

SECOND-ORDER OPTIMALITY CONDITIONS FOR THE SPARSE OPTIMAL CONTROL OF NONVISCIOUS CAHN–HILLIARD SYSTEMS

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Abstract. In this paper, we study the optimal control of an initial-boundary value problem for the classical nonviscous Cahn–Hilliard system with zero Neumann boundary conditions. Phase field systems of this type govern the evolution of diffusive phase transition processes with conserved order parameter. For such systems, optimal control problems have been studied in the past. We focus here on the situation when the cost functional of the optimal control problem contains a sparsity-enhancing nondifferentiable term like the L^1 -norm. For such cases, we establish first-order necessary and second-order sufficient optimality conditions for locally optimal controls, where in the approach to second-order sufficient conditions we employ a technique introduced by Casas, Ryll and Tröltzsch in the paper [*SIAM J. Control Optim.* **53** (2015) 2168–2202]. The main novelty of this paper is that this method, which has recently been successfully applied to systems of viscous Cahn–Hilliard type, can be adapted also to the classical nonviscous case. Since in the case without viscosity the solutions to the state and adjoint systems turn out to be considerably less regular than in the viscous case, numerous additional technical difficulties have to be overcome, and additional conditions have to be imposed. In particular, we have to restrict ourselves to the case when the nonlinearity driving the phase separation is regular, while in the presence of a viscosity term also nonlinearities of logarithmic type turn could be admitted. In addition, the implicit function theorem, which was employed to establish the needed differentiability properties of the control-to-state operator in the viscous case, does not apply in our situation and has to be substituted by other arguments.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ denote some bounded and connected open set with smooth boundary $\Gamma = \partial\Omega$ (a compact hypersurface of class C^2), unit outward normal \mathbf{n} , and associated outward normal derivative $\partial_{\mathbf{n}}$. Moreover, let $T > 0$ denote some final time, and set

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t), \quad \text{for } t \in (0, T], \quad \text{and} \quad Q := Q_T, \quad \Sigma := \Sigma_T.$$

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We then study the following optimal control problem:

(CP) Minimize the cost functional

$$\begin{aligned} \mathcal{J}(\varphi, u) &:= \frac{b_1}{2} \iint_Q |\varphi - \varphi_Q|^2 + \frac{b_2}{2} \int_\Omega |\varphi(T) - \varphi_\Omega|^2 + \frac{b_3}{2} \iint_Q |u|^2 + \kappa G(u), \\ &=: J(\varphi, u) + \kappa G(u) \end{aligned} \quad (1.1)$$

subject to the initial-boundary value system

$$\langle \partial_t \varphi, v \rangle + \int_\Omega \nabla \mu \cdot \nabla v = 0 \quad \text{for all } v \in H^1(\Omega) \text{ and a.e. in } (0, T), \quad (1.2)$$

$$-\Delta \varphi + f'(\varphi) = \mu + w \quad \text{a.e. in } Q, \quad (1.3)$$

$$\gamma \partial_t w + w = u \quad \text{a.e. in } Q, \quad (1.4)$$

$$\partial_{\mathbf{n}} \varphi = 0 \quad \text{a.e. on } \Sigma, \quad (1.5)$$

$$\varphi(0) = \varphi_0, \quad w(0) = w_0 \quad \text{a.e. in } \Omega, \quad (1.6)$$

and to the control constraint

$$\mathcal{U}_{\text{ad}} = \{u \in \mathcal{U} : \underline{u}(x, t) \leq u(x, t) \leq \bar{u}(x, t) \text{ for a.a. } (x, t) \text{ in } Q\}, \quad (1.7)$$

where the control space is specified by

$$\mathcal{U} = L^2(0, T; L^2(\Omega)). \quad (1.8)$$

The given bounds $\underline{u}, \bar{u} \in L^\infty(Q)$ satisfy $\underline{u} \leq \bar{u}$ almost everywhere in Q . Moreover, the targets $\varphi_Q, \varphi_\Omega$ are given functions, $b_1 \geq 0, b_2 \geq 0, b_3 > 0$ are constants, and $\kappa > 0$ is a constant representing the sparsity parameter. The sparsity-enhancing functional $G : L^2(Q) \rightarrow \mathbb{R}$ is nonnegative, continuous and convex. Typically, G has a nondifferentiable form like, *e.g.*,

$$G(u) = \|u\|_{L^1(Q)} = \iint_Q |u|. \quad (1.9)$$

The state equations (1.2)–(1.3) constitute a classical nonviscous Cahn–Hilliard system in which a number of physical constants have been normalized to unity. Notice that (1.2) is just the weak form of the partial differential equation $\partial_t \varphi - \Delta \mu = 0$, where, throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^1(\Omega)$ and its dual $H^1(\Omega)^*$: actually, in (1.2) also the boundary condition $\partial_{\mathbf{n}} \mu = 0$ is taken into account. The state variables φ and μ are monitored through the input variable w , which is in turn determined by the action of the control u via the linear control equation (1.4). Equation (1.4) models how the “forcing” w is generated by the external control u . We remark that (1.5) could be replaced by much more general partial differential equations modeling the relation between an L^2 -control u and a forcing w . In the system (1.2)–(1.6), φ plays the role of an *order parameter*, while μ is the associated *chemical potential*. Moreover, γ is a given (uniformly) positive function defined on Ω , and φ_0 and w_0 are given initial data. The nonlinearity f represents a smooth double-well potential whose derivative defines the local part of the thermodynamic force driving the evolution of the system. A typical form of f is

$$f(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2. \quad (1.10)$$

Starting with the seminal paper [1], there exists an abundant literature on the well-posedness and asymptotic behavior of the nonviscous Cahn–Hilliard system with zero Neumann and with dynamic boundary conditions that cannot be cited here. In spite of this large amount of related literature, we have chosen to provide a detailed well-posedness analysis of the state system (1.2)–(1.6), both for the readers’ convenience and the fact that the system (1.2)–(1.6) was apparently not studied before in this particular form in which the control contributes to the chemical potential through the quantity w .

There are also contributions devoted to the optimal control of Cahn–Hilliard type systems in various contexts. Without claiming to be exhaustive and complete, we mention now some related papers. First, let us refer to [2–6] and, in the framework of diffusive models of tumor growth, to [7–14]. Problems with dynamical boundary conditions were studied in [8, 15–22], and convective Cahn–Hilliard systems have been the subject of [19, 20, 22–25]. In addition, quite a number of works have been dedicated to the study of cases in which the Cahn–Hilliard system is coupled to other systems; in this connection, we quote Cahn–Hilliard–Navier–Stokes models (see [26–31]) and the Cahn–Hilliard–Oono (see [32, 33]), Cahn–Hilliard–Darcy (see [34, 35]), Cahn–Hilliard–Brinkman (see [12]) and Cahn–Hilliard with curvature effects (see [36]) systems.

None of the papers cited above is concerned with the aspect of *sparsity*, *i.e.*, the possibility that any locally optimal control may vanish in subregions of positive measure of the space-time cylinder Q that are controlled by the sparsity parameter κ . In this paper, we focus on sparsity, where we restrict ourselves to the case of *full sparsity* which is connected to the $L^1(Q)$ -norm functional G introduced in (1.9). Other types of sparsity such as *directional sparsity with respect to time* and *directional sparsity with respect to space* (see, *e.g.*, [37]) are not treated in this paper.

Sparsity in the optimal control theory for partial differential equations has become an active field of research. Beginning with [38], many results on sparse optimal controls for PDEs were published. We mention only very few of them with closer relation to our paper, in particular [39–41] on directional sparsity and [42] on a general theorem for second-order conditions. Moreover, we refer to some new trends in the investigation of sparsity, namely, infinite horizon sparse optimal control (see, *e.g.*, [43, 44]) and fractional order optimal control (*cf.* [45, 46]).

The abovementioned papers concentrated on the first-order optimality conditions for sparse optimal controls of single elliptic and parabolic equations. In [47, 48], first- and second-order optimality conditions have been discussed in the context of sparsity for the (semilinear) system of FitzHugh–Nagumo equations. More recently, sparsity of optimal controls for reaction-diffusion systems of Cahn–Hilliard type have been addressed in [37, 49, 50]. Moreover, we refer to the measure control of the Navier–Stokes system studied in [51]. Second-order sufficient optimality conditions for sparse controls for the viscous Cahn–Hilliard system were recently addressed in [52].

Second-order sufficient optimality conditions are based on a condition of coercivity that is required to hold for the smooth part J of \mathcal{J} in a certain *critical cone*. The nonsmooth part G contributes to sufficiency by its convexity. For the strength of sufficient conditions it is crucial that the critical cone be as small as possible. In their paper [48], Casas–Ryll–Tröltzsch devised a technique by means of which a very advantageous (*i.e.*, small) critical cone can be chosen. This method was originally introduced for a class of semilinear second-order parabolic problems with smooth nonlinearities. In the recent papers [52–54] it has been demonstrated that it can be adapted correspondingly to the sparse optimal control of Allen–Cahn systems with dynamic boundary conditions [53], to a large class of systems modeling tumor growth [54], and to the viscous Cahn–Hilliard system [52], where in all of these papers logarithmic nonlinearities could be admitted.

It is the main aim and novelty of this work to show that also the classical nonviscous Cahn–Hilliard system can be treated accordingly. This is by no means obvious, since in the nonviscous case the solutions to the state and the adjoint state systems enjoy less regularity than in the viscous one. And indeed, it turns out that the needed analytic effort is considerably larger than in the viscous case, while the optimization part of the argument changes only little.

In particular, one of the key elements of the technique is to show that the reduced cost functional associated with the smooth part J of \mathcal{J} is twice continuously differentiable, which in turn requires that the control-to-state operator is twice continuously Fréchet differentiable between $\mathcal{U} = L^2(0, T; L^2(\Omega))$ and a suitable Banach space.

For this to be the case, it seems mandatory that the phase variable φ satisfies a so-called *uniform separation property*. While such a condition can in the viscous case be shown also for logarithmic potentials, a corresponding result seems in the nonviscous case to be available only for regular potentials. We therefore have to restrict our analysis to such nonlinearities in this paper, thereby excluding logarithmic potentials.

Another difficulty arises from the fact that the first component of the solution triple to the linearized state system (see (3.3)–(3.7)) is not known to be bounded. This entails that the technique employed in [52], which is based on the good differentiability properties of Nemytskii operators on L^∞ spaces and makes use of the implicit function theorem, does not carry over to the nonviscous case. All this has the consequence that the proof of the needed twice continuous Fréchet differentiability of the control-to-state operator is technically quite challenging and sometimes a bit tedious. Nevertheless, it turns out that all difficulties can be overcome.

The paper is organized as follows. In the following section, we formulate the general assumptions and study the state system, proving the existence of a unique solution. We also show the local Lipschitz continuity of the control-to-state operator. In Section 3, we then prove that the control-to-state operator is twice continuously Fréchet differentiable between appropriate Banach spaces. The proofs in this section require the main analytical effort of this paper. In Section 4, we investigate the control problem **(CP)** with sparsity. Besides analyzing the associated adjoint problem, we derive the first-order necessary optimality conditions. The final section then brings the derivation of the announced second-order sufficient optimality conditions for controls that are locally optimal in the sense of $L^2(Q)$.

Prior to this, let us fix some notation. For any Banach space X , we denote by $\|\cdot\|_X$ and X^* the corresponding norm and its dual space, respectively. For two Banach spaces X and Y that are both continuously embedded in some topological vector space Z , we consider the linear space $X \cap Y$ that becomes a Banach space if equipped with its natural norm $\|v\|_{X \cap Y} := \|v\|_X + \|v\|_Y$ for $v \in X \cap Y$. The standard Lebesgue and Sobolev spaces defined on Ω are, for $1 \leq p \leq \infty$ and $m \in \mathbb{N} \cup \{0\}$, denoted by $L^p(\Omega)$ and $W^{m,p}(\Omega)$, respectively. If $p = 2$, we also use the usual notation $H^m(\Omega) := W^{m,2}(\Omega)$. Moreover, for convenience, we denote the norm of $L^p(\Omega)$ by $\|\cdot\|_p$ for $1 \leq p \leq \infty$, and we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_{\mathbf{n}}v = 0 \text{ on } \Gamma\},$$

where we denote by (\cdot, \cdot) the natural inner product in H . As usual, H is identified with a subspace of the dual space V^* according to the identity

$$\langle u, v \rangle = (u, v) \quad \text{for every } u \in H \text{ and } v \in V.$$

We then have the Hilbert triple $V \subset H \subset V^*$ with dense and compact embeddings.

We close this section by introducing a convention concerning the constants used in estimates within this paper: we denote by C any positive constant that depends only on the given data occurring in the state system and in the cost functional, as well as on a constant that bounds the $L^2(Q)$ -norms of the elements of \mathcal{U}_{ad} . The actual value of such generic constants C may change from formula to formula or even within formulas. Finally, the notation C_δ indicates a positive constant that additionally depends on the quantity δ .

2. PROPERTIES OF THE STATE SYSTEM

2.1. Notation and assumptions

In this section, we formulate the general assumptions for the data of the state system (1.2)–(1.6), and we introduce some known tools for later use. Throughout this paper, we generally assume:

- (A1)** $f = f_1 + f_2$, where $f_1 \in C^5(\mathbb{R})$ is a convex and nonnegative function with $f_1(0) = 0$ and $f_2 \in C^5(\mathbb{R})$ has a Lipschitz continuous first derivative f_2' on \mathbb{R} .
- (A2)** $\gamma > 0$ is a constant. Moreover, $w_0 \in V$ and $\varphi_0 \in H^3(\Omega) \cap W$.
- (A3)** $R > 0$ is a fixed constant, and $\mathcal{U}_R := \{u \in L^2(Q) : \|u\|_{L^2(Q)} < R\}$.

From the condition **(A1)** it follows that f'_1 is monotone increasing on \mathbb{R} and induces a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$. Note that **(A1)** is fulfilled if f is given by the potential (1.10), and the condition $\varphi_0 \in H^3(\Omega) \cap W$ implies that $\varphi_0 \in C^0(\bar{\Omega})$. Moreover, the mean value

$$m_0 := \frac{1}{|\Omega|} \int_{\Omega} \varphi_0, \quad (2.1)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω , belongs to a bounded interval in \mathbb{R} . In the following, we use the notation \bar{v} to denote the mean value of a generic function $v \in L^1(\Omega)$. More generally, we set

$$\bar{v} := \frac{1}{|\Omega|} \langle v, 1 \rangle \quad \text{for every } v \in V^*, \quad (2.2)$$

noting that the constant function 1 is an element of V . Clearly, \bar{v} is the usual mean value of v if $v \in H$.

Next, we recall an important tool which is commonly used when working with problems connected to the Cahn–Hilliard equation. To this end, consider the weak formulation of the Poisson equation $-\Delta z = \zeta$ with homogeneous Neumann boundary conditions. Namely, for a given $\zeta \in V^*$ (which does not necessarily belong to H), we consider the problem of finding

$$z \in V \quad \text{such that} \quad \int_{\Omega} \nabla z \cdot \nabla v = \langle \zeta, v \rangle \quad \text{for every } v \in V. \quad (2.3)$$

Since Ω is connected and smooth, it is well known that the above problem admits solutions $z \in V$ if and only if ζ has zero mean value. Hence, we can introduce the following solution operator \mathcal{N} by setting

$$\mathcal{N} : \text{dom}(\mathcal{N}) := \{\zeta \in V^* : \bar{\zeta} = 0\} \rightarrow \{z \in V : \bar{z} = 0\}, \quad \mathcal{N} : \zeta \mapsto z, \quad (2.4)$$

where z is the unique solution to (2.3) satisfying $\bar{z} = 0$. It turns out that \mathcal{N} is an isomorphism between the above spaces, and it follows that the formula

$$\|\zeta\|_*^2 := \int_{\Omega} |\nabla \mathcal{N}(\zeta - \bar{\zeta})|^2 + |\bar{\zeta}|^2 \quad \text{for every } \zeta \in V^* \quad (2.5)$$

defines a Hilbert norm in V^* that is equivalent to the standard dual norm of V^* . In particular, there is a constant $C_{\Omega} > 0$, which depends only on Ω , such that

$$|\langle \zeta, v \rangle| \leq C_{\Omega} \|\zeta\|_* \|v\|_V \quad \text{for all } \zeta \in V^* \text{ and } v \in V. \quad (2.6)$$

Moreover, from the Young, Poincaré and Sobolev inequalities, elliptic estimates, and Ehrling's lemma, we have the estimates

$$ab \leq \frac{\delta}{2}|a|^2 + \frac{1}{2\delta}|b|^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0, \quad (2.7)$$

$$\|v\|_V \leq C_{\Omega} (\|\nabla v\|_{L^2(\Omega)^3} + |\bar{v}|) \quad \text{for every } v \in V, \quad (2.8)$$

$$\|v\|_W \leq C_{\Omega} (\|\Delta v\|_H + \|v\|_*) \quad \text{for every } v \in W, \quad (2.9)$$

$$\|v\|_p \leq \delta \|\nabla v\|_{L^2(\Omega)^3} + C_{\Omega, p, \delta} \|v\|_* \quad \text{for every } v \in V, p \in [1, 6) \text{ and } \delta > 0, \quad (2.10)$$

$$\|v\|_V \leq \delta \|\Delta v\|_H + C_{\Omega, \delta} \|v\|_* \quad \text{for every } v \in W \text{ and } \delta > 0. \quad (2.11)$$

In addition, from the above properties there follow the identities

$$\int_{\Omega} \nabla \mathcal{N}\zeta \cdot \nabla v = \langle \zeta, v \rangle \quad \text{for every } \zeta \in \text{dom}(\mathcal{N}) \text{ and } v \in V, \quad (2.12)$$

$$\langle \zeta, \mathcal{N}\xi \rangle = \langle \xi, \mathcal{N}\zeta \rangle \quad \text{for every } \zeta, \xi \in \text{dom}(\mathcal{N}), \quad (2.13)$$

$$\langle \zeta, \mathcal{N}\zeta \rangle = \int_{\Omega} |\nabla \mathcal{N}\zeta|^2 = \|\zeta\|_*^2 \quad \text{for every } \zeta \in \text{dom}(\mathcal{N}). \quad (2.14)$$

Moreover, we point out the equality

$$\langle \partial_t \zeta(t), \mathcal{N}\zeta(t) \rangle = \frac{1}{2} \frac{d}{dt} \|\zeta(t)\|_*^2 \quad \text{for a.a. } t \in (0, T), \quad (2.15)$$

which holds true for every $\zeta \in H^1(0, T; V^*)$ satisfying $\bar{\zeta} = 0$ for a.e. $t \in (0, T)$.

2.2. Existence for the state system

In this section, we are going to prove an existence result for the state system. Prior to this, we notice that, thanks to the linear equation (1.4) and the second initial condition in (1.6), w can be explicitly written in terms of u by means of the variation of constants formula

$$w(x, t) = w_0(x) \exp(-t/\gamma) + \int_0^t \exp(-(t-s)/\gamma) u(x, s) ds, \quad \text{a.e. } (x, t) \in Q. \quad (2.16)$$

We have the following result.

Theorem 2.1. *Suppose that (A1)–(A3) are fulfilled. Then the state system (1.2)–(1.6) has for every $u \in L^2(0, T; H)$ a unique solution triple (φ, μ, w) satisfying*

$$\varphi \in W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \cap C^0(\bar{Q}), \quad (2.17)$$

$$\mu \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W \cap H^3(\Omega)), \quad (2.18)$$

$$w \in H^1(0, T; H). \quad (2.19)$$

In addition, there is a constant $K_1 > 0$, which depends only on $\|u\|_{L^2(0, T; H)}$ and the data of the state system, such that

$$\begin{aligned} & \|\varphi\|_{W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \cap C^0(\bar{Q})} \\ & + \|\mu\|_{H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W \cap H^3(\Omega))} + \|w\|_{H^1(0, T; H)} \leq K_1 \end{aligned} \quad (2.20)$$

whenever (φ, μ, w) is the solution to the state system associated with u .

Proof. Although the proof of the above result seems to be pretty standard by now, we carry it out for the reader's convenience. We argue by a Faedo–Galerkin approximation. To this end, consider the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ of the eigenvalue problem

$$-\Delta v = \lambda v \quad \text{in } \Omega, \quad \partial_{\mathbf{n}} v = 0 \quad \text{on } \Gamma,$$

and let $\{e_j\}_{j \in \mathbb{N}} \subset W$ be the associated eigenfunctions, normalized by $\|e_j\|_H = 1$. Then

$$0 = \lambda_1 < \lambda_2 \leq \dots, \quad \lim_{j \rightarrow \infty} \lambda_j = +\infty,$$

$$\int_{\Omega} e_j e_k = \int_{\Omega} \nabla e_j \cdot \nabla e_k = 0 \quad \text{for } j \neq k.$$

We then define the n -dimensional spaces $V_n := \text{span}\{e_1, \dots, e_n\}$ for $n \in \mathbb{N}$, where we observe that V_1 is just the space of constant functions on Ω . It is well known that the union of these spaces is dense in both H and V . Notice also that

$$\mathcal{N}v \in V_n \quad \text{for every } v \in V_n \text{ with } \bar{v} = 0. \quad (2.21)$$

Indeed, if $v \in V_n$ and $\bar{v} = 0$ then $v = \sum_{j=2}^n c_j e_j$ with suitable $c_j \in \mathbb{R}$, $j = 1, \dots, n$, and $z := \sum_{j=2}^n \lambda_j^{-1} c_j e_j \in V_n$ satisfies $\bar{z} = 0$ and $-\Delta z = v$, that is, $z = \mathcal{N}v$.

The approximating n -dimensional problem then reads as follows: find real-valued functions $\varphi_{nj}, \mu_{nj}, w_{nj}$, $j = 1, \dots, n$, defined in $[0, T]$, such that, with

$$\varphi_n(x, t) := \sum_{j=1}^n \varphi_{nj}(t) e_j(x), \quad \mu_n(x, t) := \sum_{j=1}^n \mu_{nj}(t) e_j(x), \quad w_n(x, t) := \sum_{j=1}^n w_{nj}(t) e_j(x), \quad (2.22)$$

it holds that

$$\langle \partial_t \varphi_n, v \rangle + \int_{\Omega} \nabla \mu_n \cdot \nabla v = 0 \quad \text{for all } v \in V_n, \text{ a.e. in } (0, T), \quad (2.23)$$

$$\int_{\Omega} \nabla \varphi_n \cdot \nabla v + (f'(\varphi_n), v) - (w_n, v) = (\mu_n, v) \quad \text{for all } v \in V_n, \text{ a.e. in } (0, T), \quad (2.24)$$

$$\gamma \langle \partial_t w_n, v \rangle + (w_n, v) = (u, v) \quad \text{for all } v \in V_n, \text{ a.e. in } (0, T), \quad (2.25)$$

$$\varphi_n(0) = P_n(\varphi_0), \quad w_n(0) = P_n(w_0), \quad \text{a.e. in } \Omega, \quad (2.26)$$

where P_n denotes the H -orthogonal projection onto V_n . Then $P_n(v) = \sum_{j=1}^n (v, e_j) e_j$ for every $v \in H$, and we have (see [36], formula (3.14)), with a constant $C_{\Omega} > 0$ depending only on Ω ,

$$\|P_n(v)\|_Y \leq C_{\Omega} \|v\|_Y \quad \text{for every } v \in Y, \text{ where } Y \in \{H, V, W\}. \quad (2.27)$$

Next, we insert $v = e_k$ in all of the equations (2.23)–(2.26), for $k = 1, \dots, n$, obtaining the system

$$\frac{d}{dt} \varphi_{nk} + \lambda_k \mu_{nk} = 0 \quad \text{a.e. in } (0, T), \quad (2.28)$$

$$\mu_{nk} = \lambda_k \varphi_{nk} + (f'(\varphi_n), e_k) - w_{nk} \quad \text{a.e. in } (0, T), \quad (2.29)$$

$$\gamma \frac{d}{dt} w_{nk} + w_{nk} = (u, e_k) \quad \text{a.e. in } (0, T), \quad (2.30)$$

$$\varphi_{nk}(0) = (\varphi_0, e_k), \quad w_{nk}(0) = (w_0, e_k). \quad (2.31)$$

Now insert for μ_{nk} in (2.28), using (2.29). We then arrive at an initial value problem for an explicit ODE system in the $2n$ unknowns $(\varphi_{n1}, \dots, \varphi_{nn}, w_{n1}, \dots, w_{nn})$ with locally Lipschitz continuous nonlinearities and coefficient functions in $L^2(0, T)$. By Carathéodory's theorem, this ODE system has a unique maximal solution belonging to $H^1(0, T_n; \mathbb{R}^{2n})$ for some $T_n \in (0, T]$. This solution in turn uniquely determines *via* (2.29) and (2.22) a triple $(\varphi_n, \mu_n, w_n) \in (H^1(0, T_n; V_n))^3$ that solves (2.23)–(2.26) on $\Omega \times [0, T_n]$, with the regularity of μ_n following from (2.29) and (A1). We show that one can take $T_n = T$. We do this by deriving a series of *a priori* estimates for the finite-dimensional approximations. In the following, $C > 0$ denotes constants that may depend on $\|u\|_{L^2(0, T; H)}$ and the data of the state system, but not on $n \in \mathbb{N}$.

First estimate. Testing (2.25) by $\partial_t w_n$, with the help of (2.7) we immediately get the estimate

$$\|w_n\|_{H^1(0,T_n;H)} \leq C. \quad (2.32)$$

Then, we test (2.23) by $\varphi_n \in V_n$ and (2.24) by $-\Delta\varphi_n \in V_n$, add, and integrate over $[0, t]$ where $t \in (0, T_n]$. After a cancellation and reorganisation of terms, we obtain that

$$\begin{aligned} & \frac{1}{2} \|\varphi_n(t)\|_H^2 + \iint_{Q_t} |\Delta\varphi_n|^2 + \iint_{Q_t} f_1''(\varphi_n) |\nabla\varphi_n|^2 \\ & \leq \frac{1}{2} \|\varphi_n(0)\|_H^2 - \iint_{Q_t} w_n \Delta\varphi_n + \iint_{Q_t} f_2'(\varphi_n) \Delta\varphi_n. \end{aligned}$$

By the convexity of f_1 , the last term on the left-hand side is nonnegative. Moreover, owing to **(A1)**, we have $|f_2'(\varphi_n)| \leq C(1 + |\varphi_n|)$ a.e. in $\Omega \times (0, T_n)$. In view of (2.32) and Young's inequality, the sum of the last two terms on the right-hand side is therefore bounded by

$$\frac{1}{2} \iint_{Q_t} |\Delta\varphi_n|^2 + C \int_0^t (1 + \|\varphi_n(s)\|_H^2) ds.$$

Consequently, by Gronwall's lemma, and using the estimate (2.9), we infer that

$$\|\varphi_n\|_{L^\infty(0,T_n;H) \cap L^2(0,T_n;W)} \leq C. \quad (2.33)$$

We can draw an important consequence from (2.32) and (2.33): indeed, by a standard argument it follows from these bounds that the local solution $(\varphi_{n1}, \dots, \varphi_{nn}, w_{n1}, \dots, w_{nn})$ to the ODE system resulting from (2.28)–(2.31) is uniformly bounded and thus, by its maximality, global. Therefore, it must exist on the whole interval $[0, T]$, that is, we have $T_n = T$. We will exploit this fact in the following estimates.

Second estimate. Next, we recall that the constant function $v = 1$ belongs to V_1 . Inserting it in (2.23) yields that $\overline{\partial_t \varphi_n} = 0$ a.e. in $(0, T)$, which entails that $\mathcal{N}(\partial_t \varphi_n)$ is well defined and belongs to V_n . We now insert $v = \mathcal{N}(\partial_t \varphi_n)$ in (2.23) and $v = \partial_t \varphi_n$ in (2.24), add, and integrate over $[0, t]$ where $t \in (0, T]$. Using (2.12) and (2.14), and noting the cancellation of two terms, we obtain the identity

$$\begin{aligned} & \int_0^t \|\partial_t \varphi_n(s)\|_*^2 ds + \frac{1}{2} \int_\Omega |\nabla\varphi_n(t)|^2 + \int_\Omega f_1(\varphi_n(t)) \\ & = \frac{1}{2} \|\nabla P_n(\varphi_0)\|^2 + \int_\Omega f_1(P_n(\varphi_0)) + \iint_{Q_t} w_n \partial_t \varphi_n - \iint_{Q_t} f_2'(\varphi_n) \partial_t \varphi_n. \end{aligned}$$

By **(A1)**, the third term on the left-hand side is nonnegative. Moreover, by (2.27), the first summand on the right-hand side is bounded, and we have, using the continuity of the embedding $H^2(\Omega) \subset C^0(\overline{\Omega})$, that $\|P_n(\varphi_0)\|_{C^0(\overline{\Omega})} \leq C\|P_n(\varphi_0)\|_W \leq C\|\varphi_0\|_W$. This obviously implies that the sequence $\{\int_\Omega f_1(P_n(\varphi_0))\}$ is bounded. Moreover, we obtain from (2.32), (2.33), and Young's inequality, that

$$\iint_{Q_t} w_n \partial_t \varphi_n = \int_\Omega w_n(t) \varphi_n(t) - \int_\Omega P_n(w_0) P_n(\varphi_0) - \iint_{Q_t} \varphi_n \partial_t w_n \leq C.$$

Finally, we infer from (2.6), (2.33), and Young's inequality, that

$$\begin{aligned}
& - \iint_{Q_t} f_2'(\varphi_n) \partial_t \varphi_n \leq C \int_0^t \|\partial_t \varphi_n(s)\|_* \|f_2'(\varphi_n(s))\|_V ds \\
& \leq \frac{1}{2} \int_0^t \|\partial_t \varphi_n(s)\|_*^2 ds + C \iint_{Q_t} (|f_2'(\varphi_n)|^2 + |f_2''(\varphi_n)|^2 |\nabla \varphi_n|^2) \\
& \leq \frac{1}{2} \int_0^t \|\partial_t \varphi_n(s)\|_*^2 ds + C,
\end{aligned}$$

since, owing to **(A1)**, we have $|f_2'(\varphi_n)| \leq C(1 + |\varphi_n|)$ and $|f_2''(\varphi_n)| \leq C$ a.e. in Q . Combining the above estimates, we have therefore shown that

$$\|\varphi_n\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V)} + \|f_1(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \quad (2.34)$$

Third estimate. At this point, we recall that $\overline{\partial_t \varphi_n} = 0$ a.e. in $(0, T)$, which implies that (cf. (2.1))

$$\overline{\varphi_n(t)} = \overline{P_n(\varphi_0)} = \frac{1}{|\Omega|} \int_{\Omega} (\varphi_0, e_1) e_1 = \overline{\varphi_0} \|e_1\|_H^2 = m_0 \quad \text{for all } t \in [0, T]. \quad (2.35)$$

For almost every $t \in (0, T)$, we now insert $v = \mathcal{N}(\varphi_n(t) - m_0)$ in (2.23) and $v = \varphi_n(t) - m_0$ in (2.24), and add the results. We obtain

$$\begin{aligned}
& \int_{\Omega} |\nabla(\varphi_n(t) - m_0)|^2 + \int_{\Omega} f_1'(\varphi_n(t))(\varphi_n(t) - m_0) \\
& = -\langle \partial_t \varphi_n(t), \mathcal{N}(\varphi_n(t) - m_0) \rangle + \int_{\Omega} (w_n(t) - f_2'(\varphi_n(t)))(\varphi_n(t) - m_0) \\
& \leq C \|\partial_t \varphi_n(t)\|_* \|\mathcal{N}(\varphi_n(t) - m_0)\|_V \\
& \quad + (\|w_n(t)\|_H + \|f_2'(\varphi_n(t))\|_H) (\|\varphi_n(t) - m_0\|_H).
\end{aligned} \quad (2.36)$$

Owing to (2.34) and to the bounds for $\mathcal{N}(\varphi_n(t) - m_0)$ implied by (2.33) and (2.35), it follows that the first summand on the right-hand side is bounded in $L^2(0, T)$. Moreover, the second summand is already known to be bounded in $L^\infty(0, T)$.

Now recall that f_1' is monotone increasing and that (2.35) holds. Then there exist constants $\delta_0 > 0$ and $C_0 > 0$ such that

$$f_1'(r)(r - m_0) \geq \delta_0 |f_1'(r)| - C_0 \quad \text{for every } r \in \mathbb{R}. \quad (2.37)$$

For this estimate we refer to [55], Appendix, Proposition A.1 (see also the detailed proof given in [56], p. 908). Applying (2.37), we therefore can infer from (2.36) that

$$\begin{aligned}
\delta_0 \int_{\Omega} |f_1'(\varphi_n(t))| & \leq C_0 + C \|\partial_t \varphi_n(t)\|_* \|\mathcal{N}(\varphi_n(t) - m_0)\|_V \\
& + (\|w_n(t)\|_H + \|f_2'(\varphi_n(t))\|_H) (\|\varphi_n(t) - m_0\|_H).
\end{aligned} \quad (2.38)$$

and consequently

$$\|f_1'(\varphi_n)\|_{L^2(0,T;L^1(\Omega))} \leq C. \quad (2.39)$$

Next, we insert the constant function $1 \in V_1$ in (2.24). We obtain, for a.e. $t \in (0, T)$,

$$\int_{\Omega} f_1'(\varphi_n(t)) + \int_{\Omega} (f_2'(\varphi_n(t)) - w_n(t)) = |\Omega| \overline{\mu_n(t)}. \quad (2.40)$$

By (2.39) and previous estimates, both summands on the left-hand side are bounded in $L^2(0, T)$. Then we conclude that

$$\|\overline{\mu_n}\|_{L^2(0, T)} \leq C. \quad (2.41)$$

At this point, we test (2.23) by $v = \mu_n(t) - \overline{\mu_n(t)}$, which has zero mean value. It then follows from Young's inequality and (2.34) that

$$\begin{aligned} \iint_Q |\nabla \mu_n|^2 &= - \int_0^T \langle \partial_t \varphi_n(t), \mu_n(t) - \overline{\mu_n(t)} \rangle dt \\ &\leq C \int_0^T \|\partial_t \varphi_n(t)\|_* \|(\mu_n - \overline{\mu_n})(t)\|_V dt \leq C + \frac{1}{2} \iint_Q |\nabla \mu_n|^2, \end{aligned}$$

whence, invoking also (2.41) and the Poincaré inequality (2.8), we arrive at

$$\|\mu_n\|_{L^2(0, T; V)} \leq C. \quad (2.42)$$

Fourth estimate. We now differentiate both the equations (2.23) and (2.24) with respect to t , then test the first of the resulting equations by $v = \mathcal{N}(\partial_t \varphi_n(t))$ and the second by $v = \partial_t \varphi_n(t)$. Addition and integration over $(0, t)$, where $t \in (0, T]$, and use of the properties of the operator \mathcal{N} , lead after the cancellation of two terms to the identity

$$\begin{aligned} \frac{1}{2} \|\partial_t \varphi_n(t)\|_*^2 + \iint_{Q_t} |\nabla \partial_t \varphi_n|^2 + \iint_{Q_t} f_1''(\varphi_n) |\partial_t \varphi_n|^2 \\ = \frac{1}{2} \|\partial_t \varphi_n(0)\|_*^2 - \iint_{Q_t} f_2''(\varphi_n) |\partial_t \varphi_n|^2 + \iint_{Q_t} \partial_t w_n \partial_t \varphi_n. \end{aligned} \quad (2.43)$$

By the convexity of f_1 , the third term on the left-hand side is nonnegative. Moreover, the sum of the last two terms on the right-hand side, which we denote by I , can be estimated as follows:

$$\begin{aligned} I &\leq C + C \iint_{Q_t} |\partial_t \varphi_n|^2 \leq \frac{1}{2} \iint_{Q_t} |\nabla \partial_t \varphi_n|^2 + C \int_0^t \|\partial_t \varphi_n(s)\|_*^2 ds \\ &\leq C + \frac{1}{2} \iint_{Q_t} |\nabla \partial_t \varphi_n|^2. \end{aligned}$$

Here we have used (2.32), (2.34), (A1), Young's inequality, and the compactness inequality (2.10) with $p = 2$.

For the initial value we have

$$\begin{aligned} \|\partial_t \varphi_n(0)\|_*^2 &= - \int_{\Omega} \mu_n(0) \partial_t \varphi_n(0) \\ &= - \int_{\Omega} (-\Delta(\varphi_n(0)) + f'(\varphi_n(0)) - w_n(0)) \partial_t \varphi_n(0) \\ &\leq C \|\partial_t \varphi_n(0)\|_* \|-\Delta(\varphi_n(0)) + f'(\varphi_n(0)) - w_n(0)\|_V. \end{aligned}$$

We claim that the second factor on the right-hand side in bounded. Indeed, by (2.27), we have $\|w_n(0)\|_V = \|P_n w_0\|_V \leq C\|w_0\|_V < +\infty$, since (see **(A2)**) $w_0 \in V$. Moreover, we have already shown above in the second estimate that the sequence $\{P_n(\varphi_0)\}$ is bounded in $C^0(\bar{\Omega})$. Therefore, as $\varphi_0 \in V$ and again by (2.27),

$$\begin{aligned} \|f'(\varphi_n(0))\|_V^2 &= \|f'(P_n(\varphi_0))\|_V^2 \\ &= \int_{\Omega} (|f'(P_n(\varphi_0))|^2 + |f''(P_n(\varphi_0))|^2 |\nabla P_n(\varphi_0)|^2) \\ &\leq C + C\|P_n(\varphi_0)\|_V^2 \leq C + C\|\varphi_0\|_V^2 < +\infty. \end{aligned}$$

Finally, we have

$$\begin{aligned} \Delta P_n(\varphi_0) &= \sum_{j=1}^n (\varphi_0, e_j) \Delta e_j = - \sum_{j=1}^n \lambda_j (\varphi_0, e_j) e_j, \\ \nabla \Delta P_n(\varphi_0) &= - \sum_{j=1}^n \lambda_j (\varphi_0, e_j) \nabla e_j, \end{aligned}$$

whence

$$\|\Delta P_n(\varphi_0)\|_V^2 = \sum_{j=1}^n (\lambda_j^2 + \lambda_j^3) |(\varphi_0, e_j)|^2 \leq \sum_{j=1}^{\infty} (\lambda_j^2 + \lambda_j^3) |(\varphi_0, e_j)|^2 < +\infty,$$

since, by **(A2)**, $\varphi_0 \in H^3(\Omega) \cap W$. In conclusion, we have shown that

$$\|\varphi_n\|_{W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V)} \leq C. \quad (2.44)$$

In particular, we now see that the right-hand side of (2.38) is even bounded in $L^\infty(0, T)$, so that

$$\|f'_1(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq C,$$

and it follows from (2.40) that

$$\|\overline{\mu_n}\|_{L^\infty(0,T)} \leq C. \quad (2.45)$$

At this point, we test (2.24) by $v = (\mu_n - \overline{\mu_n})(t)$, without integrating over time. As at the end of the third estimate, it then follows from (2.45) and Poincaré's inequality (2.8) that

$$\|\mu_n\|_{L^\infty(0,T;V)} \leq C. \quad (2.46)$$

Next, we test (2.24) for a.e. $t \in (0, T)$ by $v = -\Delta \varphi_n(t) \in V_n$, without integrating over time. We obtain

$$\begin{aligned} &\int_{\Omega} |\Delta \varphi_n(t)|^2 + \int_{\Omega} f''_1(\varphi_n(t)) |\nabla \varphi_n(t)|^2 \\ &= - \int_{\Omega} (w_n(t) + \mu_n(t) - f'_2(\varphi_n(t))) \Delta \varphi_n(t), \end{aligned}$$

whence, using Young's inequality, (2.32), (2.34), (2.46), and elliptic regularity,

$$\|\varphi_n\|_{L^\infty(0,T;W)} \leq C. \quad (2.47)$$

Existence. By virtue of the uniform estimates shown in the previous steps, there exists a triple (φ, μ, w) such that (possibly only on a suitable subsequence which is again labeled by $n \in \mathbb{N}$)

$$\varphi_n \rightarrow \varphi \quad \text{weakly star in } W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \quad (2.48)$$

$$\mu_n \rightarrow \mu \quad \text{weakly star in } L^\infty(0, T; V), \quad (2.49)$$

$$w_n \rightarrow w \quad \text{weakly in } H^1(0, T; H). \quad (2.50)$$

Since, owing to the compactness of the embedding $W \subset C^0(\overline{\Omega})$, it follows from [57], Section 8, Corollary 4 that $H^1(0, T; V) \cap L^\infty(0, T; W)$ is compactly embedded in $C^0(\overline{Q})$, we may also assume that

$$\varphi_n \rightarrow \varphi \quad \text{strongly in } C^0(\overline{Q}), \quad (2.51)$$

whence, by the local Lipschitz continuity of f' ,

$$f'(\varphi_n) \rightarrow f'(\varphi) \quad \text{strongly in } C^0(\overline{Q}). \quad (2.52)$$

With these strong convergence properties at hand, it follows from a standard argument (which needs no repetition here) that (φ, μ, w) is in fact a solution to the state system (1.2)–(1.6). Moreover, we can infer from the semicontinuity properties of norms and the estimates shown above that there is a constant $K_0 > 0$, which depends only on $\|u\|_{L^2(0, T; H)}$ and the data of the state system, such that

$$\|\varphi\|_{W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \cap C^0(\overline{Q})} + \|\mu\|_{L^\infty(0, T; V)} + \|w\|_{H^1(0, T; H)} \leq K_0.$$

In addition, we conclude from (1.2) and elliptic regularity that $\mu \in L^2(0, T; H^3(\Omega) \cap W)$ and

$$\|\mu\|_{L^2(0, T; H^3(\Omega) \cap W)} \leq C.$$

Besides, taking the time derivative in (1.3), we can infer from comparison that also $\mu \in H^1(0, T; V^*)$ and

$$\|\mu\|_{H^1(0, T; V^*)} \leq C.$$

With this, the existence of a solution (φ, μ, w) and of a constant $K_1 > 0$ with the asserted properties is shown. It remains to prove its uniqueness. This will be done below in Theorem 2.7 in the continuous dependence estimate. \square

An immediate consequence of the uniform bound established for $\|\varphi\|_{C^0(\overline{Q})}$ and of the uniqueness still to be proved below is the following.

Corollary 2.2. *Assume that (A1)–(A3) are fulfilled. Then there is a constant $K_2 > 0$ depending only on the data of the system and R such that*

$$\max_{0 \leq i \leq 5} \left(\max_{j=1,2} \|f_j^{(i)}(\varphi)\|_{C^0(\overline{Q})} + \|f^{(i)}(\varphi)\|_{C^0(\overline{Q})} \right) \leq K_2, \quad (2.53)$$

whenever (φ, μ, w) is the solution to the state system (1.2)–(1.6) in the sense of Theorem 2.1 associated with some $u \in \mathcal{U}_R$.

Remark 2.3. It is worth noting that for the proof of Theorem 2.1 it was not necessary to assume that $f_1(r)$ has a sufficiently strong (e.g., at least quadratic) growth as $|r| \rightarrow +\infty$. Such an assumption has been made in many papers dealing with regular potentials. The reason for this is that in our approach we avoid to test (2.24) by $\partial_t \varphi_n$ before sufficiently strong estimates for φ (here derived in the first estimate) are available to handle

the term $-\iint_{Q_t} f'_2(\varphi_n) \partial_t \varphi_n$ that arises on the right-hand side. It is well possible that a corresponding line of argumentation works also in many other cases, thus avoiding the growth assumption.

Remark 2.4. Our approach in the proof of Theorem 2.1 consisted in approximating all equations including (1.4), despite the fact that we had the explicit solution (2.16) (in terms of u) at our disposal. On the other hand, the approximation turns out to be a convenient approach especially if (1.4) would be replaced by a more complicated and possibly coupled PDE, still with the control on the right-hand side. In particular, we point out that in the paper [52], concerned with the viscous Cahn–Hilliard equation, the coefficient γ in the analog of (1.4) is allowed to depend on the space variable $x \in \Omega$, being however bounded from below by a positive constant. This setting can be considered also here without major modifications: of course, then the approximation (2.25) of (1.4) is no longer valid and should be replaced by

$$\int_{\Omega} \gamma \partial_t w_n v + (w_n, v) = (u, v) \quad \text{for all } v \in V_n, \text{ a.e. in } (0, T),$$

with the consequence that the resulting ODE system (2.30) changes into

$$\sum_{j=1}^n \left(\int_{\Omega} \gamma e_j e_k \right) \frac{dw_{nj}}{dt} + w_{nk} = (u, e_k) \quad \text{a.e. in } (0, T), \quad k = 1, \dots, n.$$

Nonetheless, note that $(\int_{\Omega} \gamma e_j e_k)_{j,k}$, $j, k = 1, \dots, n$, are the coefficients of a symmetric and positive definite (and thus invertible) matrix, so that the resulting modified ODE system (2.28)–(2.31) is still easily solvable with a time-dependent maximal solution.

2.3. An auxiliary lemma

In this section, we show the following preparatory lemma which will prove useful in numerous estimations in the following.

Lemma 2.5. *Suppose that functions*

$$a \in L^2(0, T; W^{1,4}(\Omega)), \quad g \in L^2(0, T; V), \quad h \in L^2(0, T; H) \quad (2.54)$$

are given. Then there is a unique triple (φ, μ, w) such that

$$\varphi \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad (2.55)$$

$$\mu \in L^\infty(0, T; V), \quad (2.56)$$

$$w \in H^1(0, T; H), \quad (2.57)$$

as well as

$$\langle \partial_t \varphi, v \rangle + \int_{\Omega} \nabla \mu \cdot \nabla v = 0 \quad \text{for all } v \in V \text{ and a.e. in } (0, T), \quad (2.58)$$

$$-\Delta \varphi - \mu - w = a \varphi + g \quad \text{a.e. in } Q, \quad (2.59)$$

$$\gamma \partial_t w + w = h \quad \text{a.e. in } Q, \quad (2.60)$$

$$\partial_{\mathbf{n}} \varphi = 0 \quad \text{a.e. on } \Sigma, \quad (2.61)$$

$$\varphi(0) = 0, \quad w(0) = 0, \quad \text{a.e. in } \Omega. \quad (2.62)$$

Moreover, there is some constant $K_3 > 0$, which increases monotonically with respect to the value of the norm $\|a\|_{L^2(0,T;W^{1,4}(\Omega))}$, such that

$$\begin{aligned} & \|\varphi\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\mu\|_{L^2(0,T;V)} + \|w\|_{H^1(0,T;H)} \\ & \leq K_3 \left(\|g\|_{L^2(0,T;V)} + \|h\|_{L^2(0,T;H)} \right). \end{aligned} \quad (2.63)$$

Proof. The existence proof is again performed *via* a Faedo–Galerkin approximation using the same finite-dimensional spaces as in the proof of Theorem 2.1. For the sake of brevity, we avoid writing the system explicitly here and just provide the relevant formal *a priori* estimate (2.63) for the continuous system, which in the rigorous argument has to be performed for the finite-dimensional approximations. With this *a priori* estimate (2.63) at hand, the standard limit process using weak and weak-star compactness can be carried out to prove the existence of a solution (φ, μ, w) having the regularity (2.55)–(2.57). Notice that also the uniqueness of the solution immediately follows: indeed, if (φ_i, μ_i, w_i) , $i = 1, 2$, are two solutions, then $(\varphi, \mu, w) := (\varphi_1 - \varphi_2, \mu_1 - \mu_2, w_1 - w_2)$ satisfies the system (2.58)–(2.62) with $g = h = 0$, and (2.63) yields that $\varphi = \mu = w = 0$.

To begin with, we first note that (2.58) implies that $\overline{\partial_t \varphi} = 0$ a.e. in $(0, T)$ which, thanks to the initial condition $\varphi(0) = 0$, yields that $\overline{\varphi(t)} = 0$ for all $t \in [0, T]$. We thus may test (2.58) by $\frac{1}{2}\mathcal{N}(\partial_t \varphi) + \frac{1}{2}\mu + \varphi$, (2.59) by $\partial_t \varphi - \Delta \varphi - \mu$, and (2.60) by $K \partial_t w$, where the constant $K > 0$ is yet to be determined. Addition of the resulting equations and integration over $(0, t)$, where $t \in (0, T]$, then leads to the cancellation of some terms, and upon rearranging the terms, we arrive at the identity

$$\begin{aligned} & \frac{1}{2} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + \frac{1}{2} \|\varphi(t)\|_V^2 + \frac{K}{2} \|w(t)\|_H^2 + \frac{1}{2} \iint_{Q_t} |\nabla \mu|^2 \\ & + \iint_{Q_t} (|\Delta \varphi|^2 + |\mu|^2 + K\gamma |\partial_t w|^2) \\ & = \iint_{Q_t} \nabla \varphi \cdot \nabla \mu + \iint_{Q_t} w \partial_t \varphi - \iint_{Q_t} w \Delta \varphi \\ & - \iint_{Q_t} \mu w + \iint_{Q_t} a \varphi (-\Delta \varphi - \mu) + \iint_{Q_t} a \varphi \partial_t \varphi \\ & + \iint_{Q_t} g (-\Delta \varphi - \mu) + \iint_{Q_t} g \partial_t \varphi + K \iint_{Q_t} h \partial_t w := \sum_{j=1}^9 I_j, \end{aligned} \quad (2.64)$$

with obvious meaning. Five of the terms on the right-hand side can be easily estimated using Young's inequality (2.7). Namely, we have

$$|I_1| \leq \frac{1}{4} \iint_{Q_t} |\nabla \mu|^2 + \iint_{Q_t} |\nabla \varphi|^2, \quad (2.65)$$

$$|I_3| \leq \frac{1}{4} \iint_{Q_t} |\Delta \varphi|^2 + \iint_{Q_t} |w|^2, \quad (2.66)$$

$$|I_4| \leq \frac{1}{4} \iint_{Q_t} |\mu|^2 + \iint_{Q_t} |w|^2, \quad (2.67)$$

$$|I_7| \leq \frac{1}{4} \iint_{Q_t} (|\Delta \varphi|^2 + |\mu|^2) + 2 \iint_{Q_t} |g|^2, \quad (2.68)$$

$$|I_9| \leq \frac{K\gamma}{4} \iint_{Q_t} |\partial_t w|^2 + \frac{K}{\gamma} \iint_{Q_t} |h|^2. \quad (2.69)$$

The remaining four terms, which involve $\partial_t \varphi$ and/or a , require more attention. At first, we use (2.6) and Young's inequality to see that

$$|I_8| \leq C \int_0^t \|\partial_t \varphi(s)\|_* \|g(s)\|_V ds \leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \|g\|_{L^2(0,t;V)}^2. \quad (2.70)$$

Next, we have $I_2 = \int_\Omega w(t)\varphi(t) - \iint_{Q_t} \varphi \partial_t w$, so that, by Young's inequality (2.7),

$$|I_2| \leq \frac{1}{4} \|\varphi(t)\|_H^2 + \|w(t)\|_H^2 + \frac{K\gamma}{4} \iint_{Q_t} |\partial_t w|^2 + \frac{K}{\gamma} \iint_{Q_t} |\varphi|^2. \quad (2.71)$$

Now observe that we have the continuous embedding $W^{1,4}(\Omega) \subset L^\infty(\Omega)$. Hence,

$$\|a\|_{L^2(0,T;L^\infty(\Omega))} \leq C \|a\|_{L^2(0,T;W^{1,4}(\Omega))}.$$

Therefore, by Young's inequality,

$$\begin{aligned} |I_5| &\leq \int_0^t \|a(s)\|_\infty \|\varphi(s)\|_H (\|\Delta \varphi(s)\|_H + \|\mu(s)\|_H) ds \\ &\leq \frac{1}{4} \iint_{Q_t} (|\Delta \varphi|^2 + |\mu|^2) + C \int_0^t \|a(s)\|_{W^{1,4}(\Omega)}^2 \|\varphi(s)\|_H^2 ds. \end{aligned} \quad (2.72)$$

Finally, we estimate I_6 . Using (2.6), as well as the Young and Hölder inequalities, we infer that

$$\begin{aligned} |I_6| &\leq C \int_0^t \|\partial_t \varphi(s)\|_* \|a(s)\varphi(s)\|_V ds \\ &\leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \iint_{Q_t} |a|^2 (|\varphi|^2 + |\nabla \varphi|^2) + C \iint_{Q_t} |\nabla a|^2 |\varphi|^2 \\ &\leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \int_0^t \|a(s)\|_\infty^2 \|\varphi(s)\|_V^2 ds \\ &\quad + C \int_0^t \|\nabla a(s)\|_4^2 \|\varphi(s)\|_4^2 ds \\ &\leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \int_0^t \|a(s)\|_{W^{1,4}(\Omega)}^2 \|\varphi(s)\|_V^2 ds. \end{aligned} \quad (2.73)$$

At this point, we choose $K = 4$. It then follows from (2.64)–(2.73), that there are constants $C_1 > 0$, $C_2 > 0$, which do not depend on a , h and g , such that

$$\begin{aligned} &\|\varphi(t)\|_V^2 + \|w(t)\|_H^2 + \int_0^t \|\partial_t \varphi(s)\|_*^2 ds \\ &\quad + \iint_{Q_t} (|\Delta \varphi|^2 + |\mu|^2 + |\nabla \mu|^2 + |\partial_t w|^2) \\ &\leq C_1 \int_0^t (\|g(s)\|_V^2 + \|h(s)\|_H^2) ds \\ &\quad + C_2 \int_0^t (1 + \|a(s)\|_{W^{1,4}(\Omega)}^2) (\|\varphi(s)\|_V^2 + \|w(s)\|_H^2) ds. \end{aligned}$$

Since the real-valued function $s \mapsto C_2(1 + \|a(s)\|_{W^{1,4}(\Omega)}^2)$ belongs to $L^1(0, T)$, we can apply Gronwall's lemma, whence the inequality (2.63) follows. In addition, the standard form of the Gronwall inequality ensures that the constant K_3 can be chosen to be monotonically increasing with respect to $\|a\|_{L^2(0, T; W^{1,4}(\Omega))}$. \square

Remark 2.6. We point out that the assumption $a \in L^2(0, T; W^{1,4}(\Omega))$ in (2.54) is set for convenience, to be used in the following, but it can be replaced by the more general assumption $a \in L^2(0, T; W^{1,p}(\Omega))$ with $p > 3$. Indeed, the above estimates in the proof can be repeated without major changes. In particular, the estimate (2.73) of I_6 can be arranged as follows:

$$\begin{aligned} |I_6| &\leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \int_0^t \|a(s)\|_\infty^2 \|\varphi(s)\|_V^2 ds + C \int_0^t \|\nabla a(s)\|_p^2 \|\varphi(s)\|_q^2 ds \\ &\leq \frac{1}{8} \int_0^t \|\partial_t \varphi(s)\|_*^2 ds + C \int_0^t \|a(s)\|_{W^{1,p}(\Omega)}^2 \|\varphi(s)\|_V^2 ds, \end{aligned}$$

as p is greater than the space dimension 3 and $q := 2p/(p-2) < 6$, so that $V \subset L^q(\Omega)$ with continuous embedding.

2.4. Continuous dependence and uniqueness

Next, we state a continuous dependence result that, in particular, guarantees the uniqueness of the solution provided by Theorem 2.1.

Theorem 2.7. *Suppose that the conditions (A1)–(A3) are fulfilled. Then there exists a constant $K_4 > 0$ such that the following holds true: whenever $u_i \in \mathcal{U}_R$, $i = 1, 2$, are given and (φ_i, μ_i, w_i) , $i = 1, 2$, are corresponding solutions to (1.2)–(1.6) in the sense of Theorem 2.1, then*

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{H^1(0, T; V^*) \cap C^0([0, T]; V) \cap L^2(0, T; W)} + \|\mu_1 - \mu_2\|_{L^2(0, T; V)} \\ &+ \|w_1 - w_2\|_{H^1(0, T; H)} \leq K_4 \|u_1 - u_2\|_{L^2(0, T; H)}. \end{aligned} \quad (2.74)$$

Proof. Let us set, for convenience,

$$u := u_1 - u_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \mu := \mu_1 - \mu_2, \quad w := w_1 - w_2.$$

Then $\varphi(0) = 0$ and $w(0) = 0$ a.e. in Ω , as well as $\partial_{\mathbf{n}}\varphi = 0$ a.e. on Σ . In addition, writing (1.2)–(1.4) for (φ_i, μ_i, w_i) , $i = 1, 2$, and taking the differences, we obtain that

$$\langle \partial_t \varphi v \rangle + \int_\Omega \nabla \mu \cdot \nabla v = 0 \quad \text{for every } v \in V \text{ and a.e. in } (0, T), \quad (2.75)$$

$$-\Delta \varphi - \mu - w = -(f'(\varphi_1) - f'(\varphi_2)) \quad \text{a.e. in } Q, \quad (2.76)$$

$$\gamma \partial_t w + w = u \quad \text{a.e. in } Q. \quad (2.77)$$

Now observe that

$$-(f'(\varphi_1) - f'(\varphi_2)) = - \int_0^1 \frac{d}{ds} f'(\varphi_2 + s(\varphi_1 - \varphi_2)) ds = a \varphi,$$

where

$$a := - \int_0^1 f''(\varphi_2 + s(\varphi_1 - \varphi_2)) ds. \quad (2.78)$$

With this choice of a , we see that the triple (φ, μ, w) satisfies a system of the form (2.58)–(2.62) with $g = 0$ and $h = u$. By virtue of Lemma 2.5, the assertion will thus be proved if we can show that there exists some constant $C > 0$, which depends only on the data of the system and R , such that

$$\|a\|_{L^2(0,T;W^{1,4}(\Omega))} \leq C. \quad (2.79)$$

Now recall that $u_1, u_2 \in \mathcal{U}_R$. Since the constant K_1 from (2.20) depends for controls belonging to \mathcal{U}_R only on the data and R , it follows that

$$\|\varphi_2 + s(\varphi_1 - \varphi_2)\|_{C^0(\overline{Q})} \leq K_1 \quad \text{for all } s \in [0, 1].$$

By the continuity of f'' , it then follows that $\|a\|_{C^0(\overline{Q})}$ is bounded by a constant that only depends on the data and R . The same then holds for $\|a\|_{L^2(0,T;L^4(\Omega))}$. Finally, we obviously have that $|\nabla a| \leq C(|\nabla\varphi_1| + |\nabla\varphi_2|)$ a.e. in Q . But this implies that $\|\nabla a\|_{L^\infty(0,T;L^6(\Omega)^3)}$ is bounded, which then also holds for $\|a\|_{L^2(0,T;W^{1,4}(\Omega))}$.

With this, the assertion is proved: note that the space $L^\infty(0, T; V)$ in (2.63) is replaced by $C^0([0, T]; V)$ in (2.74) since φ_1, φ_2 are known to be continuous from $[0, T]$ to V (cf. (2.20)). In particular, from (2.74), in the case when $u_1 = u_2$, it follows that $\varphi = \mu = w = 0$, which proves the uniqueness of the solution. \square

3. DIFFERENTIABILITY OF THE CONTROL-TO-STATE OPERATOR

Let us introduce the Banach spaces

$$\mathcal{X} := (H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W)) \times L^2(0, T; V) \times H^1(0, T; H), \quad (3.1)$$

$$\begin{aligned} \mathcal{Y} := & (W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \cap C^0(\overline{Q})) \\ & \times (H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W \cap H^3(\Omega))) \times H^1(0, T; H). \end{aligned} \quad (3.2)$$

From Theorem 2.1 and Theorem 2.7 we know that the control-to-state operator

$$\mathcal{S} : u \mapsto \mathcal{S}(u) = (\mathcal{S}_1(u), \mathcal{S}_2(u), \mathcal{S}_3(u)) := (\varphi, \mu, w)$$

is well defined as a mapping from $\mathcal{U} = L^2(0, T; H)$ into \mathcal{Y} and Lipschitz continuous as a mapping from \mathcal{U}_R into \mathcal{X} for every $R > 0$. In this section, we study the differentiability properties of this operator. More precisely, we want to show that under the assumptions (A1)–(A3) the operator \mathcal{S} is twice continuously Fréchet differentiable on \mathcal{U} as a mapping from \mathcal{U} into \mathcal{X} . We first show the following result.

Theorem 3.1. *Suppose that the conditions (A1)–(A3) are fulfilled. Then the control-to-state operator \mathcal{S} is for any $R > 0$ Fréchet differentiable in \mathcal{U}_R as a mapping from \mathcal{U} into \mathcal{X} . Moreover, for every $u^* \in \mathcal{U}_R$ and every increment $h \in L^2(0, T; H)$, the triple $(\xi, \eta, v) = \mathcal{S}'(u^*)[h] \in \mathcal{X}$ is the unique solution to the linearized system*

$$\langle \partial_t \xi, \rho \rangle + \int_{\Omega} \nabla \eta \cdot \nabla \rho = 0 \quad \text{for all } \rho \in V \text{ and a.e. in } (0, T), \quad (3.3)$$

$$-\Delta \xi - \eta - v = -f''(\varphi^*)\xi \quad \text{a.e. in } Q, \quad (3.4)$$

$$\gamma \partial_t v + v = h \quad \text{a.e. in } Q, \quad (3.5)$$

$$\partial_{\mathbf{n}} \xi = 0 \quad \text{a.e. on } \Sigma, \quad (3.6)$$

$$\xi(0) = 0, \quad v(0) = 0, \quad \text{a.e. in } \Omega. \quad (3.7)$$

Proof. The existence of a unique solution $(\xi, \eta, v) \in \mathcal{X}$ to the system (3.3)–(3.7) follows directly from Lemma 2.5: indeed, the system (3.3)–(3.7) is of the form (2.58)–(2.62) with $g = 0$ and $a = -f''(\varphi^*)$, and, in view of (2.20),

it is easily verified that

$$\| -f''(\varphi^*) \|_{L^2(0,T;W^{1,4}(\Omega))} \leq C,$$

with a constant $C > 0$ which depends only on the data of the state system and R . Moreover, it follows from (2.63) that the linear mapping $h \mapsto (\xi, \eta, v)$ is continuous from \mathcal{U} into \mathcal{X} .

To show the asserted Fréchet differentiability, we consider increments $h \in \mathcal{U}$ with $u^* + h \in \mathcal{U}_R$ and denote by $C > 0$ constants that may depend on the data and R , but not on the special choice of such increments. We also set $(\varphi^h, \mu^h, w^h) := \mathcal{S}(u^* + h)$, $(\varphi^*, \mu^*, w^*) := \mathcal{S}(u^*)$, and

$$y^h := \varphi^h - \varphi^* - \xi, \quad z^h := \mu^h - \mu^* - \eta, \quad \zeta^h := w^h - w^* - v.$$

We then have to show that

$$\begin{aligned} \|\mathcal{S}(u^* + h) - \mathcal{S}(u^*) - \mathcal{S}'(u^*)[h]\|_{\mathcal{X}} &= \|(y^h, z^h, \zeta^h)\|_{\mathcal{X}} = o(\|h\|_{L^2(0,T;H)}) \\ \text{as } \|h\|_{L^2(0,T;H)} &\rightarrow 0. \end{aligned} \quad (3.8)$$

This will certainly be the case if

$$\|\mathcal{S}(u^* + h) - \mathcal{S}(u^*) - \mathcal{S}'(u^*)[h]\|_{\mathcal{X}} = \|(y^h, z^h, \zeta^h)\|_{\mathcal{X}} \leq C\|h\|_{L^2(0,T;H)}^2 \quad (3.9)$$

for some suitable constant $C > 0$, which we are going to prove in the following.

Observe that (2.74) implies that

$$\begin{aligned} \|\varphi^h - \varphi^*\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\mu^h - \mu^*\|_{L^2(0,T;V)} \\ + \|w^h - w^*\|_{H^1(0,T;H)} \leq C\|h\|_{L^2(0,T;H)}. \end{aligned} \quad (3.10)$$

Moreover, the triple $(y^h, z^h, \zeta^h) \in \mathcal{X}$ is obviously a solution to the system

$$\langle \partial_t y^h, \rho \rangle + \int_{\Omega} \nabla z^h \cdot \nabla \rho = 0 \quad \text{for all } \rho \in V \text{ and a.e. in } (0, T), \quad (3.11)$$

$$-\Delta y^h - z^h - \zeta^h = -(f'(\varphi^h) - f'(\varphi^*) - f''(\varphi^*)\xi) \quad \text{a.e. in } Q, \quad (3.12)$$

$$\gamma \partial_t \zeta^h + \zeta^h = 0 \quad \text{a.e. in } Q, \quad (3.13)$$

$$\partial_{\mathbf{n}} y^h = 0 \quad \text{a.e. on } \Sigma, \quad (3.14)$$

$$y^h(0) = 0, \quad \zeta^h(0) = 0, \quad \text{a.e. in } \Omega, \quad (3.15)$$

whence it immediately follows that $\zeta^h = 0$ a.e. in Q . Moreover, we infer from Taylor's theorem with integral remainder that

$$f'(\varphi^h) - f'(\varphi^*) - f''(\varphi^*)\xi = f'''(\varphi^*)y^h + A^h(\varphi^h - \varphi^*)^2 \quad \text{a.e. in } Q, \quad (3.16)$$

with the remainder

$$A^h := \int_0^1 (1-s) f'''(\varphi^* + s(\varphi^h - \varphi^*)) ds. \quad (3.17)$$

From this we conclude that the system (3.11)–(3.15) is of the form (2.58)–(2.62) with $a := -f''(\varphi^*)$ and $g := -A^h(\varphi^h - \varphi^*)^2$.

In view of (2.20), we have $\|a\|_{L^2(0,T;W^{1,4}(\Omega))} = \|-f''(\varphi^*)\|_{L^2(0,T;W^{1,4}(\Omega))} \leq C$. It thus follows from (2.63) in Lemma 2.5 that (3.8), and thus the assertion of the theorem, is valid provided we can show that

$$\|g\|_{L^2(0,T;V)}^2 = \|-A^h(\varphi^h - \varphi^*)^2\|_{L^2(0,T;V)}^2 \leq C \|h\|_{L^2(0,T;H)}^4. \quad (3.18)$$

Now observe that, owing to (2.53), we have a.e. in Q that

$$|A^h| \leq C, \quad |\nabla A^h| \leq C (|\nabla \varphi^*| + |\nabla \varphi^h|), \quad (3.19)$$

$$\nabla (A^h(\varphi^h - \varphi^*)^2) = \nabla A^h(\varphi^h - \varphi^*)^2 + 2A^h(\varphi^h - \varphi^*)\nabla(\varphi^h - \varphi^*). \quad (3.20)$$

Therefore, we have that

$$\begin{aligned} & \|-A^h(\varphi^h - \varphi^*)^2\|_{L^2(0,T;V)}^2 \\ & \leq C \iint_Q |\varphi^h - \varphi^*|^4 + C \iint_Q (|\nabla \varphi^*|^2 + |\nabla \varphi^h|^2) |\varphi^h - \varphi^*|^4 \\ & \quad + C \iint_Q |\varphi^h - \varphi^*|^2 |\nabla(\varphi^h - \varphi^*)|^2 =: J_1 + J_2 + J_3, \end{aligned}$$

with obvious meaning. Now, owing to (2.20), (3.10), Hölder's inequality, and the continuity of the embeddings $V \subset L^6(\Omega) \subset L^4(\Omega)$,

$$J_1 \leq C \int_0^T \|(\varphi^h - \varphi^*)(t)\|_4^4 dt \leq C \|\varphi^h - \varphi^*\|_{L^\infty(0,T;V)}^4 \leq C \|h\|_{L^2(0,T;H)}^4,$$

as well as

$$\begin{aligned} J_3 & \leq C \int_0^T \|(\varphi^h - \varphi^*)(t)\|_4^2 \|\nabla(\varphi^h - \varphi^*)(t)\|_{L^4(\Omega)^3}^2 dt \\ & \leq C \|\varphi^h - \varphi^*\|_{L^\infty(0,T;V)}^2 \|\varphi^h - \varphi^*\|_{L^2(0,T;W)}^2 \leq C \|h\|_{L^2(0,T;H)}^4. \end{aligned}$$

Finally, we infer that

$$\begin{aligned} J_2 & \leq C \int_0^T (\|\nabla \varphi^*(t)\|_{L^6(\Omega)^3}^2 + \|\nabla \varphi^h(t)\|_{L^6(\Omega)^3}^2) \|(\varphi^h - \varphi^*)(t)\|_6^4 dt \\ & \leq C \|\varphi^h - \varphi^*\|_{L^\infty(0,T;V)}^4 \left(\|\varphi^*\|_{L^2(0,T;W)}^2 + \|\varphi^h\|_{L^2(0,T;W)}^2 \right) \\ & \leq C \|h\|_{L^2(0,T;H)}^4, \end{aligned}$$

which concludes the proof of the assertion. \square

At this point, we note that the first component ξ of the solution (ξ, η, v) to the linearized system (3.3)–(3.7) cannot be expected to belong to $L^\infty(Q)$, since we only have $\eta \in L^2(0,T;V)$ in the elliptic equation (3.4). This lack of regularity prevents us from establishing the differentiability properties of \mathcal{S} needed below in the control theory (namely, that \mathcal{S} is twice continuously Fréchet differentiable into a Banach space of sufficiently regular functions) by applying the implicit function theorem as it could be done in [52] in the case of the viscous Cahn–Hilliard system. Therefore, in the less regular nonviscous case considered here, we are forced to derive

the required properties directly “by hand”, which however turns out to be possible. As the next step in this direction, we now going to show that the mapping $S' : L^2(0, T; H) \rightarrow \mathcal{L}(L^2(0, T; H); \mathcal{X})$, $u \mapsto S'(u)$, is locally Lipschitz continuous. We have the following result.

Theorem 3.2. *Suppose that (A1)–(A3) are fulfilled. Then there is a constant $K_5 > 0$, which depends only on the data of the state system and on R , such that the following holds: whenever $u_i \in \mathcal{U}_R$, $i = 1, 2$, are given, then it holds for every $h \in L^2(0, T; H)$ that*

$$\|(S'(u_1) - S'(u_2))[h]\|_X \leq K_5 \|u_1 - u_2\|_{L^2(0, T; H)} \|h\|_{L^2(0, T; H)}. \quad (3.21)$$

Proof. Let $h \in L^2(0, T; H)$ be fixed. We set $(\varphi_i, \mu_i, w_i) := \mathcal{S}(u_i)$ and $(\xi_i, \eta_i, v_i) := S'(u_i)[h]$, for $i = 1, 2$, and put

$$\begin{aligned} u &:= u_1 - u_2, & \varphi &:= \varphi_1 - \varphi_2, & \mu &:= \mu_1 - \mu_2, & w &:= w_1 - w_2, \\ \xi &:= \xi_1 - \xi_2, & \eta &:= \eta_1 - \eta_2, & v &:= v_1 - v_2. \end{aligned}$$

It then easily follows that $(\xi, \eta, v) \in \mathcal{X}$ is a solution to the system

$$\langle \partial_t \xi, \rho \rangle + \int_{\Omega} \nabla \xi \cdot \nabla \rho = 0 \quad \text{for all } \rho \in V \text{ and a.e. } t \in (0, T), \quad (3.22)$$

$$-\Delta \xi - \eta - v = a\xi + g \quad \text{a.e. in } Q, \quad (3.23)$$

$$\gamma \partial_t v + v = 0 \quad \text{a.e. in } Q, \quad (3.24)$$

$$\partial_{\mathbf{n}} \xi = 0 \quad \text{a.e. on } \Sigma, \quad (3.25)$$

$$\xi(0) = 0, \quad v(0) = 0, \quad \text{a.e. in } \Omega, \quad (3.26)$$

where we have put

$$a := -f''(\varphi_1), \quad g := -(f''(\varphi_1) - f''(\varphi_2))\xi_2. \quad (3.27)$$

Again, this is a system of the form (2.58)–(2.62), and once more it is easily shown that $\|a\|_{L^2(0, T; W^{1,4}(\Omega))}$ is bounded. Hence, by Lemma 2.5, the result will be proved once we can show that

$$\|g\|_{L^2(0, T; V)} \leq C \|u\|_{L^2(0, T; H)} \|h\|_{L^2(0, T; H)}. \quad (3.28)$$

Now observe that, by Taylor’s formula,

$$f''(\varphi_1) - f''(\varphi_2) = \int_0^1 f'''(\varphi_2 + s(\varphi_1 - \varphi_2)) \, ds \varphi =: B^h \varphi,$$

which in view of (2.53) implies that, a.e. in Q ,

$$\begin{aligned} |g| &\leq |B^h| |\xi_2| |\varphi| \leq C |\xi_2| |\varphi|, \\ |\nabla g| &\leq C (|\xi_2| |\varphi| (|\nabla \varphi_1| + |\nabla \varphi_2|) + |\xi_2| |\nabla \varphi| + |\nabla \xi_2| |\varphi|). \end{aligned}$$

Next, we recall that

$$\|(\xi_i, \eta_i, v_i)\|_X = \|S'(u_i)[h]\|_X \leq C \|h\|_{L^2(0, T; H)}, \quad \text{for } i = 1, 2.$$

We therefore can conclude as follows:

$$\begin{aligned} \|g\|_{L^2(0,T;H)}^2 &\leq C \iint_Q |\xi_2|^2 |\varphi|^2 \leq C \int_0^T \|\xi_2(t)\|_4^2 \|\varphi(t)\|_4^2 dt \\ &\leq C \|\xi_2\|_{L^\infty(0,T;V)}^2 \|\varphi\|_{L^\infty(0,T;V)}^2 \leq C \|u\|_{L^2(0,T;H)}^2 \|h\|_{L^2(0,T;H)}^2, \end{aligned}$$

where we also have used (2.74). Moreover, by similar reasoning, and using the embedding $V \subset L^4(\Omega)$ once more,

$$\begin{aligned} \iint_Q |\nabla g|^2 &\leq C \iint_Q |\xi_2|^2 |\varphi|^2 (|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2) + C \iint_Q |\xi_2|^2 |\nabla \varphi|^2 \\ &\quad + C \iint_Q |\nabla \xi_2|^2 |\varphi|^2 \\ &\leq C \|\xi_2\|_{L^\infty(0,T;V)}^2 \|\varphi\|_{L^\infty(0,T;V)}^2 (\|\varphi_1\|_{L^2(0,T;W)}^2 + \|\varphi_2\|_{L^2(0,T;W)}^2) \\ &\quad + C \|\xi_2\|_{L^\infty(0,T;V)}^2 \|\varphi\|_{L^2(0,T;W)}^2 + C \|\varphi\|_{L^\infty(0,T;V)}^2 \|\xi_2\|_{L^2(0,T;W)}^2 \\ &\leq C \|\xi_2\|_{L^\infty(0,T;V) \cap L^2(0,T;W)}^2 \|\varphi\|_{L^\infty(0,T;V) \cap L^2(0,T;W)}^2 \\ &\leq C \|u\|_{L^2(0,T;H)}^2 \|h\|_{L^2(0,T;H)}^2. \end{aligned}$$

The assertion is thus proved. \square

Having shown this continuous dependence estimate, we can now proceed to prove that the control-to-state operator has a second Fréchet derivative. We have the following result:

Theorem 3.3. *Suppose that the conditions (A1)–(A3) are fulfilled. Then the control-to-state operator \mathcal{S} is for any $R > 0$ twice Fréchet differentiable in \mathcal{U}_R as a mapping from \mathcal{U} into \mathcal{X} . Moreover, for every $u^* \in \mathcal{U}_R$ and all increments $h, k \in L^2(0, T; H)$, the triple $(\psi, \nu, z) = \mathcal{S}''(u^*)[h, k] \in \mathcal{X}$ is the unique solution to the bilinearized system*

$$\langle \partial_t \psi, v \rangle + \int_\Omega \nabla \nu \cdot \nabla v = 0 \quad \text{for all } v \in V \text{ and a.e. in } (0, T), \quad (3.29)$$

$$-\Delta \psi - \nu - z = -f''(\varphi^*)\psi - f'''(\varphi^*)\xi^h \xi^k \quad \text{a.e. in } Q, \quad (3.30)$$

$$\gamma \partial_t z + z = 0 \quad \text{a.e. in } Q, \quad (3.31)$$

$$\partial_{\mathbf{n}} \psi = 0 \quad \text{a.e. on } \Sigma, \quad (3.32)$$

$$\psi(0) = 0, \quad z(0) = 0, \quad \text{a.e. in } \Omega, \quad (3.33)$$

where $(\xi^h, \eta^h, v^h) := \mathcal{S}'(u^*)[h]$ and $(\xi^k, \eta^k, v^k) := \mathcal{S}'(u^*)[k]$.

Proof. By virtue of Lemma 2.5, we first establish the existence of a unique solution $(\psi, \nu, z) \in \mathcal{X}$ to the system (3.29)–(3.33), where we immediately note that $z = 0$ a.e. in Q , due to (3.31) and (3.33). Indeed, the system (3.29)–(3.33) is of the form (2.58)–(2.62), where in this case we have that $a := -f''(\varphi^*)$ and $g := -f'''(\varphi^*)\xi^h \xi^k$. Since, again, $\|a\|_{L^2(0,T;W^{1,4}(\Omega))} \leq C$, it suffices to show that $\|g\|_{L^2(0,T;V)} \leq C$. We achieve this by proving an estimate of the form

$$\|g\|_{L^2(0,T;V)} \leq C \|h\|_{L^2(0,T;H)} \|k\|_{L^2(0,T;H)}, \quad (3.34)$$

which in view of (2.63) then also implies that the mapping $(h, k) \mapsto (\psi, \nu, z)$ is continuous from $L^2(0, T; H) \times L^2(0, T; H)$ into \mathcal{X} . To this end, recall that

$$\|\xi^h\|_{H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W)} \leq \|S'(u)[h]\|_{\mathcal{X}} \leq C \|h\|_{L^2(0, T; H)},$$

and a corresponding estimate holds true for ξ^k . Therefore, using (2.53), we have that

$$\begin{aligned} \|g\|_{L^2(0, T; H)}^2 &\leq C \iint_Q |\xi^h|^2 |\xi^k|^2 \leq C \int_0^T \|\xi^h(t)\|_4^2 \|\xi^k(t)\|_4^2 dt \\ &\leq C \|\xi^h\|_{L^\infty(0, T; V)}^2 \|\xi^k\|_{L^\infty(0, T; V)}^2 \leq C \|h\|_{L^2(0, T; H)}^2 \|k\|_{L^2(0, T; H)}^2. \end{aligned}$$

Moreover, in view of (2.53), we have a.e. in Q that

$$|\nabla g| \leq C (|\nabla \varphi^*| |\xi^h| |\xi^k| + |\nabla \xi^h| |\xi^k| + |\xi^h| |\nabla \xi^k|),$$

so that

$$\begin{aligned} \iint_Q |\nabla g|^2 &\leq C \iint_Q (|\nabla \varphi^*|^2 |\xi^h|^2 |\xi^k|^2 + |\nabla \xi^h|^2 |\xi^k|^2 + |\xi^h|^2 |\nabla \xi^k|^2) \\ &\leq C \int_0^T \|\nabla \varphi^*(t)\|_{L^6(\Omega)^3}^2 \|\xi^h(t)\|_6^2 \|\xi^k(t)\|_6^2 dt \\ &\quad + C \int_0^T \left(\|\nabla \xi^h(t)\|_{L^4(\Omega)^3}^2 \|\xi^k(t)\|_4^2 + \|\xi^h(t)\|_4^2 \|\nabla \xi^k(t)\|_{L^4(\Omega)^3}^2 \right) dt. \end{aligned}$$

Hence, it results that

$$\begin{aligned} \iint_Q |\nabla g|^2 &\leq C \|\varphi^*\|_{L^\infty(0, T; W)}^2 \|\xi^h\|_{L^\infty(0, T; V)}^2 \|\xi^k\|_{L^\infty(0, T; V)}^2 \\ &\quad + C \left(\|\xi^h\|_{L^2(0, T; W)}^2 \|\xi^k\|_{L^\infty(0, T; V)}^2 + \|\xi^h\|_{L^\infty(0, T; V)}^2 \|\xi^k\|_{L^2(0, T; W)}^2 \right) \\ &\leq C \|h\|_{L^2(0, T; H)}^2 \|k\|_{L^2(0, T; H)}^2, \end{aligned}$$

which concludes the existence and uniqueness proof.

We now show the differentiability result. For this, we have to show that

$$\begin{aligned} \sup_{\|h\|_{L^2(0, T; H)}=1} \|S'(u^* + k)[h] - S'(u^*)[h] - (\psi, \nu, z)\|_{\mathcal{X}} &= o(\|k\|_{L^2(0, T; H)}) \\ \text{as } \|k\|_{L^2(0, T; H)} &\rightarrow 0. \end{aligned} \tag{3.35}$$

To this end, let $h, k \in L^2(0, T; H)$ be given with $\|h\|_{L^2(0, T; H)} = 1$ and $u^* + k \in \mathcal{U}_R$. Next, we put $(\xi^h, \eta^h, v^h) := S'(u^*)[h]$ and $(\bar{\xi}^h, \bar{\eta}^h, \bar{v}^h) := S'(u^* + k)[h]$. We have, since $\|h\|_{L^2(0, T; H)} = 1$,

$$\|(\xi^h, \eta^h, v^h)\|_{\mathcal{X}} + \|(\bar{\xi}^h, \bar{\eta}^h, \bar{v}^h)\|_{\mathcal{X}} \leq C. \tag{3.36}$$

Moreover, it follows from (3.21) that

$$\|(\bar{\xi}^h - \xi^h, \bar{\eta}^h - \eta^h, \bar{v}^h - v^h)\|_{\mathcal{X}} \leq C \|k\|_{L^2(0, T; H)}. \tag{3.37}$$

Next, we consider the functions

$$\Phi := \bar{\xi}^h - \xi^h - \psi, \quad \rho := \bar{\eta}^h - \eta^h - \nu, \quad \omega := \bar{v}^h - v^h - z.$$

A little calculation then shows that the triple $(\Phi, \rho, \omega) \in \mathcal{X}$ solves the system

$$\langle \partial_t \Phi, v \rangle + \int_{\Omega} \nabla \Phi \cdot \nabla v = 0 \quad \text{for all } v \in V \text{ and a.e. } t \in (0, T), \quad (3.38)$$

$$-\Delta \Phi - \rho - \omega = -f''(\varphi^*)\Phi + g \quad \text{a.e. in } Q, \quad (3.39)$$

$$\gamma \partial_t \omega + \omega = 0 \quad \text{a.e. in } Q, \quad (3.40)$$

$$\partial_{\mathbf{n}} \Phi = 0 \quad \text{a.e. on } \Sigma, \quad (3.41)$$

$$\Phi(0) = 0, \quad \omega(0) = 0, \quad \text{a.e. in } \Omega, \quad (3.42)$$

where

$$\begin{aligned} g &= -(f''(\varphi^k) - f''(\varphi^*))(\bar{\xi}^h - \xi^h) - (f''(\varphi^k) - f''(\varphi^*) - f'''(\varphi^*)\xi^k)\xi^h \\ &=: g_1 + g_2, \end{aligned} \quad (3.43)$$

with obvious notation. Clearly, (3.38)–(3.42) is again of the form (2.58)–(2.62), and since $\| -f''(\varphi^*) \|_{L^2(0,T;W^{1,4}(\Omega))} \leq C$, the assertion will be proved once we can show that

$$\|g\|_{L^2(0,T;V)}^2 \leq C \|k\|_{L^2(0,T;H)}^4. \quad (3.44)$$

At first, similar estimates as above, using (2.20), (2.53), (3.36) and (3.37), yield that

$$\begin{aligned} &\|g_1\|_{L^2(0,T;V)}^2 \\ &\leq C \iint_Q |f''(\varphi^k) - f''(\varphi^*)|^2 |\bar{\xi}^h - \xi^h|^2 \\ &\quad + C \iint_Q (|f'''(\varphi^k)|^2 |\nabla(\varphi^k - \varphi^*)|^2 + |\nabla \varphi^*|^2 |f'''(\varphi^k) - f'''(\varphi^*)|^2) |\bar{\xi}^h - \xi^h|^2 \\ &\quad + C \iint_Q |f''(\varphi^k) - f''(\varphi^*)|^2 |\nabla(\bar{\xi}^h - \xi^h)|^2, \end{aligned}$$

which leads to

$$\begin{aligned} &\|g_1\|_{L^2(0,T;V)}^2 \\ &\leq C \int_0^T \|(\varphi^k - \varphi^*)(t)\|_4^2 \|(\bar{\xi}^h - \xi^h)(t)\|_4^2 dt \\ &\quad + C \int_0^T \|\nabla(\varphi^k - \varphi^*)(t)\|_{L^4(\Omega)^3}^2 \|(\bar{\xi}^h - \xi^h)(t)\|_4^2 dt \\ &\quad + C \int_0^T \|\nabla \varphi^*(t)\|_{L^6(\Omega)^3}^2 \|(\varphi^k - \varphi^*)(t)\|_6^2 \|(\bar{\xi}^h - \xi^h)(t)\|_6^2 dt \\ &\quad + C \int_0^T \|(\varphi^k - \varphi^*)(t)\|_4^2 \|\nabla(\bar{\xi}^h - \xi^h)(t)\|_{L^4(\Omega)^3}^2 dt \\ &\leq C \|k\|_{L^2(0,T;H)}^4. \end{aligned} \quad (3.45)$$

Next, observe that

$$g_2 = -\xi^h (f'''(\varphi^*) (\varphi^k - \varphi^* - \xi^k) + Q^k (\varphi^k - \varphi^*)^2)$$

where

$$Q^k := \int_0^1 (1-s) f^{(4)}(\varphi^* + s(\varphi^k - \varphi^*)) ds$$

satisfies, by virtue of (2.53),

$$|Q^k| \leq C, \quad |\nabla Q^k| \leq C (|\nabla \varphi^*| + |\nabla \varphi^k|), \quad \text{a.e. in } Q. \quad (3.46)$$

We therefore have

$$\begin{aligned} \|g_2\|_{L^2(0,T;V)}^2 &\leq C \iint_Q |\xi^h|^2 (|\varphi^k - \varphi^* - \xi^k|^2 + |\varphi^k - \varphi^*|^4) \\ &\quad + C \iint_Q |\nabla \xi^h|^2 (|\varphi^k - \varphi^* - \xi^k|^2 + |\varphi^k - \varphi^*|^4) \\ &\quad + C \iint_Q |\xi^h|^2 (|\nabla \varphi^*|^2 |\varphi^k - \varphi^* - \xi^k|^2 + |\nabla(\varphi^k - \varphi^* - \xi^k)|^2) \\ &\quad + C \iint_Q |\xi^h|^2 (|\nabla Q^k|^2 |\varphi^k - \varphi^*|^4 + |Q^k|^2 |\varphi^k - \varphi^*|^2 |\nabla(\varphi^k - \varphi^*)|^2). \end{aligned}$$

Based on this, we can infer that

$$\begin{aligned} &\|g_2\|_{L^2(0,T;V)}^2 \\ &\leq C \int_0^T \|\xi^h(t)\|_4^2 \|(\varphi^k - \varphi^* - \xi^k)(t)\|_4^2 dt + C \int_0^T \|\xi^h(t)\|_6^2 \|(\varphi^k - \varphi^*)(t)\|_6^4 dt \\ &\quad + C \int_0^T \|\nabla \xi^h(t)\|_{L^4(\Omega)^3}^2 \|(\varphi^k - \varphi^* - \xi^k)(t)\|_4^2 dt \\ &\quad + C \int_0^T \|\nabla \xi^h(t)\|_{L^6(\Omega)^3}^2 \|(\varphi^k - \varphi^*)(t)\|_6^4 dt \\ &\quad + C \int_0^T \|\xi^h(t)\|_6^2 \|\nabla \varphi^*(t)\|_{L^6(\Omega)^3}^2 \|(\varphi^k - \varphi^* - \xi^k)(t)\|_6^2 dt \\ &\quad + C \int_0^T \|\xi^h(t)\|_4^2 \|\nabla(\varphi^k - \varphi^* - \xi^k)(t)\|_{L^4(\Omega)^3}^2 dt \\ &\quad + C \int_0^T \|\xi^h(t)\|_6^2 \|\nabla Q^k(t)\|_{L^6(\Omega)^3}^2 \|(\varphi^k - \varphi^*)(t)\|_6^2 \|(\varphi^k - \varphi^*)(t)\|_\infty^2 dt \\ &\quad + C \int_0^T \|\xi^h(t)\|_6^2 \|(\varphi^k - \varphi^*)(t)\|_6^2 \|\nabla(\varphi^k - \varphi^*)(t)\|_6^2 dt \\ &=: \sum_{j=1}^8 M_j, \end{aligned} \quad (3.47)$$

with obvious notation. It remains to show that $M_j \leq C \|k\|_{L^2(0,T;H)}^4$, for $1 \leq j \leq 8$. In order not to overload the exposition, we restrict ourselves to show this for only two of the terms, leaving the check of the others to the interested reader. To this end, recall that in the proof of Theorem 3.1 we have shown the validity of (3.9) (with k replaced by h), which implies that

$$\begin{aligned} & \|\varphi^k - \varphi^* - \xi^k\|_{H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \\ & \leq \|\mathcal{S}(u^* + k) - \mathcal{S}(u^*) - \mathcal{S}'(u^*)[k]\|_X \leq C \|k\|_{L^2(0,T;H)}^2. \end{aligned} \quad (3.48)$$

By virtue of (3.36) and the continuity of the embedding $V \subset L^4(\Omega)$, we therefore conclude that

$$|M_6| \leq C \|\xi^h\|_{L^\infty(0,T;V)}^2 \|\varphi^k - \varphi^* - \xi^k\|_{L^2(0,T;W)}^2 \leq C \|k\|_{L^2(0,T;H)}^4.$$

Moreover, invoking (3.36), (3.46), (2.20), (2.74), and the continuity of the embeddings $V \subset L^6(\Omega)$ and $W \subset L^\infty(\Omega)$, we also have that

$$\begin{aligned} |M_7| & \leq C \left(\|\varphi^*\|_{L^\infty(0,T;W)}^2 + \|\varphi^k\|_{L^\infty(0,T;W)}^2 \right) \|\varphi^k - \varphi^*\|_{L^\infty(0,T;V)}^2 \|\varphi^k - \varphi^*\|_{L^2(0,T;W)}^2 \\ & \leq C \|k\|_{L^2(0,T;H)}^4. \end{aligned}$$

With this, the assertion is proved. \square

Finally, we show that the mapping $u \mapsto \mathcal{S}''(u)$ is locally Lipschitz continuous. We have the following result.

Theorem 3.4. *The mapping $\mathcal{S}'' : L^2(0,T;H) \rightarrow \mathcal{L}(L^2(0,T;H), \mathcal{L}(L^2(0,T;H), \mathcal{X}))$, $u \mapsto \mathcal{S}''(u)$, is Lipschitz continuous in the following sense: there exists a constant $K_6 > 0$, which depends only on R and the data, such that, for all controls $u_1, u_2 \in \mathcal{U}_R$ and all increments $h, k \in L^2(0,T;H)$, it holds that*

$$\|(\mathcal{S}''(u_1) - \mathcal{S}''(u_2))[h, k]\|_X \leq K_6 \|u_1 - u_2\|_{L^2(0,T;H)} \|h\|_{L^2(0,T;H)} \|k\|_{L^2(0,T;H)}. \quad (3.49)$$

Proof. Let $u_1, u_2 \in \mathcal{U}_R$ and $h, k \in L^2(0,T;H)$ be given. We put

$$\begin{aligned} (\varphi_i, \mu_i, w_i) & := \mathcal{S}(u_i), \quad (\xi_i^h, \eta_i^h, v_i^h) := \mathcal{S}'(u_i)[h], \quad (\xi_i^k, \eta_i^k, v_i^k) := \mathcal{S}'(u_i)[k], \\ (\psi_i, \nu_i, z_i) & := \mathcal{S}''(u_i)[h, k], \quad \text{for } i = 1, 2, \end{aligned}$$

where we recall (2.74), (3.21) and the fact that $\|\mathcal{S}'(u_i)[h]\|_X \leq C \|h\|_{L^2(0,T;H)}$, $i = 1, 2$, and that an analogous estimate holds true for $\mathcal{S}'(u_i)[k]$. Moreover, a little calculation shows that the triple $(\psi, \nu, z) \in \mathcal{X}$ solves the system

$$\langle \partial_t \psi, y \rangle + \int_\Omega \nabla \psi \cdot \nabla y = 0 \quad \text{for all } y \in V \text{ and a.e. in } Q, \quad (3.50)$$

$$\Delta \psi - \nu - z = a\psi + g \quad \text{a.e. in } Q, \quad (3.51)$$

$$\gamma \partial_t z + z = 0 \quad \text{a.e. in } Q, \quad (3.52)$$

$$\partial_{\mathbf{n}} \psi = 0 \quad \text{a.e. on } \Sigma, \quad (3.53)$$

$$\psi(0) = 0, \quad z(0) = 0, \quad \text{a.e. in } \Omega, \quad (3.54)$$

where $a := -f''(\varphi_2)$ is bounded in $L^2(0, T; W^{1,4}(\Omega))$ and

$$\begin{aligned} g &:= -(f''(\varphi_1) - f''(\varphi_2))\psi_1 - (f'''(\varphi_1) - f'''(\varphi_2))\xi_1^h \xi_1^k \\ &\quad - f'''(\varphi_2)(\xi_1^h - \xi_2^h)\xi_1^k - f'''(\varphi_2)\xi_2^h(\xi_1^k - \xi_2^k) \\ &=: \sum_{j=1}^4 g_j, \end{aligned} \tag{3.55}$$

with obvious notation. In view of (2.63) in Lemma 2.5, it suffices to show that

$$\|g_j\|_{L^2(0,T;V)}^2 \leq C \|u\|_{L^2(0,T;H)} \|h\|_{L^2(0,T;H)} \|k\|_{L^2(0,T;H)} \quad \text{for } j = 1, 2, 3, 4. \tag{3.56}$$

We demonstrate this only for the second and third terms. The other two terms can be treated similarly and, in order to keep the paper at a reasonable length, are left to the reader. We have, using (2.20), (2.53), (2.74), (3.21), Hölder's inequality, and the continuity of the embeddings $V \subset L^6(\Omega) \subset L^4(\Omega)$ and $W \subset L^\infty(\Omega)$,

$$\begin{aligned} &\|g_2\|_{L^2(0,T;V)}^2 \\ &\leq C \int_0^T \|(\varphi_1 - \varphi_2)(t)\|_6^2 \|\xi_1^h(t)\|_6^2 \|\xi_1^k(t)\|_6^2 dt \\ &\quad + C \int_0^T \|(\varphi_1 - \varphi_2)(t)\|_\infty^2 \|\nabla\varphi_1(t)\|_{L^6(\Omega)^3}^2 \|\xi_1^h(t)\|_6^2 \|\xi_1^k(t)\|_6^2 dt \\ &\quad + C \int_0^T \|\nabla(\varphi_1 - \varphi_2)(t)\|_{L^6(\Omega)^3}^2 \|\xi_1^h(t)\|_6^2 \|\xi_1^k(t)\|_6^2 dt \\ &\quad + C \int_0^T \|(\varphi_1 - \varphi_2)(t)\|_6^2 \|\nabla\xi_1^h(t)\|_{L^6(\Omega)^3}^2 \|\xi_1^k(t)\|_6^2 dt \\ &\quad + C \int_0^T \|(\varphi_1 - \varphi_2)(t)\|_6^2 \|\xi_1^h(t)\|_6^2 \|\nabla\xi_1^k(t)\|_{L^6(\Omega)^3}^2 dt, \end{aligned}$$

so that

$$\begin{aligned} &\|g_2\|_{L^2(0,T;V)}^2 \\ &\leq C \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V)}^2 \|\xi_1^h\|_{L^\infty(0,T;V)}^2 \|\xi_1^k\|_{L^\infty(0,T;V)}^2 \\ &\quad + C \|\varphi_1\|_{L^\infty(0,T;W)}^2 \|\xi_1^h\|_{L^\infty(0,T;V)}^2 \|\xi_1^k\|_{L^\infty(0,T;V)}^2 \|\varphi_1 - \varphi_2\|_{L^2(0,T;W)}^2 \\ &\quad + C \|\xi_1^h\|_{L^\infty(0,T;V)}^2 \|\xi_1^k\|_{L^\infty(0,T;V)}^2 \|\varphi_1 - \varphi_2\|_{L^2(0,T;W)}^2 \\ &\quad + C \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V)}^2 \|\xi_1^h\|_{L^2(0,T;W)}^2 \|\xi_1^k\|_{L^\infty(0,T;V)}^2 \\ &\quad + C \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;V)}^2 \|\xi_1^h\|_{L^\infty(0,T;V)}^2 \|\xi_1^k\|_{L^2(0,T;W)}^2 \\ &\leq C \|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{\mathfrak{X}}^2 \|\mathcal{S}'(u_1)[h]\|_{\mathfrak{X}}^2 \|\mathcal{S}'(u_1)[k]\|_{\mathfrak{X}}^2 \\ &\leq C \|u_1 - u_2\|_{L^2(0,T;H)}^2 \|h\|_{L^2(0,T;H)}^2 \|k\|_{L^2(0,T;H)}^2. \end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
\|g_3\|_{L^2(0,T;V)}^2 &\leq C \int_0^T \|(\xi_1^h - \xi_2^h)(t)\|_4^2 \|\xi_1^k(t)\|_4^2 dt \\
&\quad + C \int_0^T \|\nabla \varphi_2(t)\|_{L^6(\Omega)^3}^2 \|(\xi_1^h - \xi_2^h)(t)\|_6^2 \|\xi_1^k(t)\|_6^2 dt \\
&\quad + C \int_0^T \|\nabla(\xi_1^h - \xi_2^h)(t)\|_{L^4(\Omega)^3}^2 \|\xi_1^k(t)\|_4^2 dt \\
&\quad + C \int_0^T \|(\xi_1^h - \xi_2^h)(t)\|_4^2 \|\nabla \xi^{k_1}(t)\|_{L^4(\Omega)^3}^2 dt \\
&\leq C \|u_1 - u_2\|_{L^2(0,T;H)}^2 \|h\|_{L^2(0,T;H)}^2 \|k\|_{L^2(0,T;H)}^2.
\end{aligned}$$

With this, the assertion is proved. \square

Remark 3.5. With Theorem 3.4, we have shown that the control-to-state operator \mathcal{S} is twice continuously Fréchet differentiable as a mapping from $\mathcal{U} = L^2(0, T; H)$ into \mathcal{X} . This result paves the way to prove first-order necessary and second-order sufficient optimality conditions for the optimal control problem **(CP)** in the following section.

4. THE OPTIMAL CONTROL PROBLEM

In this section, we study the optimal control problem **(CP)** with the cost functional (1.1). Besides the general conditions **(A1)**–**(A3)**, we make the following assumptions:

(A4) It holds $b_1 \geq 0$, $b_2 \geq 0$, $b_3 > 0$, and $\kappa > 0$.

(A5) The thresholds $\underline{u}, \bar{u} \in L^\infty(Q)$ satisfy $\underline{u} \leq \bar{u}$ almost everywhere in Q , and the target functions satisfy $\varphi_Q \in L^2(Q)$ and $\varphi_\Omega \in V$.

We assume $\kappa > 0$ to include the effects of sparsity. By an obvious modification, the theory of second-order conditions remains valid also for $\kappa = 0$.

Remark 4.1. The assumption $\varphi_\Omega \in V$ is useful in order to have more regular solutions to the associated adjoint system (see below). It is not overly restrictive in view of the continuous embedding $(H^1(0, T; H) \cap L^2(0, T; W)) \subset C^0([0, T]; V)$ which implies that $\varphi(T) \in V$.

The following existence result can be shown with a standard argument.

Theorem 4.2. *Let **(A1)**–**(A5)** hold and suppose that $G : L^2(0, T; H) \rightarrow \mathbb{R}$ is nonnegative, convex and continuous. Then the optimal control problem **(CP)** admits a solution $u^* \in \mathcal{U}_{\text{ad}}$.*

4.1. The adjoint system

In the following, let $u^* \in \mathcal{U}_R$ be fixed and $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ be the associated state. The corresponding adjoint state system is formally given by:

$$-\partial_t p - \Delta q + f''(\varphi^*)q = b_1(\varphi^* - \varphi_Q) \quad \text{in } Q, \quad (4.1)$$

$$-\Delta p - q = 0 \quad \text{in } Q, \quad (4.2)$$

$$-\gamma \partial_t r + r - q = 0 \quad \text{in } Q, \quad (4.3)$$

$$\partial_{\mathbf{n}} p = \partial_{\mathbf{n}} q = 0 \quad \text{on } \Sigma, \quad (4.4)$$

$$p(T) = b_2(\varphi^*(T) - \varphi_\Omega), \quad r(T) = 0 \quad \text{in } \Omega. \quad (4.5)$$

We immediately observe that the system is decoupled in the sense that r can be directly recovered from (4.3) with the terminal condition $r(T) = 0$ once q is determined. We point out that (4.1) has to be rewritten in a weak (variational) form. We now show a well-posedness result for a slightly more general system.

Theorem 4.3. *Suppose that (A1)–(A3) are fulfilled, and assume that $u^* \in \mathcal{U}_R$ with $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$, $g_1 \in L^2(0, T; H)$ and $g_2 \in V$ are given. Then the system*

$$\langle -\partial_t p, v \rangle + \int_{\Omega} \nabla q \cdot \nabla v + \int_{\Omega} f''(\varphi^*) q v = \int_{\Omega} g_1 v$$

for all $v \in V$, a.e. in $(0, T)$,

(4.6)

$$-\Delta p - q = 0 \quad \text{a.e. in } Q, \tag{4.7}$$

$$-\gamma \partial_t r + r - q = 0 \quad \text{a.e. in } Q, \tag{4.8}$$

$$\partial_{\mathbf{n}} p = 0 \quad \text{a.e. on } \Sigma, \tag{4.9}$$

$$p(T) = g_2, \quad r(T) = 0, \quad \text{a.e. in } \Omega, \tag{4.10}$$

has a unique solution triple (p, q, r) with the regularity

$$p \in H^1(0, T; V^*) \cap C^0([0, T]; V) \cap L^2(0, T; W \cap H^3(\Omega)), \tag{4.11}$$

$$q \in L^2(0, T; V), \tag{4.12}$$

$$r \in H^1(0, T; V). \tag{4.13}$$

Moreover, there is a constant $K_7 > 0$, which depends only on R and the data, such that the two inequalities below hold:

$$\begin{aligned} & \|p\|_{C^0([0, T]; H) \cap L^2(0, T; W)} + \|q\|_{L^2(0, T; H)} + \|r\|_{H^1(0, T; H)} \\ & \leq K_7 (\|g_1\|_{L^2(0, T; H)} + \|g_2\|_H), \end{aligned} \tag{4.14}$$

$$\begin{aligned} & \|p\|_{C^0([0, T]; V) \cap L^2(0, T; W \cap H^3(\Omega))} + \|q\|_{L^2(0, T; V)} + \|r\|_{H^1(0, T; V)} \\ & \leq K_7 (\|g_1\|_{L^2(0, T; H)} + \|g_2\|_V). \end{aligned} \tag{4.15}$$

Proof. The linear initial-boundary value problem given by (4.6), (4.7), (4.9), together with the first terminal condition in (4.5), is again solved *via* a Faedo–Galerkin approximation using the same eigenvalues, eigenfunctions and n -dimensional approximating spaces V_n as in the proof of Theorem 2.1. For the sake of shortness, we avoid to write the approximating n -dimensional analogues of (4.6)–(4.7) explicitly here and just provide the relevant *a priori* estimates formally for the continuous problem. Having these estimates for the n -dimensional approximations, one can apply the standard weak and weak-star compactness arguments to pass to the limit as $n \rightarrow \infty$, thereby showing the existence of the sought solution. Uniqueness then follows immediately from the linearity and the estimate (4.14).

To this end, we insert $v = p$ in (4.6) and test (4.7) by $-q$. Then we add the resultants, noting that a cancellation of two terms occurs, and integrate over (t, T) , where $t \in [0, T)$ is arbitrary. Introducing the notation $Q^t := \Omega \times (t, T)$ for $t \in [0, T)$, we then obtain, after rearranging terms and invoking (2.53) and Young’s inequality,

$$\begin{aligned} & \frac{1}{2} \|p(t)\|_H^2 + \iint_{Q^t} |q|^2 = \frac{1}{2} \|g_2\|_H^2 - \iint_{Q^t} f''(\varphi^*) q p + \iint_{Q^t} g_1 p \\ & \leq \frac{1}{2} (\|g_1\|_{L^2(0, T; H)}^2 + \|g_2\|_H^2) + \frac{1}{2} \iint_{Q^t} |q|^2 + C \iint_{Q^t} |p|^2. \end{aligned}$$

Gronwall's lemma then yields that

$$\|p\|_{L^\infty(0,T;H)}^2 + \|q\|_{L^2(0,T;H)}^2 \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_H^2 \right). \quad (4.16)$$

In addition, we conclude from (4.7) and (4.9), invoking standard elliptic estimates, that

$$\|p\|_{L^2(0,T;W)}^2 \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_H^2 \right), \quad (4.17)$$

and comparison in (4.8) immediately shows that also

$$\|r\|_{H^1(0,T;H)}^2 \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_H^2 \right). \quad (4.18)$$

The validity of the inequality (4.14) is thus shown.

Next, we insert $v = q$ in (4.6) and test (4.7) (formally) by $-\partial_t p$. Addition and integration over (t, T) then yields

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla p(t)|^2 + \iint_{Q^t} |\nabla q|^2 &= \frac{1}{2} \int_{\Omega} |\nabla g_2|^2 + \iint_{Q^t} (g_1 q - f''(\varphi^*) q^2) \\ &\leq \|g_2\|_V^2 + \|g_1\|_{L^2(0,T;H)}^2 + C \iint_{Q^t} |q|^2 \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_V^2 \right), \end{aligned}$$

by virtue of Young's inequality, and invoking the assumption $g_2 \in V$ along with (2.53) and (4.16). Then, we deduce that

$$\|p\|_{L^\infty(0,T;V)} + \|q\|_{L^2(0,T;V)} \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_V^2 \right), \quad (4.19)$$

whence, using (4.7) and elliptic regularity,

$$\|p\|_{L^2(0,T;W \cap H^3(\Omega))} \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_V^2 \right). \quad (4.20)$$

In addition, using (4.19) and the endpoint condition $r(T) = 0$, we obtain directly from (4.8) that also

$$\|r\|_{H^1(0,T;V)}^2 \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_V^2 \right). \quad (4.21)$$

Moreover, comparison in (4.6) yields that

$$\|p\|_{H^1(0,T;V^*)} \leq C \left(\|g_1\|_{L^2(0,T;H)}^2 + \|g_2\|_V^2 \right).$$

Finally, as the embedding $(H^1(0, T; V^*) \cap L^2(0, T; H^3(\Omega))) \subset C^0([0, T]; V)$ is continuous, (4.11) follows, and the inequality (4.15) results from (4.19)–(4.21). This concludes the proof of the assertion. \square

Remark 4.4. This remark collects three different comments.

1. From the proof of Theorem 4.3 the reader may realize that a weaker existence and uniqueness result holds if

$g_1 \in L^2(0, T; H)$ and g_2 is just in H , with a solution (p, q, r) having the regularity expressed in (4.14). In fact, if we consider a weaker formulation of (4.6), namely,

$$w^* \langle -\partial_t p, v \rangle_W - \int_{\Omega} q \Delta v + \int_{\Omega} f''(\varphi^*) q v = \int_{\Omega} g_1 v \quad \text{for all } v \in W, \text{ a.e. in } (0, T),$$

then we deduce from (4.16)–(4.18) and a comparison of terms in the above equation that

$$\|\partial_t p\|_{L^2(0, T; W^*)} \leq C \left(\|g_1\|_{L^2(0, T; H)}^2 + \|g_2\|_H^2 \right),$$

which, along with (4.17), implies that $p \in C^0([0, T]; H)$ and gives a meaning to the first terminal condition in (4.10). Of course, uniqueness then follows from inequality (4.14).

2. In the case of the adjoint system (4.1)–(4.5), the one of interest for our theory, we have $g_1 := b_1(\varphi^* - \varphi_Q) \in L^2(0, T; H)$ and $g_2 := b_2(\varphi^*(T) - \varphi_{\Omega}) \in V$, so that Theorem 4.3 ensures that for every $u^* \in L^2(0, T; H)$ there is a unique solution (p^*, q^*, r^*) with the regularity (4.11)–(4.13) that satisfies (4.15).

3. If, for $i = 1, 2$, $u_i \in \mathcal{U}_R$ is given with the associated state $(\varphi_i, \mu_i, w_i) = \mathcal{S}(u_i)$ and adjoint state (p_i, q_i, r_i) , then the triple $(p_1 - p_2, q_1 - q_2, r_1 - r_2)$ solves the system (4.6)–(4.10) with $g_1 := b_1(\varphi_1 - \varphi_2) \in L^2(0, T; H)$ and $g_2 := b_2(\varphi_1(T) - \varphi_2(T)) \in V$. In view of (2.74), it therefore follows from (4.15) the estimate

$$\begin{aligned} & \|p_1 - p_2\|_{C^0([0, T]; V) \cap L^2(0, T; W \cap H^3(\Omega))} + \|q_1 - q_2\|_{L^2(0, T; V)} + \|r_1 - r_2\|_{H^1(0, T; V)} \\ & \leq C \|\varphi_1 - \varphi_2\|_{C^0([0, T]; V)} \leq C \|u_1 - u_2\|_{L^2(0, T; H)}. \end{aligned} \quad (4.22)$$

4.2. First-order necessary optimality conditions

In this section, we derive first-order necessary optimality conditions for local minima of the optimal control problem (CP). We assume that (A1)–(A5) are fulfilled and that $G : L^2(0, T; H) \rightarrow \mathbb{R}$ is a general nonnegative, convex and continuous functional. We define the reduced cost functionals associated with the functionals J and \mathcal{J} introduced in (1.1) by

$$\widehat{J}(u) := J(\mathcal{S}_1(u), u), \quad \widehat{\mathcal{J}}(u) = \mathcal{J}(\mathcal{S}_1(u), u). \quad (4.23)$$

Since $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is twice continuously Fréchet differentiable from $\mathcal{U} = L^2(0, T; H)$ into the space $C^0([0, T]; H) \times L^2(0, T; V) \times H^1(0, T; H)$ (which contains \mathcal{X}), it follows from the chain rule that the smooth part \widehat{J} of the reduced objective functional is a twice continuously Fréchet differentiable mapping from \mathcal{U} into \mathbb{R} , where, for every $u^* \in L^2(0, T; H)$ and every $h \in L^2(0, T; H)$, it holds with $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ that

$$\widehat{J}'(u^*)[h] = b_1 \iint_Q \xi^h(\varphi^* - \varphi_Q) + b_2 \int_{\Omega} \xi^h(T)(\varphi^*(T) - \varphi_{\Omega}) + b_3 \iint_Q u^* h, \quad (4.24)$$

where $(\xi^h, \eta^h, v^h) = S'(u^*)[h] \in \mathcal{Z}$ is the unique solution to the linearized system (3.3)–(3.7) associated with h .

In the following, we assume that $u^* \in \mathcal{U}_{\text{ad}}$ is a locally optimal control for (CP) in the sense of $L^\infty(Q)$. In this connection, recall that a control $u^* \in \mathcal{U}_{\text{ad}}$ is called *locally optimal in the sense of $L^p(Q)$* for some $p \in [1, +\infty]$ if and only if there is some $\varepsilon > 0$ such that

$$\widehat{\mathcal{J}}(u) \geq \widehat{\mathcal{J}}(u^*) \quad \text{for all } u \in \mathcal{U}_{\text{ad}} \text{ satisfying } \|u - u^*\|_{L^p(Q)} \leq \varepsilon. \quad (4.25)$$

It is easily seen that any locally optimal control in the sense of $L^p(Q)$ with $1 \leq p < \infty$ is also locally optimal in the sense of $L^\infty(Q)$. Therefore, a result proved for locally optimal controls in the sense of $L^\infty(Q)$ is also valid

for locally optimal controls in the sense of $L^p(Q)$ for $1 \leq p < \infty$. In particular, it is true for (globally) optimal controls.

A standard argument from convex analysis (for the details see, *e.g.*, [37], Sect. 3) then shows that there is some $\lambda^* \in \partial G(u^*) \subset L^2(0, T; H)$ such that

$$\widehat{\mathcal{J}}(u^*)[u - u^*] + \kappa \iint_Q \lambda^*(u - u^*) \geq 0 \quad \forall u \in \mathcal{U}_{\text{ad}}. \quad (4.26)$$

As usual, we simplify the expression $\widehat{\mathcal{J}}(u^*)[u - u^*]$ in (4.26) by means of the adjoint state variables defined in (4.1)–(4.5). A standard calculation using the linearized system (3.3)–(3.7) then leads to the following result.

Theorem 4.5. (Necessary optimality condition) *Suppose that (A1)–(A5) are fulfilled and that $G : L^2(0, T; H) \rightarrow \mathbb{R}$ is nonnegative, convex and continuous. Moreover, let $u^* \in \mathcal{U}_{\text{ad}}$ be a locally optimal control of (CP) in the sense of $L^\infty(Q)$ with associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and adjoint state (p^*, q^*, r^*) . Then there exists some $\lambda^* \in \partial G(u^*)$ such that, for all $u \in \mathcal{U}_{\text{ad}}$,*

$$\iint_Q (r^* + \kappa \lambda^* + b_3 u^*) (u - u^*) \geq 0. \quad (4.27)$$

4.3. Sparsity of controls

The convex function G in the objective functional accounts for the sparsity of optimal controls, *i.e.*, the possibility that any locally optimal control may vanish in some subset of the space-time cylinder Q . The form of this region depends on the particular choice of the functional G . The sparsity properties can be deduced from the variational inequality (4.27) and the particular form of the subdifferential ∂G . Here, we restrict ourselves to the case of *full sparsity* which is connected to the $L^1(Q)$ –norm functional G introduced in (1.9). Its subdifferential is given by (see [58])

$$\partial G(u) = \left\{ \lambda \in L^2(Q) : \lambda(x, t) \in \begin{cases} \{1\} & \text{if } u(x, t) > 0 \\ [-1, 1] & \text{if } u(x, t) = 0 \\ \{-1\} & \text{if } u(x, t) < 0 \end{cases} \text{ for a.e. } (x, t) \in Q \right\}.$$

With exactly the same proof as that of the corresponding result [52], Theorem 4.7, we obtain the following result.

Theorem 4.6. (Full sparsity) *Suppose that the assumptions (A1)–(A5) are fulfilled, and assume that \underline{u} and \bar{u} are constants such that $\underline{u} < 0 < \bar{u}$. Let $u^* \in \mathcal{U}_{\text{ad}}$ be a locally optimal control in the sense of $L^\infty(Q)$ for the problem (CP) with the functional*

$$G(u) = \|u\|_{L^1(Q)} = \iint_Q |u|,$$

with associated state $(\varphi^, \mu^*, w^*) = \mathcal{S}(u^*)$ solving (1.2)–(1.6) and adjoint state (p^*, q^*, r^*) solving (4.1)–(4.5). Then there exists a function $\lambda^* \in \partial G(u^*)$ satisfying (4.27), and it holds*

$$u^*(x, t) = 0 \quad \iff \quad |r^*(x, t)| \leq \kappa, \quad \text{for a.e. } (x, t) \in Q. \quad (4.28)$$

Moreover, if r^ and λ^* are given, then u^* is obtained from the projection formula*

$$u^*(x, t) = \max \{ \underline{u}, \min \{ \bar{u}, -b_3^{-1} (r^* + \kappa \lambda^*) (x, t) \} \} \text{ for a.e. } (x, t) \in Q.$$

4.4. Second-order sufficient optimality conditions

We conclude this paper with the derivation of second-order sufficient optimality conditions for functions u^* obeying the first-order necessary optimality conditions of Theorem 4.5. Second-order sufficient optimality conditions are based on a condition of coercivity that is required to hold for the smooth part J of \mathcal{J} in a certain critical cone. The nonsmooth part G contributes to sufficiency by its convexity. In the following, we generally assume that the conditions **(A1)**–**(A5)** are fulfilled. Our analysis will follow closely the lines of [52], which in turn is an adaptation of the technique used in the proof of [48], Theorem 3.4 for the sparse control of the FitzHugh–Nagumo system.

To this end, we fix a control u^* satisfying the first-order necessary optimality conditions, and we set $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$. Then the cone

$$C(u^*) = \{v \in L^2(0, T; H) \text{ satisfying the sign conditions (4.29) a.e. in } Q\},$$

where

$$v(x, t) \begin{cases} \geq 0 & \text{if } u^*(x, t) = \underline{u} \\ \leq 0 & \text{if } u^*(x, t) = \bar{u} \end{cases}, \quad (4.29)$$

is called the *cone of feasible directions*, which is a convex and closed subset of $L^2(0, T; H)$. We also need the directional derivative of the convex functional G at $u \in L^2(0, T; H)$ in the direction $v \in L^2(0, T; H)$, which is given by

$$G'(u, v) = \lim_{t \searrow 0} \frac{1}{t} (G(u + tv) - G(u)). \quad (4.30)$$

Following the definition of the critical cone (see [48], Sect. 3.1), which is inspired by finite-dimensional optimization theory, we define

$$C_{u^*} = \{v \in C(u^*) : \widehat{\mathcal{J}}'(u^*)[v] + \kappa G'(u^*, v) = 0\}, \quad (4.31)$$

which is also a closed and convex subset of $L^2(0, T; H)$. According to [48], Section 3.1, it consists of all $v \in C(u^*)$ satisfying

$$v(x, t) \begin{cases} = 0 & \text{if } |r^*(x, t) + b_3 u^*(x, t)| \neq \kappa \\ \geq 0 & \text{if } u^*(x, t) = \underline{u} \text{ or } (r^*(x, t) = -\kappa \text{ and } u^*(x, t) = 0) \\ \leq 0 & \text{if } u^*(x, t) = \bar{u} \text{ or } (r^*(x, t) = \kappa \text{ and } u^*(x, t) = 0) \end{cases}. \quad (4.32)$$

At this point, we provide an explicit expression for $\widehat{\mathcal{J}}''(u)[h, k]$ for arbitrary $u, h, k \in L^2(0, T; H)$. Arguing exactly as in the derivation of the corresponding formula [52], equation (4.52), we obtain that

$$\widehat{\mathcal{J}}''(u^*)[h, k] = \iint_Q (b_1 - f^{(3)}(\varphi^*)q^*) \xi^h \xi^k + b_2 \int_\Omega \xi^h(T) \xi^k(T) + b_3 \iint_Q h k, \quad (4.33)$$

where $(\xi^h, \eta^h, v^h) = \mathcal{S}'(u^*)[h]$ and $(\xi^k, \eta^k, v^k) = \mathcal{S}'(u^*)[k]$.

For the proof of the second-order sufficient optimality condition, we will need the following preparatory result.

Lemma 4.7. *Assume that **(A1)**–**(A5)** are satisfied and that $u^* \in \mathcal{U}_{\text{ad}}$ is given with associated state (φ^*, μ^*, w^*) and adjoint state (p^*, q^*, r^*) . Suppose that $\{\tilde{u}_j\} \subset \mathcal{U}_{\text{ad}}$ converges strongly in $L^2(0, T; H)$ to u^* , and that $\{h_j\} \subset L^2(0, T; H)$ converges weakly in $L^2(0, T; H)$ to h . In addition, let $(\tilde{\varphi}_j, \tilde{\mu}_j, \tilde{w}_j) = \mathcal{S}(\tilde{u}_j)$, and let $(\tilde{p}_j, \tilde{q}_j, \tilde{r}_j)$ be the*

associated adjoint state, for $j \in \mathbb{N}$. Moreover, let, for arbitrary $h \in L^2(0, T; H)$, $(\xi^h, \eta^h, v^h) = \mathcal{S}'(u^*)[h]$, as well as $(\tilde{\xi}^{h_j}, \tilde{\eta}^{h_j}, \tilde{v}^{h_j}) = \mathcal{S}'(\tilde{u}_j)[h_j]$, for $j \in \mathbb{N}$. Then

$$\lim_{j \rightarrow \infty} \hat{\mathcal{J}}(\tilde{u}_j)[h_j] = \hat{\mathcal{J}}(u^*)[h], \quad (4.34)$$

$$\begin{aligned} \lim_{j \rightarrow \infty} \left(\iint_Q (b_1 - f^{(3)}(\tilde{\varphi}_j)\tilde{q}_j) |\tilde{\xi}^{h_j}|^2 + b_2 \int_{\Omega} |\tilde{\xi}^{h_j}(T)|^2 \right) \\ = \iint_Q (b_1 - f^{(3)}(\varphi^*)q^*) |\xi^h|^2 + b_2 \int_{\Omega} |\xi^h(T)|^2. \end{aligned} \quad (4.35)$$

Proof. At first, notice that (4.22) yields that $\|\tilde{r}_j - r^*\|_{H^1(0, T; H)} \rightarrow 0$ as $j \rightarrow \infty$, which implies that

$$\lim_{j \rightarrow \infty} \hat{\mathcal{J}}(\tilde{u}_j)[h_j] = \lim_{j \rightarrow \infty} \iint_Q (\tilde{r}_j + b_3 \tilde{u}_j) h_j = \iint_Q (r^* + b_3 u^*) h = \hat{\mathcal{J}}(u^*)[h],$$

i.e., (4.34) is valid. Next, in order to prove (4.35), we observe that

$$(\tilde{\xi}^{h_j}, \tilde{\eta}^{h_j}, \tilde{v}^{h_j}) - (\xi^h, \eta^h, v^h) = (\mathcal{S}'(\tilde{u}_j) - \mathcal{S}'(u^*)) [h_j] + \mathcal{S}'(u^*)[h_j - h].$$

By virtue of (3.21) and the boundedness of $\{h_j\}$ in $L^2(0, T; H)$, the first summand on the right converges to zero strongly in \mathcal{X} . The second converges to zero weakly star in (cf. (3.1))

$$(H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W)) \times L^2(0, T; V) \times H^1(0, T; H).$$

Thanks to the compact embedding $V \subset L^p(\Omega)$ for $1 \leq p < 6$, the compactness result stated in [57], Section 8, Corollary 4 then ensures that

$$\tilde{\xi}^{h_j} \rightarrow \xi^h \quad \text{strongly in } C^0([0, T]; L^4(\Omega)). \quad (4.36)$$

In particular, we have that

$$\lim_{j \rightarrow \infty} \left(b_1 \iint_Q |\tilde{\xi}^{h_j}|^2 + b_2 \int_{\Omega} |\tilde{\xi}^{h_j}(T)|^2 \right) = b_1 \iint_Q |\xi^h|^2 + b_2 \int_{\Omega} |\xi^h(T)|^2. \quad (4.37)$$

Moreover, we obtain from (2.74) that $\|\tilde{\varphi}_j - \varphi^*\|_{H^1(0, T; V^*) \cap L^\infty(0, T; V)} \rightarrow 0$ as $j \rightarrow \infty$, so that we can conclude from [57], Section 8, Corollary 4, the global estimate (2.53), and (4.22), that, as $j \rightarrow \infty$,

$$\|f^{(3)}(\tilde{\varphi}_j) - f^{(3)}(\varphi^*)\|_{C^0([0, T]; L^4(\Omega))} \rightarrow 0, \quad (4.38)$$

$$\|\tilde{q}_j - q^*\|_{L^2(0, T; L^4(\Omega))} \rightarrow 0. \quad (4.39)$$

Combining this with (4.36), we readily verify that

$$\lim_{j \rightarrow \infty} \iint_Q f^{(3)}(\tilde{\varphi}_j)\tilde{q}_j |\xi^{h_j}|^2 = \iint_Q f^{(3)}(\varphi^*)q^* |\xi^h|^2, \quad (4.40)$$

which concludes the proof. \square

With Lemma 4.7 shown, the road is paved for the proof of second-order sufficient optimality conditions. To this end, we will employ the following coercivity condition:

$$\widehat{J}''(u^*)[v, v] > 0 \quad \forall v \in C_{u^*} \setminus \{0\}. \quad (4.41)$$

Condition (4.41) is a direct extension of associated conditions that are standard in finite-dimensional nonlinear optimization. In the optimal control of partial differential equation, it was first used in [42]. We have the following result.

Theorem 4.8. (Second-order sufficient condition) *Suppose that (A1)–(A5) are fulfilled. Moreover, let $u^* \in \mathcal{U}_{\text{ad}}$, together with the associated state $(\varphi^*, \mu^*, w^*) = \mathcal{S}(u^*)$ and the adjoint state (p^*, q^*, r^*) , fulfill the first-order necessary optimality conditions of Theorem 4.5. If, in addition, u^* satisfies the coercivity condition (4.41), then there exist constants $\varepsilon > 0$ and $\zeta > 0$ such that the quadratic growth condition*

$$\widehat{J}(u) \geq \widehat{J}(u^*) + \zeta \|u - u^*\|_{L^2(0, T; H)}^2 \quad (4.42)$$

holds for all $u \in \mathcal{U}_{\text{ad}}$ with $\|u - u^*\|_{L^2(0, T; H)} < \varepsilon$. Consequently, u^* is a locally optimal control in the sense of $L^2(0, T; H)$.

Proof. The proof is exactly the same as that of the corresponding [52], Theorem 4.8. In order not to seem repetitive, we therefore only sketch the argument, pointing out the places in the proof where the results of Lemma 4.7 are used. We argue by contradiction, assuming that the claim of the theorem is not true. Then there exists a sequence of controls $\{u_j\} \subset \mathcal{U}_{\text{ad}}$ such that, for all $j \in \mathbb{N}$,

$$\|u_j - u^*\|_{L^2(0, T; H)} < \frac{1}{j} \quad \text{while} \quad \widehat{J}(u_j) < \widehat{J}(u^*) + \frac{1}{2j} \|u_j - u^*\|_{L^2(0, T; H)}^2. \quad (4.43)$$

Noting that $u_j \neq u^*$ for all $j \in \mathbb{N}$, we define

$$\tau_j := \|u_j - u^*\|_{L^2(0, T; H)} \quad \text{and} \quad h_j := \frac{1}{\tau_j} (u_j - u^*).$$

Then $\|h_j\|_{L^2(0, T; H)} = 1$ and, possibly after selecting a subsequence, we can assume that

$$h_j \rightarrow h \text{ weakly in } L^2(0, T; H)$$

for some $h \in L^2(0, T; H)$. The proof is now split into three parts.

(i) $h \in C_{u^*}$: Here, one has to show that (cf. (4.31)) $\widehat{J}'(u^*)[h] + \kappa G'(u^*, h) = 0$, in particular. This follows exactly as in the corresponding step (i) in [52], where in the proof of the inequality $\widehat{J}'(u^*)[h] + \kappa G'(u^*, h) \leq 0$ the identity (4.34) is used.

(ii) $h = 0$: The proof of this claim is again exactly the same as in the corresponding step (ii) in the proof of [52], Theorem 4.8. It uses the identity (4.35) in order to show that (4.43) implies that $\widehat{J}''(u^*)[h, h] \leq 0$, whence $h = 0$ follows using (4.41).

(iii) *Contradiction*: Again, the argumentation is exactly the same as the corresponding step (iii) in the proof of [52], Theorem 4.8: we know from the previous step that $h_j \rightarrow 0$ weakly in $L^2(0, T; H)$. Now, (4.33) yields that

$$\widehat{J}_1''(u^*)[h_j, h_j] = \iint_Q (b_1 - f^{(3)}(u^*)q^*) |\xi^{h_j}|^2 + b_2 \int_\Omega |\xi^{h_j}(T)|^2 + b_3 \iint_Q |h_j|^2, \quad (4.44)$$

where we have set $(\xi^{h_j}, \eta^{h_j}, v^{h_j}) = \mathcal{S}'(u^*)[h_j]$, for $j \in \mathbb{N}$. Since $h_j \rightarrow 0$ weakly in $L^2(0, T; H)$, we find from (4.35) in Lemma 4.7 that the sum of the first two integrals on the right-hand side of (4.44) converges to zero. On the other hand, $\|h_j\|_{L^2(0, T; H)} = 1$ for all $j \in \mathbb{N}$, by construction. Therefore,

$$\liminf_{j \rightarrow \infty} \widehat{\mathcal{J}}''(u^*)[h_j, h_j] \geq \liminf_{j \rightarrow \infty} b_3 \iint_Q |h_j|^2 = b_3 > 0. \quad (4.45)$$

On the other hand, the condition (4.43) leads, using (4.35) again, to the conclusion that

$$\liminf_{j \rightarrow \infty} \widehat{\mathcal{J}}''(u^*)[h_j, h_j] \leq 0,$$

which contradicts (4.45). The assertion of the theorem is thus proved. \square

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DATA AVAILABILITY STATEMENT

This study does not involve any datasets; all results are theoretical and contained within the article.

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