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OPTIMAL CONTROL STRATEGIES FOR THE LANDAU–LIFSHITZ–GILBERT EQUATION THROUGH SPATIO-TEMPORAL CONTROL AND FIXED MAGNETIC FIELD COILS

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Abstract. The magnetization control problem for the Landau–Lifshitz–Gilbert (LLG) equation $m_t = m \times (\Delta m + u) - m \times (m \times (\Delta m + u)),\ (x,t) \in \Omega \times (0,T]$ with zero Neumann boundary data on a two-dimensional bounded domain Ω is studied when the control energy u is applied on the effective field. First, we show the existence of a weak solution, and that the magnetization vector field m satisfies an energy inequality. If a weak solution m obeys the condition that $\nabla m \in L^4(0,T;L^4(\Omega))$, then we show that it is a regular solution. The classical cost functional is modified by incorporating $L^4(0,T;L^4(\Omega))$-norm of ∇m, enabling a rigorous study of the optimal control problem. Then, we justify the existence of an optimal control and derive first-order necessary optimality conditions using an adjoint problem approach. We establish the continuous dependency and Fréchet differentiability of the control-to-state and control-to-costate operators and show the locally Lipschitz continuity of their Fréchet derivatives. Using these postulates, we derive a local second-order sufficient optimality condition. Moreover, we modified the control problem and derived a global optimality criterion. Finally, we obtain another remarkable second-order sufficient optimality condition along a cone of critical directions where the control stems from a finite number of fixed magnetic field coils.

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1. Introduction

The study of magnetization dynamics in ferromagnetic media is of great interest because of its wide range of applications, from magnetic sensors to data storage devices. In 1935, L.D. Landau and E.M. Lifshitz obtained the first dynamical model for micromagnetization phenomena occurring inside ferromagnetic materials (see [1]). Later, in 1955, T.L. Gilbert modified this equation by introducing dissipation in a phenomenological way (see [2]). This paper considers the LLG equation containing energy interactions from exchange energy and the external magnetic field.

In the context of our study, we consider a smooth, bounded region Ω within two-dimensional space, representing a material with ferromagnetic properties. The magnetization of this material is described by the vector field m, which varies over both space and time, denoted as $m : \Omega \times [0,T] \to \mathbb{R}^3$. It's important to note that

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below the Curie temperature, the magnitude of the magnetization remains constant. To account for this, we normalize the magnetization vector field by dividing it by the saturated magnetization, ensuring that our net magnetization is a unit vector field, *i.e.*, $m \in \mathbb{S}^2$, which is the unit sphere in $\mathbb{R}^3$.

The evolution of this magnetization field is governed by the Landau–Lifshitz–Gilbert (LLG) equation, given by:

$$m_t = \gamma m \times \mathcal{E}_{eff}(m) - \alpha\gamma m \times (m \times \mathcal{E}_{eff}(m)),$$

where $\times$ represents the cross product in $\mathbb{R}^3$, $\mathcal{E}_{eff}$ represents the effective field acting on the magnetization, α is a positive parameter known as the Gilbert damping constant, and γ represents the gyromagnetic factor.

The effective field $\mathcal{E}_{eff}(m)$ is determined as the negative gradient of the micromagnetism energy $\mathcal{E}(m)$, that is, $\mathcal{E}_{eff}(m) = -\nabla_m \mathcal{E}(m)$. The micromagnetism energy $\mathcal{E}(m)$ encapsulates various energy interactions within the ferromagnetic material. Specifically, $\mathcal{E} = \mathcal{E}_{ex} + \mathcal{E}_{an} + \mathcal{E}_{me} + \mathcal{E}_d + \mathcal{E}_a$, that is, $\mathcal{E}$ comprises the following energy components: exchange energy $\mathcal{E}_{ex}$, anisotropy energy $\mathcal{E}_{an}$, magnetoelastic energy $\mathcal{E}_{me}$, demagnetization field energy $\mathcal{E}_d$, and external magnetic field $\mathcal{E}_a$. For more comprehensive details on these energy interactions, we refer to [3].

In ferromagnetic materials, atomic magnetic moments tend to align themselves with neighboring moments due to exchange interactions, leading to an increase in exchange energy when they deviate from their equilibrium orientation. Assuming an external magnetic field described by a function $u : \Omega \times [0, T] \to \mathbb{R}^3$ and a purely isotropic energy field, the micromagnetism energy can be expressed as:

$$\mathcal{E} = \frac{1}{2}\int_\Omega |\nabla m|^2 \mathrm{d}x - \int_\Omega u \cdot m \ \mathrm{d}x.$$

Considering the associated magnetic fields for these energy components, the effective field is given by $\mathcal{E}_{eff}(m) = \Delta m + u$. For an extensive overview of the model and the physical interpretation of these energy terms, we refer to [4].

By taking the effective field concerning only the exchange energy and external magnetic field into account and fixing the damping constant and gyromagnetic factor to be of unit values, that is, $\alpha = \gamma = 1$, the Landau–Lifshitz–Gilbert equation becomes

$$\begin{cases} m_t = m \times (\Delta m + u) - m \times (m \times (\Delta m + u)), \quad (x,t) \in \Omega_T := \Omega \times (0, T], \\ \frac{\partial m}{\partial \eta} = 0, \qquad (x,t) \in \partial\Omega_T := \partial\Omega \times [0, T], \\ m(\cdot, 0) = m_0 \ \text{ in } \Omega, \end{cases} \tag{1.1}$$

where η is the outward unit normal vector to the boundary $\partial\Omega$ and u is the external magnetic field. Further, while going for the regular solution of system (1.1), we assume that the initial magnetic field $m_0 : \Omega \to \mathbb{R}^3$ satisfies the following conditions

$$m_0 \in H^2(\Omega), \quad \frac{\partial m_0}{\partial \eta} = 0 \ \text{ on } \partial\Omega, \quad |m_0| = 1. \tag{1.2}$$

We consider the cost functional $\mathcal{J} : \mathcal{M} \times \mathcal{U}_{ad} \to \mathbb{R}^+$ defined as

$$\begin{aligned} \mathcal{J}(m, u) := &\frac{1}{4}\int_0^T \int_\Omega |\nabla m - \nabla m_D|^4 \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{2}\int_\Omega |m(x, T) - m_\Omega(x)|^2 \ \mathrm{d}x \\ &+ \frac{1}{2}\int_0^T \int_\Omega |u(x,t)|^2 \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega |\nabla u(x,t)|^2 \ \mathrm{d}x \ \mathrm{d}t, \end{aligned} \tag{1.3}$$

where the desired evolutionary magnetic moment m_D is a map from $\Omega \times [0,T]$ to $\mathbb{R}^3$ such that $\nabla m_D \in L^4(0,T;L^4(\Omega))$ and final time target moment $m_\Omega : \Omega \to \mathbb{R}^3$ belongs to $L^2(\Omega)$. The optimal control problem can be interpreted as the search for an optimal strategy to magnetize a ferromagnetic material such that

(i) the gradient profile of the desired magnetization vector field m_D can be realized by the first term of the cost functional. This first term of $\mathcal{J}(m,u)$, which is defined with $L^4(0,T;L^4(\Omega))$ norm of $\nabla(m-m_D)$ instead of usual $L^2(0,T;L^2(\Omega))$, also plays a crucial role in showing the strong solvability of system (1.1) in Section 3.2.

(ii) The magnetization in the material should reach the target moment m_Ω at the final time, represented by the second term of the cost functional.

(iii) The last two terms ensures that we achieve desired gradient profile of evolutionary moment m_D and final target m_Ω by applying the least amount of external magnetic field in $L^2(0,T;H^1(\Omega))$ space.

The control problem finds application in optimizing the performance of magnetic hard drives and various magnetic storage devices [5]. By strategically optimizing the external magnetic field, the writing process can be significantly improved, leading to enhanced efficiency and reliability of data storage systems. The control problem is also relevant in designing and optimizing spintronic devices, such as spin valves and magnetic tunnel junctions, to achieve desired spin configurations and enhance device performance [6]. Besides, this control problem has numerous other real-world applications ranging from Magnetic Particle Imaging (MPI) in material science, magnetic hyperthermia in medical science to spin-based computing in computer science [7]. This paper explores the potential of such control optimization techniques by studying a second-order sufficient local optimality condition and an adjoint system-based global optimality condition, which are essential for numerical methods.

In [8], the authors have proved the existence of a weak solution of the Landau–Lifshitz (LL) equation on both the bounded regular domain of $\mathbb{R}^3$ and on the entire domain $\mathbb{R}^3$, but in the absence of an external magnetic field u. They first considered a penalized form, which helps to validate that $|m| = 1$, then showed the convergence of this penalized form to the weak formulation. Further, they obtained a critical result that the weak solution of the LL equation exists globally but, in general, is not unique. To address uniqueness, recent works have established weak-strong uniqueness results for variants of the LLG equation. In [9, 10], it was shown that weak solutions coincide with strong ones under appropriate regularity assumptions, including the presence of lower-order terms and coupled field effects. A visible amount of work has already been done for the strong solution of the LLG equation, but without the control parameter in the effective field. The local existence and uniqueness of regular solutions over a bounded domain in $\mathbb{R}^n, n = 2,3$ and the whole domain $\mathbb{R}^3$ were investigated in [11], [12]. The global existence under a smallness assumption on the initial data over 2D and 3D bounded domains has been analyzed in [11] and [13], respectively. Apart from the solvability of the LLG equation, minimal articles are available for the optimal control problem. The paper [14] addresses the solvability of the control problem concerning the 1D LLG equation, investigates the existence of an optimum, and provides a first-order necessary condition. The article [15] discussed the optimal control type problems associated with the LLG equation, and a necessary optimality system is derived when the magnetization is constant in space, which essentially leads to an optimization problem constraint by an ordinary differential equation. Furthermore, for more results related to the controllability of the Landau–Lifshitz equation, one may refer to [16, 17], and [18].

The works [9, 10] established the existence of weak solutions to the Maxwell–Landau–Lifshitz and related models, providing a detailed examination of the associated energy and helicity identities, and characterizing mechanisms for anomalous dissipation. A particularly important result in their work is the demonstration of weak-strong uniqueness for the Landau–Lifshitz–Gilbert and related equations, showing that strong solutions are unique within the class of weak solutions sharing the same initial data.

The main contributions of this paper are summarized as follows:

- We prove the existence of a weak solution for the LLG equation (1.1) on a bounded domain $\Omega \subset \mathbb{R}^2$, following the methodology in [8]. Addressing complications arising from the control u, we introduce a suitable penalized problem, enabling the existence of a weak solution even when the control is in $L^2(0,T;L^2(\Omega))$.

- In the context of the 1D LLG equation, as demonstrated in [14], the authors established the existence of a unique strong solution in $L^2(0,T;H^2(\Omega))$ when the control belongs to $L^2(0,T;L^2(\Omega))$. This result was achieved without imposing any restrictions based on the size of the initial data or control. However, when extending this analysis to the 2D LLG system (1.1), the severe non-linearity of the state equation made it challenging to directly show the existence of a strong solution for controls in $L^2(0,T;L^2(\Omega))$ using the classical Galerkin method. In fact, the existence of a regular solution is proven for the uncontrolled 2D LLG equation only when the initial data is sufficiently small (see [11, 19]).
  In our previous work [20], we addressed the local solvability of the 2D control problem (1.1) for all controls $u \in L^2(0,T;H^1(\Omega))$ with initial data satisfying condition (1.2). Moreover, we substantiated the existence and uniqueness of a global regular solution (see Thm. 2.2, [20]) under a smallness condition on control and initial data.
  In this paper, we make a significant contribution by demonstrating that if the magnetization field m is a weak solution of (1.1) and satisfies $\nabla m \in L^4(0,T;L^4(\Omega))$, it qualifies as a regular solution of the LLG equation (1.1), all without the need for smallness conditions imposed on the control and initial data. This result paves the way for a more comprehensive and realistic exploration of the control problem, offering broader applicability and relevance in practical scenarios.
- In Chapter 4 of [21], the author established and validated the argument that any weak solution of 3D Navier-Stokes equation, belonging to the space $L^8(0,T;L^4(\Omega))$, becomes a strong solution. Subsequently, in [22], the authors considered an appropriate norm in the cost functional, which confirmed the strong solvability of the problem and allowed for the study of the associated optimal control problem. Moreover, for a different method of solving the optimal control of 3D Navier-Stokes equations with pointwise control constraints and MHD equations with state constraints, one may refer to [23] and [24], respectively.
  In our current paper, we adopt a similar concept but with a different approach. Instead of directly employing the method from [21], which would involve a specific norm condition on Δm for the LLG equations' weak solutions to become regular, we utilize the regularity property of the local solution and derive estimate (2.2). This approach leads us to identify a condition on the gradient of the weak solution, under which it qualifies as a regular solution. This insight prompted the introduction of the term $\|\nabla m - \nabla m_D\|^4_{L^4(0,T;L^4(\Omega))}$ in the cost functional. This term ensures that any weak solution m of (1.1) satisfying $\mathcal{J}(m,u) < +\infty$ is a regular solution of (1.1). Consequently, we proceed to study the existence of an optimal control for (1.1)–(1.3) and derive a first-order optimality condition.
- Afterward, we establish a local second-order sufficient optimality condition. The local optimality criterion is defined on a certain cone (see [25]). It requires a rigorous study of the control-to-state and control-to-costate operators. We also establish the important properties that both of these operators are locally Lipschitz continuous, Fréchet differentiable, and moreover, their Fréchet derivatives are also locally Lipschitz continuous.
- Furthermore, we prove a remarkable theorem stating a global optimality condition, relying solely on the $L^2(0,T;L^2(\Omega))$ norm of the weak solution of the adjoint system (2.4). These optimality conditions are particularly valuable in non-convex or nonlinear optimal control problems and play a crucial role in demonstrating the convergence of error estimates for the numerical discretization of corresponding computational optimization problems (see [26–28]).
- Finally, from the application viewpoint, achieving optimal control as outlined above can pose significant implementation challenges. Therefore, we redirect our attention to an optimal control problem where control stems from a finite number of fixed magnetic field coils as introduced in [29, 30]. Our approach involves manipulating the intensities of the magnetic fields generated by these coils to effectively control the system. We study this optimal control problem, where the control is only a time-dependent variable, and derive a first-order necessary optimality condition. This approach also has the advantage of achieving a second-order sufficient optimality condition within the directions of a critical cone.

The paper is organized as follows: In Section 2, we present the main results of this paper. Subsection 2.4 establishes standard inequalities and essential cross product estimates that are utilized throughout the manuscript.

Section 3 addresses the existence of both weak and regular solutions for system (1.1). Section 4 investigates the existence of optimal control and presents a first-order optimality condition. This section also discusses the Fréchet differentiability and local Lipschitz continuity of the control-to-state operator, as well as the weak solvability of the adjoint system. Subsection 5.1 focuses on the Fréchet differentiability and local Lipschitz continuity of the control-to-costate operator. Subsection 5.2 is dedicated to a second-order optimality condition. Moreover, we establish a global optimality criterion in Section 6. Lastly, Section 7 addresses a second-order optimality condition for the control problem corresponding to the finite coil setting.

2. Main results and function spaces

2.1. Optimal control problem- I

In this section, we state the main results of our paper. For any regular solution of system (1.1), by taking cross product with m, we can find that (1.1) is equivalent to

$$\frac{\partial m}{\partial t} + m \times \frac{\partial m}{\partial t} = 2\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(m \times \frac{\partial m}{\partial x_i}\right) + 2\ m \times u.$$

This leads us to introduce the following notion of weak solution for system (1.1) (see [8]).

Definition 2.1. Suppose $m_0 \in H^1(\Omega)$ with $|m_0| = 1$ a.e. $x \in \Omega$ and $u \in L^2(0,T;L^2(\Omega))$. A function $m \in L^2(0,T;H^1(\Omega))$ with $m_t \in L^2(0,T;L^2(\Omega))$ and $|m| = 1$ a.e. in Ω_T is said to be a weak solution of (1.1) if the following hold:

1. $$\int_{\Omega_T} m_t \cdot \psi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} (m \times m_t)\cdot \psi \,\mathrm{d}x\,\mathrm{d}t$$ $$= -2\sum_{i=1}^{3}\int_{\Omega_T}\left(m \times \frac{\partial m}{\partial x_i}\right)\cdot\frac{\partial \psi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t + 2\int_{\Omega_T}(m \times u)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \quad \forall\ \psi \in L^2(0,T;H^1(\Omega)),$$
2. $m(x,0) = m_0(x)$ in the trace sense,
3. $\frac{1}{2}\|m_t\|^2_{L^2(0,T;L^2(\Omega))} + \sup_{t\in[0,T]}\|\nabla m(t)\|^2_{L^2(\Omega)} \le \|\nabla m_0\|^2_{L^2(\Omega)} + 2\ \|u\|^2_{L^2(0,T;L^2(\Omega))}$.

The following theorem proves the existence of a weak solution of system (1.1) in the above sense.

Theorem 2.2 (Weak Solution). *For any control $u \in L^2(0,T;L^2(\Omega))$ and initial data $m_0 \in H^1(\Omega)$ with $|m_0| = 1$, there exists a weak solution of system (1.1) in the sense of Definition 2.1.*

The proof of this theorem resembles the arguments put forth in [8]. However, in our problem, the control term appears non-linearly in the effective field. As a result, we provide a concise overview of the proof in our paper to account for this modification.

Considering the dot product of (1.1) with m and applying the cross product properties stated in Lemma 2.4, we can see that m and m_t are orthogonal in space and time, that is, $(d/\mathrm{d}t)|m(\cdot,t)|^2 = 0$. Since $m_0 \in \mathbb{S}^2$, we find that $|m(x,t)|^2 = 1$, pointwise in space and time. Consequently, for a regular solution m, expanding the cross product $m \times (m \times \Delta m)$ in (1.1) and using the identity $\Delta|m|^2 = 2(m\cdot\Delta m) + 2|\nabla m|^2$, we find an equivalent system of (1.1):

$$\begin{cases} m_t - \Delta m = |\nabla m|^2 m + m \times \Delta m + m \times u - m \times (m \times u), & (x,t) \in \Omega_T, \\ \frac{\partial m}{\partial \eta} = 0, \qquad (x,t) \in \partial\Omega_T, \\ m(\cdot,0) = m_0 \ \text{ in } \Omega. \end{cases} \tag{2.1}$$

In the absence of an explicit elliptic operator in equation (1.1), it is difficult to apply the classical techniques of Galerkin approximation. So, we will show the solvability of the equivalent system (2.1) instead of (1.1). If a function m satisfies system (2.1) almost everywhere and has an absolute value of $|m| = 1$, it will also satisfy system (1.1) almost everywhere.
Now, define the set of admissible solution spaces as

$$\mathcal{M} := W^{1,2}(0,T; H^3(\Omega), H^1(\Omega)) = \{m \in L^2(0,T; H^3(\Omega)) \mid m_t \in L^2(0,T; H^1(\Omega))\}.$$

A function $m_u \in \mathcal{M}$ that satisfies system (1.1) almost everywhere while being associated with the control function $u \in L^2(0,T; H^1(\Omega))$ is referred to as a "regular solution".

Theorem 2.3 (Global Existence of Regular Solution)**.** *Suppose the control $u \in L^2(0,T; H^1(\Omega))$ and the initial data m_0 satisfies condition (1.2). If m is any weak solution of system (1.1) and satisfies the condition $\nabla m \in L^4(0,T; L^4(\Omega))$, then m is a unique regular solution.*
Moreover, there exists a constant $C(\Omega, T) > 0$ such that the following estimate holds:

$$\begin{aligned} \|m\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m\|^2_{L^2(0,T;H^3(\Omega))} + \|m_t\|^2_{L^2(0,T;H^1(\Omega))} &\leq \left(\|\Delta m_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\right)^2 \\ &\times \exp\left(C(\Omega,T)\left[1 + \|\nabla m_0\|^4_{L^2(\Omega)} + \ \|\nabla m\|^8_{L^4(0,T;L^4(\Omega))} + \ \|u\|^4_{L^2(0,T;H^1(\Omega))}\right]\right). \end{aligned} \tag{2.2}$$

Remark 2.4. For any control $u \in L^2(0,T; H^1(\Omega))$ and initial data m_0 satisfying condition (1.2), if a regular solution exists, then it must be unique (see Thm. 2.1, [20]).

As boundedness of $\mathcal{J}(m,u)$ implies $\|\nabla m\|_{L^4(0,T;L^4(\Omega))} < \infty$, so we can get the following remark.

Remark 2.5. If m is a weak solution of (1.1) satisfying $\mathcal{J}(m,u) < +\infty$, then m is a regular solution.

Let $\mathcal{U}$ be the set of all controls in $L^2(0,T; H^1(\Omega))$ for which there exists a regular solution in $\mathcal{M}$. As a consequence of Proposition 5.2 in [20], we can see that the set of controls $\mathcal{U}$ is an open set. We aim to prove the Fréchet differentiability of the control-to-state and control-to-costate operators over this open set $\mathcal{U}$.

In this work, we also consider the admissible control variables subject to box-type constraints. Given any functions $a, b \in L^2(\Omega_T)$, we define the admissible control set as

$$\mathcal{U}_{ad} := \{u \in \mathcal{U} : a(x,t) \leq u(x,t) \leq b(x,t) \ \text{ for almost every } (x,t) \in \Omega_T\}.$$

In order to give a possible detailed analysis of the control problem, we make the following non-emptiness assumption on the set of admissible controls:

$$\mathcal{U}_{ad} \neq \emptyset. \tag{2.3}$$

Remark 2.6. The non-emptiness assumption (2.3) can be further explored as follows. From Theorem 2.7 and Proposition 5.2 of [20], it follows that under suitable initial data, the feasible control set $\mathcal{U}$ forms a non-empty open subset of $L^2(0,T; H^1(\Omega))$. In our setting, the admissible control set further incorporates box-type constraints. Although the non-emptiness of $\mathcal{U}_{ad}$ can be ensured for sufficiently small boundaries a and b by the arguments of [20], we refrain from imposing such restrictions to retain generality. We consider arbitrary initial data and box constraints, and directly assume the non-emptiness of the admissible set. For more details regarding the non-emptiness of the admissible control set in the context of the 3D Navier-Stokes equations, the reader is referred to [21, 22, 31].

A pair $(m,u) \in \mathcal{M} \times \mathcal{U}_{ad}$ is called an admissible pair if (m,u) is a regular solution of system (1.1). Let us denote the set of all admissible pairs as $\mathcal{A}$.

The optimal control problem is stated as follows:

$$\text{(OCP)} \begin{cases} \text{minimize } \mathcal{J}(m,u), \\ (m,u) \in \mathcal{A}. \end{cases}$$

Theorem 2.7 (Existence of Optimal Control)**.** *Suppose the initial data m_0 satisfies condition (1.2), final time target $m_\Omega \in L^2(\Omega)$ and $\nabla m_D \in L^4(0,T;L^4(\Omega))$. Then, under the assumption (2.3), the OCP has at least one solution.*

In the subsequent steps, we will establish a first-order necessary optimality condition for the OCP by utilizing the classical adjoint problem approach. This approach allows us to express the optimality condition in a compact form, providing a concise framework for analysis. First, by employing the formal Lagrange method [25], we derive the adjoint system associated with OCP as follows:

$$\begin{cases} \phi_t + \Delta\phi + |\nabla \widetilde{m}|^2\phi - 2\nabla\cdot\big((\widetilde{m}\cdot\phi)\nabla\widetilde{m}\big) + \Delta(\phi\times\widetilde{m}) + (\Delta\widetilde{m}\times\phi) - (\phi\times\widetilde{u}) \\ \qquad\qquad + \big((\phi\times\widetilde{m})\times\widetilde{u}\big) + \big(\phi\times(\widetilde{m}\times\widetilde{u})\big) = \nabla\cdot\big(|\nabla\widetilde{m}-\nabla m_D|^2(\nabla\widetilde{m}-\nabla m_D)\big) \quad \text{in } \Omega_T, \\ \frac{\partial\phi}{\partial\eta} = 0 \quad \text{in } \partial\Omega_T, \\ \phi(T) = \widetilde{m}(x,T) - m_\Omega(x) \quad \text{in } \Omega. \end{cases} \tag{2.4}$$

Define the weak adjoint solution space as

$$\mathcal{Z} := W^{1,2}(0,T;H^1(\Omega),H^1(\Omega)^*) = \{z\in L^2(0,T;H^1(\Omega)) \mid z_t \in L^2(0,T;H^1(\Omega)^*)\},$$

where $H^1(\Omega)^*$ is the dual of the space $H^1(\Omega)$. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)^*$. We have the following notion of weak solution for the system (2.4).

Definition 2.8 (Weak Formulation)**.** A function $\phi\in\mathcal{Z}$ is said to be a weak solution of system (2.4) if for every $\vartheta \in L^2(0,T;H^1(\Omega))$, the following holds:

$$\begin{aligned} (i)\quad & \int_0^T \langle\phi'(t),\vartheta(t)\rangle_{H^1(\Omega)^*\times H^1(\Omega)}\,\mathrm{d}t - \int_{\Omega_T}\nabla\phi\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}|\nabla\widetilde{m}|^2\phi\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\int_{\Omega_T}(\widetilde{m}\cdot\phi)\nabla\widetilde{m}\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\nabla(\phi\times\widetilde{m})\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}(\Delta\widetilde{m}\times\phi)\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{\Omega_T}(\phi\times\widetilde{u})\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\big((\phi\times\widetilde{m})\times\widetilde{u}\big)\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\big(\phi\times(\widetilde{m}\times\widetilde{u})\big)\cdot\vartheta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\int_{\Omega_T}|\nabla\widetilde{m}-\nabla m_D|^2(\nabla\widetilde{m}-\nabla m_D)\cdot\nabla\vartheta\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.5}$$

(ii) $\phi(T) = \widetilde{m}(x,T) - m_\Omega(x)$.

Theorem 2.9 (Weak Solution of Adjoint System)**.** *Suppose $(\tilde m,\tilde u)$ is an admissible pair, that is, $(\tilde m,\tilde u)\in\mathcal{A}$, $m_\Omega\in L^2(\Omega)$ and $\nabla m_D \in L^6(0,T;L^6(\Omega))$. Then, there exists a unique weak solution $\phi\in\mathcal{Z}$ of the adjoint system (2.4) in the sense of Definition 2.8. Moreover, the following estimate holds:*

$$\begin{aligned} \|\phi\|^2_{L^2(0,T;H^1(\Omega))} + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega)^*)} &\le \Big(\|\tilde m(T)-m_\Omega\|^2_{L^2(\Omega)} + \|\nabla\tilde m - \nabla m_D\|^6_{L^6(0,T;L^6(\Omega))}\Big) \\ &\times\ \exp\Big\{C\Big(\|\tilde m\|^2_{L^\infty(0,T;H^2(\Omega))}\|\tilde m\|^2_{L^2(0,T;H^3(\Omega))} + \|\tilde m\|^2_{L^\infty(0,T;H^2(\Omega))}\|\tilde u\|^2_{L^2(0,T;H^1(\Omega))}\Big)\Big\}. \end{aligned} \tag{2.6}$$

By virtue of the global existence of a regular solution, we can define a *reduced cost functional* $\mathcal{I} : \mathcal{U}_{ad} \to \mathbb{R}$ by $\mathcal{I}(u) = \mathcal{J}(G(u), u)$. Therefore, the optimal control problem (OCP) can be redefined as follows:

$$\text{(MOCP)} \begin{cases} \text{minimize} \;\; \mathcal{I}(u) \\ u \in \mathcal{U}_{ad}. \end{cases}$$

Now we are ready to state a first-order optimality condition satisfied by the optimal control $\widetilde{u} \in \mathcal{U}_{ad}$.

Theorem 2.10 (First Order Optimality Condition-I)**.** *Suppose $\tilde{u} \in \mathcal{U}_{ad}$ be an optimal control of MOCP with associated state $\tilde{m}$. Then there exists a unique element $\phi \in \mathcal{Z}$ corresponding to the admissible pair $(\tilde{m}, \tilde{u})$ such that the triplet $(\phi, \tilde{m}, \tilde{u})$ satisfies the weak adjoint formulation (2.5) and the following variational inequality holds:*

$$\begin{aligned} &\int_0^T \big(\tilde{u}, u - \tilde{u}\big)\, \mathrm{d}t + \int_0^T \big(\nabla \tilde{u}, \nabla (u - \tilde{u})\big)\, \mathrm{d}t \\ &+ \int_0^T \Big((\phi \times \tilde{m}) + \tilde{m} \times (\phi \times \tilde{m}), (u - \tilde{u})\Big)\, \mathrm{d}t \geq 0, \qquad \forall\, u \in \mathcal{U}_{ad}. \end{aligned} \tag{2.7}$$

Next, we will discuss a local second-order optimality condition that an optimal control satisfies.

Let $\widetilde{u} \in \mathcal{U}_{ad}$ satisfy the first order necessary optimality condition given by the variational inequality (2.7) and suppose $\mathcal{J}''(\widetilde{u})[h, h] > 0$ for all directions $h \in L^2(0, T; H^1(\Omega)) \setminus \{0\}$. Then $\widetilde{u}$ is a strict local minimum of the functional $\mathcal{J}$ on the set $\mathcal{U}_{ad}$. But such second-order conditions are too restrictive as they require all directions h in $L^2(0, T; H^1(\Omega))$ excluding $\{0\}$ to be taken into account. However, we can avoid this restriction by defining a cone that will suffice the needs for local optimality. For more details on these cones, one may refer to Chapter 4 in [25].

Definition 2.11. For any control $\widetilde{u} \in \mathcal{U}_{ad}$, let $\Pi(\widetilde{u})$ denotes the set of all $h \in L^2(0, T; H^1(\Omega))$ such that for almost all $(x, t) \in \Omega_T$,

$$h(x, t) = \begin{cases} \geq 0 & \text{if } \; \widetilde{u}(x, t) = a(x, t), \\ \leq 0 & \text{if } \; \widetilde{u}(x, t) = b(x, t). \end{cases}$$

In order to show the second-order optimality condition, we need the Fréchet differentiability of both the control-to-state and control-to-costate operators (defined later), and that of their Lipschitz continuity. With all these results, we established a second-order condition for local optimality as follows.

Theorem 2.12 (Second-Order Optimality Condition-I)**.** *Let $\tilde{u} \in \mathcal{U}_{ad}$ be any control satisfying the variational inequality (2.7). Suppose there exists a constant $\delta > 0$ such that $\mathcal{J}''(\tilde{u})[h, h] \geq \delta\, \|h\|^2_{L^2(0,T;H^1(\Omega))}$ for all $h \in \Pi(\tilde{u})$, that is,*

$$\begin{aligned} &\int_{\Omega_T} h^2 \,\mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} |\nabla h|^2 \,\mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} \big(\phi'_{\widetilde{u}}[h] \times m_{\widetilde{u}}\big) \cdot h \,\mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} \big(\phi_{\widetilde{u}} \times m'_{\widetilde{u}}[h]\big) \cdot h \,\mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_{\Omega_T} \big(m'_{\widetilde{u}}[h] \times (\phi_{\widetilde{u}} \times m_{\widetilde{u}})\big) \cdot h \,\mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} \big(m_{\widetilde{u}} \times (\phi'_{\widetilde{u}}[h] \times m_{\widetilde{u}})\big) \cdot h \,\mathrm{d}x\, \mathrm{d}t \\ &\quad + \int_{\Omega_T} \big(m_{\widetilde{u}} \times (\phi_{\widetilde{u}} \times m'_{\widetilde{u}}[h])\big) \cdot h \,\mathrm{d}x\, \mathrm{d}t \geq \delta \int_{\Omega_T} h^2 \,\mathrm{d}x\, \mathrm{d}t + \delta \int_{\Omega_T} |\nabla h|^2 \,\mathrm{d}x\, \mathrm{d}t \qquad \forall\, h \in \Pi(\tilde{u}). \end{aligned} \tag{2.8}$$

Then there exist $\epsilon > 0$ and $\sigma > 0$ such that for every control $u \in \mathcal{U}_{ad}$ with $\|u - \tilde{u}\|_{L^2(0,T;H^1(\Omega))} \leq \epsilon$, we have

$$\mathcal{J}(u) \geq \mathcal{J}(\tilde{u}) + \sigma\, \|u - \tilde{u}\|^2_{L^2(0,T;L^2(\Omega))}.$$

In particular, this implies that $\tilde{u}$ is a local minimum of the functional $\mathcal{J}$ on the set of admissible control $\mathcal{U}_{ad}$.

2.2. Optimal control problem- II

Although the second-order optimality conditions are very informative, the global optimality and uniqueness of such a control input remain uncertain. To address these questions, we employ a methodology inspired by the work presented in [26] for a semilinear elliptic control problem, and for related results, refer to [32, 33] and [28]. But for our system, we have modified the proof presented in these papers by incorporating the Lipschitz continuity of the control-to-state operator. This approach allows us to establish that an admissible control input satisfying a first-order variational inequality and a condition involving the adjoint solution yields a global optimal control, which is another major contribution of this paper.

In the previous optimal control problem, we had difficulties in proving the global optimality because of the lack of a uniform bound on the admissible solution set. In order to get this result, we modify the admissible control set as follows. For any fixed but arbitrary constant $R > 0$, we define a new class of admissible control set

$$\widetilde{\mathcal{U}}_{ad} := \{u \in \mathcal{U} \ : \ \mathcal{J}(m_u, u) \leq R\}.$$

Under the condition $\widetilde{\mathcal{U}}_{ad} \neq \emptyset$, the new optimal control problem is given by

$$\text{(GOCP)} \begin{cases} \text{minimize} \ \ \mathcal{I}(u) \\ u \in \widetilde{\mathcal{U}}_{ad}. \end{cases}$$

Defining such a control set has the advantage that, according to estimate (2.2), the admissible solution set will be uniformly bounded. In other words, there exists a constant C depending on R and initial data m_0, but independent of the control variable, such that the following holds:

$$\|m_u\|_{\mathcal{M}} \leq C \quad \ \forall \ u \in \widetilde{\mathcal{U}}_{ad}. \tag{2.9}$$

It is evident that the existence of optimal control (Thm. 2.7) and first-order variational inequality (Thm. 2.10) hold true for this admissible control set $\widetilde{\mathcal{U}}_{ad}$. Further, using the uniform bound (2.9), we can prove the global Lipschitz continuity of the control-to-state operator (by following Lem. 4.2), which states that for any controls $u, v \in \widetilde{\mathcal{U}}_{ad}$, there exists a constant C depending on Ω, T, R such that

$$\|m_u - m_v\|_{\mathcal{M}} \leq C \ \|u - v\|_{L^2(0,T;H^1(\Omega))}. \tag{2.10}$$

Now, we can state the global optimality result as follows.

Theorem 2.13 (Global Optimality Condition)**.** *Let $\tilde{u} \in \widetilde{\mathcal{U}}_{ad}$ be a control with the associated state $\tilde{m} \in \mathcal{M}$ and the adjoint state $\phi \in \mathcal{Z}$. Suppose that the triplet $(\tilde{u}, \tilde{m}, \phi)$ satisfies the variational inequality (2.7), and the adjoint system fulfills the following condition:*

$$C(\Omega, T) \ \left\{1 + \|\tilde{m}\|_{L^\infty(0,T;H^2(\Omega))} + \|\tilde{u}\|_{L^2(0,T;H^1(\Omega))}\right\} \ \|\phi\|_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{2}. \tag{2.11}$$

Then $\tilde{u} \in \widetilde{\mathcal{U}}_{ad}$ is a global optimal control of the functional $\mathcal{J}$. Moreover, if the inequality in condition (2.11) is strict, then the global optimum $\tilde{u}$ is unique.

2.3. Optimal control problem- III

Next, we will study the optimal control problem of the LLG system with external magnetic field (control) generated by a finite number of fixed magnetic field coils as motivated in [29, 30]. Suppose there are 'n' number of such coils, and the magnetic field of the k-th field coil is given as follows

$$u_k : \Omega_T \to \mathbb{R}^3, \qquad (x,t) \mapsto u_k(x,t) := U_k(t)B_k(x),$$

where $U_k : [0,T] \mapsto \mathbb{R}$ corresponds to the intensity of the magnetic field, which is proportional to the electric current flowing through the k-th coil. The function $B_k : \Omega \mapsto \mathbb{R}^3$ describes the geometry of the k-th field coil. Moreover, this type of control problem holds greater physical relevance due to the opportunities it presents for implementation.
Then, due to linear superposition, the total external magnetic field is given by

$$u(x,t) = \sum_{k=1}^{n} U_k(t)B_k(x) \qquad \forall\ (x,t) \in \Omega_T. \tag{2.12}$$

Now, we assume that the geometry of the field coil $B_k \in H^1(\Omega,\mathbb{R}^3)$ for each $k = 1,2,...,n$. Defining B_k in this space has the advantage that for any control $U \in L^2(0,T;\mathbb{R}^n)$, the space-time dependent function $u(x,t)$ belongs to $L^2(0,T;H^1(\Omega))$ space as

$$\|u\|^2_{L^2(0,T;H^1(\Omega))} = \left\|\sum_{k=1}^{n} U_k(t)B_k(x)\right\|^2_{L^2(0,T;H^1(\Omega))} \leq C\sum_{k=1}^{n} \|U_k\|^2_{L^2(0,T)}\ \|B_k\|^2_{H^1(\Omega)} < \infty.$$

Define the set of controls

$$\mathbb{U} := \Big\{U \in L^2(0,T;\mathbb{R}^n) \text{ s.t. } \exists \text{ a regular solution of (1.1) corresponding to } u(x,t) = \sum_{k=1}^{n} U_k(t)B_k(x)\Big\}.$$

Since the set $\mathcal{U}$ is open, it follows that the control set $\mathbb{U}$ is also open. For some functions $a,b \in L^2(0,T;\mathbb{R}^n)$, we define the following set of admissible controls

$$\mathbb{U}_{ad} := \Big\{U \in \mathbb{U}\ \big|\ a_k(t) \leq U_k(t) \leq b_k(t) \quad \text{for } k = 1,2,...,n \quad \text{and almost every } t \in [0,T]\Big\}.$$

Again, in order to make the analysis possible, we have to assume that $\mathbb{U}_{ad} \neq \emptyset$.

Remark 2.14. In addition to Remark 2.6, from the practical applications and implementation viewpoint, it is often desirable to keep the magnetic field intensities U_k as low as possible, which directly leads to the smallness in the control bounds a and b. Hence, the non-emptiness assumption on the admissible control set is not merely theoretical but also motivated by practical considerations.

For the control problem with fixed magnetic field coils, we consider the following functional

$$\widehat{\mathcal{J}}(m,U) := \frac{1}{4}\int_0^T \int_\Omega |\nabla m - \nabla m_D|^4 \ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\int_\Omega |m(x,T) - m_\Omega(x)|^2 \ \mathrm{d}x + \frac{1}{2}\int_0^T |U(t)|^2 \ \mathrm{d}t.$$

With the introduction of this new class of admissible controls $\mathbb{U}_{ad}$ and functional $\widehat{\mathcal{J}}$, we define a new optimal control problem as follows:

$$\text{(SOCP)} \begin{cases} \text{minimize } \widehat{\mathcal{J}}(m,u), \\ (m,u) \in \mathcal{M} \times \mathbb{U}_{ad}. \end{cases}$$

Consider an operator $\zeta : L^2(0,T;\mathbb{R}^n) \to L^2(0,T;H^1(\Omega))$ defined by $\zeta(U) = \sum_{k=1}^{n} U_k(t)B_k(x)$. Now, we have the following remarks corresponding to the control (2.12).

Remark 2.15.

- The existence of optimal control for the new minimization problem (SOCP) can be established similarly to the approach outlined in Theorem 2.7.
- Since ζ is a linear and bounded operator, it is continuously Fréchet differentiable with $\zeta'(\cdot)[h] = \zeta(h)$. Consequently, the cost functional $\widehat{\mathcal{J}}$ is also continuously Fréchet differentiable by using Proposition 4.3.
- We can also prove a first-order optimality condition given below in Theorem 2.16 by following a similar approach to the one used in Theorem 2.10.

Theorem 2.16 (First-Order Optimality Condition-II)**.** *Let $\widetilde{U} \in \mathbb{U}_{ad}$ be an optimal control for the SOCP with associated state $m_{\widetilde{U}}$. If $\phi \in \mathcal{Z}$ is a weak solution of the adjoint system (2.4) corresponding to the admissible pair $(m_{\widetilde{U}}, \zeta(\widetilde{U}))$, then the triplet $(\phi, m_{\widetilde{U}}, \widetilde{U})$ satisfies the following variational inequality:*

$$\int_0^T \widetilde{U} \cdot \left(U - \widetilde{U}\right) \,\mathrm{d}t + \int_{\Omega_T} \left(\phi \times m_{\widetilde{U}} + m_{\widetilde{U}} \times (\phi \times m_{\widetilde{U}})\right) \cdot \zeta(U - \widetilde{U}) \,\mathrm{d}x \,\mathrm{d}t \geq 0, \quad \forall\, U \in \mathcal{U}_{ad}. \tag{2.13}$$

Definition 2.17. (Critical Cone) For any control $\widetilde{U} \in \mathbb{U}_{ad}$, let $\Lambda(\widetilde{U})$ denotes the set of all $h \in L^2(0,T;\mathbb{R}^n)$ such that for almost all $t \in [0,T]$,

$$h(t) = \begin{cases} \geq 0 & \text{if } \ \widetilde{U}(t) = a(t), \\ \leq 0 & \text{if } \ \widetilde{U}(t) = b(t), \\ = 0 & \text{if } \ \Upsilon_{\widetilde{U}} \neq 0, \end{cases}$$

where $\Upsilon_{\widetilde{U}} := \widetilde{U} + \int_\Omega \left(\phi \times m_{\widetilde{U}} + m_{\widetilde{U}} \times (\phi \times m_{\widetilde{U}})\right) \cdot B \,\mathrm{d}x$, where B is the geometry of the coil defined in (2.12). The set of controls $\Lambda(\widetilde{U})$ is called the cone of critical directions.

Remark 2.18. The cone in Definition 2.11 is very large. But the cone defined in Definition 2.17 is a critical one in the sense that second-order sufficient conditions are the closest to the associated necessary ones. Such a cone was introduced in [27] for semilinear control problems with pointwise state constraints.

Theorem 2.19 (Second-Order Optimality Condition-II)**.** *Let $\widetilde{U} \in \mathbb{U}_{ad}$ be any control satisfying the variational inequality (2.13). Moreover, we assume that $\widehat{\mathcal{J}}''(\widetilde{U})[h,h] > 0$ for all $h \in \Lambda(\widetilde{U}) \setminus \{0\}$, that is*

$$\begin{aligned}
&\int_0^T h^2 \,\mathrm{d}t + \int_{\Omega_T} \left(\phi'_{\zeta(\widetilde{U})}[\zeta(h)] \times m_{\widetilde{U}}\right) \cdot h \,\mathrm{d}x \,\mathrm{d}t + \int_{\Omega_T} \left(\phi \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\right) \cdot h \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_{\Omega_T} \left(m'_{\zeta(\widetilde{U})}[\zeta(h)] \times \left(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\right)\right) \cdot h \,\mathrm{d}x \,\mathrm{d}t + \int_{\Omega_T} \left(m_{\widetilde{U}} \times \left(\phi'_{\zeta(\widetilde{U})}[\zeta(h)] \times m_{\widetilde{U}}\right)\right) \cdot h \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_{\Omega_T} \left(m_{\widetilde{U}} \times \left(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\right)\right) \cdot h \,\mathrm{d}x \,\mathrm{d}t > 0 \qquad \forall\, h \in \Lambda(\widetilde{U})\setminus\{0\}.
\end{aligned}$$

Then there exist $\epsilon > 0$ and $\delta > 0$ such that for every control $U \in \mathbb{U}_{ad}$ with $\|U - \widetilde{U}\|_{L^2(0,T;\mathbb{R}^n)} < \delta$,

$$\widehat{\mathcal{J}}(U) \geq \widehat{\mathcal{J}}(\widetilde{U}) + \frac{\epsilon}{2}\, \|U - \widetilde{U}\|^2_{L^2(0,T;\mathbb{R}^n)}. \tag{2.14}$$

In particular, this implies that $\widetilde{U}$ is a strict local minimum of the functional $\widehat{\mathcal{J}}$ on the set of admissible control $\mathbb{U}_{ad}$.

2.4. Function spaces and inequalities

In this subsection, we give some basic cross-product properties in Lemma 2.20, the equality of norms in Lemma 2.21, and some norm estimates in Lemma 2.23 and 2.24, which we have used throughout this paper.

Lemma 2.20. *Let a, b and c be three vectors of $\mathbb{R}^3$, then the following vector identities hold: $a\cdot(b\times c) = -(b\times a)\cdot c$, $a\cdot(a\times b)=0$, $a\times(b\times c)=(a\cdot c)b-(a\cdot b)c$. Moreover, assume that $1\le r,s\le\infty$, $(1/r)+(1/s)=1$ and $p\ge 1$, then if $f\in L^{pr}(\Omega)$ and $g\in L^{ps}(\Omega)$, we have*

$$\|f\times g\|_{L^p(\Omega)} \le \|f\|_{L^{pr}(\Omega)}\|g\|_{L^{ps}(\Omega)}. \tag{2.15}$$

The proof of estimate (2.15) can be readily derived by applying Hölder's inequality.

The L^2 theory of the Laplace operator with Neumann boundary condition leads to the following inequality of norms that will be quite useful.

Lemma 2.21 (see, [34])**.** *Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $k\in\mathbb{N}$. There exists a constant $C_{k,n}>0$ such that for all $m\in H^{k+2}(\Omega)$ and $\frac{\partial m}{\partial\eta}\big|_{\partial\Omega}=0$, it holds that*

$$\|m\|_{H^{k+2}(\Omega)} \le C_{k,n}\left(\|m\|_{L^2(\Omega)}+\|\Delta m\|_{H^k(\Omega)}\right).$$

As a consequence of Lemma 2.21, we can define an equivalent norm on $H^{k+2}(\Omega)$ as follows

$$\|m\|_{H^{k+2}(\Omega)} := \|m\|_{L^2(\Omega)}+\|\Delta m\|_{H^k(\Omega)}.$$

While showing the existence of a regular solution, we have used the following estimates.

Proposition 2.22. *Let Ω be a regular bounded subset of $\mathbb{R}^2$. There exists a constant $C>0$ depending on Ω such that for all $m\in H^2(\Omega)$ with $\frac{\partial m}{\partial\eta}\big|_{\partial\Omega}=0$, we have*

$$\|\nabla m\|_{L^s(\Omega)} \le C\ \|\Delta m\|_{L^2(\Omega)}, \quad \forall\ s\in[1,\infty), \tag{2.16}$$

$$\|D^2 m\|_{L^2(\Omega)} \le C\ \|\Delta m\|_{L^2(\Omega)}, \tag{2.17}$$

$$\|\nabla m\|_{L^6(\Omega)} \le C\ \|\nabla m\|_{L^2(\Omega)}^{\frac{1}{3}}\|\Delta m\|_{L^2(\Omega)}^{\frac{2}{3}}, \tag{2.18}$$

$$\|\nabla m\|_{L^\infty(\Omega)} \le C\ \|\nabla m\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\Delta m\|_{L^2(\Omega)}^{\frac{1}{2}}. \tag{2.19}$$

The proof of Proposition 2.22 can be found in [20].

In order to calculate the $L^2(0,T;H^1(\Omega))$ norm of different cross products and non-linear terms, we will use the following inequalities throughout the paper frequently. The constant $C>0$ may differ from one estimate to another estimate in Lemma 2.23 and 2.24.

Lemma 2.23. *Let Ω be a regular bounded domain of $\mathbb{R}^2$. Then there exists a constant $C>0$ depending on Ω and T such that*

(i) for $\xi\in L^\infty(0,T;H^2(\Omega))$ and $\zeta\in L^2(0,T;H^3(\Omega))$,

$$\|\xi\times\Delta\zeta\|^2_{L^2(0,T;H^1(\Omega))} \le C\ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\zeta\|^2_{L^2(0,T;H^3(\Omega))}, \tag{2.20}$$

(ii) for $\xi\in L^\infty(0,T;H^2(\Omega))$ and $\zeta\in L^2(0,T;H^1(\Omega))$,

$$\|\xi\times\zeta\|^2_{L^2(0,T;H^1(\Omega))} \le C\ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\zeta\|^2_{L^2(0,T;H^1(\Omega))}, \tag{2.21}$$

(iii) for $\xi \in L^2(0,T;H^3(\Omega))$, $\zeta \in L^\infty(0,T;H^2(\Omega))$ and $\omega \in L^\infty(0,T;H^2(\Omega))$,

$$\|(\nabla\xi \cdot \nabla\zeta)\ \omega\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|\xi\|^2_{L^2(0,T;H^3(\Omega))}\ \|\zeta\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\omega\|^2_{L^\infty(0,T;H^2(\Omega))}, \tag{2.22}$$

(iv) for $\xi \in L^\infty(0,T;H^2(\Omega))$, $\zeta \in L^\infty(0,T;H^2(\Omega))$ and $\omega \in L^2(0,T;H^1(\Omega))$,

$$\|\xi \times (\zeta \times \omega)\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\zeta\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\omega\|^2_{L^2(0,T;H^1(\Omega))}. \tag{2.23}$$

Proof. For the first and second estimates, applying Hölder's inequality with embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we find

$$\begin{aligned}\|\xi \times \Delta\zeta\|^2_{L^2(0,T;H^1(\Omega))} &\leq \int_0^T \|\xi\|^2_{L^\infty(\Omega)}\|\Delta\zeta\|^2_{L^2(\Omega)}\ \mathrm{d}t + 2\int_0^T \|\nabla\xi\|^2_{L^4(\Omega)}\|\Delta\zeta\|^2_{L^4(\Omega)}\mathrm{d}t \\ &\quad + 2\int_0^T \|\xi\|^2_{L^\infty(\Omega)}\|\nabla\Delta\zeta\|^2_{L^2(\Omega)}\ \mathrm{d}t \leq C\ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\zeta\|^2_{L^2(0,T;H^3(\Omega))}.\end{aligned}$$

This proves the inequality (i). The proof for (ii) can be constructed using a comparable line of reasoning as the proof for (i). Finally, for the last two estimates, implementing Hölder's inequality followed by the embeddings $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in [1,\infty)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we derive

$$\begin{aligned}\|(\nabla\xi \cdot \nabla\zeta)\ \omega\|^2_{L^2(0,T;H^1(\Omega))} &\leq \int_0^T \|\nabla\xi\|^2_{L^4(\Omega)}\|\nabla\zeta\|^2_{L^8(\Omega)}\|\omega\|^2_{L^8(\Omega)}\mathrm{d}t + 3\int_0^T \|D^2\xi\|^2_{L^4(\Omega)}\|\nabla\zeta\|^2_{L^8(\Omega)}\|\omega\|^2_{L^8(\Omega)}\mathrm{d}t \\ &\quad + 3\int_0^T \|\nabla\xi\|^2_{L^\infty(\Omega)}\|D^2\zeta\|^2_{L^2(\Omega)}\|\omega\|^2_{L^\infty(\Omega)}\mathrm{d}t + 3\int_0^T \|\nabla\xi\|^2_{L^4(\Omega)}\|\nabla\zeta\|^2_{L^8(\Omega)}\|\nabla\omega\|^2_{L^8(\Omega)}\mathrm{d}t \\ &\leq C\ \|\xi\|^2_{L^2(0,T;H^3(\Omega))}\ \|\zeta\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\omega\|^2_{L^\infty(0,T;H^2(\Omega))},\end{aligned}$$

and
$$\begin{aligned}\|\xi \times (\zeta \times \omega)\|^2_{L^2(0,T;H^1(\Omega))} &\leq \int_0^T \|\xi\|^2_{L^4(\Omega)}\|\zeta\|^2_{L^8(\Omega)}\|\omega\|^2_{L^8(\Omega)}\mathrm{d}t + 3\int_0^T \|\nabla\xi\|^2_{L^4(\Omega)}\|\zeta\|^2_{L^8(\Omega)}\|\omega\|^2_{L^8(\Omega)}\mathrm{d}t \\ &\quad + 3\int_0^T \|\xi\|^2_{L^4(\Omega)}\|\nabla\zeta\|^2_{L^8(\Omega)}\|\omega\|^2_{L^8(\Omega)}\mathrm{d}t + 3\int_0^T \|\xi\|^2_{L^\infty(\Omega)}\|\zeta\|^2_{L^\infty(\Omega)}\|\nabla\omega\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\leq C\ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\zeta\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|\omega\|^2_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Hence the proof of (iii) and (iv). ☐

Moreover, to estimate the $L^2(0,T;H^1(\Omega)^*)$ norm of various terms, we require the following lemma.

Lemma 2.24. *Let Ω be a bounded subset of $\mathbb{R}^2$ with smooth boundary. Then there exists a constant $C > 0$ such that*

(i) for $\xi \in L^\infty(0,T;H^2(\Omega))$, $\zeta \in L^\infty(0,T;H^2(\Omega))$ and $\omega \in L^2(0,T;L^2(\Omega))$,

$$\|\big(\nabla\xi \cdot \nabla\zeta\big)\omega\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\xi\|_{L^\infty(0,T;H^2(\Omega))}\|\zeta\|_{L^\infty(0,T;H^2(\Omega))}\|\omega\|_{L^2(0,T;L^2(\Omega))}, \tag{2.24}$$

(ii) for $\xi \in L^\infty(0,T;H^2(\Omega))$, $\zeta \in L^\infty(0,T;L^2(\Omega))$ and $\omega \in L^2(0,T;H^3(\Omega))$ with $\frac{\partial\omega}{\partial\eta} = 0$,

$$\|\nabla \cdot \big((\xi \cdot \zeta)\nabla\omega\big)\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\xi\|_{L^\infty(0,T;H^2(\Omega))}\|\zeta\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{L^2(0,T;H^3(\Omega))}, \tag{2.25}$$

(iii) for $\xi \in L^2(0,T;H^1(\Omega))$ and $\zeta \in L^\infty(0,T;H^2(\Omega))$ with $\frac{\partial \xi}{\partial \eta} = \frac{\partial \zeta}{\partial \eta} = 0$,

$$\|\Delta(\xi \times \zeta)\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\xi\|_{L^2(0,T;H^1(\Omega))} \|\zeta\|_{L^\infty(0,T;H^2(\Omega))}, \tag{2.26}$$

(iv) for $\xi \in L^\infty(0,T;H^2(\Omega))$ and $\zeta \in L^2(0,T;H^1(\Omega))$,

$$\|\Delta\xi \times \zeta\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\xi\|_{L^\infty(0,T;H^2(\Omega))} \|\zeta\|_{L^2(0,T;H^1(\Omega))}, \tag{2.27}$$

(v) for $\xi \in L^\infty(0,T;L^2(\Omega))$ and $\zeta \in L^2(0,T;H^1(\Omega))$,

$$\|\xi \times \zeta\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\xi\|_{L^\infty(0,T;L^2(\Omega))} \ \|\zeta\|_{L^2(0,T;H^1(\Omega))}, \tag{2.28}$$

(vi) for $\xi \in L^\infty(0,T;L^2(\Omega))$, $\zeta \in L^\infty(0,T;H^2(\Omega))$ and $\omega \in L^2(0,T;H^1(\Omega))$,

$$\|\big(\xi \times \zeta\big) \times \omega\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\xi\|_{L^\infty(0,T;L^2(\Omega))} \|\zeta\|_{L^\infty(0,T;H^2(\Omega))} \|\omega\|_{L^2(0,T;H^1(\Omega))}, \tag{2.29}$$

(vii) for $\xi \in L^\infty(0,T;L^2(\Omega))$, $\zeta \in L^\infty(0,T;H^2(\Omega))$ and $\omega \in L^2(0,T;H^1(\Omega))$,

$$\|\xi \times \big(\zeta \times \omega\big)\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\xi\|_{L^\infty(0,T;L^2(\Omega))} \|\zeta\|_{L^\infty(0,T;H^2(\Omega))} \|\omega\|_{L^2(0,T;H^1(\Omega))}, \tag{2.30}$$

(viii) for $\nabla\xi \in L^6(0,T;L^6(\Omega))$, $\nabla\zeta \in L^6(0,T;L^6(\Omega))$ and $\nabla\omega \in L^6(0,T;L^6(\Omega))$ with $\frac{\partial \omega}{\partial \eta} = 0$,

$$\|\nabla \cdot \big((\nabla\xi \cdot \nabla\zeta)\nabla\omega\big)\|_{L^2(0,T;H^1(\Omega)^*)} \leq C \ \|\nabla\xi\|_{L^6(0,T;L^6(\Omega))} \|\nabla\zeta\|_{L^6(0,T;L^6(\Omega))} \|\nabla\omega\|_{L^6(0,T;L^6(\Omega))}. \tag{2.31}$$

Proof. Let υ be an arbitrary element in $H^1(\Omega)$. For the estimates (i)-(v), we will apply Hölder's inequality followed by embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$, $H^1(\Omega) \hookrightarrow L^8(\Omega)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Finally, by taking a time integration of the square of the dual space norm, we obtain as follows:

$$\begin{aligned}
(i) \quad \langle (\nabla\xi \cdot \nabla\zeta)\ \omega, \upsilon \rangle &= \int_\Omega (\nabla\xi \cdot \nabla\zeta)\omega \cdot \upsilon \ \mathrm{d}x \leq \|\nabla\xi\|_{L^8(\Omega)} \|\nabla\zeta\|_{L^8(\Omega)} \|\omega\|_{L^2(\Omega)} \|\upsilon\|_{L^4(\Omega)} \\
&\leq C \ \|\xi\|_{H^2(\Omega)} \|\zeta\|_{H^2(\Omega)} \|\omega\|_{L^2(\Omega)} \|\upsilon\|_{H^1(\Omega)}. \\
\Longrightarrow \int_0^T \|\big(\nabla\xi \cdot \nabla\zeta\big)\ w\|^2_{H^1(\Omega)^*} \ \mathrm{d}t &\leq C \ \|\xi\|^2_{L^\infty(0,T;H^2(\Omega))} \ \|\zeta\|^2_{L^\infty(0,T;H^2(\Omega))} \ \|\omega\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Next, we give only the following duality estimates to get the proofs of (ii)–(v):

$$\begin{aligned}
(ii) \quad \langle \nabla \cdot \big((\xi \cdot \zeta)\nabla\omega\big), \upsilon \rangle &= -\int_\Omega \big((\xi \cdot \zeta)\nabla\omega\big) \cdot \nabla\upsilon \ \mathrm{d}x \\
&\leq \|\xi\|_{L^\infty(\Omega)} \|\zeta\|_{L^2(\Omega)} \|\nabla\omega\|_{L^\infty(\Omega)} \|\nabla\upsilon\|_{L^2(\Omega)}, \\
(iii) \quad \langle \Delta(\xi \times \zeta), \upsilon \rangle &= -\int_\Omega \nabla(\xi \times \zeta) \cdot \nabla\upsilon \ \mathrm{d}x \leq \big(\|\nabla\xi\|_{L^2(\Omega)} \|\zeta\|_{L^\infty(\Omega)} + \|\xi\|_{L^4(\Omega)} \|\nabla\zeta\|_{L^4(\Omega)}\big) \|\nabla\upsilon\|_{L^2(\Omega)} \\
&\leq C \ \|\xi\|_{H^1(\Omega)} \ \|\zeta\|_{H^2(\Omega)} \ \|\upsilon\|_{H^1(\Omega)}, \\
(iv) \quad \langle \Delta\xi \times \zeta, \upsilon \rangle &= \int_\Omega \big(\Delta\xi \times \zeta\big) \cdot \upsilon \ \mathrm{d}x \leq \|\Delta\xi\|_{L^2(\Omega)} \|\zeta\|_{L^4(\Omega)} \|\upsilon\|_{L^4(\Omega)} \leq C \ \|\xi\|_{H^2(\Omega)} \|\zeta\|_{H^1(\Omega)} \|\upsilon\|_{H^1(\Omega)}.
\end{aligned}$$

The proof of (v) can be established using a similar reasoning as applied in the proof of (iv). Now, for the last three estimates (vi)–(viii), again appealing to Hölder's inequality and implementing the embeddings $H^1(\Omega) \hookrightarrow L^p(\Omega)$

for $p \in [1, \infty)$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we derive

$$
\begin{aligned}
(vi) \quad \langle (\xi \times \zeta) \times \omega, \upsilon \rangle &= \int_\Omega \big((\xi \times \zeta) \times \omega \big) \cdot \upsilon \ \mathrm{d}x \le \|\xi\|_{L^2(\Omega)} \|\zeta\|_{L^\infty(\Omega)} \|\omega\|_{L^4(\Omega)} \|\upsilon\|_{L^4(\Omega)} \\
&\le C \ \|\xi\|_{L^2(\Omega)} \|\zeta\|_{H^2(\Omega)} \|\omega\|_{H^1(\Omega)} \|\upsilon\|_{H^1(\Omega)}.
\end{aligned}
$$

The proof for (vii) can be derived using an argument analogous to the one used for (vi). Finally,

$$
(viii) \quad \langle \nabla \cdot \big((\nabla\xi \cdot \nabla\zeta) \nabla\omega \big), \upsilon \rangle = - \int_\Omega \big(\nabla\xi \cdot \nabla\zeta \big) \nabla\omega \cdot \nabla\upsilon \ \mathrm{d}x \le \|\nabla\xi\|_{L^6(\Omega)} \|\nabla\zeta\|_{L^6(\Omega)} \|\nabla\omega\|_{L^6(\Omega)} \|\nabla\upsilon\|_{L^2(\Omega)}.
$$

Hence the proof. □

3. Weak solution and regular solution

3.1. Weak solution

In this section, we establish the existence of a weak solution for (1.1) in the sense of Definition 2.1, proving Theorem 2.2. We base our arguments on the approaches presented in [8] and [35]. For adopting the control u that appears non-linearly in (1.1), we define an appropriate penalized form and give a brief proof of Theorem 2.2. This penalized form allows us to verify the hypothesis $|m| = 1$.

Proof of Theorem 2.2. Consider the following penalized problem:

$$
\begin{cases}
(m^k)_t - m^k \times (m^k)_t = 2\Delta m^k - 2k \left(|m^k|^2 - 1 \right) m^k + 2u & \text{in } \Omega_T, \\
\frac{\partial m^k}{\partial \eta} = 0 & \text{on } \partial\Omega_T, \\
m^k(\cdot, 0) = m_0 & \text{in } \Omega.
\end{cases} \tag{3.1}
$$

Let $\{\xi_i\}_{i=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors for $-\Delta$ operator with vanishing Neumann boundary condition. Suppose $W_n = \operatorname{span}\{\xi_1, \xi_2, ..., \xi_n\}$ and $\mathbb{P}_n : L^2 \to W_n$ be the orthogonal projection. Then, consider the Galerkin system of (3.1)

$$
\begin{cases}
\big((m_n^k)_t, \xi_i \big) - \big(m_n^k \times (m_n^k)_t, \xi_i \big) = 2 \left(\Delta m_n^k, \xi_i \right) - 2k \left((|m_n^k|^2 - 1) m_n^k, \xi_i \right) + 2 \ (u, \xi_i), \\
\big(m_n^k(0), \xi_i \big) = (m_0, \xi_i),
\end{cases} \tag{3.2}
$$

where $m_n^k = \sum_{i=1}^n a_{ni}(t) \ \xi_i \in W_n$ and $m_n^k(0) = \mathbb{P}_n(m_0)$.

The local solvability of system (3.2) can be directly inferred from classical theories of ordinary differential equations. For more details on the solvability of such ODE, one can refer to [8] and [20]. Moreover, by an appropriate *a priori* estimate, we can find that the solution to system (3.2) exists on the entire time interval $[0, T]$.

Now, multiplying system (3.2) by $(a_{ni})_t$, summing over 0 to n and then taking an integration over 0 to t, we obtain

$$
\begin{aligned}
&\frac{1}{2} \int_0^t \int_\Omega \left| (m_n^k)_t \right|^2 dx \ \mathrm{d}t + \int_\Omega |\nabla m_n^k|^2 \mathrm{d}x + \frac{k}{2} \int_\Omega \left(|m_n^k|^2 - 1 \right)^2 \mathrm{d}x \\
&\quad \le 2 \int_0^t \|u(s)\|_{L^2(\Omega)}^2 ds + \int_\Omega |\nabla m_n^k(0)|^2 \mathrm{d}x + \frac{k}{2} \int_\Omega \left(|m_n^k(0)|^2 - 1 \right)^2 \mathrm{d}x.
\end{aligned}
$$

Using the property $\int_\Omega |m_n^k|^2 \mathrm{d}x \le \frac{1}{2}\int_\Omega \left(|m_n^k|^2-1\right)^2 \mathrm{d}x + \frac{3}{2}|\Omega|$ and $\|\mathbb{P}_n(m_0)\|_{H^1(\Omega)} \le \|m_0\|_{H^1(\Omega)}$ (see Prop. 1, [36]), we find that $\{m_n^k\}$ and $\{|m_n^k|^2-1\}$ are uniformly bounded in $L^\infty(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ respectively. Also, $\left\{(m_n^k)_t\right\}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Now, using reflexive weak compactness and Aubin-Lions-Simon Lemma, finding a subsequence and following the ideas used in [8], we can find a weak solution m^k of the penalized form (3.1) satisfying the energy estimate

$$\frac{1}{2}\int_{\Omega_T} \left|(m^k)_t\right|^2 dx\ \mathrm{d}t + \int_\Omega |\nabla m^k|^2 \mathrm{d}x + \frac{k}{2}\int_\Omega \left(|m^k|^2-1\right)^2 \mathrm{d}x \le 2\ \|u\|^2_{L^2(0,T;L^2(\Omega))} +\ \|\nabla m_0\|^2_{L^2(\Omega)}.$$

Again using the uniform bound for m^k, finding a sub-sequence that converges to m, we can show that m weakly satisfies the system (1.1) in the sense of Definition 2.1. For more details of the proof, one may refer to [8]. ☐

3.2. Existence of regular solution

Since solvability of the linearized system and the adjoint system demands more regularity than that of weak solutions, in this subsection we will prove a condition under which any weak solution of system (1.1) will be a regular solution.

Proof of Theorem 2.3. In the previous work, the authors (see Thm. 2.1, [20]), proved the existence of a unique *local regular solution* for any control in $L^2(0,T;H^1(\Omega))$ with initial data satisfying condition (1.2). Let u be any control in $L^2(0,T;H^1(\Omega))$ such that a weak solution m satisfies $\|\nabla m\|_{L^4(0,T;L^4(\Omega))} < \infty$. For such a control, suppose $T^* < T$ is the maximum time up to which the regular solution exists. Therefore, taking L^2 inner product of (2.1) with $-\Delta m$ and using the fact that $m\cdot\Delta m = -|\nabla m|^2$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\|\nabla m(t)\|^2_{L^2(\Omega)} + \int_\Omega |\Delta m(t)|^2\ \mathrm{d}x \\
&= -\int_\Omega |\nabla m|^2 m\cdot\Delta m\ \mathrm{d}x - \int_\Omega (m\times u)\cdot\Delta m\ \mathrm{d}x + \int_\Omega (m\times(m\times u))\cdot\Delta m\ \mathrm{d}x \\
&\le \|\nabla m(t)\|^4_{L^4(\Omega)} + \frac{1}{2}\|\Delta m(t)\|^2_{L^2(\Omega)} + 2\ \|u(t)\|^2_{L^2(\Omega)}, \qquad \text{for a.e. } t\in[0,T^*).
\end{aligned}$$

By considering the time integration over 0 to t, we find the following estimate for a.e. $t\in[0,T^*)$

$$\|\nabla m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta m(\tau)\|^2_{L^2(\Omega)}\mathrm{d}\tau \le \|\nabla m_0\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla m(t)\|^4_{L^4(\Omega)}\mathrm{d}t + 4\int_0^t \|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t. \tag{3.3}$$

Furthermore, in order to get the $L^2(0,T;H^3(\Omega))$ and $L^\infty(0,T;H^2(\Omega))$ regularity of the solution, we again appeal to the Galerkin approximation of system (1.1) given in Theorem 2.6 of [20]. By taking $L^2(\Omega)$ inner product of this equation with $\Delta^2 m_n$, and integrating by parts, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{\mathrm{d}t}\|\Delta m_n(t)\|^2_{L^2(\Omega)} + \int_\Omega |\nabla\Delta m_n(t)|^2\ \mathrm{d}x &= -\int_\Omega \nabla\left(|\nabla m_n|^2 m_n\right)\cdot\nabla\Delta m_n - \int_\Omega \nabla\big(m_n\times\Delta m_n\big)\cdot\nabla\Delta m_n\ \mathrm{d}x \\
&\quad - \int_\Omega \nabla\big(m_n\times u_n\big)\cdot\nabla\Delta m_n\ \mathrm{d}x + \int_\Omega \nabla\big(m_n\times(m_n\times u_n)\big)\cdot\nabla\Delta m_n\ \mathrm{d}x := \sum_{i=1}^4 E_i.
\end{aligned} \tag{3.4}$$

For the first term E_1, employing Hölder's inequlaity, and using the estimates (2.17), (2.18) and (2.19), we derive

$$E_1 = -\int_\Omega \left[2\nabla m_n(D^2 m_n)m_n\cdot\nabla\Delta m_n\ - |\nabla m_n|^2\nabla m_n\cdot\nabla\Delta m_n\right]\ \mathrm{d}x$$

$$
\begin{aligned}
&\le 2\ \|\nabla m_n(t)\|_{L^\infty(\Omega)}\|D^2 m_n(t)\|_{L^2(\Omega)}\|m_n(t)\|_{L^\infty(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} + \|\nabla m_n(t)\|^3_{L^6(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} \\
&\le C\ \|m_n(t)\|_{L^\infty(\Omega)}\|\nabla m_n(t)\|^{\frac{1}{2}}_{L^2(\Omega)}\|\Delta m_n(t)\|_{L^2(\Omega)}\|\nabla\Delta m_n(t)\|^{\frac{3}{2}}_{L^2(\Omega)} \\
&\qquad + C\ \|\nabla m_n(t)\|_{L^2(\Omega)}\|\Delta m_n(t)\|^2_{L^2(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} \\
&\le \epsilon\int_\Omega |\nabla\Delta m_n(t)|^2 dx + C(\epsilon)\ \|m_n(t)\|^4_{L^\infty(\Omega)}\ \|\nabla m_n(t)\|^2_{L^2(\Omega)}\ \|\Delta m_n(t)\|^4_{L^2(\Omega)}.
\end{aligned}
$$

For the second term E_2, applying Hölder's inequlaity, the cross-product property $(m_n(t)\times\nabla\Delta m_n(t))\cdot\nabla\Delta m_n(t) = 0$ and estimate (2.19), we find

$$
\begin{aligned}
E_2 &= -\int_\Omega (\nabla m_n \times \Delta m_n)\cdot\nabla\Delta m_n\ \mathrm{d}x \le \|\nabla m_n(t)\|_{L^\infty(\Omega)}\|\Delta m_n(t)\|_{L^2(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} \\
&\le \epsilon\int_\Omega |\nabla\Delta m_n(t)|^2 dx + C(\epsilon)\ \|\nabla m_n(t)\|^2_{L^2(\Omega)}\|\Delta m_n(t)\|^4_{L^2(\Omega)}.
\end{aligned}
$$

Similarly, we can obtain the estimates for E_3 and E_4. By implementing Hölder's inequality, followed by the estimate (2.16) and the embeddings $H^1(\Omega)\hookrightarrow L^4(\Omega)$, we deduce

$$
\begin{aligned}
E_3 &= -\int_\Omega (\nabla m_n \times u)\cdot\nabla\Delta m_n\ \mathrm{d}x - \int_\Omega (m_n\times\nabla u)\cdot\nabla\Delta m\ \mathrm{d}x \\
&\le\ \|\nabla m_n(t)\|_{L^4(\Omega)}\|u(t)\|_{L^4(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} + \|m_n(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^2(\Omega)}\|\nabla\Delta m_n(t)\|_{L^2(\Omega)} \\
&\le \epsilon\int_\Omega |\nabla\Delta m_n(t)|^2 dx + C(\epsilon)\ \left(\|m_n(t)\|^2_{L^\infty(\Omega)} + \|\Delta m_n(t)\|^2_{L^2(\Omega)}\right)\ \|u(t)\|^2_{H^1(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
E_4 &= \int_\Omega (\nabla m_n\times(m_n\times u))\cdot\nabla\Delta m_n\ \mathrm{d}x + \int_\Omega (m_n\times(\nabla m_n\times u))\cdot\nabla\Delta m_n\ \mathrm{d}x + \int_\Omega (m_n\times(m_n\times\nabla u))\cdot\nabla\Delta m_n\ \mathrm{d}x \\
&\le \big(2\ \|\nabla m_n(t)\|_{L^4(\Omega)}\|u(t)\|_{L^4(\Omega)} + \|m_n(t)\|_{L^4(\Omega)}\ \|\nabla u(t)\|_{L^2(\Omega)}\big)\ \|m_n(t)\|_{L^\infty(\Omega)}\ \|\nabla\Delta m_n(t)\|_{L^2(\Omega)} \\
&\le \epsilon\ \int_\Omega |\nabla\Delta m_n(t)|^2 dx + C(\epsilon)\ \left(\|m_n(t)\|^2_{L^\infty(\Omega)} + \|\Delta m_n(t)\|^2_{L^2(\Omega)}\right)\ \|m_n(t)\|^2_{L^\infty(\Omega)}\|u(t)\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Now, substituting the estimates for E_i for $i = 1$ to 4 in (3.4) and choosing $\epsilon = 1/8$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{\mathrm{d}t}\|\Delta m_n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla\Delta m_n(t)|^2\ \mathrm{d}x &\le C\ \Big[\|\nabla m_n(t)\|^2_{L^2(\Omega)}\left(1 + \|m_n(t)\|^4_{L^\infty(\Omega)}\right)\|\Delta m_n(t)\|^4_{L^2(\Omega)} \\
&\quad + \left(\|m_n(t)\|^2_{L^\infty(\Omega)} + \|\Delta m_n(t)\|^2_{L^2(\Omega)}\right)\left(1 + \|m_n(t)\|^2_{L^\infty(\Omega)}\right)\|u(t)\|^2_{H^1(\Omega)}\Big], \qquad \forall\ t\in[0,T^*).
\end{aligned}
$$

Next, we integrate the above inequality over 0 to t, and proceed further by using the sequential lower semi-continuity as discussed in the proof of Theorem 2.7 of [20], and the fact that $|m(x,t)| = 1,\ \forall\,(x,t)\in\Omega_{T^*}$, we conclude that

$$
\begin{aligned}
\|\Delta m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\Delta m(\tau)\|^2_{L^2(\Omega)}\ \mathrm{d}\tau &\le \|\Delta m_0\|^2_{L^2(\Omega)} + C\ \Big[\int_0^t \|\nabla m(\tau)\|^2_{L^2(\Omega)}\ \left(\|\Delta m(\tau)\|^2_{L^2(\Omega)}\right)^2\ \mathrm{d}\tau \\
&\quad + \int_0^t \left(1 + \|\Delta m(\tau)\|^2_{L^2(\Omega)}\right)\ \|u(\tau)\|^2_{H^1(\Omega)}\ \mathrm{d}\tau\Big] \qquad \text{for a.e. } t\in[0,T^*).
\end{aligned}
$$

Applying Gronwall's inequality and using the estimate (3.3), we derive

$$\begin{aligned}\|\Delta m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla \Delta m(\tau)\|^2_{L^2(\Omega)}\, \mathrm{d}\tau \le \Big(\|\Delta m_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\Big) \\ \times \exp\Big(C\ \|\nabla m\|^2_{L^\infty(0,t;L^2(\Omega))} \|\Delta m\|^2_{L^2(0,t;L^2(\Omega))} + C\ \|u\|^2_{L^2(0,T;H^1(\Omega))}\Big), \qquad \text{for a.e. } t \in [0,T^*).\end{aligned} \tag{3.5}$$

Substituting the bounds for $\|\nabla m\|^2_{L^\infty(0,t;L^2(\Omega))}$ and $\|\Delta m\|^2_{L^2(0,t;L^2(\Omega))}$ from estimate (3.3) in estimate (3.5), we obtain

$$\begin{aligned}\|\Delta m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla \Delta m(\tau)\|^2_{L^2(\Omega)}\, \mathrm{d}\tau \le \Big(\|\Delta m_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\Big) \\ \times \exp\Big(C\ \Big[1 + \|\nabla m_0\|^4_{L^2(\Omega)} +\ \|\nabla m\|^8_{L^4(0,T^*;L^4(\Omega))} +\ \|u\|^4_{L^2(0,T;H^1(\Omega))}\Big]\Big), \qquad \text{for a.e. } t \in [0,T^*).\end{aligned} \tag{3.6}$$

Therefore, the solution doesn't blow up at T^*, which contradicts our assumption that $T^* < T$ is the maximal time of existence. Hence, we conclude that $T^* = T$, in other words, the regular solution of system (1.1) exists on the entire time domain $[0,T]$.
Next, we take square of the $H^1(\Omega)$ norm of m_t in equation (1.1) and then integrate over time to obtain

$$\begin{aligned}\int_0^T \|m_t(t)\|^2_{H^1(\Omega)}\, \mathrm{d}t \le 4\Bigg(\int_0^T \|m \times \Delta m\|^2_{H^1(\Omega)}\, \mathrm{d}t + \int_0^T \|m \times u\|^2_{H^1(\Omega)}\, \mathrm{d}t \\ + \int_0^T \|m \times \big(m \times \Delta m\big)\|^2_{H^1(\Omega)}\, \mathrm{d}t + \int_0^T \|m \times \big(m \times u\big)\|^2_{H^1(\Omega)}\, \mathrm{d}t\Bigg).\end{aligned}$$

Now, estimating the terms on the right hand side using Lemma 2.23 and using the equality $|m| = 1$, we derive

$$\int_0^T \|m_t(t)\|^2_{H^1(\Omega)}\, \mathrm{d}t \le C\ \|m\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|m\|^2_{L^2(0,T;H^3(\Omega))} + C\ \|m\|^2_{L^\infty(0,T;H^2(\Omega))}\ \|u\|^2_{L^2(0,T;H^1(\Omega))}.$$

Substituting the bounds of $\|m\|^2_{L^\infty(0,T;H^2(\Omega))}$ and $\|m\|^2_{L^2(0,T;H^3(\Omega))}$ from estimate (3.6), we find

$$\begin{aligned}\|m_t\|^2_{L^2(0,T;H^1(\Omega))} \le \Big(\|\Delta m_0\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H^1(\Omega))}\Big)^2 \\ \times \exp\Big(C\ \Big[1 + \|\nabla m_0\|^4_{L^2(\Omega)} +\ \|\nabla m\|^8_{L^4(0,T^*;L^4(\Omega))} +\ \|u\|^4_{L^2(0,T;H^1(\Omega))}\Big]\Big).\end{aligned} \tag{3.7}$$

Combining estimates (3.6) and (3.7), we find the required result (2.2). Hence the proof. □

4. Existence of optimum and first order optimality condition

4.1. Existence of optimal control

For the optimal control problem to have practical relevance, it is essential to confirm the existence of at least one globally optimal solution. This assurance is provided by the following theorem.

Proof of Theorem 2.7. Since the functional $\mathcal{J}(\cdot,\cdot)$ is bounded below and $\mathcal{A} \neq \emptyset$, there exists a minimizing sequence $\big\{(m_n, u_n)\big\} \subset \mathcal{A}$ such that

$$\inf_{(m,u)\in\mathcal{A}} \mathcal{J}(m,u) = \lim_{n\to\infty} \mathcal{J}(m_n,u_n) = \alpha.$$

Since $(m_n, u_n) \in \mathcal{A}$, it follows that for each n, the pair (m_n, u_n) serves as a regular solution of the following system:

$$\begin{cases} (m_n)_t = m_n \times (\Delta m_n + u_n) - m_n \times (m_n \times (\Delta m_n + u_n)) & \text{in } \Omega_T, \\ \frac{\partial m_n}{\partial \eta} = 0 & \text{in } \partial\Omega_T, \\ m_n(\cdot, 0) = m_0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Since (m_n, u_n) is a minimizing sequence, we can extract a sub-sequence (again represented as (m_n, u_n)) such that

$$\begin{aligned} \frac{1}{4}\|\nabla m_n\|^4_{L^4(0,T;L^4(\Omega))} + \frac{1}{2}\|u_n\|^2_{L^2(0,T;H^1(\Omega))} &\leq 8\ \mathcal{J}(m_n, u_n) + 2\ \|\nabla m_D\|^4_{L^4(0,T;L^4(\Omega))} \\ &\leq 8\ \mathcal{J}(m_1, u_1) + 2\ \|\nabla m_D\|^4_{L^4(0,T;L^4(\Omega))} < \infty. \end{aligned}$$

From this estimate and the energy estimate (2.2), we find that $\{u_n\}$ and $\{m_n\}$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$ and $\mathcal{M}$ respectively. Then there exists a sub-sequence again denoted as $\{u_n\}$ such that $u_n \rightharpoonup \widetilde{u}$ weakly in $L^2(0,T;H^1(\Omega))$ for some element $\widetilde{u} \in L^2(0,T;H^1(\Omega))$. Moreover, the existence of a weakly convergent sub-sequence can be guaranteed in the same way for the state solution sequence $\{m_n\}$.

Then, by using Aubin–Lions–Simon compactness theorem, $\{m_n\}$ is relatively compact in $C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Therefore, there exists a sub-sequence (again represented as $\{(m_n, u_n)\}$) such that

$$\begin{cases} u_n \overset{w}{\rightharpoonup} \widetilde{u} & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ m_n \overset{w}{\rightharpoonup} \widetilde{m} & \text{weakly in } L^2(0,T;H^3(\Omega)), \\ (m_n)_t \overset{w}{\rightharpoonup} \widetilde{m}_t & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ m_n \overset{s}{\to} \widetilde{m} & \text{strongly in } C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \text{ as } n \to \infty. \end{cases}$$

Taking limit $n \to \infty$ in (4.1) and using the above convergences, we obtain that $(\widetilde{m}, \widetilde{u})$ is a regular solution of system (2.1), that is, $\widetilde{u} \in \mathcal{U}$. Since the set $S = \{u \in L^2(0,T;H^1(\Omega)) \mid a(x,t) \leq u(x,t) \leq b(x,t) \quad \text{for a.e. } (x,t) \in \Omega_T\}$ is a closed and convex subset of $L^2(0,T;H^1(\Omega))$, therefore it is weakly closed. Consequently, we have $\widetilde{u} \in \mathcal{U}_{ad}$.

Also, since $m_n \to \widetilde{m}$ strongly in $L^4(0,T;H^1(\Omega))$, $m_n(\cdot,T) \rightharpoonup \widetilde{m}(\cdot,T)$ weakly in $L^2(\Omega)$ and $u_n \rightharpoonup \widetilde{u}$ weakly in $L^2(0,T;H^1(\Omega))$, the functional $\mathcal{J}(\cdot,\cdot)$ is weakly lower semi-continuous, that is

$$\mathcal{J}(\widetilde{m}, \widetilde{u}) \leq \liminf_{n\to\infty} \mathcal{J}(m_n, u_n) = \lim_{n\to\infty} \mathcal{J}(m_n, u_n) = \alpha. \tag{4.2}$$

Since α is the infimum of the functional $\mathcal{J}$ over $\mathcal{A}$, so that

$$\alpha \leq \mathcal{J}(\widetilde{m}, \widetilde{u}).$$

Hence combining with (4.2), we get $\mathcal{J}(\widetilde{m}, \widetilde{u}) = \alpha = \inf_{(m,u)\in\mathcal{A}} \mathcal{J}(m, u)$. This completes the proof. $\square$

4.2. Control-to-state operator

Suppose $\widetilde{m}$ be the unique regular solution of the system (1.1) corresponding to the control $\widetilde{u} \in \mathcal{U}$ and initial data m_0 satisfying condition (1.2). Then consider the following linearized system

$$(L-LLG) \begin{cases} \mathcal{L}_{\widetilde{u}} z = f & \text{in } \Omega_T, \\ \frac{\partial z}{\partial \eta} = 0 & \text{in } \partial\Omega_T, \quad z(x,0) = z_0 \ \text{ in } \Omega, \end{cases} \tag{4.3}$$

where the operator $\mathcal{L}_{\widetilde{u}}$ is defined as

$$\mathcal{L}_{\widetilde{u}} z := z_t - \Delta z - 2(\nabla\widetilde{m}\cdot\nabla z)\widetilde{m} - |\nabla\widetilde{m}|^2 z - z\times\Delta\widetilde{m} - \widetilde{m}\times\Delta z - z\times\widetilde{u} + z\times(\widetilde{m}\times\widetilde{u}) + \widetilde{m}\times(z\times\widetilde{u}). \tag{4.4}$$

It is worth noting that as $\widetilde{m}$ is a regular solution of the problem (1.1), in the subsequent analysis we have used the inequality $1 = \frac{1}{|\Omega|}|\Omega| = \frac{1}{|\Omega|}\|\widetilde{m}\|^2_{L^2(\Omega)} \le C\ \|\widetilde{m}\|^2_{H^2(\Omega)}$ without mentioning it repeatedly.

Lemma 4.1. *For any $f \in L^2(0,T;H^1(\Omega))$, there exists a unique regular solution $z \in L^2(0,T;H^3(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ of the linearized system (4.3). Moreover, the following estimate holds:*

$$\begin{aligned}\|z\|^2_{L^\infty(0,T;H^2(\Omega))} + \|z\|^2_{L^2(0,T;H^3(\Omega))} + \|z_t\|^2_{L^2(0,T;H^1(\Omega))} &\le \Big(\|z_0\|^2_{L^2(\Omega)} + \|\Delta z_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;H^1(\Omega))}\Big)\\ \times\ \exp\Big\{C\Big(\|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widetilde{m}\|^2_{L^2(0,T;H^3(\Omega))} &+ \|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))}\Big)\Big\}.\end{aligned} \tag{4.5}$$

Proof. By appealing to the Faedo-Galerkin approximation technique, we can show the existence and uniqueness of solutions to the system (4.3). Therefore, we will only present *a priori* estimates for the solution here. Considering the L^2 inner product of (4.3) with z and employing cross product properties from Lemma 2.20, we have

$$\frac{d}{dt}\|z(t)\|^2_{L^2(\Omega)} + \|\nabla z(t)\|^2_{L^2(\Omega)} \le C\ \Big(\|\widetilde{m}(t)\|^2_{H^3(\Omega)} + \|\widetilde{u}(t)\|^2_{H^1(\Omega)}\Big)\|z(t)\|^2_{L^2(\Omega)} +\ \|f(t)\|^2_{L^2(\Omega)}. \tag{4.6}$$

By taking gradient of system (4.3) and then considering inner product with $-\nabla\Delta z$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta z(t)\|^2_{L^2(\Omega)} + \int_\Omega|\nabla\Delta z(t)|^2\mathrm{d}x &= -2\int_\Omega\nabla\big(\widetilde{m}(\nabla\widetilde{m}\cdot\nabla z)\big)\cdot\nabla\Delta z\ \mathrm{d}x - \int_\Omega\nabla\big(|\nabla\widetilde{m}|^2 z\big)\cdot\nabla\Delta z\ \mathrm{d}x\\ &- \int_\Omega\nabla\big(z\times\Delta\widetilde{m}\big)\cdot\nabla\Delta z\ \mathrm{d}x - \int_\Omega\nabla\big(\widetilde{m}\times\Delta z\big)\cdot\nabla\Delta z\ \mathrm{d}x - \int_\Omega\nabla\big(z\times\widetilde{u}\big)\cdot\nabla\Delta z\ \mathrm{d}x\\ &+ \int_\Omega\nabla\big(z\times(\widetilde{m}\times\widetilde{u})\big)\cdot\nabla\Delta z\ \mathrm{d}x + \int_\Omega\nabla\big(\widetilde{m}\times(z\times\widetilde{u})\big)\cdot\nabla\Delta z\ \mathrm{d}x - \int_\Omega\nabla f\cdot\nabla\Delta z\ \mathrm{d}x := \sum_{i=1}^{8}\Gamma_i.\end{aligned} \tag{4.7}$$

Let us estimate some of the terms on the right-hand side. For the first term Γ_1, applying Hölder's inequality and the embeddings $H^1(\Omega)\hookrightarrow L^p(\Omega)$ for $p\in[1,\infty)$ and $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, we get

$$\begin{aligned}\Gamma_1 &= -2\int_\Omega\big[\nabla\widetilde{m}(\nabla\widetilde{m}\cdot\nabla z) + \widetilde{m}(D^2\widetilde{m}\cdot\nabla z) + \widetilde{m}(\nabla\widetilde{m}\cdot D^2 z)\big]\cdot\nabla\Delta z\ \mathrm{d}x\\ &\le\ 2\ \|\nabla\widetilde{m}(t)\|_{L^4(\Omega)}\|\nabla\widetilde{m}(t)\|_{L^8(\Omega)}\|\nabla z(t)\|_{L^8(\Omega)}\|\nabla\Delta z(t)\|_{L^2(\Omega)}\\ &+\ 2\ \|\widetilde{m}(t)\|_{L^\infty(\Omega)}\big(\|D^2\widetilde{m}(t)\|_{L^4(\Omega)}\|\nabla z(t)\|_{L^4(\Omega)} + \|\nabla\widetilde{m}(t)\|_{L^\infty(\Omega)}\|D^2 z(t)\|_{L^2(\Omega)}\big)\|\nabla\Delta z(t)\|_{L^2(\Omega)}\\ &\le \epsilon\int_\Omega|\nabla\Delta z(t)|^2\mathrm{d}x +\ C(\epsilon)\ \Big(\|\widetilde{m}(t)\|^4_{H^2(\Omega)} + \|\widetilde{m}(t)\|^2_{H^2(\Omega)}\|\widetilde{m}(t)\|^2_{H^3(\Omega)}\Big)\|z(t)\|^2_{H^2(\Omega)}.\end{aligned}$$

A similar estimate for Γ_2, Γ_3 and Γ_4 can be obtained. Now, let us estimate one of the terms containing the control

$$\begin{aligned}\Gamma_6 &= \int_\Omega\Big[\nabla z\times(\widetilde{m}\times\widetilde{u}) + z\times(\nabla\widetilde{m}\times\widetilde{u}) + z\times(\widetilde{m}\times\nabla\widetilde{u})\Big]\cdot\nabla\Delta z\ \mathrm{d}x\\ &\le \|\nabla z(t)\|_{L^4(\Omega)}\|\widetilde{m}(t)\|_{L^8(\Omega)}\|\widetilde{u}(t)\|_{L^8(\Omega)}\|\nabla\Delta z(t)\|_{L^2(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&+ \ \|z(t)\|_{L^\infty(\Omega)} \left(\|\nabla \widetilde{m}(t)\|_{L^4(\Omega)}\|\widetilde{u}(t)\|_{L^4(\Omega)} + \|\widetilde{m}(t)\|_{L^\infty(\Omega)}\|\nabla \widetilde{u}(t)\|_{L^2(\Omega)}\right) \|\nabla \Delta z(t)\|_{L^2(\Omega)} \\
&\leq \epsilon \int_\Omega |\nabla \Delta z(t)|^2 \ \mathrm{d}x + C(\epsilon) \ \|\widetilde{m}(t)\|^2_{H^2(\Omega)}\|\widetilde{u}(t)\|^2_{H^1(\Omega)}\|z(t)\|^2_{H^2(\Omega)}.
\end{aligned}
$$

By substituting all these estimates in equation (4.7) and choosing a suitable value for ϵ and adding with (4.8), we get

$$
\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(\|z(t)\|^2_{L^2(\Omega)} + \|\Delta z(t)\|^2_{L^2(\Omega)}\right) + \|\nabla z(t)\|^2_{L^2(\Omega)} + \|\nabla \Delta z(t)\|^2_{L^2(\Omega)} \\
&\leq C \left[\|\widetilde{m}(t)\|^2_{H^2(\Omega)}\|\widetilde{m}(t)\|^2_{H^3(\Omega)} + \|\widetilde{m}(t)\|^2_{H^2(\Omega)}\|\widetilde{u}(t)\|^2_{H^1(\Omega)}\right] \|z(t)\|^2_{H^2(\Omega)} + C \ \|f(t)\|^2_{H^1(\Omega)}.
\end{aligned} \tag{4.8}
$$

By invoking the inequality $\|z\|_{H^2(\Omega)} \leq C \ \left(\|z\|_{L^2(\Omega)} + \|\Delta z\|_{L^2(\Omega)}\right)$ from Lemma 2.21 and applying Gronwall's inequality, we obtain

$$
\sup_{t\in[0,T]} \left(\|z(t)\|^2_{L^2(\Omega)} + \|\Delta z(t)\|^2_{L^2(\Omega)}\right) + \int_0^T \left(\|\nabla z(\tau)\|^2_{L^2(\Omega)} + \|\nabla \Delta z(\tau)\|^2_{L^2(\Omega)}\right) \ \mathrm{d}\tau \leq C_1(\Omega, T, z_0, f, \widetilde{m}, \widetilde{u}), \tag{4.9}
$$

where
$$
\begin{aligned}
C_1(\Omega, T, z_0, f, \widetilde{m}, \widetilde{u}) = &\left(\|z_0\|^2_{L^2(\Omega)} + \|\Delta z_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;H^1(\Omega))}\right) \\
&\times \ \exp\left\{C\left(\|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widetilde{m}\|^2_{L^2(0,T;H^3(\Omega))} + \|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))}\right)\right\}.
\end{aligned}
$$

Now, taking $L^2(0,T;H^1(\Omega))$ of z_t in equation (4.3) and estimating the terms using Lemma 2.23, we derive

$$
\begin{aligned}
\|z_t\|_{L^2(0,T;H^1(\Omega))} \leq &\|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \ \|\widetilde{m}\|^2_{L^2(0,T;H^3(\Omega))} \ \left(\|z\|^2_{L^\infty(0,T;H^2(\Omega))} + \|z\|^2_{L^2(0,T;H^3(\Omega))}\right) \\
&+ \|z\|^2_{L^\infty(0,T;H^2(\Omega))} \ \|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \ \|\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{4.10}
$$

Finally, substituting (4.9) in estimate (4.10), we can obtain the required result (4.5). ☐

By virtue of the definition of the control set $\mathcal{U}$, we are able to define an operator $G : \mathcal{U} \to \mathcal{M}$ called the control-to-state operator such that for every $u \in \mathcal{U}$ there exists a unique regular solution $m_u := G(u)$ of system (1.1). Now, we will prove the locally Lipschitz continuity and Fréchet differentiability of this operator.

Lemma 4.2 (Locally Lipschitz Continuity of G)**.** *The control-to-state operator $G : \mathcal{U} \to \mathcal{M}$ is locally Lipschitz continuous. That is, for each $u \in \mathcal{U}$, there exist $\epsilon > 0$ and a constant $C_2 > 0$ depending on the parameters $\Omega, T, u, m_u, \epsilon$ such that*

$$
\|G(u) - G(v)\|_{\mathcal{M}} \leq C_2 \ \|u - v\|_{L^2(0,T;H^1(\Omega))}, \tag{4.11}
$$

for all v with $\|u - v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$.

Proof. Suppose m_u and m_v are two regular solutions of system (1.1) corresponding to the controls u and v respectively. Let us define $\widehat{m} := m_u - m_v$ and $\widehat{u} := u - v$. Then $(\widehat{m}, \widehat{u})$ will satisfy the following system

$$\begin{cases} \widehat{m}_t - \Delta\widehat{m} = |\nabla m_u|^2\widehat{m} + \nabla\widehat{m}\cdot(\nabla m_u + \nabla m_v)m_v + \widehat{m}\times\Delta m_u + m_v\times\Delta\widehat{m} + m_u\times\widehat{u} \\ \qquad\qquad + \widehat{m}\times v - \widehat{m}\times(m_u\times u) - m_v\times(\widehat{m}\times u) - m_v\times(m_v\times\widehat{u}) \qquad (x,t)\in\Omega_T, \\ \frac{\partial\widehat{m}}{\partial\eta} = 0 \qquad (x,t)\in\partial\Omega_T, \\ \widehat{m}(\cdot,0) = 0 \quad \text{in } \Omega. \end{cases} \tag{4.12}$$

By estimating the $L^2(\Omega)$ inner product of equation (4.12) with $\widehat{m}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\widehat{m}\|^2_{L^2(\Omega)} + \|\nabla\widehat{m}\|^2_{L^2(\Omega)} &\le \|\widehat{u}(t)\|^2_{L^2(\Omega)} \\ &+ C\ \left(1 + \|m_u(t)\|^2_{H^2(\Omega)} + \|m_v(t)\|^2_{H^2(\Omega)} + \|u(t)\|^2_{L^2(\Omega)}\right)\ \|\widehat{m}(t)\|^2_{H^1(\Omega)}. \end{aligned} \tag{4.13}$$

Next, we apply the gradient operator to equation (4.12), followed by taking the inner product with $\nabla\Delta\widehat{m}$. This leads to the following expression

$$\begin{aligned} \frac{1}{2}\frac{d}{\mathrm{d}t}\|\Delta\widehat{m}\|^2_{L^2(\Omega)} + \|\nabla\Delta\widehat{m}\|^2_{L^2(\Omega)} &= \int_\Omega \nabla\left(|\nabla m_u|^2\widehat{m}\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x + \int_\Omega \nabla\left(\nabla\widehat{m}\cdot(\nabla m_u+\nabla m_v)m_v\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x \\ &+ \int_\Omega \nabla\left(\widehat{m}\times\Delta m_u\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x + \int_\Omega \nabla\left(m_v\times\Delta\widehat{m}\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x + \int_\Omega \nabla\left(m_u\times\widehat{u}\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x \\ &- \int_\Omega \nabla\left(\widehat{m}\times(m_u\times u)\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x - \int_\Omega \nabla\left(m_v\times(\widehat{m}\times u)\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x \\ &+ \int_\Omega \nabla\left(\widehat{m}\times v\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x - \int_\Omega \nabla\left(m_v\times(m_v\times\widehat{u})\right)\cdot\nabla\Delta\widehat{m}\ \mathrm{d}x. \end{aligned}$$

We can see that the terms in the above equation are similar to the ones in equation (4.7). Therefore, estimating in the same way as we have done for the terms Γ_i in equation (4.7) and combining with (4.13), we derive

$$\begin{aligned} \frac{d}{\mathrm{d}t}\left(\|\widehat{m}\|^2_{L^2(\Omega)} + \|\Delta\widehat{m}\|^2_{L^2(\Omega)}\right) + \|\nabla\widehat{m}\|^2_{L^2(\Omega)} + \|\nabla\Delta\widehat{m}\|^2_{L^2(\Omega)} &\le C\ \left(\|m_u\|^2_{H^2(\Omega)} + \|m_v\|^2_{H^2(\Omega)}\right)\ \|\widehat{u}\|^2_{H^1(\Omega)} \\ + C\ \left(\|m_u\|^2_{H^2(\Omega)} + \|m_v\|^2_{H^2(\Omega)}\right)\ \left(\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)} + \|m_u\|^2_{H^3(\Omega)} + \|m_v\|^2_{H^3(\Omega)}\right)\ \|\widehat{m}\|^2_{H^2(\Omega)}. \end{aligned}$$

Since the control-to-state operator is continuous (see Rem. 5.3, [20]), so for any fixed control u and constant $\sigma > 0$, we can find $\epsilon > 0$ such that for any control v with $\|u - v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$ implies $\|m_v\|_{\mathcal{M}} < \|m_u\|_{\mathcal{M}} + \sigma$. Finally, applying Grönwall's inequality, we find that

$$\|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))} \le C(\Omega, T, u, m_u, \epsilon, \sigma)\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))}.$$

Next, taking the $L^2(0,T;H^1(\Omega))$ norm of $\widehat{m}_t$ in (4.12) and estimating the terms using Lemma 2.23, we reach at our required result (4.11). ☐

Proposition 4.3. *For any control $\bar{u} \in \mathcal{U}$, let $m_{\bar{u}}$ be the unique regular solution of system (1.1). Then the following holds:*

(i) The control-to-state mapping G is Fréchet differentiable on $\mathcal{U}$, that is for any $\bar{u} \in \mathcal{U}$, there exists a bounded linear operator $G'(\bar{u}) : L^2(0,T;H^1(\Omega)) \to \mathcal{M}$ with $z := G'(\bar{u})[u]$ such that

$$\frac{\|G(\bar{u}+u) - G(\bar{u}) - G'(\bar{u})[u]\|_{\mathcal{M}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0 \qquad \text{as } \|u\|_{L^2(0,T;H^1(\Omega))} \to 0,$$

where z is the unique regular solution of the following linearized system:

$$\begin{cases} \mathcal{L}_{\bar{u}} z = m_{\bar{u}} \times u - m_{\bar{u}} \times (m_{\bar{u}} \times u) \quad \text{in } \ \Omega_T, \\ \frac{\partial z}{\partial \eta} = 0 \quad \text{on } \partial\Omega_T, \\ z(0) = 0 \quad \text{in } \Omega. \end{cases} \tag{4.14}$$

(ii) The Fréchet derivative G' is locally Lipschitz continuous, that is, for any controls $u \in \mathcal{U}$, $h \in L^2(0,T;H^1(\Omega))$ and $\epsilon > 0$, there exists a constant $C_3 > 0$ depending on $\Omega, T, u, m_u, \epsilon$ such that

$$\|G'(u)[h] - G'(v)[h]\|_{\mathcal{M}} \le C_3 \ \|u - v\|_{L^2(0,T;H^1(\Omega))} \ \|h\|_{L^2(0,T;H^1(\Omega))}, \tag{4.15}$$

for all v in $L^2(0,T;H^1(\Omega))$ with $\|u-v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$.

Proof. Suppose $\bar{u} \in \mathcal{U}$ be any fixed control and $m_{\bar{u}}$ be the corresponding regular solution. As for any control $u \in L^2(0,T;H^1(\Omega))$, the terms $m_{\bar{u}} \times u, \ m_{\bar{u}} \times (m_{\bar{u}} \times u) \in L^2(0,T;H^1(\Omega))$, therefore as a consequence of Lemma 4.1, system (4.14) has a unique regular solution $z \in \mathcal{M}$. Note that at this stage we don't know whether $z = G'(\bar{u})[u]$ or even the control-to-state operator is differentiable or not. Since $\mathcal{U}$ is an open set, so for small enough u, $\bar{u} + u$ will also belongs to $\mathcal{U}$. Let $m_{\bar{u}+u}$ be the admissible solution corresponding to the control $\bar{u}+u$, then $w := m_{\bar{u}+u} - m_{\bar{u}} - z$ solves the following system

$$\begin{cases} \mathcal{L}_{\bar{u}} w = \displaystyle\sum_{k=1}^{8} \mathcal{Q}_k \qquad \text{in } \Omega_T, \\ \frac{\partial w}{\partial \eta} = 0 \quad \text{on } \partial\Omega_T, \qquad w(0) = 0 \ \text{ in } \Omega, \end{cases} \tag{4.16}$$

where $\mathcal{L}_{\bar{u}}$ is defined in (4.4), $\widehat{m} = m_{\bar{u}+u} - m_{\bar{u}}$ and the terms $\mathcal{Q}_k$'s are given by

$$\begin{aligned} &\mathcal{Q}_1 = \big(\nabla\widehat{m} \cdot (\nabla m_{\bar{u}+u} + \nabla m_{\bar{u}})\big)\widehat{m}, \quad \mathcal{Q}_2 = |\nabla\widehat{m}|^2 m_{\bar{u}}, \qquad \mathcal{Q}_3 = \widehat{m} \times \Delta\widehat{m}, \qquad \mathcal{Q}_4 = \widehat{m} \times u, \\ &\mathcal{Q}_5 = -\widehat{m} \times (\widehat{m} \times u), \qquad \mathcal{Q}_6 = -\widehat{m} \times (\widehat{m} \times \bar{u}), \quad \mathcal{Q}_7 = -\widehat{m} \times (m_{\bar{u}} \times u), \quad \mathcal{Q}_8 = -m_{\bar{u}} \times (\widehat{m} \times u). \end{aligned}$$

Invoking Lemma 4.1 for system (4.16), we can directly find the following estimate:

$$\begin{aligned} \|w\|^2_{L^2(0,T;H^3(\Omega))} + \|w_t\|^2_{L^2(0,T;H^1(\Omega))} &\le \left(\|w(0)\|^2_{L^2(\Omega)} + \|\Delta w(0)\|^2_{L^2(\Omega)} + \sum_{k=1}^{8} \|\mathcal{Q}_k\|^2_{L^2(0,T;H^1(\Omega))} \right) \\ &\times \ \exp\left\{ C \left(\|m_{\bar{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} \|m_{\bar{u}}\|^2_{L^2(0,T;H^3(\Omega))} + \|m_{\bar{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} \|\bar{u}\|^2_{L^2(0,T;H^1(\Omega))} \right) \right\}. \end{aligned} \tag{4.17}$$

Let us estimate the terms containing $\mathcal{Q}_k$'s on the right-hand side. For the first term $\|\mathcal{Q}_1\|^2_{L^2(0,T;H^1(\Omega))}$, applying inequality (2.22) with $p = \widehat{m}$, $q = m_{\bar{u}+u} + m_{\bar{u}}$ and $r = \widehat{m}$, we find

$$\|\mathcal{Q}_1\|^2_{L^2(0,T;H^1(\Omega))} \le C \ \left(\|m_{\bar{u}+u}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m_{\bar{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} \right) \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))}.$$

Similarly, applying estimates (2.22), (2.20) and (2.21) for $\mathcal{Q}_2, \mathcal{Q}_3$ and $\mathcal{Q}_4$ respectively, we get

$$\|\mathcal{Q}_2\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))},$$
$$\|\mathcal{Q}_3\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))},$$
$$\|\mathcal{Q}_4\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|u\|^2_{L^2(0,T;H^1(\Omega))}.$$

Now, applying estimate (2.23) for the last four terms and combining it, we have

$$\sum_{k=5}^{8}\|\mathcal{Q}_k\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|\widehat{m}\|^4_{L^\infty(0,T;H^2(\Omega))}\left(\|u\|^2_{L^2(0,T;H^1(\Omega))} + \|\overline{u}\|^2_{L^2(0,T;H^1(\Omega))}\right) + C\ \|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|u\|^2_{L^2(0,T;H^1(\Omega))}.$$

Substituting all these estimates in (4.17) and using $w(0) = 0$, we find

$$\begin{aligned}\|w\|^2_{L^2(0,T;H^3(\Omega))} + \|w_t\|^2_{L^2(0,T;H^1(\Omega))} \leq \Big[&\|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|u\|^2_{L^2(0,T;H^1(\Omega))}\\ &+ \|\widehat{m}\|^4_{L^\infty(0,T;H^2(\Omega))}\left(\|u\|^2_{L^2(0,T;H^1(\Omega))} + \|\overline{u}\|^2_{L^2(0,T;H^1(\Omega))}\right)\\ &+ \left(\|m_{\overline{u}+u}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\right)\|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))}\Big]\\ &\times\ \exp\left\{C\left(\|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\|m_{\overline{u}}\|^2_{L^2(0,T;H^3(\Omega))} + \|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\overline{u}\|^2_{L^2(0,T;H^1(\Omega))}\right)\right\}.\end{aligned}$$

Now, using the locally Lipschitz continuity arguments from Lemma 4.2, we can find an uniform bound for $\|m_{\overline{u}}\|_{\mathcal{M}}$ and $\|m_{\overline{u}+u}\|_{\mathcal{M}}$ whenever $\|u\|_{L^2(0,T;H^1(\Omega))} < \epsilon$. Moreover, we will obtain the following estimate

$$\|w\|^2_{L^2(0,T;H^3(\Omega))} + \|w_t\|^2_{L^2(0,T;H^1(\Omega))} \leq C\ \|u\|^4_{L^2(0,T;H^1(\Omega))} + C\ \|u\|^6_{L^2(0,T;H^1(\Omega))}. \tag{4.18}$$

Therefore, $\frac{\|w\|_{\mathcal{M}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0$ as $\|u\|_{L^2(0,T;H^1(\Omega))} \to 0$, which gives the Frèchet differentiability of the control-to-state operator and establishes the equality $z = G'(\overline{u})[u]$. Thus, the proof of (i) is finished.

For any controls $u, v \in \mathcal{U}$ and $h \in L^2(0,T;H^1(\Omega))$, let $z_u := G'(u)[h]$ and $z_v := G'(v)[h]$ be two unique regular solutions of the linearized system (4.14). If $\widehat{z} = z_u - z_v$, $\widehat{m} = m_u - m_v$ and $\widehat{u} = u - v$, then $(\widehat{z}, \widehat{u})$ satisfies the following system:

$$\begin{cases}\mathcal{L}_u\widehat{z} = \displaystyle\sum_{k=1}^{7}\Theta_k \quad \text{ in } \Omega_T,\\ \frac{\partial\widehat{z}}{\partial\eta} = 0 \quad \text{ on } \partial\Omega_T, \quad \widehat{z}(0) = 0 \ \text{ in } \Omega,\end{cases} \tag{4.19}$$

where $\mathcal{L}_u$ is defined in (4.4) and the terms Θ_k's are given by

$$\begin{aligned}&\Theta_1 = 2\ \left(\nabla\widehat{m}\cdot\nabla z_v\right)m_u + 2\ \left(\nabla m_v\cdot\nabla z_v\right)\widehat{m}, && \Theta_2 = \left(\nabla\widehat{m}\cdot\nabla(m_u+m_v)\right)z_v,\\ &\Theta_3 = z_v\times\Delta\widehat{m} + \widehat{m}\times\Delta z_v, && \Theta_4 = z_v\times\widehat{u} + \widehat{m}\times h,\\ &\Theta_5 = -z_v\times(\widehat{m}\times u) - z_v\times(m_v\times\widehat{u}), && \Theta_6 = -\widehat{m}\times(z_v\times u) - m_v\times(z_v\times\widehat{u}),\\ &\Theta_7 = -\widehat{m}\times(m_u\times h) - m_v\times(\widehat{m}\times h).\end{aligned}$$

Again invoking Lemma 4.1 for system (4.19) and using the equality $\widehat{z}(0) = 0$, we obtain

$$\|\widehat{z}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\widehat{z}\|^2_{L^2(0,T;H^3(\Omega))} + \|\widehat{z}_t\|^2_{L^2(0,T;H^1(\Omega))} \le \sum_{k=1}^{8} \|\Theta_k\|^2_{L^2(0,T;H^1(\Omega))}$$
$$\times \ \exp\left\{ C \left(\|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} \|m_u\|^2_{L^2(0,T;H^3(\Omega))} + \|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} \|u\|^2_{L^2(0,T;H^1(\Omega))} \right) \right\}. \tag{4.20}$$

Now, let us estimate the terms containing Θ_k's on the right-hand side. Applying estimate (2.22) for the terms Θ_1 and Θ_2, we find

$$\begin{aligned}
&\|\Theta_1\|^2_{L^2(0,T;H^1(\Omega))} + \|\Theta_2\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad \le C \left(\|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m_v\|^2_{L^\infty(0,T;H^2(\Omega))} \right) \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|z_v\|^2_{L^2(0,T;H^3(\Omega))} \\
&\qquad + C \ \left(\|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m_v\|^2_{L^\infty(0,T;H^2(\Omega))} \right) \|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))} \|z_v\|^2_{L^\infty(0,T;H^2(\Omega))}.
\end{aligned}$$

Similarly, applying estimates (2.20) and (2.21) for the terms Θ_3 and Θ_4 respectively, we have

$$\begin{aligned}
\|\Theta_3\|^2_{L^2(0,T;H^1(\Omega))} &\le C \ \|z_v\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widehat{m}\|^2_{L^2(0,T;H^3(\Omega))} + C \ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|z_v\|^2_{L^2(0,T;H^3(\Omega))}, \\
\|\Theta_4\|^2_{L^2(0,T;H^1(\Omega))} &\le C \ \|z_v\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))} + C \ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|h\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Finally, applying estimate (2.23) for the terms Θ_5,Θ_6 and Θ_7, and then combining them we deduce

$$\begin{aligned}
&\|\Theta_5\|^2_{L^2(0,T;H^1(\Omega))} + \|\Theta_6\|^2_{L^2(0,T;H^1(\Omega))} + \|\Theta_7\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad \le C \ \|z_v\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|u\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad + C \ \|z_v\|^2_{L^\infty(0,T;H^2(\Omega))} \|m_v\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))} \\
&\qquad + C \ \left(\|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} + \|m_v\|^2_{L^\infty(0,T;H^2(\Omega))} \right) \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|h\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Next, we combine all these estimates and substitute them in (4.20). Then again, utilizing the locally Lipschitz continuity arguments employed in Lemma 4.2, we can establish a bound as $\|m_v\|_{\mathcal{M}} \le \|m_u\|_{\mathcal{M}} + C \ \|u - v\|_{L^2(0,T;H^1(\Omega))} \le \|m_u\|_{\mathcal{M}} + C\epsilon$ whenever $\|u - v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$. Moreover, applying these bounds in the energy estimate for z_v, we find $\|z_v\|_{\mathcal{M}} \le C \ \|h\|_{L^2(0,T;H^1(\Omega))}$. Substituting these inequalities in (4.20) and using the locally Lipschitz continuity of control-to-state operator from estimate (4.11), we derive

$$\|\widehat{z}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\widehat{z}\|^2_{L^2(0,T;H^3(\Omega))} + \|\widehat{z}_t\|^2_{L^2(0,T;H^1(\Omega))} \le C(\Omega, T, R) \ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))} \|h\|^2_{L^2(0,T;H^1(\Omega))}.$$

This leads to the proof of (4.15). □

Corollary 4.4. *The Fréchet derivative of the control-to-state operator satisfies the following estimate:*

$$\|G'(u)[v]\|_{\mathcal{M}} \le C(\Omega, T, u, m_u) \ \|v\|_{L^2(0,T;H^1(\Omega))}. \tag{4.21}$$

Next, we study the solvability of the adjoint problem. While deriving the optimality condition, we will work with the weak solution of the adjoint equation instead of the strong solution, which allows us to work with the target function $\nabla m_D \in L^6(0, T; L^6(\Omega))$. Suppose $(\widetilde{m}, \widetilde{u})$ is an admissible pair of the control problem. Then consider the following linear system:

$$(AL - LLG)\begin{cases} \mathcal{E}_{\widetilde{u}}\phi = g & \text{in } \Omega_T, \\ \frac{\partial \phi}{\partial \eta} = 0 & \text{in } \partial\Omega_T, \quad \phi(x,T) = \phi_T \ \text{ in } \Omega, \end{cases} \tag{4.22}$$

where the operator $\mathcal{E}_{\widetilde{u}}$ is defined as

$$\mathcal{E}_{\widetilde{u}}\phi := \phi_t + \Delta\phi + |\nabla\widetilde{m}|^2\phi - 2\nabla\cdot\big((\widetilde{m}\cdot\phi)\nabla\widetilde{m}\big) + \Delta(\phi\times\widetilde{m}) + (\Delta\widetilde{m}\times\phi) - (\phi\times\widetilde{u}) + \big((\phi\times\widetilde{m})\times\widetilde{u}\big) + \big(\phi\times(\widetilde{m}\times\widetilde{u})\big).$$

The weak formulation of (4.22) is the same as (2.5) with $\int_0^T \langle g, \upsilon\rangle \,\mathrm{d}t$ replacing the term on the right-hand side of (2.5).

Lemma 4.5. *Suppose $(\tilde{m}, \tilde{u})$ is an admissible pair, that is, $(\tilde{m}, \tilde{u}) \in \mathcal{A}$. Then, for any $g \in L^2(0,T;H^1(\Omega)^*)$ there exists a unique weak solution $\phi \in \mathcal{Z}$ of the linear system (4.22) in the sense of Definition 2.8. Moreover, the following estimate holds:*

$$\begin{aligned} &\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} + \|\phi_t\|^2_{L^2(0,T;H^1(\Omega)^*)} \le \left(\|\phi_T\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(0,T;H^1(\Omega)^*)}\right) \\ &\quad\times\ \exp\left\{C\left(\|\tilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\tilde{m}\|^2_{L^2(0,T;H^3(\Omega))} + \|\tilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\|\tilde{u}\|^2_{L^2(0,T;H^1(\Omega))}\right)\right\}. \end{aligned} \tag{4.23}$$

Proof. We will prove this theorem using the Galerkin method. Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigen vectors for the operator $-\Delta + I$ with vanishing Neumann boundary condition. Suppose $W_n = span\{w_1, w_2, ..., w_n\}$ and $\mathbb{P}_n : L^2 \to W_n$ be the orthogonal projection. Using the solvability of ODEs, we can find a solution $\phi_n = \sum_{j=1}^n g_{jn}(t)w_j$ of the following approximated system for each $j = 1, \cdots, n$ and $0 \le t \le T$:

$$\begin{cases} -\big(\phi_n'(t), w_j\big) + \big(\nabla\phi_n(t), \nabla w_j\big) = \big(|\nabla\widetilde{m}(t)|^2\phi_n(t), w_j\big) + 2\ \big((\widetilde{m}(t)\cdot\phi_n(t))\nabla\widetilde{m}(t), \nabla w_j\big) \\ \qquad - \big(\nabla(\phi_n(t)\times\widetilde{m}(t)), \nabla w_j\big) + \big((\Delta\widetilde{m}(t)\times\phi_n(t)), w_j\big) - \ \big((\phi_n(t)\times\widetilde{u}(t)), w_j\big) \\ \qquad + \big((\phi_n(t)\times\widetilde{m}(t))\times\widetilde{u}(t), w_j\big) + \big(\phi_n(t)\times(\widetilde{m}(t)\times\widetilde{u}(t)), w_j\big) - \ \langle g(t), w_j\rangle, \\ \phi_n(T) = \mathbb{P}_n\big(\phi_T\big). \end{cases} \tag{4.24}$$

Now, multiplying (4.24) by $g_{jn}(t)$, summing over $j = 1, ..., n$, and using the property $a\cdot(a\times b) = 0$, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{\mathrm{d}t}\|\phi_n(t)\|^2_{L^2(\Omega)} + \int_\Omega |\nabla\phi_n(t)|^2\mathrm{d}x &= \int_\Omega |\nabla\widetilde{m}|^2|\phi_n|^2\ \mathrm{d}x + 2\int_\Omega \big((\widetilde{m}\cdot\phi_n)\nabla\widetilde{m}\big)\cdot\nabla\phi_n\ \mathrm{d}x \\ &- \int_\Omega \nabla(\phi_n\times\widetilde{m})\cdot\nabla\phi_n\ \mathrm{d}x + \int_\Omega \big((\phi_n\times\widetilde{m})\times\widetilde{u}\big)\cdot\phi_n\ \mathrm{d}x - \langle g(t), \phi_n(t)\rangle. \end{aligned}$$

Applying Hölder's inequality and the embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$, $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we find

$$\begin{aligned} &-\frac{1}{2}\frac{d}{\mathrm{d}t}\|\phi_n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla\phi_n(t)|^2\mathrm{d}x \le C\ \|g(t)\|^2_{H^1(\Omega)^*} \\ &+ C\ \|\widetilde{m}(t)\|^2_{H^2(\Omega)}\left(\|\widetilde{m}(t)\|^2_{H^3(\Omega)} + \|\widetilde{u}(t)\|^2_{H^1(\Omega)}\right)\|\phi_n(t)\|^2_{L^2(\Omega)}. \end{aligned}$$

By the application of Gronwall's inequality and using the inequality $\|\phi_n(T)\|_{L^2(\Omega)} \le \|\phi(T)\|_{L^2(\Omega)}$, we derive

$$\|\phi_n(t)\|^2_{L^2(\Omega)} + \int_t^T \int_\Omega |\nabla \phi_n(s)|^2 \mathrm{d}x\, ds \le \Big(\|\phi_T\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(0,T;H^1(\Omega)^*)} \Big) \\ \times \exp \left\{ C\, \|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \int_0^T \Big(\|\widetilde{m}(s)\|^2_{H^3(\Omega)} + \|\widetilde{u}(s)\|^2_{H^1(\Omega)} \Big)\, ds \right\}, \quad \forall\, t \in [0,T). \tag{4.25}$$

Given that the right-hand side of the above estimate is independent of n, and the state $\widetilde{m} \in \mathcal{M}$, we can conclude that the sequence $\{\phi_n\}$ is uniformly bounded in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Now, proceeding in a way similar to Theorem 5.2 of [20], we find the following

$$\int_0^T \|\phi_n'(t)\|^2_{H^1(\Omega)^*} \mathrm{d}t \le C\, \Big(\|g\|^2_{L^2(0,T;H^1(\Omega)^*)} + \|\widetilde{m}\|^4_{L^\infty(0,T;H^2(\Omega))} \|\phi_n\|^2_{L^2(0,T;H^1(\Omega))} \\ + \|\widetilde{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))} \|\phi_n\|^2_{L^\infty(0,T;L^2(\Omega))} \Big).$$

Therefore, using the uniform bound for $\{\phi_n\}$ in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ from inequality (4.25) in the above estimate, we find that $\{\phi_n'\}$ is uniformly bounded in $L^2(0,T;H^1(\Omega)^*)$. Appealing to Aloglu weak* compactness and reflexive weak compactness theorems, we have

$$\begin{cases} \phi_n \xrightarrow{w} \phi \text{ weakly in } & L^2(0,T;H^1(\Omega)), \\ \phi_n \xrightarrow{w^*} \phi \text{ weak* in } & L^\infty(0,T;L^2(\Omega)), \\ \phi_n' \xrightarrow{w} \phi' \text{ weakly in } & L^2(0,T;H^1(\Omega)^*), \quad \text{as} \quad n \to \infty. \end{cases} \tag{4.26}$$

Again the Aubin-Lions-Simon lemma (see Cor. 4, [37]) establishes the existence of a sub-sequence of $\{\phi_n\}$ (again denoted as $\{\phi_n\}$) such that $\phi_n \xrightarrow{s} \phi$ strongly in $L^2(0,T;L^2(\Omega))$. Using this strong convergence along with (4.26), we can verify that ϕ satisfies equality (i) of Definition 2.8 for every $\vartheta \in \mathrm{span}(w_1, w_2, ...)$. Further, as such functions are dense in $H^1(\Omega)$, it holds true for every $\vartheta \in H^1(\Omega)$. Since for any $\alpha \in L^2(0,T)$ and $\vartheta_j \in H^1(\Omega)$, the elements of the form $\sum_{j=1}^n \alpha_j(t)\, \vartheta_j$ are dense in $L^2(0,T;H^1(\Omega))$, so equality (i) holds for every $\vartheta \in L^2(0,T;H^1(\Omega))$. □

Finally, we can conclude the proof of Theorem 2.9.

Proof of Theorem 2.9: The result can be readily derived as a direct consequence of Lemma 4.5 by making the substitution $g = \nabla \cdot \big(|\nabla \widetilde{m} - \nabla m_D|^2 \big(\nabla \widetilde{m} - \nabla m_D \big) \big)$.

For any element $\upsilon \in L^2(0,T;H^1(\Omega))$, the following estimation for the dual product of g holds:

$$\int_0^T \langle g, \upsilon \rangle_{H^1(\Omega)^* \times H^1(\Omega)}\, \mathrm{d}t = -\int_0^T \int_\Omega \big(|\nabla \widetilde{m} - \nabla m_D|^2 \big(\nabla \widetilde{m} - \nabla m_D \big) \big) \cdot \nabla \upsilon\, \mathrm{d}x\, \mathrm{d}t \\ \le \int_0^T \|\nabla \widetilde{m} - \nabla m_D\|^3_{L^6(\Omega)}\, \|\nabla \upsilon\|_{L^2(\Omega)}\, \mathrm{d}t \le C\, \|\nabla \widetilde{m} - \nabla m_D\|^3_{L^6(0,T;L^6(\Omega))}\, \|\upsilon\|_{L^2(0,T;H^1(\Omega))}. \tag{4.27}$$

Therefore, in order to make this dual product meaningful, we had to choose m_D such that $\nabla m_D \in L^6(0,T;L^6(\Omega))$. Hence, the estimate (2.6) follows from (4.23) and (4.27). □

4.3. First order optimality condition

The first-order optimality condition for non-convex problems is a basic criterion used to analyze critical points in optimization. To derive second-order sufficient and globally applicable optimality conditions, we leverage the insights obtained from the first-order optimality condition.

Proof of Theorem 2.10. Since the set of controls

$$S := \{u \in L^2(0,T;H^1(\Omega)) \mid a(x,t) \le u(x,t) \le b(x,t) \text{ for a.e. } (x,t) \in \Omega_T\}$$

is convex, so for any $u, \widetilde{u} \in S$ and $\epsilon \in (0,1)$, we have $\widetilde{u} + \epsilon(u - \widetilde{u}) \in S$. Moreover, as the set of controls in $L^2(0,T;H^1(\Omega))$ for which there exists a regular solution in $\mathcal{M}$ is open, for any control $\widetilde{u} \in \mathcal{U}$, $u \in L^2(0,T;H^1(\Omega))$ and $\epsilon > 0$ sufficiently small, we have $\widetilde{u} + \epsilon(u - \widetilde{u}) \in \mathcal{U}$. Therefore, $\widetilde{u} + \epsilon(u - \widetilde{u}) \in \mathcal{U}_{ad}$ for controls $\widetilde{u}, u \in \mathcal{U}_{ad}$ and small enough $\epsilon > 0$.

Now, as $\widetilde{u} \in \mathcal{U}_{ad}$ is an optimal control of MOCP and the control-to-state operator is Fréchet differentiable, the functional $\mathcal{I}(u) = \mathcal{J}(G(u), u)$ satisfies the following inequality:

$$D_u\mathcal{I}(\widetilde{u})[u - \widetilde{u}] = \lim_{\epsilon \to 0^+} \frac{\mathcal{I}(\widetilde{u} + \epsilon(u - \widetilde{u})) - \mathcal{I}(\widetilde{u})}{\epsilon} \ge 0, \qquad \forall\, u \in \mathcal{U}_{ad}. \tag{4.28}$$

Now, by setting $v = u - \widetilde{u}$ and applying chain rule, we have

$$\begin{aligned} D_u\mathcal{I}(\widetilde{u})[v] &= D_m\mathcal{J}(\widetilde{m}, \widetilde{u}) \circ \Big(D_uG(\widetilde{u})[v]\Big) + D_u\mathcal{J}(\widetilde{m}, \widetilde{u})[v] \\ &= \int_{\Omega_T} \widetilde{u} \cdot v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \nabla\widetilde{u} \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega} (\widetilde{m}(x,T) - m_\Omega) \cdot z(x,T)\,\mathrm{d}x \\ &\qquad + \int_{\Omega_T} |\nabla\widetilde{m} - \nabla m_D|^2 (\nabla\widetilde{m} - \nabla m_D) \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.29}$$

where $z_v := D_uG(\widetilde{u})[v] \in \mathcal{M}$ is a unique regular solution of the linearized system (4.14).

Now, we take $\vartheta = z_v$ in the weak adjoint formulation of (2.4). Then, doing a space integration by parts for the last term using $\frac{\partial z_v}{\partial \eta} = 0$, we obtain

$$\begin{aligned} &-\int_0^T (\phi(t), z_v'(t))\,\mathrm{d}t + \big(\widetilde{m}(x,T) - m_\Omega(x), z_v(T)\big) - \int_{\Omega_T} \nabla\phi \cdot \nabla z_v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla\widetilde{m}|^2 \phi \cdot z_v \,\mathrm{d}x\,\mathrm{d}t \\ &+2\int_{\Omega_T} (\widetilde{m}\cdot\phi)\nabla\widetilde{m} \cdot \nabla z_v \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \nabla(\phi \times \widetilde{m}) \cdot \nabla z_v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} (\Delta\widetilde{m} \times \phi) \cdot z_v \,\mathrm{d}x\,\mathrm{d}t \\ &-\int_{\Omega_T} (\phi \times \widetilde{u}) \cdot z_v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \big((\phi \times \widetilde{m}) \times \widetilde{u}\big) \cdot z_v \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \big(\phi \times (\widetilde{m} \times \widetilde{u})\big) \cdot z_v \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{\Omega_T} |\nabla\widetilde{m} - \nabla m_D|^2 (\nabla\widetilde{m} - \nabla m_D) \cdot \nabla z_v \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.30}$$

where we also used the following identity:

$$\int_0^T \big(z_t, \phi\big)\,\mathrm{d}t + \int_0^T \big\langle \phi_t, z\big\rangle\,\mathrm{d}t = \int_\Omega (\widetilde{m}(x,T) - m_\Omega) \cdot z(x,T)\,\mathrm{d}x. \tag{4.31}$$

On the other hand, considering the inner product of the linearized system (4.14) with ϕ, we have

$$\begin{aligned}
\int_0^T \left(z_v', \phi\right)\,\mathrm{d}t &= -\int_{\Omega_T} \nabla z_v \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla \widetilde{m}|^2 z_v \cdot \phi \,\mathrm{d}x\,\mathrm{d}t \\
&+2\int_{\Omega_T} (\widetilde{m}\cdot\phi)\big(\nabla\widetilde{m}\cdot\nabla z_v\big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} (z_v \times \Delta\widetilde{m})\cdot\phi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \nabla(\phi\times\widetilde{m})\cdot\nabla z_v\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{\Omega_T} (z_v\times\widetilde{u})\cdot\phi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \big(z_v\times(\widetilde{m}\times\widetilde{u})\big)\cdot\phi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \big(\widetilde{m}\times(z_v\times\widetilde{u})\big)\cdot\phi\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{\Omega_T} (\widetilde{m}\times v)\cdot\phi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \big(\widetilde{m}\times(\widetilde{m}\times v)\big)\cdot\phi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.32}$$

Combining equations (4.30) and (4.32), then substituting the result in (4.29), we obtain

$$D_u\mathcal{I}(\widetilde{u})\cdot v = \int_{\Omega_T} \widetilde{u}\cdot v\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \nabla\widetilde{u}\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \Big((\phi\times\widetilde{m}) + \widetilde{m}\times(\phi\times\widetilde{m})\Big)\cdot v\,\mathrm{d}x\,\mathrm{d}t.$$

At last using (4.28), applying the cross product property $(a\times b)\cdot c = b\cdot(c\times a)$ for the last integral on the right hand side, and putting $v = u - \widetilde{u}$, we get the required optimality condition (2.7). ☐

5. Second order optimality condition

In convex optimal control problems, controls that satisfy the first-order necessary optimality conditions are globally optimal. However, for non-convex optimal control problems, additional analysis, including higher derivative analysis, is required to ensure local optimality. For optimal control problems governed by the Landau–Lifshitz–Gilbert equations, second-order sufficient optimality conditions play a crucial role in the numerical analysis. These conditions are essential in guaranteeing the feasibility and local optimality of a control, considering the intricate dynamics and non-linearity of the LLG equations.

Due to the availability of a weak solution for the adjoint system, we are empowered to establish an operator that associates each control in $\mathcal{U}$ with the corresponding adjoint solution. Henceforth, we shall refer to this operator as the "control-to-costate" operator. This operator, denoted as $\Phi : \mathcal{U} \to \mathcal{Z}$, effectively maps controls from the set $\mathcal{U}$ to their respective adjoint solutions in $\mathcal{Z}$.

5.1. Control-to-costate operator

Lemma 5.1. *The control-to-costate operator $\Phi : \mathcal{U} \to \mathcal{Z}$ is locally Lipschitz continuous, that is, for each control $u \in \mathcal{U}$, there exists $\epsilon > 0$ and a constant $C_3 > 0$ depending on $\Omega, T, u, m_u, \epsilon$ such that*

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{Z}} \le C_3\, \|u - v\|_{L^2(0,T;H^1(\Omega))}, \tag{5.1}$$

for all v with $\|u - v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$.

The proof of Lemma 5.1 can be done using a comparable approach to that employed in proving Lemma 4.2. It's important to note that since the equation satisfied by the costate variable is linear, we can transform the equation satisfied by $\widehat{\phi} := \Phi(u) - \Phi(v)$ solely in terms of $\widehat{\phi}$ and $\Phi(u)$. Thus, we can carry out the proof without requiring the continuity of the control-to-costate operator beforehand.

Proposition 5.2. *Let $\bar{u} \in \mathcal{U}$ be the control and $m_{\bar{u}}$ be its corresponding regular solution. Then the following two conclusions hold:*

(i) The control-to-costate mapping Φ is Fréchet differentiable on $\mathcal{U}$, that is for any $\overline{u} \in \mathcal{U}$, there exists a bounded linear operator $\Phi'(\overline{u}) : L^2(0,T;H^1(\Omega)) \to \mathcal{Z}$ such that

$$\frac{\|\Phi(\overline{u}+u) - \Phi(\overline{u}) - \Phi'(\overline{u})[u]\|_{\mathcal{Z}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0 \qquad \text{as } \|u\|_{L^2(0,T;H^1(\Omega))} \to 0,$$

where $\phi' := \Phi'(\overline{u})[u]$ is the unique weak solution of the following system:

$$\begin{cases} \mathcal{E}_{\overline{u}}\phi' = -2\ (\nabla m_{\overline{u}} \cdot \nabla z)\phi + 2\ \nabla \cdot \big((z\cdot\phi)\nabla m_{\overline{u}}\big) + 2\ \nabla\cdot\big((m_{\overline{u}}\cdot\phi)\nabla z\big) - \Delta(\phi\times z) - (\Delta z \times \phi) \\ \qquad + (\phi\times u) - \big((\phi\times z)\times\overline{u}\big) - \big((\phi\times m_{\overline{u}})\times u\big) - \big(\phi\times(z\times\overline{u})\big) - \big(\phi\times(m_{\overline{u}}\times u)\big) \\ \qquad + \nabla\cdot\big(2\ \big((\nabla m_{\overline{u}} - \nabla m_D)\cdot\nabla z\big)(\nabla m_{\overline{u}} - \nabla m_D)\big) + \nabla\cdot\big(|\nabla m_{\overline{u}} - \nabla m_D|^2 \nabla z\big) \quad \text{in } \Omega_T, \\ \frac{\partial\phi'}{\partial\eta} = 0 \quad \text{in } \partial\Omega_T, \\ \phi'(T) = z(T) \quad \text{in } \Omega. \end{cases} \tag{5.2}$$

(ii) The Fréchet derivative Φ' is locally Lipschitz continuous. That is, for any controls $u \in \mathcal{U}$, $h \in L^2(0,T;H^1(\Omega))$ there exists $\epsilon > 0$ and a constant $C_4 > 0$ depending on $\Omega, T, u, m_u, \epsilon$ such that

$$\|\Phi'(u)[h] - \Phi'(v)[h]\|_{\mathcal{Z}} \le C_4\ \|u-v\|_{L^2(0,T;H^1(\Omega))}\ \|h\|_{L^2(0,T;H^1(\Omega))}, \tag{5.3}$$

for all v in $L^2(0,T;H^1(\Omega))$ with $\|u-v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$.

Proof. Suppose $\overline{u} \in \mathcal{U}$ be any fixed control and $m_{\overline{u}}$ be the corresponding regular solution. Let z be the unique regular solution of the linearized system (4.14) and $\phi_{\overline{u}}$ be the unique weak solution of the adjoint system (2.4). As a consequence of Lemma 4.5, we can easily show that the system (5.2) has a unique weak solution ϕ'. Note that, here ϕ' is merely a notation and does not yet signify the derivative of the control-to-costate operator. Let $\widehat{\phi} = \phi_{\overline{u}+u} - \phi_{\overline{u}}$, $\widehat{m} = m_{\overline{u}+u} - m_{\overline{u}}$ and $w = m_{\overline{u}+u} - m_{\overline{u}} - z$. Then $\xi := \phi_{\overline{u}+u} - \phi_{\overline{u}} - \phi'$ solves the following system weakly in the sense of Definition 2.8:

$$\begin{cases} \mathcal{E}_{\overline{u}}\xi = \sum_{k=1}^{8} \mathcal{S}_k \quad \text{in } \Omega_T, \\ \frac{\partial\xi}{\partial\eta} = 0 \quad \text{on } \partial\Omega_T, \quad \xi(T) = m_{\overline{u}+u}(T) - m_{\overline{u}}(T) - z(T) \ \text{ in } \Omega, \end{cases} \tag{5.4}$$

where $\mathcal{E}_{\overline{u}}$ is defined in (4.22), and the terms $\mathcal{S}_k$'s are given by

$$\begin{aligned} \mathcal{S}_1 &= -\big(\nabla\widehat{m}\cdot(\nabla m_{\overline{u}+u} + \nabla m_{\overline{u}})\big)\ \widehat{\phi} - |\nabla\widehat{m}|^2\phi_{\overline{u}} - 2\ (\nabla m_{\overline{u}}\cdot\nabla w)\ \phi_{\overline{u}}, \\ \mathcal{S}_2 &= 2\ \nabla\cdot\big((\widehat{m}\cdot\widehat{\phi})\nabla\widehat{m}\big) + 2\ \nabla\cdot\big((m_{\overline{u}}\cdot\widehat{\phi})\nabla\widehat{m}\big) + 2\ \nabla\cdot\big((\widehat{m}\cdot\phi_{\overline{u}})\nabla\widehat{m}\big) \\ &\quad + 2\ \nabla\cdot\big((\widehat{m}\cdot\widehat{\phi})\nabla m_{\overline{u}}\big) + 2\ \nabla\big((w\cdot\phi_{\overline{u}})\nabla m_{\overline{u}}\big) + 2\ \nabla\cdot\big((m_{\overline{u}}\cdot\phi_{\overline{u}})\nabla w\big), \\ \mathcal{S}_3 &= -\Delta(\widehat{\phi}\times\widehat{m}) - \Delta(\phi_{\overline{u}}\times w), \\ \mathcal{S}_4 &= -(\Delta\widehat{m}\times\widehat{\phi}) - (\Delta w\times\phi_{\overline{u}}), \\ \mathcal{S}_5 &= \widehat{\phi}\times u, \\ \mathcal{S}_6 &= -\big((\widehat{\phi}\times\widehat{m})\times(\overline{u}+u)\big) - \big((\widehat{\phi}\times m_{\overline{u}})\times u\big) - \big((\phi_{\overline{u}}\times\widehat{m})\times u\big) - \big((\phi_{\overline{u}}\times w)\times\overline{u}\big), \\ \mathcal{S}_7 &= -\big(\widehat{\phi}\times(\widehat{m}\times(\overline{u}+u))\big) - \big(\widehat{\phi}\times(m_{\overline{u}}\times u)\big) - \big(\phi_{\overline{u}}\times(\widehat{m}\times u)\big) - \big(\phi_{\overline{u}}\times(w\times\overline{u})\big), \\ \mathcal{S}_8 &= \nabla\cdot\big(\big(\nabla\widehat{m}\cdot(\nabla m_{\overline{u}+u} + \nabla m_{\overline{u}} - 2\nabla m_D)\big)\nabla\widehat{m}\big) + \nabla\cdot\big(|\nabla\widehat{m}|^2(\nabla m_{\overline{u}} - \nabla m_D)\big) \\ &\quad + 2\ \nabla\cdot\big(\big(\big((\nabla m_{\overline{u}} - \nabla m_D)\cdot\nabla w\big)(\nabla m_{\overline{u}} - \nabla m_D)\big) + \nabla\cdot(|\nabla m_{\overline{u}} - \nabla m_D|^2\nabla w). \end{aligned}$$

Using Lemma 4.5, we can find the following estimate on the solution of system (5.4):

$$\|\xi\|^2_{L^2(0,T;H^1(\Omega))} + \|\xi_t\|^2_{L^2(0,T;H^1(\Omega)^*)} \leq \left(\|m_{\overline{u}+u}(T) - m_{\overline{u}}(T) - z(T)\|^2_{L^2(\Omega)} + \sum_{k=1}^{8} \|\mathcal{S}_k\|^2_{L^2(0,T;H^1(\Omega)^*)} \right)$$
$$\times \ \exp\left\{ C \left(\|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} \|m_{\overline{u}}\|^2_{L^2(0,T;H^3(\Omega))} + \|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} \|\overline{u}\|^2_{L^2(0,T;H^1(\Omega))} \right) \right\}. \tag{5.5}$$

Applying estimate (2.24) for the $L^2(0,T;H^1(\Omega)^*)$ norm of $\mathcal{S}_1$, we obtain

$$\|\mathcal{S}_1\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \left(\|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))} + \|m_{\overline{u}+u}\|_{L^\infty(0,T;H^2(\Omega))}\right) \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} \|\widehat{\phi}\|_{L^2(0,T;L^2(\Omega))}$$
$$+ C\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \|\phi_{\overline{u}}\|_{L^2(0,T;L^2(\Omega))} + C\ \|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))} \|w\|_{L^\infty(0,T;H^2(\Omega))} \|\phi_{\overline{u}}\|_{L^2(0,T;L^2(\Omega))}.$$

Proceeding for $\mathcal{S}_2$ using estimate (2.25), we have

$$\|\mathcal{S}_2\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \left(\|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} + \|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))}\right) \|\widehat{\phi}\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{m}\|_{L^2(0,T;H^3(\Omega))}$$
$$+ C\ \left(\|\widehat{\phi}\|_{L^\infty(0,T;L^2(\Omega))} + \|\phi_{\overline{u}}\|_{L^\infty(0,T;L^2(\Omega))}\right) \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} \|\widehat{m}\|_{L^2(0,T;H^3(\Omega))}$$
$$+ C\ \|w\|_{L^\infty(0,T;H^2(\Omega))} \|\phi_{\overline{u}}\|_{L^\infty(0,T;L^2(\Omega))} \|m_{\overline{u}}\|_{L^2(0,T;H^3(\Omega))}$$
$$+ C\ \|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))} \|\phi_{\overline{u}}\|_{L^\infty(0,T;L^2(\Omega))} \|w\|_{L^2(0,T;H^3(\Omega))}.$$

Similarly using estimates (2.26), (2.27) and (2.28) to evaluate the $L^2(0,T;H^1(\Omega)^*)$ norm of $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_5$, we find

$$\|\mathcal{S}_3\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\widehat{\phi}\|_{L^2(0,T;H^1(\Omega))} \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} + C\ \|\phi_{\overline{u}}\|_{L^2(0,T;H^1(\Omega))} \|w\|_{L^\infty(0,T;H^2(\Omega))}$$
$$\|\mathcal{S}_4\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} \|\widehat{\phi}\|_{L^2(0,T;H^1(\Omega))} + C\ \|w\|_{L^\infty(0,T;H^2(\Omega))} \|\phi_{\overline{u}}\|_{L^2(0,T;H^1(\Omega))}$$
$$\|\mathcal{S}_5\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\widehat{\phi}\|_{L^\infty(0,T;L^2(\Omega))} \|u\|_{L^2(0,T;H^1(\Omega))}.$$

Now, for the terms $\mathcal{S}_6$ and $\mathcal{S}_7$ using estimates (2.29) and (2.30), we have

$$\|\mathcal{S}_6\|_{L^2(0,T;H^1(\Omega)^*)} + \|\mathcal{S}_7\|_{L^2(0,T;H^1(\Omega)^*)}$$
$$\leq C\ \|\widehat{\phi}\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} \left(\|\overline{u}\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))}\right)$$
$$+ C\ \|\widehat{\phi}\|_{L^\infty(0,T;L^2(\Omega))} \|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))} \|u\|_{L^2(0,T;H^1(\Omega))}$$
$$+ C\ \|\phi_{\overline{u}}\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{m}\|_{L^\infty(0,T;H^2(\Omega))} \|u\|_{L^2(0,T;H^1(\Omega))}$$
$$+ C\ \|\phi_{\overline{u}}\|_{L^\infty(0,T;L^2(\Omega))} \|w\|_{L^\infty(0,T;H^2(\Omega))} \|\overline{u}\|_{L^2(0,T;H^1(\Omega))}.$$

Finally, estimating $\mathcal{S}_8$ using (2.31), applying the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the inequality $\|\cdot\|_{L^6(0,T)} \leq C\ \|\cdot\|_{L^\infty(0,T)}$, we find

$$\|\mathcal{S}_8\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))} \left(\|m_{\overline{u}+u}\|_{L^\infty(0,T;H^2(\Omega))} + \|m_{\overline{u}}\|_{L^\infty(0,T;H^2(\Omega))} + \|\nabla m_D\|_{L^6(0,T;L^6(\Omega))}\right)$$
$$+ C\ \|w\|_{L^\infty(0,T;H^2(\Omega))} \left(\|m_{\overline{u}}\|^2_{L^\infty(0,T;H^2(\Omega))} + \|\nabla m_D\|^2_{L^6(0,T;L^6(\Omega))} \right).$$

Then let us combine all the above estimates, substitute it in (5.5) and divide by $\|u\|^2_{L^2(0,T;H^1(\Omega))}$. Since $\overline{u} \in \mathcal{U}$, so for small enough u we can guarantee that $\overline{u} + u \in \mathcal{U}$. Now, again using the locally Lipschitz continuity of the control-to-state operator, we can find an uniform bound for $\|m_{\overline{u}}\|_{\mathcal{M}}$ and $\|m_{\overline{u}+u}\|_{\mathcal{M}}$ for such small control function u. Similarly, from estimate (2.6) we can find a bound for $\|\phi_{\overline{u}}\|_{\mathcal{Z}}$. Moreover, from the locally Lipschitz continuity of the control-to-state and control-to-costate operators given in estimate (4.11) and (5.1) respectively, it is clear that $\|\widehat{m}\|_{\mathcal{M}} \leq C\ \|u\|_{L^2(0,T;H^1(\Omega))}$ and $\|\widehat{\phi}\|_{\mathcal{Z}} \leq C\ \|u\|_{L^2(0,T;H^1(\Omega))}$ for small enough $\|u\|_{L^2(0,T;H^1(\Omega))}$. Again, from estimate (4.18), we know that $\|w\|_{\mathcal{M}} \leq C\ \|u\|^2_{L^2(0,T;H^1(\Omega))} + C\ \|u\|^3_{L^2(0,T;H^1(\Omega))}$. Therefore, based on these estimates, we conclude that $\frac{\|\mathcal{S}_k\|_{L^2(0,T;H^1(\Omega)^*)}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0$ as $\|u\|_{L^2(0,T;H^1(\Omega))} \to 0$ for each $k = 1, 2, ..., 8$. Also, by Proposition 4.3, we have

$$
\begin{aligned}
\frac{\|m_{\overline{u}+u}(T) - m_{\overline{u}}(T) - z(T)\|_{L^2(\Omega)}}{\|u\|_{L^2(0,T;H^1(\Omega))}} &\leq \frac{\|m_{\overline{u}+u} - m_{\overline{u}} - z\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \\
&\leq \frac{\|m_{\overline{u}+u} - m_{\overline{u}} - z\|_{\mathcal{M}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \leq \frac{\|w\|_{\mathcal{M}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0 \quad \text{as } \|u\|_{L^2(0,T;H^1(\Omega))} \to 0.
\end{aligned}
$$

Hence, we conclude the Frèchet differentiability of the control-to-costate operator by showing $\frac{\|\xi\|_{\mathcal{Z}}}{\|u\|_{L^2(0,T;H^1(\Omega))}} \to 0$ as $\|u\|_{L^2(0,T;H^1(\Omega))} \to 0$.

Suppose $\phi'_u := \Phi'(u)[h]$ and $\phi'_v := \Phi'(v)[h]$ be two weak solutions of system (5.2). Let $\widehat{\psi} = \phi'_u - \phi'_v$, $\widehat{m} = m_u - m_v$ and $\widehat{u} = u - v$. Then ψ will weakly satisfy the following system:

$$
\begin{cases}
\mathcal{E}_u \widehat{\psi} = \displaystyle\sum_{k=1}^{8} \mathcal{B}_k \quad \text{ in } \Omega_T, \\
\frac{\partial \widehat{\psi}}{\partial \eta} = 0 \quad \text{ on } \partial\Omega_T, \quad \widehat{\psi}(T) = z_u(T) - z_v(T) \ \text{ in } \Omega,
\end{cases}
\tag{5.6}
$$

where $\mathcal{E}_u$ is defined in (4.22), and the terms $\mathcal{B}_k$'s are given by

$$
\begin{aligned}
\mathcal{B}_1 &= -\big(\nabla\widehat{m} \cdot (\nabla m_u + \nabla m_v)\big)\ \phi'_v - 2\ (\nabla\widehat{m} \cdot \nabla z_u)\phi_u - 2\ (\nabla m_v \cdot \nabla\widehat{z})\phi_u - 2\ (\nabla m_v \cdot \nabla z_v)\widehat{\phi}, \\
\mathcal{B}_2 &= 2\ \nabla \cdot \big((\widehat{m} \cdot \phi'_v)\nabla m_u\big) + 2\ \nabla \cdot \big((m_v \cdot \phi'_v)\nabla\widehat{m}\big) + 2\ \nabla \cdot \big((\widehat{z} \cdot \phi_u)\nabla m_u\big) + 2\ \nabla \cdot \big((z_v \cdot \widehat{\phi})\nabla m_u\big) \\
&\quad + 2\ \nabla \cdot \big((z_v \cdot \phi_v)\nabla\widehat{m}\big) + 2\ \nabla \cdot \big((\widehat{m} \cdot \phi_u)\nabla z_u\big) + 2\ \nabla \cdot \big((m_v \cdot \widehat{\phi})\nabla z_u\big) + 2\ \nabla \cdot \big((m_v \cdot \phi_v)\nabla\widehat{z}\big), \\
\mathcal{B}_3 &= -\Delta(\phi'_v \times \widehat{m}) - \Delta(\widehat{\phi} \times z_u) - \Delta(\phi_v \times \widehat{z}), \\
\mathcal{B}_4 &= -\Delta\widehat{m} \times \phi'_v - (\Delta\widehat{z} \times \phi_u) - (\Delta z_v \times \widehat{\phi}), \\
\mathcal{B}_5 &= \phi'_v \times \widehat{u} + \widehat{\phi} \times h, \\
\mathcal{B}_6 &= -(\phi'_v \times \widehat{m}) \times u - (\phi'_v \times m_v) \times \widehat{u} - (\widehat{\phi} \times z_u) \times u - (\phi_v \times \widehat{z}) \times u - (\phi_v \times z_v) \times \widehat{u} \\
&\quad - (\widehat{\phi} \times m_u) \times h - (\phi_v \times \widehat{m}) \times h, \\
\mathcal{B}_7 &= -\phi'_v \times (\widehat{m} \times u) - \phi'_v \times (m_v \times \widehat{u}) - \widehat{\phi} \times (z_u \times u) - \phi_v \times (\widehat{z} \times u) - \phi_v \times (z_v \times \widehat{u}) \\
&\quad - \widehat{\phi} \times (m_u \times h) - \phi_v \times (\widehat{m} \times h), \\
\mathcal{B}_8 &= 2\ \nabla \cdot \big((\nabla\widehat{m} \cdot \nabla z_u)(\nabla m_u - \nabla m_D)\big) + 2\ \nabla \cdot \big(((\nabla m_v - \nabla m_D) \cdot \nabla\widehat{z})(\nabla m_u - \nabla m_D)\big) \\
&\quad + 2\ \nabla \cdot \big(((\nabla m_v - \nabla m_D) \cdot \nabla z_v)\nabla\widehat{m}\big) + \nabla \cdot \big((\nabla\widehat{m} \cdot (\nabla m_u + \nabla m_v - 2\ \nabla m_D))\nabla z_u\big) \\
&\quad + \nabla \cdot \big(|\nabla m_v - \nabla m_D|^2 \nabla\widehat{z}\big).
\end{aligned}
$$

Again appealing to Lemma 4.5 we can find an estimate on the solution of the system (5.6) as follows:

$$
\begin{aligned}
\|\widehat{\psi}\|^2_{L^2(0,T;H^1(\Omega))} + \|\widehat{\psi}_t\|^2_{L^2(0,T;H^1(\Omega)^*)} &\leq \left(\|z_u(T) - z_v(T)\|^2_{L^2(\Omega)} + \sum_{k=1}^{8} \|\mathcal{B}_k\|^2_{L^2(0,T;H^1(\Omega)^*)} \right) \\
\times\ \exp\Big\{ C \Big(\|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} &\|m_u\|^2_{L^2(0,T;H^3(\Omega))} + \|m_u\|^2_{L^\infty(0,T;H^2(\Omega))} \|u\|^2_{L^2(0,T;H^1(\Omega))} \Big) \Big\}.
\end{aligned}
\tag{5.7}
$$

By doing calculations similar to Part-(i), we can find $\|\mathcal{B}_k\|_{L^2(0,T;H^1(\Omega)^*)} \leq C\,\|u-v\|_{L^2(0,T;H^1(\Omega))}\,\|h\|_{L^2(0,T;H^1(\Omega))}$ whenever $\|u-v\|_{L^2(0,T;H^1(\Omega))} < \epsilon$. Also, for the final time data, we will use the estimate

$$\|z_u(T)-z_v(T)\|^2_{L^2(\Omega)} \leq \|z_u - z_v\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\,\|u-v\|^2_{L^2(0,T;H^1(\Omega))}\,\|h\|^2_{L^2(0,T;H^1(\Omega))}.$$

Therefore, by substituting these estimates in (5.7), we obtain the required result (5.3). Hence the proof. □

Corollary 5.3. *The Fréchet derivative of the control-to-costate operator satisfies the following estimate:*

$$\|\Phi'(u)[v]\|_{\mathcal{Z}} \leq C(\Omega, T, u, m_u)\,\|v\|_{L^2(0,T;H^1(\Omega))}. \tag{5.8}$$

5.2. Second order optimality condition

In the previous subsections, we proved the Fréchet differentiability of both the control-to-state and control-to-costate operators. Using these postulates, we will evaluate the second-order derivative of the cost function and prove the sufficient optimality condition.

Proof of Theorem 2.12. Let $\widetilde{u}$ be a fixed control in the set $\mathcal{U}_{ad}$. Now, since the control set $\mathcal{U}$ is an open set, choose the value of ϵ small enough such that for $u \in \mathcal{U}_{ad}$ with $\|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} < \epsilon$ ensures that for any $\theta \in (0,1)$ the control $\widetilde{u} + \theta(u-\widetilde{u}) \in \mathcal{U}$. Now, by Proposition 4.3, it is clear that the control-to-state operator G is Fréchet differentiable over this set of controls. Indeed, we can show that it is infinitely differentiable (see Prop. 5.1 of [20]). Consequently, the reduced functional $\mathcal{I}$ is infinitely differentiable. Therefore, Taylor's expansion for the functional $\mathcal{I}$ can be given by

$$\mathcal{J}(m,u) = \mathcal{I}(u) = \mathcal{I}(\widetilde{u}) + \mathcal{I}'(\widetilde{u})[u-\widetilde{u}] + \frac{1}{2}\,\mathcal{I}''\big(\widetilde{u}+\theta(u-\widetilde{u})\big)\,[u-\widetilde{u}, u-\widetilde{u}], \tag{5.9}$$

where $\theta \in (0,1)$. From the first order optimality condition (2.7), it is clear that $\mathcal{I}'(\widetilde{u})[u-\widetilde{u}] \geq 0$. From the definition of critical cone, it also clearly appears that $u-\widetilde{u} \in \Pi(\widetilde{u})$, for all $u \in \mathcal{U}_{ad}$.

Since both the control-to-state and control-to-costate operators are Fréchet differentiable, so for any controls $u_1, u_2 \in L^2(0,T;H^1(\Omega))$, the second derivative of the functional $\mathcal{I}$ is given by

$$\begin{aligned}
\mathcal{I}''(\widetilde{u})[u_1,u_2] &= \int_{\Omega_T} u_1\cdot u_2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \nabla u_1\cdot\nabla u_2\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\big(\phi'_{\widetilde{u}}[u_2]\times m_{\widetilde{u}}\big)\cdot u_1\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T}\big(\phi_{\widetilde{u}}\times m'_{\widetilde{u}}[u_2]\big)\cdot u_1\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\big(m'_{\widetilde{u}}[u_2]\times(\phi_{\widetilde{u}}\times m_{\widetilde{u}})\big)\cdot u_1\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T}\big(m_{\widetilde{u}}\times(\phi'_{\widetilde{u}}[u_2]\times m_{\widetilde{u}})\big)\cdot u_1\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T}\big(m_{\widetilde{u}}\times(\phi_{\widetilde{u}}\times m'_{\widetilde{u}}[u_2])\big)\cdot u_1\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Next, we will show that the second derivative of the functional $\mathcal{I}$ satisfies the following continuity property

$$\big|\big(\mathcal{I}''\big(\widetilde{u}+\theta(u-\widetilde{u})\big) - \mathcal{I}''(\widetilde{u})\big)\,[u-\widetilde{u}]^2\big| \leq C\,\|u-\widetilde{u}\|^3_{L^2(0,T;H^1(\Omega))},$$

for $\theta \in (0,1)$ and for any control u with $\|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} < \epsilon$. We will only show the estimation of a few terms, and the rest of the terms can be estimated in a similar way.

Applying Hölder's inequality and then implementing the locally Lipschitz continuity of Fréchet derivative of control-to-costate operator (5.3), continuous dependency of the control-to-state operator (4.11) and estimate (5.8), we obtain

$$\left|\int_{\Omega_T}\Big(\phi'_{\widetilde{u}+\theta(u-\widetilde{u})}[u-\widetilde{u}]\times m_{\widetilde{u}+\theta(u-\widetilde{u})}\Big)\cdot(u-\widetilde{u})\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\big(\phi'_{\widetilde{u}}[u-\widetilde{u}]\times m_{\widetilde{u}}\big)\cdot(u-\widetilde{u})\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$
\begin{aligned}
&\leq \|\phi'_{\widetilde{u}+\theta(u-\widetilde{u})}[u-\widetilde{u}] - \phi'_{\widetilde{u}}[u-\widetilde{u}]\|_{L^2(0,T;L^2(\Omega))}\ \|u-\widetilde{u}\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad\qquad + \|\phi'_{\widetilde{u}}[u-\widetilde{u}]\|_{L^\infty(0,T;L^2(\Omega))}\ \|m_{\widetilde{u}+\theta(u-\widetilde{u})} - m_{\widetilde{u}}\|_{L^2(0,T;H^1(\Omega))}\ \|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\ \|u-\widetilde{u}\|^3_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
$$

Again applying Hölder's inequality, the estimates (4.11), (4.15), (4.21), (4.23) and (5.1), we get

$$
\begin{aligned}
&\left| \int_{\Omega_T} \left[\left(m'_{\widetilde{u}+\theta(u-\widetilde{u})}[u-\widetilde{u}] \times \left(\phi_{\widetilde{u}+\theta(u-\widetilde{u})} \times m_{\widetilde{u}+\theta(u-\widetilde{u})}\right)\right) - \left(m'_{\widetilde{u}}[u-\widetilde{u}] \times \left(\phi_{\widetilde{u}} \times m_{\widetilde{u}}\right)\right)\right] \cdot (u-\widetilde{u})\ \mathrm{d}x\ \mathrm{d}t \right| \\
&\leq C\ \Big[\|m'_{\widetilde{u}+\theta(u-\widetilde{u})}[u-\widetilde{u}] - m'_{\widetilde{u}}[u-\widetilde{u}]\|_{L^2(0,T;H^1(\Omega))}\ \|\phi_{\widetilde{u}+\theta(u-\widetilde{u})} - \phi_{\widetilde{u}}\|_{L^\infty(0,T;L^2(\Omega))}\ \|m_{\widetilde{u}+\theta(u-\widetilde{u})}\|_{L^\infty(0,T;H^1(\Omega))} \\
&\qquad + \|m'_{\widetilde{u}}[u-\widetilde{u}]\|_{L^\infty(0,T;H^2(\Omega))}\ \|\phi_{\widetilde{u}+\theta(u-\widetilde{u})} - \phi_{\widetilde{u}}\|_{L^2(0,T;L^2(\Omega))}\ \|m_{\widetilde{u}+\theta(u-\widetilde{u})}\|_{L^\infty(0,T;H^1(\Omega))} \\
&\qquad + \ \|m'_{\widetilde{u}}[u-\widetilde{u}]\|_{L^\infty(0,T;H^2(\Omega))}\ \|\phi_{\widetilde{u}}\|_{L^\infty(0,T;L^2(\Omega))}\ \|m_{\widetilde{u}+\theta(u-\widetilde{u})} - m_{\widetilde{u}}\|_{L^2(0,T;H^1(\Omega))}\Big]\ \|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\ \|u-\widetilde{u}\|^3_{L^2(0,T;H^1(\Omega))}, \text{whenever } \|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} < \epsilon.
\end{aligned}
$$

Using these estimates and assumption (2.8), we conclude the following:

$$
\begin{aligned}
\mathcal{I}''\big(\widetilde{u}+\theta(u-\widetilde{u})\big)[u-\widetilde{u}]^2 &= \mathcal{I}''\big(\widetilde{u}\big)[u-\widetilde{u}]^2 + \big(\mathcal{I}''\big(\widetilde{u}+\theta(u-\widetilde{u})\big)\ - \mathcal{I}''\big(\widetilde{u}\big)\big)\,[u-\widetilde{u}]^2 \\
&\geq \delta\ \|u-\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))} - C\ \|u-\widetilde{u}\|^3_{L^2(0,T;H^1(\Omega))} \geq \frac{\delta}{2}\ \|u-\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{5.10}
$$

in which we have chosen $\epsilon > 0$ small such that $\|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} \leq \frac{\delta}{2C}$, where $\delta > 0$ is the constant coming from (2.8). Finally, from (5.9)–(5.10), it is evident that $\mathcal{I}(u) \geq \mathcal{I}(\widetilde{u}) + \sigma\ \|u-\widetilde{u}\|^2_{L^2(0,T;H^1(\Omega))}$, with $\sigma = \delta/4$, provided that $\|u-\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} \leq \epsilon$ for a sufficiently small $\epsilon > 0$. This completes the proof. □

6. Global optimal control

A second-order sufficient condition primarily provides localized information and often falls short of enabling a definitive determination regarding whether a given control constitutes a global minimum or not. In this section, we prove a global optimality condition for the problem stated in Section 2.2.

Proof of Theorem 2.13. Let us define $\widehat{m} := m - \widetilde{m}$ and $\widehat{u} := u - \widetilde{u}$. To establish a global optimality of the control $\widetilde{u}$, we assert that $\mathcal{I}(u) - \mathcal{I}(\widetilde{u}) \geq 0$ for all $u \in \widetilde{\mathcal{U}}_{ad}\backslash\{\widetilde{u}\}$. By doing a straightforward calculation, we arrive at the following equality:

$$
\begin{aligned}
\mathcal{I}(u) - \mathcal{I}(\widetilde{u}) &= \frac{1}{2}\int_{\Omega_T} |\widehat{u}|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T} \widehat{u}\cdot\widetilde{u}\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\int_{\Omega_T} |\nabla\widehat{u}|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T} \nabla\widehat{u}\cdot\nabla\widetilde{u}\ \mathrm{d}x\ \mathrm{d}t \\
&+ \frac{1}{2}\int_{\Omega} |\widehat{m}(x,T)|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega} \widehat{m}(x,T)\cdot\big(\widetilde{m}(x,T) - m_\Omega(x)\big)\ \mathrm{d}x \\
&+ \frac{1}{4}\int_{\Omega_T} |\nabla\widehat{m}|^4\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T} \nabla\widehat{m}\cdot(\nabla\widetilde{m} - \nabla m_D)\ |\nabla\widetilde{m} - \nabla m_D|^2\ \mathrm{d}x\ \mathrm{d}t \\
&+ \frac{3}{2}\int_{\Omega_T} |\nabla\widehat{m}|^2\cdot|\nabla\widetilde{m} - \nabla m_D|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T} |\nabla\widehat{m}|^2\ \nabla\widehat{m}\cdot(\nabla\widetilde{m} - \nabla m_D)\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned} \tag{6.1}
$$

Now, applying the estimate $\int_{\Omega_T} |\nabla\widehat{m}|^2\ \nabla\widehat{m}\cdot(\nabla\widetilde{m}-\nabla m_D)\ \mathrm{d}x\ \mathrm{d}t \geq -\frac{3}{2}\int_{\Omega_T}|\nabla\widehat{m}|^2\ |\nabla\widetilde{m}-\nabla m_D|^2\ \mathrm{d}x\ \mathrm{d}t - \frac{1}{6}\int_{\Omega_T}|\nabla\widehat{m}|^4\ \mathrm{d}x\ \mathrm{d}t$ and the variational inequality (2.7) in (6.1), we find

$$\mathcal{I}(u)-\mathcal{I}(\widetilde{u}) \geq \frac{1}{2}\int_{\Omega_T}\big(|\widehat{u}|^2+|\nabla\widehat{u}|^2\big)\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\int_{\Omega}|\widehat{m}(x,T)|^2\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{12}\int_{\Omega_T}|\nabla\widehat{m}|^4\ \mathrm{d}x\ \mathrm{d}t + \mathcal{R}, \tag{6.2}$$

where
$$\begin{aligned}\mathcal{R} := &-\int_{\Omega_T}(\phi\times\widetilde{m})\cdot\widehat{u}\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega}\widehat{m}(x,T)\cdot\big(\widetilde{m}(x,T)-m_{\Omega}(x)\big)\ \mathrm{d}x\\ &-\int_{\Omega_T}\big(\widetilde{m}\times(\phi\times\widetilde{m})\big)\cdot\widehat{u}\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T}\nabla\widehat{m}\cdot\big(\nabla\widetilde{m}-\nabla m_D\big)\ |\nabla\widetilde{m}-\nabla m_D|^2\ \mathrm{d}x\ \mathrm{d}t.\end{aligned}$$

From the definition of $\widehat{u}$ and $\widehat{m}$, we know that $(\widehat{m},\widehat{u})$ satisfies the following system

$$\begin{cases}\mathcal{L}_{\widetilde{u}}\widehat{m} = |\nabla\widehat{m}|^2\ \widehat{m} + |\nabla\widehat{m}|^2\widetilde{m} + 2\big(\nabla\widehat{m}\cdot\nabla\widetilde{m}\big)\widehat{m} + \widehat{m}\times\Delta\widehat{m} + \widehat{m}\times\widehat{u} + \widetilde{m}\times\widehat{u} - \widehat{m}\times(\widehat{m}\times\widehat{u})\\ \qquad\qquad -\widehat{m}\times(\widehat{m}\times\widetilde{u}) - \widehat{m}\times(\widetilde{m}\times\widehat{u}) - \widetilde{m}\times(\widehat{m}\times\widehat{u}) - \widetilde{m}\times(\widetilde{m}\times\widehat{u}) \quad \text{ in } \Omega_T,\\ \frac{\partial\widehat{m}}{\partial\eta} = 0 \quad \text{ in } \partial\Omega_T, \quad \widehat{m}(x,0) = 0 \ \text{ in } \Omega,\end{cases} \tag{6.3}$$

where the operator $\mathcal{L}_{\widetilde{u}}$ is defined in (4.4).

Taking $L^2(0,T;L^2(\Omega))$ inner product of (6.3) with the weak solution ϕ of the adjoint sytem (2.4), and then doing an integration by parts, we derive

$$\begin{aligned}&\int_0^T\big\langle\mathcal{E}_{\widetilde{u}}\phi,\widehat{m}\big\rangle\ \mathrm{d}t + \int_0^T\big(|\nabla\widehat{m}|^2\ \widehat{m},\phi\big)\ \mathrm{d}t + \int_0^T\big(|\nabla\widehat{m}|^2\widetilde{m},\phi\big)\ \mathrm{d}t + 2\int_0^T\big((\nabla\widehat{m}\cdot\nabla\widetilde{m})\widehat{m},\phi\big)\ \mathrm{d}t\\ &\quad+\int_0^T\big(\widehat{m}\times\Delta\widehat{m},\phi\big)\ \mathrm{d}t + \int_0^T\big(\widehat{m}\times\widehat{u},\phi\big)\ \mathrm{d}t - \int_0^T\big(\widehat{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t\\ &\quad-\int_0^T\big(\widehat{m}\times(\widehat{m}\times\widetilde{u}),\phi\big)\ \mathrm{d}t - \int_0^T\big(\widehat{m}\times(\widetilde{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t - \int_0^T\big(\widetilde{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t\\ &\quad=\int_0^T\big(\widehat{m}(T),\phi(T)\big)\ \mathrm{d}t - \int_0^T\big(\widetilde{m}\times\widehat{u},\phi\big)\ \mathrm{d}t + \int_0^T\big(\widetilde{m}\times(\widetilde{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t,\end{aligned} \tag{6.4}$$

where we also used the identity similar to (4.31) for $\int_0^T\big(\widehat{m}_t,\phi\big)\ \mathrm{d}t$. Now, from the weak formulation of the adjoint problem (2.4), we have

$$\begin{aligned}&\int_0^T\big\langle\phi',\widehat{m}\big\rangle\ \mathrm{d}t - \int_0^T\big(\nabla\phi,\nabla\widehat{m}\big)\ \mathrm{d}t + \int_0^T\big(|\nabla\widetilde{m}|^2\ \phi,\widehat{m}\big)\ \mathrm{d}t - 2\int_0^T\big(\nabla\cdot\{(\widetilde{m}\cdot\phi)\nabla\widetilde{m}\},\widehat{m}\big)\ \mathrm{d}t\\ &\quad-\int_0^T\big(\nabla(\phi\times\widetilde{m}),\nabla\widehat{m}\big)\ \mathrm{d}t + \int_0^T\big(\Delta\widetilde{m}\times\phi,\widehat{m}\big)\ \mathrm{d}t - \int_0^T\big(\phi\times\widetilde{u},\widehat{m}\big)\ \mathrm{d}t + \int_0^T\big((\phi\times\widetilde{m})\times\widetilde{u},\widehat{m}\big)\ \mathrm{d}t\\ &\quad+\int_0^T\big(\phi\times(\widetilde{m}\times\widetilde{u}),\widehat{m}\big)\ \mathrm{d}t = -\int_0^T\big(|\nabla\widetilde{m}-\nabla m_D|^2\big(\nabla\widetilde{m}-\nabla m_D\big),\nabla\widehat{m}\big)\ \mathrm{d}t.\end{aligned} \tag{6.5}$$

Subtracting equality (6.5) from (6.4), we get

$$\begin{aligned}&\int_0^T\big(|\nabla\widehat{m}|^2\ \widehat{m},\phi\big)\ \mathrm{d}t + \int_0^T\big(|\nabla\widehat{m}|^2\widetilde{m},\phi\big)\ \mathrm{d}t + 2\int_0^T\big((\nabla\widehat{m}\cdot\nabla\widetilde{m})\ \widehat{m},\phi\big)\ \mathrm{d}t + \int_0^T\big(\widehat{m}\times\Delta\widehat{m},\phi\big)\ \mathrm{d}t\\ &\quad+\int_0^T\big(\widehat{m}\times\widehat{u},\phi\big)\ \mathrm{d}t - \int_0^T\big(\widehat{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t - \int_0^T\big(\widehat{m}\times(\widehat{m}\times\widetilde{u}),\phi\big)\ \mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \big(\widehat{m}\times(\widetilde{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t-\int_0^T \big(\widetilde{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t=\int_0^T \big(\widehat{m}(T),\phi(T)\big)\ \mathrm{d}t\\
&-\int_0^T \big(\widetilde{m}\times\widehat{u},\phi\big)\ \mathrm{d}t+\int_0^T \big(\widetilde{m}\times(\widetilde{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t+\int_0^T \big(|\nabla\widetilde{m}-\nabla m_D|^2(\nabla\widetilde{m}-\nabla m_D),\nabla\widehat{m}\big)\ \mathrm{d}t. \qquad (6.6)
\end{aligned}
$$

Therefore, as a result of equality (6.6) and $a\cdot(b\times c)=-(b\times a)\cdot c$, the value of $\mathcal{R}$ from (6.2) becomes

$$
\begin{aligned}
\mathcal{R}=&\int_0^T \big(|\nabla\widehat{m}|^2\ \widehat{m},\phi\big)\ \mathrm{d}t+\int_0^T \big(|\nabla\widehat{m}|^2\widetilde{m},\phi\big)\ \mathrm{d}t+2\int_0^T \big(\big(\nabla\widehat{m}\cdot\nabla\widetilde{m}\big)\ \mathrm{d}t\widehat{m},\phi\big)\ \mathrm{d}t\\
&+\int_0^T \big(\widehat{m}\times\Delta\widehat{m},\phi\big)\ \mathrm{d}t+\int_0^T \big(\widehat{m}\times\widehat{u},\phi\big)\ \mathrm{d}t-\int_0^T \big(\widehat{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t\\
&-\int_0^T \big(\widehat{m}\times(\widehat{m}\times\widetilde{u}),\phi\big)\ \mathrm{d}t-\int_0^T \big(\widehat{m}\times(\widetilde{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t-\int_0^T \big(\widetilde{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t. \qquad (6.7)
\end{aligned}
$$

Now, let us estimate the bounds for each term on the right-hand side of $\mathcal{R}$ individually. For the first three terms, using the fact that $|\widehat{m}|=|m-\widetilde{m}|\le|m|+|\widetilde{m}|=2$ and applying Hölder's inequality and the continuous embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$, followed by the Lipschitz continuity of control-to-state operator given by estimate (2.10), we obtain

$$
\begin{aligned}
&\int_0^T \big(|\nabla\widehat{m}|^2\ \widehat{m},\phi\big)\ \mathrm{d}t\le 2\int_0^T \|\nabla\widehat{m}(t)\|^2_{L^4(\Omega)}\ \|\phi(t)\|_{L^2(\Omega)}\ \mathrm{d}t\le C\int_0^T \|\nabla\widehat{m}(t)\|^2_{H^1(\Omega)}\ \|\phi(t)\|_{L^2(\Omega)}\ \mathrm{d}t\\
&\qquad\qquad\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))},\\
&\int_0^T \big(|\nabla\widehat{m}|^2\ \widetilde{m},\phi\big)\ \mathrm{d}t\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))},
\end{aligned}
$$

and

$$
\begin{aligned}
2\int_0^T \big(\big(\nabla\widehat{m}\cdot\nabla\widetilde{m}\big)\widehat{m},\phi\big)\ \mathrm{d}t&\le 2\int_0^T \|\nabla\widehat{m}(t)\|_{L^4(\Omega)}\ \|\nabla\widetilde{m}(t)\|_{L^8(\Omega)}\ \|\widehat{m}(t)\|_{L^8(\Omega)}\ \|\phi(t)\|_{L^2(\Omega)}\ \mathrm{d}t\\
&\le C\ \|\widetilde{m}\|_{L^2(0,T;H^2(\Omega))}\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{m}\|^2_{L^\infty(0,T;H^2(\Omega))}\\
&\le C\ \|\widetilde{m}\|_{L^2(0,T;H^2(\Omega))}\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
$$

The terms $\big(\widehat{m}\times\Delta\widehat{m},\phi\big)$ and $\big(\widehat{m}\times\widehat{u},\phi\big)$ can be estimated by the same upper bound. Further, applying the Hölder's inequality, the embedding $H^1(\Omega)\hookrightarrow L^p(\Omega)$ for $p\in[1,\infty)$ and estimate (2.10) for the subsequent terms involving the control, we derive

$$
\begin{aligned}
&\int_0^T \big(\widehat{m}\times(\widehat{m}\times\widehat{u}),\phi\big)\ \mathrm{d}t\le 2\int_0^T \|\widehat{m}(t)\|_{L^4(\Omega)}\ \|\widehat{u}(t)\|_{L^4(\Omega)}\ \|\phi(t)\|_{L^2(\Omega)}\ \mathrm{d}t\\
&\qquad\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{m}\|_{L^\infty(0,T;H^1(\Omega))}\ \|\widehat{u}\|_{L^2(0,T;H^1(\Omega))}\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \big(\widehat{m}\times(\widehat{m}\times\widetilde{u}),\phi\big)\ \mathrm{d}t\le 2\int_0^T \|\widehat{m}(t)\|^2_{L^8(\Omega)}\ \|\widetilde{u}(t)\|_{L^4(\Omega)}\ \|\phi(t)\|_{L^2(\Omega)}\ \mathrm{d}t\\
&\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widetilde{u}\|_{L^2(0,T;H^1(\Omega))}\ \|\widehat{m}\|^2_{L^\infty(0,T;H^1(\Omega))}\le C\ \|\phi\|_{L^2(0,T;L^2(\Omega))}\ \|\widetilde{u}\|_{L^2(0,T;H^1(\Omega))}\ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
$$

The rest of the terms in (6.7) can be estimated in a similar way. Now, combining all the above estimates, we find the following bound for $\mathcal{R}$:

$$|\mathcal{R}| \leq C(\Omega, T) \ \left\{1 + \|\widetilde{m}\|_{L^\infty(0,T;H^2(\Omega))} + \|\widetilde{u}\|_{L^2(0,T;H^2(\Omega))}\right\} \ \|\phi\|_{L^2(0,T;L^2(\Omega))} \ \|\widehat{u}\|^2_{L^2(0,T;H^1(\Omega))}.$$

Substituting this bound in estimate (6.2), we find our required result. The proof is thus completed. □

7. Optimal control *via* fixed magnetic field coils

Proof of Theorem 2.16. The proof follows the arguments given in [27] for general semi-linear parabolic control systems. Let us prove this theorem by the method of contradiction. Suppose that $\widetilde{U}$ does not satisfy the growth condition given by inequality (2.14). Then there exists a sequence of control functions $\{U_k\}_{k=1}^{\infty}$ in $\mathbb{U}_{ad}$ such that $U_k \to \widetilde{U}$ in $L^2(0,T;\mathbb{R}^n)$ and the following inequality will hold:

$$\widehat{\mathcal{J}}(U_k) < \widehat{\mathcal{J}}(\widetilde{U}) + \frac{1}{k} \ \|U_k - \widetilde{U}\|^2_{L^2(0,T;\mathbb{R}^n)} \quad \forall \ k. \tag{7.1}$$

Now let us define $\rho_k = \|U_k - \widetilde{U}\|_{L^2(0,T;\mathbb{R}^n)}$ and $h_k = \frac{1}{\rho_k} \ (U_k - \widetilde{U})$. Since $\|h_k\|_{L^2(0,T;\mathbb{R}^n)} = 1$, we can extract a sub-sequence, for simplicity again denoted as $\{h_k\}$, such that $h_k \rightharpoonup h$ weakly in $L^2(0,T;\mathbb{R}^n)$. First we will show $\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h] = 0$. The mean value theorem gives the equality

$$\widehat{\mathcal{J}}(U_k) = \widehat{\mathcal{J}}(\widetilde{U}) + \rho_k \frac{\partial \widehat{\mathcal{J}}}{\partial U}(v_k)[h_k],$$

where v_k is a point between $\widetilde{U}$ and U_k. Substituting this equality in estimate (7.1), we find

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(v_k)[h_k] < \frac{1}{k\rho_k}\|U_k - \widetilde{U}\|^2_{L^2(0,T;\mathbb{R}^n)} = \frac{1}{k}\|U_k - \widetilde{U}\|_{L^2(0,T;\mathbb{R}^n)}. \tag{7.2}$$

Now, the first-order variation of the functional $\widehat{\mathcal{J}}$ is given as

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(v_k)[h_k] = \int_0^T v_k \cdot h_k \ \mathrm{d}t + \int_{\Omega_T} \left(\phi_{v_k} \times m_{v_k}\right) \cdot \zeta(h_k) \ \mathrm{d}x \ \mathrm{d}t + \int_{\Omega_T} \left(m_{v_k} \times (\phi_{v_k} \times m_{v_k})\right) \cdot \zeta(h_k) \ \mathrm{d}x \ \mathrm{d}t. \tag{7.3}$$

Since $U_k \to \widetilde{U}$ strongly in $L^2(0,T;\mathbb{R}^n)$ and v_k is a point between $\widetilde{U}$ and U_k, from estimates (4.11) and (5.1), it is evident that $m_{v_k} \to m_{\widetilde{U}}$ strongly in $\mathcal{M}$ and $\phi_{v_k} \to \phi_{\widetilde{U}}$ strongly in $\mathcal{Z}$. Moreover the weak convergence of $h_k \rightharpoonup h$ in $L^2(0,T;\mathbb{R}^n)$ implies that $\zeta(h_k) \rightharpoonup \zeta(h)$ weakly in $L^2(0,T;L^2(\Omega))$. Thus, the following convergences hold:

(i) $\displaystyle\int_0^T v_k \cdot h_k \ \mathrm{d}t \to \int_0^T \widetilde{U} \cdot h \ \mathrm{d}t$,

(ii) $\displaystyle\int_{\Omega_T} \left(\phi_{v_k} \times m_{v_k}\right) \cdot \zeta(h_k) \ \mathrm{d}x \ \mathrm{d}t \to \int_{\Omega_T} \left(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\right) \cdot \zeta(h) \ \mathrm{d}x \ \mathrm{d}t$,

(iii) $\displaystyle\int_{\Omega_T} \left(m_{v_k} \times (\phi_{v_k} \times m_{v_k})\right) \cdot \zeta(h_k) \ \mathrm{d}x \ \mathrm{d}t \to \int_{\Omega_T} \left(m_{\widetilde{U}} \times (\phi_{\widetilde{U}} \times m_{\widetilde{U}})\right) \cdot \zeta(h) \ \mathrm{d}x \ \mathrm{d}t.$

Let us first prove the convergence of (ii). By applying Hölder's inequality and the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned}
&\Big| \int_{\Omega_T} \big(\phi_{v_k} \times m_{v_k}\big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t - \int_{\Omega_T} \big(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\big) \cdot \zeta(h) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\le \int_{\Omega_T} \big|\phi_{v_k} \times m_{v_k} - \phi_{\widetilde{U}} \times m_{\widetilde{U}}\big| \, \big|\zeta(h_k)\big| \, \mathrm{d}x \, \mathrm{d}t + \Big| \int_{\Omega_T} \big(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\big) \cdot \big(\zeta(h_k) - \zeta(h)\big) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\le \|\phi_{v_k} - \phi_{\widetilde{U}}\|_{L^2(0,T;L^2(\Omega))} \, \|m_{v_k}\|_{L^\infty(0,T;H^1(\Omega))} \, \|\zeta(h_k)\|_{L^2(0,T;H^1(\Omega))} \\
&\quad + \|\phi_{\widetilde{U}}\|_{L^2(0,T;H^1(\Omega))} \, \|m_{v_k} - m_{\widetilde{U}}\|_{L^\infty(0,T;H^1(\Omega))} \, \|\zeta(h_k)\|_{L^2(0,T;L^2(\Omega))} + \Big| \int_{\Omega_T} \big(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\big) \cdot \big(\zeta(h_k) - \zeta(h)\big) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\quad \to 0 \text{ as } k \to \infty.
\end{aligned}$$

The convergences (i) and (iii) can be shown analogously. Therefore, as a result of the above convergences, taking $k \to \infty$ in estimate (7.2), we can obtain

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h] = \lim_{k \to \infty} \frac{\partial \widehat{\mathcal{J}}}{\partial U}(v_k)[h_k] \le 0. \tag{7.4}$$

Also, as U_k is an admissible control, from the first-order variational inequality (2.7), it is clear that

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] = \frac{1}{\rho_k} \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[U_k - \widetilde{U}] \ge 0.$$

Therefore, taking $k \to \infty$ in the above inequality *via* (7.3) and using the weak convergence of $\{h_k\}$, we find that

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h] = \lim_{k \to \infty} \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] \ge 0. \tag{7.5}$$

Combining (7.4) and (7.5), we conclude that $\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h] = 0$.

Next, we will show that $h \in \Lambda(\widetilde{U})$. The set of all controls in $L^2(0,T;\mathbb{R}^n)$ that are non-negative if $\widetilde{U}(t) = a(t)$ and non-positive if $\widetilde{U}(t) = b(t)$, say $\mathcal{S}$ is convex and closed, whence also weakly closed. Since $U_k \in \mathbb{U}_{ad}$, clearly h_k belongs to the set $\mathcal{S}$ and so does h as a consequence of the weakly closedness of the set $\mathcal{S}$.

Define $\Upsilon_{\widetilde{U}} := \widetilde{U} + \int_\Omega \big(\phi \times m_{\widetilde{U}} + m_{\widetilde{U}} \times (\phi \times m_{\widetilde{U}})\big) \cdot B \, \mathrm{d}x$. Then the first order necessary optimality condition (2.13) can be written as

$$\int_0^T \Upsilon_{\widetilde{U}}(t) \cdot (U(t) - \widetilde{U}(t)) \, \mathrm{d}t \ge 0, \qquad \forall\, U \in \mathbb{U}_{ad}.$$

From this, we can also deduce the following condition

$$\begin{cases} \widetilde{U}(t) = a(t) & \text{if } \Upsilon_{\widetilde{U}} > 0, \\ \widetilde{U}(t) = b(t) & \text{if } \Upsilon_{\widetilde{U}} < 0. \end{cases} \tag{7.6}$$

As a result of (7.6) and $h \in \mathcal{S}$, we can find that

$$\int_0^T |\Upsilon_{\widetilde{U}}(t) \, h(t)| \, \mathrm{d}t = \int_0^T \Upsilon_{\widetilde{U}}(t) \, h(t) \, \mathrm{d}t = \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h] = 0.$$

Hence $h(t) = 0$ if $\varUpsilon_{\widetilde{U}} \neq 0$, which concludes that $h \in \Lambda(\widetilde{U})$. Next, we claim that $h = 0$. We will prove this by showing $\widehat{\mathcal{J}}''(\widetilde{U})[h, h] \leq 0$. Invoking the second order Taylor's series for $\widehat{\mathcal{J}}(U_k)$, we write the following expression:

$$\widehat{\mathcal{J}}(U_k) = \widehat{\mathcal{J}}(\widetilde{U}) + \rho_k \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] + \frac{\rho_k^2}{2} \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(w_k)[h_k, h_k],$$

where w_k is an intermediate point between $\widetilde{U}$ and U_k. As a result of this, we can write the following equality

$$\rho_k \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] + \frac{\rho_k^2}{2} \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k, h_k] = \widehat{\mathcal{J}}(U_k) - \widehat{\mathcal{J}}(\widetilde{U}) + \frac{\rho_k^2}{2} \left(\frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U}) - \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(w_k) \right) [h_k, h_k]. \tag{7.7}$$

Now, substituting $\|U_k - \widetilde{U}\|_{L^2(0,T;\mathbb{R}^n)} = \rho_k$ in estimate (7.1), we obtain

$$\widehat{\mathcal{J}}(U_k) - \widehat{\mathcal{J}}(\widetilde{U}) < \frac{\rho_k^2}{k} \qquad \text{for each } k. \tag{7.8}$$

Since $\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] \geq 0$ and $\rho_k \to 0$, we can write $\rho_k \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] \geq \rho_k^2 \frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k]$, for $k \geq k_0$, where k_0 is sufficiently large. Substituting this inequality and (7.8) in equation (7.7), and dividing by ρ_k^2, we obtain

$$\frac{\partial \widehat{\mathcal{J}}}{\partial U}(\widetilde{U})[h_k] + \frac{1}{2} \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k, h_k] < \frac{1}{k} + \frac{1}{2} \left(\frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U}) - \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(w_k) \right) [h_k, h_k]. \tag{7.9}$$

The second-order variation of the functional $\widehat{\mathcal{J}}$ is given by

$$\begin{aligned}
\frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k, h_k] &= \int_0^T h_k^2 \, dt + \int_{\Omega_T} \left(\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times m_{\widetilde{U}} \right) \cdot \zeta(h_k) \, dx \, dt \\
&\quad + \int_{\Omega_T} \left(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)] \right) \cdot \zeta(h_k) \, dx \, dt + \int_{\Omega_T} \left(m'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times \left(\phi_{\widetilde{U}} \times m_{\widetilde{U}} \right) \right) \cdot \zeta(h_k) \, dx \, dt \\
&\quad + \int_{\Omega_T} \left(m_{\widetilde{U}} \times \left(\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times m_{\widetilde{U}} \right) \right) \cdot \zeta(h_k) \, dx \, dt + \int_{\Omega_T} \left(m_{\widetilde{U}} \times \left(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)] \right) \right) \cdot \zeta(h_k) \, dx \, dt.
\end{aligned} \tag{7.10}$$

We want to show that the second term on the right-hand side of inequality (7.9) tends to 0 as $k \to \infty$. We will verify this convergence for the second term of (7.10). By applying (4.11) and (5.3), we derive

$$\begin{aligned}
&\left| \int_{\Omega_T} \left(\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times m_{\widetilde{U}} \right) \cdot \zeta(h_k) \, dx \, dt - \int_{\Omega_T} \left(\phi'_{\zeta(w_k)}[\zeta(h_k)] \times m_{w_k} \right) \cdot \zeta(h_k) \, dx \, dt \right| \\
&\quad \leq \int_{\Omega_T} \left(\left| \phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] - \phi'_{\zeta(w_k)}[\zeta(h_k)] \right| |m_{\widetilde{U}}| + \left| \phi'_{\zeta(w_k)}[\zeta(h_k)] \right| |m_{\widetilde{U}} - m_{w_k}| \right) |\zeta(h_k)| \, dx \, dt \\
&\quad \leq C \left\| \phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] - \phi'_{\zeta(w_k)}[\zeta(h_k)] \right\|_{L^2(0,T;L^2(\Omega))} \|\zeta(h_k)\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C \left\| \phi'_{\zeta(w_k)}[\zeta(h_k)] \right\|_{L^\infty(0,T;L^2(\Omega))} \|m_{\widetilde{U}} - m_{w_k}\|_{L^2(0,T;H^1(\Omega))} \|\zeta(h_k)\|_{L^2(0,T;H^1(\Omega))} \\
&\quad \leq C \left(\|\zeta(\widetilde{U}) - \zeta(w_k)\|_{L^2(0,T;H^1(\Omega))} + \|\zeta(w_k)\|_{L^2(0,T;H^1(\Omega))} \|m_{\widetilde{U}} - m_{w_k}\|_{L^2(0,T;H^1(\Omega))} \right) \|\zeta(h_k)\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad \leq C \, \|\widetilde{U} - w_k\|_{L^2(0,T;\mathbb{R}^n)} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Proceeding similarly for the rest of the terms in (7.10), we obtain that the right-hand side of (7.9) converges to 0. Now, we will look at the convergence of the second term on the left-hand side of (7.9). Since $h_k \rightharpoonup h$ weakly in $L^2(0, T; \mathbb{R}^n)$, *via* (7.10) the first term satisfies the weak sequential lower semi-continuity $\int_0^T h^2 \, dt \leq$

$\liminf_{k\to\infty} \int_0^T h_k^2 \, \mathrm{d}t$. Moreover, as $\zeta(h_k) \rightharpoonup \zeta(h)$ weakly in $L^2(0,T;L^2(\Omega))$, using the uniform boundedness, compact embeddings and the results from Propositions 4.3 and 5.2, we can find that $m'_{\zeta(\widetilde{U})}[\zeta(h_k)] \to m'_{\zeta(\widetilde{U})}[\zeta(h)]$ strongly in $L^2(0,T;H^2(\Omega))$ and $\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \to \phi'_{\zeta(\widetilde{U})}[\zeta(h)]$ strongly in $L^2(0,T;L^2(\Omega))$. Therefore, applying these convergences for the third term of (7.10), we have

$$\begin{aligned}
&\left| \int_{\Omega_T} \Big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)]\Big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t - \int_{\Omega_T} \Big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\Big) \cdot \zeta(h) \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad \le C \int_{\Omega_T} \left|\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)] - \phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\right| \, |\zeta(h_k)| \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad\qquad + \left| \int_{\Omega_T} \Big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\Big) \cdot \Big(\zeta(h_k) - \zeta(h)\Big) \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\quad \le C \, \|\phi_{\widetilde{U}}\|_{L^\infty(0,T;L^2(\Omega))} \, \left\|m'_{\zeta(\widetilde{U})}[\zeta(h_k)] - m'_{\zeta(\widetilde{U})}[\zeta(h)]\right\|_{L^2(0,T;H^1(\Omega))} \, \|\zeta(h_k)\|_{L^2(0,T;H^1(\Omega))} \\
&\qquad\qquad + \left| \int_{\Omega_T} \Big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h)]\Big) \cdot \Big(\zeta(h_k) - \zeta(h)\Big) \, \mathrm{d}x \, \mathrm{d}t \right| \to 0 \text{ as } k \to \infty.
\end{aligned}$$

By proceeding in a similar manner for the rest of the terms of (7.10), we obtain that $\dfrac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})][h,h] \le \liminf\limits_{k\to\infty} \dfrac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k,h_k]$. Finally, taking $k \to \infty$ in (7.9), we obtain that $\dfrac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h,h] \le 0$. This concludes our claim that $h = 0$.

Next, we will show that $h_k \to 0$ strongly in $L^2(0,T;\mathbb{R}^n)$. Firstly, *via* (7.10) writing the expression for $\int_0^T h_k^2 \mathrm{d}t$, and then use the inequality $\lim\limits_{k\to\infty} \dfrac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k,h_k] \le 0$ from estimate (7.9). Since $h = 0$, we have the strong convergence of $\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)]$ and $m'_{\zeta(\widetilde{U})}[\zeta(h_k)]$ to 0. By implementing these results, we find that

$$\begin{aligned}
0 < \lim_{k\to\infty} \int_0^T h_k^2 \, \mathrm{d}t &= \lim_{k\to\infty} \Bigg\{ \frac{\partial^2 \widehat{\mathcal{J}}}{\partial U^2}(\widetilde{U})[h_k,h_k] - \int_{\Omega_T} \Big(\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times m_{\widetilde{U}}\Big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\Omega_T} \Big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)]\Big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t - \int_{\Omega_T} \big(m'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times \big(\phi_{\widetilde{U}} \times m_{\widetilde{U}}\big)\big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\Omega_T} \big(m_{\widetilde{U}} \times \big(\phi'_{\zeta(\widetilde{U})}[\zeta(h_k)] \times m_{\widetilde{U}}\big)\big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t - \int_{\Omega_T} \big(m_{\widetilde{U}} \times \big(\phi_{\widetilde{U}} \times m'_{\zeta(\widetilde{U})}[\zeta(h_k)]\big)\big) \cdot \zeta(h_k) \, \mathrm{d}x \, \mathrm{d}t \Bigg\} \le 0
\end{aligned}$$

which is a contradiction as $\|h_k\|_{L^2(0,T;\mathbb{R}^n)} = 1$ for each k. Hence the proof. □

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Data availability statement

All data supporting the findings of this study are included in the article.

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