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STABILIZATION OF CONTROL SYSTEMS ASSOCIATED WITH A STRONGLY CONTINUOUS GROUP

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Abstract. This paper is devoted to the stabilization of a linear control system $y' = Ay + Bu$ and its suitable non-linear variants where $(A, \mathcal{D}(A))$ is an infinitesimal generator of a strongly continuous *group* in a Hilbert space $\mathbb{H}$, and B defined in a Hilbert space $\mathbb{U}$ is an admissible control operator with respect to the semigroup generated by A. Let $\lambda \in \mathbb{R}$ and assume that, for some *positive* symmetric, invertible $Q = Q(\lambda) \in \mathcal{L}(\mathbb{H})$, for some *non-negative*, symmetric $R = R(\lambda) \in \mathcal{L}(\mathbb{H})$, and for some *non-negative*, symmetric $W = W(\lambda) \in \mathcal{L}(\mathbb{U})$, it holds

$$AQ + QA^* - BWB^* + QRQ + 2\lambda Q = 0.$$

We then present a new approach to study the stabilization of such a system and its suitable nonlinear variants. Both the stabilization using dynamic feedback controls and the stabilization using static feedback controls in a weak sense are investigated. To our knowledge, the nonlinear case is out of reach previously when B is unbounded for both types of stabilization.

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1. Introduction

In this paper, we study the stabilization of a linear control system associated with a strongly continuous *group* and its related nonlinear systems. Let $\mathbb{H}$ and $\mathbb{U}$ be two Hilbert spaces which denote the state space and the control space, respectively. The corresponding scalar products are $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{U}}$, and the corresponding norms are $\| \cdot \|_{\mathbb{H}}$ and $\| \cdot \|_{\mathbb{U}}$. Let $\big(S(t)\big)_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ be a strongly continuous *group* on $\mathbb{H}$, *i.e.*,

$$S(0) = Id \,(\text{the identity}),$$

$$S(t_1 + t_2) = S(t_1) \circ S(t_2) \quad \forall t_1, t_2 \in \mathbb{R},$$

and

$$\lim_{t \to 0} S(t)x = x \quad \forall x \in \mathbb{H}.$$

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Here and in what follows, for two Hilbert spaces $\mathbb{X}_1$ and $\mathbb{X}_2$, we denote $\mathcal{L}(\mathbb{X}_1, \mathbb{X}_2)$ the Banach space of all bounded linear applications from $\mathbb{X}_1$ to $\mathbb{X}_2$ with the usual norm, and we simply denote $\mathcal{L}(\mathbb{X}_1, \mathbb{X}_1)$ by $\mathcal{L}(\mathbb{X}_1)$.

Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of $\big(S(t)\big)_{t \in \mathbb{R}}$ and denote $S(t)^*$ the adjoint of $S(t)$ for $t \in \mathbb{R}$. Then $\big(S(t)^*\big)_{t \in \mathbb{R}}$ is also a strongly continuous *group* of continuous linear operators and its infinitesimal generator is $(A^*, \mathcal{D}(A^*))$, which is the adjoint of $(A, \mathcal{D}(A))$. As usual, we equip the domain $\mathcal{D}(A^*)$ with the scalar product

$$\langle z_1, z_2 \rangle_{\mathcal{D}(A^*)} = \langle z_1, z_2 \rangle_{\mathbb{H}} + \langle A^* z_1, A^* z_2 \rangle_{\mathbb{H}} \text{ for } z_1, z_2 \in \mathcal{D}(A^*).$$

Then $\mathcal{D}(A^*)$ is a Hilbert space. Denote $\mathcal{D}(A^*)'$ the dual space of $\mathcal{D}(A^*)$ with respect to $\mathbb{H}$. It follows that

$$\mathcal{D}(A^*) \subset \mathbb{H} \subset \mathcal{D}(A^*)'.$$

Let

$$B \in \mathcal{L}(\mathbb{U}, \mathcal{D}(A^*)').$$

In this paper, we consider the following control system, for $T > 0$,

$$\begin{cases} y' = Ay + Bu \text{ for } t \in (0, T), \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where, at time t, the control is $u(t) \in \mathbb{U}$ and the state is $y(t) \in \mathbb{H}$, and $y_0 \in \mathbb{H}$ is an initial datum. This control setting is standard and used to model many control systems, see, *e.g.*, [1, 2]. Interesting aspects of the controllability and the stability of (1.1) can be found in [1–9] and the references therein.

As usual, see, *e.g.*, [1, 2], we assume that B is an *admissible* control operator with respect to the semigroup $\big(S(t)\big)_{t \geqslant 0}$ in the sense that, for all $u \in L^2([0, T]; \mathbb{U})$, it holds that

$$\varphi \in C([0, T]; \mathbb{H}) \text{ where } \varphi(t) := \int_0^t S(t - s) B u(s) \, \mathrm{d}s. \tag{1.2}$$

As a consequence of the closed graph theorem, see *e.g.*, [10], one has

$$\|\varphi\|_{C([0,T];\mathbb{H})} \leqslant C_T \|u\|_{L^2((0,T);\mathbb{U})}. \tag{1.3}$$

Recall that system (1.1) is called to be exactly controllable in some positive time T if for all $y_0, y_T \in \mathbb{H}$, there exists $u \in L^2((0, T); \mathbb{U})$ such that

$$y(T) = y_T,$$

where y is the unique weak solution of (1.1) (the definition of the weak solutions is recalled in Sect. 3). In this case, we also call that the pair (A, B) is exactly controllable in some positive time T. It is known that (1.1) is exactly controllable in time $T > 0$ if and only if the following observability inequality holds, see, *e.g.*, [1, 2],

$$\int_0^T \|B^* \mathrm{e}^{sA^*} x\|_{\mathbb{U}}^2 \, \mathrm{d}s \geqslant C \|x\|_{\mathbb{H}}^2 \text{ for all } x \in \mathbb{H}, \tag{1.4}$$

where C is a positive constant independent of x. Here and in what follows, if $\widetilde{A}$ is the infinitesimal generator of the semigroup $\big(\widetilde{S}(t)\big)_{t\geqslant 0}$ in a Hilbert space $\widetilde{\mathbb{H}}$, we also denote $\widetilde{S}(t)$ by $\mathrm{e}^{t\widetilde{A}}$ for $t \geqslant 0$.

Let $\lambda \in \mathbb{R}$ and assume that, for some *positive*, symmetric, invertible $Q = Q(\lambda) \in \mathcal{L}(\mathbb{H})$, for some *non-negative*, symmetric $R = R(\lambda) \in \mathcal{L}(\mathbb{H})$, and for some *non-negative*, symmetric $W = W(\lambda) \in \mathcal{L}(\mathbb{U})$, it holds

$$AQ + QA^* - BWB^* + QRQ + 2\lambda Q = 0, \tag{1.5}$$

where (1.5) is understood in the following sense

$$\langle Qx, A^* y\rangle_{\mathbb{H}} + \langle A^* x, Qy\rangle_{\mathbb{H}} - \langle WB^* x, B^* y\rangle_{\mathbb{U}} + \langle RQx, Qy\rangle_{\mathbb{H}} + 2\lambda\langle Qx, y\rangle_{\mathbb{H}} = 0 \quad \forall\, x, y \in \mathcal{D}(A^*). \tag{1.6}$$

Here and in what follows, given a Hilbert space $\widetilde{\mathbb{H}}$ and an operator $\widetilde{R} \in \mathcal{L}(\widetilde{\mathbb{H}})$ being symmetric, one says that $\widetilde{R}$ is non-negative if

$$\langle \widetilde{R}x, x\rangle_{\widetilde{\mathbb{H}}} \geqslant 0 \text{ for all } x \in \widetilde{\mathbb{H}},$$

and one says that $\widetilde{R}$ is positive if, for some positive constant C, it holds[1]

$$\langle \widetilde{R}x, x\rangle_{\widetilde{\mathbb{H}}} \geqslant C\|x\|^2_{\widetilde{\mathbb{H}}} \text{ for all } x \in \widetilde{\mathbb{H}}.$$

In this paper, we study the stabilization of (1.1) when (1.5) holds.

Several cases of identity (1.5) and their associated stabilization results appeared in the linear quadratic optimal control theory [11] (see also [12–17] and the references therein) under assumptions that are discussed now. Given a *non-negative*, symmetric $R \in \mathcal{L}(\mathbb{H})$, consider the cost function

$$J_T(u, y) = \int_0^T \langle Ry, y\rangle_{\mathbb{H}}(s) + \langle u, u\rangle_{\mathbb{U}}(s)\, \mathrm{d}s \text{ for } T \in (0, +\infty]. \tag{1.7}$$

For $0 < T < +\infty$, let $P_T \in \mathcal{L}(\mathbb{H})$ be symmetric and satisfy

$$\langle P_T y_0, y_0\rangle_{\mathbb{H}} = \inf_{u\in L^2((0,T),\mathbb{U})} J_T(u, y),$$

where y is the weak solution of (1.1) corresponding to u. Assume that the finite cost condition holds, *i.e.*,

$$\inf_{u\in L^2((0,+\infty),\mathbb{U})} J_\infty(u, y) < +\infty,$$

for all $y_0 \in \mathbb{H}$. Let u_{opt} and y_{opt} be the unique solution corresponding to the minimizing problem $\inf_{u\in L^2((0,+\infty),\mathbb{U})} J_\infty(u, y)$, *i.e.*,

$$J_\infty(u_{opt}, y_{opt}) = \inf_{u\in L^2((0,+\infty),\mathbb{U})} J_\infty(u, y), \tag{1.8}$$

where y is the weak solution of (1.1). Define

$$S_{opt}(t) y_0 = y_{opt}(t). \tag{1.9}$$

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[1] Thus positivity here means coercivity.

Then

$$S_{opt}(t)y_0 = S(t)(y_0) + \int_0^t S(t-s)Bu_{opt}(s)\,\mathrm{d}s \text{ for } t \geqslant 0. \tag{1.10}$$

Let $\big(A_{opt}, \mathcal{D}(A_{opt})\big)$ be the infinitesimal generator of $\big(S_{opt}(t)\big)_{t\geqslant 0}$. Then the pointwise limit of P_T as $T \to +\infty$ exists. Denote this limit by P_∞. It follows that $P_\infty : \mathcal{D}(A_{opt}) \to \mathcal{D}(A^*)$ and

$$u_{opt}(t) = -B^* P_\infty y_{opt}(t) \quad \text{ if } \quad y_0 \in \mathcal{D}(A_{opt}). \tag{1.11}$$

Assume also that R is invertible. Then

$$\big(S_{opt}(t)\big)_{t\geqslant 0} \text{ is exponentially stable.} \tag{1.12}$$

Assertions (1.9)–(1.12) thus give the stabilization of (1.1) by static feedback controls in a weak sense since $-B^* P_\infty$ is not defined for every element in $\mathbb{H}$ when B is not bounded or equivalently when B^* is not bounded. Assume in addition that $(S(t))_{t\in\mathbb{R}}$ is a group, and $(A^*, R^{1/2})$ and (A, B) are exactly controllable in some positive time. Then P_∞ is invertible, and $Q_\infty := P_\infty^{-1}$ satisfies the dual algebraic Riccati equation

$$AQ_\infty + Q_\infty A^* + Q_\infty R Q_\infty - BB^* = 0 \tag{1.13}$$

in the sense

$$\langle Q_\infty x, A^* z\rangle_{\mathbb{H}} + \langle A^* x, Q_\infty z\rangle_{\mathbb{H}} + \langle RQ_\infty x, Q_\infty z\rangle_{\mathbb{H}} = \langle B^* x, B^* z\rangle_{\mathbb{U}} \quad \text{ for all } x, z \in \mathcal{D}(A^*). \tag{1.14}$$

Identity (1.13) is a special case of (1.5) for which $W = I$ and $\lambda = 0$.

We have briefly mentioned so far known stabilization results related to (1.5) from the optimal control theory. We next discuss quickly known results related to (1.5) that come from Gramian operators and are also related to the optimal control theory. Let $\widetilde{\lambda} > 0$ and assume that system (1.1) is exactly controllable in time $T > 0$. Thus (1.4) holds. Set, with $T_* = T + \frac{1}{2\widetilde{\lambda}}$,

$$e(s) = \left\{ \begin{array}{ll} \mathrm{e}^{-2\widetilde{\lambda}s} & \text{in } [0, T], \\ 2\widetilde{\lambda}\mathrm{e}^{-2\widetilde{\lambda}T}(T_* - s) & \text{in } (T, T_*]. \end{array} \right. \tag{1.15}$$

It is shown in [18] (see also [19]) that (1.5) holds for $\lambda = 0$, W being the identity, and for $Q \in \mathcal{L}(\mathbb{H})$ being defined by

$$\langle Qx_1, x_2\rangle_{\mathbb{H}} = \int_0^{T_*} e(s)\langle B^*\mathrm{e}^{-sA^*}x_1, B^*\mathrm{e}^{-sA^*}x_2\rangle_{\mathbb{U}}\,\mathrm{d}s, \tag{1.16}$$

and for $R \in \mathcal{L}(\mathbb{H})$ being symmetric and defined by

$$\langle RQx, Qx\rangle_{\mathbb{H}} = -\int_0^{T_*} e'(s)\|B^*\mathrm{e}^{-sA^*}\|_{\mathbb{U}}^2\,\mathrm{d}s.$$

Previous results when B is bounded were due to Slemrod [4]. These works are inspired by the ones of Lukes [20] and Kleinman [21] where the Gramian operators were introduced in the finite-dimensional setting. In [22], Urquiza observed in the case A is skew-adjoint and $\widetilde{\lambda} > 0$ that (1.5) holds for W being identity, for $\lambda = 0$, Q being defined by

$$\langle Qx_1, x_2\rangle_{\mathbb{H}} = \int_0^\infty \mathrm{e}^{-2\widetilde{\lambda}s}\langle B^*\mathrm{e}^{-sA^*}x_1, B^*\mathrm{e}^{-sA^*}x_2\rangle_{\mathbb{U}}\,\mathrm{d}s, \tag{1.17}$$

and for $R = 2\widetilde{\lambda}Q^{-1}$. The result of Urquiza [22] was inspired by the Bass method previously discussed by Russell [23], pp. 114–115 following [1], Section 10.3. In the settings of Komornik and Urquiza, one can check that

$$Q \text{ is invertible and } (A^*, R^{1/2}) \text{ is exactly controllable.}$$

One can then apply the linear quadratic optimal control theory to conclude that system (1.1) is stabilizable by static feedback controls in the weak sense (1.11). Komornik also proved that (1.1) is stabilizable with the rate $\widetilde{\lambda}$ and Urquiza [22] also established that (1.1) is stabilizable with the rate $2\widetilde{\lambda}$ when A is skew-adjoint, both are in the weak sense. To our knowledge, these known results mentioned have not been successfully extended to the nonlinear case.

The goal of this paper is to present a new method to study the stabilization of (1.1) and its suitable nonlinear variants under condition (1.5). We also show that (1.5) appears very naturally for exactly controllable systems. We study the stabilization of (1.1) by dynamic feedback controls and by static feedback controls in a weak sense, which we call a trajectory sense. A system is called dynamically stabilizable if it can be embedded as a subsystem of a larger, exponentially stable well-posed system. This definition has been used for finite dimensions, see, *e.g.*, [1], Chapter 11, and for linear systems in infinite dimension, see, *e.g.*, [15].

Our approach is essentially based on the construction of new auxiliary dynamics for both types of stabilization (see Thm. 2.1, Thm. 2.4, Thm. 2.9, and Thm. 2.13) and “integration by parts arguments” (see Lem. 3.5 and Lem. 4.1). The new adding variable is inspired by the adjoint state in the linear quadratic optimal control theory and the way to choose controls in the Hilbert Uniqueness Method (HUM) principle. The advantage of our approach is at least twofold. First, the method works well in both linear and *nonlinear* settings. Second, a Lyapunov function is also provided for the static feedback controls. To our knowledge, the stabilization of such systems by dynamic feedback controls is new even in the linear setting. The nonlinear case is out of reach previously when B is unbounded for both types of stabilization. Concerning the static feedback controls, as far as we know, a Lyapunov function is not known even in the case where B is bounded and A is not; a Lyapunov function was previously given in the finite-dimensional case [1, 21]. Consequently, we derive that if the system is exactly controllable in some positive time, then the system is rapidly stabilizable. The techniques and ideas used in this paper have been applied and combined with the ideas in [24] to study the rapid and finite-time stabilization of the Schrödinger equation with bilinear controls [25] and of the KdV equations [26]. Concerning the bilinear control of the Schrödinger equation, the static feedback for the rapid stabilization even in the usual sense is established.

Adding a new variable is very natural and has been used a long time ago in the control theory even in finite dimensions for linear control systems, see, *e.g.*, [1], Section 11.3 and [27], Chapter 7. Coron and Pradly [28] showed that there exists a nonlinear system in finite dimensions for which the system cannot be stabilized by static feedback controls but can be stabilized by dynamic feedback ones. Dynamic feedback controls of finite dimensional nature, *i.e.*, the complement system is a system of differential equations, have been previously implemented in the infinite dimensions, see, *e.g.*, [29, 30]. It is interesting to know whether or not adding a new variable is necessary in the setting of this paper.

The paper is organized as follows. In Section 2, we state the main results of the paper on the dynamic feedback and the static feedback in the trajectory sense. Section 3 is devoted to the well-posedness and some properties of various linear systems considered in this paper. The proofs of the main results on the dynamic feedback and the static feedback are given in Section 4 and Section 5, respectively. In Section 5.3, we also discuss the infinitesimal generator of the semigroup associated with the static feedback controls given in Theorem 2.9, this in particular implies new information on $(A_{opt}, \mathcal{D}(A_{opt}))$. Finally, in Section 6, we discuss choices of Q (and also R and W) when the system is exactly controllable.

2. Statement of the main results

This section consisting of two subsections is organized as follows. In the first subsection, we discuss the stabilization (1.1) by dynamic feedback controls. In the second subsection, we discuss the stabilization of (1.1) by static feedback controls in the trajectory sense. Here and in what follows in this section, we always assume that $(A, \mathcal{D}(A))$ is an infinitesimal generator of a strongly continuous *group* in $\mathbb{H}$, and $B \in \mathcal{L}(\mathbb{U}, \mathcal{D}(A^*)')$ is an admissible control operator with respect to the semigroup generated by A.

2.1. Stabilization by dynamic feedback controls

Given an infinitesimal generator $\widetilde{A}$ of a semigroup in a Hilbert space $\widetilde{\mathbb{H}}$, set

$$\omega_0(\widetilde{A}) = \inf_{t>0} \log \|e^{t\widetilde{A}}\|_{\mathcal{L}(\widetilde{\mathbb{H}})},$$

which denotes the growth of the $e^{t\widetilde{A}}$ for $t \geqslant 0$. It is known, see, *e.g.*, [7], that

$$-\infty \leqslant \omega_0(\widetilde{A}) < +\infty.$$

Concerning the dynamic feedback controls of (1.1), we have the following result.

Theorem 2.1. *Let $\lambda \in \mathbb{R}$ and assume (1.5) with $R = 0$, and let $\lambda_1 \in \mathbb{R}$. Let $\hat{\omega}_0(A) \geqslant \omega_0(A)$ and $\hat{\omega}_0(-A^*) \geqslant \omega_0(-A^*)$ be two real constants such that, for some positive constant c,*

$$\|e^{tA}\|_{\mathcal{L}(\mathbb{H})} \leqslant c e^{t\hat{\omega}_0(A)} \text{ for } t \geqslant 0 \quad \text{ and } \quad \|e^{-tA^*}\|_{\mathcal{L}(\mathbb{H})} \leqslant c e^{t\hat{\omega}_0(-A^*)} \text{ for } t \geqslant 0, \tag{2.1}$$

and assume that

$$\lambda_1 - 2\lambda > \hat{\omega}_0(A) - \hat{\omega}_0(-A^*). \tag{2.2}$$

Given $y_0, \widetilde{y}_0 \in \mathbb{H}$ arbitrary, let $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T]; \mathbb{H})\big)^2$ be the unique weak solution of the system

$$\begin{cases} y' = Ay - BWB^*\widetilde{y} & \text{in } (0, +\infty), \\ \widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} + \lambda_1 Q^{-1}(y - Q\widetilde{y}) & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0. \end{cases} \tag{2.3}$$

Then

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C e^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big) \text{ for } t \geqslant 0, \tag{2.4}$$

where C is a positive constant independent of t and $(y_0, \widetilde{y}_0)^\mathsf{T}$. Consequently, if A is skew-adjoint and $\lambda_1 > 2\lambda$, then

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C e^{-2\lambda t} \big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big) \text{ for } t \geqslant 0. \tag{2.5}$$

Remark 2.2. The well-posedness of the weak solutions in Theorem 2.1 is established in Lemma 4.2.

Remark 2.3. In comparison with the original system (1.1), in system (2.3) a new variable $\widetilde{y}$ and its equation are added. The goal for this adding is two folds. First, the extended system (2.3) is well-posed for all initial datum. Second, the solutions of the new extended system decays exponentially suitably.

We next illustrate how this result can be extended to a nonlinear setting. Let $f : \mathbb{H} \to \mathbb{H}$ be continuous such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x)\|_{\mathbb{H}} \leqslant \varepsilon \|x\|_{\mathbb{H}} \text{ for } x \in \mathbb{H} \text{ with } \|x\|_{\mathbb{H}} < \delta, \tag{2.6}$$

and f is Lipschitz in a neighborhood of 0 in $\mathbb{H}$, *i.e.*, there exist $r > 0$ and $\Lambda > 0$ such that

$$\|f(x) - f(y)\|_{\mathbb{H}} \leqslant \Lambda \|x - y\|_{\mathbb{H}} \text{ for } x,\, y \in \mathbb{H} \text{ with } \|x\|_{\mathbb{H}},\, \|y\|_{\mathbb{H}} < r. \tag{2.7}$$

We consider the following control system

$$\begin{cases} y' = Ay + f(y) + Bu \text{ for } t \in (0,T), \\ y(0) = y_0 \in \mathbb{H}. \end{cases} \tag{2.8}$$

Concerning the local stabilization of (2.8), we have the following result.

Theorem 2.4. *Let $\lambda \in \mathbb{R}$ and assume that (1.5) holds with $R = 0$, and let $\lambda_1, \gamma \in \mathbb{R}$ be such that $\gamma < \lambda$. Let $\hat{\omega}_0(A) \geqslant \omega_0(A)$ and $\hat{\omega}_0(-A^*) \geqslant \omega_0(-A^*)$ be two real constants such that, for some positive constant c,*

$$\|\mathrm{e}^{tA}\|_{\mathcal{L}(\mathbb{H})} \leqslant c\mathrm{e}^{t\hat{\omega}_0(A)} \text{ for } t \geqslant 0 \quad \text{and} \quad \|\mathrm{e}^{-tA^*}\|_{\mathcal{L}(\mathbb{H})} \leqslant c\mathrm{e}^{t\hat{\omega}_0(-A^*)} \text{ for } t \geqslant 0. \tag{2.9}$$

Assume that

$$\lambda_1 - 2\lambda > \hat{\omega}_0(A) - \hat{\omega}_0(-A^*), \quad 2\gamma - \hat{\omega}_0(-A^*) > 0, \tag{2.10}$$

and (2.6) and (2.7) hold. There exists $\varepsilon > 0$ (small) such that for all $y_0, \widetilde{y}_0 \in \mathbb{H}$ with $\|(y_0, \widetilde{y}_0)^\mathsf{T}\|_{\mathbb{H}} \leqslant \varepsilon$, there exists a unique solution $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$ of the system

$$\begin{cases} y' = Ay + f(y) - BWB^*\widetilde{y} \quad \text{in } (0,+\infty), \\ \widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} + Q^{-1}f(Q\widetilde{y}) + \lambda_1 Q^{-1}(y - Q\widetilde{y}) \quad \text{in } (0,+\infty), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0. \end{cases} \tag{2.11}$$

Moreover, we have

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{(\hat{\omega}_0(-A^*)-2\gamma)t}\big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big) \text{ for } t \geqslant 0, \tag{2.12}$$

where C is a positive constant independent of t and $(y_0, \widetilde{y}_0)$. Consequently, if A is skew-adjoint and $\lambda_1 > 2\lambda > 2\gamma > 0$ then

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{-2\gamma t}\big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big) \text{ for } t \geqslant 0. \tag{2.13}$$

Remark 2.5. The weak solutions given in Theorem 2.4 are understood in the sense of the weak solutions where the nonlinear terms play as a part of the source term.

Remark 2.6. The well-posedness of the weak solutions in Theorem 2.4 is a part of the proof. In comparison with Theorem 2.1, λ is supposed to satisfy the condition $2\lambda - \hat{\omega}_0(-A^*) > 0$ in Theorem 2.4 to make sure that the solution remains small for large time.

As a consequence of Theorem 2.1 and Theorem 2.4 (see also Prop. 6.3), we obtain the following results.

Proposition 2.7. *Assume that system* (1.1) *is exactly controllable in some positive time. System* (1.1) *is rapidly dynamically stabilizable.*

Proposition 2.8. *Assume that system* (1.1) *is exactly controllable in some positive time, and* (2.6) *and* (2.7) *hold. System* (2.8) *is locally rapidly dynamically stabilizable.*

Recall that system (1.1) is called rapidly dynamically stabilizable if it can be dynamically exponentially stabilizable with an arbitrary decay rate. A similar meaning with suitable modifications is used for system (2.8).

2.2. Stabilization by static feedback controls

Here is the first main result on the static feedback controls of (1.1).

Theorem 2.9. *Let* $\lambda \in \mathbb{R}$ *and assume* (1.5). *Given* $y_0 \in \mathbb{H}$, *let* $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$ *be the unique weak solution of the system*

$$\begin{cases} y' = Ay - BWB^*\widetilde{y} & \text{in } (0,+\infty), \\ \widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} - RQ\widetilde{y} & \text{in } (0,+\infty), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0 := Q^{-1}y_0. \end{cases} \tag{2.14}$$

Then

$$\widetilde{y}(t) = Q^{-1}y(t) \text{ for } t \geqslant 0, \tag{2.15}$$

and

$$\|Q^{-1/2}y(t)\|_{\mathbb{H}}^2 - \|Q^{-1/2}y(\tau)\|_{\mathbb{H}}^2 = -2\lambda\int_\tau^t \|Q^{-1/2}y(s)\|_{\mathbb{H}}^2\,\mathrm{d}s - \int_\tau^t \Big(\|W^{1/2}B^*\widetilde{y}(s)\|_{\mathbb{U}}^2 + \|R^{1/2}y(s)\|_{\mathbb{H}}^2\Big)\,ds \text{ for } t \geqslant \tau \geqslant 0. \tag{2.16}$$

Consequently,

$$\|Q^{-1/2}y(t)\|_{\mathbb{H}} \leqslant \mathrm{e}^{-\lambda t}\|Q^{-1/2}y(0)\|_{\mathbb{H}} \text{ for } t \geqslant 0. \tag{2.17}$$

Some comments on Theorem 2.9 are in order. Since

$$\widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} - RQ\widetilde{y} \text{ in } (0,+\infty)$$

and $\widetilde{y}(0) \in \mathbb{H}$, it follows from Lemma 3.5 given in Section 3 that $\widetilde{y} \in C([0,T);\mathbb{H})$ is well-defined for all $T > 0$ and moreover,

$$B^*\widetilde{y} \in L^2((0,T),\mathbb{H}) \text{ for all } T > 0.$$

We thus derive that system (2.14) is well-posed and (2.16) makes sense. Combing (2.15) and the equation of y

$$y' = Ay - BWB^* \widetilde{y},$$

we have thus shown that the control system $y' = Ay + Bu$ with the *static feedback* control

$$\text{“}u = -WB^* Q^{-1} y\text{” for } t \geqslant 0, \tag{2.18}$$

is well-posed in the sense given in Theorem 2.9. We only consider (2.18) as static feedback controls in a weak sense, which we call *a trajectory sense*, since for $y \in \mathbb{H}$, it is not clear how to give the sense to the action $-WB^* Q^{-1} y$. In comparison with the static feedback controls in the sense given by (1.11), the static feedback controls given (2.18) are well-defined in the sense of Theorem 2.9 for all initial data $y_0 \in \mathbb{H}$. Theorem 2.9 can be considered as a new way to view the feedback controls given in (1.11).

It is important in Theorem 2.9 that $\widetilde{y}_0 = Q^{-1} y_0$ in (2.14). Due to this fact, one cannot derive from Theorem 2.9 that system (1.1) is dynamically stabilizable *via* the system

$$\begin{cases} y' = Ay - BWB^* \widetilde{y} & \text{in } (0, +\infty), \\ \widetilde{y}' = -A^* \widetilde{y} - 2\lambda \widetilde{y} - RQ\widetilde{y} & \text{in } (0, +\infty), \end{cases} \tag{2.19}$$

To be able to deal with (2.19) with arbitrary initial datum and to obtain the dynamic stabilization (1.1), we introduced the term $\lambda_1 Q^{-1}(y - Q\widetilde{y})$ in (2.14) of Theorem 2.1 (compare (2.14) and (2.19)).

Remark 2.10. From (2.16), the quantity $\|Q^{-1/2} y(t)\|_{\mathbb{H}}^2$ can be viewed as the Lyapunov function of the system. This fact seems new to us even in the case where B is bounded and A is not.

Remark 2.11. Assertion (2.15) was known in the case where $\lambda = 0$, $W = I$, and under the additional assumptions that $(A^*, R^{1/2})$ and (A, B) are exactly controllable in some positive time, see [11], Theorems 2.4, 2.6, and 2.7.

We next present a consequence of Theorem 2.9 in the case where A is a skew-adjoint operator and $R = 0$.

Corollary 2.12. *Let $\lambda \in \mathbb{R}$, and assume that (1.5) holds with $R = 0$ and A is skew-adjoint. Given $y_0 \in \mathbb{H}$, let $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T]; \mathbb{H})\big)^2$ be the unique weak solution of the system (2.14) with $R = 0$. Then (2.15) holds and, for some positive constants C_1, C_2, independent of y_0,*

$$C_1 \mathrm{e}^{-2\lambda t} \|y_0\|_{\mathbb{H}} \leqslant \|y(t)\|_{\mathbb{H}} \leqslant C_2 \mathrm{e}^{-2\lambda t} \|y_0\|_{\mathbb{H}} \textit{ for } t \geqslant 0. \tag{2.20}$$

Corollary 2.12 is a direct consequence of Theorem 2.9. Indeed, (2.15) is a consequence of Theorem 2.9. Since A is skew-adjoint, it follows from the equation of $\widetilde{y}$ that

$$\|\widetilde{y}(t)\| = \mathrm{e}^{-2\lambda t} \|\widetilde{y}(0)\| \text{ for } t \geqslant 0. \tag{2.21}$$

Assertion (2.20) is now a consequence of (2.15) and (2.21).

We next deal with the local stabilization of (2.8) by static feedback controls in the trajectory sense.

Theorem 2.13. *Let* $\lambda > 0$ *and assume* (1.5), (2.6), *and* (2.7). *There exists* $\varepsilon > 0$ *(small) such that for all* $y_0 \in \mathbb{H}$ *with* $\|y_0\|_{\mathbb{H}} \leqslant \varepsilon$, *there exists a unique weak solution* $(y, \widetilde{y})^{\mathsf{T}} \in \big(C([0,T]; \mathbb{H})\big)^2$ *of the system*

$$\begin{cases} y' = Ay + f(y) - BWB^*\widetilde{y} \quad \text{in } (0, +\infty), \\ \widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} - RQ\widetilde{y} + Q^{-1}f(Q\widetilde{y}) \quad \text{in } (0, +\infty), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0 := Q^{-1}y_0. \end{cases} \tag{2.22}$$

Moreover, we have

$$\widetilde{y} = Q^{-1}y \text{ for } t \geqslant 0, \tag{2.23}$$

and

$$\|Q^{-1/2}y(t)\|_{\mathbb{H}}^2 - \|Q^{-1/2}y(\tau)\|_{\mathbb{H}}^2 = -2\lambda\int_\tau^t \|Q^{-1/2}y(s)\|_{\mathbb{H}}^2 \,\mathrm{d}s \\ - \int_\tau^t \Big(\|W^{1/2}B^*\widetilde{y}(s)\|_{\mathbb{U}}^2 + \|R^{1/2}y(s)\|_{\mathbb{H}}^2\Big)\, ds + 2\int_\tau^t \langle f(y(s)), Q^{-1}y(s)\rangle \,\mathrm{d}s \text{ for } t \geqslant \tau \geqslant 0. \tag{2.24}$$

Consequently, for all $0 < \gamma < \lambda$, *there exists* ε_γ *such that for* $y_0 \in \mathbb{H}$ *with* $\|y_0\|_{\mathbb{H}} \leqslant \varepsilon_\gamma$, *it holds*

$$\|Q^{-1/2}y(t)\|_{\mathbb{H}} \leqslant \mathrm{e}^{-\gamma t}\|Q^{-1/2}y(0)\|_{\mathbb{H}} \text{ for } t \geqslant 0. \tag{2.25}$$

Remark 2.14. The weak solutions given in Theorem 2.13 are understood in the sense of the weak solutions given in Section 3 where the nonlinear term plays as a part of the source term.

Remark 2.15. In comparison with Theorem 2.1, λ is supposed to be positive in Theorem 2.4 to make sure that the solution remains small for large time.

Here is a variant of Corollary 2.16 in the nonlinear setting, which is a direct consequence of Theorem 2.13, and the proof is omitted.

Corollary 2.16. *Let* $\lambda > 0$, *and assume that* (1.5) *holds with* $R = 0$ *and* A *is skew-adjoint. Assume* (2.6) *and* (2.7). *There exists* $\varepsilon > 0$ *(small) such that for* $y_0 \in \mathbb{H}$ *with* $\|y_0\|_{\mathbb{H}} \leqslant \varepsilon$, *there exists a unique solution* $(y, \widetilde{y})^{\mathsf{T}} \in \big(C([0,T]; \mathbb{H})\big)^2$ *of the system* (2.22) *with* $R = 0$. *Moreover,* (2.23) *holds, and, for all* $0 < \gamma < \lambda$, *there exists* ε_γ *such that for* $y_0 \in \mathbb{H}$ *with* $\|y_0\|_{\mathbb{H}} \leqslant \varepsilon_\gamma$, *it holds, for some positive constants* C, *independent of* y_0,

$$\|y(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{-2\gamma t}\|y_0\|_{\mathbb{H}} \text{ for } t \geqslant 0. \tag{2.26}$$

As a consequence of Theorem 2.9 and Theorem 2.13 (see also Prop. 6.1), we obtain the following results.

Proposition 2.17. *Assume that system* (1.1) *is exactly controllable in some positive time. System* (1.1) *is rapidly (statically) stabilizable in the trajectory sense.*

Proposition 2.18. *Assume that system* (1.1) *is exactly controllable in some positive time, and* (2.6) *and* (2.7) *hold. System* (2.8) *is locally rapidly (statically) stabilizable in the trajectory sense.*

3. Preliminaries

In this section, we state and prove the well-posedness and some properties of various linear control systems considered in this paper. It is more convenient to consider a slightly more general system

$$\begin{cases} y' = Ay + f + Bu + My \text{ in } t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{3.1}$$

with $y_0 \in \mathbb{H}$, $f \in L^1((0,T);\mathbb{H})$, $u \in L^2([0,T];\mathbb{U})$, and $M \in \mathcal{L}(\mathbb{H})$. Recall that B is assumed to be an admissible control operator with respect to the semigroup $\big(S(t)\big)_{t \geqslant 0} \subset \mathcal{L}(\mathbb{H})$ generated by the operator A *throughout* the paper. In this section, we only assume that $\big(S(t)\big)_{t \geqslant 0} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous *semigroup*. A weak solution y of (3.1) is understood as an element $y \in C([0,T];\mathbb{H})$ such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\langle y, \varphi\rangle_{\mathbb{H}} = \langle Ay + f + Bu + My, \varphi\rangle_{\mathbb{H}} \text{ in } (0,T) \\ y(0) = y_0 \end{cases} \quad \text{for all } \varphi \in \mathcal{D}(A^*) \tag{3.2}$$

for which

i) the differential equation in (3.2) is understood in the distributional sense,
ii) the term $\langle Ay + f + Bu + My, \varphi\rangle_{\mathbb{H}}$ is understood as $\langle y, A^*\varphi\rangle_{\mathbb{H}} + \langle f + My, \varphi\rangle_{\mathbb{H}} + \langle u, B^*\varphi\rangle_{\mathbb{U}}$.

The convention in *ii)* will be used throughout this section.

We begin by recalling the well-posedness of (3.1), see [2], Sections 4.1 and 4.2 (in particular, [2], Rem. 4.1.2 and Prop. 4.2.5)[2], see also [25], Proposition A1.

Proposition 3.1. *Let* $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0,T);\mathbb{H})$, $u \in L^2([0,T];\mathbb{U})$, *and* $M \in \mathcal{L}(\mathbb{H})$. *Then*

i) $y \in C([0,T],\mathbb{H})$ *is a weak solution of* (3.1) *if and only if, with* $\tilde{f} := f + Bu + My$, *it holds*[3]

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\tilde{f}(s)\,\mathrm{d}s \text{ for } t \in [0,T]. \tag{3.3}$$

ii) *there exists a unique solution* $y \in C([0,T],\mathbb{H})$ *of* (3.3).

Remark 3.2. A solution of (3.3) is called a mild solution of (3.1). Part *i)* of Proposition 3.1 gives the equivalence between a weak solution of (3.1) defined by (3.2) and a mild solution of (3.1) defined by (3.3). Part *ii)* provides the well-posedness of weak solutions of (3.1) thanks to the well-posedness of mild solutions. The well-posedness of mild solutions of (3.3) can be proved using a standard fixed point argument for which the following norm is used in $C([0,T];\mathbb{H})$:

$$|||y||| = \sup_{t \in [0,T]} \mathrm{e}^{-\mu t}\|y(t)\|_{\mathbb{H}}.$$

for some large positive μ, see also the proof of Lemma 3.8 below.

Remark 3.3. As a consequence of Proposition 3.1, the control operator B is also admissible with respect to the semigroup $\mathrm{e}^{t(A+M)}$.

---

[2] There is no f in the statement of [2], Proposition 4.2.5 but the result also holds with $f \in L^1((0,T);\mathbb{H})$ and the analysis is the same.

[3] This identity is understood in $\mathcal{D}(A^*)'$, *i.e.*, $\langle y(t), \varphi\rangle_{\mathbb{H}} = \langle S(t)y_0, \varphi\rangle_{\mathbb{H}} + \int_0^t \langle S(t-s)\tilde{f}(s), \varphi\rangle_{\mathbb{H}}\,\mathrm{d}s$ in $[0,T]$ for all $\varphi \in \mathcal{D}(A^*)$.

Remark 3.4. The concept of weak solutions and mild solutions have been used in the theory of semigroups, see, *e.g.*, [2, 31]. The definitions given here are in the same spirit and adapted to the setting considered. The equivalence between weak solutions and mild solutions was first proved in the case B is bounded and $f \in C([0,T]; \mathbb{H})$ by Ball [32], see also [31], Chapter 1 of Part II for related results when B is bounded.

The unique weak solution given in Proposition 3.1 also satisfies the transposition meaning as established in the following result, which is one of the key technical results of this paper.

Lemma 3.5. *Let* $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0,T); \mathbb{H})$, $u \in L^2([0,T]; \mathbb{U})$, *and* $M \in \mathcal{L}(\mathbb{H})$, *and let* $y \in C([0,T]; \mathbb{H})$ *be the unique weak solution of* (3.1). *We have, for* $t \in (0,T]$, *for* $z_t \in \mathcal{D}(A^*)$, *and for* $g \in C([0,t]; \mathcal{D}(A^*))$,

$$\langle y(t), z_t \rangle_{\mathbb{H}} - \langle y_0, z(0) \rangle_{\mathbb{H}} = \int_0^t \langle u(s), B^* z(s) \rangle_{\mathbb{U}} \, \mathrm{d}s \\ - \int_0^t \langle g(s), y(s) \rangle_{\mathbb{H}} \, \mathrm{d}s + \int_0^t \langle f(s), z(s) \rangle_{\mathbb{H}} \, \mathrm{d}s + \int_0^t \langle M y(s), z(s) \rangle_{\mathbb{H}} \, \mathrm{d}s, \quad (3.4)$$

where $z \in C([0,t]; \mathbb{H})$ *is the unique weak solution of the backward system*

$$\begin{cases} z' = -A^* z - g \text{ in } (0,t), \\ z(t) = z_t. \end{cases} \quad (3.5)$$

Consequently, for $z_T \in \mathbb{H}$ *and* $g \in L^1((0,T); \mathbb{H})$, *the unique weak solution* $z \in C([0,T]; \mathbb{H})$ *of* (3.5) *with* $t = T$ *satisfies*

$$\| B^* z \|_{L^2((0,T); \mathbb{U})} \leqslant C_T \Big(\| g \|_{L^1((0,T); \mathbb{H})} + \| z_T \|_{\mathbb{H}} \Big), \quad (3.6)$$

and (3.4) *holds for* $z_t \in \mathbb{H}$ *and* $g \in L^1((0,t); \mathbb{H})$. *Here* C_T *denotes a position constant independent of* g, f, *and* z_T.

Remark 3.6. For $0 < T \leqslant T_0$, the constant C_T in (3.6) can be chosen independent of T. In fact, extend g by 0 for $t < 0$ and denote this extension by $\widetilde{g}$. Consider the weak solution $\widetilde{z}$ of the system

$$\begin{cases} \widetilde{z}' = -A^* \widetilde{z} - g \text{ in } (T - T_0, T), \\ \widetilde{z}(T) = z_T. \end{cases} \quad (3.7)$$

By (3.6), we have

$$\| B^* \widetilde{z} \|_{L^2((T-T_0,T); \mathbb{U})} \leqslant C_{T_0} \Big(\| \widetilde{g} \|_{L^1((T-T_0,T); \mathbb{H})} + \| z_T \|_{\mathbb{H}} \Big).$$

The desired assertion follows by noting that $\widetilde{z} = z$ in $(0,T)$ and using the definition of g.

In what follows, for notational ease, we use $\langle \cdot, \cdot \rangle$ to denote $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ or $\langle \cdot, \cdot \rangle_{\mathbb{U}}$ in a clear context. We now give the proof of Lemma 3.5.

Proof of Lemma 3.5. Let $z_t \in \mathcal{D}(A^*)$ and $g \in C([0,t]; \mathcal{D}(A^*))$, and let $z \in C([0,t]; \mathbb{H})$ be the unique weak solution of (3.5). We have, for $n \geqslant 2$,

$$\langle y(t), z(t) \rangle - \langle y(0), z(0) \rangle = \sum_{i=1}^n \Big(\langle y(t_i), z(t_i) \rangle - \langle y(t_{i-1}), z(t_{i-1}) \rangle \Big),$$

where $t_0 = 0$ and $t_i = t_{i-1} + t/n$ for $1 \leqslant i \leqslant n$.

Since $z_t \in \mathcal{D}(A^*)$ and $g \in C([0,t];\mathcal{D}(A^*))$, it follows that $z \in C\big([0,t];\mathcal{D}(A^*)\big)$. We thus obtain

$$\begin{aligned}\langle y(t_i), z(t_i)\rangle - \langle y(t_{i-1}), z(t_{i-1})\rangle &= \langle y(t_i), z(t_i) - z(t_{i-1})\rangle + \langle y(t_i) - y(t_{i-1}), z(t_{i-1})\rangle \\ &\overset{(3.1),(3.5)}{=} \langle y(t_i), \int_{t_{i-1}}^{t_i} \big(-A^* z(s) - g(s)\big)\,\mathrm{d}s\rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(t_{i-1})\rangle\,\mathrm{d}s. \end{aligned} \tag{3.8}$$

where $\tilde{f} = f + Bu + My$. Recall that the convention *ii)* in the definition of the weak solutions of (3.2) is used here. Using the fact $z \in C\big([0,t];\mathcal{D}(A^*)\big)$ and $y \in C\big([0,t];\mathbb{H}\big)$, we derive that

$$\begin{aligned}&\langle y(t_i), \int_{t_{i-1}}^{t_i} \big(-A^* z(s) - g(s)\big)\,\mathrm{d}s\rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(t_{i-1})\rangle\,\mathrm{d}s \\ &\quad = \int_{t_{i-1}}^{t_i} \langle y(s), \big(-A^* z(s) - g(s)\big)\,\mathrm{d}s\rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(s)\rangle\,\mathrm{d}s + o(t_i - t_{i-1}). \end{aligned} \tag{3.9}$$

Here the standard notation of $o(\cdot)$ is used: $o(s)/|s| \to 0$ as $s \to 0$. Combining (3.8) and (3.9) yields

$$\langle y(t_i), z(t_i)\rangle - \langle y(t_{i-1}), z(t_{i-1})\rangle = -\int_{t_{i-1}}^{t_i} \langle y(s), g(s)\,\mathrm{d}s\rangle + \int_{t_{i-1}}^{t_i} \langle \tilde{f}(s), z(s)\rangle\,\mathrm{d}s + o(t_i - t_{i-1}).$$

Using the definition of $\tilde{f}$, we derive that

$$\begin{aligned}\langle y(t_i), z(t_i)\rangle - \langle y(t_{i-1}), z(t_{i-1})\rangle &= \int_{t_{i-1}}^{t_i} \langle u(s), B^* z(s)\rangle\,\mathrm{d}s - \int_{t_{i-1}}^{t_i} \langle g(s), y(s)\rangle\,\mathrm{d}s \\ &\quad + \int_{t_{i-1}}^{t_i} \langle f(s), z(s)\rangle\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} \langle My(s), z(s)\rangle\,\mathrm{d}s + o(t_i - t_{i-1}).\end{aligned}$$

Summing with respect to n and letting $n \to +\infty$, we reach (3.4) for $z_t \in \mathcal{D}(A^*)$ and $g \in C([0,t];\mathcal{D}(A^*))$.

We next deal with (3.6). Fix $z_T \in \mathcal{D}(A^*)$ and $g \in C([0,T];\mathcal{D}(A^*))$. Let $u \in L^2((0,T);\mathbb{U})$ and let $y \in C([0,T];\mathbb{H})$ be the unique weak solution of (3.1) with $f = 0$, $y_0 = 0$, and $M = 0$. Applying (3.4) with $t = T$, we have

$$\int_0^T \langle u(s), B^* z(s)\rangle\,\mathrm{d}s = \langle y(T), z_T\rangle + \int_0^T \langle g(s), y(s)\rangle\,\mathrm{d}s. \tag{3.10}$$

Since

$$\begin{aligned}|\langle y(T), z_T\rangle| + \int_0^T |\langle g(s), y(s)\rangle|\,\mathrm{d}s &\leqslant \|y(T)\|\|z_T\| + \|g\|_{L^1((0,T);\mathbb{H})}\|y\|_{L^\infty((0,T);\mathbb{H})} \\ &\overset{(1.3),\textit{Theorem } 3.1}{\leqslant} C\|u\|_{L^2((0,T);\mathbb{U})}\Big(\|z_T\| + \|g\|_{L^1((0,T);\mathbb{H})}\Big).\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\|B^* z\|_{L^2((0,T),\mathbb{U})} \leqslant C\Big(\|z_T\| + \|g\|_{L^1((0,T);\mathbb{H})}\Big).$$

Assertion of (3.6) in the case $z_T \in \mathbb{H}$ and $g \in L^1((0,T);\mathbb{H})$ follows from this case by density.

Finally, (3.4) with $z_t \in \mathbb{H}$ and $g \in L^1((0,t);H)$ also follows from the case $z_t \in \mathcal{D}(A^*)$ and $g \in C([0,t];\mathcal{D}(A^*))$ by density. □

We now prove that the solutions in the transposition sense are also unique. Their existence is a direct consequence of Proposition 3.1 and Lemma 3.5. We first state the meaning of transposition solutions of system (3.1).

Definition 3.7. Let $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0,T);\mathbb{H})$, $u \in L^2([0,T];\mathbb{U})$, and $M \in \mathcal{L}(\mathbb{H})$. A function $y \in C([0,T];\mathbb{H})$ is called a transposition solution of (3.1) if for all $t \in (0,T]$, $z_t \in \mathbb{H}$, and $g \in L^1((0,t);H)$, identity (3.4) holds where $z \in C([0,t];\mathbb{H})$ is the unique weak solution of (3.5).

We have the following result.

Lemma 3.8. *Let $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0,T);\mathbb{H})$, and $u \in L^2((0,T);\mathbb{U})$. There exists a unique transposition solution $y \in C([0,T];\mathbb{H})$ of (3.1). Moreover,*

$$\|y(\tau)\|_{\mathbb{H}} \leqslant C_T\Big(\|y_0\| + \|f\|_{L^1((0,T);\mathbb{H})} + \|u\|_{L^2((0,T);\mathbb{U})}\Big), \tag{3.12}$$

for some positive constant C_T, independent of y_0, f, and u.

Remark 3.9. Let $0 < T \leqslant T_0$. By the arguments as in Remark 3.6, one can chose the constant C_T in (3.12) independent of T.

Proof. By Proposition 3.1 and Lemma 3.5, it suffices to prove the uniqueness. Let $\mu > 0$ be large. We equip $C([0,T];\mathbb{H})$ with the following norm

$$|||y||| = \sup_{t \in [0,T]} \mathrm{e}^{-\mu t}\|y(t)\|_{\mathbb{H}}.$$

Recall that y is a transposition solution if, for $t \geqslant 0$,

$$\langle y(t), z_t\rangle - \langle y_0, z(0)\rangle = \int_0^t \langle u(s), B^* z(s)\rangle \,\mathrm{d}s + \int_0^t \langle f(s), z(s)\rangle \,\mathrm{d}s + \int_0^t \langle y(s), M^* z(s)\rangle \,\mathrm{d}s, \tag{3.13}$$

where $z_t \in \mathbb{H}$ and z is the weak solution of the backward system

$$\begin{cases} z' = -A^* z \text{ in } (0,t), \\ z(t) = z_t. \end{cases} \tag{3.14}$$

Thus if y and $\hat{y}$ are two transposition solutions, then

$$\langle y(t) - \hat{y}(t), z_t\rangle = \int_0^t \langle y(s) - \hat{y}(s), M^* z(s)\rangle \,\mathrm{d}s. \tag{3.15}$$

This implies

$$\mathrm{e}^{-\mu t}\|y(t) - \hat{y}(t)\| \leqslant C\mathrm{e}^{-\mu t}\int_0^t \|y(s) - \hat{y}(s)\| \,\mathrm{d}s \leqslant \frac{C}{\mu}|||y - \hat{y}|||.$$

Here and in what follows in this proof, C denotes a positive constant independent of y, $\hat{y}$, and μ. Thus

$$|||y - \hat{y}||| \leqslant \frac{C}{\mu}|||y - \hat{y}|||.$$

The uniqueness follows and the proof is complete. □

Remark 3.10. The concept of transposition solutions was introduced in the literature in the case $f = 0$ and $M = 0$ for which (3.4) is required with $z \in C([0, t]; \mathbb{H})$ being a solution of (3.6) with $g = 0$, *i.e.*, $y \in C([0, T]; \mathbb{H})$ is a transposition solution of the system

$$\begin{cases} y' = Ay + Bu \text{ in } t \in (0, T), \\ \qquad y(0) = y_0, \end{cases} \tag{3.16}$$

if for all $t \in (0, T]$ and $z_t \in \mathbb{H}$, it holds

$$\langle y(t), z_t \rangle_{\mathbb{H}} - \langle y_0, z(0) \rangle_{\mathbb{H}} = \int_0^t \langle u(s), B^* z(s) \rangle_{\mathbb{U}} \, \mathrm{d}s, \tag{3.17}$$

where $z \in C([0, t]; \mathbb{H})$ is the unique weak solution of the backward system

$$\begin{cases} z' = -A^* z \text{ in } (0, t), \\ \qquad z(t) = z_t. \end{cases} \tag{3.18}$$

The well-posednesss of the transposition solutions in this case was also established, see, *e.g.*, [1], Section 2.3 of Chapter 2. The definition of transposition solutions introduced in Definition 3.7 is considered in a more general setting and the solutions have more properties even in the case $f = 0$ and $M = 0$ (compare (3.4) with $f = 0$ and $M = 0$, and (3.17)). Their well-posedness and their properties are necessary for the analysis in this paper; the known results are not enough for our purposes. Our proof is inspired by but different from the known one. As shown, Lemma 3.5 plays here an important role.

4. DYNAMIC FEEDBACK CONTROLS

This section consists of three subsections and is organized as follows. In the first subsection, we state and prove two useful lemmas, which will be used in the proofs of Theorem 2.1. The proofs of Theorem 2.1 and Theorem 2.4 are given in the last two subsections, respectively.

4.1. Two useful lemmas

Note that (1.5) can be written under an equivalent form as follows

$$A_\lambda Q + Q A_\lambda^* - BWB^* + QRQ = 0, \tag{4.1}$$

where

$$A_\lambda = A + \lambda I. \tag{4.2}$$

The meaning of (1.6) can be rewritten as follows

$$\langle Qx, A_\lambda^* y \rangle + \langle A_\lambda^* x, Qy \rangle - \langle WB^* x, B^* y \rangle + \langle RQx, Qy \rangle = 0 \quad \forall\, x, y \in \mathcal{D}(A^*). \tag{4.3}$$

We have the following result concerning (1.5).

Lemma 4.1. *Assume* (1.5)*, i.e.,* (1.6)*. Given* $x_0, y_0 \in \mathbb{H}$ *and* $f, g \in L^1((0,T);\mathbb{H})$*, let* x*,* $y \in C([0,T];\mathbb{H})$ *be the unique weak solution of the systems*

$$\begin{cases} x' = A_\lambda^* x + f \text{ in } (0,T), \\ x(0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} y' = A_\lambda^* y + g \text{ in } (0,T), \\ y(0) = y_0. \end{cases}$$

We have, for $t \in [0,T]$*,*

$$\langle Qx(t), y(t)\rangle - \langle Qx_0, y_0\rangle \\ = \int_0^t \Big(\langle WB^*x(s), B^*y(s)\rangle - \langle RQx(s), Qy(s)\rangle\Big)\, ds + \int_0^t \Big(\langle Qf(s), y(s)\rangle + \langle Qg(s), x(s)\rangle\Big)\, \mathrm{d}s. \quad (4.4)$$

Proof. We first assume that x_0, $y_0 \in \mathcal{D}(A^*)$ and f, $g \in C([0,T];\mathcal{D}(A^*))$. Then $x, y \in C([0,T];\mathcal{D}(A^*))$ and $x', y' \in C([0,T];\mathbb{H})$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Qx, y\rangle = \langle x', Qy\rangle + \langle Qx, y'\rangle = \langle A_\lambda^* x, Qy\rangle + \langle Qx, A_\lambda^* y\rangle + \langle f, Qy\rangle + \langle Qx, g\rangle.$$

Using (1.6), since Q is symmetric, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Qx, y\rangle = \langle WB^*x, B^*y\rangle - \langle RQx, Qy\rangle + \langle Qf, y\rangle + \langle Qg, x\rangle.$$

We thus obtain (4.4).

The proof in the general case is based on the previous case and a density argument. ☐

We next deal with the well-posedness of (2.14) in Theorem 2.9. It might be more convenient to consider a slightly more general system

$$\begin{cases} y' = Ay + f - BWB^*\widetilde{y} + M_1 y + M_2 \widetilde{y} \text{ for } t \in (0,T), \\ \widetilde{y}' = -A^*\widetilde{y} + \widetilde{f} + \widetilde{M}_1 \widetilde{y} + \widetilde{M}_2 y \text{ for } t \in (0,T), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0, \end{cases} \quad (4.5)$$

with $y_0, \widetilde{y}_0 \in \mathbb{H}$, $f, \widetilde{f} \in L^1((0,T);\mathbb{H})$, $M_1, M_2, \widetilde{M}_1, \widetilde{M}_2 \in \mathcal{L}(\mathbb{H})$, and $W \in \mathcal{L}(\mathbb{U})$. As usual, a weak solution $(y, \widetilde{y})^\mathsf{T}$ of (4.5) is understood as an element $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle y, \varphi\rangle_\mathbb{H} = \langle Ay + f - BWB^*\widetilde{y} + M_1 y + M_2\widetilde{y}, \varphi\rangle_\mathbb{H} \text{ in } [0,T] \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\langle \widetilde{y}, \widetilde{\varphi}\rangle_\mathbb{H} = \langle -A^*\widetilde{y} + \widetilde{f} + \widetilde{M}_1\widetilde{y} + \widetilde{M}_2 y, \widetilde{\varphi}\rangle_\mathbb{H} \text{ in } [0,T] \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0, \end{cases} \quad \text{for all } \varphi \in \mathcal{D}(A^*),\ \widetilde{\varphi} \in \mathcal{D}(A), \quad (4.6)$$

for which

i) the differential equations are understood in the distributional sense,

ii) the terms $\langle Ay + f - BWB^*\widetilde{y} + M_1 y + M_2\widetilde{y}, \varphi\rangle_\mathbb{H}$ and $\langle -A^*\widetilde{y} + \widetilde{f} + \widetilde{M}_1\widetilde{y} + \widetilde{M}_2 y, \widetilde{\varphi}\rangle_\mathbb{H}$ are understood as $\langle y, A^*\varphi\rangle_\mathbb{H} + \langle f + M_1 y + M_2\widetilde{y}, \varphi\rangle_\mathbb{H} - \langle WB^*\widetilde{y}, B^*\varphi\rangle_\mathbb{U}$ and $\langle -\widetilde{y}, A\widetilde{\varphi}\rangle_\mathbb{H} + \langle \widetilde{f} + \widetilde{M}_1\widetilde{y} + \widetilde{M}_2 y, \widetilde{\varphi}\rangle$, respectively.

Note that $B^*\widetilde{y} \in L^2(0,T;\mathbb{U})$ since B is an admissible control operator.

We have the following result on the well-posedness of (4.5).

Lemma 4.2. *Let A be an infinitesimal generator of a group, and let $M_1, M_2, \widetilde{M}_1, \widetilde{M}_2 \in \mathcal{L}(\mathbb{H})$ and $W \in \mathcal{L}(\mathbb{U})$. Let $T > 0$, $y_0, \widetilde{y}_0 \in \mathbb{H}$, $f, \widetilde{f} \in L^1((0,T);\mathbb{H})$. There exists a unique weak solution $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T],\mathbb{H})\big)^2$ of (4.5). Moreover, with $g := f - BWB^*\widetilde{y} + M_1 y + M_1\widetilde{y}$ and $\widetilde{g} := \widetilde{f} + \widetilde{M}_1\widetilde{y} + \widetilde{M}_2 y$, we have*[4]

$$y(t) = \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-s)A}g(s)\,\mathrm{d}s \text{ for } t \in [0,T], \tag{4.7}$$

and

$$\widetilde{y}(t) = \mathrm{e}^{-tA^*}\widetilde{y}_0 + \int_0^t \mathrm{e}^{-(t-s)A^*}\widetilde{g}(s)\,\mathrm{d}s \text{ for } t \in [0,T]. \tag{4.8}$$

Moreover, we have

$$\|(y(t), \widetilde{y}(t))^\mathsf{T}\|_{\mathbb{H}} \leqslant C\Big(\|(y_0, \widetilde{y}_0)^\mathsf{T}\|_{\mathbb{H}} + \|(f, \widetilde{f})^\mathsf{T}\|_{L^1((0,T);\mathbb{H})}\Big) \text{ in } [0,T],$$

for some positive constant C, independent of $y_0, \widetilde{y}_0$, f, and $\widetilde{f}$.

Remark 4.3. In Lemma 4.2, we does not require that W is symmetric (or non-negative).

Proof. We first note that $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$ is a weak solution of (4.5) if and only if $(y, \widetilde{y})^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$, and (4.7) and (4.8) hold. This is a consequence of Proposition 3.1.

We now establish the existence and uniqueness. Let $\mu > 0$ be large. We equip $C([0,T];\mathbb{H})$ the following norm

$$|||y||| = \sup_{t\in[0,T]} \mathrm{e}^{-\mu t}\|y(t)\|_{\mathbb{H}}.$$

Define $\mathcal{F} : \big(C([0,T];\mathbb{H})\big)^2 \to \big(C([0,T];\mathbb{H})\big)^2$ as follows

$$\mathcal{F}\begin{pmatrix} y(t) \\ \widetilde{y}(t) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{tA}y_0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A}g(s)\,\mathrm{d}s \\ \mathrm{e}^{-tA^*}\widetilde{y}_0 + \displaystyle\int_0^t \mathrm{e}^{-(t-s)A^*}\widetilde{g}(s)\,\mathrm{d}s \end{pmatrix} \text{ for } t \in [0,T].$$

Then, for $(y_1, \widetilde{y}_1)^\mathsf{T}, (y_2, \widetilde{y}_2)^\mathsf{T} \in \big(C([0,T];\mathbb{H})\big)^2$,

$$\mathcal{F}\begin{pmatrix} y_2(t) \\ \widetilde{y}_2(t) \end{pmatrix} - \mathcal{F}\begin{pmatrix} y_1(t) \\ \widetilde{y}_1(t) \end{pmatrix}$$
$$= \begin{pmatrix} \displaystyle\int_0^t \mathrm{e}^{(t-s)A}\Big(-BWB^*(\widetilde{y}_2 - \widetilde{y}_1) + M_1(y_2 - y_1) + M_2(\widetilde{y}_2 - \widetilde{y}_1)\Big)\,\mathrm{d}s \\ \displaystyle\int_0^t \mathrm{e}^{-(t-s)A^*}\Big(\widetilde{M}_1(\widetilde{y}_2 - \widetilde{y}_1) + \widetilde{M}_2(y_2 - y_1)\Big)\,\mathrm{d}s \end{pmatrix}.$$

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[4] These identities below are understood in $\mathcal{D}(A^*)'$ and $\mathcal{D}(-A^*)'$, respectively.

It follows from the admissibility of B with respect to A (1.3) and Lemma 3.8 that

$$\left\| \mathcal{F}\begin{pmatrix} y_2(t) \\ \widetilde{y}_2(t) \end{pmatrix} - \mathcal{F}\begin{pmatrix} y_1(t) \\ \widetilde{y}_1(t) \end{pmatrix} \right\|_{\mathbb{H}} \\ \leqslant C \left(\int_0^t \|(y_2, \widetilde{y}_2)^{\mathsf{T}}(s) - (y_1, \widetilde{y}_1)^{\mathsf{T}}(s)\|_{\mathbb{H}} \, \mathrm{d}s + \|B^*(\widetilde{y}_2 - \widetilde{y}_1)\|_{L^2((0,t);\mathbb{U})} \right).$$

Here and in what follows in this proof, C denotes a positive constant independent of solutions and μ.

This implies, by (3.6) of Lemma 3.5,

$$e^{-\mu t} \left\| \mathcal{F}\begin{pmatrix} y_2(t) \\ \widetilde{y}_2(t) \end{pmatrix} - \mathcal{F}\begin{pmatrix} y_1(t) \\ \widetilde{y}_1(t) \end{pmatrix} \right\|_{\mathbb{H}} \leqslant C e^{-\mu t} \int_0^t \|(y_2, \widetilde{y}_2)^{\mathsf{T}}(s) - (y_1, \widetilde{y}_1)^{\mathsf{T}}(s)\|_{\mathbb{H}} \, \mathrm{d}s.$$

We derive that

$$\left\| \mathcal{F}\begin{pmatrix} y_2 \\ \widetilde{y}_2 \end{pmatrix} - \mathcal{F}\begin{pmatrix} y_1 \\ \widetilde{y}_1 \end{pmatrix} \right\| \leqslant \frac{C}{\mu} \left\| \begin{pmatrix} y_2 \\ \widetilde{y}_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ \widetilde{y}_1 \end{pmatrix} \right\|.$$

By considering μ large enough, the existence and uniqueness of the weak solutions follow from a standard fixed point theorem. □

4.2. Dynamic feedback controls in the linear case - Proof of Theorem 2.1

Set, for $t \geqslant 0$,

$$y_\lambda(t) = e^{\lambda t} y(t) \quad \text{and} \quad \widetilde{y}_\lambda(t) = e^{\lambda t} \widetilde{y}(t), \tag{4.9}$$

and denote

$$A_\lambda = A + \lambda I.$$

We have

$$\begin{cases} y_\lambda' = A_\lambda y_\lambda - BWB^* \widetilde{y}_\lambda \quad \text{in } (0, +\infty), \\ \widetilde{y}_\lambda' = -A_\lambda^* \widetilde{y}_\lambda + \lambda_1 Q^{-1}(y_\lambda - Q\widetilde{y}_\lambda) \quad \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \widetilde{y}_\lambda(0) = \widetilde{y}(0). \end{cases} \tag{4.10}$$

Set, for $t \geqslant 0$,

$$Z_\lambda(t) = y_\lambda(t) - Q\widetilde{y}_\lambda(t).$$

We formally have, for $t \in (0, +\infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_\lambda = A_\lambda y_\lambda - BWB^* \widetilde{y}_\lambda + QA_\lambda^* \widetilde{y}_\lambda - \lambda_1 Z_\lambda \\ = A_\lambda (y_\lambda - Q\widetilde{y}_\lambda) + A_\lambda Q \widetilde{y}_\lambda - BWB^* \widetilde{y}_\lambda + QA_\lambda^* \widetilde{y}_\lambda - \lambda_1 Z_\lambda,$$

which yields, since (1.5) holds with $R = 0$, that

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_\lambda = A_\lambda Z_\lambda - \lambda_1 Z_\lambda. \tag{4.11}$$

We now give the proof of (4.11) (in the sense of the weak solutions). Let $\tau > 0$, $\varphi_\tau \in \mathbb{H}$ and let $\varphi \in C([0,\tau];\mathbb{H})$ be the unique weak solution of the system

$$\begin{cases} \varphi' = -A_\lambda^* \varphi \text{ in } (0,\tau), \\ \varphi(\tau) = \varphi_\tau. \end{cases} \tag{4.12}$$

Applying Lemma 3.5 for A_λ with $t = \tau$, we derive from (4.10) and (4.12) that

$$\langle y_\lambda(\tau), \varphi(\tau)\rangle - \langle y_\lambda(0), \varphi(0)\rangle = -\int_0^\tau \langle WB^* \widetilde{y}_\lambda(s), B^*\varphi(s)\rangle \,\mathrm{d}s. \tag{4.13}$$

Applying Lemma 4.1 for A_λ, $\widetilde{y}_\lambda(\tau - \cdot)$, and $\varphi(\tau - \cdot)$ (with $R = 0$), we obtain

$$\begin{aligned} &\langle Q\widetilde{y}_\lambda(0), \varphi(0)\rangle - \langle Q\widetilde{y}_\lambda(\tau), \varphi(\tau)\rangle \\ &\qquad = \int_0^\tau \langle WB^* \widetilde{y}_\lambda(\tau - s), B^*\varphi(\tau - s)\rangle \,ds - \lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s)\rangle \,\mathrm{d}s. \end{aligned} \tag{4.14}$$

Summing (4.13) and (4.14), we deduce from (4.10) and (4.12) that

$$\langle Z_\lambda(\tau), \varphi(\tau)\rangle - \langle Z_\lambda(0), \varphi(0)\rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s)\rangle \,\mathrm{d}s.$$

This yields

$$\langle Z_\lambda(\tau), \varphi(\tau)\rangle - \langle Z_\lambda(0), \mathrm{e}^{\tau A^*}\varphi(\tau)\rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \mathrm{e}^{sA^*}\varphi(\tau)\rangle \,\mathrm{d}s.$$

Since $\varphi(\tau) \in \mathbb{H}$ is arbitrary, we obtain

$$Z_\lambda(\tau) = \mathrm{e}^{\tau A} Z_\lambda(0) - \lambda_1 \int_0^\tau \mathrm{e}^{(\tau - s)A} Z_\lambda(s) \,\mathrm{d}s,$$

which implies (4.11).

We derive from (4.11) that

$$\|Z_\lambda(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))t} \|Z_\lambda(0)\|_{\mathbb{H}}, \tag{4.15}$$

which yields

$$\|y(t) - Q\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{(-\lambda_1 + \hat{\omega}_0(A))t} \|y(0) - Q\widetilde{y}(0)\|_{\mathbb{H}}. \tag{4.16}$$

Here and in what follows in this proof, C is a positive constant independent of t and $(y_0, \widetilde{y}_0)$.

Since

$$\widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} + \lambda_1 Q^{-1}(y - Q\widetilde{y}) \quad \text{ in } (0,+\infty),$$

it follows that

$$\widetilde{y}_{2\lambda}' = -A^*\widetilde{y}_{2\lambda} + f(t) \text{ in } (0,+\infty),$$

where

$$\widetilde{y}_{2\lambda}(t) = \mathrm{e}^{2\lambda}\widetilde{y}(t) \quad \text{ and } \quad f(t) = \lambda_1 \mathrm{e}^{2\lambda t} Q^{-1}(y(t) - Q\widetilde{y}(t)) \text{ in } (0,+\infty).$$

We obtain

$$\widetilde{y}_{2\lambda}(t) = \mathrm{e}^{-tA^*}\widetilde{y}_{2\lambda}(0) + \int_0^t \mathrm{e}^{-(t-s)A^*} f(s)\,\mathrm{d}s. \tag{4.17}$$

From the definition of f and (4.16), we have

$$\left\| \int_0^t \mathrm{e}^{-(t-s)A^*} f(s)\,\mathrm{d}s \right\|_{\mathbb{H}} \leqslant C \int_0^t \mathrm{e}^{\hat{\omega}_0(-A^*)(t-s)} \mathrm{e}^{\left(-\lambda_1 + \hat{\omega}_0(A) + 2\lambda\right)s} \|y(0) - Q\widetilde{y}(0)\|_{\mathbb{H}}\,\mathrm{d}s.$$

Since

$$-\hat{\omega}_0(-A^*) + \hat{\omega}_0(A) + 2\lambda - \lambda_1 \overset{(2.2)}{<} 0,$$

it follows that

$$\left\| \int_0^t \mathrm{e}^{-(t-s)A^*} f(s)\,\mathrm{d}s \right\|_{\mathbb{H}} \leqslant C\mathrm{e}^{\hat{\omega}_0(-A^*)t} \|y(0) - Q\widetilde{y}(0)\|_{\mathbb{H}}. \tag{4.18}$$

Combining (4.17) and (4.18) yields

$$\|\widetilde{y}_{2\lambda}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{\hat{\omega}_0(-A^*)t}\big(\|\widetilde{y}(0)\|_{\mathbb{H}} + \|y(0) - Q\widetilde{y}_0\|_{\mathbb{H}}\big),$$

which implies

$$\|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t}\big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big). \tag{4.19}$$

Combining (4.16) and (4.19), we obtain

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\Big(\mathrm{e}^{(-\lambda_1 + \hat{\omega}_0(A))t} + \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t}\Big)\big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big). \tag{4.20}$$

Since

$$\lambda_1 - \hat{\omega}_0(A) \overset{(2.2)}{>} 2\lambda - \hat{\omega}_0(-A^*),$$

it follows from (4.20) that

$$\|y(t)\|_{\mathbb{H}} + \|\widetilde{y}(t)\|_{\mathbb{H}} \leqslant C\mathrm{e}^{(\hat{\omega}_0(-A^*)-2\lambda)t}\big(\|y(0)\|_{\mathbb{H}} + \|\widetilde{y}(0)\|_{\mathbb{H}}\big), \tag{4.21}$$

which is (2.4).

It is clear that (2.5) is a direct consequence of (2.4).

The proof is complete. □

Remark 4.4. For an exactly controllable system, given a positive symmetric $W \in \mathcal{L}(\mathbb{H})$ and λ sufficiently large, one can choose Q such that (1.5) holds with $R = 0$ (see, *e.g.*, Proposition 6.3 in Sect. 6). It is worth noting that the assumption $R = 0$ is necessary in Theorem 2.1. Indeed, if (1.5) holds for some $R \in \mathcal{L}(\mathbb{H})$, which is not necessary to be 0, then the following system is imposed for $(y, \widetilde{y})^{\mathsf{T}}$:

$$\begin{cases} y' = Ay - BWB^*\widetilde{y} \quad \text{in } (0, +\infty), \\ \widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} - RQ\widetilde{y} + \lambda_1 Q^{-1}(y - Q\widetilde{y}) \quad \text{in } (0, +\infty), \\ y(0) = y_0, \quad \widetilde{y}(0) = \widetilde{y}_0. \end{cases} \tag{4.22}$$

(see also Thm. 2.9). We also obtain (4.11) in this case. This in turn implies (4.16). To estimate $\widetilde{y}$ to reach the conclusion, we note

$$\widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} - RQ\widetilde{y} + \lambda_1 Q^{-1}(y - Q\widetilde{y}) \quad \text{in } (0, +\infty).$$

For λ_1 being very large, using (4.16), one can heuristically ignore the term $\lambda_1 Q^{-1}(y - Q\widetilde{y})$ in the above equation of $\widetilde{y}$. One thus needs to understand the decay of $\widetilde{z}$ where $\widetilde{z}$ is a solution of the equation

$$\widetilde{z}' = -A^*\widetilde{z} - 2\lambda\widetilde{z} - RQ\widetilde{z} \quad \text{in } (0, +\infty).$$

Since the size of $\|Q\|_{\mathcal{L}(\mathbb{H})}$ is not known/given and can explode very fast as λ goes to $+\infty$ in the general case (see, *e.g.*, [25, 26]), one cannot establish the decay of $\widetilde{z}$ without further information on Q and also on R in the case $R \not\equiv 0$.

4.3. Dynamic feedback controls in the nonlinear case - Proof of Theorem 2.4

For each $T > 0$, there exists $\varepsilon_T > 0$ such that (2.11) is well-posed in the time interval $[0, T]$. The global existence and uniqueness follow for small ε provided that (2.12) is established for each fixed time interval $[0, T]$ with ε_T sufficiently small. The proof is in the same spirit of the one of Theorem 2.1 but more involved due to the nonlinearity.

Set, for $t \geqslant 0$,

$$y_\lambda(t) = \mathrm{e}^{\lambda t} y(t) \quad \text{and} \quad \widetilde{y}_\lambda(t) = \mathrm{e}^{\lambda t}\widetilde{y}(t), \tag{4.23}$$

and denote

$$A_\lambda = A + \lambda I.$$

We have

$$\left\{\begin{array}{c} y_\lambda' = A_\lambda y_\lambda + e^{\lambda \cdot} f(e^{-\lambda \cdot} y_\lambda) - BWB^* \widetilde{y}_\lambda \quad \text{in } (0, +\infty), \\ \widetilde{y}_\lambda' = -A_\lambda^* \widetilde{y}_\lambda + Q^{-1} e^{\lambda \cdot} f(e^{-\lambda \cdot} Q \widetilde{y}_\lambda) + \lambda_1 Q^{-1}(y_\lambda - Q\widetilde{y}_\lambda) \quad \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \widetilde{y}_\lambda(0) = \widetilde{y}(0). \end{array}\right. \tag{4.24}$$

Set, for $t \geqslant 0$,

$$Z_\lambda(t) = y_\lambda(t) - Q\widetilde{y}_\lambda(t).$$

As in the proof of (4.11) in the proof of Theorem 2.1, we formally derive that Z_λ is a weak solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_\lambda = A_\lambda Z_\lambda - \lambda_1 Z_\lambda + g_1, \tag{4.25}$$

where

$$g_1(t) = e^{\lambda t}\Big(f(e^{-\lambda t} y_\lambda(t)) - f(e^{-\lambda t} Q\widetilde{y}_\lambda(t))\Big) \text{ for } t \in (0, +\infty). \tag{4.26}$$

The proof of (4.25) is similar to the one of (4.11) given in the proof of Theorem 2.1. For the convenience of the reader, we present the details. Let $\tau > 0$, $\varphi_\tau \in \mathbb{H}$ and let $\varphi \in C([0, \tau]; \mathbb{H})$ be the unique weak solution of the system

$$\left\{\begin{array}{l} \varphi' = -A_\lambda^* \varphi \text{ in } (0, \tau), \\ \qquad \varphi(\tau) = \varphi_\tau. \end{array}\right. \tag{4.27}$$

Applying Lemma 3.5 for A_λ with $t = \tau$, we derive from (4.24) and (4.27) that

$$\langle y_\lambda(\tau), \varphi(\tau)\rangle - \langle y_\lambda(0), \varphi(0)\rangle = -\int_0^\tau \langle WB^*\widetilde{y}_\lambda(s), B^*\varphi(s)\rangle \,\mathrm{d}s + \int_0^\tau \langle e^{\lambda s} f(e^{-\lambda s} y_\lambda(s)), \varphi(s)\rangle \,\mathrm{d}s. \tag{4.28}$$

Applying Lemma 4.1 for A_λ, $\widetilde{y}_\lambda(\tau - \cdot)$, and $\varphi(\tau - \cdot)$ (with $R = 0$), we obtain

$$\langle Q\widetilde{y}_\lambda(0), \varphi(0)\rangle - \langle Q\widetilde{y}_\lambda(\tau), \varphi(\tau)\rangle = \int_0^\tau \langle WB^*\widetilde{y}_\lambda(\tau - s), B^*\varphi(\tau - s)\rangle ds \\ - \int_0^\tau \langle e^{\lambda(\tau - s)} f(e^{-\lambda(\tau - s)} Q\widetilde{y}_\lambda(\tau - s)), \varphi(\tau - s)\rangle \,\mathrm{d}s - \lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s)\rangle \,\mathrm{d}s. \tag{4.29}$$

Summing (4.28) and (4.29), we deduce from (4.10) and (4.27) that

$$\langle Z_\lambda(\tau), \varphi(\tau)\rangle - \langle Z_\lambda(0), \varphi(0)\rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s)\rangle \,\mathrm{d}s + \int_0^\tau \langle g_1(\tau - s), \varphi(\tau - s)\rangle \,\mathrm{d}s.$$

This yields

$$\langle Z_\lambda(\tau), \varphi(\tau)\rangle - \langle Z_\lambda(0), e^{\tau A^*}\varphi(\tau)\rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), e^{sA^*}\varphi(\tau)\rangle \,\mathrm{d}s + \int_0^\tau \langle g_1(\tau - s), e^{sA^*}\varphi(\tau)\rangle \, d\tau.$$

Since $\varphi(\tau) \in \mathbb{H}$ is arbitrary, we obtain

$$Z_\lambda(\tau) = \mathrm{e}^{\tau A} Z_\lambda(0) - \lambda_1 \int_0^\tau \mathrm{e}^{(\tau-s)A} Z_\lambda(s)\,\mathrm{d}s + \int_0^\tau \mathrm{e}^{(\tau-s)A} g_1(s)\,\mathrm{d}s,$$

which implies (4.25).

It follows from (2.9) and (4.25) that

$$\|Z_\lambda(t)\|_{\mathbb{H}} \leqslant c\mathrm{e}^{(-\lambda_1+\lambda+\hat{\omega}_0(A))t}\|Z_\lambda(0)\|_{\mathbb{H}} + c\int_0^t \mathrm{e}^{(-\lambda_1+\lambda+\hat{\omega}_0(A))(t-s)}\|g_1(s)\|_{\mathbb{H}}\,\mathrm{d}s. \tag{4.30}$$

From (4.30), we obtain

$$\begin{aligned}&\|y(t) - Q\widetilde{y}(t)\|_{\mathbb{H}}\\ &\qquad \leqslant c\mathrm{e}^{(-\lambda_1+\hat{\omega}_0(A))t}\|y(0) - Q\widetilde{y}(0)\|_{\mathbb{H}} + c\mathrm{e}^{(-\lambda_1+\hat{\omega}_0(A))t}\int_0^t \mathrm{e}^{-(-\lambda_1+\lambda+\hat{\omega}_0(A))s}\|g_1(s)\|_{\mathbb{H}}\,\mathrm{d}s.\end{aligned} \tag{4.31}$$

Since

$$\widetilde{y}' = -A^*\widetilde{y} - 2\lambda\widetilde{y} + \mathrm{e}^{\lambda\cdot}Q^{-1}f(\mathrm{e}^{-\lambda\cdot}Q\widetilde{y}_\lambda) + \lambda_1 Q^{-1}(y - Q\widetilde{y}) \quad \text{in } (0,+\infty),$$

it follows that

$$\widetilde{y}_{2\lambda}' = -A^*\widetilde{y}_{2\lambda} + h_1(t) + h(t) \text{ in } (0,+\infty),$$

where, in $(0,+\infty)$,

$$\widetilde{y}_{2\lambda} = \mathrm{e}^{2\lambda}\widetilde{y}(t), \quad h(t) = \lambda_1\mathrm{e}^{2\lambda t}Q^{-1}(y(t) - Q\widetilde{y}(t)), \quad \text{and} \quad h_1(t) = \mathrm{e}^{3\lambda t}Q^{-1}f(\mathrm{e}^{-\lambda t}Q\widetilde{y}_\lambda(t)).$$

We derive that

$$\widetilde{y}_{2\lambda}(t) = \mathrm{e}^{-tA^*}\widetilde{y}_{2\lambda}(0) + \int_0^t \mathrm{e}^{-(t-s)A^*}\big(h(s) + h_1(s)\big)\,\mathrm{d}s. \tag{4.32}$$

Using the first inequality in (2.10), we derive from (4.31) that

$$\begin{aligned}&\left\|\int_0^t \mathrm{e}^{-(t-s)A^*}h(s)\,\mathrm{d}s\right\|_{\mathbb{H}}\\ &\qquad \leqslant C_T\mathrm{e}^{t\hat{\omega}_0(-A^*)}\Big(\|y(0) - Q\widetilde{y}(0)\|_{\mathbb{H}} + \int_0^t \mathrm{e}^{-(-\lambda_1+\lambda+\hat{\omega}_0(A))s}\|g_1(s)\|_{\mathbb{H}}\,\mathrm{d}s\Big) \text{ for } t \in [0,T].\end{aligned} \tag{4.33}$$

Here and in what follows in this proof, C denotes a positive constant independent of T, t, and $(y_0, \widetilde{y}_0)^\top$ and C_T denotes a positive constant independent of t, and $(y_0, \widetilde{y}_0)^\top$ but might depend on T.

Using (2.6) and the first inequality in (2.10), we derive from (4.31), (4.32), and (4.33) that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|(y(t), \widetilde{y}(t))\|_{\mathbb{H}} \leqslant \delta$ in $[0, T]$ for some $T > 0$, then

$$\begin{aligned}\|(y(t), \widetilde{y}(t))\|_{\mathbb{H}} \leqslant C \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \|(y_0, \widetilde{y}_0)\|_{\mathbb{H}} \\ + C_T \varepsilon \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \int_0^t \mathrm{e}^{-(-\lambda_1 + \lambda + \hat{\omega}_0(A))s} \|(y(s), \widetilde{y}(s))\|_{\mathbb{H}} \, \mathrm{d}s \\ + C_T \varepsilon \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \int_0^t \mathrm{e}^{\left(2\lambda - \hat{\omega}_0(-A^*)\right)s} \|(y(s), \widetilde{y}(s))\|_{\mathbb{H}} \, \mathrm{d}s \text{ for } t \in [0, T].\end{aligned} \tag{4.34}$$

Thus, for all $T > 0$, there exists $\varepsilon_T > 0$ such that if $\|(y_0, \widetilde{y}_0)\|_{\mathbb{H}} \leqslant \varepsilon_T$ then

$$\|(y(t), \widetilde{y}(t))\|_{\mathbb{H}} \leqslant C \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \|(y_0, \widetilde{y}_0)\|_{\mathbb{H}} \text{ in } [0, T]. \tag{4.35}$$

In particular, we derive that if T is chosen sufficiently large,

$$\|(y(T), \widetilde{y}(T))\|_{\mathbb{H}} \leqslant \mathrm{e}^{(\omega_0(-A^*) - 2\gamma)T} \|(y_0, \widetilde{y}_0)\|_{\mathbb{H}}. \tag{4.36}$$

The conclusion follows from (4.35) and (4.36) by considering the time $nT \leqslant t \leqslant n(T+1)$ for $n \in \mathbb{N}$. □

Remark 4.5. The relation between the smallness of the initial data and the decay rate can be explicitly computed. Indeed, from (4.34), one can prove that, $t \in [0, T]$,

$$\|(y(t), \widetilde{y}(t))\|_{\mathbb{H}} \leqslant C \mathrm{e}^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \|(y_0, \widetilde{y}_0)\|_{\mathbb{H}} + C_T \varepsilon \max_{s \in [0,t]} \|(y(s), \widetilde{y}(s))\|_{\mathbb{H}}, \tag{4.37}$$

for some positive constant C_T which is independent of $(y, \widetilde{y})$ and can be explicitly estimated *via* T, λ, λ_1, $\hat{\omega}_0(A)$, and $\hat{\omega}(-A^*)$, and f (through (2.6) and (2.7)). From this, we derive (4.35) where ε_T can be estimated *via* T, λ, λ_1, $\hat{\omega}_0(A)$, and $\hat{\omega}(-A^*)$, and f. One thus obtains (4.36) where ε_T can be estimated as a function of λ, λ_1, $\hat{\omega}_0(A)$, and $\hat{\omega}(-A^*)$, f, and γ. The conclusion follows by taking $\varepsilon = \varepsilon_T$ for large T.

5. STATIC FEEDBACK CONTROLS IN THE TRAJECTORY SENSE

This section consisting of three subsections is organized as follows. The proofs of Theorem 2.9 and Theorem 2.13 are given in the first two subsections, respectively. In the last subsection, we study of the infinitesimal generator of the semigroup associated with the static feedback controls given in Theorem 2.9.

5.1. Static feedback controls in the linear case - Proof of Theorem 2.9

Set, for $t \geqslant 0$,

$$y_\lambda(t) = \mathrm{e}^{\lambda t} y(t) \quad \text{and} \quad \widetilde{y}_\lambda(t) = \mathrm{e}^{\lambda t} \widetilde{y}(t). \tag{5.1}$$

We then have, with $A_\lambda = A + \lambda I$,

$$\begin{cases} y_\lambda' = A_\lambda y_\lambda - BWB^* \widetilde{y}_\lambda & \text{in } (0, +\infty), \\ \widetilde{y}_\lambda' = -A_\lambda^* \widetilde{y}_\lambda - RQ\widetilde{y}_\lambda & \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \widetilde{y}_\lambda(0) = \widetilde{y}(0) (= Q^{-1} y(0)). \end{cases} \tag{5.2}$$

Set, for $t \geqslant 0$,

$$Z_\lambda(t) = y_\lambda(t) - Q\widetilde{y}_\lambda(t).$$

As in the proof of (4.11) in the proof of Theorem 2.1, we derive that Z_λ is a weak solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_\lambda = A_\lambda Z_\lambda. \tag{5.3}$$

Since $Z_\lambda(0) = 0$, it follows that

$$Z_\lambda(t) = 0 \text{ for } t \geqslant 0. \tag{5.4}$$

In other words, (2.15) holds.

We next deal with (2.16). Formally, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle y, \widetilde{y}\rangle &= \langle Ay - BWB^*\widetilde{y}, \widetilde{y}\rangle_{\mathbb{H}} + \langle y, -A^*\widetilde{y} - RQ\widetilde{y} - 2\lambda\widetilde{y}\rangle_{\mathbb{H}} \\ &\overset{(2.15)}{=} \langle Ay - BWB^*\widetilde{y}, \widetilde{y}\rangle_{\mathbb{H}} + \langle y, -A^*\widetilde{y} - Ry - 2\lambda Q^{-1}y\rangle_{\mathbb{H}} \\ &= -\|W^{1/2}B^*\widetilde{y}\|_{\mathbb{U}}^2 - \|R^{1/2}y\|_{\mathbb{H}}^2 - 2\lambda\langle Q^{-1}y, y\rangle_{\mathbb{H}},\end{aligned} \tag{5.5}$$

which yields (2.16). The rigor proof of (2.16) can be done by applying Lemma 3.5 for y and $\widetilde{y}$.

To derive (2.17) from (2.16), one just needs to set

$$\rho(t) := \langle Q^{-1}y(t), y(t)\rangle \text{ for } t \geqslant 0, \tag{5.6}$$

and note that, by (2.16),

$$\rho \in W^{1,1}(0,T) \text{ for all } T > 0 \quad \text{and} \quad \rho'(t) \leqslant -2\lambda\rho(t) \text{ for } t \geqslant 0. \tag{5.7}$$

The proof is complete. □

5.2. Static feedback controls in the nonlinear case - Proof of Theorem 2.13

For each $T > 0$ there exists $\varepsilon_T > 0$ such that (5.9) is well-posed in the time interval $[0,T]$ for $\|y_0\|_{\mathbb{H}} < \varepsilon_T$. The global existence and uniqueness follow for small ε provided that (2.23), (2.24), and (2.25) are established for each fixed time interval $[0,T]$ with ε_T sufficiently small.

We now establish (2.23), (2.24), (2.25) in $[0,T]$ for $\varepsilon < \varepsilon_T$ (small). Set, in $[0,T]$,

$$y_\lambda(t) = \mathrm{e}^{\lambda t}y(t) \quad \text{and} \quad \widetilde{y}_\lambda(t) = \mathrm{e}^{\lambda t}\widetilde{y}(t). \tag{5.8}$$

We then have, with $A_\lambda = A + \lambda I$,

$$\begin{cases} y_\lambda' = A_\lambda y_\lambda + \mathrm{e}^{\lambda t}f(\mathrm{e}^{-\lambda t}y_\lambda(t)) - BWB^*\widetilde{y}_\lambda & \text{in } (0,T), \\ \widetilde{y}_\lambda' = -A_\lambda^*\widetilde{y}_\lambda - RQ\widetilde{y}_\lambda + Q^{-1}\mathrm{e}^{\lambda t}f(\mathrm{e}^{-\lambda t}Q\widetilde{y}_\lambda) & \text{in } (0,T), \\ y_\lambda(0) = y(0), \quad \widetilde{y}_\lambda(0) = \widetilde{y}(0) (= Q^{-1}y(0)). \end{cases} \tag{5.9}$$

Set, for $t \in [0, T]$,

$$Z_\lambda(t) = y_\lambda(t) - Q\widetilde{y}_\lambda(t).$$

As in the proof of (4.11) in the proof of Theorem 2.1, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_\lambda = A_\lambda Z_\lambda + \mathrm{e}^{\lambda \cdot}\Big(f(\mathrm{e}^{-\lambda \cdot} y_\lambda) - f(\mathrm{e}^{-\lambda \cdot} Q\widetilde{y}_\lambda(\cdot))\Big). \tag{5.10}$$

Since $Z_\lambda(0) = 0$, we obtain

$$y_\lambda(t) - Q\widetilde{y}_\lambda(t) = \int_0^t \mathrm{e}^{(t-s)A_\lambda} \mathrm{e}^{\lambda s}\Big(f(\mathrm{e}^{-\lambda s} y_\lambda(s)) - f(\mathrm{e}^{-\lambda s} Q\widetilde{y}_\lambda(s))\Big) \,\mathrm{d}s.$$

Using (2.7), we deduce that

$$\| y_\lambda(t) - Q\widetilde{y}_\lambda(t)\|_{\mathbb{H}} \leqslant C_{\lambda, T} \int_0^t \| y_\lambda(s) - Q\widetilde{y}_\lambda(s)\|_{\mathbb{H}} \,\mathrm{d}s.$$

This yields

$$y_\lambda(t) = Q\widetilde{y}_\lambda(t) \text{ for } t \geqslant 0,$$

which implies (2.23).

We next deal with (2.24). The proof of (2.24) is similar to the one of (2.16) by applying Lemma 3.5 for y and $\widetilde{y}$.

What have been done so far does not require $\lambda > 0$. The fact $\lambda > 0$ is used to derive (2.25) from (2.24). Set

$$\rho(t) = \langle Q^{-1} y(t), y(t)\rangle \text{ for } t \geqslant 0,$$

Note that, by (2.24), as in the proof of (4.35) for all $T > 0$, there exists $\delta > 0$ such that if $\|y_0\|_{\mathbb{H}} \leqslant \delta$ in $[0, T]$, then

$$\rho(t) \leqslant C \mathrm{e}^{-2\lambda t} \rho(0) \text{ in } [0, T]. \tag{5.11}$$

In particular, we have, if T is chosen sufficiently large,

$$\|(y(T), \widetilde{y}(T))\|_{\mathbb{H}} \leqslant \mathrm{e}^{-2\gamma T} \|(y_0, \widetilde{y}_0)\|_{\mathbb{H}}. \tag{5.12}$$

The conclusion follows from (5.11) and (5.12) by considering the time $nT \leqslant t \leqslant n(T+1)$ for $n \in \mathbb{N}$.

The proof is complete. □

5.3. The infinitesimal generator of the semigroup associated with the static feedback controls

Here is the main result of this section on the infinitesimal generator of the semigroup associated with the static feedback controls from Theorem 2.9

Proposition 5.1. *Let* $\lambda \in \mathbb{R}$ *and assume* (1.5). *Let* $y_0 \in \mathbb{H}$, *set*

$$S^Q(t)(y_0) = y(t), \tag{5.13}$$

where $(y, \widetilde{y})$ *is the solution of* (2.14). *Then*

$$\big(S^Q(t)\big)_{t\geqslant 0} \text{ is a strongly continuous semigroup on } \mathbb{H}. \tag{5.14}$$

Moreover, the semigroup $\big(S^Q(t)\big)_{t\geqslant 0}$ *decays exponentially with the rate* λ, *i.e., there exists* $C > 0$ *such that*

$$\|S^Q(t)\| \leqslant C\mathrm{e}^{-\lambda t} \text{ for } t \geqslant 0. \tag{5.15}$$

Let $(A^Q, \mathcal{D}(A^Q))$ *be its infinitesimal generator. We have*

$$\mathcal{D}(A^Q) = Q\mathcal{D}(A^*) := \Big\{Qx; x \in \mathcal{D}(A^*)\Big\} \tag{5.16}$$

and

$$A^Q z = -QA^*Q^{-1}z - 2\lambda z - QRz \text{ for } z \in \mathcal{D}(A^Q). \tag{5.17}$$

We also have

i) if BWB^* *is bounded, i.e.,* $BWB^* \in \mathcal{L}(\mathbb{H})$, *then*

$$\mathcal{D}(A^Q) = \mathcal{D}(A) \quad \text{and} \quad A^Q x = Ax - BWB^*Q^{-1}x \text{ for } x \in \mathcal{D}(A) = \mathcal{D}(A^Q). \tag{5.18}$$

ii) if $\mathcal{D}(A^Q) = \mathcal{D}(A)$, *then* $BWB^*x \in \mathbb{H}$ *for* $x \in \mathcal{D}(A^*)$, *and*

$$\|BWB^*x\|_{\mathbb{H}} \leqslant \|AQx\|_{\mathbb{H}} + C(\|A^*x\|_{\mathbb{H}} + \|x\|_{\mathbb{H}}) \text{ for } x \in \mathcal{D}(A^*) \tag{5.19}$$

for some positive constant C *independent of* x.

Proof of Proposition 5.1. It is clear that (5.14) and (5.15) are the consequences of Theorem 2.9.

We now prove (5.16) and (5.17). Fix $y_0 \in Q\mathcal{D}(A^*)$ (arbitrary). Let $(y, \widetilde{y})$ be the unique weak solution of (2.14). Since $\widetilde{y}(0) = Q^{-1}y_0 \in \mathcal{D}(A^*)$, it follows that $\widetilde{y} \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); \mathcal{D}(A^*))$ and

$$\widetilde{y}'(0) = A^*\widetilde{y}(0) - 2\lambda\widetilde{y}(0) - RQ\widetilde{y}(0). \tag{5.20}$$

Since $y(t) = Q\widetilde{y}(t)$ for $t \geqslant 0$ by Theorem 2.9, we derive that $y'(0)$ is well-defined and

$$y'(0) = Q\widetilde{y}'(0) \overset{(5.20)}{=} -QA^*Q^{-1}y_0 - 2\lambda y_0 - QRQy_0.$$

Hence $y_0 \in \mathcal{D}(A^Q)$ and

$$A^Q y_0 = -QA^*Q^{-1}y_0 - 2\lambda y_0 - QRQy_0.$$

To complete the proof of (5.16) and (5.17), we now show that if $y_0 \in \mathcal{D}(A^Q)$ then $y_0 \in Q\mathcal{D}(A^*)$. Fix $y_0 \in \mathcal{D}(A^Q)$ (arbitrary) and let $(y, \widetilde{y})$ be the unique solution of (2.14). Since $y_0 \in \mathcal{D}(A^Q)$ and $S^Q(t)(y_0) = y(t)$, it follows that $y \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); \mathcal{D}(A^Q))$. In particular $y'(0)$ is well-defined. Since $y(t) = Q\widetilde{y}(t)$ for $t \geqslant 0$

by Theorem 2.9, it follows from the equation of $\widetilde{y}$ in (2.14) that $\widetilde{y}'(0)$ is well-defined and thus $\widetilde{y}(0) \in \mathcal{D}(A^*)$. Since $\widetilde{y}(0) = Q^{-1}y_0$, we derive that

$$Q^{-1}y_0 \in \mathcal{D}(A^*).$$

In other words, $y_0 \in Q\mathcal{D}(A^*)$.

We next establish (5.18). We first assume that $BWB^* \in \mathcal{L}(\mathbb{H})$. It follows that the generator of the semigroup $\big(S^Q(t)\big)_{t \geqslant 0}$ is $A - BWB^*Q^{-1}$ with the domain $\mathcal{D}(A)$.

We finally derive (5.19). Assume that $\mathcal{D}(A^Q) = \mathcal{D}(A)$. From (1.6), we have, for $x, y \in \mathcal{D}(A^*)$,

$$\begin{aligned}|\langle WB^*x, B^*y\rangle_{\mathbb{U}}| \leqslant & |\langle Qx, A^*y\rangle_{\mathbb{H}}| + |\langle Qy, A^*x\rangle_{\mathbb{H}}| + |\langle RQx, Qy\rangle_{\mathbb{H}}| + 2|\lambda||\langle Qx, y\rangle_{\mathbb{H}}| \\ \leqslant & (\|AQx\|_{\mathbb{H}} + C\|A^*x\|_{\mathbb{H}} + C\|x\|_{\mathbb{H}})\|y\|_{\mathbb{H}}.\end{aligned}$$

It follows that

$$\|BWB^*x\|_{\mathbb{H}} \leqslant \|AQx\|_{\mathbb{H}} + C(\|A^*x\|_{\mathbb{H}} + \|x\|_{\mathbb{H}}) \text{ for } x \in \mathcal{D}(A^*),$$

which is (5.19).

The proof is complete. □

Remark 5.2. Related results to Proposition 5.1 from the linear quadratic optimal control theory can be found in [5, 13, 15, 33, 34]. Known results established in the case $\lambda = 0$ and W being identity are connections between $\mathcal{D}(A^{Q^*})$ and $\mathcal{D}(A)$, see [34], Theorem 2.1. This is different from (5.16) where a connection between $\mathcal{D}(A^Q)$ and $\mathcal{D}(A^*)$ is established. Assertion *i)* is equivalent to the fact that B is bounded, *i.e.*, $B \in \mathcal{L}(\mathbb{U}, \mathbb{H})$ when W is positive; this case is well-known.

6. Choices of Q for exactly controllable systems

In this section, we discuss how to choose Q for exactly controllable systems. Assume that the system is exactly controllable at time T. This is equivalent to the fact that (1.4) holds. Fix $\lambda \in \mathbb{R}$ and $T_* > T$ and let $\rho : [0, T_*] \to \mathbb{R}$ be such that

$$\rho \text{ is Lipschitz, } \rho \text{ is decreasing, } \rho(0) = 1,\ \rho(T) > 0, \text{ and } \rho(T_*) = 0. \tag{6.1}$$

Let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and positive. Define $Q : \mathbb{H} \to \mathbb{H}$ as follows

$$\langle Qz_1, z_2\rangle = \int_0^{T_*} \rho(s) \mathrm{e}^{-2\lambda s} \langle WB^* \mathrm{e}^{-sA^*} z_1, B^* \mathrm{e}^{-sA^*} z_2\rangle \, \mathrm{d}s \text{ for } z_1, z_2 \in \mathbb{H}. \tag{6.2}$$

Then Q is linear, continuous, and symmetric. Moreover, since ρ is decreasing and $\rho(T) > 0$, A is an infinitesimal of a group, it follows from (1.4) that

$$Q \text{ is invertible.} \tag{6.3}$$

Let $R : \mathbb{H} \to \mathbb{H}$ be defined by

$$\langle RQz_1, Qz_2\rangle = -\int_0^{T_*} \rho'(s) \langle WB^* \mathrm{e}^{-s(A+\lambda I)^*} z_1, B^* \mathrm{e}^{-s(A+\lambda I)^*} z_2\rangle \, \mathrm{d}s. \tag{6.4}$$

For $z_1, z_2 \in \mathcal{D}(A^{*2})$, we have, from (6.2),

$$\begin{aligned}\langle Qz_1, (A+\lambda I)^* z_2\rangle + \langle (A+\lambda I)^* z_1, Qz_2\rangle \\ = \int_0^{T_*} \rho(s) \langle WB^* \mathrm{e}^{-s(A+\lambda I)^*} z_1, B^* \mathrm{e}^{-s(A+\lambda I)^*} (A+\lambda I)^* z_2\rangle \, \mathrm{d}s \\ + \int_0^{T_*} \rho(s) \langle WB^* \mathrm{e}^{-s(A+\lambda I)^*} (A+\lambda I)^* z_1, B^* \mathrm{e}^{-s(A+\lambda I)^*} z_2\rangle \, \mathrm{d}s. \quad (6.5)\end{aligned}$$

Using the fact that, for $z \in \mathcal{D}(A^{*2})$,

$$\mathrm{e}^{-s(A+\lambda I)^*} (A+\lambda I)^* z = -\frac{\mathrm{d}}{\mathrm{d}s} \left(\mathrm{e}^{-s(A+\lambda I)^*} z \right),$$

we derive from (6.5) that

$$\langle Qz_1, (A+\lambda I)^* z_2\rangle + \langle (A+\lambda I)^* Qz_1, z_2\rangle = -\int_0^{T_*} \rho(s) \frac{\mathrm{d}}{\mathrm{d}s} \left(\mathrm{e}^{-s(A+\lambda I)} BWB^* \mathrm{e}^{-s(A+\lambda I)^*} \right) \mathrm{d}s, \quad (6.6)$$

which yields, by an integration by parts,

$$\begin{aligned}\langle Qz_1, (A+\lambda I)^* z_2\rangle + \langle (A+\lambda I)^* Qz_1, z_2\rangle \\ = \langle WB^* z_1, B^* z_2\rangle + \int_0^{T_*} \rho'(s) \langle WB^* \mathrm{e}^{-s(A+\lambda I)^*} z_1, B^* \mathrm{e}^{-s(A+\lambda I)^*} z_2\rangle \, \mathrm{d}s, \quad (6.7)\end{aligned}$$

This implies (1.6) for $z_1, z_2 \in \mathcal{D}(A^{*2})$. The general case follows by density.

We have just proven the following result.

Proposition 6.1. *Assume that $(S(t))_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous group in $\mathbb{H}$, B is an admissible control operator, and system (1.1) is exactly controllable in time T for some $T > 0$. Let $\lambda \in \mathbb{R}$, $T_* > T$, and $\rho : [0, T_*] \to \mathbb{R}$ be a function satisfying (6.1), and let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and positive. Define $Q : \mathbb{H} \to \mathbb{H}$ by*

$$\langle Qz_1, z_2\rangle = \int_0^{T_*} \rho(s) \mathrm{e}^{-2\lambda s} \langle WB^* \mathrm{e}^{-sA^*} z_1, B^* \mathrm{e}^{-sA^*} z_2\rangle \, \mathrm{d}s \text{ for } z_1, z_2 \in \mathbb{H}. \quad (6.8)$$

Then Q is linear, continuous, symmetric, and invertible and (1.5) holds with R being defined by (6.4), i.e., (1.6) is valid.

Remark 6.2. Proposition 6.1 covers the setting considered by Komornik. Indeed, set, with $T_* = T + \frac{1}{2\lambda}$

$$\rho(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant T, \\ 2\lambda \mathrm{e}^{-2\lambda(T-t)} (T_* - t) & \text{for } T < t \leqslant T_*. \end{cases} \quad (6.9)$$

Then

$$e_\lambda(t) = \mathrm{e}^{\lambda t} \rho(t) \text{ in } [0, T_*].$$

Since, for $T \leqslant t \leqslant T_* = T + \frac{1}{2\lambda}$,

$$\rho(t) = e\tau \mathrm{e}^{-\tau} \text{ with } \tau = 2\lambda(T_* - t),$$

and the function $\tau \mathrm{e}^{-\tau}$ is increasing in $[0, 1]$, it follows that ρ defined in (6.9) verifies (6.1).

When A is skew-adjoint and $R = 0$, one has the following result.

Proposition 6.3. *Assume that $\big(S(t)\big)_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous group, B is an admissible control operator, and system (1.1) is exactly controllable in time T for some $T > 0$. Let $\lambda \in \mathbb{R}$ and let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and non-negative, and assume that $\lambda > \omega_0(-A^*)$. Define $Q : \mathbb{H} \to \mathbb{H}$ by*

$$\langle Q z_1, z_2 \rangle = \int_0^\infty \mathrm{e}^{-2\lambda s} \langle W B^* \mathrm{e}^{-sA^*} z_1, B^* \mathrm{e}^{-sA^*} z_2 \rangle \, \mathrm{d}s \text{ for } z_1, z_2 \in \mathbb{H}. \tag{6.10}$$

Then Q is linear, continuous, symmetric, and invertible, and (1.5) holds with $R = 0$, i.e., (1.6) is valid with $R = 0$.

Proof. The proof of (6.1) is almost the same as the one of Proposition 6.1. One just needs to note that the RHS of (6.10) is well-defined for $\lambda > \omega_0(-A^*)$. The details are omitted. □

Remark 6.4. Proposition 6.3 was previously obtained by Urquiza [22] by a different approach using results of Grabowski in [35] (see also [36]).

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No new data/codes were created or analyzed in this study.

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