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ENERGY RELEASE AND GRIFFITH'S CRITERION FOR PHASE-FIELD FRACTURE

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Abstract. Phase-field evolutions are obtained by means of time discrete schemes, providing (or selecting) at each time step an equilibrium configuration of the system, which is usually computed by descent methods for the free energy (*e.g.* staggered and monolithic schemes) under a suitable irreversibility constraint on the phase-field parameter. We consider a class of phase-field energies including both volumetric-deviatoric and spectral decomposition; we study in detail the time continuous limits of these evolutions considering monotonicity as irreversibility constraint and providing a general result, which holds independently of the scheme employed in the incremental problem. In particular, we show that in the steady state regime the limit evolution is simultaneous (in displacement and phase field parameter) and satisfies Griffith's criterion in terms of toughness and phase field energy release rate. In the unsteady regime the limit evolution may instead depend on the adopted scheme and Griffith's criterion may not hold. We prove also the thermodynamical consistency of the monotonicity constraint over the whole evolution, and we study the system of PDEs (actually, a weak variational inequality) in the steady state regime. Technically, the proof employs a suitable reparametrization of the time discrete points, whose Kuratowski limit characterizes the set of steady state propagation. The study of the quasi-static time continuous limit relies on the strong convergence of the phase-field function together with the convergence of the power identity.

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1. Introduction

Griffith's criterion [1] was originally proposed for the quasi-static propagation of straight cracks in brittle materials. Mathematically, it is a rate-independent system comprising an irreversibility constraint, an equilibrium condition, and an evolution law (or flow rule). Usually it is expressed by means of Karush-Kuhn-Tucker complementarity conditions, in terms of toughness and *energy release* or, equivalently, in terms of derivatives of the energy.

In order to briefly describe the model, we start with the sharp crack setting in the case where the crack path is known a-priori. Assume, for the moment, that $\Omega \subset \mathbb{R}^2$ is the reference configuration and that the crack path

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is given by a curve $\gamma : [0,1] \to \Omega$. Given $l \in [0,1]$ let $K(l) = \gamma([0,l])$ be the corresponding crack and let

$$E(u, K(l)) = \int_{\Omega \setminus K(l)} W(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x$$

be the elastic energy for a displacement $u \in H^1(\Omega \setminus K(l); \mathbb{R}^2)$. Let us consider a "loading history" given by the boundary datum $s \mapsto g(s)$ for $s \in [0,1]$. Given $s \in [0,1]$ and $l \in [0,1]$ the *elastic energy* at equilibrium is given by

$$\tilde{E}(s,l) = \min\{E(u, K(l)) : u \in H^1(\Omega \setminus K(l); \mathbb{R}^2) \text{ with } u = g(s) \text{ on } \partial_D \Omega\}. \tag{1.1}$$

The *energy release* is the (configurational) variation of the elastic energy $\tilde{E}(s,\cdot)$ with respect to crack growth; in this setting

$$G(s,l) = -\lim_{h \to 0^+} \frac{\tilde{E}(s,l+h) - \tilde{E}(s,l)}{L_\gamma(l+h) - L_\gamma(l)}$$

where L_γ denotes the lenght of the crack $K(l)$.

Introducing the toughness $G_c > 0$, the classical form of Griffith's criterion is provided by the following Karush-Kuhn-Tucker conditions:

$$G(s, l(s)) \le G_c, \quad \big(G(s,l(s)) - G_c\big)\, l'(s) = 0, \tag{1.2}$$

together with the monotonicity constraint $l'(s) \ge 0$, which models irreversibility. Note that in this setting the evolution is described by the scalar map $s \mapsto l(s)$, which in turn provides the crack $K(l(s))$ and the displacement $u(s)$.

At this point, it is important to notice that the rate-independent system (1.2) holds if the solution $s \mapsto l(s)$ is sufficiently regular. However, in some cases, the solution presents a jump and Griffith's criterion breaks. In a general sense, continuity corresponds to steady state (or stable) evolutions, while discontinuity to unsteady (or unstable) propagation. Within the quasi-static setting, there are several notions of quasi-static evolution which could be used to "generalize" Griffith's criterion for discontinuities, see *e.g.* [2] for a review. Dealing with these evolutions, it is useful to write the graph of the map $s \mapsto l(s)$ by means of a (Lipschitz) parametrization $t \mapsto (s(t), \ell(t))$ for $t \in [0,T]$ in such a way that discontinuities of l correspond to "vertical parts" of the parametrization. In these terms, a jump of l is represented by the map $t \mapsto (s(t), \ell(t))$ in a subinterval $[t_a, t_b] \subset [0,T]$ where the function $t \mapsto s(t)$ is constant, while the map $t \mapsto \ell(t)$ describes the transition from $\ell(t_a) = l^-(s)$ to $\ell(t_b) = l^+(s)$. For instance, if $[t_a, t_b]$ correspond to the discontinuity point s, the evolution in [3] satisfies $G(s, \ell(t_a)) = G_c$, $G(s, \ell(t)) \ge G_c$ for $t \in (t_a, t_b)$ and $G(s, \ell(t_b)) = G_c$, so that jumps provide transitions between equilibrium configurations, passing only through critical and unstable configurations. For other types of evolution, *e.g. energetic solutions* [4–6], the behaviour of discontinuous solutions is different. However, in the steady state regime any type of quasi-static evolution is consistent with Griffith's criterion; indeed, out of the discontinuity intervals, say $[t_a, t_b]$, Griffith's criterion holds in parametrized form:

$$G(s(t), \ell(t)) \le G_c, \quad \big(G(s(t), \ell(t)) - G_c\big)\, \dot{\ell}(t) = 0.$$

An alternative way, out of the quasi-static setting, could be to avoid discontinuous solutions employing dynamic or rate-independent models, see *e.g.* [7–9].

When the path of the crack is not known *a priori*, the system of KKT conditions is complemented by a *directional criterion*, whose role is to select the "geometry" of the crack. Several criteria exist in the literature, for instance, the *principle of local symmetry* and the *maximum energy release rate*, which give different but comparable predictions. Here, we mention only the latter, since it is relevant in our phase-field framework. For sake of simplicity, we do not enter into fine technical details (interesting but out of scope here). We confine ourselves to steady-state evolutions and consider (unknown) cracks K represented by (unknown) simple curves γ. Let $\gamma : [0,1] \to \Omega$ be such a curve. Given $l \in (0,1)$ let $K_\gamma(l) = \gamma([0,l]) \subset \Omega$ be the corresponding crack. For $\zeta \in \mathbb{R}^2 \setminus \{0\}$ let us consider a "virtual extension" $\tilde{\gamma}$ with $\tilde{\gamma}'_+(l) = \zeta$. Then, the release of elastic energy in direction ζ is

$$G(s, K_\gamma(l); \zeta) = - \lim_{h \to 0^+} \frac{\tilde{E}(s, K_{\tilde{\gamma}}(l+h)) - \tilde{E}(s, K_\gamma(l))}{L_{\tilde{\gamma}}(l+h) - L_\gamma(l)}, \tag{1.3}$$

where $\tilde{E}(s, \cdot)$ denotes again the elastic energy at equilibrium, cfr. (1.1), and $L_\gamma(\cdot)$ denotes the length of $K_\gamma(\cdot)$ (Here, we assume that the above limit exists and is independent of the extension.) With this notation, the *maximal energy release rate* is given by

$$G_{max}(s, K_\gamma(l)) = \max \{ G(s, K_\gamma(l); \zeta) : \zeta \in \mathbb{R}^2 \setminus \{0\} \}; \tag{1.4}$$

roughly speaking, G_{max} is the absolute value of the derivative of the elastic energy along the direction of steepest descent.

Now, let us consider a simple curve $\beta : [0, S] \to \Omega$ which gives the evolution of the crack tip as a function of the loading parameter $s \in [0, S]$. Denoting by $l(s)$ the length of the crack $K_\beta(s) = \beta([0,s])$ we have $l'(s) = |\beta'(s)|$. With this notation, in the steady state regime β satisfies Griffith's criterion and the maximal energy realease rate criterion when

$$G_{max}(s, K_\beta(s)) \le G_c, \quad (G_{max}(s, K_\beta(s)) - G_c)\, l'(s) = 0, \tag{1.5}$$

and

$$G_{max}(s, K_\beta(s)) = G(s, K_\beta(s); \beta'(s)). \tag{1.6}$$

The latter condition means that the propagation direction $\zeta = \beta'(s)$ is that of steepest descent for the elastic energy. In the above presentation of Griffith's criterion and maximal energy release criterion we did not enter into the technical details and the issues related to the convergence of time-discrete evolutions, we refer the interested reader to [10] and [11].

Now, let us turn to the *phase-field* framework. In this setting, proposed in [12], the crack is represented by the phase-field variable $v \in [0,1]$, with $v = 0$ and $v = 1$ corresponding to fracture and sound material, respectively. The total energy takes the form

$$\mathcal{F}_\varepsilon(u, v) = \mathcal{E}_\varepsilon(u, v) + G_c\, \mathcal{L}_\varepsilon(v), \tag{1.7}$$

where u is the displacement and $\varepsilon > 0$ is the internal length. Let us consider again a "loading history" $s \mapsto g(s)$ for $s \in [0,1]$. Phase-field *evolutions* are computed using incremental problems. To this end, let $s^n_k = kS/n$, for $k = 0, ..., n$. Given the initial value v_0, the discrete configurations (u^n_k, v^n_k) are separate minimizers of the energy, *i.e.*

$$\begin{cases} u^n_k \in \operatorname{argmin} \{ \mathcal{F}_\varepsilon(u, v^n_k) : u = g(s^n_k) \text{ on } \partial_D \Omega \} & \text{for } k = 0, ..., n, \\ v^n_k \in \operatorname{argmin} \{ \mathcal{F}_\varepsilon(u^n_k, v) : v \le v^n_{k-1} \} & \text{for } k = 1, ..., n. \end{cases} \tag{1.8}$$

Here, we do not specify the (sub)-scheme which drives (u^n_{k-1}, v^n_{k-1}) to (u^n_k, v^n_k). Note that, being the energy non-convex, in general the update (u^n_k, v^n_k) is non-unique and may depend on the adopted (sub)-scheme. This assumption is intentionally large, in order to include all the (sub)-schemes employed in the numerical simulations, such as staggered and monolithic [12, 13], and even energetic evolutions.

The first goal of our paper is the characterization of the time-continuous limit. To this end we employ a linear interpolation of the configurations (u^n_k, v^n_k) together with a suitable Lipschitz reparametrization $t \mapsto (s(t), u(t), v(t))$ in a finite "time" interval $I = [0, T]$. In the limit, stable and unstable branches of the continuous evolution $t \mapsto (s(t), u(t), v(t))$ correspond respectively to disjoint subsets I_s and I_u of I, as in the scalar case described above. Technically, these sets are characterized by Kuratowski convergence of the points t^n_k, corresponding to the points s^n_k. First, we provide the following characterization, in terms of derivatives of the energy. For a.e. $t \in I_s$ the configuration $(u(t), v(t))$ satisfies

$$\partial_u \mathcal{F}_\varepsilon(u(t), v(t))[\phi] = 0, \quad \partial_v \mathcal{F}_\varepsilon(u(t), v(t))[\xi] \geq 0, \quad \partial_v \mathcal{F}_\varepsilon(u(t), v(t))[\dot{v}(t)] = 0, \tag{1.9}$$

for every admissible variation ϕ and ξ. Technically, in the proof of the above conditions there are a couple of delicate points: the strong convergence of the phase-field functions (see Lem. 5.5) and the proof of the power identity (see Thm. 5.13).

Our next goal is the characterization of the steady state evolution in terms of Griffith's criterion with maximal energy release rate. First, we introduce the notions of *energy release*, in analogy with the sharp-crack setting (1.3–1.4), *i.e.* the variation of elastic energy with respect to the variation of surface energy; thus for $\xi \leq 0$ (by irreversibility) and $\xi \neq 0$ we define

$$\mathcal{G}_\varepsilon(s, v; \xi) = -\lim_{h \to 0^+} \frac{\tilde{\mathcal{E}}_\varepsilon(s, v + h\xi) - \tilde{\mathcal{E}}_\varepsilon(s, v)}{\mathcal{L}_\varepsilon(v + h\xi) - \mathcal{L}_\varepsilon(v)},$$

where $\tilde{\mathcal{E}}_\varepsilon(s, v) = \min\{\mathcal{E}_\varepsilon(u, v) : u = g(s) \text{ on } \partial_D \Omega\}$ is again the elastic energy at equilibrium (in Sect. 6.1 we will provide alternative representation and a detailed analysis of the energy release $\mathcal{G}_\varepsilon$). As before, the maximal energy release reads

$$\mathcal{G}_{max}(s, v) = \sup\{\mathcal{G}_\varepsilon(s, v; \xi) : \xi \leq 0\}. \tag{1.10}$$

In Theorem 3.10 we prove that for steady state propagation, *i.e.* for $t \in I_s$, the evolution $t \mapsto (s(t), u(t), v(t))$ satisfies Griffith's criterion

$$\mathcal{G}_{max}(s(t), v(t)) \leq G_c, \quad (\mathcal{G}_{max}(s(t), v(t)) - G_c)\, \dot{\ell}(t) = 0,$$

where $\dot{\ell}(t) = d\mathcal{L}_\varepsilon(v(t))[\dot{v}(t)]$ gives the speed of the crack, together with the maximal energy release criterion, *i.e.*

$$\mathcal{G}_{max}(s(t), v(t)) = \mathcal{G}_\varepsilon(s(t), v(t); \dot{v}(t)).$$

Finally, we show that the irreversibility condition $\dot{\ell}(t) \geq 0$ holds, actually in the whole parametrization interval I.

2. NOTATION

Assume that Ω is a bounded Lipschitz connected open set in $\mathbb{R}^2$. The boundary $\partial\Omega$ of Ω is split into $\partial_D \Omega$ (for Dirichlet boundary condition) and $\partial_N \Omega = \partial\Omega \setminus \partial\Omega_D$ (for Neumann boundary condition); we assume that $\partial_D \Omega$ is relatively open in $\partial\Omega$. We define the set of admissible displacements

$$\mathcal{U}(s) := \{u \in H^1(\Omega; \mathbb{R}^2) : u = g(s) \text{ on } \partial_D \Omega\} \quad \text{for } s \in [0, 1],$$

where $g(s) := \alpha(s)\hat{g}$ with $\hat{g} \in W^{1,q}(\Omega;\mathbb{R}^2)$, for $q > 2$, and $\alpha \in C^1([0,1],\mathbb{R}_+)$. Working in the rate-independent setting, g provides the *loading path* for the system (independently of time). The spaces of the phase-field will be

$$\mathcal{V} := H^1(\Omega) \cap L^\infty(\Omega).$$

Although the phase-field functions $v \in \mathcal{V}$ may *a prioriri* take any real values, in the evolution we will have $\dot{v} \leq 0$ (to model irreversibility) and $0 \leq v \leq 1$, with $v = 1$ corresponding to no damage and $v = 0$ corresponding to maximum damage.

In this work, we are not studying any limit as the internal length ε vanishes, so for convenience we set $\varepsilon = 1$ and omit the dependence on ε in the notation, thus the total energy takes the form

$$\mathcal{F}(u,v) := \mathcal{E}(u,v) + G_c\,\mathcal{L}(v).$$

Due to the irreversibility constraint, the natural set of admissible variations for the phase field variable is the cone $\Xi = \{\xi \in H^1(\Omega) \cap L^\infty(\Omega) : \xi \leq 0\}$, while the displacement variations are in the space $\Phi = \{\phi \in H^1(\Omega;\mathbb{R}^2) : \phi = 0 \text{ on } \partial_D\Omega\}$; thus equilibrium is characterized by

$$\partial_v \mathcal{F}(u,v)[\xi] \geq 0, \qquad \partial_u \mathcal{F}(u,v)[\phi] = 0, \qquad \text{for every } \xi \in \Xi \text{ and } \phi \in \Phi.$$

In the literature there are several options for the elastic energy $\mathcal{E}$ and the length term $\mathcal{L}$, for a comprehensive view we refer to [14–17]. Here we consider an elastic energy density of the form

$$\mathcal{E}(u,v) := \int_\Omega W(\boldsymbol{\varepsilon}(u),v)\,\mathrm{d}x, \qquad W(\boldsymbol{\varepsilon},v) = \psi(v)W_+(\boldsymbol{\varepsilon}) + W_-(\boldsymbol{\varepsilon}).$$

For the stress we employ the notation

$$\boldsymbol{\sigma}(u,v) := \partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}(u),v) = \psi(v)\,\partial_{\boldsymbol{\varepsilon}} W_+(\boldsymbol{\varepsilon}(u)) + \partial_{\boldsymbol{\varepsilon}} W_-(\boldsymbol{\varepsilon}(u)). \tag{2.1}$$

Here, the degradation function takes the form $\psi(v) := v^2 + \eta$, for $\eta > 0$. Let $\mathbb{R}^2_{sym}$ be the space of symmetric real 2×2 matrices. Combining the assumptions of [18, 19] we will assume that the energy densities $W_\pm : \mathbb{R}^{2\times 2}_{sym} \to [0,+\infty)$ are convex, of class C^1, and such that

(H1) $W_\pm(h\boldsymbol{\varepsilon}) = h^2\,W_\pm(\boldsymbol{\varepsilon})$ for $h \geq 0$,
(H2) $|\partial_{\boldsymbol{\varepsilon}} W_\pm(\boldsymbol{\varepsilon}_1) - \partial_{\boldsymbol{\varepsilon}} W_\pm(\boldsymbol{\varepsilon}_0)| \leq C_1|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0|$ for $C_1 > 0$ and for every $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{2\times 2}_{sym}$.

Moreover, we require that the energy $W : \mathbb{R}^{2\times 2}_{sym} \times [0,1] \to [0,+\infty)$ satisfies the following coercivity property: there exists $C_2 > 0$ such that

(H3) $\big(\partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}_1,v) - \partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}_0,v)\big) : \big(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0\big) \geq C_2|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0|^2$ for every $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{2\times 2}_{sym}$ and $v \in [0,1]$.

Note that, being $0 \leq \psi(v) \leq 1$, from (H1)–(H2) we easily get

(P1) $W(h\boldsymbol{\varepsilon},v) = h^2\,W(\boldsymbol{\varepsilon},v)$ for every $h \geq 0$ and $v \in [0,1]$.
(P2) $|\partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}_1,v) - \partial_{\boldsymbol{\varepsilon}} W(\boldsymbol{\varepsilon}_0,v)| \leq C_1|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_0|$ for every $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{2\times 2}_{sym}$ and $v \in [0,1]$.

These abstract hypotheses allow to embrace both the volumetric-deviatoric [20] and the spectral decomposition [21], as explained hereafter.

Example 2.1 (volumetric-deviatoric split)**.** Following [20] the elastic energy is decomposed as

$$W(\boldsymbol{\varepsilon},v) = \psi(v)W_+(\boldsymbol{\varepsilon}) + W_-(\boldsymbol{\varepsilon}), \quad W_+(\boldsymbol{\varepsilon}) := \mu|\boldsymbol{\varepsilon}_d|^2 + \kappa|\varepsilon_v^+|^2, \quad W_-(\boldsymbol{\varepsilon}) := \kappa|\varepsilon_v^-|^2,$$

where $\boldsymbol{\varepsilon}_v^+ := \frac{1}{2}\mathrm{tr}^+(\boldsymbol{\varepsilon})\boldsymbol{I}$ and $\boldsymbol{\varepsilon}_v^- := \frac{1}{2}\mathrm{tr}^-(\boldsymbol{\varepsilon})\boldsymbol{I}$ are respectively the tensile and compressive components of the strain, while $\boldsymbol{\varepsilon}_d := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_v$ is the deviatoric component (with standard notation, we denote by μ and κ the shear and bulk modulus respectively). With this split damage does not occur under compression and the non-interpenetration condition emerges in the sharp-crack limit [22]. It is easy to check (H1) while (H2)–(H3) are proven in [18], Lemma 1.

Example 2.2 (spectral decomposition)**.** Another way to distinguish between traction, compression and shear was proposed in [21]. Given $\boldsymbol{\varepsilon} \in \mathbb{R}^{2\times 2}_{sym}$ let $\zeta_i \in \mathbb{R}$ (for $i = 1, 2$) be its eigenvalues. With this notation the elastic energies $W_\pm$ are defined as

$$W_\pm(\boldsymbol{\varepsilon}) = \frac{\mu}{2}[\zeta_1 + \zeta_2]^2_\pm + \lambda\big([\zeta_1]^2_\pm + [\zeta_2]^2_\pm\big).$$

It easy to check (H1), while (H1)–(H2) follow by [19], Lemma 2.9.

As far as the length term, we employ the classical quadratic functional

$$\mathcal{L}(v) := \tfrac{1}{2}\int_\Omega (v-1)^2 + |\nabla v|^2 \mathrm{d}x,$$

known as AT_2 in the engineering literature.

Remark 2.3. In the above setting we are assuming, for the sake of simplicity, that the internal length $\varepsilon = 1$. However, we should mention that the phase-field approach relies on the rigorous base of Γ-*convergence* [23–25] as the internal lenght ε vanishes. For instance, in the volumetric deviatoric setting, the phase-field energies converge (under suitable technical assumptions [22]) to the sharp crack energy

$$\int_{\Omega\setminus S_u} \mu|\boldsymbol{\varepsilon}_d|^2 + \kappa|\boldsymbol{\varepsilon}_v|^2 \,\mathrm{d}x + G_c\mathcal{H}^1(S_u),$$

where the displacement $u \in SBD(\Omega)$ satisfies the non-interpenetration condition $[\![u]\!]\cdot\nu \geq 0$ on the discontinuity set S_u. It is well known that Γ-convergence provides the convergence of global minimizers, while not much is known on the convergence of equilibrium points, energy release rate, and evolutions. An accurate numerical study (in the steady and unsteady propagation regimes) on the convergence of energies, energy release rate and evolutions is contained in [26]. On the theoretical side, the convergence of critical points of the Ambrosio-Tortorelli energy is contained in [27] while the convergence of phase-field to sharp-crack evolutions is proven in [28, 29], actually at the price of restrictive assumptions on the geometry of the crack.

3. STATEMENT OF THE MAIN RESULTS

In the study of rate-independent processes, it is customary to employ schemes based on a time discretization or on a discretization of the loading path. Given $v_0 \in \mathcal{V}$ with $0 \leq v_0 \leq 1$, let $u_0 \in \mathrm{argmin}\,\{\mathcal{F}(u, v_0) : u \in \mathcal{U}(0)\}$. For $n \in \mathbb{N}$, $n > 0$, set $s^n_k := k/n$ for $k = 0, ..., n$ and $v^n_0 := v_0$. Let us consider (u^n_k, v^n_k) for $k = 1, ..., n$ such that

$$\begin{cases} u^n_k \in \mathrm{argmin}\,\{\mathcal{F}(u, v^n_k) : u \in \mathcal{U}(s^n_k)\} \\ v^n_k \in \mathrm{argmin}\,\{\mathcal{F}(u^n_k, v) : v \in \mathcal{V} \text{ with } v \leq v^n_{k-1}\}. \end{cases}$$

In other terms, (u^n_k, v^n_k) is a separate minimizer of the energy $\mathcal{F}$ under the constraints $u \in \mathcal{U}(s^n_k)$ and $v \leq v^n_{k-1}$.

Example 3.1. In practice, there are several "sub-schemes" to provide the update (u^n_k, v^n_k): for instance, we may employ an energy descent method, such as staggered (see Sect. 5), or global energy minimization. In the latter case,

$$(u^n_k, v^n_k) \in \operatorname{argmin} \{\mathcal{F}(u,v) : u \in \mathcal{U}(s^n_k),\ v \in \mathcal{V},\ v \le v^n_{k-1}\}.$$

In the former, we introduce the auxiliary sequences $\{u^n_{k,i}\}$ and $\{v^n_{k,i}\}$ defined by induction in the following way: $u^n_{k,0} = u^n_{k-1}$, $v^n_{k,0} = v^n_{k-1}$, and then

$$\begin{cases} u^n_{k,i+1} \in \operatorname{argmin} \{\mathcal{F}(u, v^n_{k,i}) : u \in \mathcal{U}(s^n_k)\} \\ v^n_{k,i+1} \in \operatorname{argmin} \{\mathcal{F}(u^n_{k,i+1}, v) : v \in \mathcal{V} \text{ with } v \le v^n_{k-1}\}. \end{cases}$$

Then, see Proposition 8.2, $v^n_{k,i} \to v_\infty$ in H^1 and $u^n_{k,i} \to u_\infty$ in H^1 (up to non-relabelled subsequences) where (u_∞, v_∞) satisfies (1.8); thus we set $u^n_k = u_\infty$ and $v^n_k = v_\infty$. Other choices could be constrained minimization [30], second order schemes [31] *etc.*

From (1.8) by separate minimality we have

$$\partial_u \mathcal{F}(u^n_k, v^n_k)[\phi] = 0, \quad \partial_v \mathcal{F}(u^n_k, v^n_k)[v - v^n_k] \ge 0, \quad \text{for every } \phi \in \Phi \text{ and } v \in \mathcal{V} \text{ with } v \le v^n_{k-1}. \tag{3.1}$$

As a straightforward consequence

$$\partial_v \mathcal{F}(u^n_k, v^n_k)[v^n_k - v^n_{k-1}] = 0. \tag{3.2}$$

In general, the "speed" $\|v^n_k - v^n_{k-1}\|_{H^1}/(s^n_k - s^n_{k-1})$ can be arbitrarily large; however, with a suitable time-parametrization of the loading path we can control it, as follows. Let us consider a strictly increasing parametrization $c^n : [0,T] \to [0,1]$ and the points $t^n_k = (c^n)^{-1}(s^n_k)$. We denote by $v^n : [0,T] \to H^1(\Omega)$ the piecewise affine interpolation of v^n_k in the times t^n_k.

Lemma 3.2. *There exist a fixed time T and a sequence of uniformly Lipschitz, strictly increasing parametrizations $c^n : [0,T] \to [0,1]$ such that the sequence $\{v^n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\infty}(0,T; H^1(\Omega))$. Given v^n, let $u^n(t) := \operatorname{argmin} \{\mathcal{E}(u, v^n(t)) : u \in \mathcal{U}(c^n(t))\}$; there exists $p > 2$ such that the sequence $\{u^n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\infty}(0,T; W^{1,p}(\Omega;\mathbb{R}^2))$.*

From the above result it follows that there exists a subsequence (non relabelled) such that

- $c^n \overset{*}{\rightharpoonup} c$ in $W^{1,\infty}(0,T)$, and thus uniformly in $[0,T]$,
- $v^n \overset{*}{\rightharpoonup} v$ in $W^{1,\infty}(0,T; H^1(\Omega))$,
- $u^n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(0,T; W^{1,p}(\Omega;\mathbb{R}^2))$ for some $p > 2$.

The parametrization $c : [0,T] \to [0,1]$ is Lipschitz continuous and increasing. Given the parametrization c we will write by abuse of notation $\mathcal{U}(t) = \mathcal{U}(c(t))$ *etc.* Clearly, v and u are Lipschitz continuous with the same constant of v^n and u^n respectively.

Remark 3.3. From the *mechanical* point of view, the parametrizations c^n and c provide some sort of control of the boundary condition, ensuring finite (and arbitrarily small) speed of the crack in the steady state setting, which is consistent with the intuitive idea that quasi-static evolutions are slow in time. However, even if the evolution is reparametrized and the speed in controlled in the steady state setting, in general we cannot avoid unstable branches, where the speed is infinite. From the *mathematical* point of view, parametrizations allow to distinguish clearly steady and unsteady evolutions (see below) and to work with Lipschitz maps in Sobolev spaces (such a regularity would not be possible using only monotonicity and relying on Helly's Theorem for compactness).

In order to study the limit evolution (u, v) a couple of delicate properties are needed: strong convergence of v^n and u^n and a characterization of stable and unstable regimes (in the parametrized setting). The following result provides strong convergence, putting together Lemma 5.5 and Corollary 5.11.

Lemma 3.4. *Let $t \in [0, T]$, then $v^n(t)$ converge strongly to $v(t)$ in $H^1(\Omega)$. Moreover, $u^n(t)$ converges to $u(t) := \operatorname{argmin}\{\mathcal{E}(u, v(t))\ u \in \mathcal{U}(t)\}$ strongly in $W^{1,p}(\Omega; \mathbb{R}^2)$ for some $p > 2$.*

In order to separate stable and unstable regimes we argue in the following way, inspired by the theory of visco-energetic evolutions [32]. For every $n \in \mathbb{N}$, $n > 0$, we define a suitable set of (parametrized) time discrete points $\Upsilon^n := \{t^n_k : k = 0, ..., n\} \subset [0, T]$, corresponding to the points s^n_k. There exists a subsequence (non-relabelled) such that Υ^n converge to a set $I_s \subseteq [0, T]$ in the sense of Kuratowski. As we will see, the set I_s is the set where the (parametrized) evolution is steady state (or stable), while the set $I_u = [0, T] \setminus I_s$ is the set where the evolution is unstable. We will show that I_s is a compact set of positive measure, see Corollary 5.9.

The set I_s depends on the time discrete sequence and in practice on the scheme used to provide it. For example, global minimization algorithms may predict different evolutions from those obtained by energy descent methods, see *e.g.* [3, 26]: since the energy is non-convex it may happen that, in the incremental procedure, global minimization finds a configuration (u^n_k, v^n_k) even if the potential energy is not decreasing along any path starting from (u^n_{k-1}, v^n_{k-1}). In this scenario, an energy descent algorithm would not predict (u^n_k, v^n_k). Even within energy descent algorithms, alternate minimization and monolithic schemes may predict different evolutions, see *e.g.* [31].

Remark 3.5. Linear interpolation between (u^n_{k-1}, v^n_{k-1}) and (u^n_k, v^n_k) is a very natural choice for our result, which is independent of the incremental sub-scheme. When the evolution is steady-state any interpolation would be fine, since (u^n_{k-1}, v^n_{k-1}) and (u^n_k, v^n_k) will converge to the same limit. However, in the unstable regime (u^n_{k-1}, v^n_{k-1}) and (u^n_k, v^n_k) converge to different configurations; in this case a precise description of the transition between them would require ad-hoc results depending on the specific incremental sub-scheme.

Next Theorem describes the behaviour of the evolution in the steady state regime by a sort of *evolutionary variational inequality* in Karush-Kuhn-Tucker form.

Theorem 3.6. *For every $t \in I_s$ the limit configuration $(u(t), v(t))$ is an equilibrium point for the energy $\mathcal{F}(\cdot, \cdot)$, i.e.,*

$$\partial_v \mathcal{F}(u(t), v(t))[\xi] \geq 0, \qquad \partial_u \mathcal{F}(u(t), v(t))[\phi] = 0, \qquad \text{for every } \xi \in \Xi \text{ and } \phi \in \Phi.$$

Moreover, for a.e. $t \in I_s$ it holds

$$\partial_v \mathcal{F}(u(t), v(t))[\dot{v}(t)] = 0$$

The statement of Theorem 3.6 can be recast in terms of PDEs, under sufficient regularity, as stated in the next result; this is indeed the system of evolution equations employed in several engineering papers, see *e.g.* [16].

Theorem 3.7. *For $t \in I_s$ the solution u solves the following PDE*

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}(t)) = 0 & \text{in } \Omega \\ \boldsymbol{\sigma}(t)\, \hat{n} = 0 & \text{in } \partial_N \Omega \\ u(t) = g(t) & \text{in } \partial_D \Omega, \end{cases} \tag{3.3}$$

where for simplicity $\boldsymbol{\sigma}(t)$ denotes the phase-field stress (2.1). Moreover, if $\dot v \in C(\bar\Omega)$ we have

$$\begin{cases} -G_c\Delta v(t) + G_c(v(t)-1) + 2v(t)W_+(\boldsymbol{\varepsilon}(t)) \le 0 & \text{in } \Omega \\ \dfrac{\partial v(t)}{\partial n} \le 0 & \text{in } \partial\Omega \\ \big[-G_c\Delta v(t) + G_c(v(t)-1) + 2v(t)W_+(\boldsymbol{\varepsilon}(t))\big]\dot v(t) = 0 & \text{in } \Omega \\ \dfrac{\partial v(t)}{\partial n}\dot v(t) = 0 & \text{in } \partial\Omega \end{cases} \tag{3.4}$$

where $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}(u(t))$.

The limit evolution satisfies also the following energy identity.

Proposition 3.8. *For a.e. $t \in [0,T]$ it holds*

$$\mathcal{F}(u(t),v(t)) = \mathcal{F}(u_0,v_0) + \int_0^t \mathcal{P}(s,u(s),v(s))\,\mathrm{d}s + \int_{I_u\cap(0,t)} \partial_v\mathcal{F}(u(s),v(s))[\dot v(s)]\,\mathrm{d}s,$$

where $\mathcal{P}$ is the power

$$\mathcal{P}(t,u(t),v(t)) := \int_\Omega \boldsymbol{\sigma}(u(t),v(t)) : \boldsymbol{\varepsilon}(\dot a(t)\hat g)\,\mathrm{d}x,$$

and $a(t) := \alpha(c(t))$.

Remark 3.9. Observe that the function $t \mapsto (u(t),v(t)) \mapsto \mathcal{F}(u(t),v(t))$ is Lipschitz continuous, as it is the composition of the bounded, Lipschitz continuous map $t \mapsto (u(t),v(t))$ taking values in $W^{1,p}\times H^1$ with the locally Lipschitz continuous function $(u,v)\mapsto\mathcal{F}(u,v)$ (see Lem. 4.8) defined in $W^{1,p}\times H^1$. If the evolution is steady state the set I_u is empty and thus the classic energy identity holds. On the contrary, for unstable evolutions it may happen that

$$\int_{I_u\cap(0,t)} \partial_v\mathcal{F}(u(s),v(s))[\dot v(s)]\,\mathrm{d}s \neq 0.$$

Finally, we want to show that the limit evolution satisfies Griffith's criterion. First, let us provide the notion of *(maximal) energy release* in the phase field context [33]

$$\mathcal{G}_{max}(t,v) := \sup\big\{-\partial_v\tilde{\mathcal{E}}(t,v)[\xi] : \xi\in\Xi \text{ with } d\mathcal{L}(v)[\xi]=1\big\}, \tag{3.5}$$

where $\tilde{\mathcal{E}}(t,v)$ is the reduced energy

$$\tilde{\mathcal{E}}(t,v) := \mathcal{E}(u_{t,v},v) \quad \text{ for } \quad u_{t,v}\in\operatorname{argmin}\{\mathcal{E}(u,v) : u\in\mathcal{U}(t)\}.$$

Other representations and properties of $\mathcal{G}_{max}$ are contained in Section 6. The energy release rate, as we defined it, is not a simple quantity to compute, since the derivative of the energy should be calculated in any admissible direction. Nevertheless, we will see in the sequel that there is no need to do so. The reason for introducing $\mathcal{G}_{max}$ is to prove that the evolution $t\mapsto v(t)$ satisfies Griffith criterion with this notion of energy release; this result is summarized in next Theorem, where, in order to facilitate its statement, we denote $\ell(t) := \mathcal{L}(v(t))$.

Theorem 3.10. *Assume that $v_0 \not\equiv 1$. The limit evolution v satisfies $0\le v(t)\le 1$ for every $t\in[0,T]$, the irreversibility constraint $\dot v(t)\le 0$ and the thermodynamic consistency condition $\dot\ell(t) = \mathrm{d}\mathcal{L}(v(t))[\dot v(t)] \ge 0$ a.e. in*

$[0,T]$. *The following Griffith's criterion with maximal energy release rate holds. In the steady state regime we have*

- $\mathcal{G}_{\max}(t,v(t)) \le G_c$ *everywhere in* I_s,
- $(\mathcal{G}_{\max}(t,v(t)) - G_c)\,\dot\ell(t) = 0$ *a.e. in* I_s,
- $\partial_v \tilde{\mathcal{E}}(t,v)[\dot v(t)] = -\dot\ell(t)\,\mathcal{G}_{\max}(t,v(t))$ *a.e. in* I_s.

In the unsteady regime we have instead

- $\mathcal{G}_{\max}(t,v(t)) \ge G_c$ *on a subset of positive measure of* I_u.

Remark 3.11. The irreversibility of the crack is modelled by the monotonicity of the phase field variable v. This hypothesis does not directly imply the monotonicity of the dissipated energy, which is however true for the evolution, as stated in the above Theorem 3.10. Note that in the steady state regime, the above properties correspond precisely to (1.5)–(1.6) in the sharp crack setting. The assumption $v_0 \not\equiv 1$ is needed here since the energy release is not well defined for $v_0 = 1$, in agreement with the sharp crack setting where $v_0 = 1$ corresponds to no pre-existing cracks. Note that the evolution is actually well defined for any initial datum, as stated in Theorem 3.6.

4. ENERGY VARIATIONS AND CONTINUOUS DEPENDENCE

This brief section is devoted to some preliminary results that will be widely used in the sequel, starting from some properties of the energy density W. For the proofs of Lemmata 4.3, 4.5 and 4.6 we refer to [18, 34].

Lemma 4.1. *The densities* $W_\pm$ *satisfy:*

$$\partial_{\boldsymbol\varepsilon} W_\pm(h\boldsymbol\varepsilon) = h\partial_{\boldsymbol\varepsilon} W_\pm(\boldsymbol\varepsilon) \text{ for } h \ge 0, \quad |\partial_{\boldsymbol\varepsilon} W_\pm(\boldsymbol\varepsilon)| \le C_1|\boldsymbol\varepsilon|, \quad |W_\pm(\boldsymbol\varepsilon_1) - W_\pm(\boldsymbol\varepsilon_0)| \le C_1(|\boldsymbol\varepsilon_0| + |\boldsymbol\varepsilon_1|)|\boldsymbol\varepsilon_1 - \boldsymbol\varepsilon_0|,$$

for every $\boldsymbol\varepsilon, \boldsymbol\varepsilon_i \in \mathbb{R}^{n\times n}_{sym}$. *As an immediate consequence*

$$\begin{aligned}&\boldsymbol\sigma(hu,v) = h\boldsymbol\sigma(u,v) \text{ for every } h \ge 0 \text{ and } v \in [0,1], \quad |\boldsymbol\sigma(u,v)| \le C|\boldsymbol\varepsilon(u)|,\\ &|W(\boldsymbol\varepsilon_1,v) - W(\boldsymbol\varepsilon_0,v)| \le C_2(|\boldsymbol\varepsilon_0| + |\boldsymbol\varepsilon_1|)|\boldsymbol\varepsilon_1 - \boldsymbol\varepsilon_0| \text{ for every } v \in [0,1].\end{aligned}$$

Proof. By (H1) we know that $W_\pm(h\boldsymbol\varepsilon) = h^2 W_\pm(\boldsymbol\varepsilon)$. Taking the derivative on both sides gives $h\partial_{\boldsymbol\varepsilon} W_\pm(h\boldsymbol\varepsilon) = h^2\partial_{\boldsymbol\varepsilon} W_\pm(\boldsymbol\varepsilon)$. Hence $\partial_{\boldsymbol\varepsilon} W_\pm(0) = 0$ and by (H2) it follows that $|\partial_{\boldsymbol\varepsilon} W_\pm(\boldsymbol\varepsilon)| \le C_1|\boldsymbol\varepsilon|$.

For the last estimate, by convexity we have

$$W_\pm(\boldsymbol\varepsilon_0) \ge W_\pm(\boldsymbol\varepsilon_1) + \partial_{\boldsymbol\varepsilon} W_\pm(\boldsymbol\varepsilon_1) : (\boldsymbol\varepsilon_0 - \boldsymbol\varepsilon_1) \ge W_\pm(\boldsymbol\varepsilon_1) - C_1|\boldsymbol\varepsilon_1|\,|\boldsymbol\varepsilon_1 - \boldsymbol\varepsilon_0| \ge W_\pm(\boldsymbol\varepsilon_1) - C_1(|\boldsymbol\varepsilon_1| + |\boldsymbol\varepsilon_0|)\,|\boldsymbol\varepsilon_1 - \boldsymbol\varepsilon_0|.$$

Arguing in the same way we get a lower bound for $W_\pm(\boldsymbol\varepsilon_1)$ and the proof of the last estimate is concluded. ☐

Corollary 4.2. *The elastic energy reads*

$$\mathcal{E}(u,v) = \tfrac12 \int_\Omega \boldsymbol\sigma(u,v) : \boldsymbol\varepsilon(u)\,\mathrm{d}x.$$

Proof. Consider the map $h \mapsto \mathcal{E}(hu,v)$ for $h \in [0,1]$. Being $\mathcal{E}(0,v) = 0$, by previous Lemma, we get

$$\begin{aligned}\mathcal{E}(u,v) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}h}\mathcal{E}(hu,v)\,\mathrm{d}h = \int_0^1 \left[\int_\Omega \frac{\mathrm{d}}{\mathrm{d}h} W(\boldsymbol\varepsilon(hu),v)\,\mathrm{d}x\right]\mathrm{d}h\\ &= \int_0^1 \left[\int_\Omega \boldsymbol\sigma(hu,v) : \boldsymbol\varepsilon(u)\,\mathrm{d}x\right]\mathrm{d}h = \left(\int_0^1 h\,\mathrm{d}h\right)\left(\int_\Omega \boldsymbol\sigma(u,v) : \boldsymbol\varepsilon(u)\,\mathrm{d}x\right),\end{aligned}$$

which concludes the proof. ☐

Lemma 4.3. *The functional $\mathcal{F}(\cdot,\cdot)$ is sequentially weakly lower semicontinuous in $\mathcal{U}(s)\times\mathcal{V}$. Moreover, for $v\in\mathcal{V}$ and $u\in\mathcal{U}(s)$ the partial derivative of $\mathcal{F}$ w.r.t. u is given by*

$$\partial_u\mathcal{F}(u,v)[\phi]=\partial_u\mathcal{E}(u,v)[\phi]=\int_\Omega \partial_{\boldsymbol{\varepsilon}}W(\boldsymbol{\varepsilon}(u),v):\boldsymbol{\varepsilon}(\phi)\,\mathrm{d}x=\int_\Omega \boldsymbol{\sigma}(u,v):\boldsymbol{\varepsilon}(\phi)\,\mathrm{d}x, \tag{4.1}$$

for every $\phi\in\Phi$. If $v\in\mathcal{V}$ and $u\in\mathcal{U}(s)$, then for every $\xi\in\Xi$,

$$\partial_v\mathcal{F}(u,v)\,[\xi]=\int_\Omega 2v\xi\,W_+(\boldsymbol{\varepsilon}(u))\,\mathrm{d}x+G_c\int_\Omega (v-1)\xi+\nabla v\cdot\nabla\xi\,\mathrm{d}x. \tag{4.2}$$

Corollary 4.4. *There exists $C>0$ such that, for every $s\in[0,1]$ and every v_0, $v_1\in\mathcal{V}$, it holds:*

$$|\mathcal{E}(u_1,v_0)-\mathcal{E}(u_0,v_0)|\le C\|u_1-u_0\|^2_{H^1},$$

where $u_0=\operatorname{argmin}\{\mathcal{E}(u,v_0):\,u\in\mathcal{U}(s)\}$ and $u_1\in\mathcal{U}(s)$.

Proof. Let us consider the convex combinations $u_r=ru_1+(1-r)u_0$ where $r\in[0,1]$ and write:

$$\mathcal{E}(u_1,v_0)-\mathcal{E}(u_0,v_0)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{E}(u_r,v_0)\,\mathrm{d}r=\int_0^1\partial_u\mathcal{E}(u_r,v_0)[u_r']\,\mathrm{d}r,$$

where $u_r'=u_1-u_0$. Since $\partial_u\mathcal{E}(u_0,v_0)[\phi]=0$ for every $\phi\in\Phi$, we can continue with:

$$\begin{aligned}\int_0^1\partial_u\mathcal{E}(u_r,v_0)[u_r']\,\mathrm{d}r&=\int_0^1\partial_u\mathcal{E}(u_r,v_0)[u_r']\,\mathrm{d}r-\partial_u\mathcal{E}(u_0,v_0)[u_r']\,\mathrm{d}r\\&=\int_0^1\langle\boldsymbol{\sigma}(u_r,v_0)-\boldsymbol{\sigma}(u_0,v_0),\boldsymbol{\varepsilon}(u_r')\rangle_{L^2}\,\mathrm{d}r.\end{aligned}$$

It follows that

$$|\mathcal{E}(u_1,v_0)-\mathcal{E}(u_0,v_0)|\le\int_0^1\|\boldsymbol{\sigma}(u_r,v_0)-\boldsymbol{\sigma}(u_0,v_0)\|_{L^2}\|\boldsymbol{\varepsilon}(u_r')\|_{L^2}\,\mathrm{d}r.$$

By (P2), we get:

$$\|\boldsymbol{\sigma}(u_r,v_0)-\boldsymbol{\sigma}(u_0,v_0)\|_{L^2}\le C\|\boldsymbol{\varepsilon}(u_r)-\boldsymbol{\varepsilon}(u_0)\|_{L^2}=Cr\|\boldsymbol{\varepsilon}(u_1-u_0)\|_{L^2}\le C\|u_1-u_0\|_{H^1}.$$

Since $\|\boldsymbol{\varepsilon}(u_r')\|_{L^2}=\|\boldsymbol{\varepsilon}(u_1-u_0)\|_{L^2}\le C\|u_1-u_0\|_{H^1}$, the proof is concluded. ☐

A higher integrability of the strains is often needed and it is obtained from the fact that $g(s)\in W^{1,q}(\Omega;\mathbb{R}^2)$ for $q>2$.

Lemma 4.5. *There exists $\tilde p\in(2,q]$ and $C>0$ such that for every $v\in\mathcal{V}$ with $\|v\|_{L^\infty}\le 1$ and every $s\in[0,1]$ it holds*

$$\|u\|_{W^{1,\tilde p}}\le C,$$

where $u=\operatorname{argmin}\{\mathcal{E}(u,v):\,u\in\mathcal{U}(s)\}$.

In the next two lemmata we provide two results of continuous dependence.

Lemma 4.6. *Let $\tilde{p}$ be as in Lemma 4.5. For $i = 1, 2$, let $s_i \in [0, 1]$ and $v_i \in \mathcal{V}$ with $0 \le v_i \le 1$ and denote by u_i the minimizer of $\mathcal{E}(\cdot, v_i)$ over $\mathcal{U}(s_i)$. Then*

$$\|u_2 - u_1\|_{W^{1,p}} \le C\big(|s_2 - s_1| + \|v_2 - v_1\|_{L^r}\big)$$

where $C > 0$ is independent of s_i and v_i while $\frac{1}{r} = \frac{1}{p} - \frac{1}{\tilde{p}}$ for $p \in [2, \tilde{p}]$.

Remark 4.7. Let us fix for the rest of the paper $\tilde{p} \in (2, q]$ as the exponent obtained in Lemma 4.5. Moreover, let us also fix $p \in (2, \tilde{p})$ and accordingly $r \in [1, +\infty)$ as in Lemma 4.6.

Lemma 4.8. *The energy $\mathcal{F}$ is locally Lipschitz continuous in $W^{1,p}(\Omega, \mathbb{R}^2) \times \mathcal{V}$.*

Proof. We consider the elastic and the dissipated energy separately. Regarding the latter, $\mathcal{L}(v) = \frac{1}{2}\|v - 1\|^2_{H^1}$ is locally Lipschitz continuous since the H^1-norm is Lipschitz continuous. As for the elastic energy, for every (u_1, v_1), $(u_2, v_2) \in W^{1,p}(\Omega, \mathbb{R}^2) \times H^1(\Omega)$, by Lemma 4.1 it holds:

$$\begin{aligned}
|\mathcal{E}(u_2, v_2) - \mathcal{E}(u_1, v_1)| &= |\mathcal{E}(u_2, v_2) - \mathcal{E}(u_1, v_2)| + |\mathcal{E}(u_1, v_2) - \mathcal{E}(u_1, v_1)| \\
&\le \int_\Omega \big|W(\boldsymbol{\varepsilon}(u_2), v_2) - W(\boldsymbol{\varepsilon}(u_1), v_2)\big| + \big|W(\boldsymbol{\varepsilon}(u_1), v_2) - W(\boldsymbol{\varepsilon}(u_1), v_1)\big| \, \mathrm{d}x \\
&\le \int_\Omega C(|\boldsymbol{\varepsilon}(u_1)| + |\boldsymbol{\varepsilon}(u_2)|) \, |\boldsymbol{\varepsilon}(u_2) - \boldsymbol{\varepsilon}(u_1)| \, \mathrm{d}x + \int_\Omega |v_2^2 - v_1^2| W_+(\boldsymbol{\varepsilon}(u_1)) \, \mathrm{d}x \\
&\le C(\|u_1\|_{W^{1,p'}} + \|u_2\|_{W^{1,p'}})\|u_2 - u_1\|_{W^{1,p}} + C\|v_2 - v_1\|_{L^{(p/2)'}}\|v_2 + v_1\|_{L^\infty}\|u_2\|^2_{W^{1,p/2}}.
\end{aligned}$$

From the continuous embedding of $H^1(\Omega)$ in $L^r(\Omega)$ for every $1 \le r < +\infty$, it follows that

$$|\mathcal{E}(u_2, v_2) - \mathcal{E}(u_1, v_1)| \le C\big(\|u_2 - u_1\|_{W^{1,p}} + \|v_2 - v_1\|_{H^1}\big)$$

and the lemma is proven. □

Next Lemma follows from [35], together with Lemma 4.5.

Lemma 4.9. *The reduced elastic energy $\tilde{\mathcal{E}}(t, \cdot)$ is Fréchet differentiable in $\mathcal{V}$, with respect to the H^1-norm.*

Lemma 4.10. *Let $u_1, u_2 \in W^{1,p}(\Omega; \mathbb{R}^2)$, $v_0 \in \mathcal{V}$ with $0 \le v_0 \le 1$ and*

$$v_1 = \operatorname{argmin}\{\, \mathcal{F}(u_1, v) \,:\, v \in \mathcal{V}, v \le v_0 \,\}, \tag{4.3}$$

$$v_2 = \operatorname{argmin}\{\, \mathcal{F}(u_2, v) \,:\, v \in \mathcal{V}, v \le v_0 \,\}. \tag{4.4}$$

Then

$$\|v_1 - v_2\|_{H^1} \le C\,\|u_1 - u_2\|_{H^1}\,, \tag{4.5}$$

where the constant C is independent of u_i and v_i.

Proof. We adapt the argument of [18], Proposition 1. The solutions of the constrained minimization problems (4.3) and (4.4) verify the following inequality:

$$\partial_v \mathcal{F}(u_i, v_i)\,[v - v_i] \ge 0 \qquad \text{for all } v \in \mathcal{V} \text{ such that } v \le v_0, \text{ for } i = 1, 2.$$

Therefore, for $i = 2$ and taking $v = v_1 \le v_0$, we obtain that $\partial_v \mathcal{F}(u_2, v_2)\,[v_2 - v_1] = -\partial_v \mathcal{F}(u_2, v_2)\,[v_1 - v_2] \le 0$. Moreover, for $i = 1$ and setting $v = v_2 \le v_0$, $\partial_v \mathcal{F}(u_1, v_1)\,[v_2 - v_1] \ge 0$. Hence

$$\langle \partial_v \mathcal{F}(u_2, v_2), v_2 - v_1 \rangle \le \langle \partial_v \mathcal{F}(u_1, v_1), v_2 - v_1 \rangle,$$

which yields

$$\langle \partial_v \mathcal{F}(u_2, v_2) - \partial_v \mathcal{F}(u_2, v_1), v_2 - v_1 \rangle \leq \langle \partial_v \mathcal{F}(u_1, v_1) - \partial_v \mathcal{F}(u_2, v_1), v_2 - v_1 \rangle. \tag{4.6}$$

The left hand side of (4.6) is greater than $G_c \|v_2 - v_1\|^2_{H^1}$. Indeed,

$$\begin{aligned}
&\langle \partial_v \mathcal{F}(u_2, v_2) - \partial_v \mathcal{F}(u_2, v_1), v_2 - v_1 \rangle \\
&\qquad = \int_\Omega v_2(v_2 - v_1)\, W_+(\boldsymbol{\varepsilon}(u_2))\, \mathrm{d}x + G_c \int_\Omega (v_2 - 1)(v_2 - v_1) + \nabla v_2 \cdot \nabla(v_2 - v_1)\, \mathrm{d}x \\
&\qquad\quad - \Big(\int_\Omega v_1(v_2 - v_1)\, W_+(\boldsymbol{\varepsilon}(u_2))\, \mathrm{d}x + G_c \int_\Omega (v_1 - 1)(v_2 - v_1) + \nabla v_1 \cdot \nabla(v_2 - v_1)\, \mathrm{d}x \Big) \\
&\qquad = \int_\Omega (v_2 - v_1)^2 W_+(\boldsymbol{\varepsilon}(u_2))\, \mathrm{d}x + G_c \int_\Omega (v_2 - v_1)^2 + |\nabla(v_2 - v_1)|^2\, \mathrm{d}x \;\geq G_c \|v_2 - v_1\|^2_{H^1}.
\end{aligned}$$

As for the right hand side of (4.6), we obtain:

$$\begin{aligned}
&\langle \partial_v \mathcal{F}(u_1, v_1) - \partial_v \mathcal{F}(u_2, v_1), v_2 - v_1 \rangle \\
&\qquad = \int_\Omega v_1(v_2 - v_1)\, W_+(\boldsymbol{\varepsilon}(u_1))\, \mathrm{d}x + G_c \int_\Omega (v_1 - 1)(v_2 - v_1) + \nabla v_1 \cdot \nabla(v_2 - v_1)\, \mathrm{d}x \\
&\qquad\quad - \Big(\int_\Omega v_1(v_2 - v_1)\, W_+(\boldsymbol{\varepsilon}(u_2))\, \mathrm{d}x + G_c \int_\Omega (v_1 - 1)(v_2 - v_1) + \nabla v_1 \cdot \nabla(v_2 - v_1)\, \mathrm{d}x \Big) \\
&\qquad = \int_\Omega v_1(v_2 - v_1)\big(W_+(\boldsymbol{\varepsilon}(u_1)) - W_+(\boldsymbol{\varepsilon}(u_2))\big)\, \mathrm{d}x.
\end{aligned}$$

Then, by Lemma 4.1 we get

$$\begin{aligned}
\langle \partial_v \mathcal{F}(u_1, v_1) - \partial_v \mathcal{F}(u_2, v_1), v_2 - v_1 \rangle &\leq C_2 \int_\Omega v_1 |v_2 - v_1| |\boldsymbol{\varepsilon}(u_1 - u_2)| (|\boldsymbol{\varepsilon}(u_1)| + |\boldsymbol{\varepsilon}(u_2)|)\, \mathrm{d}x \\
&\leq C_2 \|v_1\|_{L^\infty} \|v_2 - v_1\|_{L^r} \|u_1 - u_2\|_{H^1} (\|u_1\|_{W^{1,p}} + \|u_2\|_{W^{1,p}}) \\
&\leq C_2 \|v_2 - v_1\|_{H^1} \|u_1 - u_2\|_{H^1},
\end{aligned}$$

where we applied Hölder inequality with $\frac{1}{r} + \frac{1}{2} + \frac{1}{p} = 1$ and exploited the continuous embedding of $H^1(\Omega)$ in $L^r(\Omega)$ for any $1 \leq r < \infty$. This finishes the proof. ☐

5. Convergence of time discrete evolutions

5.1. Control in time and interpolation

We present a preliminary result that will then be used in the proof of Lemma 3.2:

Lemma 5.1. *There exists a constant $\tilde{C}$ such that $\sum_{k=1}^n \|v^n_k - v^n_{k-1}\|_{H^1} \leq \tilde{C}$ for every $n \in \mathbb{N}$.*

Proof. By separate minimality, $v^n_{k-1} \in \operatorname{argmin}\{\mathcal{F}(u^n_{k-1}, v) : v \in \mathcal{V} \text{ with } v \leq v^n_{k-2}\}$ and therefore, we also have $v^n_{k-1} \in \operatorname{argmin}\{\mathcal{F}(u^n_{k-1}, v) : v \in \mathcal{V} \text{ with } v \leq v^n_{k-1}\}$, since $v^n_{k-1} \leq v^n_{k-2}$. By definition $v^n_k \in \operatorname{argmin}\{\mathcal{F}(u^n_k, v) : v \in \mathcal{V} \text{ with } v \leq v^n_{k-1}\}$, hence, Lemma 4.10 yields $\|v^n_k - v^n_{k-1}\|_{H^1} \leq C\|u^n_k - u^n_{k-1}\|_{H^1}$. Moreover, by minimality of u^n_k, Lemma 4.6 provides $\|u^n_k - u^n_{k-1}\|_{W^{1,p}} \leq C(|s^n_k - s^n_{k-1}| + \|v^n_k - v^n_{k-1}\|_{L^r})$. Putting together these estimates, we get that for $k = 1, \ldots, n$

$$\|v^n_k - v^n_{k-1}\|_{H^1} \leq C\|u^n_k - u^n_{k-1}\|_{W^{1,p}} \leq C(|s^n_k - s^n_{k-1}| + \|v^n_k - v^n_{k-1}\|_{L^r}). \tag{5.1}$$

Let $\theta \in (0,1)$. Then, for $s := \frac{r(1-\theta)}{1-r\theta} \in (r,+\infty)$ and $\delta > 0$, to be chosen *a posteriori*, by interpolation in L^p-spaces we get:

$$\begin{aligned}\|v_k^n - v_{k-1}^n\|_{L^r} &\le \delta^{1-\theta}\|v_k^n - v_{k-1}^n\|_{L^s}^{1-\theta}\ \delta^{\theta-1}\|v_k^n - v_{k-1}^n\|_{L^1}^{\theta}\\ &\le (1-\theta)\delta\|v_k^n - v_{k-1}^n\|_{L^s} + \theta\delta^{1-1/\theta}\|v_k^n - v_{k-1}^n\|_{L^1},\end{aligned}$$

where in last inequality we used Young's inequality. Therefore it follows from (5.1) and from the embedding of $H^1(\Omega)$ in $L^s(\Omega)$ that

$$\|v_k^n - v_{k-1}^n\|_{H^1} \le C(\tfrac{1}{n} + C_1\delta\|v_k^n - v_{k-1}^n\|_{H^1} + C_2\delta^{1-1/\theta}\|v_k^n - v_{k-1}^n\|_{L^1})$$

and if we take δ sufficiently small (in such a way that $c := CC_1\delta < 1$) we can absorb the second term in the right-hand side into the left-hand side. Taking the sum over k, by the monotonicity of v_k^n with respect to k we finally get

$$(1-c)\sum_{k=1}^{n}\|v_k^n - v_{k-1}^n\|_{H^1} \le \sum_{k=1}^{n} C(\tfrac{1}{n} + C_2'\|v_k^n - v_{k-1}^n\|_{L^1}) \le C + C'\|v_n^n - v_0^n\|_{L^1},$$

hence the thesis follows. □

Proof of Lemma 3.2. Let $S > 0$. First of all we define the auxiliary non-uniform steps ρ_k^n according to the length $\|v_k^n - v_{k-1}^n\|_{H^1}$ at the given step s_k^n, *i.e.*, for $k = 1,\dots,n$:

$$\rho_k^n = \max\{\tfrac{1}{n}, \tfrac{1}{S}\|v_k^n - v_{k-1}^n\|_{H^1}\} = \begin{cases}\frac{1}{n} & \text{if } \|v_k^n - v_{k-1}^n\|_{H^1} \le \frac{S}{n}\\ \frac{1}{S}\|v_k^n - v_{k-1}^n\|_{H^1} & \text{otherwise.}\end{cases} \tag{5.2}$$

Note that $\rho_k^n \ge 1/n$. We define $r_0^n := 0$, and for $k = 1,\dots,n$, $r_k^n := r_{k-1}^n + \rho_k^n = \sum_{j=1}^{k}\rho_j^n$. It holds that $T^n := r_n^n$ is finite: indeed, by Lemma 5.1

$$T^n = \sum_{k=1}^{n}\rho_k^n \le \sum_{k=1}^{n}\big(\tfrac{1}{n} + \tfrac{1}{S}\|v_k^n - v_{k-1}^n\|_{H^1}\big) \le 1 + \tfrac{\tilde{C}}{S}.$$

Note that $T^n \ge 1$ since $\rho_k^n \ge 1/n$. Let $w^n : [0,T^n] \to [0,1]$ be the piecewise affine map which interpolates s_k^n in r_k^n. Clearly w^n is monotone increasing and bounded. Moreover, being $\rho_k^n \ge 1/n$, we have that for $r \in [r_{k-1}^n, r_k^n]$:

$$|\dot{w}^n(r)| = \frac{s_k^n - s_{k-1}^n}{r_k^n - r_{k-1}^n} = \frac{s_k^n - s_{k-1}^n}{\rho_k^n} \le n(s_k^n - s_{k-1}^n) = 1, \tag{5.3}$$

and thus w^n is 1-Lipschitz. There exists a (non-relabelled) subsequence of T^n converging to a certain $T \ge 1$. We reparametrize each of the corresponding (non-relabelled) parametrizations w^n on the interval $[0,T]$ as follows: $c^n(t) := w^n(t\frac{T^n}{T})$. In this way $c^n(0) = 0$ and $c^n(T) = 1$. Moreover, the parametrizations c^n are uniformly Lipschitz, since

$$\big|\dot{c}^n(t)\big| = \big|\dot{w}^n(t\tfrac{T^n}{T})\tfrac{T^n}{T}\big| \le \sup\{T^n\} =: C'. \tag{5.4}$$

Set $t_k^n := r_k^n\frac{T}{T^n}$ and $\tau_k^n := t_k^n - t_{k-1}^n = \rho_k^n\frac{T}{T^n}$, we define the piecewise affine interpolate of v_k^n in the times t_k^n as

$$v^n(t) := v_{k-1}^n + (t - t_{k-1}^n)\frac{v_k^n - v_{k-1}^n}{\tau_k^n} \qquad \text{for } t \in [t_{k-1}^n, t_k^n) \tag{5.5}$$

and observe that it has controlled speed. Indeed, for $t \in [t^n_{k-1}, t^n_k)$

$$\|\dot v^n(t)\|_{H^1} = \frac{\left\|v^n_k - v^n_{k-1}\right\|_{H^1}}{\tau^n_k}.$$

If $\|v^n_k - v^n_{k-1}\|_{H^1} \le \frac{S}{n}$, then $\tau^n_k = \frac{1}{n}\frac{T}{T^n}$, and thus $\|\dot v^n(t)\|_{H^1} \le \frac{S/n}{1/n}\frac{T^n}{T} \le SC'$. Otherwise, $\|v^n_k - v^n_{k-1}\|_{H^1} = S\rho^n_k$ and

$$\|\dot v^n(t)\|_{H^1} = \frac{\left\|v^n_k - v^n_{k-1}\right\|_{H^1}}{\tau^n_k} = \frac{S\rho^n_k}{\rho^n_k T/T^n} = S\tfrac{T^n}{T} \le SC'.$$

This finishes the first part of the proof. Let us now check that the sequence $\{u^n\}_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,\infty}(0,T;W^{1,p}(\Omega;\mathbb{R}^2))$. For $t_1, t_2 \in [0,T]$, by minimality and Lemma 4.6

$$\|u^n(t_2) - u^n(t_1)\|_{W^{1,p}} \le C|c^n(t_2) - c^n(t_1)| + C\left\|v^n(t_2) - v^n(t_1)\right\|_{L^r}.$$

This, by Sobolev embedding of $H^1(\Omega)$ in $L^r(\Omega)$, concludes the proof. □

Remark 5.2. Clearly there exists a subsequence (not relabelled) of c^n that converges uniformly in $[0,T]$. The limit parametrization $c : [0,T] \to [0,1]$ is Lipschitz continuous, increasing, and surjective. As anticipated in Section 3, given the parametrization c we will write, by abuse of notation, $\mathcal{U}(t) = \mathcal{U}(c(t))$ *etc.*

Differently from [33] we do not employ a reparametrization with respect to the arc length (in the intrinsic norms), rather we provide a time parametrization of the loading path. Note that, thanks to the control c^n introduced in Lemma 3.2, the evolutions are Lipschitz continuous in time.

5.2. Compactness

Lemma 5.3. *The piecewise affine interpolates v^n converge weakly* (up to non-relabelled subsequences) to a limit v in $W^{1,\infty}(0,T;H^1(\Omega))$; therefore v^n converge to v in $L^r(\Omega)$ for all $r < +\infty$ uniformly in $[0,T]$. Moreover, u^n converge weakly* to a limit u in $W^{1,\infty}(0,T;W^{1,p}(\Omega;\mathbb{R}^2))$.*

Proof. By Lemma 3.2, the piecewise affine interpolate v^n weakly* converges (up to non-relabelled subsequences) to an evolution v in $W^{1,\infty}([0,T];H^1(\Omega))$, that, by Aubin-Lions Lemma, is compactly embedded in $C([0,T];L^r(\Omega))$ for all $r < +\infty$. Therefore v^n converges to v uniformly in $L^r(\Omega)$. In the same way, u^n weakly* converges (up to non-relabelled subsequences) to a limit u in $W^{1,\infty}([0,T];W^{1,p}(\Omega,\mathbb{R}^2))$. □

Next, we will prove strong convergence in $H^1(\Omega)$ of the phase-field variable. Note that this property does not follow from Aubin-Lions lemma. Moreover, it cannot follow solely from unilateral minimality, since at the moment we know that the obstacles v^n_k converge weakly. The proof follows instead from unilateral minimality together with a (delicate) construction of back-in-time admissible competitors, inspired by [36], Proposition 2.13. To this purpose, we must first introduce a partition of the time set $[0,T]$ as follows. For every $n \in \mathbb{N}$, we define $\Upsilon^n := \{t^n_k : k = 0, ..., n\} \subset [0,T]$. We call $I_s \subseteq [0,T]$ the Kuratowski limit of this sequence of sets (up to non-relabelled subsequences). This is a compact set and therefore $I_u := [0,T] \setminus I_s = \bigcup_{j\in\mathbb{N}}(a_j, b_j)$ is the countable union of disjoint open intervals. For a brief overview on Kuratowski convergence, we refer to Appendix A.

The following Lemma immediately follows from Lemma 4.6 and will be widely employed in the sequel.

Lemma 5.4. *Let p and r be as in Remark 4.7. If $t^m \to t$ and v^m converges to v in $L^r(\Omega)$, then $u^m :=$ $\operatorname{argmin}\{\mathcal{F}(u, v^m) : u \in \mathcal{U}(t^m)\}$ converges to $u \in \operatorname{argmin}\{\mathcal{F}(u, v) : u \in \mathcal{U}(t)\}$ in $W^{1,p}(\Omega\ \mathbb{R}^2)$.*

We are now ready to present the main result on the strong convergence of the sequences $\{v^n(t)\}_{n\in\mathbb{N}}$ and $\{u^n(t)\}_{n\in\mathbb{N}}$ as they are defined in Lemma 3.2.

Lemma 5.5. *Let p and r be as in Remark 4.7. Let $t \in I_s$, then $v^n(t)$ converge strongly to $v(t)$ in $H^1(\Omega)$ and $u^n(t)$ converges to $u(t)$ strongly in $W^{1,p}(\Omega;\mathbb{R}^2)$.*

Proof. For sake of simplicity we divide the proof in three parts: we start by showing estimate (5.6), that will be used, in the second part, to prove that $v^n(t^n_{k_n})$ strongly converges to $v(t)$ in $H^1(\Omega)$, for every $t \in I_s$ and every $\{t^n_{k_n}\}_{n\in\mathbb{N}} \in \Upsilon^n$ converging to t. Finally, in the third part of the proof, we will prove the strong convergence of $v^n(t)$ and $u^n(t)$.

I. For $1 < s < \frac{\tilde{p}}{2}$ let $\mathcal{W} = \{W^s_+(\boldsymbol{\varepsilon}(u^n_k)) : n \in \mathbb{N}, 0 \le k \le n\}$. By Lemma 4.5 and by de la Vallée-Poussin Lemma we know that the family $\mathcal{W}$ is equi-integrable: indeed, being $u \in W^{1,\tilde{p}}(\Omega)$ and $W_+(\boldsymbol{\varepsilon}(u)) \le C_2|\boldsymbol{\varepsilon}(u)|^2$ (by Lem. 4.1), it holds that for $s < \frac{\tilde{p}}{2}$, there exists $c > 1$ such that $\int_\Omega W^{cs}_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x \le C$. Therefore, there exists a modulus of continuity $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to 0^+}\psi(x) = 0$ and $\int_A W^s_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x \le \psi(|A|)$ for every measurable set $A \subset \Omega$ and every choice of $n \in \mathbb{N}$ and $0 \le k \le n$.

We claim that there exists $C > 0$ such that the following holds: let $z \in \mathcal{V}$ such that $z \le v_0$, then for every $n \in \mathbb{N}$ and every index $0 \le m \le n$ we have

$$\int_{\{v^n_m < z\}} |\nabla v^n_m|^2\,\mathrm{d}x \le \int_{\{v^n_m < z\}} |\nabla z|^2\,\mathrm{d}x + C\psi^{1/s}\big(|\{v^n_m < z\}|\big). \tag{5.6}$$

Let $j := \max\{k : v^n_k \ge z\}$ (remember that $v^n_0 = v_0 \ge z$ and thus $j \ge 0$). If $m \le j$ then the set $\{v^n_m < z\}$ is empty and there is nothing to prove in (5.6). Otherwise, we provide a construction of admissible competitors going back-in-time up to t^n_j. For $j \le k \le m$ let $A^n_k = \{v^n_k < z\}$. Note that $A^n_j = \emptyset$, moreover, the monotonicity of v^n_k (with respect to k) implies that the sets A^n_k are increasing (with respect to k); in particular the family $\{A^n_k \setminus A^n_{k-1} :$ for $k = j+1, ..., m\}$ is a disjoint partition of the set $A^n_m = \{v^n_m < z\}$, which appears in (5.6).

- Let $k = j+1$. We introduce the auxiliary function $z^n_k = \max\{z, v^n_k\}$. Note that $z^n_k \le v^n_{k-1}$ since $v^n_k \le v^n_{k-1}$ (by monotonicity) and $z \le v^n_{k-1} = v^n_j$ (by definition of j). Hence z^n_k is an admissible competitor for the incremental problem and by minimality we get $\mathcal{F}(u^n_k, v^n_k) \le \mathcal{F}(u^n_k, z^n_k)$, *i.e.*,

$$\int_\Omega ((v^n_k)^2 + \eta)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x + \tfrac{1}{2}G_c\int_\Omega (v^n_k - 1)^2 + |\nabla v^n_k|^2\,\mathrm{d}x$$
$$\le \int_\Omega ((z^n_k)^2 + \eta)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x + \tfrac{1}{2}G_c\int_\Omega (z^n_k - 1)^2 + |\nabla z^n_k|^2\,\mathrm{d}x.$$

As $v^n_k \le z^n_k$ we get $(v^n_k - 1)^2 \ge (z^n_k - 1)^2$ and thus

$$\int_\Omega ((v^n_k)^2 + \eta)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x + \tfrac{1}{2}G_c\int_\Omega |\nabla v^n_k|^2\,\mathrm{d}x$$
$$\le \int_\Omega ((z^n_k)^2 + \eta)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x + \tfrac{1}{2}G_c\int_\Omega |\nabla z^n_k|^2\,\mathrm{d}x.$$

Note that $\{z^n_k \ne v^n_k\} = \{v^n_k < z\} = A^n_k \subset A^n_m$. Therefore, we can "localize" the above inequality in A^n_k, where $z^n_k = z$, obtaining

$$\begin{aligned}
\tfrac{1}{2}G_c\int_{A^n_k} |\nabla v^n_k|^2\,\mathrm{d}x &\le \tfrac{1}{2}G_c\int_{A^n_k} |\nabla z|^2\,\mathrm{d}x + \int_{A^n_k} \big(z^2 - (v^n_k)^2\big)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x \\
&\le \tfrac{1}{2}G_c\int_{A^n_k} |\nabla z|^2\,\mathrm{d}x + \int_{A^n_m} \big((v^n_{k-1})^2 - (v^n_k)^2\big)W_+(\boldsymbol{\varepsilon}(u^n_k))\,\mathrm{d}x \\
&\le \tfrac{1}{2}G_c\int_{A^n_k} |\nabla z|^2\,\mathrm{d}x + \tfrac{1}{2}C\|v^n_{k-1} - v^n_k\|_{L^{s'}}\|W_+(\boldsymbol{\varepsilon}(u^n_k))\|_{L^s(A^n_m)} \\
&\le \tfrac{1}{2}G_c\int_{A^n_k} |\nabla z|^2\,\mathrm{d}x + C\|v^n_{k-1} - v^n_k\|_{L^{s'}}\psi^{1/s}(|A^n_m|).
\end{aligned} \tag{5.7}$$

- For $j+1 < k \leq m$ we introduce the auxiliary functions

$$z_k^n = \min\{\max\{z, v_k^n\}, v_{k-1}^n\}.$$

Note that

$$\begin{cases} v_k^n \leq v_{k-1}^n < z & \text{in } A_{k-1}^n, \\ v_k^n < z \leq v_{k-1}^n & \text{in } A_k^n \setminus A_{k-1}^n, \\ z \leq v_k^n \leq v_{k-1}^n & \text{in } \Omega \setminus A_k^n. \end{cases} \tag{5.8}$$

Hence the function z_k^n can be equivalently written as

$$z_k^n = \begin{cases} v_{k-1}^n & \text{in } A_{k-1}^n, \\ z & \text{in } A_k^n \setminus A_{k-1}^n, \\ v_k^n & \text{in } \Omega \setminus A_k^n. \end{cases} \tag{5.9}$$

By construction $v_k^n \leq z_k^n \leq v_{k-1}^n$, hence z_k^n is again an admissible competitor in the incremental problem and by minimality we get first $\mathcal{F}(u_k^n, v_k^n) \leq \mathcal{F}(u_k^n, z_k^n)$ and then (arguing as above)

$$\begin{aligned} &\int_\Omega ((v_k^n)^2 + \eta) W_+(\boldsymbol{\varepsilon}(u_k^n))\,\mathrm{d}x + \tfrac{1}{2} G_c \int_\Omega |\nabla v_k^n|^2\,\mathrm{d}x \\ &\qquad\qquad \leq \int_\Omega ((z_k^n)^2 + \eta) W_+(\boldsymbol{\varepsilon}(u_k^n))\,\mathrm{d}x + \tfrac{1}{2} G_c \int_\Omega |\nabla z_k^n|^2\,\mathrm{d}x. \end{aligned}$$

Using (5.9), the "localization" of the previous inequality in $A_k^n \subset A_m^n$ becomes

$$\begin{aligned} \tfrac{1}{2} G_c \int_{A_k^n} |\nabla v_k^n|^2\,\mathrm{d}x &\leq \tfrac{1}{2} G_c \int_{A_{k-1}^n} |\nabla v_{k-1}^n|^2\,\mathrm{d}x + \tfrac{1}{2} G_c \int_{A_k^n \setminus A_{k-1}^n} |\nabla z|^2\,\mathrm{d}x \\ &\quad + \int_{A_k^n} \big((v_{k-1}^n)^2 - (v_k^n)^2\big) W_+(\boldsymbol{\varepsilon}(u_k^n))\,\mathrm{d}x \\ &\leq \tfrac{1}{2} G_c \int_{A_{k-1}^n} |\nabla v_{k-1}^n|^2\,\mathrm{d}x + \tfrac{1}{2} G_c \int_{A_k^n \setminus A_{k-1}^n} |\nabla z|^2\,\mathrm{d}x \\ &\quad + C \|v_{k-1}^n - v_k^n\|_{L^{s'}} \psi^{1/s}(|A_m^n|). \end{aligned}$$

Note that the above estimate actually holds also for $k = j+1$, indeed $A_j^n = \emptyset$ and thus it is equivalent to (5.7). Hence for every $j+1 \leq k \leq m$ we have

$$\int_{A_k^n} |\nabla v_k^n|^2\,\mathrm{d}x - \int_{A_{k-1}^n} |\nabla v_{k-1}^n|^2\,\mathrm{d}x \leq \int_{A_k^n \setminus A_{k-1}^n} |\nabla z|^2\,\mathrm{d}x + C \|v_{k-1}^n - v_k^n\|_{L^{s'}} \psi^{1/s}(|A_m^n|).$$

Let us take the sum for $j+1 \leq k \leq m$. Remember that: $\{A_k^n \setminus A_{k-1}^n\}$ provide a disjoint partition of $A_m^n = \{v_m^n < z\}$, $A_j^n = \emptyset$, and that v^n is of bounded variation in $L^r(\Omega)$ for any $1 \leq r < +\infty$, since it is uniformly Lipschitz continuous in $H^1(\Omega)$. Hence we get (5.6).

II. Since $t \in I_s$, there exists a sequence $\{t_{k_n}^n\}_{n \in \mathbb{N}}$ such that $t_{k_n}^n \to t$. Even though each index k_n obviously depends on n, we omit this dependence for simplicity of notation, setting $k := k_n$. As $v^n \overset{*}{\rightharpoonup} v$ in

$W^{1,\infty}(0,T;H^1(\Omega))$ we have $v_k^n = v^n(t_k^n) \rightharpoonup v(t)$ in $H^1(\Omega)$; by Sobolev embedding $v_k^n \to v(t)$ in $L^r(\Omega)$ for every $r \in [1,+\infty)$ and thus $v_k^n \to v(t)$ in measure, *i.e.*, for every $\delta > 0$

$$|\{|v_k^n - v(t)| > \delta\}| \to 0. \tag{5.10}$$

By minimality we have $u_k^n \in \operatorname{argmin}\{\mathcal{E}(u,v_k^n) : u \in \mathcal{U}(t_k^n)\}$. Since $v_k^n \to v(t)$ in $L^r(\Omega)$ and $t_k^n \to t$, by Lemma 5.4, $u_k^n \to u(t)$ strongly in $W^{1,p}(\Omega;\mathbb{R}^2)$ where $p = \frac{r\tilde{p}}{\tilde{p}+r}$ and

$$u(t) \in \operatorname{argmin}\{\mathcal{E}(u,v(t)) : u \in \mathcal{U}(t)\}. \tag{5.11}$$

Let us show that $v_k^n \to v(t)$ strongly in $H^1(\Omega)$. By lower semicontinuity of the norm we have

$$\int_\Omega |\nabla v(t)|^2 \,\mathrm{d}x \le \liminf_{n\to+\infty} \int_\Omega |\nabla v_k^n|^2 \,\mathrm{d}x,$$

therefore it is enough to show that

$$\int_\Omega |\nabla v(t)|^2 \,\mathrm{d}x \ge \limsup_{n\to+\infty} \int_\Omega |\nabla v_k^n|^2 \,\mathrm{d}x,$$

so that $\lim_{n\to+\infty} \int_\Omega |\nabla v_k^n|^2 \,\mathrm{d}x = \int_\Omega |\nabla v(t)|^2 \,\mathrm{d}x$. Indeed, combining the convergence in norm with the fact that $v_k^n \rightharpoonup v(t)$ in $H^1(\Omega)$, we get strong convergence in $H^1(\Omega)$.

For $\delta > 0$ let $z := \max\{0, v(t) - \delta\}$ and $z^- := \min\{z, v_k^n\}$. Let us also introduce the set

$$Z_k^n = \{z^- < z\} = \{v_k^n < z = \max\{0, v(t)-\delta\}\} = \{v_k^n < v(t) - \delta\}.$$

Note that $|Z_k^n| \to 0$ by convergence in measure (5.10). Moreover, since $z^- \le v_k^n \le v_{k-1}^n$, it turns out that z^- is a competitor for the problem

$$\min\{\mathcal{F}(v,u_k^n) : v \in \mathcal{V},\ v \le v_{k-1}^n\}$$

which is solved by v_k^n. Therefore we have $\mathcal{F}(v_k^n,u_k^n) \le \mathcal{F}(z^-,u_k^n)$, *i.e.*,

$$\begin{aligned}\int_\Omega ((v_k^n)^2 + \eta) W_+(\boldsymbol{\varepsilon}(u_k^n)) \,\mathrm{d}x + \tfrac{1}{2} G_c \int_\Omega (v_k^n - 1)^2 + |\nabla v_k^n|^2 \,\mathrm{d}x \\ \le \int_\Omega ((z^-)^2 + \eta) W_+(\boldsymbol{\varepsilon}(u_k^n)) \,\mathrm{d}x + \tfrac{1}{2} G_c \int_\Omega (z^- - 1)^2 + |\nabla z^-|^2 \,\mathrm{d}x.\end{aligned}$$

Since $v_k^n \ge z^-$ we can write

$$\int_\Omega (v_k^n - 1)^2 + |\nabla v_k^n|^2 \,\mathrm{d}x \le \int_\Omega (z^- - 1)^2 + |\nabla z^-|^2 \,\mathrm{d}x.$$

In the set Z_k^n we have $v_k^n < z$ and thus $z^- = v_k^n$. Hence, we can “localize” the previous integral inequality in the set $\Omega \setminus Z_k^n = \{z^- \ne v_k^n\} = \{z^- = z\}$ and re-arranging the terms we get

$$\begin{aligned}\int_{\Omega\setminus Z_k^n} |\nabla v_k^n|^2 \,\mathrm{d}x &\le \int_{\Omega\setminus Z_k^n} |\nabla z^-|^2 \,\mathrm{d}x + \int_{\Omega\setminus Z_k^n} (z^- - 1)^2 - (v_k^n - 1)^2 \,\mathrm{d}x \\ &= \int_{\Omega\setminus Z_k^n} |\nabla z|^2 \,\mathrm{d}x + \int_{\Omega\setminus Z_k^n} (z - 1)^2 - (v_k^n - 1)^2 \,\mathrm{d}x.\end{aligned}$$

Note that $Z^n_k = \{v^n_k < z\}$, hence by (5.6) with $m = k$ we get

$$\int_{Z^n_k} |\nabla v^n_k|^2 \,\mathrm{d}x \le \int_{Z^n_k} |\nabla z|^2 \,\mathrm{d}x + C\psi^{1/s}\big(|Z^n_k|\big).$$

Taking the sum of the previous inequalities and remembering that $\nabla z = \nabla v(t)$ provides

$$\int_\Omega |\nabla v^n_k|^2 \,\mathrm{d}x \le \int_\Omega |\nabla v(t)|^2 \,\mathrm{d}x + \int_{\Omega\setminus Z^n_k} (z-1)^2 - (v^n_k - 1)^2 \,\mathrm{d}x + C\psi^{1/s}\big(|Z^n_k|\big).$$

Clearly last term is infinitesimal. For the second term we write

$$|(1-z)^2 - (1-v^n_k)^2| = |(v^n_k - z)(2 - z - v^n_k)| \le C|v^n_k - z| \le C|v^n_k - v(t)| + C\delta.$$

Hence

$$\limsup_{n\to\infty} \int_{\Omega\setminus Z^n_k} (z-1)^2 - (v^n_k - 1)^2 \,\mathrm{d}x \le \limsup_{n\to\infty} \int_\Omega |v^n_k - v(t)| \,\mathrm{d}x + C\delta|\Omega| \le C'\delta.$$

We conclude by the arbitrariness of $\delta > 0$.

III. In conclusion, applying Lemma 3.2, we get:

$$\|v^n(t) - v(t)\|_{H^1} \le \|v^n(t) - v^n(t^n_k)\|_{H^1} + \|v^n(t^n_k) - v(t)\|_{H^1} \le C|t^n_k - t| + \|v^n_k - v(t)\|_{H^1} \to 0.$$

The strong convergence of $u^n(t)$ follows by Lemma 4.6. □

Remark 5.6. We recall that the set I_u, being the complement of a compact subset of $[0,T]$, is the (at most) countable union of its connected components, denoted (a_j, b_j) for $j \in \mathbb{N}$. Therefore, the set of right or left-isolated points $I_{isol} = \left(\bigcup_{j\in\mathbb{N}} a_j\right) \cup \left(\bigcup_{j\in\mathbb{N}} b_j\right)$ has zero measure.

We will now show that on each component of I_u the limit parametrization c is constant and the limit evolution v is affine. To do so, let us state some preliminary results.

Lemma 5.7. *Let us consider a connected component (a,b) of I_u. There exists a sequence $\{t^n_{k_n} \in \Upsilon^n\}_{n\in\mathbb{N}}$ such that $t^n_{k_n} \to a$ and $t^n_{k_n+1} \to b$.*

Proof. First of all, observe that for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\Upsilon^n \cap (a+\varepsilon, b-\varepsilon) = \emptyset$ for all $n > N_\varepsilon$. Indeed, if there was a sequence $\{t^{n_i}_{k_i}\}_{i\in\mathbb{N}}$ such that $t^{n_i}_{k_i} \in \Upsilon^{n_i} \cap (a+\varepsilon, b-\varepsilon)$, we could take a subsequence converging to a limit $t \in [a+\varepsilon, b-\varepsilon]$; this t would be in Kuratowski limit superior of Υ^n, that is exactly I_s. However, by definition $[a+\varepsilon, b-\varepsilon] \subset I_u = [0,T] \setminus I_s$.

Now, set $a < a' < b$ and define $a_n := \max\{t^n_k \in \Upsilon^n \,:\, t^n_k \le a'\}$. This sequence must converge to a; let's prove it by contradiction. If there was a sequence $\{n_i\}_{i\in\mathbb{N}}$ and $\varepsilon > 0$ (sufficiently small) such that $a_{n_i} > a + \varepsilon$ then $a_{n_i} \in (\Upsilon^{n_i} \cap (a+\varepsilon, a']) \subset (\Upsilon^{n_i} \cap (a+\varepsilon, b-\varepsilon))$ which is empty for all $n > N_\varepsilon$. On the other hand, if there was a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that $a_{n_i} < a - \varepsilon$, by definition of a_n, $\Upsilon^{n_i} \cap [a-\varepsilon, a') = \emptyset$; hence, there wouldn't exist elements $t^{n_i}_{k_i} \in \Upsilon^{n_i}$ converging (as $i \to \infty$) to a, that, however, belongs to the Kuratowski limit of Υ^n.

As we have seen, $\Upsilon^n \cap (a+\varepsilon, b-\varepsilon) = \emptyset$ for $n \ge N_\varepsilon$. Let k_n be such that $t^n_{k_n} = a_n$, then $t^n_{k_n+1} \ge b - \varepsilon$ at least definitely. In particular, for $n \ge N_\varepsilon$, $t^n_{k_n+1} = \min\{t^n_k \in \Upsilon^n \,:\, t^n_k \ge b - \varepsilon\}$. We thus proceed as we did above and obtain that $t^n_{k_n+1} \to b$. □

Lemma 5.8. *The function c (appearing in Lem. 3.2) is constant on each of the connected components of I_u.*

Proof. Let's consider the connected component (a, b) of I_u. From Lemma 5.7 we know that there exists a sequence $\{t^n_{k_n} \in \Upsilon^n\}_{n\in\mathbb{N}}$ such that $t^n_{k_n} \to a$ and $t^n_{k_n+1} \to b$. Now, since $c^n \to c$ uniformly in $[0, T]$, it holds that $s^n_{k_n} = c^n(t^n_{k_n}) \to c(a)$ and $s^n_{k_n+1} = c^n(t^n_{k_n+1}) \to c(b)$. Hence $c(b) - c(a) = \lim_{n\to\infty} s^n_{k_n+1} - s^n_{k_n} = \lim_{n\to\infty} 1/n = 0$, and the Lemma is proven. ☐

Corollary 5.9. *The set I_s has positive measure.*

Proof. By Lemma 5.8, $\dot c = 0$ in the open set I_u therefore $\operatorname{supp}(\dot c) \subset I^c_u = I_s$. Now, the set $\operatorname{supp}(\dot c)$ has positive measure, indeed,

$$\int_{(0,T)} \dot c(t)\, \mathrm{d}t = c(T) - c(0) = 1$$

and, being $0 \le \dot c \le 1$,

$$1 = \int_{(0,T)} \dot c(t)\, \mathrm{d}t = \int_{\operatorname{supp}(\dot c)} \dot c(t)\, \mathrm{d}t \le |\operatorname{supp}(\dot c)|.$$

The proof is concluded. ☐

Proposition 5.10. *Let (a, b) be a connected component of the set I_u. The limit evolution v is the affine interpolate between $v(a)$ and $v(b)$.*

Proof. Let $t \in (a, b)$ and write it as $t = \theta b + (1 - \theta)a$ for a certain $\theta \in (0, 1)$. Let $t^n_{k_n}$ be as in Lemma 5.7 and, as above, let us write t^n_k for simplicity. As shown in step II of the proof of Lemma 5.5, $v^n(t^n_k) \to v(a)$ and $v^n(t^n_{k+1}) \to v(b)$ strongly in $H^1(\Omega)$. For n sufficiently large, $t \in (t^n_k, t^n_{k+1})$, so we can write it as $t = \theta^n t^n_{k+1} + (1 - \theta^n)t^n_k$, where

$$\theta^n = \frac{t - t^n_k}{t^n_{k+1} - t^n_k} \to \frac{t - a}{b - a} = \theta.$$

By definition

$$v^n(t) = \theta^n v^n(t^n_{k+1}) + (1 - \theta^n)v^n(t^n_k) \to \theta v(b) + (1 - \theta)v(a)$$

strongly in $H^1(\Omega)$, which is indeed the affine interpolate between $v(a)$ and $v(b)$. ☐

From Proposition 5.10 together with Lemma 5.5 we get the following result.

Corollary 5.11. *Let $t \in [0, T] \setminus I_s$, then $v^n(t)$ converge strongly to $v(t)$ in $H^1(\Omega)$ and $u^n(t)$ converge strongly to $u(t)$ in $W^{1,p}(\Omega; \mathbb{R}^2)$.*

This Corollary, together with Lemma 5.5 gives Lemma 3.4.

5.3. Equilibrium

Theorem 5.12. *For every $t \in I_s$ the limit configuration $(u(t), v(t))$ is an equilibrium point for the energy $\mathcal{F}(\cdot, \cdot)$, which means that*

$$\partial_u \mathcal{F}(u(t), v(t))[\phi] = 0 \quad \textit{for all } \phi \in \Phi, \qquad \partial_v \mathcal{F}(u(t), v(t))\,[\xi] \ge 0 \quad \textit{for all } \xi \in \Xi. \tag{5.12}$$

Proof. By definition of Kuratowski limit, there exist a sequence $\{t^n_{k_n} \in \Upsilon^n\}_{n\in\mathbb{N}}$ such that $t^n_{k_n} \to t$. As usual, we write for simplicity t^n_k. We have seen in the proof of Lemma 5.5 that $v^n(t^n_k) \to v(t)$ in $H^1(\Omega)$, and thus in

$L^r(\Omega)$ for any $r < \infty$, and that $u^n(t_k^n) \to u(t) = \operatorname{argmin}\{\mathcal{E}(u, v(t)) : u \in \mathcal{U}(t)\}$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ for $p \in (2, \tilde{p})$; in light of these results, from (3.1) we obtain that

$$\partial_u \mathcal{F}(u(t), v(t))[\phi] = \int_\Omega \boldsymbol{\sigma}(u(t), v(t)) : \boldsymbol{\varepsilon}(\phi)\,\mathrm{d}x = 0 \qquad \text{for all } \phi \in \mathcal{V}. \tag{5.13}$$

By definition $v^n(t_k^n) \in \operatorname{argmin}\{\mathcal{F}(u_k^n, v) : v \in \mathcal{V} \text{ with } v \le v_{k-1}^n\}$, and hence:

$$\int_\Omega v^n(t_k^n)\xi\, W_+(\boldsymbol{\varepsilon}(u_k^n))\,\mathrm{d}x + G_c \int_\Omega (v^n(t_k^n) - 1)\xi + \nabla v^n(t_k^n)\cdot\nabla\xi\,\mathrm{d}x \ge 0 \qquad \text{for all } \xi \in \Xi. \tag{5.14}$$

As aforementioned $v^n(t_k^n)$ converges to $v(t)$ strongly in $H^1(\Omega)$ and $u^n(t_k^n) \to u(t)$ in $W^{1,p}(\Omega;\mathbb{R}^2)$, therefore by Lemma 4.1 we get $W_+(\boldsymbol{\varepsilon}(u_k^n)) \to W_+(\boldsymbol{\varepsilon}(u(t))$ in $L^{\frac{p}{2}}(\Omega)$, where $\frac{p}{2} > 1$. Thereby, passing to the limit in (5.14), we get:

$$\partial_v \mathcal{F}(u(t), v(t))\,[\xi] = \int_\Omega v(t)\xi\, W_+(\boldsymbol{\varepsilon}(u(t))\,\mathrm{d}x + G_c\int_\Omega (v(t)-1)\xi + \nabla v(t)\cdot\nabla\xi\,\mathrm{d}x \ge 0 \tag{5.15}$$

for all $\xi \in \Xi$. □

5.4. Power identity

In this section we will prove Proposition 3.8, *i.e.*, for every $t \in [0, T]$

$$\mathcal{F}(u(t), v(t)) = \mathcal{F}(u_0, v_0) + \int_0^t \mathcal{P}(s, u(s), v(s))\,\mathrm{d}s + \int_{I_u \cap (0,t)} \partial_v \mathcal{F}(u(s), v(s))[\dot{v}(s))]\,\mathrm{d}s.$$

We recall the definition of power (of external forces)

$$\mathcal{P}(t, u(t), v(t)) := \int_\Omega \boldsymbol{\sigma}(u(t), v(t)) : \boldsymbol{\varepsilon}(\dot{a}(t)\hat{g})\,\mathrm{d}x, \tag{5.16}$$

where $a(t) := \alpha(c(t))$. Notice that this is a good definition for the power: indeed, being $u(t)$ a minimum for the energy, by Theorem 3.7 and by Green formula, if u is sufficiently regular it holds

$$\begin{aligned}\mathcal{P}(t, u(t), v(t)) &= {}_{H^{-1/2}(\partial_D\Omega)}\langle \boldsymbol{\sigma}(u(t), v(t))\cdot n, \dot{a}(t)\hat{g}\rangle_{H^{1/2}(\partial_D\Omega)} \\ &= \int_{\partial_D\Omega} (\boldsymbol{\sigma}(u(t), v(t))\cdot n)\cdot\gamma(\dot{a}(t)\hat{g})\,\mathrm{d}s,\end{aligned}$$

this is indeed the product between force and the velocity imposed on the boundary. Finally note that

$$\mathcal{P}(t, u(t), v(t)) = \partial_u \mathcal{F}(u(t), v(t))[\dot{u}(t)], \tag{5.17}$$

since by Lemma 3.2 we can write $\dot{u}(t)$ as the sum of $\dot{g}(t) = \dot{a}(t)\hat{g}$ and a function $w(t) \in \Phi$.

Theorem 5.13. *For a.e. $t \in I_s$, it holds*

$$\dot{\mathcal{F}}(u(t), v(t)) = \mathcal{P}(t, u(t), v(t)).$$

Proof. By Theorem 5.12, $(u(t), v(t))$ is an equilibrium point for the total energy $\mathcal{F}(\cdot,\cdot)$, and since $\dot v(t)$ is in $H^1(\Omega)$ and $\dot v(t) \le 0$, by (5.12) the following estimate for the total derivative of $\mathcal{F}$ holds almost everywhere in I_s:

$$\dot{\mathcal{F}}(u(t), v(t)) = \partial_u \mathcal{F}(u(t), v(t))[\dot u(t)] + \partial_v \mathcal{F}(u(t), v(t))[\dot v(t)] \ge \mathcal{P}(t, u(t), v(t)), \tag{5.18}$$

where we used (5.17). We recall Remark 3.9, where we proved that $t \mapsto \mathcal{F}(u(t), v(t))$ is Lipschitz continuous, and thus a.e. differentiable. To derive the thesis it is now sufficient to prove the inverse inequality.

I. In this first part of the proof we "go back" to the time discrete setting, in order to prove that

$$\mathcal{F}(u^n_{k+1}, v^n_{k+1}) \le \mathcal{F}(u^n_k, v^n_k) + \int_{t^n_k}^{t^n_{k+1}} \mathcal{P}^n(t, u^n(t), v^n(t))\,\mathrm{d}t + C|t^n_{k+1} - t^n_k|^2, \tag{5.19}$$

where

$$\mathcal{P}^n(t, u, v) := \int_\Omega \boldsymbol{\sigma}(u, v) : \boldsymbol{\varepsilon}(\dot a^n(t)\hat g)\,\mathrm{d}x.$$

We set $a^n_k := \alpha(s^n_k) = \alpha \circ c^n(t^n_k)$ for $k = 0, ..., n$ and denote by a^n the piecewise affine interpolant of a^n_k in t^n_k. By minimality $\mathcal{F}(u^n_{k+1}, v^n_{k+1}) \le \mathcal{F}(u^n_{k+1}, v^n_k) = \mathcal{E}(u^n_{k+1}, v^n_k) + G_c\mathcal{L}(v^n_k)$. We define for any $j = 0, ..., n$, $\hat u^n_j := \operatorname{argmin}\{\mathcal{E}(u, v^n_j) : u = \hat g \text{ on } \partial_D\Omega\}$ and note that, by Lemma 4.1, we can express $\mathcal{E}(u^n_{k+1}, v^n_k)$ as $(a^n_{k+1})^2\mathcal{E}(\hat u^n_{k+1}, v^n_k)$. According to Corollary 4.4, $\mathcal{E}(\hat u^n_{k+1}, v^n_k) \le \mathcal{E}(\hat u^n_k, v^n_k) + C\|\hat u^n_{k+1} - \hat u^n_k\|^2_{H^1}$ and by Lemma 4.6, last term is bounded by $C\|v^n_{k+1} - v^n_k\|^2_{H^1} \le C|t^n_{k+1} - t^n_k|^2$, with the inequality following from Lemma 3.2. Summarizing,

$$\mathcal{E}(u^n_{k+1}, v^n_k) \le (a^n_{k+1})^2\mathcal{E}(\hat u^n_k, v^n_k) + C'|t^n_{k+1} - t^n_k|^2. \tag{5.20}$$

We write the first addendum as

$$\begin{aligned}
(a^n_{k+1})^2\mathcal{E}(\hat u^n_k, v^n_k) &= (a^n_k)^2\mathcal{E}(\hat u^n_k, v^n_k) + (a^n_{k+1} - a^n_k)\left(2a^n_k + (a^n_{k+1} - a^n_k)\right)\mathcal{E}(\hat u^n_k, v^n_k)\\
&= \mathcal{E}(u^n_k, v^n_k) + \Big(2a^n_k(a^n_{k+1} - a^n_k) + (a^n_{k+1} - a^n_k)^2\Big)\mathcal{E}(\hat u^n_k, v^n_k)\\
&\le \mathcal{E}(u^n_k, v^n_k) + 2a^n_k(a^n_{k+1} - a^n_k)\mathcal{E}(\hat u^n_k, v^n_k) + C|t^n_{k+1} - t^n_k|^2,
\end{aligned}$$

where the last inequality comes from the fact that $\mathcal{E}(\hat u^n_k, v^n_k)$ is bounded and a^n is Lipschitz continuous. Observe that, by Corollary 4.2 and by the definition of $\hat u^n_k$, $\mathcal{E}(\hat u^n_k, v^n_k) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(\hat u^n_k, v^n_k) : \boldsymbol{\varepsilon}(\hat g)\mathrm{d}x$, since, being $\hat u^n_k - \hat g$ an admissible variation, $\partial_u\mathcal{E}(u^n_k, v^n_k)[\hat u^n_k - \hat g] = 0$. By bilinearity we hence get that

$$2a^n_k(a^n_{k+1} - a^n_k)\mathcal{E}(\hat u^n_k, v^n_k) = \int_\Omega \boldsymbol{\sigma}(u^n_k, v^n_k) : \boldsymbol{\varepsilon}((a^n_{k+1} - a^n_{k+1})\hat g)\,\mathrm{d}x = \int_{t^n_k}^{t^n_{k+1}} \mathcal{P}^n(t, u^n_k, v^n_k)\,\mathrm{d}t$$

and (5.20) can finally be rewritten as:

$$\mathcal{E}(u^n_{k+1}, v^n_k) \le \mathcal{E}(u^n_k, v^n_k) + \int_{t^n_k}^{t^n_{k+1}} \mathcal{P}^n(t, u^n_k, v^n_k)\,\mathrm{d}t + C|t^n_{k+1} - t^n_k|^2. \tag{5.21}$$

Now, in order to obtain (5.19), we add and subtract $\mathcal{P}^n(t,u^n(t),v^n(t))$ to $\mathcal{P}^n(t,u^n_k,v^n_k)$:

$$\int_\Omega \Big(\boldsymbol{\sigma}(u^n_k,v^n_k)-\boldsymbol{\sigma}(u^n(t),v^n(t))\Big):\boldsymbol{\varepsilon}(\dot a^n(t)\hat g)\mathrm{d}x+\mathcal{P}^n(t,u^n(t),v^n(t)).$$

We estimate the first term as follows (writing ± to add and subtract):

$$\begin{aligned}\int_\Omega \Big(\boldsymbol{\sigma}(u^n_k,v^n_k)-\boldsymbol{\sigma}(u^n(t),v^n(t))\Big):\boldsymbol{\varepsilon}(\dot a^n(t)\hat g)\mathrm{d}x \pm \int_\Omega \boldsymbol{\sigma}(u^n(t),v^n_k):\boldsymbol{\varepsilon}(\dot a^n(t)\hat g)\,\mathrm{d}x \\ =\dot a^n(t)\Big[\int_\Omega \Big(\boldsymbol{\sigma}(u^n_k,v^n_k)-\boldsymbol{\sigma}(u^n(t),v^n_k)\Big):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x \\ +\int_\Omega \Big(\boldsymbol{\sigma}(u^n(t),v^n_k)-\boldsymbol{\sigma}(u^n(t),v^n(t))\Big):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x\Big].\end{aligned} \tag{5.22}$$

By hypothesis (H2), the first addendum is lower than $C\|u^n_k-u^n(t)\|_{H^1}$. As for the second addendum, by Lemma 4.1 we rewrite it explicitly:

$$\begin{aligned}\int_\Omega \big((v^n_k)^2-v^n(t)^2\big)\partial_{\boldsymbol{\varepsilon}}W_+(\boldsymbol{\varepsilon}(u^n(t))):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x \\ \le C_1\|v^n_k-v^n(t)\|_{L^s}\|v^n_k+v^n(t)\|_{L^\infty}\|u^n(t)\|_{W^{1,p}}\|\hat g\|_{W^{1,q}},\end{aligned}$$

where $\frac{1}{s}+\frac{1}{p}+\frac{1}{q}=1$. By the continuous embedding of $H^1(\Omega)$ in $L^s(\Omega)$,

$$(5.22)\le C\big(\|u^n_k-u^n(t)\|_{H^1}+\|v^n_k-v^n(t)\|_{H^1}\big)\le C'|t-t^n_k|,$$

where we used the fact that u^n and v^n are Lipschitz continuous (see Lem. 3.2). As a consequence,

$$\int_{t^n_k}^{t^n_{k+1}}\mathcal{P}^n(t,u^n_k,v^n_k)\,\mathrm{d}t\le\int_{t^n_k}^{t^n_{k+1}}\mathcal{P}^n(t,u^n(t),v^n(t))\,\mathrm{d}t+C'|t^n_{k+1}-t^n_k|^2$$

and (5.19) is proven.

II. Starting from the estimate obtained in the discrete setting, let us now see what happens in the limit. Since the set of isolated points of I_s has zero measure, we can state that for a.e. $t\in I_s$, there exists a sequence $\{t_j\in I_s\}_{j\in\mathbb{N}}$ such that $t_j\to t^+$ or $t_j\to t^-$ with $t_j\neq t$. Let us consider the first case (we use the same reasoning for the second). Let $t^n_l,t^n_h\in\Upsilon^n$ be such that $t^n_l\to t_j$ and $t^n_h\to t$ (as usual we omit the dependence on n in the indices l and h). Then, taking the sum of (5.19) for $k=h,\dots,l-1$, yields

$$\mathcal{F}(u^n_l,v^n_l)\le\mathcal{F}(u^n_h,v^n_h)+\int_{t^n_h}^{t^n_l}\mathcal{P}^n(t,u^n(t),v^n(t))\,\mathrm{d}t+C\sum_{k=h}^{l-1}|t^n_{k+1}-t^n_k|^2. \tag{5.23}$$

Let us pass the limit as $n\to+\infty$. By strong convergence, seen in the proof of Lemma 5.5, we infer that $\mathcal{F}(u^n_l,v^n_l)\to\mathcal{F}(u(t_j),v(t_j))$ and $\mathcal{F}(u^n_h,v^n_h)\to\mathcal{F}(u(t),v(t))$; indeed, we can write $v^n_l=(v^n(t^n_l)-v^n(t_j))+v^n(t_j)$, where $v^n(t_j)\to v(t_j)$ in $H^1(\Omega)$ by Lemma 3.4, and $\|v^n(t^n_l)-v^n(t_j)\|_{H^1}\le C|t^n_l-t_j|\to 0$ by Lipschitz continuity. Similarly we get that $u^n_l\to u(t_j)$ in $W^{1,p}(\Omega)$. We can apply the same reasoning to (u^n_h,v^n_h). If we take j sufficiently large that $|t_j-t|<\varepsilon$, then $|t^n_{k+1}-t^n_k|<\varepsilon$ and thus

$$\sum_{k=h}^{l-1}|t^n_{k+1}-t^n_k|^2\le\varepsilon|t^n_l-t^n_h|.$$

It now remains to prove that

$$\lim_{n\to+\infty}\int_{t_h^n}^{t_l^n}\mathcal{P}^n(t,u^n(t),v^n(t))\,\mathrm{d}t=\int_t^{t_j}\mathcal{P}(t,u(t),v(t))\,\mathrm{d}t.$$

To this end, let us write

$$\mathcal{P}^n(t,u^n(t),v^n(t)):=\dot a^n(t)\int_\Omega\boldsymbol{\sigma}(u^n(t),v^n(t)):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x:=\dot a^n(t)P^n(t).$$

We will now prove that $a^n\overset{*}{\rightharpoonup}a:=\alpha\circ c$ in $W^{1,\infty}(0,T)$ and that $P^n\to P$ strongly in $L^1(0,T)$, where $P(t):=\int_\Omega\boldsymbol{\sigma}(u(t),v(t)):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x$. Let us start from a^n. It is enough to check the pointwise convergence of a^n to a in $[0,T]$. We start observing what happens on the points $t_k^n\in\Upsilon^n$:

$$|a^n(t_k^n)-a(t_k^n)|=|\alpha(c^n(t_k^n))-\alpha(c(t_k^n))|\le C|c^n(t_k^n)-c(t_k^n)|,\tag{5.24}$$

where we used the fact that α is Lipschitz continuous. Let us now see what happens on I_u. As we have proven in Lemma 5.8, c is constant on the connected components of I_u, and so it is $a=\alpha\circ c$. For $t\in I_u$, there exists n sufficiently big such that $t\in(t_k^n,t_{k+1}^n)\subset I_u$ (with k depending on n). By definition of a^n, $a^n(t)=(1-\theta^n)a^n(t_k^n)+\theta^na^n(t_{k+1}^n)$ for some $\theta^n\in(0,1)$, while a, being constant on $[t_k^n,t_{k+1}^n]$, can be written as any convex combination of $a(t_k^n)$ and $a(t_{k+1}^n)$. Hence:

$$|a^n(t)-a(t)|=(1-\theta^n)|a^n(t_k^n)-a(t_k^n)|+\theta^n|a^n(t_{k+1}^n)-a(t_{k+1}^n))|\to0,$$

following from (5.24) and the fact that $c^n\to c$ uniformly in $[0,T]$. Lastly, for $t\in I_s$, we can take a sequence $\{t_k^n\in\Upsilon^n\}_{n\in\mathbb{N}}$ (with k depending on n) converging to t and therefore:

$$\begin{aligned}|a^n(t)-a(t)|&\le|a^n(t)-a^n(t_k^n)|+|a^n(t_k^n)-a(t_k^n)|+|a(t_k^n)-a(t)|\\&\le C|t-t_k^n|+|a^n(t_k^n)-a(t_k^n)|\to0,\end{aligned}$$

where we used the fact that a^n and a are Lipschitz continuous together with (5.24) and the fact that $c^n\to c$ uniformly in $[0,T]$.

Let us now check that $P^n\to P$ strongly in $L^1(0,T)$. We know that $v^n(t)\to v(t)$ strongly in $L^r(\Omega)$ for any $1\le r<+\infty$, and that $u^n(t)\to u(t)$ strongly in $W^{1,p}(\Omega)$, therefore by Lemma 4.1

$$\begin{aligned}\big|\boldsymbol{\sigma}(u^n(t),v^n(t))-\boldsymbol{\sigma}(u(t),v(t))\big|&\le\big|\boldsymbol{\sigma}(u^n(t),v^n(t))-\boldsymbol{\sigma}(u^n(t),v(t))\big|+\big|\boldsymbol{\sigma}(u^n(t),v(t))-\boldsymbol{\sigma}(u(t),v(t))\big|\\&\le\big|(v^n(t))^2-v^2(t)\big|\,\big|\partial_{\boldsymbol{\varepsilon}}W_+(\boldsymbol{\varepsilon}(u^n(t)))\big|+C|\boldsymbol{\varepsilon}(u^n(t))-\boldsymbol{\varepsilon}(u(t))|\\&\le C|v^n(t)-v(t)|\,|\boldsymbol{\varepsilon}(u^n(t))|+C|\boldsymbol{\varepsilon}(u^n(t))-\boldsymbol{\varepsilon}(u(t))|.\end{aligned}$$

As a consequence $\boldsymbol{\sigma}(u^n(t),v^n(t))\to\boldsymbol{\sigma}(u(t),v(t))$ in L^2 and then

$$\int_\Omega\boldsymbol{\sigma}(u^n(t),v^n(t)):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x\to\int_\Omega\boldsymbol{\sigma}(u(t),v(t)):\boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x.$$

All in all, we get

$$\mathcal{F}(u(t_j),v(t_j))\le\mathcal{F}(u(t),v(t))+\int_t^{t_j}\int_\Omega\boldsymbol{\sigma}(u(s),v(s)):\boldsymbol{\varepsilon}(\dot a(s)\hat g)\,\mathrm{d}x\,\mathrm{d}s+\varepsilon|t_j-t|.$$

Dividing by $|t_j - t|$ and passing to the limit as $t_j \to t$ gives $\dot{\mathcal{F}}(t) \le \mathcal{P}(t, u(t), v(t)) + \varepsilon$ and we conclude by arbitrariness of $\varepsilon > 0$. ☐

Corollary 5.14. *$\partial_v \mathcal{F}(u(t), v(t))[\dot{v}(t)] = 0$ a.e. in I_s.*

Proof. This is a straightforward consequence of Theorem 5.13. Indeed:

$$\dot{\mathcal{F}}(u(t), v(t)) = \partial_u \mathcal{F}(u(t), v(t))[\dot{u}(t)] + \partial_v \mathcal{F}(u(t), v(t))[\dot{v}(t)] = \partial_u \mathcal{F}(u(t), v(t))[\dot{u}(t)]$$

a.e. in I_s and hence $\partial_v \mathcal{F}(u(t), v(t))[\dot{v}(t)] = 0$. ☐

6. GRIFFITH'S CRITERION

In this section, we consider the elastic energy $W(\boldsymbol{\varepsilon}, v) = \psi(v)(W_+(\boldsymbol{\varepsilon}) + \delta) + W_-(\boldsymbol{\varepsilon})$ where $\delta \ge 0$ and $W_\pm$ satisfy the properties (H1)–(H3) listed in Section 2. For $\delta = 0$ we recover the classical phase-field elastic energy, while for $\delta > 0$ (arbitrarily small) a better description of the energy release holds (see Prop. 6.2). Adding a constant to the elastic energy does not change the constitutive law, but it will change the energy release (by an arbitrarily small amount).

6.1. Energy release

In this section we provide comments and properties on the notion of energy release employed in Theorem 3.10. In literature, the definition of *phase-field energy release* $\mathcal{G}_{\max}(t, v)$ is not univocal and in the following we will give it in the same vein as [33]. Along the lines of the aforementioned article, let us define the reduced energy

$$\tilde{\mathcal{E}}(t, v) := \mathcal{E}(u_{t,v}, v) \qquad \text{for} \qquad u_{t,v} \in \operatorname{argmin}\{\mathcal{E}(u, v) : u \in \mathcal{U}(t)\},$$

that, as proven in the following Lemma, is differentiable as a function of v.

Lemma 6.1. *The reduced energy functional $\tilde{\mathcal{E}}$ is differentiable as a function of v with respect to variations $\xi \in \Xi$. Its derivative has the following integral representation:*

$$\partial_v \tilde{\mathcal{E}}(t, v)[\xi] = \partial_v \mathcal{E}(u_{t,v}, v)[\xi] = \int_\Omega v\xi\big(W_+(\boldsymbol{\varepsilon}(u_{t,v})) + \delta\big)\mathrm{d}x. \tag{6.1}$$

It follows that $\partial_v \tilde{\mathcal{E}}(t, v)[\xi] \le 0$ for every $\xi \in \Xi$.

Proof. The directional derivative of $\tilde{\mathcal{E}}$ at v is

$$\partial_v \tilde{\mathcal{E}}(t, v)[\xi] = \lim_{h \to 0^+} \frac{\tilde{\mathcal{E}}(t, v + h\xi) - \tilde{\mathcal{E}}(t, v)}{h} = \lim_{h \to 0^+} \frac{\mathcal{E}(u_{t,v+h\xi}, v + h\xi) - \mathcal{E}(u_{t,v}, v)}{h}$$

Adding and subtracting the term $\mathcal{E}(u_{t,v+h\xi}, v)$, we obtain

$$\lim_{h \to 0^+} \left(\frac{\mathcal{E}(u_{t,v+h\xi}, v) - \mathcal{E}(u_{t,v}, v)}{h} + \frac{\mathcal{E}(u_{t,v+h\xi}, v + h\xi) - \mathcal{E}(u_{t,v+h\xi}, v)}{h} \right).$$

We examine the two addenda separately. The first term is infinitesimal; indeed, using Corollary 4.4 and Lemma 4.6, we obtain

$$\frac{1}{h}\big|\mathcal{E}(u_{t,v+h\xi}, v) - \mathcal{E}(u_{t,v}, v)\big| \le \frac{C}{h}\|u_{t,v+h\xi} - u_{t,v}\|^2_{H^1} \le \frac{C}{h}\|h\xi\|^2_{L^r},$$

that tends to zero as $h \to 0$. As for the second term, we write it as follows:

$$\begin{aligned}\frac{1}{2h}\int_\Omega \big((v+h\xi)^2 - v^2\big)\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) + \delta\big)\,\mathrm{d}x &= \frac{1}{2h}\int_\Omega \big(h^2\xi^2 + 2v\xi h\big)\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) + \delta\big)\,\mathrm{d}x \\ &= \int_\Omega \big(\tfrac{1}{2}h\xi^2 + v\xi\big)\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) + \delta\big)\,\mathrm{d}x.\end{aligned}$$

From Lemma 4.5 we know that $u_{t,v+h\xi}$ is bounded in $W^{1,\tilde p}(\Omega,\mathbb{R}^2)$ uniformly with respect to h and ξ. Choosing $1 \le q < +\infty$ such that $\frac{2}{q} + \frac{2}{\tilde p} = 1$ we get

$$\tfrac{1}{2}\int_\Omega h\xi^2\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) + \delta\big)\,\mathrm{d}x \le \tfrac{1}{2}h\big(||\xi||^2_{L^q}||u_{t,v+h\xi}||^2_{W^{1,\tilde p}} + \delta||\xi||^2_{L^2}\big) \le Ch \to 0.$$

Now consider $\int_\Omega v\xi\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) + \delta\big)\,\mathrm{d}x$ and observe that it converges to (6.1). By Lemmas 4.1 and 4.6, indeed,

$$\begin{aligned}||v\xi W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) - v\xi W_+(\boldsymbol{\varepsilon}(u_{t,v}))||_{L^1} &= \int_\Omega |v\xi\big(W_+(\boldsymbol{\varepsilon}(u_{t,v+h\xi})) - W_+(\boldsymbol{\varepsilon}(u_{t,v}))\big)|\,\mathrm{d}x \\ &\le C\big(\|u_{t,v+h\xi}\|_{H^1} + \|u_{t,v}\|_{H^1}\big)\|u_{t,v+h\xi} - u_{t,v}\|_{H^1} \to 0.\end{aligned}$$

The proof is conlcuded. □

We are now ready to define the phase-field energy release rate for $v \in \mathcal{V}$ and $t \in [0,T]$ as:

$$\mathcal{G}_{max}(t,v) := \sup\big\{-\partial_v\tilde{\mathcal{E}}(t,v)[\xi] : \xi \in \hat\Xi(v)\big\},$$

where $\hat\Xi(v)$ is the set of *normalized admissible variations* with respect to the "phase-field crack length", *i.e.*,

$$\hat\Xi(v) = \{\xi \in \Xi : d\mathcal{L}(v)[\xi] = 1\}.$$

First, note that $\mathcal{L}(v) = \frac{1}{2}\|v-1\|^2_{H^1}$ and thus $d\mathcal{L}(v)[\xi] = \langle v-1,\xi\rangle_{\mathcal{V}} = 0$ for every ξ when $v \equiv 1$. Therefore, we assume that $v \not\equiv 1$, so that $\hat\Xi(v) \neq \emptyset$. Clearly $\mathcal{G}_{max}(t,v) \ge 0$ since $\partial_v\tilde{\mathcal{E}}(t,v)[\xi] \le 0$ for every $\xi \in \hat\Xi(v)$.

To better understand this definition, it is important to observe that it is given in analogy with the sharp crack setting, where the maximal energy release is defined as the steepest descent of the (reduced) elastic energy with respect to crack elongation among any admissible direction. Indeed, if we consider the admissible variations $\xi \in \Xi$ such that $d\mathcal{L}(v)[\xi] > 0$ and for each of them we define a sort of forward derivative of $\tilde{\mathcal{E}}$ with respect to the phase-field crack length, the following identities hold:

$$\begin{aligned}&\sup\left\{-\lim_{h\to0^+}\frac{\tilde{\mathcal{E}}(t,v+h\xi) - \tilde{\mathcal{E}}(t,v)}{\mathcal{L}(v+h\xi) - \mathcal{L}(v)} : \xi \in \Xi \text{ with } d\mathcal{L}(v)[\xi] > 0\right\} \\ &\qquad = \sup\left\{-\frac{\partial_v\tilde{\mathcal{E}}(t,v)[\xi]}{\mathrm{d}\mathcal{L}(v)[\xi]} : \xi \in \Xi \text{ with } d\mathcal{L}(v)[\xi] > 0\right\} \\ &\qquad = \sup\left\{-\partial_v\tilde{\mathcal{E}}(t,v)[\xi] : \xi \in \Xi \text{ with } d\mathcal{L}(v)[\xi] = 1\right\} = \mathcal{G}_{max}(t,v).\end{aligned} \tag{6.2}$$

The second equality follows from linearity with respect to ξ, which allows to consider admissible variations normalized with respect to the phase-field crack elongation.

In principle, given an arbitrary function $v \in \mathcal{V}$ it may happen that $d\mathcal{L}(v)[\xi] \le 0$ for some $\xi \in \Xi$; these variations would not be admissible in the representation of $\mathcal{G}_{max}$ given above. However, if (u,v) is an equilibrium configuration of the energy (which is the case in quasi-static evolutions) we have $d\mathcal{L}(v)[\xi] \ge 0$ for every $\xi \in \Xi$

(independently of $\delta \geq 0$); moreover, if $\delta > 0$ (but arbitrarily small) and if (u, v) is an equilibrium configuration then $d\mathcal{L}(v)[\xi] > 0$ for every $\xi \in \Xi$ with $\xi \neq 0$, as stated in next Proposition.

Proposition 6.2. *Let (u, v) be an equilibrium configuration for the energy, i.e.*

$$\partial_u \mathcal{F}(u, v)[\phi] = 0 \quad \text{for all } \phi \in \Phi, \qquad \partial_v \mathcal{F}(u, v)\,[\xi] \geq 0 \quad \text{for all } \xi \in \Xi.$$

Then $d\mathcal{L}(v)[\xi] \geq 0$ for every $\xi \in \Xi$. Moreover, if $\delta > 0$ then $d\mathcal{L}(v)[\xi] > 0$ for every $\xi \in \Xi$ with $\xi \neq 0$.

Proof. Since $\partial_v \mathcal{F}(u, v)[\xi] = \partial_v \mathcal{E}(u, v)[\xi] + G_c d\mathcal{L}(v)[\xi] \geq 0$, it follows that $G_c d\mathcal{L}(v)[\xi] \geq -\partial_v \mathcal{E}(u, v)[\xi] \geq 0$ for every $\xi \in \Xi$.

Let $\delta > 0$. Assume by contradiction that $d\mathcal{L}(v)[\xi] = 0$ for some $\xi \in \Xi$ with $\xi \neq 0$. It follows that $\partial_v \mathcal{E}(u, v)[\xi] \geq 0$ and then $\partial_v \mathcal{E}(u, v)[\xi] = 0$. However,

$$\partial_v \mathcal{E}(u, v)[\xi] = \int_\Omega v\xi(W_+(\boldsymbol{\varepsilon}(u)) + \delta)\,\mathrm{d}x = 0$$

implies that $v\xi = 0$ a.e. in Ω since $W_+(\boldsymbol{\varepsilon}(u)) + \delta > 0$. Since $v \in H^1(\Omega)$ we have $\nabla v = 0$ a.e. in the set $\{v = 0\}$ (including the cases in which the measure of the set vanishes) and, similarly, $\nabla \xi = 0$ a.e. in the set $\{\xi = 0\}$. As $v\xi = 0$ a.e. in Ω it follows that $\nabla v \cdot \nabla \xi = 0$ a.e. in Ω. As a consequence,

$$d\mathcal{L}(v)[\xi] = \int_\Omega (v - 1)\xi + \nabla v \cdot \nabla \xi \,\mathrm{d}x = \int_\Omega -\xi \,\mathrm{d}x = 0,$$

which implies $\xi = 0$ a.e. in Ω since $\xi \leq 0$. $\square$

From the previous Lemma we get the following representation of the energy release.

Proposition 6.3. *Let $\delta > 0$ and let (u, v) be an equilibrium configuration for the energy. Then*

$$\begin{aligned}
\mathcal{G}_{max}(t, v) &= \sup\left\{-\partial_v \tilde{\mathcal{E}}(t, v)[\xi] \;:\; \xi \in \Xi \text{ with } d\mathcal{L}(v)[\xi] = 1\right\} \\
&= \sup\left\{-\frac{\partial_v \tilde{\mathcal{E}}(t, v)[\xi]}{\mathrm{d}\mathcal{L}(v)[\xi]} \;:\; \xi \in \Xi,\, \xi \neq 0\right\} \qquad (6.3) \\
&= \limsup_{z \to v^-} \frac{[\tilde{\mathcal{E}}(t, z) - \tilde{\mathcal{E}}(t, v)]_-}{\mathcal{L}(z) - \mathcal{L}(v)}, \qquad (6.4)
\end{aligned}$$

where $z \to v^-$ means that $z \to v$ in $L^1(\Omega)$ with $0 \leq z \leq v$ and $z \neq v$.

Proof. The first identity follows immediately from the previous Proposition and (6.2). As $0 \leq z \leq v$ we have $\tilde{\mathcal{E}}(t, z) \leq \tilde{\mathcal{E}}(t, v)$. Therefore

$$\limsup_{z \to v^-} \frac{[\tilde{\mathcal{E}}(t, z) - \tilde{\mathcal{E}}(t, v)]_-}{\mathcal{L}(z) - \mathcal{L}(v)} = \limsup_{z \to v^-} \frac{\tilde{\mathcal{E}}(t, v) - \tilde{\mathcal{E}}(t, z)}{\mathcal{L}(z) - \mathcal{L}(v)}.$$

Note also that $\mathcal{L}(z) > \mathcal{L}(v)$ since $\mathcal{L}$ is quadratic and thus

$$\mathcal{L}(z) = \mathcal{L}(v) + d\mathcal{L}(v)[z - v] + \tfrac{1}{2}\|z - v\|^2_{H^1},$$

where $d\mathcal{L}(v)[z - v] > 0$ by Proposition 6.2 (unless $z = v$).

To check that

$$\sup \left\{ - \frac{\partial_v \tilde{\mathcal{E}}(t,v)[\xi]}{\mathrm{d}\mathcal{L}(v)[\xi]} \; : \; \xi \in \Xi, \, \xi \neq 0 \right\} \le \limsup_{z \to v^-} \frac{\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t,z)}{\mathcal{L}(z) - \mathcal{L}(v)},$$

it is not restrictive to assume that $\partial_v \tilde{\mathcal{E}}(t,v)[\xi] < 0$, otherwise $\partial_v \tilde{\mathcal{E}}(t,v)[\xi] = 0$ and there is nothing to prove. Given $\xi \in \Xi$ let us choose $z = [v + h\xi]_+$ and let $h \to 0^+$. First, note that $z \neq v$; indeed if $[v + h\xi]_+ = v$ then ξ would be supported in the set $\{v = 0\}$, and thus $\partial_v \tilde{\mathcal{E}}(t,v)[\xi] = 0$. Then, $\mathcal{L}(v) < \mathcal{L}(z) \le \mathcal{L}(v + h\xi)$ and thus

$$\frac{1}{\mathcal{L}(v + h\xi) - \mathcal{L}(v)} \le \frac{1}{\mathcal{L}(z) - \mathcal{L}(v)}.$$

Moreover, $(v + h\xi)^2 \ge z^2$ and thus $\tilde{\mathcal{E}}(t, v + h\xi) \ge \tilde{\mathcal{E}}(t,z)$. Hence,

$$\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t, v + h\xi) \le \tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t,z) = [\tilde{\mathcal{E}}(t,z) - \tilde{\mathcal{E}}(t,v)]_-.$$

In conclusion,

$$\frac{\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t, v + h\xi)}{\mathcal{L}(v + h\xi) - \mathcal{L}(v)} \le \frac{[\tilde{\mathcal{E}}(t,z) - \tilde{\mathcal{E}}(t,v)]_-}{\mathcal{L}(z) - \mathcal{L}(v)}.$$

Passing to the limsup as $h \to 0^+$ gives the required inequality, by arbitrariness of ξ.

It remains to prove the opposite inequality. To this end, it is not restrictive to assume that

$$L = \limsup_{z \to v^-} \frac{\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t,z)}{\mathcal{L}(z) - \mathcal{L}(v)} > 0.$$

First, consider the case $L < +\infty$. For $\varepsilon > 0$ (sufficiently small) there exists $z_n \to v^-$ such that

$$\frac{\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t,z_n)}{\mathcal{L}(z_n) - \mathcal{L}(v)} \ge L - \varepsilon > 0.$$

Note that $\tilde{\mathcal{E}}(t,z_n) \to \tilde{\mathcal{E}}(t,v)$, and thus $\mathcal{L}(z_n) \to \mathcal{L}(v)$ (otherwise the limit of the left-hand side would vanish). As a consequence $z_n \to v$ strongly in $H^1(\Omega)$ and thus in $L^q(\Omega)$ for any $1 \le q < +\infty$. Setting $u_n \in \operatorname{argmin} \{\mathcal{E}(u, z_n) : u \in \mathcal{U}(t)\}$ and $u \in \operatorname{argmin} \{\mathcal{E}(u,v) : u \in \mathcal{U}(t)\}$, let us write

$$\begin{aligned} \tilde{\mathcal{E}}(t,z_n) &= \mathcal{E}(u_n, z_n) - \mathcal{E}(u, z_n) + \mathcal{E}(u, z_n) \ge \mathcal{E}(u, z_n) - C\|u_n - u\|^2_{H^1} \\ &\ge \mathcal{E}(u, z_n) - C\|z_n - v\|^2_{L^r}, \end{aligned}$$

where once again we used Lemma 4.4 and Lemma 4.6. By convexity

$$\mathcal{E}(u, z_n) \ge \mathcal{E}(u,v) + \partial_v \mathcal{E}(u,v)[z_n - v] = \tilde{\mathcal{E}}(t,v) + \partial_v \tilde{\mathcal{E}}(t,v)[z_n - v].$$

Hence,

$$\tilde{\mathcal{E}}(t,v) - \tilde{\mathcal{E}}(t,z_n) \le -\partial_v \tilde{\mathcal{E}}(t,v)[z_n - v] + C\|z_n - v\|^2_{L^r}.$$

Since $\mathcal{L}$ is quadratic we can write

$$\mathcal{L}(z_n) - \mathcal{L}(v) = d\mathcal{L}(v)[z_n - v] + \tfrac{1}{2}\|z_n - v\|^2_{H^1}$$

and then

$$L - \varepsilon \leq \frac{-\partial_v \tilde{\mathcal{E}}(t,v)[z_n - v] + C\|z_n - v\|^2_{L^r}}{d\mathcal{L}(v)[z_n - v] + \frac{1}{2}\|z_n - v\|^2_{H^1}}. \tag{6.5}$$

Note that $-\partial_v \tilde{\mathcal{E}}(t,v)[z_n - v] \geq 0$ and let $\xi_n = (z_n - v)/\|z_n - v\|_{L^r}$, so that $\|\xi_n\|_{L^r} = 1$.

- If $\|\xi_n\|_{H^1} \leq C$ then (up to non-relabelled subsequences) $\xi_n \rightharpoonup \xi$ in $H^1(\Omega)$ and thus $\xi_n \to \xi$ in $L^r(\Omega)$ by Sobolev embedding. In particular $\xi \neq 0$ since $\|\xi\|_{L^r} = 1$. From (6.5) we get

$$\begin{aligned} L - \varepsilon &\leq \frac{-\partial_v \tilde{\mathcal{E}}(t,v)[z_n - v]}{d\mathcal{L}(v)[z_n - v]} + C\frac{\|z_n - v\|^2_{L^r}}{d\mathcal{L}(v)[z_n - v]} \\ &\leq \sup\left\{-\frac{\partial_v \tilde{\mathcal{E}}(t,v)[\xi]}{\mathrm{d}\mathcal{L}(v)[\xi]} : \xi \in \Xi,\, \xi \neq 0\right\} + C\frac{\|z_n - v\|_{L^r}}{d\mathcal{L}(v)[\xi_n]}. \end{aligned}$$

  In the limit as $n \to +\infty$ the last term vanishes since $d\mathcal{L}(v)[\xi_n] \to d\mathcal{L}(v)[\xi] > 0$, by Proposition 6.2, while $z_n \to v$ in $L^r(\Omega)$.
- If $\|\xi_n\|_{H^1} \to +\infty$ for a (non-relabelled) subsequence then (6.5) yields

$$\begin{aligned} L - \varepsilon &\leq \frac{-\partial_v \tilde{\mathcal{E}}(t,v)[z_n - v]}{d\mathcal{L}(v)[z_n - v]} + C\frac{\|z_n - v\|^2_{L^r}}{\|z_n - v\|^2_{H^1}} \\ &\leq \sup\left\{-\frac{\partial_v \tilde{\mathcal{E}}(t,v)[\xi]}{\mathrm{d}\mathcal{L}(v)[\xi]} : \xi \in \Xi,\, \xi \neq 0\right\} + \frac{C}{\|\xi_n\|^2_{H^1}}. \end{aligned}$$

  Clearly last term vanishes in the limit as $n \to +\infty$. In both cases we conclude by the arbitrariness on ε.

In the case $L = +\infty$ it is enough to replace $L - \varepsilon$ with an arbitrary large value. ☐

Remark 6.4. Note that writing the energy release as the "slope"

$$\mathcal{G}_{max}(t,v) = \limsup_{z \to v^-} \frac{[\tilde{\mathcal{E}}(t,z) - \tilde{\mathcal{E}}(t,v)]_-}{\mathcal{L}(z) - \mathcal{L}(v)},$$

gives the most general way of looking at variations of energy with respect to variations of phase-field crack length. In perspective, this would be the natural definition of energy release for the Γ-limit functional in the space SBD, where directional derivatives seems not general enough to describe all the possible unilateral variations of the crack.

6.2. Griffith's criterion

Before presenting the phase-field version of Griffith's criterion satisfied by the limit evolution v, given by Lemma 3.4, we shall study the energy functional on the set I_u.

Lemma 6.5. *Let (a,b) be a connected component of I_u, then $\mathcal{F}(u(b),v(b)) \leq \mathcal{F}(u(a),v(a))$. Moreover*

$$\dot{\mathcal{F}}(u(a^+),v(a^+)) \geq 0\,, \quad \dot{\mathcal{F}}(u(b^-),v(b^-)) = 0.$$

Proof. We recall that v is affine in (a,b) and denote for convenience

$$f(t) := \mathcal{F}(u(t),v(t)), \quad e(t) := \tilde{\mathcal{E}}(t,v(t)) = \mathcal{E}(u(t),v(t)), \quad \ell(t) := \mathcal{L}(v(t)). \tag{6.6}$$

In this notation, we will show that the function f satisfies $f(b) \le f(a)$, $\dot f(a^+) \ge 0$ and $\dot f(b^-) = 0$.

As we have already mentioned, since a, $b \in I_s$ by strong convergence (seen in the proof of Lem. 5.5) combined with Lemma 5.7 there exists $u^n(t^n_{k+1}) \to u(b)$ (as usual $k = k(n)$) and $u^n(t^n_k) \to u(a)$ strongly in $W^{1,p}(\Omega;\mathbb{R}^2)$, $v^n(t^n_k) \to v(a)$ and $v^n(t^n_{k+1}) \to v(b)$ strongly in $H^1(\Omega)$. It follows that $\mathcal{F}(u(t^n_k),v(t^n_k)) \to \mathcal{F}(u(a),v(a)) = f(a)$ and $\mathcal{F}(u(t^n_{k+1}),v(t^n_{k+1})) \to \mathcal{F}(u(b),v(b)) = f(b)$. By minimality together with (5.21)

$$\mathcal{F}(u^n_{k+1},v^n_{k+1}) \le \mathcal{F}(u^n_k,v^n_k) + \int_{t^n_k}^{t^n_{k+1}} \mathcal{P}^n(t,u^n_k,v^n_k)\,\mathrm{d}t + C|t^n_{k+1}-t^n_k|^2, \tag{6.7}$$

where

$$\mathcal{P}^n(t,u,v) := \int_\Omega \boldsymbol{\sigma}(u,v) : \boldsymbol{\varepsilon}(\dot a^n(t)\hat g)\,\mathrm{d}x.$$

Now, by definition of a^n, we have that, for $t \in (t_k,t_{k+1})$, $\dot a^n(t) = \frac{\alpha(s^n_{k+1})-\alpha(s^n_k)}{\tau_{k+1}}$ and thus the integral of the power in (6.7) reads

$$\int_\Omega \big(\alpha(s^n_{k+1})-\alpha(s^n_k)\big)\boldsymbol{\sigma}(u^n_k,v^n_k) : \boldsymbol{\varepsilon}(\hat g)\,\mathrm{d}x \;\to 0$$

since α is continuous and $s^n_{k+1} = s^n_k + \frac{1}{n} \to s^n_k$. Therefore

$$f(b) = \lim_{n\to+\infty} \mathcal{F}(u^n_{k+1},v^n_{k+1}) \le \lim_{n\to+\infty} \mathcal{F}(u^n_k,v^n_k) = f(a).$$

Let's now prove that $\dot f(a^+) = 0$. We know that $v^n_k \to v(a)$ and $v^n_{k+1} \to v(b)$ strongly in $H^1(\Omega)$. Hence

$$\dot v^n_{k+1} = \frac{v^n_{k+1}-v^n_k}{t^n_{k+1}-t^n_k} \longrightarrow \frac{v(b)-v(a)}{b-a} = \dot v(t) \quad \text{strongly in } H^1(\Omega) \text{ for all } t \in (a,b).$$

Being $(u(a),v(a))$ an equilibrium configuration (see Thm. 5.12) and being $\dot v(a^+) \le 0$ an admissible variation, (5.12) yields $\dot f(a) \ge 0$.

It remains to prove that $\dot f(b^-) = 0$. Recall that, by (3.2), $\partial_v \mathcal{F}(u^n_{k+1},v^n_{k+1})[\dot v^n_{k+1}] = 0$. Its explicit form is:

$$\partial_v \mathcal{F}(u^n_{k+1},v^n_{k+1})[\dot v^n_{k+1}] = \int_\Omega v^n_{k+1}\dot v^n_{k+1}\big(W_+(\boldsymbol{\varepsilon}(u^n_{k+1})+\delta\big)\,\mathrm{d}x + G_c \int_\Omega (v^n_{k+1}-1)\dot v^n_{k+1} + \nabla v^n_{k+1}\nabla \dot v^n_{k+1} = 0.$$

The strong convergence of v^n_{k+1} and $\dot v^n_{k+1}$ combined to the fact that $u^n_{k+1} \to u(b)$ strongly in $W^{1,p}(\Omega)$, gives

$$0 = \partial_v \mathcal{F}(u^n_{k+1},v^n_{k+1})[\dot v^n_{k+1}] \to \partial_v \mathcal{F}(u(b),v(b))[\dot v(b^-)] = \dot f(b^-),$$

which concludes the proof. ☐

As we anticipated, neither the energy release nor the thermodynamic consistency enter into the definition of the time discrete evolution (1.8), however, the limit (time continuous) evolution satisfies both Griffith's criterion, in terms of the phase-field energy release, and thermodynamic consistency, as stated in Theorem 3.10 (recalled hereafter for reader's convenience).

Assume that $v_0 \not\equiv 1$. *The limit evolution* v*, obtained in the limit of the time discrete evolution* (1.8)*, satisfies* $0 \le v(t) \le 1$ *for every* $t \in [0,T]$*, the irreversibility constraint* $\dot v(t) \le 0$ *and the thermodynamic consistency condition* $\dot\ell(t) := \mathrm{d}\mathcal{L}(v(t))[\dot v(t)] \ge 0$ *a.e. in* $[0,T]$*. Moreover, an extended version of Griffith's criterion holds:*

- $\mathcal{G}_{max}(t, v(t)) \le G_c$ *everywhere in* I_s*,*
- $(\mathcal{G}_{max}(t, v(t)) - G_c)\,\dot\ell(t) = 0$ *a.e. in* I_s*,*
- $\partial_v \tilde{\mathcal{E}}(t, v)[\dot v(t)] = -\dot\ell(t)\,\mathcal{G}_{max}(t, v(t))$ *a.e. in* I_s*,*
- $\mathcal{G}_{max}(t, v(t)) \ge G_c$ *on a subset of positive measure of* I_u*.*

Proof of Theorem 3.10. **I.** The validity of the thermodynamic consistency condition on I_s is a straightforward consequence of Corollary 5.14. Indeed, by definition $\partial_v\mathcal{F}(u(t),v(t))[\dot v(t)] = \partial_v\mathcal{E}(u(t),v(t))[\dot v(t)] + G_c\, d\mathcal{L}(v(t))[\dot v(t)]$. Now, from the aforementioned Corollary, we know that $\partial_v\mathcal{F}(u(t),v(t))[\dot v(t)] = 0$ a.e. in I_s and therefore:

$$G_c\, d\mathcal{L}(v(t))[\dot v(t)] = -\partial_v\mathcal{E}(u(t),v(t))[\dot v(t)] = -\int_\Omega v(t)\,\dot v(t)\,W_+(\boldsymbol{\varepsilon}(u(t))\,\mathrm{d}x \ge 0.$$

To prove the same property on I_u, we proceed as follows. Let (a,b) be a connected component of I_u. The function $t \mapsto e(t) = \tilde{\mathcal{E}}(t, v(t)) = \mathcal{E}(u(t), v(t))$ is non increasing in (a,b). Indeed, if $t_1 < t_2$ then $v(t_1) \ge v(t_2)$ and hence:

$$\begin{aligned}\mathcal{E}(u(t_1),v(t_1)) &= \int_\Omega \big(v^2(t_1)+\eta\big)W_+(\boldsymbol{\varepsilon}(u(t_1))\,\mathrm{d}x \ge \int_\Omega \big(v^2(t_2)+\eta\big)W_+(\boldsymbol{\varepsilon}(u(t_1))\,\mathrm{d}x\\ &\ge \int_\Omega \big(v^2(t_2)+\eta\big)W_+(\boldsymbol{\varepsilon}(u(t_2))\,\mathrm{d}x = \mathcal{E}(u(t_2),v(t_2)),\end{aligned}$$

where the last inequality follows from the minimality of $u(t_2)$ (we recall that in (a,b) the boundary condition is fixed since the control c is constant, see Lemma 5.8). Now, writing $f(t) = e(t) + G_c\ell(t)$ as in (6.6), from Lemma 6.5 we get

$$0 = \dot f(b^-) = \dot e(b^-) + G_c\,\dot\ell(b^-).$$

Therefore, since the energy $t \mapsto e(t)$ is non increasing $d\mathcal{L}(v(b))[\dot v(b^-)] = \dot\ell(b^-) = -\dot e(b^-)/G_c \ge 0$. Similarly, from the equilibrium condition in a

$$0 \le \dot f(a^+) = \dot e(a^+) + G_c\,\dot\ell(a^+)$$

and from the fact that $\dot e(a^+) \le 0$, it follows that $d\mathcal{L}(v(a))[\dot v(a^+)] = \dot\ell(a^+) \ge -\dot e(a^+)/G_c \ge 0$. Now, since ℓ is quadratic, its derivative is linear and since $\dot\ell(t) \ge 0$ for both $t = a$ and $t = b$, it must be so also for every $t \in (a,b)$. We thus have that $\dot\ell(t) = \partial_v\mathcal{L}(v(t))[\dot v(t)] \ge 0$ also for all $t \in I_u$.

II. We will now prove that a classical version of Griffith's criterion holds on I_s, following [33].

- To prove that $\mathcal{G}_{max}(t, v(t)) \le G_c$ in I_s, we use the fact that, by (5.12),

  $$\partial_v\mathcal{F}(u(t),v(t))[\xi] = \partial_v\mathcal{E}(u(t),v(t))[\xi] + G_c\, d\mathcal{L}(v(t))[\xi] \ge 0 \qquad \text{for every } \xi \in \Xi.$$

  In particular, if we take $\hat\xi \in \hat\Xi(v(t))$ we get $G_c \ge -\partial_v\tilde{\mathcal{E}}(t, v(t))[\hat\xi]$ and thus

  $$G_c \ge \sup\big\{ -\partial_v\tilde{\mathcal{E}}(t,v(t))[\hat\xi] : \hat\xi \in \hat\Xi(v(t))\big\} = \mathcal{G}_{max}(t, v(t)).$$

- Finally, let us prove that $(\mathcal{G}_{max}(t, v(t)) - G_c)\,\dot{\ell}(t) = 0$ a.e. in I_s. Clearly, if $\dot{\ell}(t) = 0$ there is nothing to prove. Consider $\dot{\ell}(t) = d\mathcal{L}(v(t))[\dot{v}(t)] > 0$. Again by Corollary 5.14 we have that $\partial_v \mathcal{F}(u(t), v(t))[\lambda \dot{v}(t)] = 0$ for every $\lambda > 0$ a.e. in I_s. On the other hand, $\partial_v \mathcal{F}(u(t), v(t))[\xi] \geq 0$ for every $\xi \in \Xi$ and thus for every $\lambda > 0$

  $$\begin{aligned} \lambda \dot{v}(t) &\in \operatorname{argmin}\{\partial_v \mathcal{F}(u(t), v(t))[\xi] : \xi \in \Xi\} \\ &\in \operatorname{argmin}\{\partial_v \tilde{\mathcal{E}}(t, v(t))[\xi] + G_c d\mathcal{L}(v(t))[\xi] : \xi \in \Xi\}. \end{aligned}$$

  Choosing $\lambda = 1/d\mathcal{L}(v)[\dot{v}(t)] = 1/\dot{\ell}(t)$, in such a way that $d\mathcal{L}(v)[\lambda \dot{v}(t)] = 1$ (so that $\lambda \dot{v}(t) \in \hat{\Xi}(v(t)) \subset \Xi$), we get that

  $$\begin{aligned} \lambda \dot{v}(t) &\in \operatorname{argmin}\{\partial_v \tilde{\mathcal{E}}(t, v(t))[\hat{\xi}] + G_c\, d\mathcal{L}(v(t))[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}(v(t))\} \\ &\in \operatorname{argmin}\{\partial_v \tilde{\mathcal{E}}(t, v(t))[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}(v(t))\}, \end{aligned}$$

  since $G_c\, d\mathcal{L}(v(t))[\hat{\xi}] = G_c$ is constant in $\hat{\Xi}(v(t))$. Therefore, by definition

  $$\partial_v \tilde{\mathcal{E}}(t, v)[\lambda \dot{v}(t)] = -\mathcal{G}_{max}(t, v(t))$$

  and

  $$\partial_v \mathcal{F}(u(t), v(t))[\lambda \dot{v}(t)] = -\mathcal{G}_{max}(t, v(t)) + G_c = 0,$$

  which concludes the proof.

III. Let's now focus on the set of instability points I_u. Let (a, b) be a connected component of I_u. By Lemma 6.5 we have $f(a) \geq f(b)$, therefore the subset of (a, b) where $\dot{f} \leq 0$ is of positive measure.

First, note the following: if the evolution is not identically constant (in which case there is nothing to prove) then $\dot{\ell}(t) > 0$ in I_u. Indeed, being $\dot{\ell}(t)$ linear and non-negative, if $\dot{\ell}(t) = 0$ for some $t \in (a, b)$, then $\dot{\ell}$ should be zero everywhere and ℓ should be constant on (a, b). This means that v would belong to a contour line of $\mathcal{L}(v) = \frac{1}{2}\|v - 1\|^2_{H^1(\Omega)}$. Now, by uniform convexity of $H^1(\Omega)$, if $v(a) \neq v(b)$, then for any $t \in (a, b)$ we would have $\|v(t) - 1\|^2_{H^1} < \|v(a) - 1\|^2_{H^1} = \|v(b) - 1\|^2_{H^1}$, *i.e.*, $v(t)$ could not belong to the contour line of $\mathcal{L}$, that is absurd.

We will show that $\mathcal{G}(t, v(t)) \geq G_c$ where $\dot{f}(t) \leq 0$. Write

$$\dot{f}(t) = \dot{e}(t) + G_c\, \dot{\ell}(t) \leq 0.$$

As we have seen, $\lambda = \dot{\ell}(t) > 0$, so we can divide, obtaining

$$G_c = G_c \frac{\dot{\ell}(t)}{\lambda} \leq -\frac{\dot{e}(t)}{\lambda} = -\partial_v \tilde{\mathcal{E}}(t, v(t))\Big[\frac{\dot{v}(t)}{\lambda}\Big] \leq \mathcal{G}_{max}(t, v(t)), \tag{6.8}$$

where in the last inequality we used the defintion of $\mathcal{G}_{max}$ and the fact that $\dot{v}(t)/\lambda \in \hat{\Xi}(v(t))$, since by construction $\dot{\ell}(t) = d\mathcal{L}(v(t))[\dot{v}(t)] = \lambda$. □

7. SYSTEM OF PDEs

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, Lipschitz domain. Let $\zeta \in (H^1(\Omega))^*$ such that $\langle \zeta, \xi \rangle \geq 0$ for every $\xi \in H^1(\Omega)$ with $\xi \leq 0$. Then ζ is "represented" by $\mu \in \mathcal{M}^-(\bar{\Omega})$ (the set of finite, negative, Radon measures* *supported in $\bar{\Omega}$). More precisely,*

$$\langle \zeta, \xi \rangle = \int_{\Omega} \xi d\mu_{|\Omega} + \int_{\partial\Omega} \xi d\mu_{|\partial\Omega} \quad \textit{for every } \xi \in H^1(\Omega) \cap C(\bar{\Omega}).$$

Proof. Let $\tilde{\Omega}$ be an open set with $\bar{\Omega} \subset \tilde{\Omega}$ and consider the extension $\tilde{\zeta} \in H^{-1}(\tilde{\Omega})$ given by $\langle \tilde{\zeta}, \phi \rangle = \langle \zeta, \phi_{|\Omega} \rangle$. Then $\langle \tilde{\zeta}, \phi \rangle \geq 0$ for every $\phi \in C^{\infty}_c(\tilde{\Omega})$ with $\phi \leq 0$, since $\xi = \phi_{|\Omega} \in H^1(\Omega)$ and $\xi \leq 0$. As a consequence $\tilde{\zeta}$ (as a distribution) is represented by $\tilde{\mu} \in \mathcal{M}^-_{\text{loc}}(\tilde{\Omega})$. Clearly, the support of $\tilde{\mu}$ is contained in $\bar{\Omega}$ and thus, being $\tilde{\mu}$ locally finite, it turns out that $\tilde{\mu} = \tilde{\mu}_{|\bar{\Omega}}$ is finite.

Let $\xi \mapsto \tilde{\xi}$ be a (linear and continuous) extension from $H^1(\Omega)$ to $H^1_0(\tilde{\Omega})$. Then $\langle \zeta, \xi \rangle = \langle \zeta, \tilde{\xi}_{|\Omega} \rangle = \langle \tilde{\zeta}, \tilde{\xi} \rangle$. In particular, the integral representation holds for $\xi \in H^1(\Omega) \cap C(\bar{\Omega})$. ☐

If $z \in H^1(\Omega)$ with $\Delta z(t) \in L^2(\Omega)$ then $\partial z / \partial n$ (in weak form) denotes the operator given by Green's formula

$$\langle \partial z / \partial n, w \rangle = \int_{\Omega} w \, \Delta z + \nabla w \cdot \nabla z \, \mathrm{d}x$$

for $w \in H^1(\Omega)$. In a similar way, if $z \in H^1(\Omega)$ such that Δz (as a distribution) is represented by a finite measure, we define the linear operator

$$\langle \partial z / \partial n, w \rangle = \int_{\Omega} w \, d(\Delta z) + \int_{\Omega} \nabla w \cdot \nabla z \, \mathrm{d}x, \tag{7.1}$$

for $w \in H^1(\Omega) \cap C(\bar{\Omega})$. Note that the first integral is well defined since $w \in C(\bar{\Omega})$.

Let us denote by $\zeta(t) \in (H^1(\Omega))^*$ the linear operator $\langle \zeta(t), \xi \rangle = \partial_v F(t, u(t), v(t))[\xi]$. For $t \in I_s$ we have that $\langle \zeta(t), \xi \rangle \geq 0$ for every $\xi \in H^1(\Omega)$ with $\xi \leq 0$ and thus, by Lemma 7.1, $\zeta(t)$ is represented by a measure $\mu(t) \in \mathcal{M}^-(\bar{\Omega})$. Let us write $\mu(t) = \mu_{|\Omega}(t) + \mu_{|\partial\Omega}(t)$.

Lemma 7.2. *For $t \in I_s$ we have $\Delta v(t) \in \mathcal{M}(\Omega)$ (the space of finite Radon measures in Ω). Moreover,*

$$\mu_{|\Omega}(t) = -G_c \, \Delta v(t) + 2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big) + G_c(v(t) - 1) \leq 0,$$
$$\mu_{|\partial\Omega}(t) = G_c \, \partial v(t) / \partial n \leq 0.$$

Proof. For $\phi \in C^{\infty}_c(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} \phi \, d\mu_{|\Omega}(t) &= \langle \zeta(t), \phi \rangle = \partial_v F(t, u(t), v(t))[\phi] \\ &= \int_{\Omega} 2v(t) \phi \, W_+\big(\boldsymbol{\varepsilon}(u(t))\big) \, \mathrm{d}x + G_c \int_{\Omega} \nabla v(t) \cdot \nabla \phi + (v(t) - 1) \phi \, \mathrm{d}x \\ &= G_c \; {}_{\mathcal{D}'}\langle -\Delta v(t), \phi \rangle_{\mathcal{D}} + \int_{\Omega} [2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big) + G_c(v(t) - 1)] \phi \, \mathrm{d}x. \end{aligned}$$

Hence

$$G_c \; {}_{\mathcal{D}'}\langle \Delta v(t), \phi \rangle_{\mathcal{D}} = \int_{\Omega} [2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big) + G_c(v(t) - 1)] \phi \, \mathrm{d}x - \int_{\Omega} \phi \, d\mu_{|\Omega}(t). \tag{7.2}$$

It follows that $\Delta v(t) \in \mathcal{M}(\Omega)$ and that

$$\mu_{|\Omega}(t) = -G_c \Delta v(t) + 2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big) + G_c(v(t) - 1). \tag{7.3}$$

Hence, for every $\xi \in H^1(\Omega) \cap C(\bar\Omega)$ we can write

$$\begin{aligned}\langle \zeta(t), \xi\rangle = \partial_v F(t, u(t), v(t))[\xi] &= \int_\Omega G_c \nabla v(t) \cdot \nabla \xi + 2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big)\xi + G_c(v(t)-1)\xi \,\mathrm{d}x \\ &= G_c \langle \partial v(t)/\partial n, \xi\rangle - G_c \int_\Omega \xi \, d(\Delta v(t)) + \int_\Omega 2v(t) W_+\big(\boldsymbol{\varepsilon}(u(t))\big)\xi + G_c(v(t)-1)\xi \,\mathrm{d}x \\ &= G_c \langle \partial v(t)/\partial n, \xi\rangle + \int_\Omega \xi \, d\mu_{|\Omega}(t).\end{aligned}$$

Since

$$\langle \zeta(t), \xi\rangle = \int_{\partial\Omega} \xi \, d\mu_{|\partial\Omega}(t) + \int_\Omega \xi \, d\mu_{|\Omega}(t),$$

it follows that $G_c \, \partial v(t)/\partial n$ is represented by the negative measure $\mu_{|\partial\Omega}(t)$. □

Proposition 7.3. *Let $t \in I_s$ such that $\partial_v F(t, v(t), u(t))[\dot v(t)] = 0$. If $\dot v(t) \in H^1(\Omega) \cap C(\bar\Omega)$ then*

$$\big(- G_c \Delta v(t) + 2v(t) W_+(\boldsymbol{\varepsilon}(u(t))) + G_c(v(t)-1)\big)\dot v(t) = 0, \qquad (\partial v(t)/\partial n)\dot v(t) = 0, \tag{7.4}$$

in the sense of measures.

Proof. By Lemma 7.1

$$\partial_v F(t, v(t), u(t))[\dot v(t)] = \int_\Omega \dot v(t) d\mu_{|\Omega}(t) + \int_{\partial\Omega} \dot v(t) d\mu_{|\partial\Omega}(t) = 0.$$

Since $\mu_{|\Omega}(t) \le 0$, $\mu_{|\partial\Omega}(t) \le 0$ and $\dot v(t) \le 0$ it follows that both the above integrals are non-negative, and thus they both vanish. By Proposition 7.2 we finally get (7.4). □

Propositions 7.2 and 7.3 give Theorem 3.7.

8. Alternate minimization

Given $v_0 \in \mathcal{V}$, with $0 \le v_0 \le 1$ let $u_0 \in \operatorname{argmin}\{\mathcal{F}(u, v_0) : u \in \mathcal{U}(0)\}$. For $n \in \mathbb{N}$ set $s^n_k := k/n$ for $k = 0, ..., n$ and $v^n_0 := v_0$. In this section we consider the alternate minimization scheme [12] at s^n_k with initial condition v^n_{k-1} (defined in s^n_{k-1} and obtained at the previous iteration), *i.e.*, set $v^n_{k,0} := v^n_{k-1}$ and $u^n_{k,0} := u^n_{k-1}$, then we define by induction for $i \ge 0$:

$$\begin{cases} u^n_{k,i+1} \in \operatorname{argmin}\{\mathcal{F}(u, v^n_{k,i}) : u \in \mathcal{U}(s^n_k)\} \\ v^n_{k,i+1} \in \operatorname{argmin}\{\mathcal{F}(u^n_{k,i+1}, v) : v \in \mathcal{V} \text{ with } v \le v^n_{k,0} = v^n_{k-1}\}. \end{cases} \tag{8.1}$$

The updates u^n_k and v^n_k are respectively defined as $u^n_k := \lim_{i\to\infty} u^n_{k,i}$ and $v^n_k := \lim_{i\to\infty} v^n_{k,i}$; existence of the limits (up to non-relabelled subsequences) is proved in the following Proposition.

Remark 8.1. As in the original and mostly used version of the alternate minimization algorithm, we require that $v^n_{k+1} \le v^n_k$ for every $k = 0, \dots n-1$, and not $v^n_{k,i+1} \le v^n_{k,i}$ for every $i \in \mathbb{N}$ (*i.e.*, monotonicity over each staggered iteration) as in [33].

Proposition 8.2. *For every step s^n_k, there exists a subsequence of $\{(u^n_{k,i}, v^n_{k,i})\}_{i\in\mathbb{N}}$ converging strongly in $W^{1,p}(\Omega,\mathbb{R}^2)\times H^1(\Omega)$; the limit, denoted (u^n_k, v^n_k), is a separate minimizer for $\mathcal{F}$, i.e.,*

$$\begin{cases} u^n_k \in \operatorname{argmin}\{\mathcal{F}(u, v^n_k) : u \in \mathcal{U}(s^n_k)\}, \\ v^n_k \in \operatorname{argmin}\{\mathcal{F}(u^n_k, v) : v \in \mathcal{V} \textit{ with } v \le v^n_{k,0} = v^n_{k-1}\}. \end{cases} \tag{8.2}$$

Proof. By minimality

$$0 \le \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1}) \le \mathcal{F}(u^n_{k,i+1}, v^n_{k,i}) \le \mathcal{F}(u^n_{k,i}, v^n_{k,i}) \le \cdots \le \mathcal{F}(u^n_{k,0}, v^n_{k,0}) := C.$$

Now, since $\mathcal{F}(\cdot, u)$ is quadratic we have that

$$\begin{aligned} \mathcal{F}(u^n_{k,i+1}, v^n_{k,i}) = \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1}) + \partial_v \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1})[v^n_{k,i} - v^n_{k,i+1}] \\ + \tfrac{1}{2}\partial^2_{vv}\mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1})[v^n_{k,i} - v^n_{k,i+1}, v^n_{k,i} - v^n_{k,i+1}]. \end{aligned} \tag{8.3}$$

Again by minimality $\partial_v \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1})[v^n_{k,i} - v^n_{k,i+1}] \ge 0$, while

$$\begin{aligned} \partial^2_{vv}\mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1})[v^n_{k,i} - v^n_{k,i+1}, v^n_{k,i} - v^n_{k,i+1}] = \int_\Omega 2(v^n_{k,i} - v^n_{k,i+1})^2 W_+(\varepsilon(u^n_{k,i+1}))\mathrm{d}x \\ + G_c \int_\Omega (v^n_{k,i} - v^n_{k,i+1})^2 + |\nabla v^n_{k,i} - \nabla v^n_{k,i+1}|^2 dx \ge G_c \left\| v^n_{k,i} - v^n_{k,i+1} \right\|^2_{H^1}. \end{aligned}$$

Therefore, from (8.3)

$$\tfrac{1}{2}\left\| v^n_{k,i} - v^n_{k,i+1}\right\|^2_{H^1} \le \mathcal{F}(u^n_{k,i+1}, v^n_{k,i}) - \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1}) \le \mathcal{F}(u^n_{k,i}, v^n_{k,i}) - \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1}),$$

and summing over i it follows that

$$\tfrac{1}{2}\sum_{i=0}^{+\infty}\left\| v^n_{k,i} - v^n_{k,i+1}\right\|^2_{H^1} \le \sum_{i=0}^{+\infty} \mathcal{F}(u^n_{k,i}, v^n_{k,i}) - \mathcal{F}(u^n_{k,i+1}, v^n_{k,i+1}) \le \mathcal{F}(u^n_{k,0}, v^n_{k,0}) \le C.$$

As a consequence $\|v^n_{k,i} - v^n_{k,i+1}\|^2_{H^1} \to 0$ as $i \to \infty$ and from the continuous embedding of $H^1(\Omega)$ in $L^r(\Omega)$, for every $1 \le r < +\infty$, we have

$$\|v^n_{k,i} - v^n_{k,i+1}\|^2_{L^r} \to 0. \tag{8.4}$$

Since the energies $\mathcal{F}(u^n_{k,i}, v^n_{k,i})$ are uniformly bounded, by coercivity the sequence $\{v^n_{k,i}\}$ turns out to be bounded in $H^1(\Omega)$, and hence there exists a subsequence $\{v^n_{k,i_j}\}_{j\in\mathbb{N}}$ weakly converging in $H^1(\Omega)$ to a limit, that we call v^n_k. By compact embedding this subsequence strongly converges in $L^r(\Omega)$ for every $1 \le r < +\infty$; as a consequence of (8.4), also $\{v^n_{k,i_j-1}\}_{j\in\mathbb{N}}$ converges to the same function. By definition

$$u^n_{k,i_j} \in \operatorname{argmin}\{\mathcal{F}(u, v^n_{k,i_j-1}) \,:\, u \in \mathcal{U}(s^n_k)\}.$$

By Lemma 4.6, the subsequence $\{u^n_{k,i_j}\}_{j\in\mathbb{N}}$ converges in $W^{1,p}(\Omega,\mathbb{R}^2)$ to $u^n_k \in \operatorname{argmin}\{\mathcal{F}(u, v^n_k) \,:\, u \in \mathcal{U}(s^n_k)\}$. Analogously, since

$$v^n_{k,i_j} \in \operatorname{argmin}\{\mathcal{F}(u^n_{k,i_j}, v) \,:\, v \le v^n_k\},$$

by Lemma 4.10 the subsequence $\{v^n_{k,i_j}\}_{j\in\mathbb{N}}$ converges in $H^1(\Omega)$ to $v^n_k \in \operatorname{argmin}\{\mathcal{F}(u^n_k, v) : v \le v^n_k\}$, which concludes the proof. ☐

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Data availability statement

The research data associated with this article are included in the article.

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Appendix A. Kuratowski convergence

Kuratowski convergence is a notion of convergence for sequences of subsets of a topological space [37]. Here, we briefly present the notion of Kuratowski limit of a sequence of subsets of a metric space (X, d).

Definition A.1. Given a sequence of subsets $\{A_n\}_{n\in\mathbb{N}}$ of X, the Kuratowski limit inferior of A_n as $n \to +\infty$ is:

$$LiA_n := \{x \in X \,:\, \limsup_{n\to+\infty} d(x, A_n) = 0\} = \{x \in X \,:\, \forall n \in \mathbb{N}, \exists\, a_n \in A_n \,:\, \lim_{n\to+\infty} d(x, a_n) = 0\}$$

while the Kuratowski limit superior of A_n as $n \to +\infty$ is:

$$LsA_n := \{x \in X \,:\, \liminf_{n\to+\infty} d(x, A_n) = 0\} = \{x \in X \,:\, \exists\, \{n_i\}_{i\in\mathbb{N}} \text{ and } a_{n_i} \in A_{n_i} \,:\, \lim_{i\to+\infty} d(x, a_{n_i}) = 0\}$$

In general $LiA_n \subset LsA_n$ and if they are equal, the common set is called Kuratowski limit of A_n.

Note that both LiA_n and LsA_n are closed subsets of X. Moreover, we have the following compactness result.

Theorem A.2. *Let (X, d) be a separable metric space and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of closed sets. There exists a subsequence of $\{A_n\}_{n\in\mathbb{N}}$ converging in the sense of Kuratowski.*