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A GENERAL MAXIMUM PRINCIPLE FOR PARTIALLY OBSERVED STOCHASTIC EVOLUTION CONTROL SYSTEMS

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Abstract. In this paper, we study an optimal control problem of partially observed stochastic evolution equation control system. The observation process is an infinite-dimensional stochastic process, and the control variable is allowed to enter the diffusion term of the state process and the drift term of the observation process. The control domain need not be convex. Combined with Girsanov theorem, we establish the related stochastic maximum principle in the sense of transposition solution.

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1. Introduction

We shall consider the following general partially observed optimal stochastic control problem: finding the $\bar{u}(\cdot) \in \mathcal{U}[0,T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} \mathcal{J}(u(\cdot)).$$

The cost functional takes the following form

$$\mathcal{J}(u(\cdot)) = \mathbb{E}^u\Big[\int_0^T g(t,x(t),u(t))\,\mathrm{d}t + c(x(T))\Big], \tag{1.1}$$

subject to the stochastic evolution equation

$$\begin{cases} \mathrm{d}x(t) = (Ax(t) + a(t,x(t),u(t)))\,\mathrm{d}t + b(t,x(t),u(t))\,\mathrm{d}W_1(t), \\ x(0) = x_0, \qquad t\in(0,T], \end{cases} \tag{1.2}$$

Keywords and phrases: Maximum principle, stochastic evolution equation, partial observation, transposition solution, second order adjoint process.

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with the partial observation

$$y(t) = \int_0^t h(s, x(s), u(s))\,\mathrm{d}s + \widehat{W}(t). \tag{1.3}$$

Here, the observation process y is an infinite-dimensional stochastic process.

Maximum principle was first formulated by Pontryagin in the 1950s and it converted the optimization problems into maximizing the corresponding Hamiltonian functions. Bismut [1] introduced the linear backward stochastic differential equations (BSDEs) as the adjoint equations, which was a milestone in the development of this theory. The general stochastic maximum principle was obtained by Peng in [2] by introducing the second order adjoint equations, which allowed the control enter in both the drift and diffusion coefficients while the control domain was nonconvex. Subsequently, there have been extensive studies on the maximum principle for other control systems. For example, Hu and Peng [3] established the maximum principle for semilinear stochastic evolution control systems; Tao and Wu [4] derived necessary and sufficient conditions for optimal control in forward–backward regime-switching systems; Zhang and Shi [5] obtained the maximum principle for forward–backward doubly stochastic control systems, and these results were further extended in the work of Zhang, Zhou, and Yang [6], among others. These studies have made significant contributions to the theory of stochastic maximum principle.

In the infinite-dimensional case of Refs. [3, 7–10], scholars have studied the maximum principle under the following assumptions respectively: 1) the control domain was convex, 2) the diffusion coefficient did not depend on the control, and 3) the state equation and cost functional were both linear in the state variable. If we aim to establish an infinite-dimensional generalized stochastic maximum principle corresponding to Peng [2], it becomes necessary to introduce an operator-valued backward stochastic evolution equation as the second-order adjoint equation. In contrast to the finite-dimensional case, the existence proof for solutions to this equation is highly challenging. To overcome this difficulty, researchers have conducted numerous studies. Lü and Zhang [11–13] employ the stochastic transposition method to introduce a concept of transposition solution/relaxed transposition solution to the desired operator-valued backward stochastic evolution equation, and obtain the general stochastic maximum principle in the sense of transposition solution. Du and Meng [14] and Fuhrman, Hu, and Tessitore [15] define P *via* a stochastic bilinear form derived from the limit of the quadratic terms in the variational calculation, and obtain general stochastic maximum principle. In addition, Guatteri and Tessitore [16, 17] and Stannat and Wessels [18] use some other methods to represent the second-order adjoint processes.

As mentioned above, they all assume that the overall information is available, *i.e.*, decision makers can obtain all information of the state. However, in practice, decision makers can only get partial information in most cases. There is a rich literature on finite-dimensional partial information control system. Bensoussan [19] and Haussmann [20] employed the Zakai equation to transform finite-dimensional partially observed control problems into infinite-dimensional fully observed control problems. Zhou [10] investigated optimal control problems for a class of stochastic partial differential equations, with the obtained results being applicable to control problem involving Zakai equations. Li and Tang [21] investigated optimal control problems in partially observed stochastic control systems where the control variables enter into both the drift term of the observation equation and the diffusion term of the state equation, with a non-convex control domain. Under the framework where the observation process was initially modeled as Brownian motion, they established a generalized stochastic maximum principle for such systems using a purely probabilistic approach, in which the Girsanov theorem played a pivotal role. Wang and Wu [22] and Wu [23] studied the partially observed optimal control of forward–backward stochastic systems with convex and non-convex control domain, respectively. Wang, Zhang and Zhang [24] concerned with a partially observed optimal control problem, whose novel feature is that the cost functional is of mean-field type, [31–33] also represent valuable references on partial information problems. In addition, there are some recent studies on partially observed control problems. For instance, Chen and Wu [25] studied the progressive optimal control for partially observed stochastic system of mean-field type with random jumps, Li and Wu [26] developed the second-order maximum principle for partially observed singular stochastic control systems.

Compared to the literature on partially observed stochastic maximum principle (SMP) problems in finite dimensions, the literature addressing the infinite-dimensional case is notably scarce. While Bensoussan and Viot [27] investigated partially observed problems for infinite-dimensional linear systems, its methodology is not applicable to general nonlinear systems. Our work fills this gap. In this paper, we extend the approach from [21] to the infinite-dimensional setting. By leveraging Girsanov theorem in Hilbert spaces, we transform the partially observed problem into a fully observed one. In doing so, we overcome the challenge posed by the potential inapplicability of the infinite-dimensional Itô's formula in certain scenarios. Subsequently, we introduce the concept of the transposition solution to characterize the adjoint process. This ultimately leads us to derive three adjoint equations and establish the maximum principle. The rest part is organized as follows. Section 2 presents some definitions and lemmas. Section 3 gives a description of the problem and the conditions. Section 4 constructs the adjoint equations and obtains the general stochastic maximum principle. In Section 5, we discuss an example.

We finish the introduction with several notations. Let $p, q \geq 1, T > 0$, $(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P})$ (with $\mathbf{F} = \{\mathscr{F}_t\}_{t\in[0,T]}$) is a fixed filtered probability space satisfying the usual condition, and we denote by $\mathbb{F}$ the progressive σ-field w.r.t. $\mathbf{F}$; V, H and H' are Hilbert spaces.

- $\mathscr{L}(V;H)$: the space of all bounded linear operators from V to H.
- $\mathscr{L}_2(V;H)$: the space of all Hilbert–Schmidt operators from V into H. An operator $F \in \mathscr{L}(V;H)$ is said to be a Hilbert–Schmidt operator if $\sum_{j=1}^{\infty} |Fe_j|_H^2 < \infty$, $\{e_j\}_{j=1}^{\infty}$ is the orthonormal basis of V.
- $\mathscr{L}(H,H;H')$: the space of all bilinear maps from H to H', with the canonic norm.
- $L_{\mathbb{F}}^{p,loc}([0,T];H)$: the space of H-valued progressive processes $x(\cdot)$ such that $\int_0^T |x(s)|_H^p \,\mathrm{d}s < \infty$ a.s.
- $L_{\mathbb{F}}^p(\Omega; L^q([0,T];H))$: the space of H-valued progressive processes $x(\cdot)$ such that $\mathbb{E}[\int_0^T |x(s)|_H^q \,\mathrm{d}s]^{\frac{p}{q}} < \infty$.
- $L_{\mathbb{F}}^p([t,T]; L^q(\Omega;H))$: the space of H-valued progressive processes $x(\cdot)$ such that $\int_t^T (\mathbb{E}|x(s)|_H^q)^{\frac{p}{q}} \,\mathrm{d}s < \infty$, $t \in [0,T]$.
- $C_{\mathbb{F}}([0,T]; L^p(\Omega;H))$: the space of H-valued progressive processes $x(\cdot)$ such that $x(\cdot) : [0,T] \to L^p_{\mathscr{F}_T}(\Omega;H)$ is continuous, and $\sup_{t\in[0,T]} \mathbb{E}|x(t)|_H^p < \infty$.
- $D_{\mathbb{F}}([0,T]; L^p(\Omega;H))$: the space of H-valued progressive processes $x(\cdot)$ such that $x(\cdot) : [0,T] \to L^p_{\mathscr{F}_T}(\Omega;H)$ is càdlàg, and $\operatorname{ess\,sup}_{t\in[0,T]} \mathbb{E}|x(t)|_H^p < \infty$.

2. Preliminaries

In this section, we will introduce the concepts of transposition solution and relaxed transposition solution. Lü and Zhang employed the stochastic transposition method (developed first in their paper [28] for backward stochastic differential equations) to introduce another concept of solution, *i.e.*, the transposition solution (relaxed transposition solution), to the equation (2.7) (Eq. (2.13)). This class of solutions is defined through the dual equations of these BSEEs (Backward Stochastic Evolution Equations), and its specific formulation offers significant advantages in deriving the maximum principle. Crucially, for highly intractable second-order adjoint equation (operator-valued BSEEs), we can solve them by defining relaxed transpose solutions, thereby establishing a general maximum principle.

In this part, $T > 0$, $(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P})$ (with $\mathbf{F} = \{\mathscr{F}_t\}_{t\in[0,T]}$) is a fixed filtered probability space satisfying the usual condition, and we denote by $\mathbb{F}$ the progressive σ-field w.r.t. $\mathbf{F}$; H and V are two separable Hilbert spaces; we denote by I the identity operator on H, and write $\mathscr{L}_2^0 = \mathscr{L}_2(V;H)$. $W(\cdot)$ is a V-valued, cylindrical Brownian motion.

For any $h \in H$, $h^* \in H^*$ and $f \in \mathscr{L}(V;H)$, we define two maps $\langle\langle f, h\rangle\rangle_H(\cdot)$, $\langle\langle h^*, f\rangle\rangle_H(\cdot) : V \to \mathbb{R}$ as follows:

$$\langle\langle f, h\rangle\rangle_H(v) = \langle f(v), h\rangle_H,\ \langle\langle h^*, f\rangle\rangle_H(v) = \langle h^*, f(v)\rangle_{H^*,H} = h^*(f(v)), \qquad \forall v \in V. \tag{2.1}$$

The main purpose of this symbol is to represent random integrals of the form $\int_0^t \langle f \,\mathrm{d}W(s), h\rangle_H$ in the more standard form $\int_0^t \langle\langle f, h\rangle\rangle_H \,\mathrm{d}W(s)$. It is easy to check that $\langle\langle f, h\rangle\rangle_H$, $\langle\langle h^*, f\rangle\rangle_H \in \mathscr{L}_2(V;\mathbb{R})$.

In this section, Girsanov theorem (Lem. 2.12) and Proposition 2.13 are quoted from Da Prato and Zabczyk [29], the remainder (especially the definitions and lemmas related to transposition solution) is quoted from Lü and Zhang [12].

2.1. Well-posedness of stochastic evolution equations

We consider the following stochastic evolution equation:

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + F(t, X(t)))\,\mathrm{d}t + \widetilde{F}(t, X(t))\,\mathrm{d}W(t), \\ X(0) = X_0, \ t \in (0, T]. \end{cases} \tag{2.2}$$

Here $X_0 : \Omega \to H$ is an $\mathscr{F}_0$-measurable random variable, A generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H, and $F(\cdot,\cdot) : [0,T]\times\Omega\times H \to H$ and $\widetilde{F}(\cdot,\cdot) : [0,T]\times\Omega\times H \to \mathscr{L}_2^0$ are two given functions. Let's start with a few definitions:

Definition 2.1 ([12], Defi. 3.8)**.** An H-valued, F-adapted, continuous stochastic process $X(\cdot)$ is called a weak solution to (2.2) if $F(\cdot, X(\cdot)) \in L^1([0,T]; H)$ *a.s.*, $\widetilde{F}(\cdot, X(\cdot)) \in L^{2,loc}_{\mathbb{F}}([0,T]; \mathscr{L}_2^0)$, and for any $t \in [0,T]$ and $\xi \in D(A^*)$,

$$\begin{aligned} \langle X(t), \xi\rangle_H = \langle X_0, \xi\rangle_H &+ \int_0^t (\langle X(s), A^*\xi\rangle_H + \langle F(s, X(s)), \xi\rangle_H)\,\mathrm{d}s \\ &+ \int_0^t \langle\langle \widetilde{F}(s, X(s)), \xi\rangle\rangle_H\,\mathrm{d}W(s), \quad a.s. \end{aligned} \tag{2.3}$$

Definition 2.2 ([12], Defi. 3.9)**.** An H-valued, F-adapted, continuous stochastic process $X(\cdot)$ is called a mild solution to (2.2) if $F(\cdot, X(\cdot)) \in L^1([0,T]; H)$ *a.s.*, $\widetilde{F}(\cdot, X(\cdot)) \in L^{2,loc}_{\mathbb{F}}([0,T]; \mathscr{L}_2^0)$, and for any $t \in [0,T]$

$$\begin{aligned} X(t) = S(t)X_0 &+ \int_0^t S(t-s)F(s, X(s))\,\mathrm{d}s \\ &+ \int_0^t S(t-s)\widetilde{F}(s, X(s))\,\mathrm{d}W(s), \quad a.s. \end{aligned} \tag{2.4}$$

Condition 2.3. *$F(\cdot,\cdot) : [0,T]\times\Omega\times H \to H$ and $\widetilde{F}(\cdot,\cdot) : [0,T]\times\Omega\times H \to \mathscr{L}_2^0$ are two given functions satisfying*
1) Both $F(\cdot, x)$ and $\widetilde{F}(\cdot, x)$ are $\mathbf{F}$-adapted for each $x \in H$;
2) There exist two nonnegative (real-valued) functions $L_1(\cdot) \in L^1(0,T)$ and $L_2(\cdot) \in L^2(0,T)$ such that, for any $x, y \in H$ and a.e. $t \in [0,T]$,

$$\begin{cases} |F(t,x) - F(t,y)|_H \leq L_1(t)|x-y|_H, \\ |\widetilde{F}(t,x) - \widetilde{F}(t,y)|_{\mathscr{L}_2^0} \leq L_2(t)|x-y|_H, \quad a.s.. \end{cases} \tag{2.5}$$

Then, we have the following well-posedness result for (2.2).

Lemma 2.4 ([12], Thm. 3.14)**.** *Let Condition 2.3 hold, and $F(\cdot,0) \in L^p_{\mathbb{F}}(\Omega; L^1([0,T]; H))$ and $\widetilde{F}(\cdot,0) \in L^p_{\mathbb{F}}(\Omega; L^2([0,T]; \mathscr{L}_2^0))$ for some $p \geq 2$. Then, for any $X_0 \in L^p_{\mathscr{F}_0}(\Omega; H)$, the equation (2.2) admits a unique* *mild solution* $X(\cdot) \in C_{\mathbb{F}}([0,T]; L^p(\Omega; H))$. *Moreover,*

$$\begin{aligned} &|X(\cdot)|_{C_{\mathbb{F}}([0,T];L^p(\Omega;H))} \\ &\leq \mathcal{C}\big(|X_0|_{L^p_{\mathscr{F}_0}(\Omega;H)} + |F(\cdot,0)|_{L^p_{\mathbb{F}}(\Omega;L^1([0,T];H))} + |\widetilde{F}(\cdot,0)|_{L^p_{\mathbb{F}}(\Omega;L^2([0,T];\mathscr{L}_2^0))}\big). \end{aligned} \tag{2.6}$$

The following result shows the equivalence between weak and mild solutions to (2.2).

Lemma 2.5 ([12], Thm. 3.10)**.** *An H-valued, F-adapted, continuous stochastic process $X(\cdot)$ is a weak solution to (2.2) if and only if it is a mild solution to the same equation.*

Remark 2.6 ([12], Thm. 3.18, 3.21)**.** If the C_0-semigroup satisfies $|S(t)|_{\mathscr{L}(H)} \leq \mathrm{e}^{ct}$ for some constant $c \in \mathbb{R}$ and any $t \geq 0$, the mild solution in Lemma 2.4 can be path-continuous.

2.2. Backward stochastic evolution equations

We consider the following backward stochastic evolution equation:

$$\begin{cases} \mathrm{d}y(t) = -Ay(t)\,\mathrm{d}t + F(t, y(t), Y(t))\,\mathrm{d}t + Y(t)\,\mathrm{d}W(t), \\ y(T) = y_T, \qquad t \in [0,T). \end{cases} \tag{2.7}$$

Here, $y_T : \Omega \to H$ is an F_T-measurable random variable, A generates a C_0-semigroup $S(t)_{t\geq 0}$ on H, and $F : [0,T] \times \Omega \times H \times \mathscr{L}_2^0 \to H$ is a given function. Throughout this section, we assume that $y_T \in L^p_{\mathscr{F}_T}(\Omega; H)$ for a given $p \in (1,2]$ unless other stated.

In order to define the transposition solution to (2.7), for a given $\eta \in L^q_{\mathscr{F}_t}(\Omega; H)$ and $v_1 \in L^1_{\mathbb{F}}([t,T]; L^q(\Omega; H))$, $v_2 \in L^q_{\mathbb{F}}(\Omega; L^2([t,T]; \mathscr{L}_2^0))$, we introduce the following (forward) stochastic evolution equation:

$$\begin{cases} \mathrm{d}z = (A^* z + v_1)\,\mathrm{d}s + v_2\,\mathrm{d}W(s), \\ z(t) = \eta, \qquad s \in (t,T], \end{cases} \tag{2.8}$$

where $t \in [0,T]$ and $q = \dfrac{p}{p-1}$. This equation admits a unique mild solution $z \in C_{\mathbb{F}}([t,T]; L^q(\Omega; H))$, and

$$\begin{aligned} &|z|_{C_{\mathbb{F}}([t,T];L^q(\Omega;H))} \\ &\leq \mathcal{C}\big(|\eta|_{L^q_{\mathscr{F}_t}(\Omega;H)} + |v_1|_{L^1_{\mathbb{F}}([t,T];L^q(\Omega;H))} + |v_2|_{L^q_{\mathbb{F}}(\Omega;L^2([t,T];\mathscr{L}_2^0))}\big). \end{aligned} \tag{2.9}$$

We now introduce the following notion.

Definition 2.7 ([12], Defi. 4.13)**.** We call $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}([0,T]; L^p(\Omega; \mathscr{L}_2^0))$ a transposition solution to the equation (2.7) if for any $t \in [0,T]$, $v_1(\cdot) \in L^1_{\mathbb{F}}([t,T]; L^q(\Omega; H))$, $v_2(\cdot) \in L^q_{\mathbb{F}}(\Omega; L^2([t,T]; \mathscr{L}_2^0))$, $\eta \in L^q_{\mathscr{F}_t}(\Omega; H)$ and the corresponding solution $z \in C_{\mathbb{F}}([t,T]; L^q(\Omega; H))$ to (2.8), it holds that $\langle z(\cdot), F(\cdot, y(\cdot), Y(\cdot))\rangle_H \in L^1_{\mathbb{F}}(\Omega; L^1([t,T]))$, and

$$\begin{aligned} &\mathbb{E}\langle z(T), y_T\rangle_H - \mathbb{E}\int_t^T \langle z(s), F(s, y(s), Y(s))\rangle_H\,\mathrm{d}s \\ &= \mathbb{E}\langle \eta, y_t\rangle_H + \mathbb{E}\int_t^T \langle v_1(s), y(s)\rangle_H\,\mathrm{d}s + \mathbb{E}\int_t^T \langle v_2(s), Y(s)\rangle_{\mathscr{L}_2^0}\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

Condition 2.8. *$F : [0,T] \times \Omega \times H \times \mathscr{L}_2^0 \to H$ is a given function satisfying*
1) $F(\cdot, y, z)$ is $\mathbf{F}$-adapted for each $(y,z) \in H \times \mathscr{L}_2^0$;

2) There exist two nonnegative functions $L_1(\cdot) \in L^1([0,T])$ and $L_2(\cdot) \in L^2([0,T])$ such that, for any $y_1, y_2 \in H$, $z_1, z_2 \in \mathscr{L}_2^0$, a.e. $t \in [0,T]$,

$$
\begin{aligned}
&|F(t,y_1,z_1) - F(t,y_2,z_2)|_H \\
&\leq L_1(t)|y_1 - y_2|_H + L_2(t)|z_1 - z_2|_{\mathscr{L}_2^0} \quad a.s.
\end{aligned}
\tag{2.11}
$$

Then,we have the following well-posedness result for (2.7) in the sense of transposition solution.

Lemma 2.9 ([12], Thm. 4.19)**.** *Let Condition 2.8 hold and $F(\cdot,0,0) \in L^1_{\mathbb{F}}([0,T]; L^p(\Omega;H))$. Then, for any $y_T \in L^p_{\mathscr{F}_T}(\Omega;H)$, the equation (2.7) admits one and only one transposition solution $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0,T]; L^p(\Omega;H)) \times L^2_{\mathbb{F}}([0,T]; L^p(\Omega;\mathscr{L}_2^0))$. Furthermore,*

$$
\begin{aligned}
&|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([0,T];L^p(\Omega;H)) \times L^2_{\mathbb{F}}([0,T];L^p(\Omega;\mathscr{L}_2^0))} \\
&\leq \mathcal{C}\big(|y_T|_{L^p_{\mathscr{F}_T}(\Omega;H)} + |F(\cdot,0,0)|_{L^1_{\mathbb{F}}([0,T];L^p(\Omega;H))}\big).
\end{aligned}
\tag{2.12}
$$

2.3. Operator-valued backward stochastic evolution equations

In this section, we are going to study the following formally $\mathscr{L}(H)$-valued backward stochastic evolution equation ($p \in (1,2]$):

$$
\begin{cases}
\mathrm{d}P = -(A^* + J^*)P\,\mathrm{d}t - P(A+J)\,\mathrm{d}t - K^*PK\,\mathrm{d}t \\
\qquad\quad - (K^*Q + QK)\,\mathrm{d}t + F\,\mathrm{d}t + Q\,\mathrm{d}W(t), \\
P(T) = P_T, \qquad t \in [0,T),
\end{cases}
\tag{2.13}
$$

where ($q = p/(p-1)$),

$$
J \in L^{2q}_{\mathbb{F}}([0,T]; L^\infty(\Omega;\mathscr{L}(H))), \qquad K \in L^{2q}_{\mathbb{F}}([0,T]; L^\infty(\Omega;\mathscr{L}(H;\mathscr{L}_2^0))),
\tag{2.14}
$$

$$
F \in L^1_{\mathbb{F}}([0,T]; L^p(\Omega;\mathscr{L}(H))), \qquad P_T \in L^p_{\mathscr{F}_T}(\Omega;\mathscr{L}(H)).
\tag{2.15}
$$

Let us introduce the following two (forward) stochastic evolution equations:

$$
\begin{cases}
\mathrm{d}x_1 = (A+J)x_1\,\mathrm{d}s + u_1\,\mathrm{d}s + Kx_1\,\mathrm{d}W(s) + v_1\,\mathrm{d}W(s), \\
x_1(t) = \xi_1, \qquad s \in (t,T],
\end{cases}
\tag{2.16}
$$

and

$$
\begin{cases}
\mathrm{d}x_2 = (A+J)x_2\,\mathrm{d}s + u_2\,\mathrm{d}s + Kx_2\,\mathrm{d}W(s) + v_2\,\mathrm{d}W(s), \\
x_2(t) = \xi_2, \qquad s \in (t,T].
\end{cases}
\tag{2.17}
$$

Here $\xi_1, \xi_2 \in L^{2q}_{\mathscr{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; H))$ and $v_1, v_2 \in L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; \mathscr{L}_2^0))$. Also, we need to introduce the solution space for (2.13). For this purpose, write

$$
\begin{aligned}
& D_{\mathbb{F},\omega}([0,T]; L^p(\Omega; \mathscr{L}(H))) \\
& = \Big\{ P(\cdot,\cdot) | P(\cdot,\cdot) \in \mathscr{L}_{pd}(L^2_{\mathbb{F}}([0,T]; L^{2q}(\Omega; H)); L^2([0,T]; L^{\frac{2p}{p+1}}_{\mathbb{F}}(\Omega; H))), \\
& \quad P(\cdot,\cdot)\xi \in D_{\mathbb{F}}([t,T]; L^{\frac{2p}{p+1}}(\Omega; H)) \ and \\
& \quad |P(\cdot,\cdot)\xi|_{D_{\mathbb{F}}([t,T]; L^{\frac{2p}{p+1}}(\Omega;H))} \leq C|\xi|_{L^{2q}_{\mathscr{F}_t}(\Omega;H)} \\
& \quad for\ every\ t \in [0,T]\ and\ \xi \in L^{2q}_{\mathscr{F}_t}(\Omega; H) \Big\},
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
\mathcal{H}_t = & L^{2q}_{\mathscr{F}_t}(\Omega, H) \times L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; H)) \times L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; \mathscr{L}_2^0)), \quad \forall t \in [0,T), \\
\mathcal{Q}^p[0,T] = & \big\{ (Q^{(\cdot)}, \hat{Q}^{(\cdot)}) | Q^{(t)}, \hat{Q}^{(t)} \in \mathscr{L}(\mathcal{H}_t; L^2_{\mathbb{F}}([t,T]; L^{\frac{2p}{p+1}}(\Omega; \mathscr{L}_2^0))) \\
& \ and\ Q^{(t)}(0,0,\cdot)^* = \hat{Q}^{(t)}(0,0,\cdot)\ for\ any\ t \in [0,T) \big\}.
\end{aligned}
\tag{2.19}
$$

$\mathscr{L}_{pd}(L^2_{\mathbb{F}}([0,T]; L^{2q}(\Omega; H)); L^2([0,T]; L^{\frac{2p}{p+1}}_{\mathbb{F}}(\Omega; H)))$ is the space of all bounded, pointwise defined linear operators $\mathscr{L}$ from $L^2_{\mathbb{F}}([0,T]; L^{2q}(\Omega; H))$ to $L^2([0,T]; L^{\frac{2p}{p+1}}_{\mathbb{F}}(\Omega; H))$, *i.e.*, for a.e. $(t,\omega) \in [0,T] \times \Omega$, there exists an $L(t,\omega) \in \mathscr{L}(H)$ satisfying that $(\mathscr{L}u(\cdot))(t,\omega) = L(t,\omega)u(t,\omega)$, $\forall u(\cdot) \in L^2_{\mathbb{F}}([0,T]; L^{2q}(\Omega; H))$.

We now define the notion of relaxed transposition solution to (2.13) as follows:

Definition 2.10 ([12], Defi. 12.8)**.** We call $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)}) \in D_{\mathbb{F},\omega}([0,T]; L^p(\Omega; \mathscr{L}(H))) \times \mathcal{Q}^p[0,T]$ a relaxed transposition solution to the equation (2.13) if for any $t \in [0,T]$, $\xi_1, \xi_2 \in L^{2q}_{\mathscr{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; H))$ and $v_1, v_2 \in L^2_{\mathbb{F}}([t,T]; L^{2q}(\Omega; \mathscr{L}_2^0))$, it holds that

$$
\begin{aligned}
& \mathbb{E}\langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E}\int_t^T \langle F(s)x_1(s), x_2(s) \rangle_H \, \mathrm{d}s \\
& = \mathbb{E}\langle P(t)\xi_1, \xi_2 \rangle_H + \mathbb{E}\int_t^T \langle P(s)u_1(s), x_2(s) \rangle_H \, \mathrm{d}s \\
& + \mathbb{E}\int_t^T \langle P(s)x_1(s), u_2(s) \rangle_H \, \mathrm{d}s + \mathbb{E}\int_t^T \langle P(s)K(s)x_1(s), v_2(s) \rangle_{\mathscr{L}_2^0} \, \mathrm{d}s \\
& + \mathbb{E}\int_t^T \langle P(s)v_1(s), K(s)x_2(s) + v_2(s) \rangle_{\mathscr{L}_2^0} \, \mathrm{d}s \\
& + \mathbb{E}\int_t^T \langle v_1(s), \hat{Q}^{(t)}(\xi_2, u_2, v_2)(s) \rangle_{\mathscr{L}_2^0} \, \mathrm{d}s \\
& + \mathbb{E}\int_t^T \langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_{\mathscr{L}_2^0} \, \mathrm{d}s,
\end{aligned}
\tag{2.20}
$$

where $x_1(\cdot)$ and $x_2(\cdot)$ solve respectively (2.16) and (2.17).

Lemma 2.11 ([12], Thm. 12.9)**.** *Assume that $p \in (1,2]$ and the Banach space $L^p_{\mathscr{F}_T}(\Omega; \mathbb{C})$ is separable. Then, for any J, K, F and P_T satisfying (2.14)–(2.15), the equation (2.13) admits one and only one relaxed transposition* *solution* $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)}) \in D_{\mathbb{F},\omega}([0,T]; L^p(\Omega; \mathscr{L}(H))) \times \mathcal{Q}^p[0,T]$. *Furthermore,*

$$
\begin{aligned}
&|P|_{\mathscr{L}(L^2_{\mathbb{F}}([0,T];L^{2q}(\Omega;H));L^2([0,T];L_{\mathbb{F}}^{\frac{2p}{p+1}}(\Omega;H)))} \\
&+ \sup_{t\in[0,T)} |(Q^{(t)}, \hat{Q}^{(t)})|_{\mathscr{L}(\mathcal{H}_t;L^2_{\mathbb{F}}([t,T];L^{\frac{2p}{p+1}}(\Omega;\mathscr{L}_2^0)))^2} \\
&\le \mathcal{C}\big(|F|_{L^1_{\mathbb{F}}([0,T];L^p(\Omega;\mathscr{L}(H)))} + |P_T|_{L^p_{\mathscr{F}_T}(\Omega;\mathscr{L}(H))}\big).
\end{aligned}
\tag{2.21}
$$

2.4. Girsanov theorem and Itô's formula

Lemma 2.12 ([29], Thm. 10.14)**.** *Assume that* $\psi(\cdot)$ *is a* V*-valued* $\mathscr{F}_t$*-predictable process such that*

$$
\mathbb{E}\big(e^{\int_0^T \langle \psi(s),\, dW(s)\rangle_V - \frac{1}{2}\int_0^T |\psi(s)|_V^2\, ds}\big) = 1. \tag{2.22}
$$

Then the process

$$
\widehat{W}(t) = W(t) - \int_0^t \psi(s)\, ds, \qquad t \in [0,T] \tag{2.23}
$$

is a cylindrical Brownian motion with respect to $\{\mathscr{F}_t\}_{t\ge 0}$ *on the probability space* $(\Omega, \mathscr{F}, \widehat{\mathbb{P}})$ *where*

$$
d\widehat{\mathbb{P}}(\omega) = e^{\int_0^T \langle \psi(s),\, dW(s)\rangle_V - \frac{1}{2}\int_0^T |\psi(s)|_V^2\, ds}\, d\mathbb{P}(\omega). \tag{2.24}
$$

In order for (2.22) to hold, we have

Proposition 2.13 ([29], Prop. 10.17)**.** *One of following conditions is sufficient in order for (2.22) to hold:*
1) $\mathbb{E}(e^{\frac{1}{2}\int_0^T |\psi(t)|_V^2\, dt}) < +\infty$,
2) there exists $\delta > 0$ *such that* $\sup_{t\in[0,T]} \mathbb{E}(e^{\delta|\psi(t)|_V^2\, ds}) < +\infty$.

Definition 2.14 ([12], Defi. 2.138)**.** For any $b(\cdot) \in L^{1,loc}_{\mathbb{F}}([0,T]; H)$, cylindrical Brownian motion $W(\cdot)$ and $\sigma(\cdot) \in L^{2,loc}_{\mathbb{F}}([0,T]; \mathscr{L}_2^0)$, the following form of $\mathbf{F}$-adapted process

$$
X(t) = X(0) + \int_0^t b(s)\, ds + \int_0^t \sigma(s)\, dW(s), \qquad t \in [0,T], \tag{2.25}
$$

is called an H-valued Itô process, or simply an Itô process (if the meaning is clear from the context).

Lemma 2.15 ([12], Thm. 2.139)**.** *Let* $X(\cdot)$ *be given by (2.25). Let* $F : [0,T] \times H \to R$ *be a function such that its partial (Fréchet) derivatives* F_t, F_x *and* F_{xx} *are uniformly continuous on any bounded subsets of* $[0,T] \times H$. *Then, a.s., for all* $t \in [0,T]$

$$
\begin{aligned}
F(t, X(t)) =& F(0, X(0)) + \int_0^t \langle\langle F_x(s, X(s)), \sigma(s)\rangle\rangle_H\, dW(s) \\
&+ \int_0^t [F_t(s, X(s)) + \langle F_x(s, X(s)), b(s)\rangle_{H^*,H} \\
&+ \frac{1}{2} F_{xx}(s, X(s))(\sigma(s), \sigma(s))_{\mathscr{L}_2^0}]\, ds,
\end{aligned}
\tag{2.26}
$$

where $F_{xx}(s, X(s))(\sigma(s), \sigma(s))_{\mathscr{L}_2^0} = \sum_{j=1}^\infty F_{xx}(s, X(s))(\sigma(s)e_j, \sigma(s)e_j)$, $\{e_j\}_{j=1}^\infty$ *is the orthonormal basis of* V.

3. Statement of the problem

H and V are two separable Hilbert spaces, let $T>0$, $W(\cdot)$ is a V-valued, cylindrical Brownian motion, $\{\mathscr{F}_t\}$ is the natural filtration generated by $W(\cdot)$, $(\Omega,\mathscr{F},\mathbf{F},\mathbb{P})$ (with $\mathbf{F}=\{\mathscr{F}_t\}_{t\in[0,T]}$) is a fixed filtered probability space satisfying the usual condition, and we denote by $\mathbb{F}$ the progressive σ-field w.r.t. $\mathbf{F}$.

$\{h_j\}_{j=1}^{\infty}$ and $\{e_1,e_2,e_3\dots r_1,r_2,r_3\dots\}$ are respectively orthonormal basis of H and V. V_1, V_2 are linear closed subspace spanned by $\{e_j\}_{j=1}^{\infty}$, $\{r_j\}_{j=1}^{\infty}$, respectively, and$\langle u_1,v_1\rangle_{V_1}=\langle u_1,v_1\rangle_V, \langle u_2,v_2\rangle_{V_2}=\langle u_2,v_2\rangle_V$ $(v_1,u_1\in V_1; v_2,u_2\in V_2)$, then V_1,V_2 are also Hilbert spaces. Moreover, for any $v\in V$ there are unique $v_1\in V_1, v_2\in V_2$, such that $v=v_1+v_2$. If $T_1\in\mathscr{L}_2(V_1;H), T_2\in\mathscr{L}_2(V_1;H)$, we can find a linear operator $T:V\to H$ such that

$$T(v)=T_1(v_1)+T_2(v_2),\qquad T\in\mathscr{L}_2(V;H).$$

Conversely, if $T\in\mathscr{L}_2(V;H)$, we can also find $T_1\in\mathscr{L}_2(V_1;H), T_2\in\mathscr{L}_2(V_1;H)$, such that $T_1(v_1)=T(v_1)$ and $T_2(v_2)=T(v_2)$. $W_1(\cdot)$ and $y(\cdot)$ are cylindrical Brownian motions with values on V_1 and V_2 respectively, and satisfy $W(t,\omega)=W_1(t,\omega)+y(t,\omega)$, $\quad\forall(t,\omega)\in[0,T]\times\Omega$. $\{\mathscr{F}_t^w\}$ and $\{\mathscr{F}_t^y\}$ are filtration generated by $W_1(\cdot)$ and $y(\cdot)$. Let A be an unbounded linear operator (with domain $D(A)$ on H), which generates a C_0-semigroup $\{S(t)\}_{t\ge 0}$ satisfies $|S(t)|_{\mathscr{L}(H)}\le \mathrm{e}^{ct}$ for some constant $c\in\mathbb{R}$ and any $t\ge 0$. Let U be a separable metric space with a metric $d(\cdot,\cdot)$. Put

$$\mathcal{U}[0,T]=\Big\{u(\cdot):[0,T]\times\Omega\to U|u(\cdot)\ is\ \{\mathscr{F}_t^y\}_{t\in[0,T]}-predictable\Big\},$$

$\mathcal{C}_L>0$ is a generic constant, and we assume the following condition:

Condition 3.1. *Suppose that $a:[0,T]\times\Omega\times H\times U\to H$ and $b:[0,T]\times\Omega\times H\times U\to\mathscr{L}_2(V_1;H)$ are two (vector-valued) functions satisfying:*
1) For any $(x,u)\in H\times U$, the functions $a(\cdot,x,u):[0,T]\times\Omega\to H$ and $b(\cdot,x,u):[0,T]\times\Omega\to\mathscr{L}_2(V_1;H)$ are $\mathbb{F}$-measurable;
2) For any $x\in H$ and a.e. $(t,\omega)\in(0,T)\times\Omega$, the functions $a(t,x,\cdot):U\to H$ and $b(t,x,\cdot):U\to\mathscr{L}_2(V_1;H)$ are continuous;
3) For any $(x_1,x_2,u)\in H\times H\times U$ and a.e. $(t,\omega)\in(0,T)\times\Omega$,

$$\begin{cases}|a(t,x_1,u)-a(t,x_2,u)|_H+|b(t,x_1,u)-b(t,x_2,u)|_{\mathscr{L}_2(V_1;H)}\le\mathcal{C}_L|x_1-x_2|_H,\\ |a(t,0,u)|_H+|b(t,0,u)|_{\mathscr{L}_2(V_1;H)}\le\mathcal{C}_L.\end{cases}\tag{3.1}$$

Condition 3.2. *Suppose that $g:[0,T]\times\Omega\times H\times U\to\mathbb{R}$ and $c:\Omega\times H\to\mathbb{R}$ are two functions satisfying:*
1) For any $(x,u)\in H\times U$, the function $g(\cdot,x,u):[0,T]\times\Omega\to\mathbb{R}$ is $\mathbb{F}$-measurable, the function $c(x):\Omega\to\mathbb{R}$ is $\mathscr{F}_T$-measurable;
2) For any $x\in H$ and a.e. $(t,\omega)\in(0,T)\times\Omega$, the function $g(t,x,\cdot):U\to\mathbb{R}$ is continuous;
3) For any $(x_1,x_2,u)\in H\times H\times U$ and a.e. $(t,\omega)\in(0,T)\times\Omega$,

$$\begin{cases}|g(t,x_1,u)-g(t,x_2,u)|+|c(x_1)-c(x_2)|\le\mathcal{C}_L|x_1-x_2|_H,\\ |g(t,0,u)|+|c(0)|\le\mathcal{C}_L.\end{cases}\tag{3.2}$$

Since the control domain is the general case, we need to further assume that:

Condition 3.3. *For any $u\in U$ and a.e. $(t,\omega)\in(0,T)\times\Omega$, the functions $a(t,\cdot,u):H\to H, b(t,\cdot,u):H\to\mathscr{L}_2(V_1;H), g(t,\cdot,u):H\to\mathbb{R}$ and $c(\cdot):H\to\mathbb{R}$ are C^2. For any $x\in H$ and a.e. $(t,\omega)\in(0,T)\times\Omega$, the functions $a_x(t,x,\cdot):U\to\mathscr{L}(H)$, $b_x(t,x,\cdot):U\to\mathscr{L}(H;\mathscr{L}_2(V_1;H))$, $g_x(t,x,\cdot):U\to H$, $a_{xx}(t,x,\cdot):U\to\mathscr{L}(H,H;H)$, $b_{xx}(t,x,\cdot):U\to\mathscr{L}(H,H;\mathscr{L}_2(V_1;H))$ and $g_{xx}(t,x,\cdot):U\to\mathscr{L}(H)$ are continuous. Moreover, for any $(x,u)\in$*

$H \times U$ *and a.e.* $(t,\omega) \in (0,T) \times \Omega$,

$$\begin{cases} |a_x(t,x,u)|_{\mathscr{L}(H)} + |b_x(t,x,u)|_{\mathscr{L}(H;\mathscr{L}_2(V_1;H))} + |g_x(t,x,u)|_H + |c_x(x)|_H \leq \mathcal{C}_L, \\ |a_{xx}(t,x,u)|_{\mathscr{L}(H,H;H)} + |b_{xx}(t,x,u)|_{\mathscr{L}(H,H;\mathscr{L}_2(V_1;H))} \\ \quad + |g_{xx}(t,x,u)|_{\mathscr{L}(H)} + |c_{xx}(x)|_{\mathscr{L}(H)} \\ \leq \mathcal{C}_L \ . \end{cases} \tag{3.3}$$

For the coefficient h in the observation equation, we also assume:

Condition 3.4. *Suppose that* $h : [0,T] \times \Omega \times H \times U \to V_2$ *is a (vector-valued) function satisfying:*
1) For any $(x,u) \in H \times U$, *the functions* $h(\cdot,x,u) : [0,T] \times \Omega \to V_2$ *is* $\mathscr{F}_t$*-predictable;*
2) For any $x \in H$ *and a.e.* $(t,\omega) \in (0,T) \times \Omega$, *the functions* $h(t,x,\cdot) : U \to V_2$ *is continuous;*
3) For any $u \in U$ *and a.e.* $(t,\omega) \in (0,T) \times \Omega$, *the functions* $h(t,\cdot,u) : H \to V_2$ *is* C^2*. For any* $x \in H$ *and a.e.* $(t,\omega) \in (0,T) \times \Omega$, *the functions* $h_x(t,x,\cdot) : U \to \mathscr{L}(H;V_2)$ *and* $h_{xx}(t,x,\cdot) : U \to \mathscr{L}(H,H;V_2)$ *are continuous.*
4) For any $(x,u) \in H \times U$ *and a.e.* $(t,\omega) \in (0,T) \times \Omega$,

$$|h(t,x,u)|_{V_2} + |h_x(t,x,u)|_{\mathscr{L}(H;V_2)} + |h_{xx}(t,x,u)|_{\mathscr{L}(H,H;V_2)} \leq \mathcal{C}_L. \tag{3.4}$$

Let us consider the following controlled stochastic evolution equation:

$$\begin{cases} \mathrm{d}x(t) = (Ax(t) + a(t,x(t),u(t)))\,\mathrm{d}t + b(t,x(t),u(t))\,\mathrm{d}W_1(t), \\ x(0) = x_0, \qquad t \in (0,T], \end{cases} \tag{3.5}$$

with partial observation

$$y(t) = \int_0^t h(s,x(s),u(s))\,\mathrm{d}s + \widehat{W}(t), \tag{3.6}$$

where $u(\cdot) \in \mathcal{U}[0,T]$ is the control variable, $x(\cdot)$ is the state variable, and the (given) initial state $x_0 \in L^\infty_{\mathscr{F}_0}(\Omega;H)$, that is, $x_0 \in L^p_{\mathscr{F}_0}(\Omega;H)$ for any given $p \geq 2$. In the rest of this paper, we shall denote by $\mathcal{C}$ a generic constant, depending on T, A, p, $\mathcal{C}_L$ or some fixed coefficients to be introduced later.

When the above conditions hold, for $u(\cdot) \in \mathcal{U}[0,T]$, the equation (1.2) admits a unique mild solution $x(\cdot) = x(\cdot;x_0,u) \in C_{\mathbb{F}}([0,T];L^p(\Omega;H))$, which will be denoted by $x^u(\cdot)$. let:

$$Z^u(t) = \mathrm{e}^{\int_0^t \langle h(s,x^u(s),u(s)),\,\mathrm{d}y(s)\rangle_{V_2} - \frac{1}{2}\int_0^t |h(s,x^u(s),u(s))|^2_{V_2}\,\mathrm{d}s}. \tag{3.7}$$

According to Remark 2.6, Condition 3.4 and Proposition 2.13, we can obtain that $h(\cdot,x^u(\cdot),u(\cdot))$ is a V_2-valued $\mathscr{F}_t$-predictable process and $\mathbb{E}[Z^u(T)] = 1$. Then, according to the Girsanov theorem (Lem. 2.12), the process $\widehat{W}(t) = y(t) - \int_0^t h(s,x^u(s),u(s))\,\mathrm{d}s$ is a cylindrical Brownian motion with respect to $\{\mathscr{F}^y_t\}$ and $\mathrm{d}\mathbb{P}^u(\omega) = Z^u(T)\,\mathrm{d}\mathbb{P}(\omega)$. The cost functional takes the following form

$$\begin{aligned} \mathcal{J}(u(\cdot)) &= \mathbb{E}^u\Big[\int_0^T g(t,x(t),u(t))\,\mathrm{d}t + c(x(T))\Big] \\ &= \mathbb{E}\Big\{Z^u(T)\Big[\int_0^T g(t,x(t),u(t))\,\mathrm{d}t + c(x(T))\Big]\Big\}, \end{aligned} \tag{3.8}$$

where $\mathbb{E}^u$ denotes expectation with respect to probability space $(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P}^u)$, and the problem is finding the $\bar{u}(\cdot) \in \mathcal{U}[0,T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} \mathcal{J}(u(\cdot)).$$

Since Hilbert space is isomorphic to it's dual space, we also write $\langle h(s, x^u(s), u(s)), \mathrm{d}y(s)\rangle_{V_2}$ as $h(s, x^u(s), u(s))\,\mathrm{d}y(s)$. Then we use Itô's Formula for (3.7), and get

$$\begin{aligned} Z^u(t) = 1 + &\int_0^t Z^u(s) h(s, x^u(s), u(s))\,\mathrm{d}y(s) \\ &+ \int_0^t \frac{1}{2}\Big[- Z^u(s)|h(s, x^u(s), u(s))|_{V_2}^2 + Z^u(s)\Big(\sum_{j=1}^{\infty} |\langle h(s, x^u(s), u(s)), r_j\rangle_V|^2\Big)\Big]\,\mathrm{d}s \\ = 1 + &\int_0^t Z^u(s) h(s, x^u(s), u(s))\,\mathrm{d}y(s), \end{aligned} \tag{3.9}$$

and according to the partial integral formula, we have

$$\mathcal{J}(\bar{u}(\cdot)) = \mathbb{E}\Big[\int_0^T Z^u(t) g(t, x^u(t), u(t))\,\mathrm{d}t + Z^u(T) c(x(T))\Big].$$

4. Maximum principle

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of the problem. For each $\varepsilon \in (0,T)$ and $t_0 \in [0, T-\varepsilon]$, let $E_\varepsilon = (t_0, t_0+\varepsilon]$, put

$$u^\varepsilon(t) = \begin{cases} \bar{u}(t) & t \in [0,T] \setminus E_\varepsilon, \\ u(t) & t \in E_\varepsilon, \end{cases} \tag{4.1}$$

where $u(\cdot)$ is an arbitrarily given element in $\mathcal{U}[0,T]$. Let $x^\varepsilon(\cdot)$ be the state process of the system (1.2) corresponding to the control $u_\varepsilon(\cdot)$.

We introduce some notations which will be used later. For $\psi = a, b, g, h$,

$$\begin{cases} \psi(t) = \psi(t, \bar{x}(t), \bar{u}(t)), \\ \psi_1(t) = \psi_x(t, \bar{x}(t), \bar{u}(t)), \quad \psi_{11}(t) = \psi_{xx}(t, \bar{x}(t), \bar{u}(t)), \\ \psi_1^\varepsilon(t) = \displaystyle\int_0^1 \psi_x(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t))\,\mathrm{d}\sigma, \\ \psi_{11}^\varepsilon(t) = 2\displaystyle\int_0^1 (1-\sigma)\psi_{xx}(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t))\,\mathrm{d}\sigma, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \delta\psi(t) = \psi(t, \bar{x}(t), u(t)) - \psi(t, \bar{x}(t), \bar{u}(t)), \\ \delta\psi_1(t) = \psi_x(t, \bar{x}(t), u(t)) - \psi_x(t, \bar{x}(t), \bar{u}(t)), \\ \delta\psi_{11}(t) = \psi_{xx}(t, \bar{x}(t), u(t)) - \psi_{xx}(t, \bar{x}(t), \bar{u}(t)). \end{cases} \tag{4.3}$$

4.1. Variational equations and variational inequalities

Let $x_1^\varepsilon(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$, $Z_1(\cdot) = Z^\varepsilon(\cdot) - \bar{Z}(\cdot)$, where $Z^\varepsilon(\cdot)$, $\bar{Z}(\cdot)$ are the variables corresponding to the controls u^ε, $\bar{u}$, respectively. So, we have

$$\begin{cases} \mathrm{d}x_1^\varepsilon = (Ax_1^\varepsilon + a_1^\varepsilon(t)x_1^\varepsilon + I_{E_\varepsilon}(t)\delta a(t))\,\mathrm{d}t + (b_1^\varepsilon(t)x_1^\varepsilon + I_{E_\varepsilon}(t)\delta b(t))\,\mathrm{d}W_1(t), \\ x_1^\varepsilon(0) = 0, \qquad t \in (0,T], \end{cases} \tag{4.4}$$

and

$$\begin{cases} \mathrm{d}Z_1(t) = [Z_1(t)h(t, x^\varepsilon(t), u^\varepsilon(t)) + \bar{Z}(t)(h_1^\varepsilon(t)x_1^\varepsilon + I_{E_\varepsilon}(t)\delta h(t))]\,\mathrm{d}y(t), \\ Z_1(0) = 0, \qquad t \in (0,T]. \end{cases} \tag{4.5}$$

Consider the following stochastic evolution equations:

$$\begin{cases} \mathrm{d}x_2^\varepsilon = (Ax_2^\varepsilon + a_1(t)x_2^\varepsilon)\,\mathrm{d}t + (b_1(t)x_2^\varepsilon + I_{E_\varepsilon}(t)\delta b(t))\,\mathrm{d}W_1(t), \\ x_2^\varepsilon(0) = 0, \qquad t \in (0,T], \end{cases} \tag{4.6}$$

$$\begin{cases} \mathrm{d}Z_2 = [(Z_2 h(t)) + \bar{Z}(t)(h_1(t)x_2^\varepsilon + I_{E_\varepsilon}(t)\delta h(t))]\,\mathrm{d}y(t), \\ Z_2(0) = 0, \qquad t \in (0,T], \end{cases} \tag{4.7}$$

$$\begin{cases} \mathrm{d}x_3^\varepsilon = (Ax_3^\varepsilon + a_1(t)x_3^\varepsilon + I_{E_\varepsilon}(t)\delta a(t) + \dfrac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon))\,\mathrm{d}t \\ \qquad\qquad + (b_1(t)x_3^\varepsilon + I_{E_\varepsilon}(t)\delta b_1(t)x_2^\varepsilon + \dfrac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon))\,\mathrm{d}W_1(t), \\ x_3^\varepsilon(0) = 0, \qquad t \in (0,T], \end{cases} \tag{4.8}$$

$$\begin{cases} \mathrm{d}Z_3 = [Z_3 h(t) + \bar{Z}(t)(h_1(t)x_3^\varepsilon + I_{E_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon + \dfrac{1}{2}h_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)) \\ \qquad\qquad + Z_2(t)(I_{E_\varepsilon}(t)\delta h(t) + h_1(t)x_2^\varepsilon)]\,\mathrm{d}y(t), \\ Z_3(0) = 0, \qquad t \in (0,T]. \end{cases} \tag{4.9}$$

Remark 4.1. Equations (4.5), (4.7) and (4.9) can be regarded as stochastic development equations in the case of $A = 0$ and C_0-semigroup $S(t) = I$. Therefore there exists unique mild solutions to these equations and are denoted respectively as Z_1, Z_2, and Z_3.

We observe that the present formulation is very similar to the completely observed control problem; the only difference lies in the set of admissible controls. Hence, we reduce, in the following, most of the variational calculations for the partially observed case to the counterpart of the completely observed one.

Analogous to [12], we can establish the following two lemmas.

Lemma 4.2. *Let $x_4^\varepsilon = x_1^\varepsilon - x_2^\varepsilon$, $x_5^\varepsilon = x_4^\varepsilon - x_3^\varepsilon$, $\forall p \geq 2$ we have the following estimates:*

$$1)\ \sup_{t\in[0,T]} \mathbb{E}|x_1^\varepsilon(t)|^p \leq \mathcal{C}\varepsilon^{\frac{p}{2}}, \qquad 2)\ \sup_{t\in[0,T]} \mathbb{E}|x_2^\varepsilon(t)|^p \leq \mathcal{C}\varepsilon^{\frac{p}{2}},$$

$$3)\ \sup_{t\in[0,T]} \mathbb{E}|x_3^\varepsilon(t)|^p \leq \mathcal{C}\varepsilon^{p}, \qquad 4)\ \sup_{t\in[0,T]} \mathbb{E}|x_4^\varepsilon(t)|^p \leq \mathcal{C}\varepsilon^{p},$$

$$5)\ \sup_{t\in[0,T]} \mathbb{E}|x_5^\varepsilon(t)|^p = o(\varepsilon^{p}).$$

Proof. Use Lemma 2.4 for equations satisfied by $\bar{x}$ and x^ε, we have $\sup_{t\in[0,T]}\mathbb{E}|\bar{x}(t)|^p \le \mathcal{C}$ and $\sup_{t\in[0,T]}\mathbb{E}|x^\varepsilon(t)|^p \le \mathcal{C}$. $\mathcal{C}$ is a generic constant, depending on p, x_0, T, A, $\mathcal{C}_L$, *etc.* This proof is quite standard. As an illustration, we only give the proof of 5).

We claim that $x_5^\varepsilon(\cdot)$ solves the following equation:

$$\begin{cases} \mathrm{d}x_5^\varepsilon = \big[Ax_5^\varepsilon + a_1(t)x_5^\varepsilon + I_{\mathbb{E}_\varepsilon}(t)\delta a_1(t)x_1^\varepsilon + \dfrac{1}{2}(a_{11}^\varepsilon(t) - a_{xx}(t,\bar{x}(t),u^\varepsilon(t)))(x_1^\varepsilon,x_1^\varepsilon) \\ \qquad + \dfrac{1}{2}I_{\mathbb{E}_\varepsilon}(t)\delta a_{11}(t)(x_1^\varepsilon,x_1^\varepsilon) + \dfrac{1}{2}a_{11}(t)(x_1^\varepsilon,x_1^\varepsilon) - \dfrac{1}{2}a_{11}(t)(x_2^\varepsilon,x_2^\varepsilon)\big]\,\mathrm{d}t \\ \qquad + \big[b_1(t)x_5^\varepsilon + I_{\mathbb{E}_\varepsilon}(t)\delta b_1(t)x_4^\varepsilon + \dfrac{1}{2}(b_{11}^\varepsilon(t) - b_{xx}(t,\bar{x}(t),u^\varepsilon(t)))(x_1^\varepsilon,x_1^\varepsilon) \\ \qquad + \dfrac{1}{2}I_{\mathbb{E}_\varepsilon}(t)\delta b_{11}(t)(x_1^\varepsilon,x_1^\varepsilon) + \dfrac{1}{2}b_{11}(t)(x_1^\varepsilon,x_1^\varepsilon) - \dfrac{1}{2}b_{11}(t)(x_2^\varepsilon,x_2^\varepsilon)\big]\,\mathrm{d}W_1(t), \\ x_5^\varepsilon(0) = 0, \qquad t\in(0,T]. \end{cases} \tag{4.10}$$

Applying Lemma 2.4, we see that

$$\begin{aligned} &\sup_{t\in[0,T]}\mathbb{E}|x_5^\varepsilon(t)|_H^p \\ &\le \mathcal{C}\mathbb{E}\Big\{\Big(\int_0^T \big[I_{\mathbb{E}_\varepsilon}(s)(|\delta a_1(s)x_1^\varepsilon(s)|_H + |\delta a_{11}(s)(x_1^\varepsilon(s),x_1^\varepsilon(s))|_H) \\ &\quad + |(a_{11}^\varepsilon(s) - a_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|_H \\ &\quad + |a_{11}(s)(x_1^\varepsilon(s),x_1^\varepsilon(s)) - a_{11}(s)(x_2^\varepsilon(s),x_2^\varepsilon(s))|_H\big]\,\mathrm{d}s\Big)^p \\ &\quad + \Big(\int_0^T \big[I_{\mathbb{E}_\varepsilon}(s)(|\delta b_1(s)x_4^\varepsilon(s)|^2_{\mathscr{L}_2(V_1;H)} + |\delta b_{11}(s)(x_1^\varepsilon(s),x_1^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)}) \\ &\quad + |(b_{11}^\varepsilon(s) - b_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)} \\ &\quad + |b_{11}(s)(x_1^\varepsilon(s),x_1^\varepsilon(s)) - b_{11}(s)(x_2^\varepsilon(s),x_2^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)}\big]\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big\} \\ &\le o(\varepsilon^p) + \mathcal{C}\mathbb{E}\Big\{\Big(\int_0^T \big[|(a_{11}^\varepsilon(s) - a_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|_H \\ &\quad + |a_{11}(s)(x_4^\varepsilon(s),x_1^\varepsilon(s)) + a_{11}(s)(x_2^\varepsilon(s),x_4^\varepsilon(s))|_H\big]\,\mathrm{d}s\Big)^p \\ &\quad + \Big(\int_0^T \big[|(b_{11}^\varepsilon(s) - b_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)} \\ &\quad + |b_{11}(s)(x_4^\varepsilon(s),x_1^\varepsilon(s)) + b_{11}(s)(x_2^\varepsilon(s),x_4^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)}\big]\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big\} \\ &\le o(\varepsilon^p) + \mathcal{C}\mathbb{E}\Big\{\Big(\int_0^T |(a_{11}^\varepsilon(s) - a_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|_H\,\mathrm{d}s\Big)^p \\ &\quad + \Big(\int_0^T |(b_{11}^\varepsilon(s) - b_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|^2_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big\}. \end{aligned} \tag{4.11}$$

Since (replacing a with b is the same)

$$\begin{aligned} &|a_{11}^\varepsilon(s) - a_{xx}(s,\bar{x}(s),u^\varepsilon(s))|_{\mathscr{L}(H,H;H)} \\ &\le \mathcal{C}\Big(\int_0^1 |a_{xx}(s,\bar{x}(s)+\sigma x_1^\varepsilon(s),u^\varepsilon(s)) - a_{xx}(s,\bar{x}(s),\bar{u}(s))|_{\mathscr{L}(H,H;H)}d\sigma + I_{\mathbb{E}_\varepsilon}(s)\Big), \end{aligned} \tag{4.12}$$

we end up with

$$
\begin{aligned}
&\mathbb{E}\Big\{\Big(\int_0^T |(a^\varepsilon_{11}(s)-a_{xx}(s,\bar{x}(s),u^\varepsilon(s)))(x_1^\varepsilon(s),x_1^\varepsilon(s))|_H\,\mathrm{d}s\Big)^p\Big\}\\
&\le \mathcal{C}\varepsilon^p\int_0^T\Big(\mathbb{E}\int_0^1 |a_{xx}(s,\bar{x}(s)+\sigma x_1^\varepsilon(s),u^\varepsilon(s))-a_{xx}(s,\bar{x}(s),\bar{u}(s))|^{2p}_{\mathscr{L}(H,H;H)}\,\mathrm{d}\sigma+I_{\mathbb{E}_\varepsilon}(s)\Big)^{\frac{1}{2}}\mathrm{d}s\\
&=o(\varepsilon^p).
\end{aligned}
\tag{4.13}
$$

The other part is similarly estimated, and we get 5). □

Lemma 4.3. *Let $Z_4=Z_1-Z_2$, $Z_5=Z_4-Z_3$, $\forall p\ge 2$ we have the following estimates:*

$$
1)\ \sup_{t\in[0,T]}\mathbb{E}|Z_1(t)|^p\le\mathcal{C}\varepsilon^{\frac{p}{2}},\qquad 2)\ \sup_{t\in[0,T]}\mathbb{E}|Z_2(t)|^p\le\mathcal{C}\varepsilon^{\frac{p}{2}},
$$

$$
3)\ \sup_{t\in[0,T]}\mathbb{E}|Z_3(t)|^p\le\mathcal{C}\varepsilon^{p},\qquad 4)\ \sup_{t\in[0,T]}\mathbb{E}|Z_4(t)|^p\le\mathcal{C}\varepsilon^{p},
$$

$$
5)\ \sup_{t\in[0,T]}\mathbb{E}|Z_5(t)|^p=o(\varepsilon^{p}).
$$

Proof. Use Lemma 2.4 for equations satisfied by $\bar{Z}$ and Z^ε, we have $\sup_{t\in[0,T]}\mathbb{E}|\bar{Z}(t)|^p\le\mathcal{C}$ and $\sup_{t\in[0,T]}\mathbb{E}|Z^\varepsilon(t)|^p\le\mathcal{C}$. This proof is also standard, so similar to the Lemma 4.2, we only give the proof of 5) as an example.

By the definition of Z, we have

$$
\begin{aligned}
dZ_5&=\big[Z_5h(t)+\bar{Z}(t)(h^\varepsilon(t)-h(t,\bar{x}(t),u^\varepsilon(t))-h_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))\\
&\quad-I_{\mathbb{E}_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon(t)-\frac{1}{2}h_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)))\\
&\quad+Z_4(t)(h_1^\varepsilon(t)x_1^\varepsilon(t)+I_{\mathbb{E}_\varepsilon}(t)\delta h(t))\\
&\quad+Z_2(t)((h_1^\varepsilon(t)-h_1(t))x_1^\varepsilon(t)+h_1(t)x_4^\varepsilon(t))\big]\,\mathrm{d}W_2(t)\\
&=\Big\{Z_5h(t)+\bar{Z}(t)\big[I_{\mathbb{E}_\varepsilon}(t)\delta h_1(t)x_4^\varepsilon(t)+h_1(t)x_5^\varepsilon(t)+\frac{1}{2}(h_{11}(t)(x_4^\varepsilon(t),x_1^\varepsilon(t))+h_{11}(t)(x_2^\varepsilon(t),x_4^\varepsilon(t)))\big]\\
&\quad+Z_4(t)(h_1^\varepsilon(t)x_1^\varepsilon(t)+I_{\mathbb{E}_\varepsilon}(t)\delta h(t))\\
&\quad+Z_2(t)((h_1^\varepsilon(t)-h_1(t))x_1^\varepsilon(t)+h_1(t)x_4^\varepsilon(t))\Big\}\,\mathrm{d}W_2(t).
\end{aligned}
\tag{4.14}
$$

Then, use Lemma 2.4 for equations satisfied by Z_5, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}|Z_5(t)|^p\\
&\le\mathcal{C}\mathbb{E}\Big[\int_0^T\Big(|\bar{Z}(s)\big[I_{\mathbb{E}_\varepsilon}(s)\delta h_1(s)x_4^\varepsilon(s)+h_1(s)x_5^\varepsilon(s)+\frac{1}{2}(h_{11}(s)(x_4^\varepsilon(s),x_1^\varepsilon(s))+h_{11}(s)(x_2^\varepsilon(s),x_4^\varepsilon(s)))\big]|^2_{\mathscr{L}_2(V_2;H)}\\
&\quad+|Z_4(s)(h_1^\varepsilon(s)x_1^\varepsilon(s)+I_{\mathbb{E}_\varepsilon}(s)\delta h(s))|^2_{\mathscr{L}_2(V_2;H)}+|Z_2(s)((h_1^\varepsilon(s)-h_1(s))x_1^\varepsilon(s)+h_1(s)x_4^\varepsilon(s))|^2_{\mathscr{L}_2(V_2;H)}\Big)\,\mathrm{d}s\Big]^{\frac{p}{2}}\\
&=o(\varepsilon^p).
\end{aligned}
\tag{4.15}
$$

□

We now compute $\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot))$. By the definition of $\mathcal{J}(\cdot)$, we obtain that

$$
\begin{aligned}
&\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot))\\
&= \mathbb{E}\Big[\int_0^T (Z^\varepsilon(t) g(t, x^\varepsilon(t), u^\varepsilon(t)) - \bar{Z}(t) g(t, \bar{x}(t), \bar{u}(t)))\,\mathrm{d}t\Big]\\
&\quad + \mathbb{E}\big[Z^\varepsilon(T) c(x^\varepsilon(T)) - \bar{Z}(T) c(\bar{x})\big]\\
&= \mathbb{E}\Big[\int_0^T \Big(Z_1 g(t) + Z^\varepsilon(t)(I_{E_\varepsilon}(t)\delta g(t) + \langle g_x(t, \bar{x}(t), u^\varepsilon(t)), x_1^\varepsilon(t)\rangle_H\\
&\quad + \int_0^1 \langle (1-\sigma) g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) x_1^\varepsilon(t), x_1^\varepsilon(t)\rangle_H\,\mathrm{d}\sigma)\Big)\,\mathrm{d}t\Big]\\
&\quad + \mathbb{E}\Big[Z_1(T) c(\bar{x}(T)) + Z^\varepsilon(T)\Big(c_x(\bar{x}(T)) x_1^\varepsilon(T)\\
&\quad + \int_0^1 \langle (1-\sigma) c_{xx}(\bar{x}(T) + \sigma x_1^\varepsilon(T)) x_1^\varepsilon(T), x_1^\varepsilon(T)\rangle_H\,\mathrm{d}\sigma\Big)\Big].
\end{aligned}
\tag{4.16}
$$

Let's look at the integral part first. Together with the definition of $x_1^\varepsilon(\cdot)$, $x_2^\varepsilon(\cdot)$, $x_3^\varepsilon(\cdot)$, $x_4^\varepsilon(\cdot)$ and $x_5^\varepsilon(\cdot)$, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\Big(I_{E_\varepsilon}(t)\delta g(t) + \langle g_x(t, \bar{x}(t), u^\varepsilon(t)), x_1^\varepsilon(t)\rangle_H\\
&\quad + \int_0^1 \langle (1-\sigma) g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) x_1^\varepsilon(t), x_1^\varepsilon(t)\rangle_H\,\mathrm{d}\sigma\Big)\,\mathrm{d}t\Big]\\
&= \mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\Big(I_{E_\varepsilon}(t)\delta g(t) + I_{E_\varepsilon}(t)\delta g_1(t) x_1^\varepsilon(t)\\
&\quad + g_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + g_1(t) x_5^\varepsilon(t)\\
&\quad + \int_0^1 \langle (1-\sigma)(g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) - g_{xx}(t, \bar{x}(t), u^\varepsilon(t))) x_1^\varepsilon(t), x_1^\varepsilon(t)\rangle_H\,\mathrm{d}\sigma\\
&\quad + \frac{1}{2}\langle I_{E_\varepsilon}(t)\delta g_{11}(t) x_1^\varepsilon(t), x_1^\varepsilon(t)\rangle_H + \frac{1}{2}\langle g_{11}(t) x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H\\
&\quad + \frac{1}{2}\langle g_{11}(t) x_4^\varepsilon(t), x_1^\varepsilon(t) + x_2^\varepsilon(t)\rangle_H\Big)\,\mathrm{d}t\Big].
\end{aligned}
\tag{4.17}
$$

Because of for a.e. $t \in [0, T]$,

$$
\begin{aligned}
&\Big|\int_0^1 (1-\sigma)(g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) - g_{xx}(t, \bar{x}(t), u^\varepsilon(t)))\,\mathrm{d}\sigma\Big|_{\mathscr{L}(H)}\\
&\le \mathcal{C}\Big(\int_0^1 |g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), \bar{u}(t)) - g_{xx}(t, \bar{x}(t), \bar{u}(t))|_{\mathscr{L}(H)}\,\mathrm{d}\sigma + I_{E_\varepsilon}(t)\Big),
\end{aligned}
\tag{4.18}
$$

then we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\int_0^1 \langle (1-\sigma)(g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) - g_{xx}(t, \bar{x}(t), u^\varepsilon(t))) x_1^\varepsilon(t), x_1^\varepsilon(t)\rangle_H\,\mathrm{d}\sigma\,\mathrm{d}t\Big]\\
&\le \mathcal{C}\mathbb{E}\Big[\int_0^T |Z^\varepsilon(t)||x_1^\varepsilon(t)|_H^2\Big(\int_0^1 |g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), \bar{u}(t)) - g_{xx}(t, \bar{x}(t), \bar{u}(t))|_{\mathscr{L}(H)}\,\mathrm{d}\sigma + I_{E_\varepsilon}(t)\Big)\,\mathrm{d}t\Big]\\
&\le \mathcal{C}\mathbb{E}\Big[\int_{E_\varepsilon} |Z^\varepsilon(t)||x_1^\varepsilon(t)|_H^2\,\mathrm{d}t\Big] + \mathcal{C}\Big(\mathbb{E}\int_0^T |x_1^\varepsilon(t)|_H^4\,\mathrm{d}t\Big)^{\frac{1}{2}} C_\varepsilon\\
&= o(\varepsilon),
\end{aligned}
\tag{4.19}
$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$. In addition,

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\big(I_{E_\varepsilon}(t)\delta g_1(t)x_1^\varepsilon(t)+\frac{1}{2}\langle I_{E_\varepsilon}(t)\delta g_{11}(t)x_1^\varepsilon(t),x_1^\varepsilon(t)\rangle_H\big)\,\mathrm{d}t\Big]\\&\le \mathcal{C}\Big(\int_{E_\varepsilon}(\mathbb{E}|Z^\varepsilon(t)|^2)^{\frac{1}{2}}C_\varepsilon\,\mathrm{d}t\Big)\\&=o(\varepsilon),\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\Big(g_1(t)x_5^\varepsilon(t)+\frac{1}{2}\langle g_{11}(t)x_4^\varepsilon(t),x_1^\varepsilon(t)+x_2^\varepsilon(t)\rangle_H\Big)\,\mathrm{d}t\Big]\\&\le \mathcal{C}\Big(\mathbb{E}\int_0^T|Z^\varepsilon(t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T|x_5^\varepsilon(t)|_H^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\\&\quad+\mathcal{C}C_\varepsilon\Big(\mathbb{E}\int_0^T|Z^\varepsilon(t)|^4\,\mathrm{d}t\Big)^{\frac{1}{4}}\Big(\mathbb{E}\int_0^T|x_4^\varepsilon(t)|_H^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\\&=o(\varepsilon),\end{aligned}\tag{4.21}$$

are holds, so we can get

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\Big(I_{E_\varepsilon}(t)\delta g(t)+\langle g_x(t,\bar{x}(t),u^\varepsilon(t)),x_1^\varepsilon(t)\rangle_H\\&\quad+\int_0^1\langle(1-\sigma)g_{xx}(t,\bar{x}(t)+\sigma x_1^\varepsilon(t),u^\varepsilon(t))x_1^\varepsilon(t),x_1^\varepsilon(t)\rangle_H\,\mathrm{d}\sigma\Big)\,\mathrm{d}t\Big]\\&=\mathbb{E}\Big[\int_0^T Z^\varepsilon(t)\big(I_{E_\varepsilon}(t)\delta g(t)+g_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))\\&\quad+\frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t),x_2^\varepsilon(t)\rangle_H\big)\,\mathrm{d}t\Big]+o(\varepsilon).\end{aligned}\tag{4.22}$$

Now let's look at the whole integral part:

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T\Big(Z_1g(t)+Z^\varepsilon(t)\big(I_{E_\varepsilon}(t)\delta g(t)+g_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))\\&\quad+\frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t),x_2^\varepsilon(t)\rangle_H\big)\Big)\,\mathrm{d}t\Big]\\&=\mathbb{E}\Big[\int_0^T\Big((Z_2+Z_3)g(t)+Z_5g(t)+(Z_1+\bar{Z}(t))\big(I_{E_\varepsilon}(t)\delta g(t)\\&\quad+g_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))+\frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t),x_2^\varepsilon(t)\rangle_H\big)\Big)\,\mathrm{d}t\Big].\end{aligned}\tag{4.23}$$

We can see

$$\mathbb{E}\Big[\int_0^T Z_5g(t)\,\mathrm{d}t\Big]\le\Big(\mathbb{E}\int_0^T|Z_5(t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T|g(t)|^2\,\mathrm{d}t\Big)^{\frac{1}{2}}=o(\varepsilon),\tag{4.24}$$

$$\mathbb{E}\Big[\int_0^T Z_1I_{E_\varepsilon}(t)\delta g(t)\,\mathrm{d}t\Big]\le\int_{E_\varepsilon}(\mathbb{E}|Z_1(t)|^2)^{\frac{1}{2}}(\mathbb{E}|\delta g(t)|^2)^{\frac{1}{2}}\,\mathrm{d}t=o(\varepsilon),\tag{4.25}$$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T Z_1\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H \,\mathrm{d}t\Big] \\
&\leq \mathcal{C}\Big(\mathbb{E}\int_0^T |Z_1(t)|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T |x_2^\varepsilon(t)|_H^4 \,\mathrm{d}t\Big)^{\frac{1}{2}} \\
&= o(\varepsilon)
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T (Z_1 g_1(t)x_3^\varepsilon(t) + (Z_1 - Z_2)g_1(t)x_2^\varepsilon(t))\,\mathrm{d}t\Big] \\
&\leq \mathcal{C}\bigg(\Big(\mathbb{E}\int_0^T |Z_1(t)|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T |x_3^\varepsilon(t)|_H^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} + \Big(\mathbb{E}\int_0^T |Z_4(t)|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\mathbb{E}\int_0^T |x_2^\varepsilon(t)|_H^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}\bigg) \\
&= o(\varepsilon),
\end{aligned}
\tag{4.27}
$$

then we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big(Z_1 g(t) + Z^\varepsilon(t)(I_{E_\varepsilon}(t)\delta g(t) + g_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) \\
&\quad + \frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H)\Big)\,\mathrm{d}t\Big] \\
&= \mathbb{E}\Big[\int_0^T \Big((Z_2 + Z_3)g(t) + Z_2 g_1(t)x_2^\varepsilon(t) + \bar{Z}(t)(I_{E_\varepsilon}(t)\delta g(t) \\
&\quad + g_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + \frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H)\Big)\,\mathrm{d}t\Big] + o(\varepsilon).
\end{aligned}
\tag{4.28}
$$

After a similar calculation for the terminal value part, we end up with:

$$
\begin{aligned}
&\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\
&= \mathbb{E}\Big[\int_0^T \bar{Z}(t)\Big(I_{E_\varepsilon}(t)\delta g(t) + g_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + \frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H\Big)\,\mathrm{d}t \\
&\quad + \bar{Z}(T)\Big(c_x(\bar{x}(T))(x_2^\varepsilon(T) + x_3^\varepsilon(T)) + \frac{1}{2}\langle c_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T)\rangle_H\Big) \\
&\quad + \int_0^T \Big((Z_2(t) + Z_3(t))g(t) + Z_2(t)g_1(t)x_2^\varepsilon(t)\Big)\,\mathrm{d}t \\
&\quad + (Z_2(T) + Z_3(T))c(\bar{x}(T)) + Z_2(T)c_x(\bar{x}(T))x_2^\varepsilon(T)\Big] + o(\varepsilon) \\
&= \bar{\mathbb{E}}\Big[\int_0^T \Big(I_{E_\varepsilon}(t)\delta g(t) + g_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + \frac{1}{2}\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H\Big)\,\mathrm{d}t \\
&\quad + c_x(\bar{x}(T))(x_2^\varepsilon(T) + x_3^\varepsilon(T)) + \frac{1}{2}\langle c_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T)\rangle_H \\
&\quad + \int_0^T \bar{Z}^{-1}(t)\Big((Z_2(t) + Z_3(t))g(t) + Z_2(t)g_1(t)x_2^\varepsilon(t)\Big)\,\mathrm{d}t \\
&\quad + \bar{Z}^{-1}(T)((Z_2(T) + Z_3(T))c(\bar{x}(T)) + Z_2(T)c_x(\bar{x}(T))x_2^\varepsilon(T))\Big] + o(\varepsilon),
\end{aligned}
\tag{4.29}
$$

where $\bar{\mathbb{E}}$ is the expectation of the probability measure $\bar{\mathbb{P}}$ obtained by $\bar{Z}(\cdot)$, involving Girsanov theorem.

4.2. Adjoint equations

In this section, we will characterize a total of three adjoint processes by utilizing the definitions of transposition solution and relaxed transposition solution. Specifically, two of these processes will be characterized as transposition solutions to vector-valued backward stochastic evolution equations (BSEEs), while the third will be characterized as a relaxed transposition solution to an operator-valued BSEE.

Let

$$W_2(t) = y(t) - \int_0^t h(s, \bar{x}(s), \bar{u}(s)) \, \mathrm{d}s, \qquad t \in [0, T],$$

we give the following two equations:

$$\begin{cases} \mathrm{d}\bar{Z}^{-1}(t)Z_2(t) = \Big(h_1(t)x_2^\varepsilon(t) + I_{E_\varepsilon}(t)\delta h(t)\Big) \, \mathrm{d}W_2(t), \\ \mathrm{d}\bar{Z}^{-1}(t)Z_3(t) = \Big(h_1(t)x_3^\varepsilon(t) + I_{E_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon(t) + \dfrac{1}{2}h_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \\ \qquad\qquad + \bar{Z}^{-1}(t)Z_2(t)(I_{E_\varepsilon}(t)\delta h(t) + h_1(t)x_2^\varepsilon(t))\Big) \, \mathrm{d}W_2(t), \\ \bar{Z}^{-1}(0)Z_2(0) = \bar{Z}^{-1}(0)Z_3(0) = 0, \qquad t \in (0, T], \end{cases} \tag{4.30}$$

then we have

$$\begin{cases} \mathrm{d}\bar{Z}^{-1}(t)(Z_2(t) + Z_3(t)) = \Big(h_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + I_{E_\varepsilon}(t)\delta h(t) \\ \qquad\qquad + I_{E_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon(t) + \dfrac{1}{2}h_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \\ \qquad\qquad + \bar{Z}^{-1}(t)Z_2(t)\big(I_{E_\varepsilon}(t)\delta h(t) + h_1(t)x_2^\varepsilon(t)\big)\Big) \, \mathrm{d}W_2(t), \\ \bar{Z}^{-1}(0)(Z_2(0) + Z_3(0)) = 0, \qquad t \in (0, T]. \end{cases} \tag{4.31}$$

Now we introduce an equation

$$\begin{cases} \mathrm{d}z(t) = -g(t) \, \mathrm{d}t + z_2(t) \, \mathrm{d}W_2(t), \\ z(T) = c(\bar{x}(T)). \end{cases} \tag{4.32}$$

Since Lemma 2.9 does not require the filtration to be a natural filtration ([12]), it follows from Lemma 2.9 as well as Definition 2.7 that

$$\begin{aligned} &\bar{\mathbb{E}}\bar{Z}^{-1}(T)(Z_2(T) + Z_3(T))c(\bar{x}(T)) + \bar{\mathbb{E}} \int_0^T \bar{Z}^{-1}(t)(Z_2(t) + Z_3(t))g(t) \, \mathrm{d}s \\ &= \bar{\mathbb{E}} \int_0^T \Big\langle h_1(t)(x_2^\varepsilon(t) + x_3^\varepsilon(t)) + I_{E_\varepsilon}(t)\delta h(t) \\ &+ I_{E_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon(t) + \frac{1}{2}h_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \\ &+ \bar{Z}^{-1}(t)Z_2(t)(I_{E_\varepsilon}(t)\delta h(t) + h_1(t)x_2^\varepsilon(t)), z_2(t) \Big\rangle_{V_2} \, \mathrm{d}t, \end{aligned} \tag{4.33}$$

and combining this with

$$
\begin{aligned}
&\bar{\mathbb{E}}\int_0^T \langle I_{E_\varepsilon}(t)\delta h_1(t)x_2^\varepsilon(t)+\bar{Z}^{-1}(t)Z_2(t)I_{E_\varepsilon}(t)\delta h(t), z_2(t)\rangle_{V_2}\,\mathrm{d}t\\
&\leq \mathcal{C}\Big[\int_{E_\varepsilon}(\bar{\mathbb{E}}|x_2^\varepsilon(t)|_H^2)^{\frac12}(\bar{\mathbb{E}}|z_2(t)|_{V_2}^2)^{\frac12}\,\mathrm{d}t+\\
&\Big(\int_{E_\varepsilon}\bar{\mathbb{E}}|z_2(t)|_{V_2}^2\,\mathrm{d}t\Big)^{\frac12}\Big(\int_{E_\varepsilon}(\bar{\mathbb{E}}|\bar{Z}^{-1}(t)|^4)^{\frac12}(\bar{\mathbb{E}}|Z_2(t)|^4)^{\frac12}\,\mathrm{d}t\Big)^{\frac12}\Big]\\
&=o(\varepsilon),
\end{aligned}
\tag{4.34}
$$

(4.29) becomes

$$
\begin{aligned}
&\mathcal{J}(u^\varepsilon(\cdot))-\mathcal{J}(\bar{u}(\cdot))\\
&=\bar{\mathbb{E}}\Big[\int_0^T\Big(I_{E_\varepsilon}(t)\delta g(t)+\langle I_{E_\varepsilon}(t)\delta h(t), z_2(t)\rangle_{V_2}+\langle g_1(t)+h_1^*(t)z_2(t), x_2^\varepsilon(t)+x_3^\varepsilon(t)\rangle_H\\
&+\frac12\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H+\frac12\langle h_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)), z_2(t)\rangle_{V_2}\Big)\,\mathrm{d}t\\
&\quad+c_x(\bar{x}(T))(x_2^\varepsilon(T)+x_3^\varepsilon(T))+\frac12\langle c_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T)\rangle_H\\
&\quad+\int_0^T\langle g_1(t)+h_1^*(t)z_2(t), \bar{Z}^{-1}(t)Z_2(t)x_2^\varepsilon(t)\rangle_H\,\mathrm{d}t\\
&\quad+\langle c_x(\bar{x}(T)), \bar{Z}^{-1}(T)Z_2(T)x_2^\varepsilon(T)\rangle_H\Big]+o(\varepsilon).
\end{aligned}
\tag{4.35}
$$

Set $X(t)=\bar{Z}^{-1}(t)Z_2(t)x_2^\varepsilon(t)$. We now aim to derive the equation satisfied by $X(\cdot)$. In the finite-dimensional case (*e.g.*, as seen in [21]), this step is achieved through the direct application of Itô's formula. However, in the infinite-dimensional setting, the mild solution to equation (4.6) does not satisfy the applicability conditions of Itô's formula. Consequently, we cannot directly apply Itô's formula in this context. According to the definition of a weak solution (Def. 2.1), Lemma 2.5 and the formula of integration by parts, for any $h'\in D(A^*)$ we have

$$
\begin{aligned}
\bar{Z}^{-1}(t)Z_2(t)\langle x_2^\varepsilon(t), h'\rangle_H &= \bar{Z}^{-1}(t)Z_2(t)\Big[\int_0^t(\langle x_2^\varepsilon(s), A^*h'\rangle_H+\langle a_1(s)x_2^\varepsilon(s), h'\rangle_H)\,\mathrm{d}s\\
&\qquad+\int_0^t\langle\langle b_1(s)x_2^\varepsilon(s)+I_{E_\varepsilon}(t)\delta b(s), h'\rangle\rangle_H\,\mathrm{d}W_1(s)\Big]\\
&=\int_0^t(\langle X(s), A^*h_j\rangle_H+\langle a_1(s)X(s), h'\rangle_H)\,\mathrm{d}s\\
&\qquad+\int_0^t\langle\langle b_1(s)X(s)+\bar{Z}^{-1}(s)Z_2(s)I_{E_\varepsilon}(t)\delta b(s), h'\rangle\rangle_H\,\mathrm{d}W_1(s)\\
&\qquad+\int_0^t\langle\langle (h_1(s)x_2^\varepsilon(s)+I_{E_\varepsilon}(s)\delta h(s))\otimes x_2^\varepsilon(s), h'\rangle\rangle_H\,\mathrm{d}W_2(s)\\
&=\langle X(t), h'\rangle_H,
\end{aligned}
\tag{4.36}
$$

where $v\otimes h$, $v\in V, h\in H$ is a bounded linear operator from V to H, defined by

$$(v\otimes h)W=\langle v, W\rangle_V h,\qquad \forall W\in V.$$

Equation (4.36) means that, $X(\cdot)$ is a weak solution to the following equation

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + a_1(t)X(t))\,\mathrm{d}t + (b_1(t)X(t) + \bar{Z}^{-1}(t)Z_2(t)I_{E_\varepsilon}(t)\delta b(t))\,\mathrm{d}W_1(t) \\ \qquad\quad + (h_1(t)x_2^\varepsilon(t) + I_{E_\varepsilon}(t)\delta h(t)) \otimes x_2^\varepsilon(t)\,\mathrm{d}W_2(t), \\ X(0) = 0,\ t \in (0, T]. \end{cases} \tag{4.37}$$

By Lemma 2.5, it is also a mild solution to the same equation. Again, we introduce an equation:

$$\begin{cases} \mathrm{d}q(t) = -Aq(t)\,\mathrm{d}t - (g_1(t) + h_1^*(t)z_2(t) + a_1^*(t)q(t) + b_1^*(t)q_1(t))\,\mathrm{d}t \\ \qquad\qquad + q_1(t)\,\mathrm{d}W_1(t) + q_2(t)\,\mathrm{d}W_2(t), \\ q(T) = c_x(\bar{x}(T)), \qquad t \in [0, T), \end{cases} \tag{4.38}$$

it follows from Lemma 2.9 as well as Definition 2.7 that

$$\begin{aligned} &\bar{\mathbb{E}}\langle X(T), c_x(\bar{x}(T))\rangle_H \\ &\quad + \bar{\mathbb{E}}\int_0^T \langle X(t), g_1(t) + h_1^*(t)z_2(t) + a_1^*(t)q(t) + b_1^*(t)q_1(t)\rangle_H\,\mathrm{d}t \\ &= \bar{\mathbb{E}}\int_0^T \langle a_1(t)X(t), q(t)\rangle_H\,\mathrm{d}t + \bar{\mathbb{E}}\int_0^T \langle b_1(t)X(t) + \bar{Z}^{-1}(t)Z_2(t)I_{E_\varepsilon}(t)\delta b(t), q_1(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t \\ &\quad + \bar{\mathbb{E}}\int_0^T \langle (h_1(t)x_2^\varepsilon(t) + I_{E_\varepsilon}(t)\delta h(t)) \otimes x_2^\varepsilon(t), q_2(t)\rangle_{\mathscr{L}_2(V_2;H)}\,\mathrm{d}t. \end{aligned} \tag{4.39}$$

Further, note the following estimates:

$$\begin{aligned} &\bar{\mathbb{E}}\int_0^T \langle \bar{Z}^{-1}(t)Z_2(t)I_{E_\varepsilon}(t)\delta b(t), q_1(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t \\ &\le \mathcal{C}(x_0)\Big(\int_{E_\varepsilon} \bar{\mathbb{E}}|q_1(t)|^2_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{E_\varepsilon} (\bar{\mathbb{E}}|\bar{Z}^{-1}(t)|^4)^{\frac{1}{2}}(\bar{\mathbb{E}}|Z_2(t)|^4)^{\frac{1}{2}}\,\mathrm{d}t\Big)^{\frac{1}{2}} \\ &= o(\varepsilon), \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} &\bar{\mathbb{E}}\int_0^T \langle I_{E_\varepsilon}(t)\delta h(t) \otimes x_2^\varepsilon(t), q_2(t)\rangle_{\mathscr{L}_2(V_2;H)}\,\mathrm{d}t \\ &\le \mathcal{C}\Big(\int_{E_\varepsilon} \bar{\mathbb{E}}|q_2(t)|^2_{\mathscr{L}_2(V_2;H)}\,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_{E_\varepsilon} \bar{\mathbb{E}}|x_2^\varepsilon(t)|^2_H\,\mathrm{d}t\Big)^{\frac{1}{2}} \\ &= o(\varepsilon), \end{aligned} \tag{4.41}$$

equation (4.29) becomes

$$\begin{aligned}&\mathcal{J}(u^\varepsilon(\cdot))-\mathcal{J}(\bar u(\cdot))\\&=\bar{\mathbb{E}}\Big[\int_0^T\Big(I_{E_\varepsilon}(t)\delta g(t)+\langle I_{E_\varepsilon}(t)\delta h(t),z_2(t)\rangle_{V_2}+\langle g_1(t)+h_1^*(t)z_2(t),x_2^\varepsilon(t)+x_3^\varepsilon(t)\rangle_H\\&+\frac12\langle g_{11}(t)x_2^\varepsilon(t),x_2^\varepsilon(t)\rangle_H+\frac12\langle h_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),z_2(t)\rangle_{V_2}+\langle h_1(t)x_2^\varepsilon(t)\otimes x_2^\varepsilon(t),q_2(t)\rangle_{\mathscr{L}_2(V_2;H)}\Big)\,\mathrm{d}t\\&\quad+c_x(\bar x(T))(x_2^\varepsilon(T)+x_3^\varepsilon(T))+\frac12\langle c_{xx}(\bar x(T))x_2^\varepsilon(T),x_2^\varepsilon(T)\rangle_H\Big]+o(\varepsilon).\end{aligned}\tag{4.42}$$

Combining (4.6) and (4.8) and applying the transposed solution of (4.38) again, we obtain:

$$\begin{aligned}&\bar{\mathbb{E}}\langle x_2^\varepsilon(T)+x_3^\varepsilon(T),c_x(\bar x(T))\rangle_H\\&\quad+\bar{\mathbb{E}}\int_0^T\big\langle x_2^\varepsilon(t)+x_3^\varepsilon(t),g_1(t)+h_1^*(t)z_2(t)+a_1^*(t)q(t)+b_1^*(t)q_1(t)\big\rangle_H\,\mathrm{d}t\\&=\bar{\mathbb{E}}\int_0^T\big\langle a_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))+I_{E_\varepsilon}(t)\delta a(t)+\frac12 a_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),q(t)\big\rangle_H\,\mathrm{d}t\\&\quad+\bar{\mathbb{E}}\int_0^T\big\langle b_1(t)(x_2^\varepsilon(t)+x_3^\varepsilon(t))+I_{E_\varepsilon}(t)\delta b(t)+I_{E_\varepsilon}(t)\delta b_1(t)x_2^\varepsilon(t)\\&\quad+\frac12 b_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),q_1(t)\big\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t.\end{aligned}\tag{4.43}$$

Taking into consideration the estimate

$$\begin{aligned}&\bar{\mathbb{E}}\int_0^T\langle I_{E_\varepsilon}(t)\delta b_1(t)x_2^\varepsilon(t),q_1(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t\\&\le\mathcal{C}\Big(\int_{E_\varepsilon}\bar{\mathbb{E}}|q_1(t)|^2_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t\Big)^{\frac12}\Big(\int_{E_\varepsilon}\bar{\mathbb{E}}|x_2^\varepsilon(t)|^2_H\,\mathrm{d}t\Big)^{\frac12}\\&=o(\varepsilon),\end{aligned}\tag{4.44}$$

we arrive at the following:

$$\begin{aligned}&\mathcal{J}(u^\varepsilon(\cdot))-\mathcal{J}(\bar u(\cdot))\\&=\bar{\mathbb{E}}\Big[\int_0^T\Big(\langle I_{E_\varepsilon}(t)\delta a(t),q(t)\rangle_H+\langle I_{E_\varepsilon}(t)\delta b(t),q_1(t)\rangle_{\mathscr{L}_2(V_1;H)}+I_{E_\varepsilon}(t)\delta g(t)+\langle I_{E_\varepsilon}(t)\delta h(t),z_2(t)\rangle_{V_2}\\&\quad+\frac12\langle a_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),q(t)\rangle_H+\frac12\langle b_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),q_1(t)\rangle_{\mathscr{L}_2(V_1;H)}+\frac12\langle g_{11}(t)x_2^\varepsilon(t),x_2^\varepsilon(t)\rangle_H\\&\quad+\frac12\langle h_{11}(t)(x_2^\varepsilon(t),x_2^\varepsilon(t)),z_2(t)\rangle_{V_2}+\langle h_1(t)x_2^\varepsilon(t)\otimes x_2^\varepsilon(t),q_2(t)\rangle_{\mathscr{L}_2(V_2;H)}\Big)\,\mathrm{d}t\\&\quad+\frac12\langle c_{xx}(\bar x(T))x_2^\varepsilon(T),x_2^\varepsilon(T)\rangle_H\Big]+o(\varepsilon).\end{aligned}\tag{4.45}$$

Next, we will utilize the concept of relaxed transposition solutions to characterize the second-order adjoint process. Define the Hamiltonian function

$$\begin{aligned}H(t,x,u,q,q_1,z_2)=&\langle q,a(t,x,u)\rangle_H+\langle q_1,b(t,x,u)\rangle_{\mathscr{L}_2(V_1;H)}\\&+g(t,x,u)+\langle z_2,h(t,x,u)\rangle_{V_2},\end{aligned}\tag{4.46}$$

and consider this equation

$$\begin{cases} \mathrm{d}P = -(A^* + a_1^*(t))P\,\mathrm{d}t - P(A + a_1(t))\,\mathrm{d}t - b_1^*(t)Pb_1(t)\,\mathrm{d}t - (b_1^*(t)Q + Qb_1(t))\,\mathrm{d}t \\ \qquad\quad - (H_{xx}(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t)) + q_2(t)h_1(t) + h_1^*(t)q_2^*(t))\,\mathrm{d}t + Q\,\mathrm{d}W_1(t), \\ P(T) = c_{xx}(\bar{x}(T)), \qquad t \in [0,T). \end{cases} \tag{4.47}$$

By the definition of the relaxed transposition solution to the equation (4.47) and Lemma 2.11, we obtain that

$$\begin{aligned} &\mathbb{E}\langle c_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T)\rangle_H \\ &+ \mathbb{E}\int_0^T \langle (H_{xx}(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t)) + q_2(t)h_1(t) + h_1^*(t)q_2^*(t))x_2^\varepsilon(t), x_2^\varepsilon(t)\rangle_H\,\mathrm{d}t \\ &= \mathbb{E}\int_0^T \langle P(t)b_1(t)x_2^\varepsilon(t), I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t \\ &+ \mathbb{E}\int_0^T \langle P(t)I_{E_\varepsilon}(t)\delta b(t), b_1(t)x_2^\varepsilon(t) + I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t \\ &+ \mathbb{E}\int_0^T \langle I_{E_\varepsilon}(t)\delta b(t), \hat{Q}^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t \\ &+ \mathbb{E}\int_0^T \langle Q^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t), I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\,\mathrm{d}t. \end{aligned} \tag{4.48}$$

We note that

$$\begin{aligned} &\Big|\bar{\mathbb{E}}\int_0^T \Big[\langle P(t)b_1(t)x_2^\varepsilon(t), I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)} + \langle P(t)I_{E_\varepsilon}(t)\delta b(t), b_1(t)x_2^\varepsilon(t)\rangle_{\mathscr{L}_2(V_1;H)}\Big]\,\mathrm{d}t\Big| \\ &= o(\varepsilon), \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} \langle h_1(t)x_2^\varepsilon(t)\otimes x_2^\varepsilon(t), q_2(t)\rangle_{\mathscr{L}_2(V_2;H)} &= \sum_{j=1}^\infty \langle h_1(t)x_2^\varepsilon(t), r_j\rangle_{V_2}\langle x_2^\varepsilon(t), q_2(t)r_j\rangle_H \\ &= \langle h_1(t)x_2^\varepsilon(t), q_2^*(t)x_2^\varepsilon(t)\rangle_{V_2}, \end{aligned} \tag{4.50}$$

then we can get

$$\begin{aligned} &\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\ &= \bar{\mathbb{E}}\Big[\int_0^T \Big(I_{E_\varepsilon}(t)(H(t,\bar{x}(t),u(t),q(t),q_1(t),z_2(t)) - H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t))) \\ &+ \frac{1}{2}\langle P(t)I_{E_\varepsilon}(t)\delta b(t), I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)} + \frac{1}{2}\langle I_{E_\varepsilon}(t)\delta b(t), \hat{Q}^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t)\rangle_{\mathscr{L}_2(V_1;H)} \\ &+ \frac{1}{2}\langle Q^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t), I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\Big)\,\mathrm{d}t\Big] + o(\varepsilon). \end{aligned} \tag{4.51}$$

Now, we shall get rid of the terms containing $Q^{(0)}$ and $\hat{Q}^{(0)}$. We give the following conclusion directly:

$$\begin{aligned}&\Big|\bar{\mathbb{E}}\int_0^T\Big(\langle I_{E_\varepsilon}(t)\delta b(t),\hat{Q}^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t)\rangle_{\mathscr{L}_2(V_1;H)}+\langle Q^{(0)}(0,0,I_{E_\varepsilon}\delta b)(t),I_{E_\varepsilon}(t)\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\Big)\,\mathrm{d}t\Big|\\&=o(\varepsilon).\end{aligned}\tag{4.52}$$

The proof of (4.52) is the same as under complete information and is omitted here due to its complexity. Details can be found on pp. 452–456, [12].

Eventually we get:

$$\begin{aligned}&\mathcal{J}(u^\varepsilon(\cdot))-\mathcal{J}(\bar{u}(\cdot))\\&=\bar{\mathbb{E}}\Big[\int_0^T I_{E_\varepsilon}(t)\Big(H(t,\bar{x}(t),u(t),q(t),q_1(t),z_2(t))-H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t))\\&\quad+\frac{1}{2}\langle P(t)\delta b(t),\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\Big)\,\mathrm{d}t\Big]\geq o(\varepsilon).\end{aligned}\tag{4.53}$$

Divide by ε on both sides of the inequality and make $\varepsilon\to 0$. At this point, due to the arbitrariness of t_0, we can get:

$$\begin{aligned}&\bar{\mathbb{E}}\Big[H(t,\bar{x}(t),u(t),q(t),q_1(t),z_2(t))-H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t))\\&\quad+\frac{1}{2}\langle P(t)\delta b(t),\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)}\Big]\geq 0,\qquad a.e.\ t\in[0,T].\end{aligned}\tag{4.54}$$

For $\forall A\in\mathscr{F}_t^y$, we can take $u_A(t)=\bar{u}(t)+I_A(u(t)-\bar{u}(t))$, *i.e.*, for $\forall A\in\mathscr{F}_t^y$, there is

$$\begin{aligned}&\bar{\mathbb{E}}\Big[I_A(H(t,\bar{x}(t),u(t),q(t),q_1(t),z_2(t))-H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t))\\&\quad+\frac{1}{2}\langle P(t)\delta b(t),\delta b(t)\rangle_{\mathscr{L}_2(V_1;H)})\Big]\geq 0,\qquad a.e.\ t\in[0,T].\end{aligned}\tag{4.55}$$

So, we have

$$\begin{aligned}&\bar{\mathbb{E}}\Big[H(t,\bar{x}(t),u,q(t),q_1(t),z_2(t))-H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t))\\&\quad+\frac{1}{2}\langle P(t)(b(t,\bar{x}(t),u)-b(t,\bar{x}(t),\bar{u}(t))),(b(t,\bar{x}(t),u)-b(t,\bar{x}(t),\bar{u}(t)))\rangle_{\mathscr{L}_2(V_1;H)}\Big|\mathscr{F}_t^y\Big]\geq 0,\\&\quad\forall u\in U\ a.e.\ t\in[0,T]\ ,\ a.s.\ .\end{aligned}\tag{4.56}$$

4.3. The maximum principle

Theorem 4.4. *Suppose that H is a separable Hilbert space, $L^p_{\mathscr{F}_T}(\Omega;\mathbb{R})$ $(1\leq p<\infty)$ is a separable Banach space, U is a separable metric space, and $x_0\in L^\infty_{\mathscr{F}_0}(\Omega;H)$. Let the Condition 3.1–3.4 hold, and let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be an optimal pair of the problem. Let $(z(\cdot),z_2(\cdot))$ and $(q(\cdot),q_1(\cdot),q_2(\cdot))$ be the transposition solutions to (4.32) and (4.38), respectively. Assume that $(P(\cdot),Q^{(\cdot)},\hat{Q}^{(\cdot)})$ is the relaxed transposition solution to the equation (4.47).*

Then,

$$
\begin{aligned}
&\bar{\mathbb{E}}\Big[H(t,\bar{x}(t),u,q(t),q_1(t),z_2(t)) - H(t,\bar{x}(t),\bar{u}(t),q(t),q_1(t),z_2(t)) \\
&\quad + \frac{1}{2}\big\langle P(t)(b(t,\bar{x}(t),u) - b(t,\bar{x}(t),\bar{u}(t))),(b(t,\bar{x}(t),u) - b(t,\bar{x}(t),\bar{u}(t)))\big\rangle_{\mathscr{L}_2(V_1;H)}\Big|\mathscr{F}_t^y\Big] \geq 0, \\
&\quad \forall u \in U \ a.e.\ t \in [0,T]\ ,\ a.s.\ .
\end{aligned}
\tag{4.57}
$$

5. Example

We present an example of controlled SPDEs that fits our framework.

Example 5.1. Given a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ with a C^∞ boundary $\partial\mathcal{O}$, consider a partially observed optimal control problem for the following controlled stochastic wave equation:

$$
\begin{cases}
\mathrm{d}y_t(x,t) - \triangle y(x,t)\,\mathrm{d}t = a_1(x,t,u(t))y(x,t)\,\mathrm{d}t + b_1(x,t,u(t))y(x,t)\,\mathrm{d}W_1(t), \\
y(x,t) = 0, \qquad\qquad\qquad (x,t) \in \partial\mathcal{O} \times (0,T), \\
y(0) = y_0, \quad y_t(0) = y_1,
\end{cases}
\tag{5.1}
$$

with the partial observation

$$
Y(t) = \int_0^t h_1(s,y(s),u(s))\,\mathrm{d}s + \widehat{W}(t)
\tag{5.2}
$$

and the cost functional

$$
\mathcal{J}(y_0,y_1;u(\cdot)) = \mathbb{E}^u\Big[\int_0^T \int_{\mathcal{O}} g_1(x,t,u(t))y(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{O}} c_1(x)y(x,T)\,\mathrm{d}x\Big].
\tag{5.3}
$$

Here a_1, b_1, h_1, g_1 and c_1 are given coefficients satisfying appropriate assumptions, $y_0, y_1 \in H_0^1(\mathcal{O}) \cap L^2(\mathcal{O})$. $W(t), Y(t)$ are mutually independent Brownian motion. U is a separable metric space and does not necessarily have to be convex, u is a predictable U-valued stochastic process with respect to $\{\mathscr{F}_t^y\}$, the filtration generated by $Y(t)$. Our optimal control problem is to find $\bar{u}$ that minimizes the cost functional $\mathcal{J}(y_0,y_1;u(\cdot))$.

This problem is a concrete example of the problem discussed in this paper in the following setting:

- $H = H_0^1(\mathcal{O}) \times L^2(\mathcal{O})$ and $V_1, V_2 = \mathbb{R}$.
- The operator A is defined as follows:

$$
\begin{cases}
D(A) = [H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})] \times H_0^1(\mathcal{O}), \\
A(\phi_1,\phi_2)^\top = (\phi_2, \triangle\phi_1)^\top, \quad \forall(\phi_1,\phi_2)^\top \in D(A);
\end{cases}
\tag{5.4}
$$

- $a(t,(y(t),y_t(t))^\top,u(t)) = (0, a_1(t,u(t))y(t))^\top$ and $b(t,(y(t),y_t(t))^\top,u(t)) = (0, b_1(t,u(t))y(t))^\top$ takes values in H,.
- $h(t,(y(t),y_t(t))^\top,u(t)) = h_1(t,y(t),u(t))$ takes values in $\mathbb{R}$.
- $g(t,(y(t),y_t(t))^\top,u(t)) = \int_{\mathcal{O}} g_1(x,t,u(t))y(x,t)\,\mathrm{d}x$ and $c(y(T),y_t(T)) = \int_{\mathcal{O}} c_1(x)y(x,T)\,\mathrm{d}x$.

Here's a concise explanation of the mathematical statement: The C_0-semigroup generated by A satisfies $|S(t)|_{\mathscr{L}(H)} \leq \mathrm{e}^{ct}$ for some constant $c \in \mathbb{R}$ and any $t \geq 0$.

Given the inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H_0^1(\mathcal{O})} + \langle\cdot,\cdot\rangle_{L^2(\mathcal{O})}$ on the space $H_0^1(\mathcal{O}) \times L^2(\mathcal{O})$, it follows that there exists a $\lambda > 0$ such that the operator $(A - (\lambda-1)I)$ is dissipative. Given $F = (f,g)^\top \in H$, combined with the equation $(\lambda I - A)U = F$, *i.e.*

$$\begin{cases} \lambda u - v = f, \\ \lambda v - \triangle u = g. \end{cases} \tag{5.5}$$

Then, we can conclude that there exists a solution $U = (u, v)^\top \in D(A)$ satisfying the equation with some $\lambda > 0$. Thus, we conclude that the range of $(\lambda I - A)$ is the entire space H. Finally, combining with the Phillips-Lumer theorem ([30], Chapter IX. 8), we conclude that for some $\lambda > 0$, the operator $(A - (\lambda - 1)I)$ generates a C_0-contraction semigroup, thereby establishing $|S(t)|_{\mathscr{L}(H)} \leq \mathrm{e}^{(\lambda-1)t}$.

Therefore, by Theorem 4.4, we can obtain the maximum principle for the aforementioned stochastic optimal control problem:

Proposition 5.2. *Let $(z(\cdot), z_2(\cdot))$ and $(q(\cdot), q_1(\cdot), q_2(\cdot))$ be the transposition solutions of the equations obtained by substituting the coefficients $(A, a, b, h, g, c$, etc.$)$ from Example 5.1 into equations (4.32) and (4.38), respectively. Let $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)})$ be the relaxed transposition solution of the equation obtained by substituting the aforementioned coefficients into equation (4.47). Then, the optimal pair $((\bar{y}(\cdot), \bar{y}_t(\cdot))^\top, \bar{u}(\cdot))$ of Example 5.1 satisfies:*

$$\begin{aligned} \bar{\mathbb{E}}\Big[& H(t, (\bar{y}(t), \bar{y}_t(t))^\top, u, q(t), q_1(t), z_2(t)) - H(t, (\bar{y}(t), \bar{y}_t(t))^\top, \bar{u}(t), q(t), q_1(t), z_2(t)) \\ & + \frac{1}{2} \big\langle P(t)(0, (b_1(t, u) - b_1(t, \bar{u}(t)))\bar{y}(t))^\top, (0, (b_1(t, u) - b_1(t, \bar{u}(t)))\bar{y}(t))^\top \big\rangle_{\mathscr{L}_2(\mathbb{R}; H)} \Big| \mathscr{F}_t^y \Big] \geq 0, \\ & \forall u \in U \ \text{a.e. } t \in [0, T] \ , \ \text{a.s.} \ . \end{aligned} \tag{5.6}$$

Here, the Hamiltonian function is defined as:

$$\begin{aligned} H(t, (y_1, y_2)^\top, u, q, q_1, z_2) = & \langle q, (0, a_1(t, u)y_1)^\top \rangle_H + \langle q_1, (0, b_1(t, u)y_1)^\top \rangle_{\mathscr{L}_2(\mathbb{R}; H)} \\ & + g(t, (y_1, y_2)^\top, u) + z_2 h_1(t, y_1, u). \end{aligned} \tag{5.7}$$

6. Conclusion

In this paper, we derive a general maximum principle for partially observed stochastic evolution control systems to the infinite dimensional case by virtue of some idea of [21]. It is well known that the infinite-dimensional case has many differences as well as some difficulties compared to the finite-dimensional case. The stringent conditions for the infinite-dimensional Itô's formula prevent its direct application to derive certain equations obtainable *via* Itô's formula in finite dimensions ([21]), creating difficulties. Characterizing the second-order adjoint processes in infinite dimensions is also a major challenge. We addressed these issues by employing the definition of weak solution (distinct from weak solutions for finite-dimensional SDEs) and transposition solution/relaxed transposition solution, ultimately obtaining three adjoint equations and establishing the maximum principle. Subsequently, we can build on this to further consider the case with jumps (*e.g.*, Markov chains) in near future research.

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Data availability statement

No new data/codes were created or analyzed in this study.

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