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TURNPIKE PROPERTY OF NONZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES

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Abstract. This paper investigates the turnpike properties of deterministic nonzero-sum linear-quadratic (LQ) differential games. Under certain assumptions on the Hamiltonian matrix of the nonzero-sum LQ differential game, we establish the solvability of both the coupled non-symmetric differential Riccati equation (DRE) and the algebraic Riccati equation (ARE). Moreover, we identify the convergence relationship between the DRE and ARE, which is essential for understanding the turnpike properties. Over a finite but sufficiently long time horizon, the open-loop Nash equilibrium is shown to remain exponentially close to the solution of a two-objective optimization problem for the majority of the time horizon.

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1. Introduction

Consider the following controlled linear differential equation on a finite horizon $[0,T]$:

$$\begin{cases}\dot{X}(t)=AX(t)+B_1u_1(t)+B_2u_2(t)+b, \quad t\in[0,T],\\ X(0)=x,\end{cases} \tag{1.1}$$

where $A\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times m_i}$, $i=1,2$, and $b\in\mathbb{R}^n$ are given constant matrices. The cost functionals of Player 1 and Player 2 are given by

$$\begin{aligned}J_1^{(T)}(x;u_1(\cdot),u_2(\cdot)) \triangleq \frac{1}{2}\Big\{&\langle G_1X(T),X(T)\rangle+2\langle g_1,X(T)\rangle\\ &+\int_0^T\Big[\langle Q_1X(t),X(t)\rangle+\langle R_{11}u_1(t),u_1(t)\rangle+\langle R_{12}u_2(t),u_2(t)\rangle\\ &+2\langle q_1,X(t)\rangle+2\langle r_{11},u_1(t)\rangle+2\langle r_{12},u_2(t)\rangle\Big]\mathrm{d}t\Big\},\end{aligned} \tag{1.2}$$

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and

$$J_2^{(T)}(x;u_1(\cdot),u_2(\cdot)) \triangleq \frac{1}{2}\bigg\{\langle G_2X(T),X(T)\rangle + 2\langle g_2,X(T)\rangle + \int_0^T \Big[\langle Q_2X(t),X(t)\rangle + \langle R_{21}u_1(t),u_1(t)\rangle + \langle R_{22}u_2(t),u_2(t)\rangle + 2\langle q_2,X(t)\rangle + 2\langle r_{21},u_1(t)\rangle + 2\langle r_{22},u_2(t)\rangle\Big]\mathrm{d}t\bigg\}, \tag{1.3}$$

respectively, where $G_1,G_2,Q_1,Q_2 \in \mathbb{R}^{n\times n}$, $R_{11},R_{21} \in \mathbb{R}^{m_1\times m_1}$, $R_{12},R_{22} \in \mathbb{R}^{m_2\times m_2}$, $g_1,g_2,q_1,q_2 \in \mathbb{R}^n$, $r_{11},r_{21} \in \mathbb{R}^{m_1}$, and $r_{12},r_{22} \in \mathbb{R}^{m_2}$ are given constant matrices or vectors with G_1,G_2,Q_1,Q_2 and $R_{11},R_{12},R_{21},R_{22}$ being symmetric. The function $u_i(\cdot)$ represents the control of Player i, which is selected from the following space:

$$\mathcal{U}_i[0,T] \triangleq \left\{\varphi:[0,T]\to\mathbb{R}^{m_i} \;\Big|\; \int_0^T |\varphi(t)|^2\mathrm{d}t<\infty\right\}, \quad i=1,2. \tag{1.4}$$

The problem considered in the paper is to find, for each initial state $x\in\mathbb{R}^n$, a pair $(\tilde{u}_1^{(T)}(\cdot),\tilde{u}_2^{(T)}(\cdot)) \in \mathcal{U}_1[0,T]\times\mathcal{U}_2[0,T]$ such that

$$\begin{aligned} J_1^{(T)}(x;\tilde{u}_1^{(T)}(\cdot),\tilde{u}_2^{(T)}(\cdot)) &\leqslant J_1^{(T)}(x;u_1(\cdot),\tilde{u}_2^{(T)}(\cdot)), \quad \forall\, u_1(\cdot)\in\mathcal{U}_1[0,T],\\ J_2^{(T)}(x;\tilde{u}_1^{(T)}(\cdot),\tilde{u}_2^{(T)}(\cdot)) &\leqslant J_2^{(T)}(x;\tilde{u}_1^{(T)}(\cdot),u_2(\cdot)), \quad \forall\, u_2(\cdot)\in\mathcal{U}_2[0,T]. \end{aligned} \tag{1.5}$$

This problem, denoted by Problem $(\mathrm{DG})_T$, is called a *deterministic nonzero-sum linear-quadratic (LQ) differential game*. The pair $(\tilde{u}_1^{(T)}(\cdot),\tilde{u}_2^{(T)}(\cdot))$ (if exists) in (1.5) is called an *open-loop Nash equilibrium* of Problem $(\mathrm{DG})_T$ for the initial state x, the corresponding state trajectory $\tilde{X}^{(T)}(\cdot)$ is called an *open-loop optimal state trajectory*. Furthermore,

$$V_i^{(T)}(x) \triangleq J_i^{(T)}(x;\tilde{u}_1^{(T)}(\cdot),\tilde{u}_2^{(T)}(\cdot)), \quad x\in\mathbb{R}^n$$

is called the *value function* of Player i.

The history of nonzero-sum differential games can be traced back to the work of Nash [1], with some early contributions found in [2, 3]. In this context, each player has their own cost function, and even in adversarial situations, the objectives of opposing players may not be directly opposed. Mathematically, the open-loop Nash equilibrium for Problem $(\mathrm{DG})_T$ can be expressed in a closed-loop form by solving a system of non-symmetric Riccati equations. This closed-loop representation may differ from the solution obtained for the closed-loop Nash equilibrium in general. Several significant results have been established for the open-loop Nash equilibrium over both finite and infinite time horizons [4]. In recent decades, numerous studies have explored stochastic differential games (see, *e.g.*, [5–9]).

The turnpike property refers to the phenomenon in many optimal control problems over finite, but long-time horizons, where optimal trajectories tend to approach a steady state of the system and remain near it for most of the time horizon. In practice, directly computing the optimal solution for complex problems is often impractical or inefficient with existing methods. The turnpike property provides a powerful alternative approach: by solving an associated static optimization problem, one obtains a reliable approximation of the optimal state-control pair without requiring a full analytical solution. This insight is highly beneficial for the design and acceleration of numerical algorithms for optimal control.

The study of turnpike phenomena began with the work of von Neumann in economics [10], and the term was first coined by Dorfman, Samuelson, and Solow in 1958 [11], referring to the American English word for a highway. In recent decades, the turnpike property has garnered significant attention from researchers in various fields (see, *e.g.*, [12–14]), as it often provides a critical insight into the nature of the optimal solution without the need for a full analytical solution. As noted by [15], the exponential turnpike property arises from a general phenomenon of hyperbolicity, which naturally occurs in classical LQ optimal control problems under certain stability and detectability conditions. This property also significantly simplifies numerical methods for solving such optimal control problems. A substantial body of research has been developed on the turnpike property for both finite and infinite time horizons in discrete-time and continuous-time systems (see, *e.g.*, [15–25], and the references therein). It is also worth noting that recent groundbreaking research has explored the turnpike property in stochastic optimal control problems [26, 27].

For the deterministic LQ optimal control problem (a single-player game), denoted by Problem $(\mathrm{DLQ})_T$, the exponential turnpike property has been established in [28] and [15]. This property states that the optimal pair converges exponentially during the transient phase (as $T \to \infty$) to the minimum point (x^*, u^*), called the *turnpike limit*, of a specific static optimization problem. That is, there exist positive constants K and λ, independent of T, such that the optimal pair $(\tilde{X}^{(T)}(\cdot), \tilde{u}^{(T)}(\cdot))$ of Problem $(\mathrm{DLQ})_T$ satisfies

$$|\tilde{X}^{(T)}(t) - x^*| + |\tilde{u}^{(T)}(t) - u^*| \leqslant K\Big[\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda(T-t)}\Big], \quad \forall t \in [0, T].$$

In this paper, we establish the exponential turnpike property for Problem $(\mathrm{DG})_T$. Unlike the single-player case, in the Problem $(\mathrm{DG})_T$, each player can only influence the state process solely *via* their own control in order to optimize their individual cost functional. Further, each player must continuously adjust their control strategy in response to the actions of another player, iteratively seeking a Nash equilibrium. In contrast, the single-player case does not involve such interactive decision-making.

The primary challenges involve constructing the convergence relationship between the coupled non-symmetric differential Riccati equation (DRE) and its corresponding algebraic Riccati equation (ARE), as well as accurately formulating the static optimization problem. In the context of optimal control, the convergence of the Riccati equation can be derived under stabilizability conditions on the state equation over $[0, \infty)$ and positive definiteness conditions on the weighting coefficients in the cost functional. However, the non-symmetric nature of the solution causes the Riccati equation to lose its monotonicity. Moreover, the static problem in this case is a two-objective optimization problem, setting it apart from the turnpike properties established in the existing literature. By introducing additional assumptions, we address these challenges and establish key properties essential to the turnpike phenomenon. We also prove the integral and mean-square turnpike properties for Problem $(\mathrm{DG})_T$ and show that the value functions of both players converge to the objective values of the static optimization problem in a time-averaged sense. Additionally, an extension for the N-player case is given. These findings offer valuable insights into optimal strategies without relying on explicit analytical solutions and may enhance numerical methods for solving nonzero-sum LQ differential games.

The rest of this paper is structured as follows. In Section 2, we present preliminaries and relevant results on nonzero-sum LQ differential games. Section 3 addresses the solvability of the coupled non-symmetric ARE under suitable assumptions. In Section 4, we establish the exponential convergence rate of the coupled non-symmetric DRE as the time horizon approaches infinity. Section 5 focuses on the exponential turnpike property for Problem $(\mathrm{DG})_T$, including the integral and mean-square turnpike properties, as well as the N-player case. Examples are given in Section 6.

2. Preliminaries

Let $\mathbb{R}^{n\times m}$ be the space of real $n \times m$ matrices, equipped with the Frobenius inner product

$$\langle M, N\rangle \triangleq \mathrm{tr}\,(M^\top N), \quad M, N \in \mathbb{R}^{n\times m},$$

where the superscript $\top$ denotes matrix transposition, and $\mathrm{tr}\,(M^\top N)$ represents the trace of the matrix $M^\top N$. The norm induced by this inner product is denoted by $|\cdot|$. For a subset $\mathbb{H}$ of $\mathbb{R}^{n\times m}$, we denote by $C([0, T]; \mathbb{H})$

the space of all continuous functions from $[0, T]$ into $\mathbb{H}$. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices, and $\mathbb{S}^n_+$ the subset of $\mathbb{S}^n$ consisting of positive definite matrices. For $M, N \in \mathbb{S}^n$, we write $M \geqslant N$ (or $M > N$) if $M - N$ is positive semidefinite (or positive definite). The identity matrix of size n is denoted by I_n. The spectrum of a matrix X, *i.e.*, the set of its eigenvalues, is denoted by $\sigma(X)$. We use $\mathbb{C}^-$ and $\mathbb{C}^+$ to represent the sets of complex numbers with negative and positive real parts, respectively. If all eigenvalues of X lie in $\mathbb{C}^-$ (or $\mathbb{C}^+$), we write $\sigma(X) \subseteq \mathbb{C}^-$ (or $\sigma(X) \subseteq \mathbb{C}^+$). Let $Re(c)$ denote the real part of constant c.

For notational simplicity, we let $m \triangleq m_1 + m_2$ and

$$\widehat{A} \triangleq \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \widehat{B} \triangleq \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad B \triangleq (B_1, B_2), \qquad q \triangleq \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad g \triangleq \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$
$$G \triangleq \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \qquad Q \triangleq \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \qquad \widehat{R} \triangleq \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix}, \quad r \triangleq \begin{pmatrix} r_{11} \\ r_{22} \end{pmatrix}.$$

Let

$$S \triangleq (S_1, S_2),$$

where $S_i \triangleq B_i R_{ii}^{-1} B_i^\top, i = 1, 2$. Then we have $S = B\widehat{R}^{-1}\widehat{B}^\top$. We introduce the following matrix, commonly referred to as the *Hamilton matrix* for Problem $(\mathrm{DG})_T$:

$$M \triangleq \begin{pmatrix} -A & S \\ Q & \widehat{A}^\top \end{pmatrix} \in \mathbb{R}^{3n\times 3n}. \tag{2.1}$$

The following lemma provides the necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium for Problem $(\mathrm{DG})_T$. For a detailed proof, see [4].

Lemma 2.1. *Suppose that*

$$G_i \geqslant 0, \quad Q_i \geqslant 0, \quad R_{ii} > 0, \quad i = 1, 2.$$

Then, for any initial state $x \in \mathbb{R}^n$, Problem $(DG)_T$ has a unique open-loop Nash equilibrium if and only if the matrix

$$\begin{pmatrix} I_n & 0 & 0 \end{pmatrix} \mathrm{e}^{MT} \begin{pmatrix} I_n \\ G_1 \\ G_2 \end{pmatrix} \tag{2.2}$$

is invertible.

Next, we introduce the following hypotheses, which will be imposed throughout the paper.

(A1) For $i = 1, 2$, $G_i \geqslant 0, Q_i \geqslant 0$ and $R_{ii} > 0$.
(A2) The matrix in (2.2) is invertible for all $T \geqslant 0$, and the inverse of (2.2) is bounded in $T \in [0, \infty)$.

Remark 2.2. The boundedness of the inverse of (2.2) is instrumental in establishing the convergence of the solution to the DRE (2.5) as $T \to \infty$ in Section 4. There are sufficient conditions available that ensure the boundedness of the inverse of (2.2); see [4].

The following lemma summarizes several results related to Problem $(\mathrm{DG})_T$. For proofs, please refer to [4, 8].

Lemma 2.3. *Let (A1)–(A2) hold. Then the following hold:*

(i) *For any initial state $x \in \mathbb{R}^n$, Problem $(DG)_T$ admits a unique open-loop Nash equilibrium.*

(ii) $\tilde{u}^{(T)}(\cdot) \triangleq \begin{pmatrix}\tilde{u}_1^{(T)}(\cdot)\\ \tilde{u}_2^{(T)}(\cdot)\end{pmatrix} \in \mathcal{U}_1[0,T]\times\mathcal{U}_2[0,T]$ *is an open-loop Nash equilibrium of Problem(DG)$_T$ for x if and only if the solution pair $(\tilde{X}^{(T)}(\cdot), \tilde{Y}_i^{(T)}(\cdot))$ to the following forward-backward ordinary differential equation (FBODE) on $[0,T]$:*

$$\begin{cases}\dot{\tilde{X}}^{(T)}(t) = A\tilde{X}^{(T)}(t) + B_1\tilde{u}_1^{(T)}(t) + B_2\tilde{u}_2^{(T)}(t) + b,\\ \dot{\tilde{Y}}_i^{(T)}(t) = -\Big[A^\top \tilde{Y}_i^{(T)}(t) + Q_i\tilde{X}^{(T)}(t) + q_i\Big], \quad i=1,2,\\ \tilde{X}^{(T)}(0)=x, \quad \tilde{Y}_i^{(T)}(T) = G_i\tilde{X}^{(T)}(T) + g_i,\end{cases} \tag{2.3}$$

satisfies the stationarity condition:

$$\widehat{B}^\top \tilde{Y}^{(T)}(t) + \widehat{R}\tilde{u}^{(T)}(t) + r = 0, \quad a.e.\ t\in[0,T], \tag{2.4}$$

where $\tilde{Y}^{(T)}(\cdot) \triangleq \begin{pmatrix}\tilde{Y}_1^{(T)}(\cdot)\\ \tilde{Y}_2^{(T)}(\cdot)\end{pmatrix}$.

(iii) *The following DRE*

$$\begin{cases}\dot{\varPi}^{(T)}(t) + \widehat{A}^\top \varPi^{(T)}(t) + \varPi^{(T)}(t)A + Q - \varPi^{(T)}(t)S\varPi^{(T)}(t) = 0,\\ \varPi^{(T)}(T) = G,\end{cases} \tag{2.5}$$

admits a unique solution $\varPi^{(T)}(\cdot) \equiv \begin{pmatrix}\varPi_1^{(T)}(\cdot)\\ \varPi_2^{(T)}(\cdot)\end{pmatrix} \in C([0,T];\mathbb{R}^{2n\times n})$, *given by*

$$\begin{cases}\varPi_1^{(T)}(t) = \begin{pmatrix}0 & I_n & 0\end{pmatrix} \mathrm{e}^{M(T-t)}\begin{pmatrix}I_n\\ G_1\\ G_2\end{pmatrix}\left[\begin{pmatrix}I_n & 0 & 0\end{pmatrix}\mathrm{e}^{M(T-t)}\begin{pmatrix}I_n\\ G_1\\ G_2\end{pmatrix}\right]^{-1},\\ \varPi_2^{(T)}(t) = \begin{pmatrix}0 & 0 & I_n\end{pmatrix} \mathrm{e}^{M(T-t)}\begin{pmatrix}I_n\\ G_1\\ G_2\end{pmatrix}\left[\begin{pmatrix}I_n & 0 & 0\end{pmatrix}\mathrm{e}^{M(T-t)}\begin{pmatrix}I_n\\ G_1\\ G_2\end{pmatrix}\right]^{-1}.\end{cases} \tag{2.6}$$

(iv) *Let $\varPi^{(T)}(\cdot)$ be the solution of (2.5). The following ODE admits a unique solution* $\eta^{(T)}(\cdot) \equiv \begin{pmatrix}\eta_1^{(T)}(\cdot)\\ \eta_2^{(T)}(\cdot)\end{pmatrix}$:

$$\begin{cases}\dot{\eta}^{(T)}(t) + [\widehat{A}^\top - \varPi^{(T)}(t)S]\eta^{(T)}(t) - \varPi^{(T)}(t)B\widehat{R}^{-1}r + \varPi^{(T)}(t)b + q = 0,\\ \eta^{(T)}(T) = g.\end{cases} \tag{2.7}$$

(v) *Let $\varPi^{(T)}(\cdot)$ be the solution of (2.5) and $\eta^{(T)}(\cdot)$ the solution of (2.7). Set*

$$\varTheta^{(T)}(t) \triangleq -\widehat{R}^{-1}\widehat{B}^\top \varPi^{(T)}(t), \quad t\in[0,T]. \tag{2.8}$$

Then the open-loop Nash equilibrium of Problem (DG)$_T$ admits the following closed-loop representation:

$$\tilde{u}^{(T)}(t) = \varTheta^{(T)}(t)\tilde{X}^{(T)}(t) - \widehat{R}^{-1}\Big[\widehat{B}^\top\eta^{(T)}(t) + r\Big], \quad t\in[0,T]. \tag{2.9}$$

Remark 2.4. Equation (2.5) can be rewritten componentwise as follows:

$$\begin{cases}\dot{\mathit{\Pi}}_1^{(T)}+\mathit{\Pi}_1^{(T)}A+A^\top\mathit{\Pi}_1^{(T)}+Q_1-\mathit{\Pi}_1^{(T)}S_1\mathit{\Pi}_1^{(T)}-\mathit{\Pi}_1^{(T)}S_2\mathit{\Pi}_2^{(T)}=0,\\ \dot{\mathit{\Pi}}_2^{(T)}+\mathit{\Pi}_2^{(T)}A+A^\top\mathit{\Pi}_2^{(T)}+Q_2-\mathit{\Pi}_2^{(T)}S_1\mathit{\Pi}_1^{(T)}-\mathit{\Pi}_2^{(T)}S_2\mathit{\Pi}_2^{(T)}=0,\\ \mathit{\Pi}_1^{(T)}(T)=G_1,\quad \mathit{\Pi}_2^{(T)}(T)=G_2.\end{cases} \tag{2.10}$$

Thus, the equations for $\mathit{\Pi}_1^{(T)}(\cdot)$ and $\mathit{\Pi}_2^{(T)}(\cdot)$ are coupled and non-symmetric, indicating that they are generally non-symmetric matrices. We can also express (2.7) and (2.8) componentwise as follows:

$$\begin{cases}\dot{\eta}_1+(A^\top-\mathit{\Pi}_1^{(T)}S_1)\eta_1-\mathit{\Pi}_1^{(T)}S_2\eta_2-\mathit{\Pi}_1^{(T)}B_1R_{11}^{-1}r_{11}-\mathit{\Pi}_1^{(T)}B_2R_{22}^{-1}r_{22}+\mathit{\Pi}_1^{(T)}b+q_1=0,\\ \dot{\eta}_2+(A^\top-\mathit{\Pi}_2^{(T)}S_2)\eta_2-\mathit{\Pi}_2^{(T)}S_1\eta_1-\mathit{\Pi}_2^{(T)}B_1R_{11}^{-1}r_{11}-\mathit{\Pi}_2^{(T)}B_2R_{22}^{-1}r_{22}+\mathit{\Pi}_2^{(T)}b+q_2=0,\\ \eta_1(T)=g_1,\quad \eta_2(T)=g_2,\end{cases}$$

$$\mathit{\Theta}^{(T)}\equiv\begin{pmatrix}\mathit{\Theta}_1^{(T)}\\ \mathit{\Theta}_2^{(T)}\end{pmatrix}=-\begin{pmatrix}R_{11}^{-1}B_1^\top\mathit{\Pi}_1^{(T)}\\ R_{22}^{-1}B_2^\top\mathit{\Pi}_2^{(T)}\end{pmatrix}.$$

3. Stabilizability and the algebraic Riccati equation

In this section, we will make some additional preparations. Consider the following ordinary differential system:

$$\dot{X}(t)=AX(t),\quad t\geqslant 0.$$

The system, or equivalently the matrix A, is said to be *stable* if all the eigenvalues of A have negative real parts. In this case, there exists constants $K,\lambda>0$ such that the fundamental solution $\Phi(t)=\mathrm{e}^{At}$ satisfies the following bound:

$$|\mathrm{e}^{At}|\leqslant K\mathrm{e}^{-\lambda t},\quad \forall t\geqslant 0.$$

Recall the notation $B\triangleq(B_1,B_2)$. The controlled system

$$\dot{X}(t)=AX(t)+Bu(t), \tag{3.1}$$

denoted by $[A;B]$, is said to be *stabilizable*, if there exists a matrix $\mathit{\Theta}\in\mathbb{R}^{m\times n}$ such that $A+B\mathit{\Theta}$ is stable. In this case, $\mathit{\Theta}$ is called a *stabilizer* of $[A;B]$.

When considering the infinite horizon version of the nonzero-sum LQ Nash differential game, we encounter the following system of AREs:

$$\begin{cases}\mathit{\Pi}_1A+A^\top\mathit{\Pi}_1+Q_1-\mathit{\Pi}_1S_1\mathit{\Pi}_1-\mathit{\Pi}_1S_2\mathit{\Pi}_2=0,\\ \mathit{\Pi}_2A+A^\top\mathit{\Pi}_2+Q_2-\mathit{\Pi}_2S_1\mathit{\Pi}_1-\mathit{\Pi}_2S_2\mathit{\Pi}_2=0.\end{cases} \tag{3.2}$$

Several studies, such as [4, 29–32], have demonstrated that the eigenvalues and eigenstructure of the Hamiltonian matrix (2.1) play a crucial role in solving (3.2). Let $\mathit{\Pi}\triangleq\begin{pmatrix}\mathit{\Pi}_1\\ \mathit{\Pi}_2\end{pmatrix}$, then we can rewrite (3.2) in a more compact form as follows:

$$\widehat{A}^\top\mathit{\Pi}+\mathit{\Pi}A+Q-\mathit{\Pi}S\mathit{\Pi}=0. \tag{3.3}$$

Let the Hamilton matrix M, defined in (2.1), have the Jordan canonical form

$$M = EJE^{-1} = E \begin{pmatrix} J_{c_1}(\nu_1) & & & \\ & J_{c_2}(\nu_2) & & \\ & & \ddots & \\ & & & J_{c_s}(\nu_s) \end{pmatrix} E^{-1},$$

where

$$J_{c_i}(\nu_i) = \begin{pmatrix} \nu_i & 1 & & & \\ & \nu_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \nu_i & 1 \\ & & & & \nu_i \end{pmatrix}, \quad i = 1, 2, \cdots, s$$

is a $c_i \times c_i$ Jordan block, with $c_1 + c_2 + \cdots + c_s = 3n$. Without loss of generality, we assume that the diagonal elements $\mu_1, \mu_2, \cdots, \mu_{3n}$ of J satisfy $Re(\mu_1) \geqslant Re(\mu_2) \geqslant \cdots \geqslant Re(\mu_{3n})$. Denote

$$\begin{aligned} \mathcal{A}(\mathit{\Pi}) &\triangleq A - S\mathit{\Pi} = A - S_1\mathit{\Pi}_1 - S_2\mathit{\Pi}_2, \\ \mathcal{B}(\mathit{\Pi}) &\triangleq \widehat{A}^\top - \mathit{\Pi} S = \begin{pmatrix} A^\top - \mathit{\Pi}_1 S_1 & -\mathit{\Pi}_1 S_2 \\ -\mathit{\Pi}_2 S_1 & A^\top - \mathit{\Pi}_2 S_2 \end{pmatrix}. \end{aligned}$$

We introduce the following definition.

Definition 3.1. A solution $\mathit{\Pi}$ of ARE (3.3) is called *strongly stabilizing* if $\sigma(\mathcal{A}(\mathit{\Pi})) \subseteq \mathbb{C}^-$ and $\sigma(\mathcal{B}(\mathit{\Pi})) \subseteq \mathbb{C}^-$, *i.e.*, both $\mathcal{A}(\mathit{\Pi})$ and $\mathcal{B}(\mathit{\Pi})$ are stable.

Remark 3.2. The matrix $\mathcal{B}(\mathit{\Pi})$ will frequently appear throughout our analysis. Its properties will aid in analyzing the non-homogeneous term in the unique Nash equilibrium, particularly in relation to the ODE $\eta^{(T)}$ defined in (2.7). The stability of the matrix $\mathcal{B}(\mathit{\Pi})$ will facilitate establishing the connection between the DRE (2.5) and the ARE (3.3) in Section 4, as well as in demonstrating the exponential turnpike property of Problem $(\mathrm{DG})_T$ in Section 5.

The following lemma concerning the solution of the Sylvester equation is introduced here, as it will be needed in later sections. For the proof, see [30].

Lemma 3.3. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$, and $C \in \mathbb{C}^{n\times m}$. The Sylvester equation*

$$AX + XB = C$$

has a unique solution X if and only if the matrices $-A$ and B have no common eigenvalue.

The following result establishes the unique solvability of the ARE (3.3). While the eigenvalue and eigenstructure of M are taken from [31], our condition is more straightforward to verify.

Theorem 3.4. *The ARE (3.3) admits a unique real strongly stabilizing solution $\mathit{\Pi}$ if and only if $Re(\mu_n) > 0 > Re(\mu_{n+1})$ and the matrix*

$$\begin{pmatrix} I_n & 0 & 0 \end{pmatrix} E \begin{pmatrix} I_n \\ 0 \\ 0 \end{pmatrix} \tag{3.4}$$

is invertible.

Proof. Necessity. Suppose that $\mathit{\Pi}$ is the strongly stabilizing solution of (3.3). Since $\mathcal{A}(\mathit{\Pi})$ and $\mathcal{B}(\mathit{\Pi})$ are stable, we see from Lemma 3.3 that the Sylvester-type equation

$$\mathcal{A}(\mathit{\Pi})P + P\mathcal{B}(\mathit{\Pi}) - S = 0. \tag{3.5}$$

admits a unique solution P. A straightforward computation shows that

$$M\begin{pmatrix} I_n & P \\ \mathit{\Pi} & I_{2n} + \mathit{\Pi}P \end{pmatrix} = \begin{pmatrix} I_n & P \\ \mathit{\Pi} & I_{2n} + \mathit{\Pi}P \end{pmatrix}\begin{pmatrix} -\mathcal{A}(\mathit{\Pi}) & 0 \\ 0 & \mathcal{B}(\mathit{\Pi}) \end{pmatrix}.$$

Let $-\mathcal{A}(\mathit{\Pi})$ and $\mathcal{B}(\mathit{\Pi})$ admit the following Jordan canonical form

$$-\mathcal{A}(\mathit{\Pi}) = E_1 J_1 E_1^{-1}, \quad \mathcal{B}(\mathit{\Pi}) = E_2 J_2 E_2^{-1},$$

where J_i $(i = 1, 2)$ are the corresponding Jordan blocks, and $\sigma(J_1) \subseteq \mathbb{C}^+$, $\sigma(J_2) \subseteq \mathbb{C}^-$. Then

$$M = \begin{pmatrix} E_1 & PE_2 \\ \mathit{\Pi}E_1 & E_2 + \mathit{\Pi}PE_2 \end{pmatrix}\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}\begin{pmatrix} E_1 & PE_2 \\ \mathit{\Pi}E_1 & E_2 + \mathit{\Pi}PE_2 \end{pmatrix}^{-1}.$$

This proves the necessity part.

Sufficiency. We can rewrite the Jordan canonical form as follows:

$$M\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \tag{3.6}$$

where $\sigma(J_1) = \{\mu_1, \mu_2, \cdots, \mu_n\} \subseteq \mathbb{C}^+$, $\sigma(J_2) = \{\mu_{n+1}, \mu_{n+2}, \cdots, \mu_{3n}\} \subseteq \mathbb{C}^-$, $E_{11} \in \mathbb{C}^{n\times n}$, $E_{12} \in \mathbb{C}^{n\times 2n}$, $E_{21} \in \mathbb{C}^{2n\times n}$, $E_{22} \in \mathbb{C}^{2n\times 2n}$, and the matrix E_{11} is invertible. Let $\mathit{\Pi} = E_{21}E_{11}^{-1}$. Since

$$\begin{aligned} -AE_{11} + SE_{21} &= E_{11}J_1, \\ QE_{11} + \widehat{A}^\top E_{21} &= E_{21}J_1, \end{aligned}$$

we have

$$Q + \widehat{A}^\top \mathit{\Pi} = E_{21}E_{11}^{-1}E_{11}J_1E_{11}^{-1} = -\mathit{\Pi}\mathcal{A}(\mathit{\Pi}) = -\mathit{\Pi}(A - S\mathit{\Pi}).$$

Thus $\mathit{\Pi} = E_{21}E_{11}^{-1}$ is the solution of the ARE (3.3). Moreover, $\mathcal{A}(\mathit{\Pi})$ is stable. To see that $\mathit{\Pi}$ is a strongly stabilizing solution, consider the rest part in (3.6):

$$\begin{aligned} -AE_{12} + SE_{22} &= E_{12}J_2, \\ QE_{12} + \widehat{A}^\top E_{22} &= E_{22}J_2. \end{aligned}$$

Then we have

$$\begin{aligned} (E_{22} - \mathit{\Pi}E_{12})J_2 &= QE_{12} + \widehat{A}^\top E_{22} - \mathit{\Pi}(-AE_{12} + SE_{22}) \\ &= (Q + \mathit{\Pi}A)E_{12} + (\widehat{A}^\top - \mathit{\Pi}S)E_{22} \\ &= (\mathit{\Pi}S - \widehat{A}^\top)\mathit{\Pi}E_{12} + (\widehat{A}^\top - \mathit{\Pi}S)E_{22} \\ &= \mathcal{B}(\mathit{\Pi})(E_{22} - \mathit{\Pi}E_{12}). \end{aligned}$$

Since $E_{22} - \mathit{\Pi} E_{12} = E_{22} - E_{21}E_{11}^{-1}E_{12}$ is invertible, it follows that $\mathcal{B}(\mathit{\Pi})$ is stable. Consequently, $\mathit{\Pi}$ is strongly stabilizing.

Finally, to see that the solution is real and unique, let us consider the conjugate of (3.6). We have

$$M \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{pmatrix} = \begin{pmatrix} \bar{E}_{11} & \bar{E}_{12} \\ \bar{E}_{21} & \bar{E}_{22} \end{pmatrix} \begin{pmatrix} \bar{J}_1 & 0 \\ 0 & \bar{J}_2 \end{pmatrix},$$

where the matrices $\bar{E}_{ij},\ \bar{J}_i,\ i,j = 1,2$, denote the conjugates of $E_{ij},\ J_i$, respectively. Since $-\bar{J}_1$ and $\bar{J}_2$ are stable, we can use the same argument as before to show that $\bar{\mathit{\Pi}} = \bar{E}_{21}\bar{E}_{11}^{-1} = \overline{E_{21}E_{11}^{-1}}$, the conjugate of $\mathit{\Pi}$, is also a strongly stabilizing solution to the ARE (3.3). Now suppose that $\tilde{\mathit{\Pi}}_1$ and $\tilde{\mathit{\Pi}}_2$ are two strongly stabilizing solutions of (3.3). Then

$$\begin{aligned} 0 &= \widehat{A}^\top(\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2) + (\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2)A - \tilde{\mathit{\Pi}}_1 S \tilde{\mathit{\Pi}}_1 + \tilde{\mathit{\Pi}}_2 S \tilde{\mathit{\Pi}}_2 \\ &= (\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2)(A - S\tilde{\mathit{\Pi}}_1) + (\widehat{A}^\top - \tilde{\mathit{\Pi}}_2 S)(\mathit{\Pi}_1 - \mathit{\Pi}_2) \\ &= (\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2)\mathcal{A}(\tilde{\mathit{\Pi}}_1) + \mathcal{B}(\tilde{\mathit{\Pi}}_2)(\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2). \end{aligned} \tag{3.7}$$

Since $\mathcal{A}(\tilde{\mathit{\Pi}}_1)$ and $\mathcal{B}(\tilde{\mathit{\Pi}}_2)$ are stable, it follows from the uniqueness part of Lemma 3.3 that $\tilde{\mathit{\Pi}}_1 - \tilde{\mathit{\Pi}}_2 \equiv 0$. Consequently $\mathit{\Pi} = \bar{\mathit{\Pi}}$ is the unique real solution. ☐

We now further introduce the following assumption.

(A3) The Hamilton matrix M satisfies $Re(\mu_n) > 0 > Re(\mu_{n+1})$, and the matrix (3.4) is invertible.

Remark 3.5. Assumption (A3) plays several key roles in establishing the turnpike property of Problem $(\mathrm{DG})_T$. As shown in Theorem 3.4, (A3) is the sufficient and necessary condition for the existence of a unique real strongly stabilizing solution to the ARE (3.3). This is essential for deriving the convergence relationship between the solutions of the DRE (2.5) and the ARE (3.3). Further, (A3) guarantees the solvability of the Problem (O) in Section 5.

Remark 3.6. Let (A1) and (A3) hold. Then the matrix

$$\mathit{\Theta} \triangleq -\widehat{R}^{-1}\widehat{B}^\top \mathit{\Pi} = -\begin{pmatrix} R_{11}^{-1}B_1^\top \mathit{\Pi}_1 \\ R_{22}^{-1}B_2^\top \mathit{\Pi}_2 \end{pmatrix} \tag{3.8}$$

is a stabilizer of the system (3.1). This can be seen from the fact $\mathcal{A}(\mathit{\Pi}) = A - S\mathit{\Pi} = A + B\mathit{\Theta}$.

It should be noted that there has been some similar research on the stabilizing solution of the nonsymmetric ARE (3.3), see [4, 31]. We will briefly introduce their work and highlight its connection to our assumption (A3). First, we present the following definition.

Definition 3.7. We say that a matrix $N \in \mathbb{R}^{3n\times 3n}$ has an n-dimensional *stable invariant subspace*, if there exist a matrix $\mathit{\Gamma} \in \mathbb{R}^{n\times n}$ with $\sigma(\mathit{\Gamma}) \subseteq \mathbb{C}^-$ and a full column rank matrix $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ with $Z_1 \in \mathbb{R}^{n\times n}$ and $Z_2 \in \mathbb{R}^{2n\times n}$ such that

$$M \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \mathit{\Gamma}. \tag{3.9}$$

In addition, if Z_1 is invertible, we say that the range of $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ is a *stable graph subspace*.

The following assumption was used to characterize the unique strongly stabilizing solution of the ARE (3.3) in Theorem 7.10 of [31].

(A3′) The matrix $-M$ possesses an n-dimensional stable graph subspace and has $2n$ eigenvalues with positive real parts.

Remark 3.8. In fact, the original assumption in [31] requires that $-M$ has $2n$ eigenvalues with non-negative real parts. Under this condition, the ARE (3.3) admits a unique stabilizing solution $\varPi$ such that $\mathcal{A}(\varPi)$ is stable, but $\mathcal{B}(\varPi)$ may not be. To ensure the stability of $\mathcal{B}(\varPi)$, we specifically consider the case where $-M$ has $2n$ eigenvalues with positive real parts.

The following result shows that assumption (A3′) is equivalent to assumption (A3). It is evident that (A3) is easier to verify than (A3′). Therefore, we will use (A3) instead of (A3′) throughout this article, even though both can be employed to characterize the unique strongly stabilizing solution of the ARE (3.3).

Proposition 3.9. *For the Hamilton matrix M, (A3′) holds if and only if (A3) holds.*

Proof. Necessity. Suppose that $-M$ has an n-dimensional stable graph subspace. Then there is a matrix $\varGamma \in \mathbb{R}^{n\times n}$ with $\sigma(\varGamma) \subseteq \mathbb{C}^+$ such that (3.9) holds. Since Z_1 is invertible, we have

$$M\begin{pmatrix} I_n \\ Z_2 Z_1^{-1} \end{pmatrix} = \begin{pmatrix} I_n \\ Z_2 Z_1^{-1} \end{pmatrix} Z_1 \varGamma Z_1^{-1}.$$

Let $Z \triangleq Z_2 Z_1^{-1}$. Then

$$\begin{pmatrix} -Z & I_{2n} \end{pmatrix} M \begin{pmatrix} I_n \\ Z \end{pmatrix} = \begin{pmatrix} -Z & I_{2n} \end{pmatrix} \begin{pmatrix} I_n \\ Z \end{pmatrix} Z_1 \varGamma Z_1^{-1}.$$

A simple calculation yields

$$\widehat{A}^\top Z + ZA + Q - ZSZ = 0,$$

which implies that Z is a solution of (3.3). Moreover, it is straightforward to verify that

$$M\begin{pmatrix} I_n & 0 \\ Z & I_{2n} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ Z & I_{2n} \end{pmatrix}\begin{pmatrix} -\mathcal{A}(Z) & S \\ 0 & \mathcal{B}(Z) \end{pmatrix}.$$

Thus, $\sigma(\mathcal{A}(Z)) \subseteq \mathbb{C}^-$ and $\sigma(\mathcal{B}(Z)) \subseteq \mathbb{C}^-$, showing that Z is strongly stabilizing. Consequently, (A3) holds by Theorem 3.4.

Sufficiency. Let (A3) hold. From the Jordan canonical form (3.6), we obtain

$$M\begin{pmatrix} E_{11} \\ E_{21} \end{pmatrix} = \begin{pmatrix} E_{11} \\ E_{21} \end{pmatrix} J_1,$$

where $-J_1$ is stable. Since E_{11} is invertible, we have

$$M\begin{pmatrix} I_n \\ E_{21} E_{11}^{-1} \end{pmatrix} = \begin{pmatrix} I_n \\ E_{21} E_{11}^{-1} \end{pmatrix} E_{11} J_1 E_{11}^{-1}.$$

By Theorem 3.4, $E_{21}E_{11}^{-1}$ is a real matrix. It follows that $-M$ has an n-dimensional stable graph space, *i.e.*, (A3′) holds. □

4. Convergence of the Riccati equation

In this section, we explore the connection between the solutions to the DRE (2.5) and the ARE (3.3). The main result of this section is as follows, which plays a crucial role in establishing the turnpike property of Problem $(\mathrm{DG})_T$.

Theorem 4.1. *Let (A1)–(A3) hold. Let $\varPi^{(T)}(\cdot)$ be the unique solution of (2.5), and let $\varPi$ be the unique strongly stabilizing solution of (3.3). Then there exist positive constants $K, \lambda > 0$ such that*

$$|\varPi^{(T)}(t) - \varPi| \leqslant K\mathrm{e}^{-\lambda(T-t)}, \quad \forall 0 \leqslant t \leqslant T < \infty. \tag{4.1}$$

Proof. Let

$$\varUpsilon^{(T)}(t) \triangleq \varPi^{(T)}(T-t) = \begin{pmatrix} \varPi_1^{(T)}(T-t) \\ \varPi_2^{(T)}(T-t) \end{pmatrix}, \quad t \in [0, T].$$

Note that $\varUpsilon^{(T)}(t)$ does not depend on the choice of $T \geqslant t$, and we can express $\varUpsilon^{(T)}(t)$ as

$$\varUpsilon^{(T)}(t) = \begin{pmatrix} \varUpsilon_1^{(T)}(t) \\ \varUpsilon_2^{(T)}(t) \end{pmatrix}, \quad t \in [0, T].$$

This vector satisfies the following ODE:

$$\begin{cases} \dot{\varUpsilon}_1^{(T)} - \varUpsilon_1^{(T)} A - A^\top \varUpsilon_1^{(T)} - Q_1 + \varUpsilon_1^{(T)} S_1 \varUpsilon_1^{(T)} + \varUpsilon_1^{(T)} S_2 \varUpsilon_2^{(T)} = 0, \\ \dot{\varUpsilon}_2^{(T)} - \varUpsilon_2^{(T)} A - A^\top \varUpsilon_2^{(T)} - Q_2 + \varUpsilon_2^{(T)} S_1 \varUpsilon_1^{(T)} + \varUpsilon_2^{(T)} S_2 \varUpsilon_2^{(T)} = 0, \\ \varUpsilon_1^{(T)}(0) = G_1, \quad \varUpsilon_2^{(T)}(0) = G_2. \end{cases} \tag{4.2}$$

Step 1. We first show that $\lim_{t\to\infty} \varUpsilon^{(T)}(t) = \varPi$.

According to Lemma 2.3, the solution of (4.2) can be represented as follows:

$$\begin{aligned} \varUpsilon_1^{(T)}(t) &= \begin{pmatrix} 0 & I_n & 0 \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G_1 \\ G_2 \end{pmatrix} \left[\begin{pmatrix} I_n & 0 & 0 \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G_1 \\ G_2 \end{pmatrix} \right]^{-1}, \\ \varUpsilon_2^{(T)}(t) &= \begin{pmatrix} 0 & 0 & I_n \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G_1 \\ G_2 \end{pmatrix} \left[\begin{pmatrix} I_n & 0 & 0 \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G_1 \\ G_2 \end{pmatrix} \right]^{-1}. \end{aligned} \tag{4.3}$$

We can rewrite (4.3) as

$$\varUpsilon^{(T)}(t) = \begin{pmatrix} 0_{2n\times n} & I_{2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} \left[\begin{pmatrix} I_n & 0_{n\times 2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} \right]^{-1}. \tag{4.4}$$

According to the proof of Theorem 3.4, M can be represented as

$$M = \tilde{E} \begin{pmatrix} -\mathcal{A}(\varPi) & 0 \\ 0 & \mathcal{B}(\varPi) \end{pmatrix} \tilde{E}^{-1},$$

where

$$\tilde{E} = \begin{pmatrix} I_n & P \\ \mathit{\Pi} & I_{2n} + \mathit{\Pi} P \end{pmatrix}, \quad \tilde{E}^{-1} = \begin{pmatrix} I_n + P\mathit{\Pi} & -P \\ -\mathit{\Pi} & I_{2n} \end{pmatrix},$$

with $\mathit{\Pi}$ and P being the solutions of (3.3) and (3.5), respectively. Then

$$\mathrm{e}^{Mt} = \tilde{E} \begin{pmatrix} \mathrm{e}^{-\mathcal{A}(\mathit{\Pi})t} & 0 \\ 0 & \mathrm{e}^{\mathcal{B}(\mathit{\Pi})t} \end{pmatrix} \tilde{E}^{-1}.$$

A simple calculation shows that

$$\begin{cases} \begin{pmatrix} 0_{2n\times n} & I_{2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} = \mathit{\Pi} \mathrm{e}^{-\mathcal{A}(\mathit{\Pi})t}[I_n + P(\mathit{\Pi} - G)] - (I_{2n} + \mathit{\Pi} P)\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G), \\ \begin{pmatrix} I_n & 0_{n\times 2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} = \mathrm{e}^{-\mathcal{A}(\mathit{\Pi})t}[I_n + P(\mathit{\Pi} - G)] - P\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G). \end{cases}$$

Then as $t \to \infty$, we have

$$\begin{aligned} &\begin{pmatrix} 0_{2n\times n} & I_{2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} - \mathit{\Pi} \begin{pmatrix} I_n & 0_{n\times 2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix} \\ &\quad = \mathit{\Pi} \mathrm{e}^{-\mathcal{A}(\mathit{\Pi})t}[I_n + P(\mathit{\Pi} - G)] - (I_{2n} + \mathit{\Pi} P)\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G) \\ &\qquad - \mathit{\Pi} \Big\{ \mathrm{e}^{-\mathcal{A}(\mathit{\Pi})t} \Big[I_n + P(\mathit{\Pi} - G) \Big] - P\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G) \Big\} \\ &\quad = -(I_{2n} + \mathit{\Pi} P)\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G) + \mathit{\Pi} P\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G) \\ &\quad = -\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}(\mathit{\Pi} - G) \to 0. \end{aligned}$$

Since the inverse of $\begin{pmatrix} I_n & 0_{n\times 2n} \end{pmatrix} \mathrm{e}^{Mt} \begin{pmatrix} I_n \\ G \end{pmatrix}$ is bounded in $t \in [0, \infty)$, it follows that $\lim_{t\to\infty} \varUpsilon^{(T)}(t) = \mathit{\Pi}$.

Step 2. Set $\varDelta(t) = \varUpsilon^{(T)}(t) - \mathit{\Pi}$. Then

$$\begin{aligned} \dot{\varDelta}(t) &= \widehat{A}^\top \varDelta(t) + \varDelta(t) A - \varUpsilon S \varUpsilon + \mathit{\Pi} S \mathit{\Pi} \\ &= (\widehat{A}^\top - \mathit{\Pi} S)\varDelta(t) + \varDelta(t)(A - S\mathit{\Pi}) - \varDelta(t) S \varDelta(t) \\ &= \mathcal{B}(\mathit{\Pi})\varDelta(t) + \varDelta(t)\mathcal{A}(\mathit{\Pi}) - \varDelta(t) S \varDelta(t). \end{aligned}$$

Let $\Lambda(t) \triangleq -\varDelta(t) S \varDelta(t)$. Then there exists a constant $\gamma > 0$ such that

$$|\Lambda(t)| \leqslant \gamma |\varDelta(t)|^2. \tag{4.5}$$

By the variation of constants formula, for $0 \leqslant s \leqslant t$,

$$\varDelta(t) = \mathrm{e}^{\mathcal{B}(\mathit{\Pi})(t-s)} \varDelta(s) \mathrm{e}^{\mathcal{A}(\mathit{\Pi})(t-s)} + \int_s^t \mathrm{e}^{\mathcal{B}(\mathit{\Pi})(t-r)} \Lambda(r) \mathrm{e}^{\mathcal{A}(\mathit{\Pi})(t-r)} \mathrm{d}r. \tag{4.6}$$

Since the matrices $\mathcal{A}(\mathit{\Pi})$ and $\mathcal{B}(\mathit{\Pi})$ are stable, there exist positive constants α_i, β_i $(i = 1, 2)$ such that

$$|\mathrm{e}^{\mathcal{A}(\mathit{\Pi})t}| \leqslant \alpha_1 \mathrm{e}^{-\beta_1 t}, \quad |\mathrm{e}^{\mathcal{B}(\mathit{\Pi})t}| \leqslant \alpha_2 \mathrm{e}^{-\beta_2 t}, \quad \forall t \geqslant 0.$$

Take $\alpha=\max\{\alpha_1,\alpha_2\}$ and $\beta=\min\{\beta_1,\beta_2\}$ so that

$$|\mathrm{e}^{\mathcal{A}(\varPi)t}|\leqslant\alpha\mathrm{e}^{-\beta t},\quad|\mathrm{e}^{\mathcal{B}(\varPi)t}|\leqslant\alpha\mathrm{e}^{-\beta t},\quad\forall t\geqslant 0.\tag{4.7}$$

From (4.5), (4.6) and (4.7), we have

$$|\varDelta(t)|\leqslant\alpha^2\mathrm{e}^{-2\beta(t-s)}\bigg\{|\varDelta(s)|+\gamma\int_s^t\mathrm{e}^{-2\beta(s-r)}|\varDelta(r)|^2\mathrm{d}r\bigg\}.$$

Set $h(t)=\mathrm{e}^{2\beta(t-s)}|\varDelta(t)|$ with $s\geqslant 0$ being fixed. Then we have

$$h(t)\leqslant\alpha^2|\varDelta(s)|+\gamma\alpha^2\int_s^t\mathrm{e}^{-2\beta(r-s)}h^2(r)\mathrm{d}r,\quad t\geqslant s.$$

Set

$$\kappa(t)=\alpha^2|\varDelta(s)|+\gamma\alpha^2\int_s^t\mathrm{e}^{-2\beta(r-s)}h^2(r)\mathrm{d}r,\quad p(t)=\frac{1}{\kappa(t)},\quad t\geqslant s.$$

Then

$$\dot p(t)=-\frac{\dot\kappa(t)}{\kappa^2(t)}=-\frac{\gamma\alpha^2\mathrm{e}^{-2\beta(t-s)}h^2(t)}{\kappa^2(t)}\geqslant-\gamma\alpha^2\mathrm{e}^{-2\beta(t-s)}.$$

Since $\lim_{s\to\infty}\varDelta(s)=0$, we can choose a large enough s such that

$$p(s)=\frac{1}{\kappa(s)}=\frac{1}{\alpha^2|\varDelta(s)|}\geqslant 1+\frac{\gamma\alpha^2}{2\beta}.$$

Then

$$p(t)\geqslant p(s)-\int_s^t\gamma\alpha^2\mathrm{e}^{-2\beta(r-s)}\mathrm{d}r=p(s)+\frac{\gamma\alpha^2}{2\beta}\Big[\mathrm{e}^{-2\beta(t-s)}-1\Big]\geqslant 1.$$

Thus

$$h(t)\leqslant\kappa(t)=\frac{1}{p(t)}\leqslant 1,\quad\forall t\geqslant s.$$

It follows that

$$|\varDelta(t)|=\mathrm{e}^{-2\beta(t-s)}h(t)\leqslant K\mathrm{e}^{-\lambda t},\quad\forall t\geqslant 0,$$

for some constants $K,\lambda>0$. Note that $\varUpsilon^{(T)}(t)=\varPi^{(T)}(T-t)$, the desired result follows. □

5. The turnpike property

We present the main results of the paper in this section. First, we now introduce the following static optimization problem, whose solution can be viewed as the turnpike limit of Problem $(\mathrm{DG})_T$. Let

$$\mathscr{V}\triangleq\big\{(x,u_1,u_2)\in\mathbb{R}^n\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\mid Ax+B_1u_1+B_2u_2+b=0\big\},$$

and

$$
\begin{aligned}
F_1(x,u_1,u_2) \triangleq & \frac{1}{2}\Big[\langle Q_1x,x\rangle + \langle R_{11}u_1,u_1\rangle + \langle R_{12}u_2,u_2\rangle \\
& + 2\langle q_1,x\rangle + 2\langle r_{11},u_1\rangle + 2\langle r_{12},u_2\rangle\Big], \\
F_2(x,u_1,u_2) \triangleq & \frac{1}{2}\Big[\langle Q_2x,x\rangle + \langle R_{21}u_1,u_1\rangle + \langle R_{22}u_2,u_2\rangle \\
& + 2\langle q_2,x\rangle + 2\langle r_{21},u_1\rangle + 2\langle r_{22},u_2\rangle\Big].
\end{aligned}
$$

We consider the following two-objective static optimization problem.

Problem (O). Find a triple $(x^*,u_1^*,u_2^*)\in\mathscr{V}$ such that

$$
\begin{aligned}
\tilde{V}_1 &\equiv F_1(x^*,u_1^*,u_2^*) \leqslant F_1(x,u_1,u_2^*), \quad \forall (x,u_1,u_2^*)\in\mathscr{V}, \\
\tilde{V}_2 &\equiv F_2(x^*,u_1^*,u_2^*) \leqslant F_2(x,u_1^*,u_2), \quad \forall (x,u_1^*,u_2)\in\mathscr{V}.
\end{aligned} \tag{5.1}
$$

For the above static optimization problem, we have the following result.

Proposition 5.1. *Let (A1) and (A3) hold. The Problem (O) admits a unique solution. Moreover, $(x^*,u_1^*,u_2^*)\in\mathscr{V}$ is the solution if and only if for some Lagrange multipliers $\lambda_1^*,\lambda_2^*\in\mathbb{R}^n$, the following hold:*

$$
\begin{cases}
Ax^* + B_1u_1^* + B_2u_2^* + b = 0, \\
Q_ix^* + A^\top\lambda_i^* + q_i = 0, \\
R_{ii}u_i^* + B_i^\top\lambda_i^* + r_{ii} = 0, \quad i=1,2.
\end{cases} \tag{5.2}
$$

Proof. According to Remark 3.6, the system $[A;(B_1,B_2)]$ is stabilizable. Then by the Hautus lemma for stabilizability, the matrix $(A-\lambda I_n, B_1, B_2)$ is of full rank for any λ with $Re(\lambda)\geqslant 0$. Now, by taking $\lambda=0$, it easy to see that the feasible set $\mathscr{V}$ of Problem (O) is non-empty.

Suppose that (x^*,u_1^*,u_2^*) is a solution of Problem (O). For a given u_2^*, we can just consider the following static optimization problem

$$
\begin{cases}
\text{Minimize } F_1(x,u_1,u_2^*), \\
\text{subject to } Ax + B_1u_1 + B_2u_2^* + b = 0.
\end{cases}
$$

This is a single-objective optimization problem, which consisting of objective function $F_1(x,u_1,u_2^*)$ and the constraint condition $(x,u_1,u_2^*)\in\mathscr{V}$. It can be solved by the Lagrange multiplier method. We can construct a Lagrange function for some multipliers $\lambda_1^*\in\mathbb{R}^n$ as

$$
L_1(x,u_1,\lambda_1) \triangleq F_1(x,u_1,u_2^*) + \lambda_1^\top(Ax + B_1u_1 + B_2u_2^* + b).
$$

Thus, (x^*,u_1^*,λ_1^*) is the optimal solution for $L_1(x,u_1,\lambda_1)$ if and only if

$$
\begin{cases}
0 = Q_1x^* + A^\top\lambda_1^* + q_1, \\
0 = R_{11}u_1^* + B_1^\top\lambda_1^* + r_{11}.
\end{cases}
$$

Similarly, we can also construct a Lagrange function for some multipliers $\lambda_2^*\in\mathbb{R}^n$ as

$$
L_2(x,u_2,\lambda_2) \triangleq F_2(x,u_1^*,u_2) + \lambda_2^\top(Ax + B_1u_1^* + B_2u_2 + b),
$$

then $(x^*, u_2^*, \lambda_2^*)$ is the optimal solution for $L_2(x, u_2, \lambda_2)$ if and only if

$$\begin{cases} 0 = Q_2 x^* + A^\top \lambda_2^* + q_2, \\ 0 = R_{22} u_2^* + B_2^\top \lambda_2^* + r_{22}. \end{cases}$$

Thus, we obtain the system of (5.2).

Next, we consider the unique solvability of the system (5.2). Since $R_{ii} \in \mathbb{S}_+^{m_i}$ for $i = 1, 2$, then we have

$$u_i^* = -R_{ii}^{-1}(B_i^\top \lambda_i^* + r_{ii}).$$

Substituting the above equations into (5.2), then we have

$$M \begin{pmatrix} x^* \\ \lambda_1^* \\ \lambda_2^* \end{pmatrix} = - \begin{pmatrix} B_1 R_{11}^{-1} r_{11} + B_2 R_{22}^{-1} r_{22} - b \\ q_1 \\ q_2 \end{pmatrix}.$$

Under (A3), all of the eigenvalues of M are non-zero, which implies that the matrix M is invertible. As a result, the above equation is uniquely solvable, leading to the uniqueness of optimal tuple $(x^*, u_1^*, u_2^*) \in \mathscr{V}$. $\square$

Let $\tilde{X}^{(T)}(\cdot)$ be the open-loop optimal state process, $\tilde{u}^{(T)}(\cdot) \triangleq \begin{pmatrix} \tilde{u}_1^{(T)}(\cdot) \\ \tilde{u}_2^{(T)}(\cdot) \end{pmatrix}$ be the open-loop Nash equilibrium of Problem $(\text{DG})_T$ for the given initial state x. Let $\tilde{Y}^{(T)}(\cdot) = \begin{pmatrix} \tilde{Y}_1^{(T)}(\cdot) \\ \tilde{Y}_2^{(T)}(\cdot) \end{pmatrix}$ be the solution to the corresponding equation in (2.3). Let (x^*, u_1^*, u_2^*) be the unique solution of Problem (O) and $\lambda_1^*, \lambda_2^* \in \mathbb{R}^n$ be the corresponding Lagrange multipliers. Define

$$\tilde{X}(t) = \tilde{X}^{(T)}(t) - x^*, \quad \tilde{u}_i(t) = \tilde{u}_i^{(T)}(t) - u_i^*, \quad \tilde{Y}_i(t) = \tilde{Y}_i^{(T)}(t) - \lambda_i^*, \quad i = 1, 2. \tag{5.3}$$

Note that

$$\tilde{Y}_i(T) = G_i \tilde{X}^{(T)}(T) + g_i - \lambda_i^* = G_i \tilde{X}(T) + G_i x^* + g_i - \lambda_i^* = G_i \tilde{X}(T) + g_i^*, \quad i = 1, 2,$$

with $g^* = \begin{pmatrix} g_1^* \\ g_2^* \end{pmatrix} \triangleq \begin{pmatrix} G_1 x^* + g_1 - \lambda_1^* \\ G_2 x^* + g_2 - \lambda_2^* \end{pmatrix}$. Taking the notation (5.3) into (2.3), (2.4), and using the equalities in (5.2), we can obtain by a straightforward calculation that

$$\begin{cases} \dot{\tilde{X}}(t) = A\tilde{X}(t) + B_1 \tilde{u}_1(t) + B_2 \tilde{u}_2(t), \\ \dot{\tilde{Y}}_i(t) = -\Big[A^\top \tilde{Y}_i(t) + Q_i \tilde{X}(t)\Big], \\ \tilde{X}(0) = x - x^*, \quad \tilde{Y}_i(T) = G_i \tilde{X}(T) + g_i^*, \\ B_i^\top \tilde{Y}_i(t) + R_{ii} \tilde{u}_i(t) = 0, \quad t \in [0, T], \quad i = 1, 2, \end{cases} \tag{5.4}$$

Now, by comparing the coefficients of (5.4) to (2.3) and (2.4) in Lemma 2.3(ii), we can see that $\tilde{X}(\cdot)$ and $\tilde{u}(\cdot) \triangleq \begin{pmatrix} \tilde{u}_1(\cdot) \\ \tilde{u}_2(\cdot) \end{pmatrix}$ are the optimal state process and open-loop Nash equilibrium of a nonzero-sum LQ differential games with state equation:

$$\begin{cases}\dot{X}(t)=AX(t)+B_1u_1(t)+B_2u_2(t), \quad t\in[0,T],\\ X(0)=x-x^*\equiv\tilde{x},\end{cases}$$

and cost functionals

$$\begin{aligned}J_i(x;u_1(\cdot),u_2(\cdot))=&\frac{1}{2}\Big\{\langle G_iX(T),X(T)\rangle+2\langle g_i^*,X(T)\rangle\\ &+\int_0^T\Big[\langle Q_iX(t),X(t)\rangle+\langle R_{ii}u_i(t),u_i(t)\rangle\Big]\mathrm{d}t\Big\}, \quad i=1,2.\end{aligned}$$

Then we have the following result.

Theorem 5.2. *Let (A1)–(A3) hold. Let $\varPi^{(T)}(\cdot)$ be the solution to DRE (2.5). Define*

$$\tilde{\varTheta}^{(T)}(t)\triangleq-\widehat{R}^{-1}\widehat{B}^\top\varPi^{(T)}(t), \tag{5.5}$$

and let $\varphi(\cdot)$ be the solution to the ODE

$$\begin{cases}\dot{\varphi}(t)+[\widehat{A}^\top-\varPi^{(T)}(t)S]\varphi(t)=0,\\ \varphi(T)=g^*.\end{cases} \tag{5.6}$$

Then the process $\tilde{u}(\cdot)$ defined in (5.3) admits a closed-representation as

$$\tilde{u}(\cdot)=\tilde{\varTheta}^{(T)}(\cdot)\tilde{X}(\cdot)-\widehat{R}^{-1}\widehat{B}^\top\varphi(\cdot). \tag{5.7}$$

To prove the turnpike property of Problem $(\mathrm{DG})_T$, we also need the following lemma.

Lemma 5.3. *Let (A1)–(A3) hold. The solution $\varphi(\cdot)$ to the ODE (5.6) satisfies*

$$|\varphi(t)|\leqslant K\mathrm{e}^{-\lambda(T-t)}, \quad \forall 0\leqslant t\leqslant T, \tag{5.8}$$

for some constants $K,\lambda>0$ independent of T.

Proof. Let $\varPi$ be the strongly stabilizing solution of ARE (3.3). Notice that

$$\widehat{A}^\top-\varPi^{(T)}(t)S=\mathcal{B}[\varPi^{(T)}(t)].$$

Then we can write (5.6) as

$$\begin{cases}\dot{\varphi}(t)=-\mathcal{B}(\varPi)\varphi(t)-\Big\{\mathcal{B}\Big[\varPi^{(T)}(t)\Big]-\mathcal{B}(\varPi)\Big\}\varphi(t),\\ \varphi(T)=g^*.\end{cases}$$

By the variation of constants formula, we have

$$\varphi(t)=\mathrm{e}^{\mathcal{B}(\varPi)(T-t)}\Big[g^*+\int_t^T\mathrm{e}^{\mathcal{B}(\varPi)(s-T)}\rho(s)\mathrm{d}s\Big], \tag{5.9}$$

where $\rho(s)=\{\mathcal{B}[\Pi^{(T)}(s)]-\mathcal{B}(\Pi)\}\varphi(s)$. By Theorem 4.1 and $\mathcal{B}(\Pi)$ is stable, so there exist constants $K_1,\lambda>0$ independent of T such that

$$|e^{\mathcal{B}(\Pi)t}|\leqslant K_1e^{-\lambda t},\quad |\Pi^{(T)}(t)-\Pi|\leqslant K_1e^{-2\lambda(T-t)},\quad \forall 0\leqslant t\leqslant T<\infty.$$

Observe that

$$\mathcal{B}\Big[\Pi^{(T)}(s)\Big]-\mathcal{B}(\Pi)=-[\Pi^{(T)}(s)-\Pi]S,$$

it follows that

$$|\rho(s)|\leqslant K_2e^{-2\lambda(T-s)}|\varphi(s)|$$

for some $K_2>0$. Let $h(t)=e^{\lambda(T-t)}|\varphi(t)|$. Then by (5.9), take $K_3=\max\{K_1|\lambda^*|,K_1^2K_2\}$ so that

$$\begin{aligned}h(t)&\leqslant K_1\Big[|g^*|+\int_t^T K_1e^{-\lambda(s-T)}|\rho(s)|ds\Big]\\&\leqslant K_1\Big[|g^*|+\int_t^T K_1K_2e^{-\lambda(s-T)}e^{-2\lambda(T-s)}|\varphi(s)|ds\Big]\\&\leqslant K_3+K_3\int_t^T e^{-\lambda(T-s)}|\varphi(s)|ds\\&\leqslant K_3+K_3\int_t^T e^{-2\lambda(T-s)}h(s)ds.\end{aligned}$$

Applying Gronwall's inequality, we obtain that for some $K>0$ independent of T,

$$h(t)\leqslant K,\quad \forall t\in[0,T].$$

The desired result then follows. □

We are ready for the main result, which establishes the *exponential turnpike property* of Problem $(\mathrm{DG})_T$.

Theorem 5.4. *Let (A1)–(A3) hold. Let $\tilde X^{(T)}(\cdot)$ be the open-loop optimal state process, $\tilde u^{(T)}(\cdot)$ be the open-loop Nash equilibrium of Problem (DG)$_T$ for the given initial state x. Let $\tilde Y^{(T)}(\cdot)$ be the solution to the corresponding equation in (2.3). Let (x^*,u_1^*,u_2^*) be the unique solution of Problem (O) and λ^* be the corresponding Lagrange multiplier. Then there exist constants $K,\lambda>0$, independent of T, such that*

$$|\tilde X^{(T)}(t)-x^*|+|\tilde u^{(T)}(t)-u^*|+|\tilde Y^{(T)}(t)-\lambda^*|\leqslant K\Big[e^{-\lambda t}+e^{-\lambda(T-t)}\Big],\quad \forall\ t\in[0,T]. \tag{5.10}$$

Proof. Substituting (5.7) into the ODE for $\tilde X^{(T)}(\cdot)$ in (5.4) yields

$$\begin{cases}\dot{\tilde X}(t)=\Big[A+B\tilde\Theta^{(T)}(t)\Big]\tilde X(t)-B\widehat R^{-1}\widehat B^\top\varphi=\mathcal{A}(\Pi)\tilde X(t)+\xi(t),\\ \tilde X(0)=\tilde x,\end{cases} \tag{5.11}$$

where

$$\mathcal{A}(\Pi)=A+B\Theta,\quad \xi(t)\triangleq B[\tilde\Theta^{(T)}(t)-\Theta]\tilde X(t)-B\widehat R^{-1}\widehat B^\top\varphi(t),$$

and Θ is the stabilizer defined by (3.8). We have by the variation of constants formula that

$$\tilde{X}(t)=\mathrm{e}^{\mathcal{A}(\Pi)t}\tilde{x}+\int_0^t \mathrm{e}^{\mathcal{A}(\Pi)(t-s)}\xi(s)\mathrm{d}s.$$

By Theorem 4.1 and since $\mathcal{A}(\Pi)$ is stable, there exist constants $K_4,\lambda>0$ independent of T such that

$$|\mathrm{e}^{\mathcal{A}(\Pi)t}|\leqslant K_4\mathrm{e}^{-\lambda t},\quad |\tilde{\Theta}^{(T)}(t)-\Theta|\leqslant K_4\mathrm{e}^{-\lambda(T-t)},\quad \forall 0\leqslant t\leqslant T<\infty.$$

Recalling Lemma 5.3, we obtain

$$|\xi(t)|\leqslant K_5\mathrm{e}^{-\lambda(T-t)}\Big[|\tilde{X}(t)|+1\Big]$$

for some $K_5>0$. Taking $K_6=\max\{K_4|\tilde{x}|,K_4K_5\}$,

$$\begin{aligned}|\tilde{X}(t)|&\leqslant K_4\mathrm{e}^{-\lambda t}|\tilde{x}|+K_4K_5\int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{e}^{-\lambda(T-s)}\Big[|\tilde{X}(s)|+1\Big]\mathrm{d}s\\&\leqslant K_6\Big[\mathrm{e}^{-\lambda t}+\mathrm{e}^{-\lambda(T-t)}\Big]+K_6\int_0^t \mathrm{e}^{-\lambda(T+t-2s)}|\tilde{X}(s)|\Big]\mathrm{d}s\\&\leqslant K_6\Big[\mathrm{e}^{-\lambda t}+\mathrm{e}^{-\lambda(T-t)}\Big]+K_6\int_0^t \mathrm{e}^{-\lambda(T-s)}|\tilde{X}(s)|\mathrm{d}s.\end{aligned}$$

Applying the Gronwall's inequality,

$$|\tilde{X}(t)|\leqslant K\Big[\mathrm{e}^{-\lambda t}+\mathrm{e}^{-\lambda(T-t)}\Big],\quad \forall t\in[0,T], \tag{5.12}$$

for some $K>0$ independent of T. It follows that

$$\begin{aligned}|\tilde{u}(t)|=&|\bar{u}^{(T)}(t)-u^*|\\&\leqslant|\tilde{\Theta}(t)\tilde{X}(t)|+|\widehat{R}^{-1}\widehat{B}^\top\varphi(t)|\\&\leqslant K\Big[|\tilde{X}(t)|+|\varphi(t)|\Big]\\&\leqslant K\Big[\mathrm{e}^{-\lambda t}+\mathrm{e}^{-\lambda(T-t)}\Big],\quad \forall t\in[0,T],\end{aligned}\tag{5.13}$$

for some positive constants $K,\lambda>0$ independent of T. Finally, we can verify that the following relation holds:

$$\tilde{Y}(t)=\Pi^{(T)}(t)\tilde{X}(t)+\varphi(t),$$

from which it follows that

$$|\tilde{Y}(t)|\leqslant K\Big[\mathrm{e}^{-\lambda t}+\mathrm{e}^{-\lambda(T-t)}\Big],\quad \forall t\in[0,T]. \tag{5.14}$$

Combining (5.12)–(5.14), we get the desired result. □

Remark 5.5. The proofs presented in Theorem 5.3 and 5.4 exhibit a lot of similarities to [26, 27]. Nevertheless, the technical methodologies are classical. In the LQ situations, the optimal controls are always linear feedbacks to the state process. Consequently, the difference between the optimal state trajectory and the turnpike limit (or analogous pairs) satisfies a linear ODE whose coefficients comprise stable constant components and exponential decay terms. Thus, this difference can be effectively estimated by using the variation of constants formula and the Grownwall's inequality. In our setting, the technical difficulties mainly focus on the convergence relationship between the DREs and AREs, as well as the construction of the correct static optimization problem.

Remark 5.6. The stochastic differential games may rise to several non-trivial potential challenges, which can be summarized as follows: First, the solvability of relevant DREs and AREs presents significant difficulties, as illustrated in [8]. Second, the convergence behavior from the solution of DRE to the solution of ARE is also challenging. Third, characterizing the turnpike limit will no longer be straightforward, as it requires to solve a nonzero-sum stationary distribution differential game.

We have the following corollary as an immediate consequence of Theorem 5.4, which shows that the integral and the mean-square turnpike properties also hold for Problem $(\mathrm{DG})_T$.

Corollary 5.7. *Let (A1)–(A3) hold. Then as $T\to\infty$,*

$$\begin{aligned}&\frac{1}{T}\int_0^T \tilde{X}^{(T)}(t)\mathrm{d}t \to x^*, \quad \frac{1}{T}\int_0^T |\tilde{X}^{(T)}(t)-x^*|^2\mathrm{d}t\to 0,\\ &\frac{1}{T}\int_0^T \tilde{u}^{(T)}(t)\mathrm{d}t \to u^*, \quad \frac{1}{T}\int_0^T |\tilde{u}^{(T)}(t)-u^*|^2\mathrm{d}t\to 0.\end{aligned}$$

Next, it is easy to show that for any initial state x, the value $\mathbf{V}^{(T)}(x)=\begin{pmatrix}V_1^{(T)}(x)\\ V_2^{(T)}(x)\end{pmatrix}$ of Problem $(\mathrm{DG})_T$ converges to the minimum value $\mathbf{V}=\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\end{pmatrix}$ of Problem (O) in the time-average sense.

Corollary 5.8. *Let (A1)–(A3) hold. Then*

$$\lim_{T\to\infty}\frac{1}{T}\mathbf{V}^{(T)}(x)=\mathbf{V},\quad \forall x\in\mathbb{R}^n.$$

Now, we present an extension when there are N players. Since this case are similar to the two-person case, we only demonstrate the problems, the assumptions and corresponding results without proofs. Specifically, the controlled linear state equation is given by

$$\begin{cases}\dot{X}(t)=AX(t)+B_1u_1(t)+\cdots+B_Nu_N(t)+b, \quad t\in[0,T],\\ X(0)=x\in\mathbb{R}^n,\end{cases}\tag{5.15}$$

where $u_i(\cdot)$, selected from the admissible set $\mathcal{U}_i[0,T]$ defined similarly to (1.4), is the control for player i whose corresponding cost functional is

$$\begin{aligned}J_i^{(T)}(x;u_1(\cdot),\cdots,u_N(\cdot))\triangleq\frac{1}{2}\Big\{&\langle G_iX(T),X(T)\rangle+2\langle g_i,X(T)\rangle\\ &+\int_0^T\Big[\langle Q_iX(t),X(t)\rangle+\langle R_{i1}u_1(t),u_1(t)\rangle+\cdots+\langle R_{iN}u_N(t),u_N(t)\rangle\\ &+2\langle q_i,X(t)\rangle+2\langle r_{i1},u_1(t)\rangle+\cdots+2\langle r_{iN},u_N(t)\rangle\Big]\mathrm{d}t\Big\},\quad i=1,\cdots,N,\end{aligned}\tag{5.16}$$

where coefficients in (5.15) and (5.16) are suitable dimensional matrices/vectors with $G_i, Q_i, R_{ii}, i = 1, \cdots, N$ being symmetric. Then the open-loop Nash equilibrium of N-player nonzero-sum LQ differential game problem can be described as follows.

Problem (NDG)$_T$. For any fixed $x \in \mathbb{R}^n$, find a tuple $(\tilde{u}_1^{(T)}(\cdot), \cdots, \tilde{u}_N^{(T)}) \in \mathcal{U}_1[0,T] \times \cdots \times \mathcal{U}_N[0,T]$ such that for each $i = 1, \cdots, N$, it holds that

$$J_i(x; \tilde{u}_1^{(T)}, \cdots, \tilde{u}_N^{(T)}(\cdot)) \leqslant J_i^{(T)}(x; \tilde{u}_1^{(T)}(\cdot), \cdots, \tilde{u}_{i-1}^{(T)}(\cdot), u_i(\cdot), u_{i+1}^{(T)}(\cdot), \cdots, \tilde{u}_N^{(T)}), \quad \forall u_i(\cdot) \in \mathcal{U}_i[0,T]. \tag{5.17}$$

Now, let $m = m_1 + \cdots + m_N$, $S_i = B_i R_{ii}^{-1} B_i^\top$ and redefine the previous notation as

$$\begin{aligned}
&G = (G_1, \cdots, G_N)^\top, \qquad Q = (Q_1, \cdots, Q_N)^\top, \qquad B = (B_1, \cdots, B_N), \qquad S = (S_1, \cdots, S_N), \\
&\widehat{A} = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix}, \quad \widehat{R} = \begin{pmatrix} R_{11} & & \\ & \ddots & \\ & & R_{NN} \end{pmatrix}, \quad M = \begin{pmatrix} -A & S \\ Q & \widehat{A}^\top \end{pmatrix}, \\
&g = (g_1^\top, \cdots, g_N^\top)^\top, \qquad q = (q_1^\top, \cdots, q_N^\top)^\top, \qquad r = (r_{11}^\top, \cdots, r_{NN}^\top)^\top.
\end{aligned}$$

Similar to (A1)–(A3), we need the following new assumptions.

(A4) For $i = 1, \cdots, N$, $G_i \geqslant 0$, $Q_i \geqslant 0$ and $R_{ii} > 0$.
(A5) The matrix

$$\begin{pmatrix} I_n & 0_{n \times Nn} \end{pmatrix} \mathrm{e}^{MT} \begin{pmatrix} I_n \\ G \end{pmatrix}, \tag{5.18}$$

is invertible for all $T \geqslant 0$, and the inverse of (5.18) is bounded uniformly in $T \in [0, \infty)$.
(A6) The assumption (A3) holds for the new $M \in \mathbb{R}^{(N+1)n \times (N+1)n}$.

Then the relation between the open-loop Nash equilibrium $\tilde{u}^{(T)}(\cdot) \triangleq (\tilde{u}_1^{(T)}(\cdot), \cdots, \tilde{u}_N^{(T)}(\cdot))^\top$ of (5.17) and its corresponding state process $\tilde{X}^{(T)}(\cdot)$ can be described as follows.

Proposition 5.9. *Let (A4)–(A6) hold. Then (5.17) admits a unique open-loop Nash equilibrium*

$$\tilde{u}^{(T)}(t) = -\widehat{R}^{-1} \widehat{B}^\top \Pi^{(T)}(t) \tilde{X}^{(T)}(t) - \widehat{R}^{-1} \Big[\widehat{B}^\top \eta^{(T)}(t) + r \Big],$$

where

$$\Pi^{(T)}(t) = \begin{pmatrix} 0_{Nn \times n} & I_{Nn} \end{pmatrix} \mathrm{e}^{M(T-t)} \begin{pmatrix} I_n \\ G \end{pmatrix} \left[\begin{pmatrix} I_n & 0_{n \times Nn} \end{pmatrix} \mathrm{e}^{M(T-t)} \begin{pmatrix} I_n \\ G \end{pmatrix} \right]^{-1}$$

and $\eta^{(T)}(\cdot)$ is the unique solution to (2.7) with the new notation.

Next, we give the turnpike limit for problem (5.17). Let

$$\mathscr{V} \triangleq \big\{ (x, u_1, \cdots, u_N) \in \mathbb{R}^n \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N} \mid Ax + B_1 u_1 + \cdots + B_N u_N + b = 0 \big\},$$

and

$$\begin{aligned}
F_i(x, u_1, \cdots, u_N) \triangleq \frac{1}{2} \Big[&\langle Q_i x, x \rangle + \langle R_{i1} u_1, u_1 \rangle + \cdots + \langle R_{iN} u_N, u_N \rangle \\
&+ 2\langle q_i, x \rangle + 2\langle r_{i1}, u_1 \rangle + \cdots + 2\langle r_{iN}, u_N \rangle \Big], \ i = 1, 2, \cdots, N.
\end{aligned}$$

We consider the following N-objective static optimization problem.

Problem (NO). Find a $(x^*, u_1^*, \cdots, u_N^*) \in \mathscr{V}$ such that for each $i = 1, \cdots, N$, it holds that

$$F_i(x^*, u_1^*, \cdots, u_N^*) \leqslant F_i(x, u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^* \cdots, u_N^*), \quad \forall (x, u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_N^*) \in \mathscr{V}.$$

Let $u^* \triangleq ((u_1^*)^\top, \cdots, (u_N^*)^\top)^\top$. Similar to the proof of Proposition 5.1, we can obtain the unique solvability of Problem (NO). Then we can establish the N-player version of Theorem 5.4 as follows.

Theorem 5.10. *Let (A4)–(A6) hold. Then there exist constants $K, \lambda > 0$, independent of T, such that*

$$|\tilde{X}^{(T)}(t) - x^*| + |\tilde{u}^{(T)}(t) - u^*| \leqslant K\Big[\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda(T-t)}\Big], \quad \forall\, t \in [0, T].$$

At the end of this section, we give some remarks for other types of differential game.

Remark 5.11. As shown above in Theorem 5.4 and 5.10, the open-loop Nash equilibrium admits the exponential turnpike property. However, such property for the closed-loop Nash equilibrium is unknown, since the convergence of the related differential Riccati system is still not available. We refer [30], Chapter 6.6 for details.

Remark 5.12. In (1.2) and (1.3), if $J_1^{(T)} + J_2^{(T)} = 0$, then problem (1.5) becomes the two-person zero-sum differential game, which differs from assumption (A1). In the stochastic framework, the turnpike properties for this game have been established in [33] under suitable assumptions. Now, for the deterministic case, results can be directly derived by this existing literature.

6. Some examples

In this section, we present several examples illustrating the results established in the previous sections.

The following example shows that the assumption (A3) plays an important role in the turnpike property of the nonzero-sum LQ differential games. We may consider the singular matrix M, in which case the static optimization Problem (O) does not admit a unique optimal solution.

Example 6.1. Consider the two-dimensional state equation

$$\begin{cases} \dot{X}(t) = u_1(t) + u_2(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \in [0, T], \\ X(0) = x, \end{cases}$$

and the cost functionals

$$\begin{aligned} J_i^{(T)}(x; u_1(\cdot), u_2(\cdot)) \triangleq \int_0^T \Big[&\langle X(t), X(t)\rangle + \langle u_i(t), u_i(t)\rangle \\ &+ 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, X(t)\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_i(t)\rangle \Big] \mathrm{d}t, \ i = 1, 2. \end{aligned}$$

We see that in this case

$$\begin{cases} A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_1 = Q_2 = B_1 = B_2 = R_{11} = R_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ b = r_{11} = r_{22} = q_1 = q_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{cases}$$

and the other coefficients are 0. A simple calculation yields that

$$S_1 = S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then the Hamiltonian matrix

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of M are $\{\sqrt{2}, \sqrt{2}, 0, 0, -\sqrt{2}, -\sqrt{2}\}$, and the matrix of eigenvectors is

$$E_3 = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

A directly calculation yields

$$\begin{pmatrix} I_2 & 0 & 0 \end{pmatrix} \mathrm{e}^{MT} \begin{pmatrix} I_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{e}^{2\sqrt{2}T}+1}{2\mathrm{e}^{\sqrt{2}T}} & 0 \\ 0 & \frac{\mathrm{e}^{2\sqrt{2}T}+1}{2\mathrm{e}^{\sqrt{2}T}} \end{pmatrix}.$$

This means that M only fails to (A3).

The corresponding AREs read:

$$\begin{cases} \varPi_1\varPi_1 + \varPi_1\varPi_2 - I_2 = 0, \\ \varPi_2\varPi_1 + \varPi_2\varPi_2 - I_2 = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} (\varPi_1 + \varPi_2)^2 - 2I_2 = 0, \\ (\varPi_1 - \varPi_2)(\varPi_1 + \varPi_2) = 0. \end{cases}$$

Hence the solution is $\varPi_1 = \varPi_2 = \frac{\sqrt{2}}{2} I_2$. In this case, $\mathcal{A}(\varPi)$ is stable, but $\mathcal{B}(\varPi)$ is not.

The Problem (O) is equivalent to

$$\begin{cases} \text{Minimize } \frac{1}{2}\Big[\langle x, x\rangle + \langle u_1, u_1\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_1\rangle\Big], \\ \text{Minimize } \frac{1}{2}\Big[\langle x, x\rangle + \langle u_2, u_2\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2\rangle\Big], \\ \text{subject to } u_1 + u_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \end{cases}$$

By Proposition 5.1, the optimal triple (x^*, u_1^*, u_2^*) of Problem (O) satisfies

$$\begin{cases} u_1^* + u_2^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ x^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ u_i^* + \lambda_i^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \; i = 1, 2. \end{cases}$$

Thus we have

$$x^* = -\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$u_1^* = \lambda_2^*, \; u_2^* = \lambda_1^*.$$

Thus (u_1^*, u_2^*) satisfies the equation

$$u_1^* + u_2^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

it means that (u_1^*, u_2^*) has infinite many solution. Thus the static Problem (O) is not well defined.

The following example illustrates the turnpike property of nonzero-sum LQ differential games under assumptions (A1)–(A3).

Example 6.2. Consider the two-dimensional state equation (1.1) and the cost functionals (1.2) and (1.3) with

$$\begin{cases} A = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q_1 = Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ R_{11} = R_{22} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad b = r_{11} = r_{22} = q_1 = q_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{cases}$$

and the other coefficients are 0. We see that in this case

$$S_1 = S_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then the Hamiltonian matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of M is $\{\sqrt{5}, 1, -1, -1, -1, -\sqrt{5}\}$, and the matrix of eigenvectors is

$$E_4 = \begin{pmatrix} -1+\sqrt{5} & 0 & 0 & 1 & 2 & -\sqrt{5}-1 \\ 1+\sqrt{5} & 2 & 0 & 0 & 0 & 1-\sqrt{5} \\ 2-\sqrt{5} & -1 & -1 & -3 & -6 & 2+\sqrt{5} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2-\sqrt{5} & -1 & 1 & 0 & 0 & 2+\sqrt{5} \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

A directly calculation yields

$$\begin{aligned} &\begin{pmatrix} I_2 & 0 & 0 \end{pmatrix} \mathrm{e}^{MT} \begin{pmatrix} I_2 \\ 0 \\ 0 \end{pmatrix} \\ =&\frac{1}{20\mathrm{e}^{(1+\sqrt{5})T}} \begin{pmatrix} (5+3\sqrt{5})\mathrm{e}^{(1+2\sqrt{5})T} + 10\mathrm{e}^{\sqrt{5}T} + (5-3\sqrt{5})\mathrm{e}^{T} & (5-\sqrt{5})\mathrm{e}^{(1+2\sqrt{5})T} - 10\mathrm{e}^{\sqrt{5}T} + (5+\sqrt{5})\mathrm{e}^{T} \\ (15+7\sqrt{5})\mathrm{e}^{(1+2\sqrt{5})T} - 30\mathrm{e}^{(2+\sqrt{5})T} + (15-7\sqrt{5})\mathrm{e}^{T} & (5+\sqrt{5})\mathrm{e}^{(1+2\sqrt{5})T} + 10\mathrm{e}^{(2+\sqrt{5})T} + (5-\sqrt{5})\mathrm{e}^{T} \end{pmatrix}. \end{aligned}$$

We can verify that M satisfies the assumptions (A2) and (A3). Then the corresponding AREs read:

$$\begin{cases} \Pi_1 A + A^\top \Pi_1 + I_2 - \Pi_1 S_1 \Pi_1 - \Pi_1 S_2 \Pi_2 = 0, \\ \Pi_2 A + A^\top \Pi_2 + I_2 - \Pi_2 S_1 \Pi_1 - \Pi_2 S_2 \Pi_2 = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} (\Pi_1 + \Pi_2)(A - S_1\Pi_1 - S_2\Pi_2) + A^\top(\Pi_1 + \Pi_2) + 2I_2 = 0, \\ (\Pi_1 - \Pi_2)(A - S_1\Pi_1 - S_2\Pi_2) + A^\top(\Pi_1 - \Pi_2) = 0. \end{cases}$$

Thus, the unique strongly stabilizing solution solution is $\Pi_1 = \Pi_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. The Problem (O) is equivalent to

$$\begin{cases} \text{Minimize } \frac{1}{2}\Big[\langle x, x\rangle + \langle u_1, u_1\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_1\rangle\Big], \\ \text{Minimize } \frac{1}{2}\Big[\langle x, x\rangle + \langle u_2, u_2\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x\rangle + 2\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2\rangle\Big], \\ \text{subject to } \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u_1 + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \end{cases}$$

By Proposition 5.1, the optimal triple (x^*, u_1^*, u_2^*) of Problem (O) satisfies

$$\begin{cases} \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} x^* + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u_1^* + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u_2^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ x^* + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \lambda_i^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \\ u_i^* + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \lambda_i^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \ i = 1, 2. \end{cases}$$

Hence the turnpike limit is

$$x^* = u_1^* = u_2^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

with

$$\lambda_1^* = \lambda_2^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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Data availability statement

The research data associated with this artical are included in the article.

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