mathcal{P}(k)$ is satisfied, and let us check $\mathcal{P}(k+1)$.
From iii) of remark 4.2, we know that, if system (2.1) is differential observability and property $\mathcal{P}(k)$ is satisfied, then $L_g L_f^{k-1}(h) = G_k \circ \varphi_k$. Consequently, we have $L_g L_f^{j-1}(h) = G_j \circ \varphi_j$, for $1 \leq j \leq k$. Combining this last property with the fact that property $\mathfrak{C}(k+1)$ is satisfied, lemma 5.4 implies that $\mathcal{X}(k)$ satisfies the $(\varphi_k, \varphi_{k+1}, X)$-composite function property, for $X = f, g$. As a consequence, $\mathcal{P}(k+1)$ is then fulfilled. Which ends the proof of proposition 4.1.

□

6. Conclusion

As a corollary of the work stated in [25], [11], we noted that if a system (f, g, h) is uniformly observable on a manifold $\mathcal{M}$ of dimension m and if $\varphi_{m+2} = (h, \ldots, L_f^{m+1}(h))$ is injective and that the map $\varphi_m = (h, \ldots, L_f^{m-1}(h))$ is everywhere a local diffeomorphism, then φ_{m+2} globally transforms the system (f, g, h) into a normal form of observability. In the presence of singularities, an extension of this result (see [29]) consists of assuming that φ_{m+2} is injective and that if $\varphi_{k+1}(\underline{x}) = \varphi_{k+1}(\underline{x}')$, then we can approximate $(\underline{x}, \underline{x}')$ by a pair (x, x') such that $\varphi_{k+1}(x) = \varphi_{k+1}(x')$ and such that x or x' is a regular point of φ_k, for $1 \leq k \leq m$. The proof is based on an approach similar to that established in [11]. Although this result is original, it suffers from two things: i) the first is that the proposed hypothesis excludes certain autonomous (uncontrolled) systems, for which the injectivity of φ_{m+2} is sufficient to transform them into normal forms, ii) the proposed hypothesis does not allow immersion to be extended to normal forms in dimensions greater than $m+2$. In this paper, we have proposed a sufficient condition that takes into account points i) and ii). The approach proposed to show our result is quite close to that of [25] and therefore completely different from that used in [11] and [29].

Data availability statement

The research data associated with this article are included within the article.

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