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STABILIZATION OF LINEAR DISTRIBUTED SYSTEMS UNDER SWITCHING CONTROL CONSTRAINT

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Abstract. We consider a class of infinite-dimensional linear control systems with controls subject to possible saturation. The aim is to study the stabilization under the additional switching control constraint which means that only one actuator is active. The appropriate feedback turns out to be multivalued and the existence of the solution to the resulting differential inclusion is established by using nonlinear semigroup theory. The switching control property occurs whenever the set of instants at which the trajectory intersects some specified subsets of the state space is of null measure. In the multivalued feedback framework, asymptotic stability and asymptotic output stability are obtained from LaSalle invariance principle. We establish also other stabilization results by constructing suitable state dependent switched systems where only one control is activated in each subsystem. Applications to the simultaneous stabilization under switching control constraint are treated for various partial differential equations. The latter includes heat, plate and wave systems.

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1. Problem formulation and preliminaries

1.1. Introduction

Control systems in real applications are usually endowed with several actuators. Applications from civil engineering lead to pumps, valves and various control elements. It is often desirable to design switching control strategies guaranteeing that, at each instant of time, only one control is activated. In order to treat the resulting modeling process, we are led to consider discontinuous dynamical systems whose motions are not continuous with respect to time.

Regarding this aspect, switching refers to a change of the actuator that is activated at each time instant. Switching may also refer to the possibility of the state equation itself to change from one configuration to another at some time instants. See [1, 2] for an introduction and the survey articles [3, 4]. We refer to such systems as switched systems. Our main goal concerns the stabilization of a distributed system defined by a state equation given *a priori* under the requirement that the feedback control satisfies the switching between the controllers. We shall see later that we are led to consider the two meaning of switching mentioned above: switching for the actuators and switching for the state equation configurations.

Keywords and phrases: Distributed systems, switching control, multivalued feedback, asymptotic stabilization, switched systems.

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Recall that stability under arbitrary switching can be related to differential inclusions and it is mainly achieved by constructing common Lyapunov functions. Such a construction is often complicated and it can lead to functions with set-valued derivatives. See, for instance, [5] and the references there in. Note also that while the stability analysis of specific classes of such systems has far been concerned primarily with finite dimensional dynamical systems determined by ordinary differential equations, more recently, infinite-dimensional dynamical systems have also been treated [6–13].

In this article, we consider the linear distributed system

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where the operator A generates a linear semigroup of contractions on the Hilbert space Y (state space), the operator B is linear bounded from a Hilbert space U (control space) to Y. We suppose that the control space U is a product of Hilbert spaces specified by $U = U_1 \times U_2 \times ... \times U_p$. Our objective concerns the asymptotic stabilization with (possibly) constrained control function $u = (u_1, u_2, ..., u_p)$ whose components are subject to switching strategies. In order to specify the framework of constrained stabilizability with switching control, we suppose that the control operator has the form

$$B = (B_1, B_2, ..., B_p),\ B_j \in \mathcal{L}(U_j, Y),\ j = 1, 2, ..., p, \tag{1.2}$$

and for any $j = 1, 2, ..., p$, B_j will be referred to as a controller operator component. Then the state equation is

$$\begin{cases} y'(t) = Ay(t) + \sum\limits_{j=1}^{p} B_j u_j(t), \\ y(0) = y_0, \end{cases} \tag{1.3}$$

and our objective is to design feedback control strategies of the form $u(t) = u(y(t))$ guaranteeing that, at each instant of time, only one actuator is active so that

$$\|u_i(y(t))\|_{U_i} \|u_j(y(t))\|_{U_j} = 0 \text{ a.e on } (0, \infty), \tag{1.4}$$

for all $i, j = 1, 2, ..., p$, $i \neq j$, where $\|.\|_{U_j}$ denotes the norm on U_j. Also, we shall take into account the realistic situation where each control component u_j is subject to saturation constraint (possibly after re-scaling) such as

$$\|u_j(y(t))\|_{U_j} \leq 1. \tag{1.5}$$

Thus, beside the possible saturation assumption, there is a supplementary constraint to the activity of the control components $u_1, ..., u_p$ in the sense expressed by (1.4). Subject to this additional constraint, we treat the question of stabilizing the system (1.1) (or (1.3)) with an appropriate feedback. Any feedback accomplishing this task will be referred to as switching control feedback. Note that while our initial object is contained in the control switching level, searching an appropriate feedback will affect the state equation and consequently the change in the latter is inevitable. It follows that we shall be concerned with switched systems governed by continuous one-parameter nonlinear semigroups in Hilbert spaces. This problem has been treated recently in the context of ordinary differential equations in [14]. To the best of our knowledge, this issue has not been treated until now in the framework of infinite dimensional systems.

Remark 1.1. The open loop version of this issue can be formulated as null controllability problem in a prescribed finite time $T > 0$. A systematic way of building switching control based on variational methods has been treated in [15, 16]. While in the first reference the controls are supposed unconstrained, the second one treats also the case of saturated controls in the context of finite dimensional systems.

In order to motivate our method based on multi-valued feedback approach, we exclude at the outset some misleading attracting situations. Suppose that we have at our disposal convenient partition $t_0 = 0 < t_1 < ... < t_i < t_{i+1} < ...$ of the time interval $(0, \infty)$ such that on each part (t_i, t_{i+1}) one may stabilize a system of the form

$$y'(t) = Ay(t) + B_j u_j(t),\ t_i < t < t_{i+1}, \tag{1.6}$$

for some $j \in \{1, 2, ..., p\}$. Thus, we are led to a situation where the state equation is switched. However, it must be emphasized that the following facts should be taken into account. On the one hand, even if we are in presence of stable sub-systems, it is well known that the stability of the overall system is not guaranteed in the sense that it is still possible to construct a divergent trajectory from any initial state for such a switched system. On the other hand, if one single controller operator B_j suffices for stabilizing the system, condition (1.4) may be trivially achieved whenever the other ones are subject to vanish. In order to avoid such a trivial situation, we shall suppose that every subsystem (1.6) is not supposed to be stabilizable by one controller. This situation occurs naturally for the simultaneous stabilization of finitely many systems by using a switching control strategy. Indeed, let us consider, for $j = 1, ..., p$, the systems

$$y_j'(t) = A_j y_j(t) + \tilde{B}_j u_j(t),$$

defined on the Hilbert spaces Y_j, where the operator A_j generates a semigroup of contractions on Y_j. The operator $\tilde{B}_j : U_j \to Y_j$ is supposed linear bounded. Then the problem of stabilizing simultaneously all these systems can be reduced to consider on the product space $Y := Y_1 \times Y_2 \times ... \times Y_p$ a system similar to (1.3) with $A : D(A_1) \times A_2 \times ... \times D(A_p) \to Y$ defined by

$$A\left(y_1, ..., y_p\right) = \left(A_1 y_1, ..., A_p y_2\right),$$

and $B_j : U_j \to Y$ defined by

$$B_1(u_1) = \left(\tilde{B}_1 u_1, 0, ..., 0\right), ...,\ B_p(u_p) = \left(0, ..., 0, \tilde{B}_p u_p\right).$$

In such a situation, it is clear that the system with only one controller, as in (1.6), can not be stabilized with respect to the extended space Y. In other words, the system (1.3) should be stabilized "collectively".

Throughout this paper, we shall use the following notations. On any Hilbert space H, the inner product and its associated norm are denoted by $\langle ., . \rangle_H$ and $\|.\|_H$ respectively. The adjoint of any linear bounded operator D will be denoted by D^*. In particular, for $B^* : Y \to U_1 \times U_2 \times ... \times U_p$ and $y \in Y$, we shall need the following expression

$$B^* y = \left(B_1^* y, B_2^* y, ..., B_p^* y\right). \tag{1.7}$$

Given two Hilbert spaces H_1 and H_2, the space of linear bounded operators from H_1 to H_2 is denoted by $\mathcal{L}(H_1, H_2)$. The norm on this space is defined by

$$\|D\|_{\mathcal{L}(H_1,H_2)} := \sup_{\|y\|_{H_1} \le 1} \|Dy\|_{H_2},\ D \in \mathcal{L}(H_1, H_2).$$

As in [14], we shall develop a systematic method allowing one to get switching controls by using a multi-valued feedback approach. The latter is based on nonsmooth convex analysis tools and nonlinear semigroup theory generated by maximal monotone operators. Our method is inspired by the stabilization of infinite dimensional systems with possible saturation constraint on the control. Indeed, in [17] it has been established that many types of stabilizing feedback can be recast as monotone feedback of the form

$$u(y(t)) = -\Psi'(B^* y(t)), \tag{1.8}$$

where the mapping $\Psi : U \to \mathbb{R}$ is convex and Gâteaux differentiable with gradient Ψ' satisfying $\Psi'(0) = 0$. We shall see that a solution to our switching control feedback problem can be obtained in a first step by adopting a nonsmooth context in the sense that the equality in (1.8) is relaxed to the following inclusion

$$(u_1(y(t)), u_2(y(t)), ..., u_p(y(t))) \in -\partial(\Psi) B^* y(t), \tag{1.9}$$

where $\partial\Psi$ denotes the subgradient of an appropriate convex function Ψ. We point out that the choice of such a suitable function constitutes a main preliminary contribution of this paper. Moreover, this function will be presented in an abstract setting which will enable us to cover the case where the control components may be subject to distinct constraints. Hence, we can consider situations where two different controls u_i and u_j are not necessarily subject to the same type of constraints. This fact will turn out to be useful in order to construct switched single valued feedback systems such that the switching control constraint holds for each subsystem, and the overall system is stable. Combining (1.1) and (1.9) leads to the following differential inclusion:

$$\begin{cases} y'(t) - Ay(t) + B\partial(\Psi)B^* y(t) \ni 0, \\ y(0) = y_0. \end{cases} \tag{1.10}$$

Recall that the system (1.10) is said to be stable in the sense of Lyapunov if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|y_0\|_Y \leq \delta \Rightarrow \|y(t)\|_Y \leq \epsilon \text{ for all } t \geq 0.$$

We shall be concerned with two types of asymptotic stability. The first one is the usual asymptotic stability. The second one supposes that we have at our disposal observations which are available from the following output equation

$$z(t) = (z_1(t), ..., z_p(t)) = B^* y(t). \tag{1.11}$$

The precise definitions are as follows.

Definition 1.2. Assume that the system (1.10) is stable in the sense of Lyapunov. This system is said to be asymptotically stable if every solution of (1.10) satisfies $y(t) \to 0$ as $t \to \infty$. The system (1.10) is said to be asymptotically output stable if every solution of (1.10) satisfies $B^* y(t) \to 0$ as $t \to \infty$.

Let us specify the main contributions of this paper.

- The second section is devoted to the introduction of the multivalued feedback setting leading to the differential inclusion (1.10). We shall see that the switching behavior motivates naturally such a model. The latter is treated in the framework of nonlinear semigroup of contractions as stated in Theorem 2.4. The content of Proposition 2.2 is the main technical contribution in view to establish the existence and the uniqueness of the solution. Moreover, we show that the switching control property in feedback form is of local nature with respect to the time variable. As applications, we introduce various partial differential equations (PDEs) for which the problem of simultaneous stabilization under switching control constraint will be treated. Here the explicit form of the multi-valued feedbacks are given.
- In Section 3, asymptotic stability and asymptotic output stability results are established in the multi-valued feedback setting resulting from the switching control constraint (1.4). The LaSalle invariance

principle enables us to treat directly the stability in a differential inclusion context. Applications to the PDEs introduced in Section 2 are treated.

- In Section 4, we exploit the flexibility provided by the multi-valued feedback setting and the fact that the switching control property (1.4) is of local nature with respect to the time variable. The difficulty of providing a single-valued feedback valid for switching control will be circumvented by using, in implicit state feedback form, appropriate switched systems where only one control is activated on appropriate state dependent time intervals. The construction of these systems, stated in Theorem 4.1, is performed by defining explicitly an increasing sequence of transition times $\{t_n\}_n$. Unlike the construction performed in [14], here the construction is more general and is based on "collective" obervability condition which will be crucial for the choice of the first activation instant. Moreover, in this paper, the notion of periodic-like intermittent activation relative to each control component u_j, introduced in [14], is dispensed with. We shall obtain in Theorem 4.5, Theorem 4.6 and Theorem 4.7 various types of stability according to the behavior of this sequence. By analogy, this behavior is treated in the light of the familiar switched systems terminology developed in [1, 4] and specified by Zeno behavior, fast switching and slow switching with dwell time. Considering again the PDEs introduced in Section 2, we give explicitly the single-valued feedback corresponding to each sub-interval with the corresponding transition rule.
- In Section 5, we present open problems related to our work. We hope that these issues lead to interesting developments in the future.

1.2. Preliminaries on nonsmooth analysis

Below we present some nonsmooth analysis notions which will be used in the sequel. Let H be a Hilbert space and let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a given extended-valued function. The effective domain of f, denoted $\mathrm{dom} f$, is the set

$$\mathrm{dom} f = \{x \in H \,:\, f(x) < \infty\}\,.$$

The function f is called proper when $\mathrm{dom} f \neq \emptyset$. Let f be a proper function and let $x \in \mathrm{dom} f$. An element ξ of H is called a subgradient of f at x (in the sense of convex analysis) if it satisfies the following subgradient inequality:

$$f(y) - f(x) \geq \langle \xi, y - x \rangle_H \text{ for all } y \in H. \tag{1.12}$$

The (possibly empty) set of all subgradients of f at x is denoted by $\partial f(x)$, and referred to as the subdifferential of f at x. Note that if f is convex and $x \in \mathrm{dom} f$ is a point of continuity of f, then $\partial f(x)$ is nonempty and weakly compact. See ([18], p. 62). Moreover, the directional derivative $f'(x; h)$ defined by

$$f'(x;h) = \lim_{t \searrow 0} \frac{f(x+th) - f(x)}{t},$$

exists for every $h \in H$, with values in $[-\infty, +\infty]$. When f is convex, the subgradient and the directional derivative are related by

$$\partial f(x) = \{\xi \in H \,:\, f'(x;h) \geq \langle \xi, h \rangle_H\,, \forall h \in H\}\,. \tag{1.13}$$

See Proposition 4.3 in ([18], p. 61). We mention also that when f is bounded above on a neighborhood of x, then for all $h \in H$

$$f'(x;h) = \max\{\langle \xi, h \rangle_H \,:\, \xi \in \partial f(x)\}\,. \tag{1.14}$$

This fact can be deduced from Proposition 2.1.2 in ([19], p. 27) and Proposition 2.2.7 in ([19], p. 36). A useful application of either (1.13) or (1.14) yields the subgradient of the norm function $N_H(v) := \|v\|_H$ on H at the origin given by

$$\partial N_H(0) = \mathcal{B}_H, \tag{1.15}$$

where $\mathcal{B}_H$ denotes the closed unit ball in H. Below, for any real $r > 0$ and $j \in \{1, 2, ..., p\}$, $\mathcal{B}_j(0, r)$ denotes the closed ball in U_j defined by

$$\mathcal{B}_j(0, r) = \left\{ u_j \in U_j \,:\, \|u_j\|_{U_j} \leq r \right\}. \tag{1.16}$$

In particular, the closed unit ball $\mathcal{B}_j(0, 1)$ will be denoted $\mathcal{B}_j$ and we shall use the fact that $\mathcal{B}_j(0, r) = r\mathcal{B}_j$. A multivalued operator $T : H \to H$ is determined by its domain

$$D(T) = \{v \in H \,:\, Tv \neq \emptyset\},$$

and its graph

$$G(T) = \{(v, w) \,:\, w \in T(v)\} \subset H \times H.$$

It is said to be monotone if

$$\langle v_1 - v_2, w_1 - w_2 \rangle_H \geq 0 \text{ whenever } w_1 \in T(v_1),\, w_2 \in T(v_2). \tag{1.17}$$

It is called a maximal monotone operator if, in addition, its graph is not properly contained in the graph of any other monotone operator. Let Id denote the identity operator on H. It is known (*e.g.* ([20], p. 23)) that T is maximal monotone if and only if, the single-valued resolvent operator $(Id - T)^{-1}$ is a contraction on H. The minimal section defined by the element of minimum norm on Tv is denoted by $T^0 v$. T is said to be bounded if, for any bounded subset $\mathcal{H} \subset H$, the set $\underset{y \in \mathcal{H}}{\cup} T(y)$ is bounded. A (generally nonlinear) semigroup on H is a family of continuous maps $S(t) : H \to H$, $t \geq 0$, satisfying (i) $S(0) =$ identity, (ii) $S(t + s) = S(t)S(s)$, for all $t, s \geq 0$, (iii) $S(t)v \to v$ strongly as $t \searrow 0$ for all $v \in H$. The semigroup is said to be of contractions if for all $t \geq 0$

$$\|S(t)v_0 - S(t)v_1\|_H \leq \|v_0 - v_1\|_H \;\forall v_0, v_1 \in H. \tag{1.18}$$

Given a subset $E \subset H$, $\operatorname{conv}(E)$ denotes the convex hull of the set E. For a subset $W \subset H$ and $\lambda \in \mathbb{R}$, we define the set λW by

$$\lambda W := \{\lambda w \,:\, w \in W\}. \tag{1.19}$$

Also, given C and D subsets contained in H, we denote by $C + D$ the set given by

$$C + D := \{c + d \,:\, c \in C,\, d \in D\}. \tag{1.20}$$

2. A multivalued feedback setting

In order to facilitate the presentation of the abstract setting, we introduce some feedbacks related to significant types of constraint on each control component u_j. We shall see that every constraint can be naturally derived from an appropriate convex function.

2.1. Abstract motivating examples and terminology

Since we are seeking appropriate feedback for any activated control component, we consider for each $j \in \{1, 2, ..., p\}$ the state equation of the system (1.6). In principle, the main goal in stabilizing this system is to find a feedback control

$$u_j(t) = u_j(y(t))$$

which yields $y(t) \to 0$ as $t \to \infty$. If we compute formally the time rate of change of the "energy" $\frac{1}{2} \|y(t)\|_Y^2$, then by using the fact that $\langle Av, v\rangle_Y \leq 0$ for all $v \in D(A)$, we get formally

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|y(t)\|_Y^2 = \langle Ay(t), y(t)\rangle_Y + \langle B_j u_j(t), y(t)\rangle_Y$$

$$\leq \left\langle u_j(t), B_j^* y(t)\right\rangle_{U_j}.$$

In order to obtain a nonincreasing energy, we are led to consider a feedback of the form

$$u_j(y(t)) = -f_j(B_j^* y(t)), \tag{2.1}$$

where $f_j : U_j \to U_j$ satisfies $\langle f_j(u_j), u_j\rangle_{U_j} \geq 0$ for all $u_j \in U_j$. Having in mind a unified presentation which encompasses in a convex framework the typical examples below, we shall derive f_j from an auxiliary functional $F_j : U_j \to \mathbb{R}$ by setting formally

$$f_j(u_j) = F_j'(u_j). \tag{2.2}$$

Moreover, this functional will be defined by

$$F_j(u_j) = \int_0^{\|u_j\|_{U_j}} w_j(\tau)\mathrm{d}\tau, \tag{2.3}$$

where $w_j : \mathbb{R}^+ \to \mathbb{R}$ is an appropriate function characterizing the type of constraint on the component control u_j. Later, the functional F_j and the function w_j will be referred to as auxiliary feedback functional and auxiliary feedback function respectively. Note that whenever $u_j \neq 0$, we have

$$f_j(u_j) = \frac{w_j\left(\|u_j\|_{U_j}\right)}{\|u_j\|_{U_j}} u_j. \tag{2.4}$$

Let us mention also other general saturating feedbacks based on locally Lipschitz functions f_j satisfying $f_j(0) = 0$. This approach has been developed in [21]. However, it seems that the Lipschitz framework is inappropriate whenever the multivalued feedback is unavoidable as in (1.9) and in the bang-bang feedback given below. Moreover, even in the single-valued feedback case, the convex approach turns out to be suitable in the case where the control operator B_j is unbounded (see [17] for details).

(i) Unconstrained control

In the unconstrained context, the most simple feedback can be reduced to the following canonical linear one

$$u_j(y(t)) = -B_j^* y(t). \tag{2.5}$$

This amounts to consider as feedback function

$$f_j(u_j) = f_q(u_j) := u_j \text{ for all } j \in \{1, 2, ..., p\}. \tag{2.6}$$

The corresponding auxiliary functional would be the quadratic functional given by

$$F_j(u_j) = F_q(u_j) := \frac{1}{2} \|u_j\|_{U_j}^2 \text{ for all } j \in \{1, 2, ..., p\}, \tag{2.7}$$

and the corresponding auxiliary function is defined by

$$w_j(s) = w_q(s) := s \text{ for all } j \in \{1, 2, ..., p\}. \tag{2.8}$$

(ii) Saturated control

Following the analogous control systems literature on saturated control subject to (1.5) for ordinary differential equations (see, for instance, [22]), a natural control law which comes at mind is given by:

$$u_j(y(t)) = \begin{cases} -\frac{B_j^* y(t)}{\|B_j^* y(t)\|_{U_j}} \text{ if } \|B_j^* y(t)\|_{U_j} \geq 1, \\ -B_j^* y(t) \text{ if } \|B_j^* y(t)\|_{U_j} \leq 1, \end{cases} \tag{2.9}$$

for all $j \in \{1, 2, ..., p\}$. Among other saturating feedbacks satisfying (1.5), let us mention the following simpler one given by

$$u_j(y(t)) = -\frac{B_j^* y(t)}{1 + \|B_j^* y(t)\|_{U_j}}. \tag{2.10}$$

More generally and following [23], let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $r\rho(r)$ nondecreasing and

$$\rho(r) > 0 \text{ for all } r \geq 0, \tag{2.11}$$

$$\rho(r) \leq \min(1, \frac{1}{r}) \text{ for all } r > 0. \tag{2.12}$$

If we consider for any $j = 1, 2, ..., p$ the auxiliary feedback function

$$w_j(s) = w_\rho(s) := \rho(s)s, \ s \geq 0, \tag{2.13}$$

and the associated feedback function given by

$$f_\rho(u_j) = \rho\left(\|u_j\|_{U_j}\right) u_j, \tag{2.14}$$

then it is easy to see that f_ρ satisfies the control saturation (1.5). Furthermore, the feedbacks (2.9) and (2.10) are contained in our setting since they correspond respectively to the auxiliary feedback functions

$$w_{\rho_0}(s) = \rho_0(s)s, \ \ \rho_0(r) = \min(1, \frac{1}{r}), \tag{2.15}$$

$$w_{\rho_1}(s) = \rho_1(s)s, \ \ \rho_1(r) = \frac{1}{1 + r}. \tag{2.16}$$

The assumptions on $\rho(.)$ above are readily satisfied by ρ_0 and ρ_1. Note also that the corresponding feedback is given by

$$u_j(y(t)) = -\rho\left(\left\|B_j^* y(t)\right\|_{U_j}\right) B_j^* y(t), \tag{2.17}$$

and the corresponding auxiliary function, denoted by F_ρ, would read

$$F_\rho(u_j) = \int_0^{\|u_j\|_{U_j}} \tau\rho(\tau)\mathrm{d}\tau,\ u_j \in U_j. \tag{2.18}$$

Moreover, the unconstrained control feedback (2.5) can be interpreted as a degenerate case in which $\rho \equiv 1$.
(iii) Bang-bang control
Under the saturation constraint (1.5) and in the context of the feedback defined by (2.1), (2.2) and (2.3), the appropriate auxiliary feedback function, denoted by w_{bb}, can be defined by

$$w_{bb}(s) \equiv 1 \text{ for all } s \geq 0, \tag{2.19}$$

The resulting auxiliary function, denoted by F_{bb}, is defined by

$$F_{bb}(u_j) = \|u_j\|_{U_j} \text{ for all } j \in \{1, 2, ..., p\}. \tag{2.20}$$

It follows that the corresponding bang-bang feedback is the following multivalued feedback

$$u_j(y(t)) = -F'_{bb}(B_j^* y(t)) = -\begin{cases} \frac{B_j^* y(t)}{\left\|B_j^* y(t)\right\|_{U_j}} \text{ if } B_j^* y(t) \neq 0, \\ \mathcal{B}_j \text{ if } B_j^* y(t) = 0. \end{cases} \tag{2.21}$$

Remark 2.1. (i) Note that in the bang-bang case, the auxiliary feedback function w_{bb} corresponds to the degenerate case of w_ρ where

$$s\rho(s) \equiv 1,\ \forall s > 0.$$

(ii) In the case of scalar controls $u_1, ..., u_p$ with $B_j u_j = u_j b_j$ and $b_j \in Y$, we obtain as feedback analogous to (2.21)

$$u_j(y(t)) = \begin{cases} -1 \text{ if } \langle b_j, y(t)\rangle_Y > 0, \\ 1 \text{ if } \langle b_j, y(t)\rangle_Y < 0, \\ [-1, 1] \text{ if } \langle b_j, y(t)\rangle_Y = 0. \end{cases} \tag{2.22}$$

It follows that this bang-bang feedback gives another switching type between the extremal values of the control. We note also that it has been obtained naturally in multivalued form.

2.2. Switching control feedback and differential inclusion

Let us introduce a convenient framework leading to the multivalued feedback system (1.10). For $j = 1, 2, ..., p$, let $w_j : \mathbb{R}^+ \to \mathbb{R}$ be a given auxiliary feedback function and let e_j denote the projection operator $e_j : U = U_1 \times U_2 \times ... \times U_p \to U$ such that $e_j(u_1, u_2, ..., u_j, .., u_p) = (0, ..., 0, u_j, 0, .., 0)$. Considering the three types of feedback components presented above, we note that our abstract model contains situations where the control components u_j may be subject independently to distinct types of constraints.

Proposition 2.2. *Suppose that the auxiliary feedback functions w_j are continuous, nondecreasing and satisfying $w_j(s) > 0$ for all $s > 0$. Consider the mapping $\Psi : U \to \mathbb{R}$ defined by*

$$\Psi(u_1, u_2, ..., u_p) = \max\left(F_1(u_1), F_2(u_2), ..., F_p(u_p)\right), \tag{2.23}$$

where the associated auxiliary functionals $F_j : U_j \to \mathbb{R}$ are given by (2.3). Then the subgradient $\partial\Psi$ is well defined and its explicit expression is given as follows. (i) For any $u = (u_1, u_2, ..., u_p) \in U$ such that $(u_1, u_2, ..., u_p) \neq (0, 0, ..., 0)$, we have

$$\partial\Psi(u) = conv\left\{\frac{w_j\left(\|u_j\|_{U_j}\right)}{\|u_j\|_{U_j}} e_j(u) \,:\, j \in I(u)\right\}, \tag{2.24}$$

where

$$I(u) = \{k \in \{1, 2, ..., p\} \,:\, \Psi(u_1, u_2, ..., u_p) = F_k(u_k)\}. \tag{2.25}$$

(ii) For $u = 0$ we have

$$\partial\Psi(0) = w_1(0)\mathcal{B}_1 \times w_2(0)\mathcal{B}_2 \times ... \times w_p(0)\mathcal{B}_p. \tag{2.26}$$

Proof. Considering the functions F_j as functions of the variables $u_1, ..., u_p$, the resulting functions, denoted by $\widetilde{F}_j$, are defined by $\widetilde{F}_j : U \to \mathbb{R}$ and

$$\widetilde{F}_j(u_1, ..., u_p) := F_j(u_j). \tag{2.27}$$

Then Ψ can be reformulated by

$$\Psi(u_1, ..., u_p) = \max\left(\widetilde{F}_1(u_1, ..., u_p), ..., \widetilde{F}_p(u_1, ..., u_p))\right). \tag{2.28}$$

On the other hand, since the functions w_j are nondecreasing, it is easy to see that the auxiliary functions F_j are convex. It follows that Ψ is also convex on U. Hence by adapting Proposition 2.3.12 in ([19], p. 47), we get

$$\partial\Psi(u) = \text{conv}\left\{\partial\widetilde{F}_j(u) \,:\, j \in I(u)\right\}.$$

(i) For any $u = (u_1, u_2, ..., u_p) \in U$ such that $(u_1, u_2, ..., u_p) \neq (0, 0, ..., 0)$, we have clearly

$$u_j \neq 0 \text{ for all } j \in I(u),$$

and whenever $j \in I(u)$

$$\partial\widetilde{F}_j(u) = e_j(0, ..., 0, F_j'(u_j), 0, .., 0).$$

This yields easily (2.24) since

$$F_j'(u_j) = \frac{w_j\left(\|u_j\|_{U_j}\right)}{\|u_j\|_{U_j}} u_j.$$

(ii) For $u = 0$ we have

$$\partial\Psi(0) = \text{conv}\left\{\partial\widetilde{F}_j(0)\ :\ j = 1, 2, ..., p\right\},$$

so that we are led to determine $\partial\widetilde{F}_j(0)$ for each j fixed. To this end, we consider the decomposition $\widetilde{F}_j := K_j \circ N_j$ where the function $K_j : \mathbb{R}^+ \to \mathbb{R}$ is defined by $K_j(s) = \int_0^s w_j(r)dr$ and N_j denotes the norm component function on U given by $N_j(u_1, ..., u_p) = \|u_j\|_{U_j}$. Then by applying the chain rule result in ([19], p. 42), it follows that

$$\partial\widetilde{F}_j(0) = \text{conv}\left\{\alpha_j\xi_j\ :\ \alpha_j \in \partial K_j(0),\ \xi_j \in \partial N_j(0)\right\}. \tag{2.29}$$

We deduce from (1.15) that

$$\partial N_j(0) = \{0\} \times ... \times \mathcal{B}_j \times ... \times \{0\}.$$

Then (2.26) is a ready consequence of (2.29) and the fact that $\partial K_j(0) = \{w_j(0)\}$. This completes the proof of the proposition. □

Remark 2.3. In the case where $(u_1, u_2, ..., u_p) \neq (0, 0, ..., 0)$, we have

$$\left\{\langle v, u\rangle_U\ :\ v \in \partial\Psi(u)\right\} = \left\{\sum_{j\in I(u)} \alpha_j \|u_j\|_{U_j} w_j\left(\|u_j\|_{U_j}\right)\ :\ 0 \le \alpha_j \le 1,\ \sum_{j\in I(u)} \alpha_j = 1\right\}. \tag{2.30}$$

It follows that, whenever $v \in \partial\Psi(u)$, then

$$\langle v, u\rangle_U = 0\ \Rightarrow\ u = 0. \tag{2.31}$$

This result will turn out to be useful for the stability issue treated in Section 3.

Here we give a representation of the solution of the differential inclusion system (1.10) that is based upon nonlinear semigroup theory.

Theorem 2.4. *Suppose that the assumptions of Proposition 2.2 hold and consider the operator $A_\Psi := A - B\partial(\Psi)B^*$, where the functional Ψ is defined by (2.3) and (2.23). Then A_Ψ generates a nonlinear semigroup of contractions on Y denoted by e^{tA_Ψ}. Moreover, for any $y_0 \in Y$, the feedback system (1.10) admits a unique solution defined by*

$$y(t) = \mathrm{e}^{tA_\Psi} y_0. \tag{2.32}$$

Proof. Consider the mapping $\Phi = \Psi \circ B^*$. Since Ψ is a lower semicontinuous proper convex function on Y, then Φ is also a lower semicontinuous proper convex function and its subgradient $\partial\Phi$ is maximal monotone as a graph on Y with $D(\partial\Phi) = Y$. See, for instance, [24]. Furthermore, since A generates a linear semigroup of contractions, the operator $-A$ is also maximal monotone. It follows that $-A + \partial\Phi$ admits as domain $D(A)$ and it is also maximal monotone [25]. On the other hand, by using ([18], p. 64), we obtain $\partial\Phi = B(\partial\Psi)B^*$ so that the operator $-A + B(\partial\Psi)B^*$ is maximal monotone. As a consequence of the general theory developed in [20], the feedback system (1.10) admits a unique solution $y \in C(0, \infty; Y)$ defined by the nonlinear semigroup generated by A_Ψ. □

2.3. Properties of solutions to the differential inclusions

Let us mention the usual properties of the solution of (1.10). See, for instance, Theorem 3.1 in ([20], p. 54). Whenever $y_0 \in D(A)$, then $y(t) \in D(A)$ for any $t > 0$. Furthermore, the right derivative, denoted by $\frac{\mathrm{d}^+ y}{\mathrm{d}t}(t)$, is well defined for $t \geq 0$ and the following properties hold.
(i) Because the semigroup $(\mathrm{e}^{tA_\Psi})_t$ is of contractions and $\mathrm{e}^{tA_\Psi}(0) = 0$, it follows that $\|y(t)\|_Y \leq \|y_0\|_Y$ for all $t \geq 0$ and the system (1.10) is stable in the sense of Lyapunov. Moreover, the function $t \to \|y(t)\|_Y$ is nonnegative nonincreasing so that the limits of $\|y(t)\|_Y$ as $t \to \infty$ is a well defined non-negative number.
(ii) the function $t \mapsto \mathrm{e}^{tA_\Psi} y_0$ is Lipschitz continuous and almost everywhere differentiable on $[0, \infty[$ with

$$\begin{cases} y'(t) - Ay(t) + Bg(t) = 0, \\ g(t) \in \partial(\Psi) B^* y(t) \text{ a.e, } \ t \geq 0, \end{cases} \tag{2.33}$$

$$\frac{\mathrm{d}^+ y}{\mathrm{d}t}(t) + A_\Psi^0 y(t) = 0. \tag{2.34}$$

(iii) The function $t \mapsto A_\Psi^0 y(t)$ is right continuous everywhere and the function $t \mapsto \left\|A_\Psi^0 y(t)\right\|_Y$ is nonincreasing.

Remark 2.5. Let us give explicitly the multivalued feedback resulting from (1.10) such that

$$u(y(t)) = (u_1(y(t)), ..., u_p(y(t))) \in -(\partial\Psi) B^* y(t).$$

It follows from Proposition 2.2 that if $B^* y(t) \neq 0$, or, equivalently, $B_j^* y(t) \neq 0$ for some $j \in \{1, 2, ..., p\}$, then

$$u(y(t)) \in -\mathrm{conv} \left\{ \frac{w_j\left(\left\|B_j^* y(t)\right\|_{U_j}\right)}{\left\|B_j^* y(t)\right\|_{U_j}} e_j(B^* y(t)) \ : \ j \in I(B^* y(t)) \right\}, \tag{2.35}$$

where

$$I(B^* y(t)) = \left\{ k \in \{1, ..., p\} \ : \ F_k\left(B_k^* y(t)\right) = \max_j F_j\left(B_j^* y(t)\right) \right\}. \tag{2.36}$$

When $B_j^* y(t) = 0$ for all $j \in \{1, 2, ..., p\}$, then

$$u(y(t)) \in w_1(0)\mathcal{B}_1 \times w_2(0)\mathcal{B}_2 \times ... \times w_p(0)\mathcal{B}_p. \tag{2.37}$$

2.4. Switching control property in the multivalued feedback setting

Taking into account Remark 2.5, we are in a position to give sufficient conditions guaranteeing that the solution of (1.10) satisfies the switching conditions (1.4). To this end, we introduce for $i, j \in \{1, 2, ..., p\}$, $i \neq j$, the state space subsets

$$S_{ij} := \left\{ y \in Y \ : \ F_i\left(B_i^* y\right) = F_j\left(B_j^* y\right) \right\}, \tag{2.38}$$

and their corresponding time subsets

$$\mathcal{T}_{ij}(y_0) := \left\{ t \in (0, \infty) \ : \ y(t) \in S_{ij} \right\}, \tag{2.39}$$

$$\mathcal{T}(y_0) := \underset{i,j}{\cup} \mathcal{T}_{ij}(y_0). \tag{2.40}$$

Then, by combining Proposition 2.2 and Remark 2.5, we get:

Corollary 2.6. *Suppose that the assumptions of Proposition 2.2 hold. Then the abstract feedback system (1.10) possesses the switching control property (1.4) provided that the set $\mathcal{T}(y_0)$ is of null measure.*

Proof. Suppose that $\mathcal{T}(y_0)$ is of null measure, then for almost all $t > 0$ we have $t \notin \mathcal{T}(y_0)$ so that $I(B^*y(t))$ is necessarily reduced to some singleton $\{j\}$ satisfying

$$F_j\left(B_j^*y(t)\right) > F_k\left(B_k^*y(t)\right) \text{ for all } k \neq j.$$

It follows that the corresponding feedback in (2.35) would read

$$u_k(y(t)) = \begin{cases} -\frac{w_j\left(\left\|B_j^*y(t)\right\|_{U_j}\right)}{\left\|B_j^*y(t)\right\|_{U_j}} B_j^*y(t) \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases} \tag{2.41}$$

Hence the switching control property (1.4) is fulfilled at time $t > 0$ almost everywhere. $\square$

The following result specifies the local nature of the switching control property (1.4) with respect to the time variable. This fact will enable us to obtain in feedback form switched stabilizing systems satisfying this property. The construction of such systems will be treated in Section 4.

Proposition 2.7. *Suppose that the assumptions of Proposition 2.2 hold. Assume that for some instant $t_a > 0$ the solution of the system (1.10) satisfies for some $j \in \{1, 2, ..., p\}$*

$$F_j\left(B_j^*y(t_a)\right) > F_k\left(B_k^*y(t_a)\right) \text{ for all } k \neq j. \tag{2.42}$$

Then there exists $\tau_a > 0$ such that

$$0 < t_a - \tau_a \leq t < t_a + \tau_a \Rightarrow F_j\left(B_j^*y(t)\right) > F_k\left(B_k^*y(t)\right) \text{ for all } k \neq j, \tag{2.43}$$

so that u_j is the sole activated feedback control on the time interval $(t_a - \tau_a, t_a + \tau_a)$ in which the state equation is given by

$$y'(t) = Ay(t) - \frac{w_j\left(\left\|B_j^*y(t)\right\|_{U_j}\right)}{\left\|B_j^*y(t)\right\|_{U_j}} B_jB_j^*y(t),\ t_a - \tau_a \leq t < t_a + \tau_a. \tag{2.44}$$

Proof. We note that for any $k \neq j$ the function

$$\theta_k : t \mapsto F_j\left(B_j^*y(t)\right) - F_k\left(B_k^*y(t)\right)$$

satisfies $\theta_k(t_a) > 0$ and it is continuous at t_a. It follows that there exists $\eta_k > 0$ such that

$$0 < t_a - \eta_k \leq t < t_a + \eta_k \Rightarrow \theta_k(t) > 0.$$

If we set $\tau_a := \min\limits_{k \neq j} \eta_k$, then we have

$$t_a - \tau_a \leq t < t_a + \tau_a \Rightarrow \theta_k(t) > 0 \text{ for all } k \neq j.$$

This implies (2.43) and from the expressions established in (2.35) and (2.36) we get

$$u_j(y(t)) = -\frac{w_j\left(\left\|B_j^* y(t)\right\|_{U_j}\right)}{\left\|B_j^* y(t)\right\|_{U_j}} B_j^* y(t).$$

This completes the proof of the proposition. □

2.5. Applications to partial differential equations (I)

Recall that the problem of simultaneous stabilization under switching control constraint has been already mentioned in the introduction. In this subsection we shall consider this problem for various examples of distributed parameter systems governed by PDEs. Our objective will consist on giving explicitly, for each case, the differential inclusion resulting from the multivalued feedback setting stated in Proposition 2.2 and Theorem 2.4. In order to give a presentation based on a unified framework, we consider the general abstract state equation

$$y'(t) = Ay(t) + B_1 u_1(t) + B_2 u_2(t), \tag{2.45}$$

and the related two abstract systems

$$y_1'(t) = A_1 y_1(t) + \tilde{B}_1 u_1(t), \tag{2.46}$$

$$y_2'(t) = A_2 y_2(t) + \tilde{B}_2 u_2(t). \tag{2.47}$$

The equations (2.46) and (2.47) are assumed to be defined in the Hilbert spaces Y_1 and Y_2 where the operators A_1 and A_2 generate semigroups of contractions on Y_1, Y_2 respectively. For $i = 1, 2$, the operator $\tilde{B}_i : U_i \to Y_i$ is supposed linear bounded. Then the problem of stabilizing simultaneously the two systems amounts to consider on the product space $Y := Y_1 \times Y_2$ the system

$$y'(t) = A_s y(t) + B_1 u_1(t) + B_2 u_2(t), \tag{2.48}$$

with $y(t) := (y_1(t), y_2(t))$, $A_s : D(A_1) \times D(A_2) \to Y$ defined by

$$A_s\left(y_1, y_2\right) := \left(A_1 y_1, A_2 y_2\right), \tag{2.49}$$

and $B_i : U_i \to Y$ defined by

$$B_1(u_1) = \left(\tilde{B}_1 u_1, 0\right),\ B_2(u_2) = \left(0, \tilde{B}_2 u_2\right). \tag{2.50}$$

In view to obtain readily for our next examples the appropriate multivalued feedbacks, we consider the general setting defined by (2.46)–(2.50). Then, by adapting (2.35)–(2.37), the appropriate multi-valued feedback can be expressed as follows.

- If $\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1} + \left\|\tilde{B}_2^* y_2(t)\right\|_{U_2} > 0$ and $F_1\left(\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}\right) = F_2\left(\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}\right)$, then

$$(u_1(t), u_2(t)) \in -\mathrm{conv}\left\{\left(\frac{w_1\left(\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}\right)}{\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}}\tilde{B}_1^* y_1(t), 0\right), \left(0, \frac{w_2\left(\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}\right)}{\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}}\tilde{B}_2^* y_2(t)\right)\right\}. \tag{2.51}$$

- If $F_1\left(\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}\right) > F_2\left(\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}\right)$, then

$$(u_1(t),u_2(t)) = \left(-\frac{w_1\left(\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}\right)}{\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}}\tilde{B}_1^* y_1(t), 0\right). \tag{2.52}$$

- If $F_2\left(\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}\right) > F_1\left(\left\|\tilde{B}_1^* y_1(t)\right\|_{U_1}\right)$, then

$$(u_1(t),u_2(t)) = \left(0, -\frac{w_2\left(\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}\right)}{\left\|\tilde{B}_2^* y_2(t)\right\|_{U_2}}\tilde{B}_2^* y_2(t)\right). \tag{2.53}$$

- If $\tilde{B}_1^* y_1(t) = 0$ and $\tilde{B}_2^* y_2(t) = 0$, then

$$(u_1(t),u_2(t)) \in w_1(0)\mathcal{B}_1 \times w_2(0)\mathcal{B}_2. \tag{2.54}$$

In the evolution PDEs below, the usual derivative denotes the partial derivative with respect to the time variable.

(i) *Multi-valued feedback for a heat-heat system.* We consider the problem of simultaneous stabilization of two heat systems under switching control. Let $\mathcal{O} \subset \mathbb{R}^N$ and $\tilde{\mathcal{O}} \subset \mathbb{R}^{\tilde{N}}$ be two open bounded domains with sufficiently smooth boundaries denoted by $\partial\mathcal{O}$ and $\partial\tilde{\mathcal{O}}$ respectively. We denote by $\frac{\partial}{\partial \nu}$ the exterior normal derivative on each boundary. Letting $\omega_1 \subset \mathcal{O}$ and $\omega_2 \subset \tilde{\mathcal{O}}$ denote non-empty open subsets, we shall regard $L^2(\omega_1)$ and $L^2(\omega_2)$ as subspaces of $L^2(\mathcal{O})$ and $L^2(\tilde{\mathcal{O}})$ respectively and we denote by χ_{ω_1} and χ_{ω_2} their respective characteristic functions. Then we introduce the following heat systems

$$\left\{\begin{array}{c} \theta' = \Delta\theta + u_1(t)\chi_{\omega_1} \text{ on } (0,\infty)\times\mathcal{O}, \\ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } (0,\infty)\times\Gamma_{\mathcal{O}}, \\ \theta(0,x) = \theta_0(x) \text{ on } \mathcal{O}, \end{array}\right. \tag{2.55}$$

$$\left\{\begin{array}{c} \tilde{\theta}' = \Delta\tilde{\theta} + u_2(t)\chi_{\omega_2} \text{ on } (0,\infty)\times\tilde{\mathcal{O}}, \\ \frac{\partial\tilde{\theta}}{\partial\nu} = 0 \text{ on } (0,\infty)\times\Gamma_{\tilde{\mathcal{O}}}, \\ \tilde{\theta}(0,x) = \tilde{\theta}_0(x) \text{ on } \tilde{\mathcal{O}}. \end{array}\right. \tag{2.56}$$

These systems have the forms (2.46) and (2.47) with $Y_1 := L^2(\mathcal{O})$, $Y_2 := L^2(\tilde{\mathcal{O}})$ as state spaces and $U_1 := L^2(\omega_1)$, $U_2 := L^2(\omega_2)$ as control ones. Moreover, the appropriate control operators $\tilde{B}_1 : L^2(\omega_1) \to L^2(\mathcal{O})$, $\tilde{B}_2 : L^2(\omega_2) \to L^2(\tilde{\mathcal{O}})$ are given by

$$\tilde{B}_i u_i = \chi_{\omega_i} u_i \text{ for } i = 1,2,$$

and the generators A_1 and A_2 are defined by

$$A_1\theta = \Delta\theta,\ D(A_1) = \left\{\theta \in L^2(\mathcal{O}) \mid \Delta\theta \in L^2(\mathcal{O}),\ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial\mathcal{O}\right\}, \tag{2.57}$$

$$A_2\tilde{\theta} = \Delta\tilde{\theta},\ D(A_2) = \left\{ \tilde{\theta} \in L^2(\tilde{\mathcal{O}}) \,|\, \Delta\tilde{\theta} \in L^2(\tilde{\mathcal{O}}),\ \frac{\partial\tilde{\theta}}{\partial\nu} = 0 \text{ on } \partial\tilde{\mathcal{O}} \right\}. \tag{2.58}$$

Then, considering as state $y(t) := (\theta(t), \tilde{\theta}(t))$, we get from (2.51)–(2.54) the following expressions for the feedback.

- If $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} + \left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)} > 0$ and $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) = F_2\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right)$, then

$$(u_1(t), u_2(t))$$

$$\in -\text{conv}\left\{ \left(\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}} \chi_{\omega_1}\theta(t), 0 \right), \left(0, \frac{w_2\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}} \chi_{\omega_2}\tilde{\theta}(t) \right) \right\}. \tag{2.59}$$

- If $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) > F_2\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right)$, then

$$(u_1(t), u_2(t)) = \left(-\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}} \chi_{\omega_1}\theta(t), 0 \right). \tag{2.60}$$

- If $F_2\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right) > F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)$, then

$$(u_1(t), u_2(t)) = \left(0, -\frac{w_2\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}} \chi_{\omega_2}\tilde{\theta}(t) \right). \tag{2.61}$$

- If $\chi_{\omega_2}\tilde{\theta}(t) = 0$ and $\chi_{\omega_1}\theta(t) = 0$, then

$$(u_1(t), u_2(t)) \in w_1(0)\mathcal{B}_{\omega_1} \times w_2(0)\mathcal{B}_{\omega_2}, \tag{2.62}$$

  where $\mathcal{B}_{\omega_1}$ and $\mathcal{B}_{\omega_2}$ are the unit closed balls in $L^2(\omega_1)$ and $L^2(\omega_2)$ respectively.

(ii) *Multi-valued feedback for a heat-plate system.* Here and later, we shall use similar notations as the ones introduced above. We consider an open bounded domain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. Let $\tilde{\omega} \subset \Omega$ denote a non-empty open subset. Then beside the heat equation (2.55), we consider the plate equation

$$\begin{cases} \tilde{v}'' = -\Delta^2\tilde{v} + u_2(t)\chi_{\tilde{\omega}} \text{ on } (0,\infty) \times \Omega, \\ \tilde{v} = \frac{\partial\tilde{v}}{\partial\nu} = 0 \text{ on } (0,\infty) \times \partial\Omega, \\ \tilde{v}(0,x) = \tilde{v}_0(x),\ \frac{\partial\tilde{v}}{\partial t}(0,x) = \tilde{v}_1(x) \text{ on } \Omega. \end{cases} \tag{2.63}$$

These systems have the forms (2.46) and (2.47) with $Y_1 := L^2(\mathcal{O})$, $Y_2 := H_0^2(\Omega) \times L^2(\Omega)$ as state spaces and $U_1 := L^2(\omega_1)$, $U_2 := L^2(\tilde{\omega})$ as control ones. Moreover, the appropriate control operator $B_2 : L^2(\tilde{\omega}) \to Y$ is given by

$$\tilde{B}_2 u_2 = (0, \chi_{\tilde{\omega}} u_2),$$

and the generators A_1 and A_2 are defined by (2.57) and

$$D(A_2) = \left\{(x_0, x_1) \in H_0^2(\Omega) \times L^2(\Omega) \, : \, x_1 \in H_0^2(\Omega), \, \Delta^2 x_0 \in L^2(\Omega)\right\},$$

$$A_2(x_0, x_1) = (x_1, -\Delta^2 x_0),$$

respectively. Then, considering as state $y(t) := (\theta(t), (\tilde{v}(t), \tilde{v}'(t)))$, we get from (2.35)-(2.37) above the following expressions for the feedback.

- If $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} + \|\chi_{\tilde{\omega}}\tilde{v}'(t)(t)\|_{L^2(\tilde{\omega})} > 0$ and $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) = F_2\left(\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right)$, then

$$(u_1(t), u_2(t))$$

$$\in -\mathrm{conv}\left\{\left(\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\theta(t), 0\right), \left(0, \frac{w_2\left(\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right)}{\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}}\chi_{\tilde{\omega}}\tilde{v}'(t)\right)\right\}. \tag{2.64}$$

- If $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) > F_2\left(\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right)$, then

$$(u_1(t), u_2(t)) = \left(\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\theta(t), 0\right). \tag{2.65}$$

- If $F_2\left(\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right) > F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)$, then

$$(u_1(t), u_2(t)) = \left(0, -\frac{w_2\left(\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right)}{\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}}\chi_{\tilde{\omega}}\tilde{v}'(t)\right). \tag{2.66}$$

- If $\chi_{\omega_1}\theta(t) = \chi_{\tilde{\omega}_2}\tilde{v}'(t) = 0$, then

$$(u_1(t), u_2(t)) \in w_1(0)\mathcal{B}_{\omega_1} \times w_2(0)\mathcal{B}_{\tilde{\omega}}, \tag{2.67}$$

  where $\mathcal{B}_{\omega_1}$ and $\mathcal{B}_{\tilde{\omega}}$ are the unit closed balls in $L^2(\omega_1)$ and $L^2(\tilde{\omega})$ respectively.

(iii) *Multi-valued feedback for a heat-wave system.* Beside the heat equation (2.55), we consider the wave equation

$$\left\{\begin{array}{c} v'' = \Delta v + u_2\chi_\omega \text{ on } (0,\infty) \times \tilde{\Omega}, \\ v = 0 \text{ on } (0,\infty) \times \partial\tilde{\Omega}, \\ v(0,x) = v_0(x), \, \frac{\partial v}{\partial t}(0,x) = v_1(x) \text{ on } \tilde{\Omega}, \end{array}\right. \tag{2.68}$$

where $\tilde{\Omega} \subset \mathbb{R}^{\tilde{d}}$ is an open bounded domain with sufficiently smooth boundary $\partial\tilde{\Omega}$, and $\omega \subset \tilde{\Omega}$ denotes a non-empty open subset. These systems have the forms (2.46) and (2.47) with $Y_1 := L^2(\mathcal{O})$, $Y_2 := H_0^1(\Omega) \times L^2(\Omega)$

as state spaces and $U_1 := L^2(\omega_1)$, $U_2 := L^2(\omega)$ as control ones. Moreover, the appropriate control operator $\tilde{B}_2 : L^2(\omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is given by

$$\tilde{B}_2 u_2 = (0, \chi_\omega u_2),$$

and the generators A_1 and A_2 are defined by (2.57) and

$$D(A_2) = \left\{ (x_0, x_1) \in H_0^1(\tilde{\Omega}) \times L^2(\tilde{\Omega}) \,:\, x_1 \in H_0^1(\tilde{\Omega}),\, \Delta x_0 \in L^2(\tilde{\Omega}) \right\},$$

$$A(x_0, x_1) = (x_1, \Delta x_0),$$

respectively. Then, considering as state $y(t) := (\theta(t), (v(t), v'(t)))$ we get from (2.35)–(2.37) above the following expressions for the feedback.

- If $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} + \|\chi_\omega v'(t)\|_{L^2(\omega)} > 0$ and $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) = F_2\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right)$, then

$$(u_1(t), u_2(t))$$

$$\in -\mathrm{conv}\left\{ \left(\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}} \chi_{\omega_1}\theta(t), 0 \right), \left(0, \frac{w_2\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right)}{\|\chi_\omega v'(t)\|_{L^2(\omega)}} \chi_\omega v'(t) \right) \right\}. \tag{2.69}$$

- If $F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right) > F_2\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right)$, then

$$(u_1(t), u_2(t)) = \left(-\frac{w_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}} \chi_{\omega_1}\theta(t), 0 \right). \tag{2.70}$$

- If $F_2\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right) > F_1\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)$, then

$$(u_1(t), u_2(t)) = \left(0, -\frac{w_2\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right)}{\|\chi_\omega v'(t)\|_{L^2(\omega)}} \chi_\omega v'(t) \right). \tag{2.71}$$

- If $\chi_\omega v'(t) = 0$ and $\chi_{\omega_1}\theta(t) = 0$, then

$$(u_1(t), u_2(t)) \in w_1(0)\mathcal{B}_{\omega_1} \times w_2(0)\mathcal{B}_\omega, \tag{2.72}$$

  where $\mathcal{B}_{\omega_1}$ and $\mathcal{B}_\omega$ are the unit closed balls in $L^2(\omega_1)$ and $L^2(\omega)$ respectively.

3. Stability in the multivalued feedback setting

3.1. General stability results

We recall some basic ideas on nonlinear semigroups and LaSalle invariance principle for evolution equations. Let $S(t)$ be a (generally nonlinear) semigroup on Y. For any $x \in Y$, the positive orbit through x is defined by $O^+(x) = \{S(t)x : \ t \geq 0\}$. The ω-limit set of x, denoted by $\omega(x)$, is the (possibly empty) set given by those $\varphi \in Y$ such that there exists a sequence $t_n \to \infty$ as $n \to \infty$ for which $S(t_n)x \to \varphi$ in Y as $n \to \infty$. The following lemmas will enable us to obtain preliminary properties of the ω-limit set $\omega(y_0)$ resulting from the semigroup e^{tA_Ψ} introduced in Theorem 2.4.

Lemma 3.1. *Suppose that the resolvent $(Id - A)^{-1}$ is compact and consider the operator $A_\Psi = A - B\partial(\Psi)B^*$ where Ψ is defined by (2.3) and (2.23). Then the single-valued resolvent operator $(Id - A_\Psi)^{-1}$ is also compact.*

Proof. Note that the compactness of the resolvent $(Id - A_\Psi)^{-1}$ amounts to the fact that any subset $K \subset D(A)$ is relatively compact whenever $(Id - A_\Psi)(K)$ is bounded. More precisely, if the set

$$(Id - A_\Psi)(K) = \bigcup_{y \in K} \{(y - \xi) : \xi \in (Id - A_\Psi)(y)\} \tag{3.1}$$

is bounded, then necessarily K is relatively compact. Under the assumption that the resolvent $(Id - A)^{-1}$ is compact, it is sufficient to establish

$$(Id - A_\Psi)(K) \text{ bounded } \ \Rightarrow (Id - A)(K) \text{ bounded }.$$

Without loss of generality, we assume that $0 \in K$ and, for some $R_K > 0$, we have for any $y \in K$

$$\|y - Ay + \beta\|_Y \leq R_K \text{ for all } \beta \in B(\partial(\Psi)B^* y). \tag{3.2}$$

Moreover, from $y \in (Id - A_\Psi)^{-1}(Id - A_\Psi)(y)$ it follows that $y = (Id - A_\Psi)^{-1}\xi$ for some $\xi \in y - Ay + B(\partial(\Psi)B^* y)$. In the same way, $0 = (Id - A_\Psi)^{-1}\xi_0$ for some $\xi_0 \in B\partial\Psi(0)$. Since $(Id - A_\Psi)^{-1}$ is a contraction, we get

$$\|y - 0\|_Y = \|y\|_Y \leq \|\xi - \xi_0\|_Y$$

$$\leq \|\xi\|_Y + \|\xi_0\|_Y \leq 2R_K.$$

Hence, the set K is necessarily bounded. On the other hand, the inequality (3.2) yields:

$$\|y - Ay\|_Y \leq \sup\{\|\beta\|_Y \ : \ \beta \in B(\partial(\Psi)B^* y)\} + R_K. \tag{3.3}$$

Since the linear operators B and B^* are bounded, the proof is reduced to the fact that the subgradient $\partial\Psi$, introduced in Proposition 2.2, is bounded. To this end, let $r > 0$ and consider the set

$$\Gamma_r := \left\{u = (u_1, ..., u_p) \in U \ : \ \|u_j\|_{U_j} \leq r, \, \forall j\right\}.$$

Consider $v \in \partial\Psi(u)$ with $u \neq 0$. Then, since w_j is nondecreasing we get from (2.24) and (2.25):

$$\|v\|_U \leq \max_{1 \leq j \leq p} w_j(r).$$

As for the case $v \in \partial\Psi(0)$, we obtain from (2.26):

$$\|v\|_U \leq \sqrt{p} \max_{1 \leq j \leq p} w_j(0) \text{ for all } v \in \partial\Psi(0).$$

Thus, the set $\partial\Psi(\Gamma_r)$ is bounded. This completes the proof of the lemma. $\square$

Lemma 3.2. *Suppose that the resolvent $(Id - A)^{-1}$ is compact and the evolution equation (1.10) holds under the assumptions given in Proposition 2.2 and Theorem 2.4. Then the ω-limit set $\omega(y_0)$, with respect to the system (1.10), is nonempty. Moreover, we have $B^*\varphi_0 = 0$ for any $\varphi_0 \in \omega(y_0)$.*

Proof. Following [26], the orbit $O^+(y_0)$ is relatively compact, and the set $\omega(y_0)$ is nonempty and compact provided that the continuous single-valued resolvent operator $(Id - A_\Psi)^{-1}$ is compact. From Lemma 3.1 it follows that the compactness property of the resolvent $(Id-A)^{-1}$ implies that $\omega(y_0)$ is nonempty. Let $\varphi_0 \in \omega(y_0)$ and let us introduce the function $\varphi(t) = \mathrm{e}^{tA_\Psi}\varphi_0$. Then we have

$$\|\varphi(t)\|_Y^2 - \|\varphi_0\|_Y^2 \leq -2\int_0^t \langle g(s), B^*\varphi(s)\rangle_U \,\mathrm{d}\tau, \tag{3.4}$$

where $g(s) \in \partial(\Psi)B^*\varphi(s)$. Indeed, suppose at the outset that $\varphi_0 \in D(A)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi(t)\|_Y^2\right) = 2\,\langle \varphi'(t), \varphi(t)\rangle_Y$$

$$= \langle A\varphi(t) - Bg(t), \varphi(t)\rangle_Y\,.$$

The contraction property of the semigroup generated by A yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi(t)\|_Y^2\right) \leq -\,\langle g(t), B^*\varphi(t)\rangle_U\,,$$

and (3.4) follows by integration. The case $\varphi_0 \in Y$ can be deduced from a density argument. Furthermore, because $0 \in \partial\Psi(0)$, $g(t) \in (\partial\Psi)B^*\varphi(t)$, and $\partial\Psi$ is monotone, we have

$$\langle g(t) - 0, B^*\varphi(t) - 0\rangle_U = \langle g(t), B^*\varphi(t)\rangle_U \geq 0.$$

Combining the fact that $\omega(y_0)$ is invariant and the property (i) in Section 2.3, we get for all $t > 0$

$$\|\varphi(t)\|_Y = \|\varphi_0\|_Y\,, \tag{3.5}$$

so that (3.4) gives

$$\int_0^t \langle g(s), B^*\varphi(s)\rangle_U \,\mathrm{d}s = 0,$$

and

$$\langle g(t), B^*\varphi(t)\rangle_U = 0 \text{ for all } t \geq 0. \tag{3.6}$$

Thus, taking into account Remark 2.3, we get

$$B^*\varphi(t) = 0 \text{ for all } t \geq 0. \tag{3.7}$$

This completes the proof of the lemma. ☐

Then we obtain the following asymptotic output stabilization result.

Theorem 3.3. *Suppose that the assumptions of Lemma 3.2 hold. Then for any initial state $y_0 \in Y$, the system (1.10) is asymptotically output stable.*

Proof. Let $L := \limsup_{t\to\infty} \|z(t)\|_U$, where $z(t)$ is defined by (1.11). We shall see that $L = 0$. By using the contraction property in (i) Section 2.3 we get for any $j = 1, ..., p$,

$$\left\|B_j^* y(t)\right\|_{U_j} \leq \left\|B_j^*\right\|_{\mathcal{L}(Y,U_j)} \|y_0\|_Y, \forall t \geq 0,$$

so that

$$\|z(t)\|_U \leq \max_j \left\|B_j^*\right\|_{\mathcal{L}(Y,U_j)} \|y_0\|_Y, \forall t \geq 0.$$

Hence the set $\{z(t)\}_{t\geq 0}$ is bounded in U and $0 \leq L < \infty$. Moreover, there exists a sequence $t_n \to \infty$ as $n \to \infty$ for which $\|z(t_n)\|_U \to L$. On the other hand, from Lemma 3.2 we deduce that the compactness property of the resolvent $(Id - A)^{-1}$ implies that the corresponding sequence of states $y(t_n)$ is contained in some compact subset in Y. Thus there exists a subsequence, still denoted by $y(t_n)$, such that $y(t_n) \to \varphi_0$ in Y as $n \to \infty$ for some $\varphi_0 \in \omega(y_0)$. Then $z(t_n) = B^* y(t_n) \to B^* \varphi_0$. From Lemma 3.2 we deduce that $z(t_n) \to 0$ so that $L = 0$. This completes the proof of the theorem. ☐

Under some particular types of constraints on the controls, we can obtain a more precise description of $\omega(y_0)$ leading to an asymptotic stability result . To this end, we introduce the following sets defined by

$$Y_u := \left\{x \in Y \,:\, \left\|e^{tA} x\right\|_Y = \|x\|_Y \text{ for all } t \geq 0\right\}, \tag{3.8}$$

$$M = \left\{\psi \in Y \,:\, B^* e^{tA} \psi = 0 \text{ for all } t \geq 0\right\}. \tag{3.9}$$

In (3.8), the subscript "u" refers to the unitary property with respect to the semigroup $(e^{tA})_{t\geq 0}$. Then we have:

Theorem 3.4. *Suppose that the assumptions of Lemma 3.2 hold. Assume also that*

$$w_j(0) = 0 \textit{ for all } j \in \{1, 2, ..., p\}. \tag{3.10}$$

Then for any initial state $y_0 \in Y$, we have $\omega(y_0) \subset M \cap Y_u$.

Proof. Let $\varphi_0 \in \omega(y_0)$ and consider again the function $\varphi(t) = e^{tA_\Psi} \varphi_0$. The property (3.7) established above combined with (3.10) implies by virtue of (2.37)

$$(\partial \Psi) B^* \varphi(t) = \{0\} \text{ for all } t > 0.$$

It follows that $e^{tA_\Psi} \varphi_0 = e^{tA} \varphi_0$. Thus the properties (3.7) and (3.5) yield $\omega(y_0) \subset M$ and $\omega(y_0) \subset Y_u$ respectively. This completes the proof of the theorem. ☐

Corollary 3.5. *Suppose that (3.10) and the assumptions of Lemma 3.2 hold. Then for any initial state $y_0 \in Y$, the solution of (1.10) satisfies $y(t) \to 0$ as $t \to \infty$ if and only if $M \cap Y_u = \{0\}$.*

Proof. Suppose that there exists a nonzero $y_0 \in M \cap Y_u$. Then the hypothesis (3.10) implies that $y(t) = e^{tA} y_0$ is a solution of (1.10) with

$$\|y(t)\|_Y^2 = \|y_0\|_Y^2 > 0 \text{ for all } t > 0. \tag{3.11}$$

The converse part is a ready consequence of Theorem 3.4. ☐

3.2. Application to the abstract motivating examples

(i) Unconstrained control
Considering the auxiliary functions in (2.7), the appropriate convex functional, denoted by Ψ_q, is given by

$$\Psi_q(u_1, u_2, ..., u_p) = \frac{1}{2}\max\left(\|u_1\|^2_{U_1}, \|u_2\|^2_{U_2}, ..., \|u_p\|^2_{U_p}\right). \tag{3.12}$$

The associated multivalued feedback is

$$\begin{cases} y'(t) - Ay(t) + B(\partial\Psi_q)B^* y(t) \ni 0, \\ y(0) = y_0, \end{cases} \tag{3.13}$$

Since $f_q(0) = 0$, it follows from Corollary 3.1 that the feedback system (3.13) is asymptotically stable if and only if $M \cap Y_u = \{0\}$.
(ii) Saturated control
Considering the auxiliary functional in (2.18), the appropriate functional, denoted by Ψ_ρ, is given by

$$\Psi_\rho(u_1, u_2, ..., u_p) = \max\left(F_\rho(u_1), F_\rho(u_2), ..., F_\rho(u_p)\right). \tag{3.14}$$

The associated multivalued feedback is

$$\begin{cases} y'(t) - Ay(t) + B(\partial\Psi_\rho)B^* y(t) \ni 0, \\ y(0) = y_0. \end{cases} \tag{3.15}$$

Since $f_\rho(0) = 0$, it follows from Corollary 3.1 that the feedback system (3.15) is asymptotically stable if and only if $M \cap Y_u = \{0\}$.
(iii) Bang-bang control
Considering the auxiliary functional in (2.20), the appropriate functional, denoted by Ψ_{bb}, is given by

$$\Psi_{bb}(u_1, ..., u_p) = \max\left(\|u_1\|_{U_1}, ..., \|u_p\|_{U_p}\right). \tag{3.16}$$

The associated multivalued feedback is

$$\begin{cases} y'(t) - Ay(t) + B(\partial\Psi_{bb})B^* y(t) \ni 0, \\ y(0) = y_0. \end{cases} \tag{3.17}$$

From (2.19) it follows that (3.10) is not verified. However, we can apply Theorem 3.3 to the solution of (3.17). Hence for bang-bang feedback control, we have asymptotic output stabilization.

3.3. Applications to partial differential equations (II)

Let us make more precise the notations in the context of the simultaneous stabilization of the systems (2.46) and (2.47). We are led to consider $Y_u := Y_1^u \times Y_2^u$, $M := M_1 \times M_2$ where

$$Y_j^u := \left\{x \in Y_j \,:\, \left\|e^{tA_j}x\right\|_{Y_j} = \|x\|_{Y_j} \text{ for all } t \geq 0\right\}, \tag{3.18}$$

$$M_j = \left\{\psi \in Y_j \,:\, \tilde{B}_j^* e^{tA_j}\psi = 0 \text{ for all } t \geq 0\right\}. \tag{3.19}$$

(i) *Switching stabilization of the heat-heat system.* Here we consider the feedback system specified by the differential inclusion resulting from (2.55), (2.56), and (2.59)-(2.62). By virtue of the properties of the Laplacian operator established in ([27], Chapter VII, p. 96), the resolvent of the operators specified by (2.57) and (2.58) is compact and we have

$$Y_1^u = \{c\chi_{\mathcal{O}} \,:\, c \in \mathbb{R}\} = \left\{\psi \in L^2(\mathcal{O}) \,:\, \mathrm{e}^{tA_1}\psi = \psi \text{ for all } t \geq 0\right\}, \tag{3.20}$$

$$Y_2^u = \{c\chi_{\tilde{\mathcal{O}}} \,:\, c \in \mathbb{R}\} = \left\{\psi \in L^2(\tilde{\mathcal{O}}) \,:\, \mathrm{e}^{tA_2}\psi = \psi \text{ for all } t \geq 0\right\}. \tag{3.21}$$

Moreover, from the facts that

$$M_1 = \left\{\psi \in L^2(\mathcal{O}) \,:\, \chi_{\omega_1}\mathrm{e}^{tA_1}\psi = 0 \text{ for all } t \geq 0\right\}, \tag{3.22}$$

$$M_2 = \left\{\psi \in L^2(\tilde{\mathcal{O}}) \,:\, \chi_{\omega_2}\mathrm{e}^{tA_2}\psi = 0 \text{ for all } t \geq 0\right\}, \tag{3.23}$$

it follows clearly that $Y_u \cap M = \{(0,0)\}$. Hence, whenever $w_1(0) = w_2(0) = 0$, we deduce from Corollary 3.5 that the feedback system is asymptotically stable. Otherwise, as in the bang-bang control case where $w_1 = w_2 \equiv 1$, this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\left\|\tilde{\theta}(t)\right\|_{L^2(\omega_2)} \to 0$ as $t \to \infty$. Note also that, in this case, the system is obviously asymptotically stable provided that $\omega_1 = \mathcal{O}$ and $\omega_2 = \tilde{\mathcal{O}}$.

(ii) *Switching stabilization of the heat-plate system.* Let us consider the feedback system defined by the differential inclusion given by (2.55), (2.63), and (2.64)–(2.67). We refer to ([28], Chapter 2) for the main results concerned with Petrowsky systems. Beside the set Y_1^u already defined in (3.20), the standard energy conservation property of the plate equation implies that $Y_2^u = H_0^2(\Omega) \times L^2(\Omega)$. Moreover, Rellich theorem implies that the corresponding operator A_2 has compact resolvent. While the set M_1 is given by (3.22), the set M_2 is given by the initial conditions $(\varphi_0, \varphi_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfying

$$\varphi'' + \Delta^2\varphi = 0 \text{ on } (0,\infty) \times \Omega,\; \varphi(0) = \varphi_0,\; \varphi'(0) = \varphi_1, \tag{3.24}$$

and $\varphi'(t) = 0$ on $(0,\infty) \times \tilde{\omega}$. Hence the fact that $M_2 \cap Y_2^u = \{(0,0)\}$ is reduced to a unique continuation property which is a ready consequence of the Holmgren's theorem under the sole condition that $\tilde{\omega}$ is a nonempty open subset of Ω. We refer to ([29], Chapter I), for a discussion of this problem. Hence, whenever $w_1(0) = w_2(0) = 0$, we deduce from Corollary 3.5 that the feedback system is asymptotically stable. Otherwise, as in the bang-bang control case where $w_1 = w_2 \equiv 1$, this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|\tilde{v}'(t)\|_{L^2(\tilde{\omega})} \to 0$ as $t \to \infty$.

(iii) *Switching stabilization of the heat-wave system.* Let us consider the feedback system specified by the differential inclusion resulting from (2.55), (2.68), and (2.69)–(2.72). The set Y_1^u has been defined in (3.20) and, from standard energy conservation property of the wave equation, we get $Y_u^2 = H_0^1(\tilde{\Omega}) \times L^2(\tilde{\Omega})$. While the set M_1 is given by (3.22), the set M_2 can be viewed as the initial conditions $(\varphi_0, \varphi_1) \in H_0^1(\tilde{\Omega}) \times L^2(\tilde{\Omega})$ satisfying

$$\varphi'' - \Delta\varphi = 0 \text{ on } (0,\infty) \times \Omega,\; \varphi(0) = \varphi_0,\; \varphi'(0) = \varphi_1, \tag{3.25}$$

and $\varphi'(t) = 0$ on $(0,\infty) \times \omega$. Hence $M_2 = \{(0,0)\}$ amounts to a unique continuation property. By using again Holmgren's uniqueness theorem it can be easily seen that this property holds. It follows from Corollary 3.5 that whenever $w_1(0) = w_2(0) = 0$, the feedback system is asymptotically stable. Otherwise, as in the bang-bang control case where $w_1 = w_2 \equiv 1$, this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|v'(t)\|_{L^2(\omega)} \to 0$ as $t \to \infty$.

4. Stabilization by switched systems feedback

4.1. Construction of switched multivalued feedback systems

In this subsection we introduce a family of multi-valued feedback systems defined by the index set

$$\mathcal{P} := \{1, 2, ..., p\}, \tag{4.1}$$

and a fixed parameter $0 < \alpha < 1$ as follows. Let $w : \mathbb{R}^+ \to \mathbb{R}$ be a continuous nondecreasing function satisfying $w(s) > 0$ for all $s > 0$. Let $j \in \mathcal{P}$ be fixed, we consider for $k \in \mathcal{P}$ the auxiliary constraint functionals $G_k^j : U_k \to \mathbb{R}$ defined by

$$G_k^j(u_k) = \begin{cases} \int_0^{\|u_j\|_{U_j}} w(\tau)\mathrm{d}\tau \text{ if } k = j, \\ \int_0^{\alpha\|u_k\|_{U_k}} w(\tau)\mathrm{d}\tau \text{ if } k \neq j. \end{cases} \tag{4.2}$$

We note that for $k \neq j$ and $u_j \neq 0$, we have

$$\|u_j\|_{U_j} = \|u_k\|_{U_k} \;\Rightarrow\; G_j^j(u_j) > G_k^j(u_k). \tag{4.3}$$

Then the appropriate functional, analogous to Ψ in (2.23), and denoted by $\mathcal{F}_j$, is such that

$$\mathcal{F}_j(u_1, ..., u_p) = \max\left(G_1^j(u_1), ..., G_p^j(u_p)\right). \tag{4.4}$$

For $1 \leq j \leq p$, we shall consider the switched system whose j-th constituent subsystem is the multi-valued feedback system given by

$$y'(t) - Ay(t) + B\partial(\mathcal{F}_j)B^*y(t) \ni 0. \tag{4.5}$$

We shall exploit Proposition 2.7 in order to introduce successive time intervals (t_n, t_{n+1}) such that only one control is activated at each instant $t \in (t_n, t_{n+1})$ and the corresponding state equation has the form in (2.44) for some $j \in \{1, ..., p\}$. This issue will be detailed in the next subsection. By this alternative method, we shall see that it is possible to carry out a controller selection procedure with the help of logic-based switching among a family of control laws. In other words, a switched systems strategy will be used as an additional feedback control and it will enable us to obtain switching controls.

4.2. Construction of switched single valued feedback systems

Let us recall some familiar terminology related to switching systems as presented, for instance, in [1]. The logical rule that orchestrates switching between the resulting single-valued feedback subsystems generates switching signals, which will be described as classes of piecewise constant maps $\sigma : \mathbb{R}^+ \to \mathcal{P}$. The index $j = \sigma(t)$ is called the active mode at the time instant t. Here, the active mode at t will depend not only on the time instant t, but also on the current state $y(t)$. Accordingly, our switching logic can be classified as state-dependent. The following theorem gives an explicit construction of our switched single-valued feedback subsystems.

Theorem 4.1. *Consider the index set defined by (4.1) and the family of multi-valued feedback systems (4.5) where the functionals $\mathcal{F}_j$ are given by (4.2) and (4.4). Suppose that, for some $T_0 > 0$, the observability condition*

$$(B^*\mathrm{e}^{tA}y = 0,\ 0 < t < T_0) \;\Rightarrow\; y = 0 \tag{4.6}$$

holds. Then there exists a switched system

$$\Sigma_{\mathcal{S}} \,:\, y'(t) = Ay(t) - \frac{w\left(\left\|B^*_{\sigma(t)}y(t)\right\|_{U_{\sigma(t)}}\right)}{\left\|B^*_{\sigma(t)}y(t)\right\|_{U_{\sigma(t)}}} B_{\sigma(t)}B^*_{\sigma(t)}y(t), \tag{4.7}$$

whose subsystems are single-valued feedback systems where only one control is activated in feedback form. These subsystems are given by

$$\Sigma_{\mathcal{S}_j} \,:\, y'(t) = Ay(t) - \frac{w\left(\left\|B^*_j y(t)\right\|_{U_j}\right)}{\left\|B^*_j y(t)\right\|_{U_j}} B_j B^*_j y(t) \tag{4.8}$$

for $j \in \mathcal{P}$ and they are defined by (possibly finite) sequence of transition times $\{t_n\}_n$ with $t_0 \geq 0$. The switching signal $\sigma : \mathbb{R}^+ \to \mathcal{P}$ is constant between any successive switching times. The resulting modes given by

$$\sigma(t) = j_m \text{ for all } t_m \leq t < t_{m+1},$$

obey the following logic defined for $t_m \leq t < t_{m+1}$ by

$$\sigma(t) = j_m \;\Leftrightarrow\; \left\|B^*_{j_m}y(t)\right\|_{U_{j_m}} > \alpha \left\|B^*_k y(t)\right\|_{U_k} \text{ for all } k \neq j_m, \tag{4.9}$$

and

$$j_{m+1} := \min\left\{j \in \mathcal{P} \,:\, \left\|B^*_{j_m}y(t_{m+1})\right\|_{U_{j_m}} = \alpha \left\|B^*_j y(t_{m+1})\right\|_{U_j}\right\}. \tag{4.10}$$

Proof. We note at the outset that the observability property implies that for each non null initial state y_0, we have necessarily $B^*\mathrm{e}^{t_0 A}y_0 \neq 0$ for some $0 < t_0 < T_0$. In order to include the possibility $B^*y_0 \neq 0$, we suppose that the first transition time t_0 satisfies $t_0 \geq 0$ and we proceed in two steps:
Step 1. Construction of the transition times $0 \leq t_0 < t_1 < t_2$. Since $B^*\mathrm{e}^{t_0 A}y_0 \neq 0$, it follows that

$$\max_k \left\|B^*_k\mathrm{e}^{t_0 A}y_0\right\|_{U_k} > 0.$$

Letting

$$j_0 := \min\left\{j \in \mathcal{P} \,:\, \left\|B^*_j\mathrm{e}^{t_0 A}y_0\right\|_{U_j} = \max_k \left\|B^*_k\mathrm{e}^{t_0 A}y_0\right\|_{U_k}\right\}, \tag{4.11}$$

we have

$$\left\|B^*_{j_0}\mathrm{e}^{t_0 A}y_0\right\|_{U_{j_0}} > \alpha \left\|B^*_k\mathrm{e}^{t_0 A}y_0\right\|_{U_k} \text{ for all } k \neq j_0. \tag{4.12}$$

Then by exploiting (4.3) we get

$$G^{j_0}_{j_0}\left(B^*_{j_0}\mathrm{e}^{t_0 A}y_0\right) > G^{j_0}_k\left(B^*_k\mathrm{e}^{t_0 A}y_0\right) \text{ for all } k \neq j_0. \tag{4.13}$$

Let us consider, for $t \geq t_0$, the solution of the multi-valued feedback system

$$y'(t) - Ay(t) + B(\partial \mathcal{F}_{j_0})B^* y(t) \ni 0, \tag{4.14}$$

such that $y(t_0) = \mathrm{e}^{t_0 A} y_0$. We note that, for any $k \neq j_0$, the mapping

$$\gamma_k : t \mapsto G_{j_0}^{j_0}\left(B_{j_0}^* y(t)\right) - G_k^{j_0}\left(B_k^* y(t)\right)$$

satisfies $\gamma_k(t_0) > 0$. Moreover, it is right continuous at t_0 so that there exists $\delta_k > 0$ such that

$$t_0 \leq t < \delta_k \ \Rightarrow\ \gamma_k(t) > 0. \tag{4.15}$$

By proceeding as in the proof of Proposition 2.7, we set $\tau_0 := \min\limits_{k \neq j_0} \delta_k$. Then we have

$$t_0 \leq t < \tau_0 \ \Rightarrow\ \gamma_k(t) > 0 \text{ for all } k \neq j_0. \tag{4.16}$$

From the analogous of the expressions established in (2.41) we get

$$u_{j_0}(y(t)) = -\frac{w\left(\left\|B_{j_0}^* y(t)\right\|_{U_{j_0}}\right)}{\left\|B_{j_0}^* y(t)\right\|_{U_{j_0}}} B_{j_0}^* y(t), \tag{4.17}$$

so that u_{j_0} is the sole activated feedback control on the time interval $[t_0, \tau_0)$ in which the state equation is given by

$$y'(t) = Ay(t) - \frac{w\left(\left\|B_{j_0}^* y(t)\right\|_{U_{j_0}}\right)}{\left\|B_{j_0}^* y(t)\right\|_{U_{j_0}}} B_{j_0} B_{j_0}^* y(t),\ t_0 < t < \tau_0. \tag{4.18}$$

Let $\mathcal{T}_0$ denote the set of instants $\tau_0 > t_0$ satisfying (4.16). Note that $\sup \mathcal{T}_0 = \infty$ means that we can activate only the control u_{j_0}. Otherwise, we suppose that $\sup \mathcal{T}_0 < \infty$ and we set $t_1 = \sup \mathcal{T}_0$. It follows that for some $j \in \mathcal{P}$

$$\left\|B_{j_0}^* y(t_1)\right\|_{U_{j_0}} = \alpha \left\|B_j^* y(t_1)\right\|_{U_j} \geq \alpha \left\|B_k^* y(t_1)\right\|_{U_k} \text{ for all } k \neq j_0.$$

Let

$$j_1 := \min\left\{ j \in \mathcal{P} \,:\, \left\|B_{j_0}^* y(t_1)\right\|_{U_{j_0}} = \alpha \left\|B_j^* y(t_1)\right\|_{U_j} \right\}. \tag{4.19}$$

Then

$$\left\|B_{j_1}^* y(t_1)\right\|_{U_{j_1}} = \frac{1}{\alpha} \left\|B_{j_0}^* y(t_1)\right\|_{U_{j_0}} > \left\|B_{j_0}^* y(t_1)\right\|_{U_{j_0}},$$

so that

$$\left\|B_{j_1}^* y(t_1)\right\|_{U_{j_1}} > \alpha \left\|B_k^* y(t_1)\right\|_{U_k} \text{ for all } k \neq j_1,$$

and consequently we obtain as above

$$G_{j_1}^{j_1}\left(B_{j_1}^* y(t_1)\right) > G_k^{j_1}\left(B_k^* y(t_1)\right) \text{ for all } k \neq j_1. \tag{4.20}$$

Then we consider for $t \geq t_1$ the multi-valued feedback system

$$y'(t) - Ay(t) + B\partial(\mathcal{F}_{j_1})B^* y(t) \ni 0,\, t \geq t_1. \tag{4.21}$$

By proceeding as above, there exists $\tau_1 > 0$ such that the solution of (4.21) satisfies

$$t_1 \leq t < t_1 + \tau_1 \;\Rightarrow\; \left\|B_{j_1}^* y(t)\right\|_{U_{j_1}} > \alpha \left\|B_k^* y(t)\right\|_{U_k} \text{ for all } k \neq j_1, \tag{4.22}$$

so that u_{j_1} is the sole activated feedback control on the time interval $[t_1, t_1 + \tau_1)$ in which the state equation is given by

$$y'(t) = Ay(t) - \frac{w\left(\left\|B_{j_1}^* y(t)\right\|_{U_{j_1}}\right)}{\left\|B_{j_1}^* y(t)\right\|_{U_{j_1}}} B_{j_1} B_{j_1}^* y(t),\, t_1 \leq t < t_1 + \tau_1. \tag{4.23}$$

Hence, we have obtained in feedback form the switching controls u_{j_0} and u_{j_1}. Moreover, the switching instant t_1 is defined by the rule $t_1 = \sup \mathcal{T}_0$. In the same way, we can define the next possible switching instant t_2. Let $\mathcal{T}_1$ denote the set of instants $\tau_1 > 0$ satisfying (4.22). Note that $\sup \mathcal{T}_1 = \infty$ means that we can activate only the switching controls u_{j_0} (given by (4.17)) and u_{j_1} defined by

$$u_{j_1}(y(t)) = -\frac{w\left(\left\|B_{j_1}^* y(t)\right\|_{U_{j_1}}\right)}{\left\|B_{j_1}^* y(t)\right\|_{U_{j_1}}} B_{j_1}^* y(t), \tag{4.24}$$

with t_1 as unique switching instant between the systems (4.18) and (4.23). Otherwise, we suppose that $\sup \mathcal{T}_1 < \infty$ and we set $t_2 = t_1 + \sup \mathcal{T}_1$.

Step 2. Construction of the transition times $\{t_n\}_{n \geq 2}$. More generally, suppose that we have at hand switching instants $t_0 < t_1 < t_2 < ... < t_n$ corresponding to respective modes $j_0, j_1, j_2, ..., j_{n-1}$ in $\mathcal{P}$. In accordance with the system (4.5) and the switched systems (4.7), for $m = 0, ..., n-1$, we are led to consider on each interval (t_m, t_{m+1}) a feedback system defined by

$$y'(t) - Ay(t) + B(\partial\mathcal{F}_{j_m})B^* y(t) \ni 0,\, t_m \leq t < t_{m+1}. \tag{4.25}$$

The switching times and the corresponding modes are related by the following rules

$$t_m \leq t < t_{m+1} \;\Rightarrow\; \left\|B_{j_m}^* y(t)\right\|_{U_{j_m}} > \alpha \left\|B_k^* y(t)\right\|_{U_k} \text{ for all } k \neq j_m, \tag{4.26}$$

$$j_{m+1} := \min\left\{ j \in \mathcal{P} \,:\, \left\|B_{j_m}^* y(t_{m+1})\right\|_{U_{j_m}} = \alpha \left\|B_j^* y(t_{m+1})\right\|_{U_j} \right\}. \tag{4.27}$$

Moreover, for $m = 0, ..., n-1$, the multivalued feedback subsystem (4.25) admits the following single-valued feedback version

$$y'(t) = Ay(t) - \frac{w\left(\left\|B_{j_m}^* y(t)\right\|_{U_{j_m}}\right)}{\left\|B_{j_m}^* y(t)\right\|_{U_{j_m}}} B_{j_m} B_{j_m}^* y(t),\ t_m \leq t < t_{m+1}. \tag{4.28}$$

In particular, the last switching instant t_n and the corresponding feedback control u_{n-1} defined on $t_{n-1} < t < t_n$ are characterized by

$$t_{n-1} \leq t < t_n \ \Rightarrow\ \left\|B_{j_{n-1}}^* y(t)\right\|_{U_{j_{n-1}}} > \alpha \left\|B_k^* y(t)\right\|_{U_k} \text{ for all } k \neq j_{n-1}, \tag{4.29}$$

$$u_{j_{n-1}}(y(t)) = -\frac{w\left(\left\|B_{j_{n-1}}^* y(t)\right\|_{U_{j_{n-1}}}\right)}{\left\|B_{j_{n-1}}^* y(t)\right\|_{U_{j_{n-1}}}} B_{j_{n-1}}^* y(t),\ t_{n-1} \leq t < t_n, \tag{4.30}$$

and

$$\left\|B_{j_{n-1}}^* y(t_n)\right\|_{U_{j_{n-1}}} = \alpha \left\|B_j^* y(t_n)\right\|_{U_j}, \tag{4.31}$$

for some $j \in \mathcal{P}$. This enables us to introduce the next mode j_n by

$$j_n := \min\left\{ j \in \mathcal{P} \,:\, \left\|B_{j_{n-1}}^* y(t_n)\right\|_{U_{j_{n-1}}} = \alpha \left\|B_j^* y(t_n)\right\|_{U_j} \right\}, \tag{4.32}$$

with

$$\left\|B_{j_n}^* y(t_n)\right\|_{U_{j_n}} > \alpha \left\|B_k^* y(t_n)\right\|_{U_k} \text{ for all } k \neq j_n.$$

Thus we are led to consider the appropriate functionals $G_k^{j_n}$ which satisfies

$$G_{j_n}^{j_n}\left(B_{j_n}^* y(t_n)\right) > G_k^{j_n}\left(B_k^* y(t_n)\right) \text{ for all } k \neq j_n. \tag{4.33}$$

In order to define the (possible) next transition time, we introduce for $t \geq t_n$ the multi-valued feedback system

$$y'(t) - Ay(t) + B\partial(\mathcal{F}_{j_n})B^* y(t) \ni 0,\ t \geq t_n. \tag{4.34}$$

By using again a simple right continuity argument at t_n, there exists $\tau_n > 0$ such that

$$t_n \leq t < t_n + \tau_n \ \Rightarrow\ \left\|B_{j_n}^* y(t)\right\|_{U_{j_n}} > \alpha \left\|B_k^* y(t)\right\|_{U_k} \text{ for all } k \neq j_n, \tag{4.35}$$

and u_{j_n} is the sole activated feedback control on the time interval $[t_n, t_n + \tau_n)$ in which the state equation (4.34) becomes

$$y'(t) = Ay(t) - \frac{w\left(\left\|B_{j_n}^* y(t)\right\|_{U_{j_n}}\right)}{\left\|B_{j_n}^* y(t)\right\|_{U_{j_n}}} B_{j_n} B_{j_n}^* y(t),\ t_n \leq t < t_n + \tau_n. \tag{4.36}$$

In the same way, we can define the next possible switching instant t_{n+1}. Let $\mathcal{T}_n$ denote the set of instants $\tau_n > 0$ satisfying (4.35). Note that $\sup \mathcal{T}_n = \infty$ means that we can activate only the switching control u_{j_n} for $t > t_n$ with t_n as the last switching instant. Otherwise, we get $\sup \mathcal{T}_n < \infty$ and we set $t_{n+1} = t_n + \sup \mathcal{T}_n$. The resulting switching controls u_{j_0}, u_{j_1},..., u_{j_n} given in state feedback form are such that

$$u_{j_i}(y(t)) = -\frac{w_{j_i}\left(\left\|B_{j_i}^* y(t)\right\|_{U_{j_i}}\right)}{\left\|B_{j_i}^* y(t)\right\|_{U_{j_i}}} B_{j_i}^* y(t),\ t_i \leq t < t_{i+1} \quad \text{for } i = 0, 1, ..., n. \tag{4.37}$$

Moreover, the corresponding sequence of the transition times $(t_n)_n$ satisfies one of the following possibilities:

- $t_{\overline{n}} = \infty$ for some integer $\overline{n}$ so that we can activate only the switching control $u_{j_{\overline{n}}}$ for $t > t_{\overline{n}}$ with $t_{\overline{n}}$ as the last switching instant.
- $(t_n)_n$ is an increasing sequence of positive numbers.

This completes the proof of the theorem. □

Remark 4.2. **(i)** It should be emphasized that the sequence of the switching times $\{t_n\}_n$ depends implicitly on the initial state y_0. Let $t_\infty := \sup_n \{t_n\}$. We note that the switched systems make sense only on the time interval (t_0, t_∞). In the sequel, subject to the logic rules (4.9) and (4.10), these subsystems will be referred to as the overall switched system (4.7).
(ii) In the light of the notations used in Theorem 2.4, the solution of the overall switched system (4.7) can be expressed by using the solutions of the sub-systems in (4.5). Indeed, we have readily

$$y(t) = \mathrm{e}^{(t-t_n)A_{\mathcal{F}_{j_n}}} y(t_n) \text{ for all } t_n \leq t \leq t_{n+1}. \tag{4.38}$$

(iii) According to the first step in the construction of the sequence $\{t_n\}_n$, some attention has to be addressed to the first instant of activation t_0. The latter would satisfy $0 \leq t_0 < T_0$ so that difficulties may arise whenever T_0 is not identified precisely. Note that this problem can be avoided when the observability condition (4.6) is verified for arbitrarily short time T_0.

The following lemma gives some collected technical ingredients which will be useful for the stability issue treated in the next subsection.

Lemma 4.3. *Let $y_0 \in D(A)$, suppose that the observability condition (4.6) holds and consider the associated sequence $\{t_n\}_n$ constructed in Theorem 4.1. Then the solution of the overall switched system (4.7) satisfies the following properties.*
(i) For some positive constant C_0 depending on the initial state y_0, we have for any $t_0 \leq t < t_\infty$

$$\left\|\frac{\mathrm{d}^+ y}{\mathrm{d}t}(t)\right\|_Y \leq C_0. \tag{4.39}$$

(ii) The limit of $\|y(t)\|_Y$, as $t \to t_\infty$, is a well defined non negative number.

Proof. (i) Letting

$$A_{\mathcal{F}_{j_n}} := A - B(\partial \mathcal{F}_{j_n})B^*,$$

we have Clearly $D(A_{\mathcal{F}_{j_n}}) = D(A)$ and $y_0 \in D(A_{\mathcal{F}_{j_n}})$ for all n. Exploiting the properties stated in Section 2.3, it follows that the solution of the switched system (4.7) satisfies $y(t) \in D(A)$ for $0 < t < \sup_n \{t_n\}$. On the other hand, the fact that the function $t \mapsto \left\| A^0_{\mathcal{F}_{j_n}} y(t) \right\|_Y$ is non-increasing on (t_n, t_{n+1}), combined with (2.34), implies

$$\left\| \frac{\mathrm{d}^+ y}{\mathrm{d}t}(t) \right\|_Y = \left\| A^0_{\mathcal{F}_{j_n}} y(t) \right\|_Y \leq \left\| A^0_{\mathcal{F}_{j_n}} y(t_n) \right\|_Y \text{ for all } t_n \leq t < t_{n+1}.$$

Thus

$$\left\| \frac{\mathrm{d}^+ y}{\mathrm{d}t}(t) \right\|_Y \leq \left\| A^0_{\mathcal{F}_{j_0}} y_0 \right\|_Y \text{ for all } t \geq 0.$$

From Theorem 4.1 we deduce

$$A^0_{\mathcal{F}_{j_0}} \mathrm{e}^{t_0 A} y_0 = A\mathrm{e}^{t_0 A} y_0 - \frac{w\left(\left\| B^*_{j_0} \mathrm{e}^{t_0 A} y_0 \right\|_{U_{j_0}} \right)}{\left\| B^*_{j_0} \mathrm{e}^{t_0 A} y_0 \right\|_{U_{j_0}}} B_{j_0} B^*_{j_0} \mathrm{e}^{t_0 A} y_0.$$

Hence we get readily (4.39) by setting

$$C_0 = \left\| A\mathrm{e}^{t_0 A} y_0 \right\|_Y + \sup_{j \in \mathcal{P}} \left\{ w\left(\left\| B^*_j \mathrm{e}^{t_0 A} y_0 \right\|_{U_j} \right) \left\| B_j \right\|_{\mathcal{L}(U_j, Y)} \right\}.$$

(ii) By virtue of the property (i) in Section 2.3, it follows that the function $t \to \|y(t)\|_Y$ is non-increasing on each interval (t_n, t_{n+1}) so that

$$\|y(t_{n+1})\|_Y \leq \|y(t_n)\|_Y \text{ for all } n \in \mathbb{N}. \tag{4.40}$$

Hence, for the solution of the overall system, the function $t \to \|y(t)\|_Y$ is non-increasing on $(0, t_\infty)$ and, consequently, the limit of $\|y(t)\|$, as $t \to t_\infty$ is a well defined non negative number. □

4.3. Stability of the overall switched single valued feedback system

4.3.1. Preliminaries

In the sequel, it is assumed that the initial state satisfies $y_0 \in D(A)$ and the observabilty condition (4.6) is verified. We suppose also that the corresponding sequence of the switching times $\{t_n\}_n$ defined in Theorem 4.1 is infinite and increasing. Then we relate the stabilization of the system (1.3) under switching control constraint (1.4) to the stability of the overall switched system (4.7). In principle, we would be in presence of one of the two following occurrences.

- The sequence $\{t_n\}_n$ is bounded so that the limit t_∞ is an accumulation point. Such a possibility turns out to be an example of the well-known Zeno behavior. In [1], the bouncing ball is introduced as prototypical model. Whenever such a circumstance occurs, the limit t_∞ will be denoted by t_ζ.
- $t_n \to \infty$ as $n \to \infty$ so that we will be in presence of a typical asymptotic behavior.

We are led to specify the notion of stability for the overall system in the light of the behavior of the sequence of transition times. More precisely, we shall be concerned with the following types of stability.

Definition 4.4. Let $y_0 \in D(A)$, suppose that the observability condition (4.6) is verified and consider the associated sequence $\{t_n\}_n$ constructed in Theorem 4.1. Then:

- If the Zeno behavior holds, the overall switched system (4.7) is said to be output stable in finite time if the output defined by (1.11) satisfies $z(t) \to 0$ as $t \to t_\zeta$.

- If the asymptotic behavior holds, the overall switched system (4.7) is said to be asymptotically output stable if the output defined by (1.11) satisfies $z(t) \to 0$ as $t \to \infty$. Moreover, it is said to be asymptotically stable if the solution satisfies $y(t) \to 0$ as $t \to \infty$.

Following the switched systems literature and proceeding by analogy, we are led to distinguish between drastically different possibilities. The first one concerns the so-called fast switching and it can be expressed by

$$\lim_{n\to\infty}(t_{n+1} - t_n) = 0. \tag{4.41}$$

We note that this condition is naturally verified in the Zeno behavior case. The second one is more familiar and it is more convenient in the case where each subsystem is asymptotically stable and the switching is sufficiently slow so that $t_{n+1} - t_n > \tau$ for some constant $\tau > 0$. Hence one can maintain stability under the possibility of staying long enough and switching less frequently. This idea is proved to be reasonable and is captured by the concept of dwell time introduced in the switching systems literature. See, for instance, [4]. In our context, the dwell time is defined by analogy as the positive constant

$$\tau_d := \inf_{n\geq 0}(t_{n+1} - t_n). \tag{4.42}$$

Moreover, considering carefully the underlying setting developed in Theorem 4.1, the following facts would be taken into account. On the one hand, considering the switched subsystems in the form of single-valued feedback systems as in (4.8), we will be in principle in presence of unstable subsystems. On the other hand, in the case where these subsystems are viewed as multi-valued feedback ones as in (4.5), we will be in presence of possibly asymptotically stable subsystems by virtue of the results established in Section 3. Later, the dwell time notion will relate fruitfully the two points of view in order to obtain asymptotic stability.

In order to treat the output stability issue, it would be natural to suppose that each controller is activated intermittently as the time variable t increases. More precisely, if we consider for each mode $j \in \mathcal{P}$ the sequence of intermittent activation instants of the controller B_j given by

$$\mathcal{I}_j := \{t_n \,:\, \sigma(t) = j,\, t_n \leq t < t_{n+1}\}, \tag{4.43}$$

then this fact can be characterized by the property that this set is infinite. Consequently, for each mode $j \in \mathcal{P}$, we shall present naturally $\mathcal{I}_j$ under the form of the subsequence of activation instants $\left\{t_{n(k)}\right\}_k$ such that

$$n(k+1) - n(k) = q_k. \tag{4.44}$$

Then we introduce the activated modes between the successive instants $t_{n(k)}$ and $t_{n(k+1)}$ as follows:

$$\sigma((t_{n(k)}, t_{n(k)+1})) = \{j\} =: \left\{j_k^{(0)}\right\}, \tag{4.45}$$

$$\left\{j_k^{(i)}\right\} := \sigma\big((t_{n(k)+i}, t_{n(k)+i+1})\big),\ 1 \leq i \leq q_k - 1, \tag{4.46}$$

$$\sigma\big((t_{n(k)+q_k}, t_{n(k+1)+1})\big) = \sigma\big((t_{n(k+1)}, t_{n(k+1)+1})\big) = \{j\} =: j_k^{(q_k)}. \tag{4.47}$$

Note that the integers $\{q_k\}_k$ depend on j and satisfy $q_k \geq 2$ for all k. Furthermore, the activated modes $\left\{j_k^{(i)}\right\}_{0\leq i\leq q_k}$ verify implicitly the compatibility conditions:

$$j \neq j_k^{(i)} \text{ for all } 1 \leq i \leq q_k - 1, \tag{4.48}$$

$$j_k^{(i)} \neq j_k^{(i+1)} \text{ for all } 0 \leq i \leq q_k - 1. \tag{4.49}$$

4.3.2. Zeno behavior case

In the case where the Zeno behavior holds, we have:

Theorem 4.5. *Let $y_0 \in D(A)$, suppose that the observability condition (4.6) is verified and consider the associated sequence $\{t_n\}_n$ constructed in Theorem 4.1. Suppose that $t_n \to t_\zeta$ as $n \to \infty$. Then the overall switched system (4.7) is output stable in finite time.*

Proof. The output stability amounts to the fact that $\|B^* y(t)\|_U \to 0$ as $t \to t_\zeta$. Considering the components of $B^* y(t)$ deduced from (1.7), we are led to establish that, for all $j \in \mathcal{P}$, we have $\left\|B_j^* y(t)\right\|_{U_j} \to 0$ as $t \to t_\zeta$. At the outset we shall establish that $y(t) \to y_\zeta$ as $t \to t_\zeta$ for some state $y_\zeta \in Y$. Taking into account (4.39) in Lemma 4.3, we get

$$\left\|\frac{\mathrm{d}^+ y}{\mathrm{d}t}(t)\right\|_Y \leq C_0 \text{ for all } t_0 < t < t_\zeta,$$

where C_0 is a positive constant depending on y_0. The mean value theorem implies that for all $t, s \in (0, t_\zeta)$

$$\|y(t) - y(s)\|_Y \leq C_0 \, |t - s| \, .$$

Hence the limit $y_\zeta \in Y$ of $y(t)$ as $t \to t_\zeta$ is well-defined. Furthermore, we have for all $j \in \mathcal{P}$

$$B_j^* y(t) \to B_j^* y_\zeta \text{ as } t \to t_\zeta, \tag{4.50}$$

and consequently

$$B_j^* y(t_n) \to B_j^* y_\zeta \text{ as } n \to \infty.$$

By virtue of the transition rules specified in (4.10) we get from (4.45)–(4.47)

$$\left\|B_j^* y(t_{n(k)+1})\right\|_{U_j} = \alpha \left\|B^*_{j_k^{(1)}} y(t_{n(k)+1})\right\|_{U_{j_k^{(1)}}}, \tag{4.51}$$

$$\left\|B^*_{j_k^{(i)}} y(t_{n(k)+i+1})\right\|_{U_{j_k^{(i)}}} = \alpha \left\|B^*_{j^{(i+1)}} y(t_{n(k)+i+1})\right\|_{U_{j_k^{(i+1)}}}, \; 1 \leq i \leq q_k - 1. \tag{4.52}$$

By exploiting (4.47), we get in particular for $i = q_k - 1$

$$\left\|B^*_{j_k^{(q_k-1)}} y(t_{n(k+1)})\right\|_{U_{j_k^{(q_k-1)}}} = \alpha \left\|B^*_{j_k^{(q_k)}} y(t_{n(k+1)})\right\|_{U_{j_k^{(q_k)}}},$$

and consequently

$$\left\|B^*_{j_k^{(q_k-1)}} y(t_{n(k+1)})\right\|_{U_{j_k^{(q_k-1)}}} = \alpha \left\|B_j^* y(t_{n(k+1)})\right\|_{U_j}. \tag{4.53}$$

Then the equalities (4.51)–(4.53) give as $k \to \infty$

$$\left\| B_j^* y_\zeta \right\|_{U_j} = \lim_{k\to\infty} \alpha \left\| B_{j_k^{(1)}}^* y(t_{n(k)+1}) \right\|_{U_{j_k^{(1)}}} \tag{4.54}$$

$$\lim_{k\to\infty} \left\| B_{j_k^{(i)}}^* y(t_{n(k)+i+1}) \right\|_{U_{j_k^{(i)}}} = \lim_{k\to\infty} \alpha \left\| B_{j_k^{(i+1)}}^* y(t_{n(k)+i+1}) \right\|_{U_{j_k^{(i+1)}}}, \ 1 \le i \le q_k - 1, \tag{4.55}$$

$$\lim_{k\to\infty} \left\| B_{j_k^{(q_k-1)}}^* y(t_{n(k+1)}) \right\|_{U_{j^{(q_k-1)}}} = \lim_{k\to\infty} \alpha \left\| B_j^* y(t_{n(k+1)}) \right\|_{U_j}. \tag{4.56}$$

It follows that

$$\left\| B_j^* y_\zeta \right\|_{U_j} = \lim_{k\to\infty} \alpha^{q_k} \left\| B_j^* y(t_{n(k+1)}) \right\|_{U_j}. \tag{4.57}$$

Because $0 < \alpha < 1$ and $q_k \geq 2$, we have

$$\left\| B_j^* y(t_{n(k)+1}) \right\|_{U_j} = \alpha^{q_k} \left\| B_j^* y(t_{n(k+1)}) \right\|_{U_j}$$

$$< \alpha \left\| B_j^* y(t_{n(k+1)}) \right\|_{U_j}.$$

Hence

$$\lim_{k\to\infty} \left\| B_j^* y(t_{n(k)+1}) \right\|_{U_j} \le \alpha \lim_{k\to\infty} \left\| B_j^* y(t_{n(k+1)}) \right\|_{U_j},$$

so that

$$\left\| B_j^* y_\zeta \right\|_{U_j} \le \alpha \left\| B_j^* y_\zeta \right\|_{U_j}.$$

It follows that necessarily $B_j^* y_\zeta = 0$ for all $j \in \mathcal{P}$ and we can easily conclude by using again (4.50). □

4.3.3. The asymptotic case

In the asymptotic case with fast switching, we shall obtain also an output stability property for the overall switched system (4.7).

Theorem 4.6. *Let $y_0 \in D(A)$, suppose that the observability condition (4.6) holds and consider the associated sequence $\{t_n\}_n$ constructed in Theorem 4.1. Suppose that (4.41) holds and $t_n \to \infty$ as $n \to \infty$. Then the overall switched system (4.7) is asymptotically output stable.*

Proof. Here again the proof is reduced to show that $\left\| B_j^* y(t) \right\|_{U_j} \to 0$ as $t \to \infty$ for all $j \in \mathcal{P}$. To this end, we shall establish that for j fixed, we have $\left\| B_j^* y(t_{n(k)}) \right\|_{U_j} \to 0$ as $k \to \infty$. Let $L_j := \limsup_{k\to\infty} \left\| B_j^* y(t_{n(k)}) \right\|_{U_j}$. Since the subsequence $\left\{ \left\| B_j^* y(t_{n(k)}) \right\|_{U_j} \right\}_k$ is non-negative, it is sufficient to verify that $L_j = 0$. By definition, this subsequence admits a subsequence denoted by $\left\{ t_{n(k')} \right\}_{k'}$ such that $\left\| B_j^* y(t_{n(k')}) \right\|_{U_j} \to L_j$ as $k' \to \infty$. Let us introduce the activated modes between the successive instants $t_{n(k')}$ and $t_{n(k'+1)}$ given by

$$n(k'+1) - n(k') = q_{k'} \text{ for all } k' \in \mathbb{N}, \tag{4.58}$$

and, for each $k' \in \mathbb{N}$, the intermediary modes $\left\{j_{k'}^{(i)}\right\}_{0\leq i\leq q_{k'}}$ given by

$$\sigma((t_{n(k')+i}, t_{n(k')+i+1})) = \left\{j_{k'}^{(i)}\right\},\ 0 \leq i \leq q_{k'}, \tag{4.59}$$

Note that

$$j_{k'}^{(0)} = j_{k'}^{(q_{k'})} = j.$$

By virtue of the transition rules specified in Theorem 4.1, we get

$$\left\|B_j^* y(t_{n(k')+1})\right\|_{U_j} = \alpha \left\|B^*_{j_{k'}^{(1)}} y(t_{n(k')+1})\right\|_{U_{j_{k'}^{(1)}}} \tag{4.60}$$

$$\left\|B^*_{j_{k'}^{(i)}} y(t_{n(k')+i+1})\right\|_{U_{j_{k'}^{(i)}}} = \alpha \left\|B^*_{j_{k'}^{(i+1)}} y(t_{n(k')+i+1})\right\|_{U_{j_{k'}^{(i+1)}}},\ 1 \leq i \leq q_{k'} - 1. \tag{4.61}$$

We get in particular for $i = q_{k'} - 1$

$$\left\|B^*_{j_{k'}^{(q_{k'}-1)}} y(t_{n(k'+1)})\right\|_{U_{j_{k'}^{(q_{k'}-1)}}} = \alpha \left\|B^*_{j_{k'}^{q_{k'}}} y(t_{n(k'+1)})\right\|_{U_{j_{k'}^{(q_{k'})}}},$$

so that

$$\left\|B^*_{j_{k'}^{(q_{k'}-1)}} y(t_{n(k'+1)})\right\|_{U_{j_{k'}^{(q_{k'}-1)}}} = \alpha \left\|B_j^* y(t_{n(k'+1)})\right\|_{U_j}. \tag{4.62}$$

On the other hand, from

$$\begin{aligned}\left|\left\|B^*_{j_{k'}^{(i)}} y(t_{n(k')+i+1})\right\|_{U_{j_{k'}^{(i)}}} - \left\|B^*_{j_{k'}^{(i)}} y(t_{n(k')+i})\right\|_{U_{j_{k'}^{(i)}}}\right| &\leq \left\|B^*_{j_{k'}^{(i)}} y(t_{n(k')+i+1}) - B^*_{j_{k'}^{(i)}} y(t_{n(k')+i})\right\|_{U_{j_{k'}^{(i)}}} \\ &\leq \left\|B^*_{j_{k'}^{(i)}}\right\|_{\mathcal{L}(Y,U_{j_{k'}^{(i)}})} \left\|y(t_{n(k')+i+1}) - y(t_{n(k')+i})\right\|_Y \\ &\leq \max_{1\leq q\leq p} \left\|B_q^*\right\|_{\mathcal{L}(Y,U_q)} \left\|y(t_{n(k')+i+1}) - y(t_{n(k')+i})\right\|_Y .\end{aligned}$$

Then Lemma 4.3, combined with the mean value theorem, yields

$$\left\|y(t_{n(k')+i+1}) - y(t_{n(k')+i})\right\|_Y \leq C_0 \left|t_{n(k')+i+1} - t_{n(k')+i}\right|,$$

and (4.41) implies

$$\left\| B^*_{j^{(i)}_{k'}} y(t_{n(k')+i+1}) \right\|_{U_{j^{(i)}_{k'}}} - \left\| B^*_{j^{(i)}_{k'}} y(t_{n(k')+i}) \right\|_{U_{j^{(i)}_{k'}}} \to 0 \text{ as } k' \to \infty. \tag{4.63}$$

Then the transition rules combined with (4.63) give

$$\lim_{k'\to\infty} \left\| B^*_j y(t_{n(k')+1}) \right\|_{U_j} = \lim_{k'\to\infty} \left\| B^*_j y(t_{n(k')}) \right\|_{U_j}$$

$$= \lim_{k'\to\infty} \alpha \left\| B^*_{j^{(1)}_{k'}} y(t_{n(k')+1}) \right\|_{U_{j^{(1)}_{k'}}},$$

$$\lim_{k'\to\infty} \left\| B^*_{j^{(i)}_{k'}} y(t_{n(k')+i+1}) \right\|_{U_{j^{(i)}_{k'}}} = \lim_{k'\to\infty} \alpha \left\| B^*_{j^{(i+1)}_{k'}} y(t_{n(k')+i+1}) \right\|_{U_{j^{(i+1)}_{k'}}}, \ 0 \le i \le q_{k'} - 2,$$

and

$$\lim_{k'\to\infty} \left\| B^*_{j^{(q_{k'}-1)}_{k'}} y(t_{n(k'+1)}) \right\|_{U_{j^{(q_{k'}-1)}_{k'}}} = \lim_{k'\to\infty} \alpha \left\| B^*_j y(t_{n(k'+1)}) \right\|_{U_j}.$$

It follows that

$$\lim_{k'\to\infty} \left\| B^*_j y(t_{n(k')}) \right\|_{U_j} = \lim_{k'\to\infty} \alpha^{q_{k'}} \left\| B^*_j y(t_{n(k'+1)}) \right\|_{U_j}. \tag{4.64}$$

Because $0 < \alpha < 1$, $q_{k'} \ge 2$, and

$$\lim_{k'\to\infty} \left\| B^*_j y(t_{n(k')}) \right\|_{U_j} = \lim_{k'\to\infty} \left\| B^*_j y(t_{n(k'+1)}) \right\|_{U_j} = L_j,$$

we deduce as in the Zeno behavior case that necessarily $L_j = 0$ for all $j \in \mathcal{P}$. On the other hand, the assumption (4.41) implies also $(t_{n(k)+1} - t_{n(k)}) \to 0$ as $k \to \infty$. Hence for any $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that for any $k \ge k_\epsilon$ we have

$$t_{n(k)+1} - t_{n(k)} < \epsilon,$$

$$\left\| B^*_j y(t_{n(k)}) \right\|_{U_j} < \epsilon.$$

Let us consider $t > t_{n(k_\epsilon)}$ and $k \ge k_\epsilon$ such that $t_{n(k)} \le t < t_{n(k)+1}$. Then the decomposition

$$B^*_j y(t) = B^*_j \left(y(t) - y(t_{n(k)}) \right) + B^*_j y(t_{n(k)})$$

yields

$$\left\| B^*_j y(t) \right\|_{U_j} \le \left\| B^*_j \left(y(t) - y(t_{n(k)}) \right) \right\|_{U_j} + \left\| B^*_j y(t_{n(k)}) \right\|_{U_j}.$$

By adapting techniques similar to those used in the proof of Theorem 4.5, we get

$$\left\|B_j^* y(t)\right\|_{U_j} \leq C_0 \left|t - t_{n(k)}\right| + \epsilon$$

$$\leq C_0 \left|t_{n(k)+1} - t_{n(k)}\right| + \epsilon < (C_0 + 1)\epsilon,$$

for some positive constant C_0. Consequently, we conclude that $\left\|B_j^* y(t)\right\|_{U_j} \to 0$ as $t \to \infty$ for all $j \in \mathcal{P}$. This completes the proof of the theorem. □

In the asymptotic case with slow switching, we shall see that the asymptotic stability is possible under the circumstance where, for each $j \in \mathcal{P}$, the multivalued feedback system (4.5) is asymptotically stable. To this end, we shall suppose that the dwell time is large enough so that some significant dissipation can be obtained in each switching interval. The magnitude of the needed dissipation will be specified by considering for each transition time t_n the values $\|y(t_n + \tau_d)\|_Y$ and $\left\|\mathrm{e}^{(t_n - t_0)A_{\mathcal{F}_j}}\left(\mathrm{e}^{t_0 A} y_0\right)\right\|_Y$ $(j \in \mathcal{P})$. Roughly, for at least one mode j, the dissipation for the state $y(t)$ at $t_n + \tau_d$ has to be more important than the one obtained by the solution of the (stable) multivalued feedback system (4.5) at t_n. The precise statement is as follows.

Theorem 4.7. *Let $y_0 \in D(A)$, suppose that the observability condition (4.6) holds and consider the associated sequence $\{t_n\}_n$ constructed in Theorem 4.1. Assume that the dwell time τ_d given by (4.42) is positive and is such that the solution of the overall switched system (4.7) satisfies*

$$\|y(t_n + \tau_d)\|_Y \leq \max_{j \in \mathcal{P}} \left\|\mathrm{e}^{(t_n - t_0)A_{\mathcal{F}_j}}\left(\mathrm{e}^{t_0 A} y_0\right)\right\|_Y, \tag{4.65}$$

for infinitely many integers n. Assume also that $w(0) = 0$. Then the overall switched system (4.7) is asymptotically stable.

Proof. The proof is based on the setting used in Corollary 3.5. We note that the observability condition (4.6) yields $M \cap Y_u = \{0\}$. Combined with the assumption $w(0) = 0$, it follows that, for each mode j, the multivalued feedback system (4.5) is asymptotically stable. Let

$$v_n := \max_{j \in \mathcal{P}} \left\|\mathrm{e}^{(t_n - t_0)A_{\mathcal{F}_j}}\left(\mathrm{e}^{t_0 A} y_0\right)\right\|_Y, \tag{4.66}$$

then clearly $v_n \to 0$ as $n \to \infty$. Moreover, from the assumption (4.65) and taking into account

$$t_n < t_n + \tau_d \leq t_{n+1},$$

we can extract a subsequence $\left\{t_{n(m)}\right\}_m$ such that

$$\left\|y(t_{n(m)+1})\right\|_Y \leq \left\|y(t_{n(m)} + \tau_d)\right\|_Y \leq v_{n(m)}.$$

Hence we have $\left\|y(t_{n(m)})\right\|_Y \to 0$ as $m \to \infty$ so that by using Lemma 4.3, we get also both $\|y(t_n)\|_Y \to 0$ as $n \to \infty$ and $\|y(t)\|_Y \to 0$ as $t \to \infty$. This completes the proof of the theorem. □

4.4. Applications to partial differential equations (III)

We shall consider the PDEs introduced in subsection 2.5 and we treat their switching stabilization in the framework developed in Theorem 4.1. Recall that the construction of the sequence of transition times $\{t_n\}_n$ is based on the observability condition (4.6). Moreover, following Remark 4.2 (iii), the first activation time t_0 is

closely related to the observability time T_0. Having in mind the switched system (4.7) subject to the transition rules (4.9)–(4.10), we suppose implicitly that these systems are not activated for $0 \le t \le t_0$. Then we give the expression of the resulting switched single-valued feedback for each subsystem on the interval (t_n, t_{n+1}). The formulations in the PDEs context of the various stability results stated in Theorem 4.5, Theorem 4.6 and Theorem 4.7 are similar to the ones given in Section 3.3. The study will be presented in detail for the heat-heat system. As for the heat-plate and the heat-wave systems, in order to avoid redundancy, their corresponding results are mentioned briefly.
(i) *Switched systems for the heat-heat system.* Here we consider the heat-heat system (2.55)-(2.56). Let $(\mathrm{e}^{t\Delta})_{t\ge 0}$ denote in unified way the semigroups generated by both the operators A_1 and A_2 given by (2.57) and (2.58). By virtue of Theorem 4.1 we are led to check that the observability conditions:

$$(\chi_{\omega_1}\mathrm{e}^{t\Delta}\theta = 0 \text{ on } (0,T_0)\times\mathcal{O}, \tag{4.67}$$

$$(\chi_{\omega_2}\mathrm{e}^{t\Delta}\tilde{\theta} = 0 \text{ on } (0,T_0)\times\tilde{\mathcal{O}} \tag{4.68}$$

imply both $\theta = 0$ and $\tilde{\theta} = 0$. By using analyticity arguments similar to the ones developed in ([30], Part III, Chapter 1), it can be seen that these observability conditions hold for T_0 arbitrarily small. It follows that the first activation time t_0 can be chosen arbitrarily short. In order to present the study of the stabilization of the heat-heat system under switching constraint, we proceed in four steps:
Step 1. Construction of the single-valued feedback subsystems. The construction will be made by assuming, for instance, that the initial conditions satisfy $\left\|\chi_{\omega_1}\mathrm{e}^{t_0\Delta}\theta_0\right\|_{L^2(\omega_1)} \ge \left\|\chi_{\omega_2}\mathrm{e}^{t_0\Delta}\tilde{\theta}_0\right\|_{L^2(\omega_2)}$. Then by applying the transition rules (4.9)–(4.10), we get the following single-valued feedback as follows. Letting $k\in\mathbb{N}$, for $t_{2k}\le t < t_{2k+1}$ we have $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} > \alpha\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}$ and

$$(u_1(t),u_2(t)) = \left(-\frac{w\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\theta(t), 0\right), \tag{4.69}$$

with $\|\chi_{\omega_1}\theta(t_{2k+1})\|_{L^2(\omega_1)} = \alpha\left\|\chi_{\omega_2}\tilde{\theta}(t_{2k+1})\right\|_{L^2(\omega_2)}$. For $t_{2k+1}\le t < t_{2k+2}$ we have $\alpha\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} < \left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}$ and

$$(u_1(t),u_2(t)) = \left(0, -\frac{w\left(\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{\theta}(t)\right\|_{L^2(\omega_2)}}\chi_{\omega_2}\tilde{\theta}(t)\right), \tag{4.70}$$

with $\alpha\|\chi_{\omega_1}\theta(t_{2k+2})\|_{L^2(\omega_1)} = \left\|\chi_{\omega_2}\tilde{\theta}(t_{2k+2})\right\|_{L^2(\omega_2)}$.
Step 2. Output stability of the overall heat-heat switched system . The appropriate output stability results can be obtained by itemizing as follows.

- If the sequence $\{t_n\}_n$ is bounded, Theorem 4.5 implies that this system is output stable in finite time so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\left\|\tilde{\theta}(t)\right\|_{L^2(\omega_2)} \to 0$ as $t \nearrow \sup\limits_n t_n$. Note also that, in this case, the system is obviously stable in finite time provided that $\omega_1 = \mathcal{O}$ and $\omega_2 = \tilde{\mathcal{O}}$.

- If the sequence $\{t_n\}_n$ is unbounded and satisfies the fast switching hypothesis (4.41), from Theorem 4.6 we deduce that this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\left\|\tilde{\theta}(t)\right\|_{L^2(\omega_2)} \to 0$ as $t \to \infty$. Note also again that, in this case, the system is obviously asymptotically stable provided that $\omega_1 = \mathcal{O}$ and $\omega_2 = \tilde{\mathcal{O}}$.

Step 3. Construction of the multi-valued feedback heat-heat subsystems. This construction will enable us to apply Theorem 4.7. Let us introduce the family of multi-valued feedback systems defined by the index set $\mathcal{P} := \{1, 2\}$ and the corresponding subsystems analogous to (4.5). At the outset, for $j = 1$ we have to consider the functionals $G_1^1 : U_1(= L^2(\omega_1)) \to \mathbb{R}$ and $G_2^1 : U_2(= L^2(\omega_2)) \to \mathbb{R}$ given by

$$G_1^1(u_1) = \int_0^{\|u_1\|_{L^2(\omega_1)}} w(\tau)\mathrm{d}\tau, \tag{4.71}$$

$$G_2^1(u_2) = \int_0^{\alpha\|u_2\|_{L^2(\omega_2)}} w(\tau)\mathrm{d}\tau. \tag{4.72}$$

Note that a simple change of variables gives

$$G_2^1(u_2) = \int_0^{\|u_2\|_{L^2(\omega_2)}} \alpha w(\alpha\tau)\mathrm{d}\tau.$$

Then the appropriate multi-valued feedback subsystem corresponding to (4.5) is based on the functional

$$\mathcal{F}_1(u_1, u_2) := \max\left(G_1^1(u_1), G_2^1(u_2)\right). \tag{4.73}$$

Let $y(t) = (\vartheta(t), \tilde{\vartheta}(t))$ denote the state of such a system. Then by adapting the forms of the multi-valued feedback system already established in Section 2.5 for the heat-heat system, we get the following state equations

$$\begin{cases} \vartheta' = \Delta\vartheta + u_1(t)\chi_{\omega_1} \text{ on } (0,\infty) \times \mathcal{O}, \\ \frac{\partial\vartheta}{\partial\nu} = 0 \text{ on } (0,\infty) \times \Gamma_{\mathcal{O}}, \end{cases} \tag{4.74}$$

$$\begin{cases} \tilde{\vartheta}' = \Delta\tilde{\vartheta} + u_2(t)\chi_{\omega_2} \text{ on } (0,\infty) \times \tilde{\mathcal{O}}, \\ \frac{\partial\tilde{\vartheta}}{\partial\nu} = 0 \text{ on } (0,\infty) \times \Gamma_{\tilde{\mathcal{O}}}, \end{cases} \tag{4.75}$$

subject to the following multi-valued feedbacks.

- If $\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)} + \left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)} > 0$ and $\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)} = \alpha\left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}$ then

$$(u_1(t), u_2(t))$$

$$\in -\mathrm{conv}\left\{\left(\frac{w\left(\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\vartheta(t), 0\right), \left(0, \frac{\alpha w\left(\alpha\left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}}\chi_{\omega_2}\tilde{\vartheta}(t)\right)\right\} \tag{4.76}$$

- if $\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)} > \alpha \left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}$ then

$$(u_1(t), u_2(t)) = \left(-\frac{w\left(\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\vartheta(t), 0\right) \tag{4.77}$$

- if $\alpha \left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)} > \|\chi_{\omega_1}\vartheta(t)\|_{L^2(\omega_1)}$ then

$$(u_1(t), u_2(t)) = \left(0, -\frac{\alpha w\left(\alpha\left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{\vartheta}(t)\right\|_{L^2(\omega_2)}}\chi_{\omega_2}\tilde{\vartheta}(t)\right) \tag{4.78}$$

- if $\chi_{\omega_2}\tilde{\vartheta}(t) = 0$ and $\chi_{\omega_1}\vartheta(t) = 0$, then

$$(u_1(t), u_2(t)) \in w(0)\mathcal{B}_{\omega_1} \times \alpha w(0)\mathcal{B}_{\omega_2}, \tag{4.79}$$

In the same way, for $j = 2$ we have to consider the functionals $G_2^2 : U_2(= L^2(\omega_2)) \to \mathbb{R}$ and $G_1^2 : U_1(= L^2(\omega_1)) \to \mathbb{R}$ given by

$$G_2^2(u_2) = \int_0^{\|u_2\|_{L^2(\omega_2)}} w(\tau)\mathrm{d}\tau \tag{4.80}$$

$$G_1^2(u_1) = \int_0^{\alpha\|u_1\|_{L^2(\omega_1)}} w(\tau)\mathrm{d}\tau \tag{4.81}$$

$$= \int_0^{\|u_1\|_{L^2(\omega_1)}} \alpha w(\alpha\tau)\mathrm{d}\tau.$$

Moreover, let

$$\mathcal{F}_2(u_1, u_2) := \max\left(G_1^2(u_1), G_2^2(u_2)\right), \tag{4.82}$$

and let $y(t) = (z(t), \tilde{z}(t))$ denote the state of the multi-valued feedback system analogous to (4.5) with $j = 2$. Then, we get as above the state equations

$$\left\{\begin{array}{c} z' = \Delta z + u_1(t)\chi_{\omega_1} \text{ on } (0,\infty)\times\mathcal{O}, \\ \frac{\partial z}{\partial \nu} = 0 \text{ on } (0,\infty)\times\Gamma_{\mathcal{O}}, \end{array}\right. \tag{4.83}$$

$$\left\{\begin{array}{c} \tilde{z}' = \Delta \tilde{z} + u_2(t)\chi_{\omega_2} \text{ on } (0,\infty)\times\tilde{\mathcal{O}}, \\ \frac{\partial \tilde{z}}{\partial \nu} = 0 \text{ on } (0,\infty)\times\Gamma_{\tilde{\mathcal{O}}}, \end{array}\right. \tag{4.84}$$

subject to the following multi-valued feedbacks.

- If $\|\chi_{\omega_1}z(t)\|_{L^2(\omega_1)} + \|\chi_{\omega_2}\tilde{z}(t)\|_{L^2(\omega_2)} > 0$ and $\alpha\|\chi_{\omega_1}z(t)\|_{L^2(\omega_1)} = \|\chi_{\omega_2}\tilde{z}(t)\|_{L^2(\omega_2)}$ then

$$(u_1(t), u_2(t))$$

$$\in -\mathrm{conv}\left\{\left(\frac{\alpha w\left(\alpha\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)}\right)}{\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)}}\chi_{\omega_1} z(t), 0\right), \left(0, \frac{w\left(\left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)}}\chi_{\omega_2}\tilde{z}(t)\right)\right\}. \tag{4.85}$$

- If $\alpha\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)} > \left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)}$ then

$$(u_1(t), u_2(t)) = \left(\frac{\alpha w\left(\alpha\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)}\right)}{\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)}}\chi_{\omega_1} z(t), 0\right). \tag{4.86}$$

- If $\left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)} > \alpha\left\|\chi_{\omega_1} z(t)\right\|_{L^2(\omega_1)}$ then

$$(u_1(t), u_2(t)) = \left(0, -\frac{w\left(\left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)}\right)}{\left\|\chi_{\omega_2}\tilde{z}(t)\right\|_{L^2(\omega_2)}}\chi_{\omega_2}\tilde{z}(t)\right). \tag{4.87}$$

- If $\chi_{\omega_2}\tilde{z}(t) = 0$ and $\chi_{\omega_1} z(t) = 0$, then

$$(u_1(t), u_2(t)) \in \alpha w(0)\mathcal{B}_{\omega_1} \times w(0)\mathcal{B}_{\omega_2}, \tag{4.88}$$

Step 4. asymptotic stability of the overall heat-heat system under slow switching. Considering carefully the inequality (4.65), we have to consider for $t \geq t_0$ the solutions $(\vartheta(.), \tilde{\vartheta}(.))$ and $(z(.), \tilde{z}(.))$ satisfying

$$(\vartheta(t_0), \tilde{\vartheta}(t_0)) = (\mathrm{e}^{t_0\Delta}\theta_0, \mathrm{e}^{t_0\Delta}\tilde{\theta}_0), \tag{4.89}$$

$$(z(t_0), \tilde{z}(t_0)) = (\mathrm{e}^{t_0\Delta}\theta_0, \mathrm{e}^{t_0\Delta}\tilde{\theta}_0). \tag{4.90}$$

Then the assumption (4.65) would read

$$\left\|\left\{\theta(t_n+\tau_d), \tilde{\theta}(t_n+\tau_d)\right\}\right\|_{L^2(\mathcal{O})\times L^2(\tilde{\mathcal{O}})}$$

$$\leq \max\left(\left\|\left\{\vartheta(t_n), \tilde{\vartheta}(t_n)\right\}\right\|_{L^2(\mathcal{O})\times L^2(\tilde{\mathcal{O}})}, \left\|\{z(t_n), \tilde{z}(t_n)\}\right\|_{L^2(\mathcal{O})\times L^2(\tilde{\mathcal{O}})}\right) \tag{4.91}$$

By virtue of Theorem 4.7, it follows that whenever $w(0) = 0$ and the dwell time τ_d is positive satisfying (4.91) for infinitely many integers n, then the overall switched heat-heat system constructed in the step 1 is asymptotically stable so that

$$\left\|\left\{\theta(t), \tilde{\theta}(t)\right\}\right\|_{L^2(\mathcal{O})\times L^2(\tilde{\mathcal{O}})} \to 0 \text{ as } t \to \infty.$$

(ii) *Switched systems for the heat-plate system.* Let us consider the systems defined by the heat equation (2.55) and the plate equation defined by (2.63). Considering (4.67) and the solution of

$$\begin{cases} \varphi'' = -\Delta^2\varphi \text{ on } (0, T_0)\times\Omega, \\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } (0, T_0)\times\partial\Omega, \\ \varphi(0) = \varphi_0,\ \varphi'(0) = \varphi_1 \text{ on } \Omega, \end{cases}$$

subject to $\varphi'(t) = 0$ on $(0, T_0) \times \tilde{\omega}$, the observability condition amounts to whether we have necessarily both $\theta = 0$ and $\varphi = 0$. The case $\theta = 0$ has been treated in (i) above. Following [31], there exists some positive constant $C > 0$ such that, for any $T_0 > 0$, this solution satisfies the stronger observability inequality

$$\int_0^{T_0} \int_{\tilde{\omega}} |\varphi|^2 \, dx dt \geq C \left(\|\varphi_0\|^2_{L^2(\Omega)} + \|\varphi_1\|^2_{H^{-2}(\Omega)} \right),$$

provided that $\tilde{\omega} = \Omega_b \cap \Omega$ for some open subset $\Omega_b \subset \mathbb{R}^d$ containing $\partial\Omega$. Furthermore, the desired unique continuation property can be easily obtained consequently. Hence the first activating instant t_0 can be arbitrarily small. Assume, for instance, that $\|\chi_{\omega_1}\theta(t_0)\|_{L^2(\omega_1)} \geq \|\chi_{\tilde{\omega}}\tilde{v}'(t_0)\|_{L^2(\tilde{\omega})}$. Then by applying the transition rules (4.9)–(4.10), we get the following single-valued feedback as follows. Letting $k \in \mathbb{N}$, for $t_{2k} \leq t < t_{2k+1}$ we have $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} > \alpha \|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}$ and

$$(u_1(t), u_2(t)) = \left(-\frac{w\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}} \chi_{\omega_1}\theta(t), 0 \right) \tag{4.92}$$

with $\|\chi_{\omega_1}\theta(t_{2k+1})\|_{L^2(\omega_1)} = \alpha \|\chi_{\tilde{\omega}}\tilde{v}'(t_{2k+1})\|_{L^2(\tilde{\omega})}$. For $t_{2k+1} \leq t < t_{2k+2}$ we have $\alpha \|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} < \|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}$ and

$$(u_1(t), u_2(t)) = \left(0, -\frac{w\left(\|\chi_{\omega}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}\right)}{\|\chi_{\tilde{\omega}}\tilde{v}'(t)\|_{L^2(\tilde{\omega})}} \chi_{\tilde{\omega}}\tilde{v}'(t) \right) \tag{4.93}$$

with $\alpha \|\chi_{\omega_1}\theta(t_{2k+2})\|_{L^2(\omega_1)} = \|\chi_{\tilde{\omega}_2}\tilde{v}'(t_{2k+2})\|_{L^2(\tilde{\omega})}$. Then we get the appropriate stability results by itemizing as follows.

- If the sequence $\{t_n\}_n$ is bounded, Theorem 4.5 implies that this system is output stable in finite time so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|\tilde{v}'(t)\|_{L^2(\tilde{\omega})} \to 0$ as $t \nearrow \sup_n t_n$.
- If the sequence $\{t_n\}_n$ is unbounded and satisfies (4.41), from Theorem 4.6 we deduce that this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|\tilde{v}'(t)\|_{L^2(\tilde{\omega})} \to 0$ as $t \to \infty$.
- Suppose that $w(0) = 0$ and the dwell time τ_d given by (4.42) satisfies the analogous of (4.65) which results from adapting to the heat-plate system the framework developed in step 3 and step 4 in the context of the heat-heat system. Then Theorem 4.7 implies that the switched heat-plate system is asymptotically stable so that $\|\theta(t)\|_{L^2(\mathcal{O})} \to 0$ and $\|\{\tilde{v}(t), \tilde{v}'(t)\}\|_{H_0^2(\Omega)\times L^2(\Omega)} \to 0$ as $t \to \infty$.

(iii) *Switched systems for the heat-wave system.* Let us consider the systems defined by the heat equation (2.55) and the wave equation (2.68). The corresponding observability condition is reduced to see whether (4.67) and the solution of

$$\begin{cases} \varphi'' = \Delta\varphi \text{ on } (0, T_0) \times \tilde{\Omega}, \\ \varphi = 0 \text{ on } (0, T_0) \times \partial\tilde{\Omega}, \\ \varphi(0) = \varphi_0, \ \varphi'(0) = \varphi_1 \text{ on } \tilde{\Omega}. \end{cases}$$

subject to $\varphi'(t) = 0$ on $(0, T_0) \times \omega$ imply both $\theta = 0$ and $\varphi = 0$. The case $\theta = 0$ has been treated in (i) above. Let us mention the various disseminated results devoted to observability inequalities for the wave equation. Under geometrical conditions relating ω and T_0, we have at our disposal the inequality

$$\int_0^{T_0}\int_\omega |\varphi|^2\,\mathrm{d}x\mathrm{d}t \geq C\left(\|\varphi_0\|^2_{L^2(\tilde\Omega)} + \|\varphi_1\|^2_{H^{-1}(\tilde\Omega)}\right),$$

See the survey article [32]. For the one dimensional case $\tilde\Omega = (0, L)$, it can be deduced from [33] that the inequality holds for $T_0 \geq 2L$ and ω arbitrary nonempty open sub-interval in $(0, L)$. It follows that the first transition instant t_0 has to be chosen carefully such that $t_0 \leq T_0$. However, in the case $\omega = \tilde\Omega$, there exists a positive constant C such that for any $T_0 > 0$ we have

$$\int_0^{T_0}\int_{\tilde\Omega} |\varphi|^2\,\mathrm{d}x\mathrm{d}t \geq C\left(\|\varphi_0\|^2_{L^2(\tilde\Omega)} + \|\varphi_1\|^2_{H^{-1}(\tilde\Omega)}\right),$$

and

$$\int_0^{T_0}\int_{\tilde\Omega} |\varphi'|^2\,\mathrm{d}x\mathrm{d}t \geq C\left(\|\varphi_0\|^2_{H^1_0(\tilde\Omega)} + \|\varphi_1\|^2_{L^2(\tilde\Omega)}\right).$$

See ([29], Chapter VII). Hence, t_0 can be chosen arbitrarily small. Then by applying the transition rules (4.9)–(4.10), we get the following single-valued feedback as follows. Assume, for instance, that $\|\chi_{\omega_1}\theta(t_0)\|_{L^2(\omega_1)} \geq \|\chi_\omega v'(t_0)\|_{L^2(\tilde\omega)}$. Then by applying the transition rules (4.9)–(4.10), we get the following single-valued feedback as follows. Letting $k \in \mathbb{N}$, for $t_{2k} \leq t < t_{2k+1}$ we have $\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} > \alpha\,\|\chi_\omega v'(t)\|_{L^2(\omega)}$ and

$$(u_1(t), u_2(t)) = \left(-\frac{w\left(\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}\right)}{\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)}}\chi_{\omega_1}\theta(t), 0\right) \tag{4.94}$$

with $\|\chi_{\omega_1}\theta(t_{2k+1})\|_{L^2(\omega_1)} = \alpha\,\|\chi_\omega v'(t_{2k+1})\|_{L^2(\omega)}$. For $t_{2k+1} \leq t < t_{2k+2}$ we have $\alpha\,\|\chi_{\omega_1}\theta(t)\|_{L^2(\omega_1)} < \|\chi_\omega v'(t)\|_{L^2(\omega)}$ and

$$(u_1(t), u_2(t)) = \left(0, -\frac{w\left(\|\chi_\omega v'(t)\|_{L^2(\omega)}\right)}{\|\chi_\omega v'(t)\|_{L^2(\omega)}}\chi_\omega v'(t)\right) \tag{4.95}$$

with $\alpha\,\|\chi_{\omega_1}\theta(t_{2k+2})\|_{L^2(\omega_1)} = \|\chi_\omega v'(t_{2k+2})\|_{L^2(\omega)}$. Then we get the appropriate stability results by itemizing as follows.

- If the sequence $\{t_n\}_n$ is bounded, Theorem 4.5 implies that this system is output stable in finite time so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|v'(t)\|_{L^2(\omega)} \to 0$ as $t \nearrow \sup_n t_n$.
- If the sequence $\{t_n\}_n$ is unbounded and satisfies (4.41), from Theorem 4.6 we deduce that this system is asymptotically output stable so that $\|\theta(t)\|_{L^2(\omega_1)} \to 0$ and $\|v'(t)\|_{L^2(\omega)} \to 0$ as $t \to \infty$.
- Suppose that $w(0) = 0$ and the dwell time τ_d given by (4.42) satisfies the analogous of (4.65) which results from adapting to the heat-wave system the framework developed in step 3 and step 4 in the context of the heat-heat system. Then Theorem 4.7 implies that the switched heat-plate system is asymptotically stable so that $\|\theta(t)\|_{L^2(\mathcal{O})} \to 0$ and $\|\{v(t), v'(t)\}\|_{H^1_0(\tilde\Omega)\times L^2(\tilde\Omega)} \to 0$ as $t \to \infty$.

5. Conclusion and further comments

For a class of linear distributed systems with possible control saturation, we treated the stabilization problem under the additional constraint that only one control is activated. We used an abstract multivalued feedback strategy leading to a differential inclusion system. Existence and properties of solutions were established in the framework of nonlinear semigroup generated by maximal monotone operators. The asymptotic stability and output stability were derived from the LaSalle invariance principle. Moreover, it has been shown that the switching control condition can be ensured whenever the orbit does not intersect some specified subsets of the state space. Exploiting the local nature of this fact with respect to the time variable, we constructed state-dependent switched systems with explicit transition rule such that, in each subsystem, only one control is activated. Then sufficient conditions guaranteeing the stability of the overall system were given. Along the paper, the results have been applied to various PDEs including heat, plate and wave systems. A possible route for future research can be considered in connection with the results and methods developed in this paper. Among them we mention the following ones.

Having dealt with the case where the control operators B_j in the state equation (1.3) are bounded, it would be interesting to extend the results of this paper to the situation in which these operators are unbounded. Such a problem seems challenging in the case of systems governed by partial differential equations with boundary control since unbounded control operators are inevitable in this case. Let us mention that multivalued feedback systems in this context have been treated for some specified partial differential equations. See, for instance, [34–36].

An interesting issue would consist on treating the stabilization under switching control constraint for bilinear control systems such as

$$y'(t) = Ay(t) + \sum_{j=1}^{p} u_j(t) C_j y(t),$$

where the controls u_j are real-valued and the operators $C_j : Y \to Y$ are linear. This problem seems to be open even for multivariable systems governed by ordinary differential equations.

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