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OBSERVABILITY AND UNIQUE CONTINUATION INEQUALITIES FOR THE SCHRÖDINGER EQUATIONS WITH INVERSE-SQUARE POTENTIALS

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Abstract. In this paper, we focus on the Schrödinger equations with inverse-square potentials in dimension one; these special potentials play an important role in the field of mathematical physics. We study several observability and unique continuation inequalities at one time point or at two time points for these equations. These observability and unique continuation inequalities are some new types of quantitative estimates which have appeared in recent literature. Their proofs essentially rely on the representation of the solution, a Nazarov-type uncertainty principle for the Hankel transform, and an interpolation inequality for functions whose Hankel transforms have compact support. Meanwhile, these inequalities can be applied to the controllability of these Schrödinger equations.

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1. Introduction

In this paper, we will present several observability and unique continuation inequalities (at either two points in time or one point in time) for the following Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) = \left(-\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}\right) u(x,t),\ x \in \mathbb{R}^+,\ t > 0, \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+), \end{cases} \tag{1.1}$$

with the Dirichlet boundary condition $u(0,t) = 0$ for $t > 0$. Here $\mathbb{R}^+$ denotes the half line $(0,\infty)$, and ν is a fixed non-negative constant, *i.e.*, $\nu \geq 0$. (Here and in what follows, $L^2(\mathbb{R}^+) = L^2(\mathbb{R}^+;\mathbb{C})$. The same is said about $C_0^\infty(\mathbb{R}^+)$.) The Schrödinger equation (1.1) with inverse-square potential is of interest in quantum mechanics. The family of differential operators $-\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}$ is particularly notable and is commonly referred to as the Bessel operator (see, for example, [1]). These operators appear in numerous applications, *e.g.*, as the radial part of the Laplacian in any dimension. Their generalized eigenfunctions can be expressed in terms of Bessel-type functions, and they have a rich and intricate theory, see [2–4]. Moreover, the heat flow associated with the inverse-square potential has been studied in the context of combustion theory (see [5] and references therein). The mathematical interest in these equations mainly arises from the fact that the potential term is homogeneous

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of degree -2, meaning it scales exactly the same as the Laplacian. This makes perturbation methods ineffective in analyzing the effect of this potential. Indeed, the decay is in some sense the borderline case for the existence of global-in-time estimates for the Schrödinger equation with a potential (see [6]). In particular, it is known that a negative potential V decaying more slowly than the inverse-square potential causes the spectrum of the associated Schrödinger operator to be unbounded from below (see [7], Sect. XIII, pp. 87–88). We denote by H_ν the Friedrichs extensions of differential operators $-\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}$ on $C_0^\infty(\mathbb{R}^+)$, see Section 2 for a more detailed discussion. By Stone's theorem, equation (1.1) admits a unique weak solution in the sense that $u(x,t) \in C(\mathbb{R}_t; L^2(\mathbb{R}^+))$, defined *via* the semigroup e^{-itH_ν} acting on the initial data $u_0(x)$. Specifically, the solution is given by $u(x,t) = \mathrm{e}^{-itH_\nu}u_0(x)$. For a more comprehensive discussion on extensions involving general parameters $\nu \in \mathbb{C}$ and some properties of the family of operators H_ν, we refer the readers to [8, 9].

To proceed with our introduction, we need to recall some background on the observability inequality for the Schrödinger equation in a more general framework. For a given manifold $\mathbb{M}$, let H denote the self-adjoint extension of the Schrödinger operator $-\Delta_g + V$ on the Hilbert space $L^2(\mathbb{M})$, where $-\Delta_g$ is the Laplace-Beltrami operator and V represents the potential term. The classical observability inequality for the Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) = Hu(x,t), \ (x,t) \in \mathbb{M} \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \in L^2(\mathbb{M}), \end{cases} \tag{1.2}$$

reads that when u solves (1.2)

$$\int_{\mathbb{M}} |u(x,0)|^2 \mathrm{d}x \le C_{obs} \int_0^T \int_\Omega |u(x,t)|^2 dx \mathrm{d}t, \tag{1.3}$$

where $T > 0$, Ω is a subset of the measure space $\mathbb{M}$, and the constant $C_{obs} > 0$ is called the observable constant or cost constant. This is the observability inequality for the Schrödinger equation involving observation over a time interval. This type of observability has been extensively studied in the literature on compact Riemannian manifolds, we refer readers to [10–17] for results on some specific compact Riemannian manifolds. For the Schrödinger equation on non-compact Riemannian manifolds, there are relatively few existing results, as new difficulties arise due to the unbounded spatial domain. But recently, there has been growing interest in observability for the Schrödinger equation in Euclidean space. For example, when $\mathbb{M} = \mathbb{R}^n$, $V = 0$, the inequality (1.3) holds in all dimensions $n \ge 1$ if $\Omega = \{x \in \mathbb{R}^n : |x| \ge r\}$ for each $r > 0$, and a sharp result in [18] shows that the inequality (1.3) holds if and only if Ω is thick (a more general set class) in dimension one. For more general observability estimates in Euclidean space, we refer the readers to [19–22].

Recently, Wang, Wang and Zhang [23] proved the following new type of observability inequality in the Euclidean setting, that is, when $\mathbb{M} = \mathbb{R}^n$: Given $x_1, x_2 \in \mathbb{R}^n$, $r_1, r_2 > 0$ and $T > S \ge 0$, there is a positive constant C depending only on n such that for all $u(x,t)$ solving (1.2) with $V = 0$,

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \mathrm{d}x \le C\mathrm{e}^{Cr_1 r_2 \frac{1}{T-S}} \left(\int_{\mathbb{R}^n \setminus B_{r_1}(x_1)} |u(x,S)|^2 \mathrm{d}x + \int_{\mathbb{R}^n \setminus B_{r_2}(x_2)} |u(x,T)|^2 \mathrm{d}x \right), \tag{1.4}$$

where $B_r(x)$ denotes the closed ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. This improves inequality (1.3), as only two time points appear on the right-hand side of inequality (1.4). For this reason, (1.4) is called the observability inequality at two time points. The proof of (1.4) in [23] is based on the following basic identity:

$$(2it)^{\frac{n}{2}} \mathrm{e}^{-i|x|^2/4t} u(x,t) = \widehat{\mathrm{e}^{i|\cdot|^2/4t} u_0}(x/2t), \ \text{ for a.e. } x \in \mathbb{R}^n, \ t > 0, \tag{1.5}$$

where $\widehat{\cdot}$ denotes the Fourier transform (see (3.2) for the definition). With identity (1.5) in hand, the authors proved in [23] that the estimate (1.4) is equivalent to the following Nazarov's uncertainty principle established in [24] (see also [25, 26]): If A, B are subsets of $\mathbb{R}^n$ of finite measure, then

$$\int_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}x \le C(n,A,B)\left(\int_{\mathbb{R}^n\setminus A} |f(x)|^2 \mathrm{d}x + \int_{\mathbb{R}^n\setminus B} |\hat{f}(\xi)|^2 \mathrm{d}\xi\right),\ f\in L^2(\mathbb{R}^n), \tag{1.6}$$

with

$$C(n,A,B) := C\mathrm{e}^{C\min\{|A||B|,|B|^{1/n}\omega(A),|A|^{1/n}\omega(B)\}},$$

where $\omega(A)$ denotes the mean width of A, and C is a positive constant that depends only on n. We need to point out that the identity (1.5) is a crucial tool in [23]. Based on this identity and other methods, the authors obtained more quantitative estimates beyond (1.4).

Three natural questions were raised in [23]: can their results be extended to the following situations? (a) Schrödinger equations with nonzero potentials. (b) Homogeneous Schrödinger equations on a bounded domain. (c) Schrödinger equations on the half-space $\mathbb{R}^n_+$ (where $\mathbb{R}^n_+ := \{(x_1,...,x_n)\in\mathbb{R}^n : x_n>0\}$). Recently, for question (a), Huang and Soffer [27] established observability inequalities similar to (1.4) at two points in time for $H=-\Delta+V$ in $\mathbb{R}^n$, where V belongs to a class of decaying potentials satisfying additional regularity conditions. Due to the generality of the potential and lack of a similar identity (1.5), the observability inequality similar to (1.4) established in [27] is restricted to the case $x_1=x_2=0$ and $r_1r_2\sim T$. Their proof is based on an operator-type Nazarov uncertainty principle and minimal escape velocity estimates. For potentials that increase to infinity as $|x|\to\infty$, to our best knowledge, there is only one result that was established for the Hermite Schrödinger equation in [18]. Further observability inequalities at two time points can be found in [28, 29] for the linear KdV equation, and in [30] for nonlinear Schrödinger equation.

Now returning to our model (1.1), we can consider it a positive answer to questions (a) and (c) mentioned above in dimension one. We first establish the following identity (see Lem. 2.2 below) similar to (1.5):

$$(2t)^{\frac{1}{2}}\mathrm{e}^{\frac{i(\nu+1)\pi}{2}}\mathrm{e}^{-i|x|^2/4t}u(x,t) = F_\nu(\mathrm{e}^{i|\cdot|^2/4t}u_0)(x/2t), \qquad \text{for all } t>0, \tag{1.7}$$

where $u(x,t)$ solves (1.1), and F_ν denotes the well known Hankel transform (see Sect. 2 for more details)

$$F_\nu(f)(x) := \int_0^\infty \sqrt{xy}J_\nu(xy)f(y)\mathrm{d}y,\ x\in\mathbb{R}^+, \tag{1.8}$$

here $J_\nu(x)$ is the standard Bessel function of the first kind. With the identity (1.7) in hand, we can extend all the results in [23] to our equation (1.1).

Prior to stating the main theorems of this paper, we introduce unified notation that will be used throughout. Whenever a statement includes the prior assumption "*For every* $\nu\ge 0$", we consistently write either $u(x,t;u_0)$ (with $(x,t)\in\mathbb{R}^+\times(0,\infty)$) or $\mathrm{e}^{-itH_\nu}u_0(x)$ (with $t\ge 0$) for the solution to the Schrödinger equation (1.1) with operator H_ν and initial data $u(x,0)=u_0(x)$, unless explicitly stated otherwise. Let $\Gamma(\cdot)$ represent the Gamma function. For a measurable set $A\subset\mathbb{R}^+$, we denote by A^c its complement in $\mathbb{R}^+$, by $|A|$ its Lebesgue measure, by $\chi_A(\cdot)$ its characteristic function, and by $\mu_\nu(A) := \int_A x^{2\nu+1}\,\mathrm{d}x$ the associated weighted measure. Define $a\wedge b := \min\{a,b\}$ for all real numbers $a,b\in\mathbb{R}$. Let $\mathbb{N} := \{0,1,2,\dots\}$ and $\mathbb{N}^+ := \{1,2,3,\dots\}$ denote the sets of non-negative and positive integers, respectively. The notation $[a]$ denotes the integer part of $a\in\mathbb{R}$, and $C(\cdot)$ (similarly $C_1(\cdot)$, *etc.*) denotes a positive constant depending on the parameters within parentheses, whose value may vary across contexts.

There are three main theorems in this paper. The first one gives an observability inequality at two points in time for equation (1.1).

Theorem 1.1. *For every* $\nu\ge 0$*, let* A *and* B *be two measurable sets in* $\mathbb{R}^+$ *with* $0<|A|<\infty$ *and* $0<|B|<\infty$*, and let* $T>S\ge 0$*. Then there is a positive constant* $C=C(\nu,A,B,T-S)$*, such that for all* $u_0\in L^2(\mathbb{R}^+)$*, we* *have*

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \left(\int_{A^c} |u(x,S;u_0)|^2 \mathrm{d}x + \int_{B^c} |u(x,T;u_0)|^2 \mathrm{d}x \right). \tag{1.9}$$

In particular, in the following three cases, the constant $C(\nu, A, B, T-S)$ can be more explicit:

(i) If $\nu = \frac{k}{2}$, $k \in \mathbb{N}$, then there exists a positive constant $C = C(\nu)$ such that

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \mathrm{e}^{C \frac{\mu_\nu(A)\mu_\nu(B)}{(T-S)^{2(\nu+1)}}} \left(\int_{A^c} |u(x,S;u_0)|^2 \mathrm{d}x + \int_{B^c} |u(x,T;u_0)|^2 \mathrm{d}x \right). \tag{1.10}$$

(ii) If $|A||B| < C_\nu = \left(\frac{2\,\Gamma(2\nu)}{\Gamma(\nu+\frac{1}{2})} + 2^{\nu+1} \right)^{-2}$, then

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le 2 \left(\frac{2\sqrt{2C_\nu(T-S)} - \sqrt{|A||B|}}{\sqrt{2C_\nu(T-S)} - \sqrt{|A||B|}} \right)^2 \left(\int_{A^c} |u(x,S;u_0)|^2 \mathrm{d}x + \int_{B^c} |u(x,T;u_0)|^2 \mathrm{d}x \right). \tag{1.11}$$

(iii) If $A = (0,a]$, $B = (0,b]$, a, $b > 0$, then there exists a positive constant $C = C(\nu)$ such that

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \mathrm{e}^{C(1+\frac{ab}{T-S})} \left(\int_{(0,a]^c} |u(x,S;u_0)|^2 \mathrm{d}x + \int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right). \tag{1.12}$$

The second one gives a unique continuation inequality at one time point for equation (1.1), when the initial data have exponential decay at infinity. It is interesting that this class of initial data is consistent with the free case and independent of the constant ν.

Theorem 1.2. *For every $\nu \ge 0$, let λ, b, $T > 0$. Then the following conclusions hold:*

(i) There exist constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$ such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$, we have

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \left(1 + \frac{b^{2\nu+2}}{(\lambda T)^{2\nu+2}} \right) \left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{\theta^{1+b/(\lambda T)}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x} |u_0(x)|^2 \mathrm{d}x \right)^{1-\theta^{1+b/(\lambda T)}}. \tag{1.13}$$

(ii) There exists a constant $C = C(\nu) > 0$ such that for any $\beta > 1$ and $\gamma \in (0,1)$ and all $u_0 \in C_0^\infty(\mathbb{R}^+)$, we have

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \mathrm{e}^{\left(\frac{C^\beta b^\beta}{\lambda(1-\gamma)T^\beta} \right)^{1/(\beta-1)}} \left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right)^\gamma \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x^\beta} |u_0(x)|^2 \mathrm{d}x \right)^{1-\gamma}. \tag{1.14}$$

(iii) Let $\alpha(s)$, $s \in \mathbb{R}^+$, be an increasing function with $\lim_{s\to\infty} \alpha(s)/s = 0$. Then for each $\gamma \in (0,1)$, there is no positive constant C such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right)^\gamma \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda \alpha(x)} |u_0(x)|^2 \mathrm{d}x \right)^{1-\gamma}. \tag{1.15}$$

The third one gives another unique continuation inequality at one time point for equation (1.1).

Theorem 1.3. *For every $\nu \geq 0$, let $A = [a_1, a_2] \subset \mathbb{R}^+$ and $B = [b_1, b_2] \subset \mathbb{R}^+$, with $a := a_2 - a_1 > 0$ and $b := b_2 - b_1 > 0$, and let $\lambda, T > 0$. Then there exist constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$ such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$,*

$$\int_A |u(x,T;u_0)|^2 \mathrm{d}x \leq C(a_2^{2\nu+2} - a_1^{2\nu+2})((\lambda T) \wedge b)^{-(2\nu+2)} \left(\int_B |u(x,T;u_0)|^2 \mathrm{d}x\right)^{\theta^p} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x} |u_0(x)|^2 \mathrm{d}x\right)^{1-\theta^p} \tag{1.16}$$

with

$$p := 1 + \frac{|x_0 - x_1| + \frac{a}{2} + \frac{b}{2}}{(\lambda T) \wedge \frac{b}{2}},$$

where x_0 and x_1 are the centers of A and B, respectively.

We make the following remarks related to the above three theorems:

($\mathbf{a}_1$) Theorem 1.1 can be explained in the following two perspectives. From the unique continuation perspective, Theorem 1.1 is a unique continuation inequality at two time points for equation (1.1). From (1.9), we find that

$$u(x,S;u_0) = 0 \text{ on } A^c, \quad u(x,T;u_0) = 0 \text{ on } B^c \implies u(x,t;u_0) = 0 \text{ on } \mathbb{R}^+ \times [0,\infty).$$

From the observability perspective, Theorem 1.1 is an observability inequality at two time points for equation (1.1). Observing a solution at two different points at time, each time outside a set of finite measure, one can recover the solution.
In addition, we elaborate on the sharpness of Theorem 1.1 in the following sense: First, one cannot recover the solution by observing it at two different points in time if one observation is made outside a bounded interval and the other inside (see Thm. 5.1(i)). Second, for any $T > 0$, one cannot recover the solution by observing it at one point in time outside a bounded interval and another point within the time interval $[0,T]$ inside a bounded interval (see Thm. 5.1(ii)). Third, one cannot recover the solution by observing it at one point in time over a subset $A \subset \mathbb{R}^+$ if $|A^c| > 0$ (see Thm. 5.1(iii)).

($\mathbf{a}_2$) Theorem 1.2 is a unique continuation inequality at one time point for equation (1.1).
From (1.13), we find that

$$\mathrm{e}^{\lambda x/2} u_0(x) \in L^2(\mathbb{R}^+), \quad u(x,T;u_0) = 0 \text{ on } (0,b]^c \implies u(x,t;u_0) = 0 \text{ on } \mathbb{R}^+ \times [0,\infty).$$

From (1.14), we find that when $\beta > 1$,

$$\mathrm{e}^{\lambda x^\beta/2} u_0(x) \in L^2(\mathbb{R}^+), \quad u(x,T;u_0) = 0 \text{ on } (0,b]^c \implies u(x,t;u_0) = 0 \text{ on } \mathbb{R}^+ \times [0,\infty).$$

In addition, we elaborate on the sharpness of Theorem 1.2 in the following sense: First, when $\beta \in (0,1)$, there are no constants $C > 0$ and $\gamma \in (0,1)$ such that the inequality (1.14) holds (see (1.15)). Hence, when we expect by observing solutions at one time point and outside a bounded interval, we have requirements for the decay rate of the initial data. Second, when $(0,b]^c$ is replaced by $(0,b]$, (1.13) and (1.14) do not hold (see Thm. 5.3(i)). So even though the initial data have any exponential decay at infinity, we cannot expect to recover the solution by observing it at one time point and inside a bounded interval.

($\mathbf{a}_3$) Theorem 1.3 is a unique continuation inequality at one time point for equation (1.1). From (1.16), we find that

$$e^{\lambda x/2}u_0(x) \in L^2(\mathbb{R}^+), \quad u(x,T;u_0) = 0 \text{ on } B = [b_1, b_2] \implies u(x,t;u_0) = 0 \text{ on } \mathbb{R}^+ \times [0,\infty).$$

For unique continuation properties of Schrödinger equations, we refer the readers to [31–34] and the references therein.

There are three consequences of the above main theorems.

Theorem 1.4. *For every $\nu \geq 0$, let b, $T > 0$. Then the following conclusions hold:*

(i) There exists a constant $C = C(\nu) > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |u_0(x)|^2 dx \leq Ce^{C\frac{b^2}{T}} \int_0^T \int_{(0,b]^c} |u(x,t;u_0)|^2 dxdt. \tag{1.17}$$

Moreover, the growth rate of the cost constant $Ce^{C\frac{b^2}{T}}$ is sharp as $T \to 0$: For every $0 < T \leq \frac{b^2}{2}$, there exists $u_T \in L^2(\mathbb{R}^+)$ such that

$$c(\nu)e^{-\frac{b^2}{2T}} \leq \frac{\int_0^T \int_{(0,b]^c} |u(x,t;u_T)|^2 dxdt}{\int_{\mathbb{R}^+} |u_T(x)|^2 dx} \leq \widetilde{C}(\nu)e^{-\frac{b^2}{8T}} \tag{1.18}$$

for some positive constants $c(\nu)$, $\widetilde{C}(\nu)$.

(ii) If the initial data u_0 satisfies $\operatorname{supp} u_0 \subset [0, N]$ for some $N > 0$, then the solution can be observed outside the interval $(0, b]$ at a fixed time. Precisely, we have

$$\int_{\mathbb{R}^+} |u_0(x)|^2 dx \leq e^{C\left(1+\frac{bN}{T}\right)} \int_{(0,b]^c} |u(x,T;u_0)|^2 dx, \tag{1.19}$$

where $C = C(\nu) > 0$.

Theorem 1.5. *For every $\nu \geq 0$, let $B = [b_1, b_2] \subset \mathbb{R}^+$ with $b := b_2 - b_1 > 0$, and let $\lambda_1, \lambda_2, T > 0$. Then there exists a constant $C = C(\nu) > 0$ such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$ and $\varepsilon \in (0, 1)$,*

$$\int_{\mathbb{R}^+} e^{-\lambda_2 x}|u(x,T;u_0)|^2 dx \leq C(x_0, b, \lambda_1, \lambda_2, T)\left(\varepsilon \int_{\mathbb{R}^+} e^{\lambda_1 x}|u_0(x)|^2 dx + \varepsilon e^{\varepsilon^{-1-\frac{C\lambda_2^{-1}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \int_B |u(x,T;u_0)|^2 dx\right), \tag{1.20}$$

where $C(x_0, b, \lambda_1, \lambda_2, T) := \exp\left\{C\left(1 + \frac{x_0+\frac{b}{2}+\lambda_2^{-1}}{(\lambda_1 T)\wedge\frac{b}{2}}\right)\right\}$, x_0 is the center of B.

It is of interest to consider what happens when we replace $\int_{\mathbb{R}^+} e^{-\lambda_2 x}|u(x,T;u_0)|^2 dx$ with $\int_{\mathbb{R}^+} |u(x,T;u_0)|^2 dx$. The following conclusion provides an answer to this question.

Theorem 1.6. *For every $\nu \geq 0$, let $B = [b_1, b_2] \subset \mathbb{R}^+$ with $b := b_2 - b_1 > 0$, and let $\lambda, T > 0$. Then there exists a constant $C = C(\nu) > 0$ such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$ and $\varepsilon \in (0, 1)$,*

$$\begin{aligned}\int_{\mathbb{R}^+} |u_0(x)|^2 dx \leq C(x_0, b, \lambda, T)\Bigg(\varepsilon\left(\int_{\mathbb{R}^+} e^{\lambda x}|u_0(x)|^2 dx + \|u_0\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)} + \int_{\mathbb{R}^+} \frac{1}{x^{4([\nu]+3)}}|u_0|^2 dx\right) \\ + \varepsilon e^{e^{\varepsilon^{-2}}} \int_B |u(x,T;u_0)|^2 dx\Bigg),\end{aligned} \tag{1.21}$$

where $C(x_0,b,\lambda,T) := \left(T+\frac{1}{T}\right)^{[\nu]+3}(1+T)^{4([\nu]+3)}\exp\left(C^{1+\frac{x_0+\frac{b}{2}+1}{(\lambda T)\wedge\frac{b}{2}}}\right)$, x_0 *is the center of* B *and* $\|\cdot\|_{H^{4([\nu]+3)}(\mathbb{R}^+)}$ *denotes the standard Sobolev norm on the positive real line.*

We make the following remarks related to the above three theorems:

($\mathbf{b}_1$) Conclusion (i) of Theorem 1.4 establishes that, for the domain $\Omega = \{x \in \mathbb{R}^+ : x \geq b\}$, the standard observability inequality (1.3) holds for solutions $u(x,t)$ of equation (1.1), with a sharp constant. We emphasize that the quantitative constant obtained in [35] is of the form $\mathrm{e}^{\frac{C}{T^2}}$, which grows faster than $C\mathrm{e}^{C\frac{b^2}{T}}$ as $T \to 0$. Another interesting question is to determine the sharp geometric condition on the subset Ω for (1.3) to hold. In [36], we proved that (1.3) holds if and only if Ω is thick (as defined in [18]). For related results on bounded domains in $\mathbb{R}^n$, we refer the readers to [37] and references therein, which address exact observability estimates for Schrödinger equations with inverse-square potentials.

($\mathbf{b}_2$) While Theorem 5.1 (iii) shows that recovery from observation at a single point in time over the domain $\Omega = \{x \in \mathbb{R}^+ : x \geq b\}$ is impossible for all initial data $u_0 \in L^2(\mathbb{R}^+)$, conclusion (ii) of Theorem 1.4 demonstrates that such recovery is possible at a fixed time for initial data $u_0 \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} u_0 \subset [0,N]$.

($\mathbf{b}_3$) The inequalities in Theorems 1.5–1.6 are different kinds of unique continuation at one time point for the Schrödinger equation. From the observability perspective, these theorems demonstrate that given supplementary information, the solution of (1.1) can be recovered on a bounded interval at a fixed time. They correspond to different kinds of controllability, which are nonstandard (see Sect. 6.2).

Finally, we would like to emphasize that the approach presented in this paper is primarily based on the research in [18, 23, 28]. Technically, thanks to the identity (1.7), we need to transition from harmonic analysis based on the Fourier transform to harmonic analysis associated with the Hankel transform, which will introduce new difficulties. For example, to prove observability inequalities at two time points similar to (1.4), we need to prove a Nazarov-type uncertainty principle (see Lem. 3.4 below) for the Hankel transform F_ν. To prove Theorems 1.2–1.3, we need to establish the interpolation inequality stated in Lemma 3.11. To prove Theorem 1.6, we need a regularity estimate Lemma 4.2 for equation (1.1). And due to the complexity of proving Theorems 1.4–1.6, for the sake of completeness, we provide all the details of their proof. To the best of our knowledge, no identity analogous to (1.7) exists in higher dimensions, which limits the scope of this paper to the one-dimensional case; see Remarks 3.6 and 3.7. We look forward to exploring higher-dimensional Schrödinger equations with inverse-square potentials in our future work.

Plan of the paper. The rest of the paper is organized as follows: Section 2 provides some preliminaries, which will be used throughout the proofs of our theorems. Section 3 gives the proofs of Theorems 1.1–1.3. Section 4 presents the proofs of Theorems 1.4–1.6. Section 5 demonstrates the sharpness of Theorems 1.1 and 1.2. Section 6 is devoted to applications concerning the controllability of the Schrödinger equation.

2. Preliminaries

In this section, we provide further remarks on the Friedrichs extensions H_ν and present the proof of the identity (1.7). Our equation (1.1) involves the following family of differential operators:

$$H_\alpha = -\partial_x^2 + \frac{\alpha}{x^2}, \;\; \alpha \geq -\frac{1}{4}. \tag{2.1}$$

The well-known classical Hardy inequality (see [38]) says, for every $u \in C_0^\infty(\mathbb{R}^+)$:

$$\frac{1}{4}\int_0^\infty \frac{|u(x)|^2}{x^2}\mathrm{d}x \leq \int_0^\infty |u'(x)|^2\mathrm{d}x, \tag{2.2}$$

and $\frac{1}{4}$ is the best constant.

This inequality shows that the operator $-\partial_x^2 + \frac{\alpha}{x^2}$ is form-bounded from below on $C_0^\infty(\mathbb{R}^+)$ if and only if $\alpha \geq -\frac{1}{4}$. Consequently, a Friedrichs extension of this operator exists. Following [8], we write $\alpha = \nu^2 - \frac{1}{4}$ with $\nu \geq 0$. For $\nu \geq 1$, the differential operator $L_\nu = -\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}$, with domain $C_0^\infty(\mathbb{R}^+)$, is essentially self-adjoint. We denote its closure as H_ν, which corresponds to the Friedrichs extension of L_ν. However, for $0 \leq \nu < 1$, the operator L_ν is not essentially self-adjoint. When $0 < \nu < 1$, this operator has exactly two distinct homogeneous extensions, the Friedrichs and Krein extensions of L_ν, denoted by H_ν and $H_{-\nu}$, respectively. When $\nu = 0$, the Friedrichs and Krein extensions of L_0 coincide; we denote this common extension by H_0. In this paper, we only consider the Friedrichs extension H_ν of $-\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}$ on $C_0^\infty(\mathbb{R}^+)$. This extension is equivalent to the self-adjoint extension of $-\partial_x^2 + (\nu^2 - \frac{1}{4})\frac{1}{x^2}$ in $L^2(\mathbb{R}^+)$, with the Dirichlet boundary condition at $x = 0$. For the general theory of the Friedrichs extension, we refer the readers to [39], Section X.3.

Next, we will prove the identity (1.7). We begin by introducing the Hankel transform F_ν. We define the Hankel transform $F_\nu(f)$ of order ν ($\nu \geq 0$) of any function $f \in L^1(\mathbb{R}^+)$ by:

$$F_\nu(f)(x) := \int_0^\infty \sqrt{xy} J_\nu(xy) f(y) \mathrm{d}y, \quad x \in \mathbb{R}^+, \tag{2.3}$$

where

$$J_\nu(z) := \sum_{k=0}^\infty \frac{(-1)^k}{k! \, \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{\nu + 2k}$$

is the Bessel function of the first kind. The Plancherel theorem for the Hankel transform states that F_ν extends to an isometric isomorphism of $L^2(\mathbb{R}^+)$ onto itself. And the inverse Hankel transform has the symmetric form (see, for example, [40])

$$f(x) := \int_0^\infty \sqrt{xy} J_\nu(xy) F_\nu(f)(y) \mathrm{d}y, \quad x \in \mathbb{R}^+.$$

The spectral theory of the Schrödinger operator H_ν has been studied explicitly in [8], and the authors provide an explicit spectral representation of H_ν in terms of the Hankel transform F_ν, as presented in the following lemma.

Lemma 2.1. *[8], Theorem 5.2 Let $\nu \geq 0$. The Hankel transform F_ν is a unitary involution on $L^2(0, \infty)$ diagonalizing the operator H_ν, more precisely*

$$F_\nu H_\nu F_\nu^{-1} = Q^2, \tag{2.4}$$

where the multiplication operator Q^2 is defined by $Q^2 f(x) = x^2 f(x)$.

With Lemma 2.1 in hand, we can derive the integral representation of the unitary group e^{-itH_ν}. The proof of the following result can be found in the proof of Theorem 2.4 in [41], which establishes a dispersive estimate for equation (1.1).

Lemma 2.2. *For every $\nu \geq 0$, let $t > 0$. Then for every $f \in L^2(\mathbb{R}^+)$, we have*

$$\begin{aligned}(\mathrm{e}^{-itH_\nu} f)(x) &= \frac{1}{2it} \int_0^\infty \sqrt{xy} J_\nu\left(\frac{xy}{2t}\right) \mathrm{e}^{-\frac{x^2+y^2}{4it}} \mathrm{e}^{\frac{-i\nu\pi}{2}} f(y) \mathrm{d}y \\ &= (2t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{i(\nu+1)\pi}{2}} \mathrm{e}^{-\frac{x^2}{4it}} F_\nu(\mathrm{e}^{-\frac{y^2}{4it}} f(y))(x/2t).\end{aligned} \tag{2.5}$$

Proof. Let $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ with compact support. By the spectral theorem in [42], Theorem 3.1

$$(\mathrm{e}^{-itH_\nu} f)(x) = \lim_{\varepsilon \to 0+} (\mathrm{e}^{-(\varepsilon+it)H_\nu} f)(x).$$

In view of (2.4), we thus get

$$\begin{aligned}
\lim_{\varepsilon \to 0+} (\mathrm{e}^{-(\varepsilon+it)H_\nu} f)(x) &= \lim_{\varepsilon \to 0^+} \left(F_\nu^{-1} \left(\mathrm{e}^{-(\varepsilon+it)p^2} F_\nu f(p) \right) \right)(x) \\
&= \lim_{\varepsilon \to 0+} \int_0^\infty \sqrt{xp} J_\nu(xp) \mathrm{e}^{-(\varepsilon+it)p^2} F_\nu(f)(p) dp \\
&= \lim_{\varepsilon \to 0^+} \int_0^\infty \sqrt{xp}\, J_\nu(xp)\, \mathrm{e}^{-(\varepsilon+it)p^2} \left(\int_0^\infty \sqrt{py}\, J_\nu(py)\, f(y)\, \mathrm{d}y \right) dp \\
&= \lim_{\varepsilon \to 0^+} \int_0^\infty \sqrt{x} \left(\int_0^\infty \sqrt{p}\, J_\nu(xp)\, \mathrm{e}^{-(\varepsilon+it)p^2} \sqrt{py}\, J_\nu(py)\, dp \right) f(y)\, \mathrm{d}y \\
&= \lim_{\varepsilon \to 0+} \frac{1}{2(\varepsilon+it)} \int_0^\infty \sqrt{xy} I_\nu \left(\frac{xy}{2(\varepsilon+it)} \right) \mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+it)}} f(y) \mathrm{d}y,
\end{aligned} \tag{2.6}$$

where $I_\nu(z) = \mathrm{e}^{-i\nu\pi/2} J_\nu(iz)$ is the modified Bessel function of the first kind (see [43], Eq. (9.6.3)). The last equality follows from the following equation in [44], p. 51,

$$F_\nu \left(\sqrt{x}\, \mathrm{e}^{-\beta x^2} J_\nu(ax) \right)(y) = \frac{\sqrt{y}}{2\beta} \exp\left(-\frac{a^2+y^2}{4\beta} \right) I_\nu \left(\frac{ay}{2\beta} \right), \quad \text{for } \operatorname{Re}\beta > 0,\ \operatorname{Re}\nu > -1. \tag{2.7}$$

Here, Re denotes the real part of a complex number. Moreover, from [43], equation (9.6.18), it follows that the function $I_\nu \left(\frac{xy}{2(\varepsilon+it)} \right) \mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+it)}}$ is bounded on every compact interval uniformly with respect to $\varepsilon > 0$. Since the support of f is compact, we can use the dominated convergence theorem and interchange the limit and integration in (2.6). Taking the limit $\varepsilon \to 0$ and using the identity $I_\nu(z) = \mathrm{e}^{-i\nu\pi/2} J_\nu(iz)$, and then using the definition of the Hankel transform F_ν, we obtain (2.5) for $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ with compact support. For $f \in L^2(\mathbb{R}^+)$, equation (2.5) is understood in the following sense:

$$(\mathrm{e}^{-itH_\nu} f)(x) = \frac{1}{2it} \lim_{R \to \infty} \int_0^R \sqrt{xy} J_\nu \left(\frac{xy}{2t} \right) \mathrm{e}^{-\frac{x^2+y^2}{4it}} \mathrm{e}^{\frac{-i\nu\pi}{2}} f(y) \mathrm{d}y,$$

where the limit is taken in the L^2-norm as R tends to infinity (see also [39], p. 11). Thus, (2.5) is established for all functions in $L^2(\mathbb{R}^+)$. This ends the proof of Lemma 2.2. □

To proceed, we need to introduce the modified Hankel transform M_ν. For $1 \le p < \infty$ and $\nu \ge 0$, we denote by $L^p_\nu(\mathbb{R}^+)$ the Banach space consisting of measurable functions f on $\mathbb{R}^+$ equipped with the norm

$$\|f\|_{L^p_\nu(\mathbb{R}^+)} := \left(\int_0^\infty |f(x)|^p \,\mathrm{d}\mu_\nu(x) \right)^{1/p},$$

where $\mathrm{d}\mu_\nu(x) = x^{2\nu+1}\mathrm{d}x$. For $f \in L^1_\nu(\mathbb{R}^+)$, the modified Hankel transform is defined by

$$M_\nu(f)(x) := \int_0^\infty (xy)^{-\nu} J_\nu(xy) f(y) \mathrm{d}\mu_\nu(y), \quad x \in \mathbb{R}^+.$$

It is well known that M_ν extends to an isometric isomorphism of $L^2_\nu(\mathbb{R}^+)$ onto itself, with symmetric inverse: $M_\nu^{-1} = M_\nu$ (see, for example, [45]). It follows directly from the definitions of F_ν and M_ν that

$$F_\nu(f)(x) = x^{\nu+\frac{1}{2}} M_\nu\big(y^{-\nu-\frac{1}{2}} f(y)\big)(x), \quad x \in \mathbb{R}^+. \tag{2.8}$$

3. Proofs of main theorems

In this section, we will prove Theorems 1.1–1.3.

3.1. Proof of Theorem 1.1

This subsection is devoted to proving Theorem 1.1. We start with introducing the uncertainty principle of the modified Hankel transform M_ν. Next, by the relationship (2.8), we get the uncertainty principle of the Hankel transform F_ν in Lemma 3.4. Then, in Lemma 3.5, we show the equivalence between the uncertainty principle and the observability at two points in time. Finally, we give the proof of Theorem 1.1.

Theorem 3.1. *[46] Theorem 4.3. Let A, B be a pair of measurable subsets of $\mathbb{R}^+$ with $0 < |A|, |B| < \infty$. Then there is a positive constant $C = C(\nu, A, B)$ such that for each $f \in L^2_\nu(\mathbb{R}^+)$,*

$$\|f\|^2_{L^2_\nu(\mathbb{R}^+)} \leq C \left(\|f\|^2_{L^2_\nu(A^c)} + \|M_\nu(f)\|^2_{L^2_\nu(B^c)} \right). \tag{3.1}$$

We give two remarks on the constant $C = C(\nu, A, B)$.

Remark 3.2. It is worth noting that when $\nu = \frac{n}{2} - 1$ with $n = 2, 3, 4, \cdots$, the constant in (3.1) has the following explicit form:

$$C(\nu, A, B) = C\mathrm{e}^{C\mu_\nu(A)\mu_\nu(B)},$$

where $C = C(\nu)$ is a positive constant. In fact, recall that the Fourier transform is defined for $f \in L^1(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-ix\cdot\xi} \mathrm{d}x \tag{3.2}$$

and extended to $L^2(\mathbb{R}^n)$ in the usual way. Thus, if $f(x) = g(|x|)$ is a radial function on $\mathbb{R}^n$, then $\mathcal{F}(f)(\xi) = M_{\frac{n}{2}-1}(g)(|\xi|)$. As for the Fourier transform, estimate (1.6) holds for $f \in L^2(\mathbb{R}^n)$ with a constant

$$C(n, A_n, B_n) = C(n)\mathrm{e}^{C(n)|A_n||B_n|},$$

where $A_n, B_n \subset \mathbb{R}^n$ are sets of finite Lebesgue measure. Now if we define A and B as

$$A_n = \{x \in \mathbb{R}^n : |x| \in A\} \quad \text{and} \quad B_n = \{x \in \mathbb{R}^n : |x| \in B\},$$

then there exists a constant $C = C(n) > 0$ such that

$$\|g\|^2_{L^2_{n/2-1}(\mathbb{R}^+)} \leq C\mathrm{e}^{C\mu_{n/2-1}(A)\mu_{n/2-1}(B)} \left(\|g\|^2_{L^2_{n/2-1}(A^c)} + \|M_{n/2-1}(g)\|^2_{L^2_{n/2-1}(B^c)} \right).$$

Remark 3.3. Unfortunately, for general $\nu \geq 0$, Theorem 3.1 does not yield an explicit constant for arbitrary finite measure sets A and B. However, when the Lebesgue measures of A and B are sufficiently small, an explicit estimate is available. More precisely, inspired by Lemma 4.2 in [46], if $|A||B| < k_\nu^{-2}$, then the constant in (3.1) can be taken as

$$C(\nu, A, B) = 2\left(1 + \frac{1}{1 - k_\nu\sqrt{|A||B|}}\right)^2,$$

where k_ν is a positive constant such that the following decay estimate holds:

$$\left|\frac{J_\nu(x)}{x^\nu}\right| \le k_\nu x^{-\nu-\frac{1}{2}}. \tag{3.3}$$

We claim that Theorem 3.1 is also true for the Hankel transform F_ν.

Lemma 3.4. *Let A, B be a pair of measurable subsets of $\mathbb{R}^+$ with $0 < |A|, |B| < \infty$. Then there is a positive constant $C = C(\nu, A, B)$ such that for each $f \in L^2(\mathbb{R}^+)$,*

$$\|f\|^2_{L^2(\mathbb{R}^+)} \le C\left(\|f\|^2_{L^2(A^c)} + \|F_\nu(f)\|^2_{L^2(B^c)}\right). \tag{3.4}$$

Proof. Let $T_\nu f(x) := x^{-\nu-\frac{1}{2}} f(x)$. Then for all $f \in L^2(\mathbb{R}^+)$, we have

$$\|T_\nu f\|^2_{L^2_\nu(\mathbb{R}^+)} = \|f\|^2_{L^2(\mathbb{R}^+)}.$$

Define $g(x) := T_\nu f(x) \in L^2_\nu(\mathbb{R}^+)$. Applying Theorem 3.1 to g, we obtain

$$\|g\|^2_{L^2_\nu(\mathbb{R}^+)} \le C\left(\|g\|^2_{L^2_\nu(A^c)} + \|M_\nu(g)\|^2_{L^2_\nu(B^c)}\right). \tag{3.5}$$

Moreover, we observe that

$$\|g\|^2_{L^2_\nu(\mathbb{R}^+)} = \|f\|^2_{L^2(\mathbb{R}^+)}, \quad \|g\|^2_{L^2_\nu(A^c)} = \|f\|^2_{L^2(A^c)}.$$

Using the identity (2.8), we also have

$$\|M_\nu(g)\|^2_{L^2_\nu(B^c)} = \|x^{-\nu-\frac{1}{2}} F_\nu(f)\|^2_{L^2_\nu(B^c)} = \|F_\nu(f)\|^2_{L^2(B^c)}.$$

Inserting the identities above into (3.5) completes the proof of (3.4). □

Lemma 3.5. *For every $\nu \ge 0$, let A and B be measurable subsets of $\mathbb{R}^+$. Then the following statements are equivalent:*
(i) There exists a positive constant $C_1(\nu, A, B)$ such that for each $f \in L^2(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x \le C_1(\nu, A, B)\left(\int_A |f(x)|^2 \mathrm{d}x + \int_B |F_\nu(f)(x)|^2 \mathrm{d}x\right). \tag{3.6}$$

(ii) There exists a positive constant $C_2(\nu, A, B)$ such that for each $T > 0$ and each $u_0 \in L^2(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C_2(\nu, A, B)\left(\int_A |u_0(x)|^2 \mathrm{d}x + \int_{2TB} |u(x, T; u_0)|^2 \mathrm{d}x\right). \tag{3.7}$$

Furthermore, when one of the above two statements holds, the constants $C_1(\nu, A, B)$ and $C_2(\nu, A, B)$ can be chosen to be the same number.

Proof. The proof of this lemma is the same as the proof of Lemma 2.3 in [23]. We just mention that one needs to replace formula (2.6) in [23] with formula (2.5) in this paper. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $T > S \geq 0$ and let A, B be measurable sets in $\mathbb{R}^+$ with finite measure. By Lemma 3.4, We have (3.6) with (A, B) replaced by $\left(A^c, \frac{B^c}{2(T-S)}\right)$ and $C_1(\nu, A, B)$ replaced by $C\left(\nu, A, \frac{B}{2(T-S)}\right)$, where $C\left(\nu, A, \frac{B}{2(T-S)}\right)$ is given in (3.4). Thus, we can apply Lemma 3.5 to get (3.7) with (A, B) replaced by $\left(A^c, \frac{B^c}{2(T-S)}\right)$ and $C_2(\nu, A, B)$ replaced by $C\left(\nu, A, \frac{B}{2(T-S)}\right)$. So we have

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq C\left(\nu, A, \frac{B}{2(T-S)}\right)\left(\int_{A^c} |u_0(x)|^2 \mathrm{d}x + \int_{B^c} |u(x, T-S; u_0)|^2 \mathrm{d}x\right). \tag{3.8}$$

Finally, by (3.8), we get

$$\int_{\mathbb{R}^+} |u(x, S; u_0)|^2 \mathrm{d}x \leq C\left(\nu, A, \frac{B}{2(T-S)}\right)\left(\int_{A^c} |u(x, S; u_0)|^2 \mathrm{d}x + \int_{B^c} |u(x, T; u_0)|^2 \mathrm{d}x\right).$$

By the conservation law for the Schrödinger equation, we get the inequality (1.9) in Theorem 1.1.

The conclusions (i) and (ii) of Theorem 1.1 follow directly from Lemma 3.4 and Lemma 3.5, in combination with Remark 3.2 and Remark 3.3, respectively. For (ii), we only need to give an estimate of the constant k_ν appearing in inequality (3.3). In fact, it is well-known that the Bessel function has the following estimate (see [47], Appendix B.7). When $\nu > 1/2$, we have

$$|J_\nu(x)| \leq 2\frac{(x/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} 2^{\nu - \frac{3}{2}}\left(\frac{\Gamma(2\nu)}{x^{2\nu}} + 2^\nu \frac{\Gamma(\nu + \frac{1}{2})}{x^{\nu + \frac{1}{2}}}\right).$$

When $1/2 \geq \nu \geq 0$, we have

$$|J_\nu(x)| \leq 2\frac{(x/2)^\nu}{x^{\nu + \frac{1}{2}}\Gamma(\frac{1}{2})}.$$

These estimates yield that for $\nu \geq 0$ and $x \geq 1$, we get

$$|J_\nu(x)| \leq 2\left(\frac{\Gamma(2\nu)}{\Gamma(\nu + \frac{1}{2})} + 2^\nu\right) x^{-\frac{1}{2}}. \tag{3.9}$$

For $\nu \geq 0$ and $0 < x < 1$, by (3.23) below, we know that

$$|J_\nu(x)| \leq x^{-\frac{1}{2}}.$$

This, along with (3.9) indicates that we can take $k_\nu = 2\left(\frac{\Gamma(2\nu)}{\Gamma(\nu + \frac{1}{2})} + 2^\nu\right)$.

Finally, we prove the conclusion (iii) of Theorem 1.1. The proof presented here is inspired by the approach used in [28], Theorem 1.1. Let $U(t) = \mathrm{e}^{-itH_\nu}$ be the unitary group given by the equation (1.1). In other words, the solution of (1.1) can be written as

$$u(x,t) = U(t)u_0(x), \quad x \in \mathbb{R}^+, \ t \in [0,\infty).$$

Then, we have

$$\int_{\mathbb{R}^+} |u(x,T-S)|^2 \mathrm{d}x \le 2\int_{\mathbb{R}^+} |U(T-S)(\chi_{(0,a]}(x)u_0(x))|^2 \mathrm{d}x + 2\int_{\mathbb{R}^+} |U(T-S)(\chi_{(0,a]^c}(x)u_0(x))|^2 \mathrm{d}x. \tag{3.10}$$

For the first term, by Theorem 1.4(ii) (the theorem will be proved later), we get

$$\begin{aligned}
&\int_{\mathbb{R}^+} |U(T-S)(\chi_{(0,a]}(x)u_0(x))|^2 \mathrm{d}x \\
&\le \mathrm{e}^{C(1+\frac{ab}{T-S})}(\int_{(0,b]^c} |U(T-S)(\chi_{(0,a]}(x)u_0(x))|^2 \mathrm{d}x) \\
&\le 2\mathrm{e}^{C(1+\frac{ab}{T-S})}\left(\int_{(0,b]^c} |u(x,T-S)|^2 \mathrm{d}x + \int_{(0,b]^c} |U(T-S)(\chi_{(0,a]^c}(x)u_0(x))|^2 \mathrm{d}x\right),
\end{aligned} \tag{3.11}$$

where $C = C(\nu) > 0$.
Inserting (3.11) into (3.10), we have

$$\int_{\mathbb{R}^+} |u(x,T-S)|^2 \mathrm{d}x \le C\mathrm{e}^{C(1+\frac{ab}{T-S})}\left(\int_{(0,b]^c} |u(x,T-S)|^2 \mathrm{d}x + \int_{\mathbb{R}^+} |U(T-S)(\chi_{(0,a]^c}(x)u_0(x))|^2 \mathrm{d}x\right). \tag{3.12}$$

By (3.12) and the conservation law, we have

$$\int_{\mathbb{R}^+} |u(x,T)|^2 \mathrm{d}x \le C\mathrm{e}^{C(1+\frac{ab}{T-S})}\left(\int_{(0,b]^c} |u(x,T)|^2 \mathrm{d}x + \int_{(0,a]^c} |u(x,S)|^2 \mathrm{d}x\right).$$

Thus, the above leads to (1.12) and ends the proof of (iii). □

Remark 3.6. For a subset $A \subset \mathbb{R}^+$, we denote $A^2 = \{x^2 : x \in A\}$. We can reformulate the uncertainty principle (3.4) as an estimate involving two projection operators on $L^2(\mathbb{R}^+)$; that is, there exists a positive constant $C = C(\nu, A, B)$ such that for each $u \in L^2(\mathbb{R}^+)$,

$$\|u\|^2_{L^2(\mathbb{R}^+)} \le C\left(\|\chi_{A^c}(x)u\|^2_{L^2(\mathbb{R}^+)} + \|\chi_{(B^c)^2}(H_\nu)u\|^2_{L^2(\mathbb{R}^+)}\right), \tag{3.13}$$

where $\chi_{(B^c)^2}(H_\nu)$ denotes the spectral projection operator associated with H_ν, defined *via* the spectral theorem [48], Theorem VIII.4 and Lemma 2.1 as

$$\chi_{(B^c)^2}(H_\nu)\, u = F_\nu\left(\chi_{(B^c)^2}(\xi^2)F_\nu(u)(\xi)\right). \tag{3.14}$$

Now, it is straightforward to see that

$$\begin{aligned}
\|\chi_{(B^c)^2}(H_\nu)u\|^2_{L^2(\mathbb{R}^+)} &= \|F_\nu\left(\chi_{(B^c)^2}(\xi^2)F_\nu(u)(\xi)\right)\|^2_{L^2(\mathbb{R}^+)} \\
&= \|\chi_{(B^c)^2}(\xi^2)F_\nu(u)(\xi)\|^2_{L^2(\mathbb{R}^+)} \\
&= \|F_\nu(u)\|^2_{L^2(B^c)}.
\end{aligned}$$

In [27], an uncertainty estimate was established for Schrödinger operator with inverse-square potentials in dimensions $n \geq 3$. Precisely, when $n \geq 3$, Corollary 2.4 in [27] states that for $H = -\Delta + V$ with $V = \frac{\lambda}{|x|^2}$, and $\lambda > -\frac{(n-2)^2}{4}$, then for any $R > 0$, there exist constants $C > 0$ and $\delta > 0$ such that

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leq C\left(\|\chi_{A^c}(|x|)u\|^2_{L^2(\mathbb{R}^n)} + \|\chi_{(B^c)^2}(H)u\|^2_{L^2(\mathbb{R}^n)}\right), \quad \forall u \in L^2(\mathbb{R}^n), \tag{3.15}$$

where $A = (0, R]$, $B = (0, \sqrt{\delta}R^{-1}]$. Compared (3.13) with (3.15), we can see that (3.15) is restricted to the subcritical Hardy potential and the special case $A = (0, r_1]$, $B = (0, r_2]$, with $r_1 = R$ and $r_2 = \sqrt{\delta}R^{-1}$, indicating that r_1 and r_2 are not independent. However, (3.13) allows for any measurable subset A, $B \subset \mathbb{R}^+$ with finite Lebesgue measure, including the critical Hardy potential. Furthermore, while taking $A = (0, r_1]$, $B = (0, r_2]$ in (3.13), where r_1 and r_2 can be taken independently, the constant C in (3.13) has the explicit form $C(\nu)\mathrm{e}^{C(\nu)r_1 r_2}$, as seen in the conclusion (iii) of Theorem 1.1. In contrast, the constant in their high-dimensional estimate (3.15) is less precise, since there is an implicit parameter δ. Finally, we should point out that it is unknown whether an observability inequality at two time points can be directly deduced from the uncertainty estimate (3.15) for the Schrödinger equation with inverse-square potentials in higher dimensions.

Remark 3.7. We now make additional remarks regarding the two-dimensional case. Consider the operator $-\Delta + \frac{a}{|x|^2}$, defined on $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ for $a > 0$, and let $\mathcal{L}_a$ denote its Friedrichs extension, as studied in [49, 50]. For technical reasons, the methods in [27] fail to establish an analogue of the estimate (3.15) for $\mathcal{L}_a$. Indeed, the proof of the uncertainty estimate (3.15) in [27] relies on two key tools: First, a Hardy-type inequality:

$$\||x|^{-1}(-\Delta)^{-\frac{1}{2}}\|_{L^2 \to L^2} \leq C, \tag{3.16}$$

and second, the L^2-boundedness of the Riesz-type operator:

$$\|(-\Delta)^{\frac{1}{2}} H^{-\frac{1}{2}}\|_{L^2 \to L^2} \leq C,$$

where C denotes a generic constant. While the L^2-boundedness of the analogous operator $(-\Delta)^{\frac{1}{2}}\mathcal{L}_a^{-\frac{1}{2}}$ still holds (see [50], Prop. 1.4), the Hardy-type inequality (3.16) fails in two dimensions (see, for example, [50], Rem. 1.6 and [5]). Therefore, proving the two-dimensional analogue of (3.15) requires a more delicate analytical approach than in dimensions three and higher, as we have done in the one-dimensional case.

3.2. Proofs of Theorems 1.2 and 1.3

The proofs of Theorems 1.2 and 1.3 are mainly based on an interpolation inequality (see Lem. 3.11 below). To proceed, we first present the following two lemmas. Lemma 3.8 below is a modified version of Theorem 1.3 in [51].

Lemma 3.8. *Let f be analytic on $[a, b]$, with $b > a \geq 0$. Let E be a subinterval of $[a, b]$. Assume that there exist positive constants M and ρ such that*

$$\left|\frac{d^k f(x)}{dx^k}\right| \leq Mk!(\rho(b-a))^{-k}, \text{ for all } x \in [a, b],\ k \in \mathbb{N}.$$

Then there are constants $N = N(\rho, \nu, \frac{|E|}{b-a}) > 0$ and $\gamma = \gamma(\rho, \nu, \frac{|E|}{b-a}) \in (0, 1)$ such that

$$\|f\|_{L^\infty([a,b])} \leq N\left(\frac{1}{\mu_\nu(E)}\int_E |f| \mathrm{d}\mu_\nu(x)\right)^\gamma M^{1-\gamma}. \tag{3.17}$$

Proof. Define the linear transform $Ax = (b-a)x + a$ and let $G(x) = f \circ A(x)$. Set $E_0 = A^{-1}E$. Then $G(x)$ is analytic in $[0,1]$, and $|E_0| = \frac{|E|}{b-a}$. By Lemma 3.3 in [51], there exist constants $N = N(\rho, \frac{|E|}{b-a}) > 0$, and $\gamma = \gamma(\rho, \frac{|E|}{b-a}) \in (0,1)$ such that

$$\|f\|_{L^\infty([a,b])} \leq N(\|f\|_{L^\infty(E)})^\gamma M^{1-\gamma}. \tag{3.18}$$

Define

$$E_1 = \left\{ x \in E : \frac{|f(x)|}{2} \leq \frac{1}{\mu_\nu(E)} \int_E |f| \mathrm{d}\mu_\nu(x) \right\}.$$

Then we have

$$|E_1| \geq \frac{|E|}{4(\nu+1)}, \ \ \|f\|_{L^\infty(E_1)} \leq \frac{2}{\mu_\nu(E)} \int_E |f| \mathrm{d}\mu_\nu(x). \tag{3.19}$$

In fact,

$$\int_{E \backslash E_1} |f| \mathrm{d}\mu_\nu(x) \geq \int_{E\backslash E_1} \left(\frac{2}{\mu_\nu(E)} \int_E |f| \mathrm{d}\mu_\nu(x) \right) \mathrm{d}\mu_\nu(x) = \frac{2\mu_\nu(E\backslash E_1)}{\mu_\nu(E)} \int_E |f| \mathrm{d}\mu_\nu(x).$$

Then we obtain $\mu_\nu(E\backslash E_1) \leq \frac{\mu_\nu(E)}{2}$, which implies $\mu_\nu(E_1) \geq \frac{\mu_\nu(E)}{2}$. Since $E \subset [a,b]$ is an interval, denote $E = [a_1, b_1]$, we thus get

$$\int_{E_1} x^{2\nu+1} \mathrm{d}x \geq \frac{1}{2} \int_{[a_1,b_1]} x^{2\nu+1} \mathrm{d}x. \tag{3.20}$$

Since $x^{2\nu+1}$ is monotonically increasing with respect to x and $E_1 \subset [a_1, b_1]$, the left hand of (3.20) is controlled by $\int_{E_1} b_1^{2\nu+1} \mathrm{d}t$, *i.e.*,

$$\int_{E_1} b_1^{2\nu+1} \mathrm{d}x \geq \int_{E_1} x^{2\nu+1} \mathrm{d}x. \tag{3.21}$$

The right hand of (3.20) is equal to

$$\frac{b_1^{2\nu+2} - a_1^{2\nu+2}}{4(\nu+1)}. \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\int_{E_1} \mathrm{d}x \geq \frac{1}{4(\nu+1)} \frac{b_1^{2\nu+2} - a_1^{2\nu+2}}{b_1^{2\nu+1}}.$$

Noting that

$$\frac{a_1^{2\nu+2}}{b_1^{2\nu+1}} \leq \frac{a_1^{2\nu+2}}{a_1^{2\nu+1}},$$

we get

$$\int_{E_1} \mathrm{d}x \geq \frac{1}{4(\nu+1)}(b_1 - a_1) \Leftrightarrow |E_1| \geq \frac{1}{4(\nu+1)}|E|.$$

This completes the proof of (3.19). Finally, applying (3.18) to the set E_1 and using (3.19), we deduce the inequality (3.17). □

Remark 3.9. Fix $\nu \geq 0$ and let $E = [a, b] \subset \mathbb{R}^+$. We consider both the Lebesgue measure and the μ_ν-measure of E. Using the inequality $x^p + y^p \leq (x+y)^p$ for $x, y \geq 0$ and $p \geq 1$, we obtain $|E|^{2(\nu+1)} \leq 2(\nu+1)\mu_\nu(E)$. However, the converse inequality does not generally hold unless $b \geq 2a > 0$. In fact, when $y \geq x > 0$ and $p \geq 1$, we have the inequality $(x+y)^p \leq 2^p(x^p + y^p) \leq 2^{p+1}y^p + x^p$, which implies

$$\frac{b^{2(\nu+1)} - a^{2(\nu+1)}}{2(\nu+1)} \leq \frac{2^{2\nu+3}(b-a)^{2(\nu+1)}}{2(\nu+1)},$$

when $b \geq 2a > 0$, that is,

$$\mu_\nu(E) \leq \frac{2^{2\nu+3}|E|^{2(\nu+1)}}{2(\nu+1)}.$$

Lemma 3.10. *For every $\nu \geq 0$ and every nonnegative integer $k \in \mathbb{N}$, we have the estimate*

$$\left| \frac{\partial^k}{\partial x^k} \left(\frac{J_\nu(xy)}{(xy)^\nu} \right) \right| \leq y^k, \qquad x,\, y \in \mathbb{R}^+. \tag{3.23}$$

Proof. By the Poisson representation formula (see [47], Appendix B.1), we have

$$\frac{J_\nu(xy)}{(xy)^\nu} = \frac{1}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} \mathrm{e}^{ixys}(1-s^2)^\nu \frac{\mathrm{d}s}{\sqrt{1-s^2}}.$$

Then for every nonnegative integer $k \in \mathbb{N}$, we get

$$\begin{aligned}
\left| \frac{\partial^k}{\partial x^k} \left(\frac{J_\nu(xy)}{(xy)^\nu} \right) \right| &= \left| \frac{y^k}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} (is)^k \mathrm{e}^{ixys}(1-s^2)^\nu \frac{\mathrm{d}s}{\sqrt{1-s^2}} \right| \\
&\leq \frac{y^k}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \left| \int_{-1}^{+1} \frac{(1-s^2)^\nu \mathrm{d}s}{\sqrt{1-s^2}} \right| = \frac{y^k \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\nu+1)} \\
&= \frac{y^k}{2^\nu \Gamma(\nu+1)} \leq y^k, \text{ for every } x,\, y \in \mathbb{R}^+, \text{ and } \nu \geq 0.
\end{aligned}$$

□

Next, we show an interpolation inequality for a class of L^2-functions whose Hankel transforms have compact supports. For the subsequent lemma, we denote the closed half-line by $\overline{\mathbb{R}^+} := [0, \infty)$.

Lemma 3.11. *Given any intervals $A = [a_1, a_2] \subset \overline{\mathbb{R}^+}$ and $B = [b_1, b_2] \subset \overline{\mathbb{R}^+}$, $a = a_2 - a_1 > 0$, $b = b_2 - b_1 > 0$ and any constant $\lambda > 0$. Then for each $f \in L^2(\mathbb{R}^+)$ with $F_\nu(f) \in C_0^\infty(\mathbb{R}^+)$, there exist constants $C = C(\nu) > 0$*

and $\theta = \theta(\nu) \in (0,1)$ *such that*

$$\int_A |f(x)|^2 \mathrm{d}x \le C(a_2^{2(\nu+1)} - a_1^{2(\nu+1)})(\lambda^{-2(\nu+1)} + b^{-2(\nu+1)}) \times \left(\int_B |f(x)|^2 \mathrm{d}x\right)^{\theta^p} \left(\int_{\mathbb{R}^+} |F_\nu(f)(y)|^2 \mathrm{e}^{\lambda y} \mathrm{d}y\right)^{1-\theta^p}, \tag{3.24}$$

where $p := 1 + \frac{|x_0 - x_1| + \frac{a}{2} + \frac{b}{2}}{\lambda \wedge \frac{b}{2}}$, x_0 *and* x_1 *are the centers of* A *and* B*, respectively.*

Proof. We organize the proof by two steps.
Step 1. We show that (3.24) holds for $\lambda = 1$.
For $F_\nu(f) \in C_0^\infty(\mathbb{R}^+)$, we have

$$f(x) = \int_{\mathbb{R}^+} \sqrt{xy} J_\nu(xy) F_\nu(f)(y) \mathrm{d}y,$$

then

$$x^{-\nu-1/2} f(x) = \int_{\mathbb{R}^+} \frac{J_\nu(xy)}{(xy)^\nu} F_\nu(f)(y) y^{\nu+1/2} \mathrm{d}y.$$

Now let $g(x) = x^{-\nu-1/2} f(x)$ and we know that $g(x)$ is an analytic even function on the whole line $\mathbb{R}$ (see [46]), and for each $k \in \mathbb{N}$,

$$\frac{d^k g(x)}{dx^k} = \int_{\mathbb{R}^+} \frac{\partial^k}{\partial x^k} \left(\frac{J_\nu(xy)}{(xy)^\nu}\right) F_\nu(f)(y) y^{\nu+1/2} \mathrm{d}y.$$

Hence, by the Hölder inequality, for each $k \in \mathbb{N}$,

$$\begin{aligned} \left\|\frac{d^k g}{dx^k}\right\|_{L^\infty(\mathbb{R}^+)} &\le \sqrt{\int_{\mathbb{R}^+} \left|\frac{\partial^k}{\partial x^k}\left(\frac{J_\nu(xy)}{(xy)^\nu}\right)\right|^2 \mathrm{e}^{-y} y^{2\nu+1} \mathrm{d}y} \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)|^2 \mathrm{e}^y \mathrm{d}y} \\ &\le \sqrt{\sqrt{\int_{\mathbb{R}^+} \left|\frac{\partial^k}{\partial x^k}\left(\frac{J_\nu(xy)}{(xy)^\nu}\right)\right|^4 \mathrm{e}^{-y} \mathrm{d}y} \sqrt{\int_{\mathbb{R}^+} \mathrm{e}^{-y} y^{4\nu+2} \mathrm{d}y}} \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)|^2 \mathrm{e}^y \mathrm{d}y}, \end{aligned}$$

then by (3.23), we have

$$\begin{aligned} \left\|\frac{d^k g}{dx^k}\right\|_{L^\infty(\mathbb{R}^+)} &\le \sqrt{\sqrt{\int_{\mathbb{R}^+} y^{4k} \mathrm{e}^{-y} \mathrm{d}y} \sqrt{\int_{\mathbb{R}^+} \mathrm{e}^{-y} y^{4\nu+2} \mathrm{d}y}} \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)|^2 \mathrm{e}^y \mathrm{d}y} \\ &= \Gamma(4\nu+3)^{\frac{1}{4}} (4k!)^{\frac{1}{4}} \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)|^2 \mathrm{e}^y \mathrm{d}y}. \end{aligned}$$

We next claim that there is an absolute constant $C > 1$ such that

$$(4k!)^{\frac{1}{4}} \le k! C^k, \text{ for all } k \in \mathbb{N}.$$

When $k=0$, the above estimate clearly holds. We therefore consider the case $k \in \mathbb{N}^+$. In fact, using Stirling's approximation for factorials

$$\ln(m!) = m\ln m - m + O(\ln m),\ \forall m \in \mathbb{N}^+,$$

we see that for all $k \in \mathbb{N}^+$,

$$\ln((4k!)^{\frac{1}{4}}) = \frac{1}{4}(4k\ln(4k) - 4k + O(\ln(4k))) = \ln k! + k\ln 4 + O(\ln k).$$

Thus, there exists an absolute constant $C > 1$ such that

$$(4k!)^{\frac{1}{4}} \le \exp(\ln k! + k\ln C) = k!C^k,\ \text{ for all } k \in \mathbb{N}^+.$$

By this, we get

$$\left\|\frac{d^k g}{dx^k}\right\|_{L^\infty(\mathbb{R}^+)} \le \Gamma(4\nu+3)^{\frac{1}{4}} k!C^k \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)|^2 \mathrm{e}^y \mathrm{d}y}.$$

Let $A = [a_1, a_2] \subset \overline{\mathbb{R}^+}$ and $B = [b_1, b_2] \subset \overline{\mathbb{R}^+}$ be two intervals. Denote by x_0 and x_1 the centers of A and B, respectively, and set $a := a_2 - a_1$, $b := b_2 - b_1$ as their lengths. For any $r > 0$, we use $B_r(x)$ to denote a closed interval centered at x with length $2r$.

Define

$$M = \Gamma(4\nu+3)^{\frac{1}{4}} \sqrt{\int_{\mathbb{R}^+} |F_\nu(f)(y)|^2 \mathrm{e}^y\, \mathrm{d}y}, \qquad r_0 = \frac{C^{-1} \wedge \frac{b}{2}}{5} < 1, \tag{3.25}$$

we obtain that

$$\left|\frac{d^k g}{dx^k}\right| \le M\frac{k!}{(5r_0)^k}, \qquad x \in B_{4r_0}(x_1).$$

Then by Lemma 3.8, there exist constants $C_1 = C_1(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$ such that

$$\|g\|_{L^\infty(B_{2r_0}(x_1))} \le C_1 M^{1-\theta} (\mu_\nu(B_{r_0}(x_1))^{-1} \|g\|_{L^1_\nu(B_{r_0}(x_1))})^\theta.$$

The Hölder inequality yields

$$\|g\|_{L^\infty(B_{2r_0}(x_1))} \le C_1 M^{1-\theta} (\mu_\nu(B_{r_0}(x_1))^{-\frac{1}{2}} \|g\|_{L^2_\nu(B_{r_0}(x_1))})^\theta.$$

Since $B_{r_0}(x_1) \subset B$ and $\|g\|_{L^2_\nu(B_{r_0}(x_1))} = \|f\|_{L^2(B_{r_0}(x_1))}$, we have

$$\|g\|_{L^\infty(B_{2r_0}(x_1))} \le C_1 M^{1-\theta} (\mu_\nu(B_{r_0}(x_1))^{-\frac{1}{2}} \|f\|_{L^2(B)})^\theta.$$

Just as in the proof in [23], let $D_r(z)$ denote the closed disk in the complex plane, centered at z with radius r. It is clear that $D_{r_0}((m+1)r_0) \subset D_{2r_0}(mr_0)$, $m \in \mathbb{N}^+$. Define $G(s) = \frac{1}{M} g(x_1 + s)$, $s \in \mathbb{R}^+$. Then G can be extended to an analytic function on

$$\Omega_{r_0} := \{x + iy \in \mathbb{C} : x, y \in \mathbb{R},\ |y| < 5r_0\}.$$

In addition, G has the property that $\|G\|_{L^\infty(\Omega_{r_0})} \le 1$. The function $G(4r_0 z)$ is analytic on $D_1(0)$ and $\sup_{z\in D_1(0)} |G(4r_0 z)| \le 1$. Then apply Lemma 3.2 in [51] to find that there are constants $C_2 > 0$ and $\theta_1 \in (0,1)$ such that

$$\sup_{z\in D_{1/2}(0)} |G(4r_0 z)| \le C_2 \sup_{x\in\mathbb{R},|x|\le 1/5} |G(4r_0 x)|^{\theta_1}.$$

We obtain

$$\|G\|_{L^\infty(D_{2r_0}(0))} \le C_2 \left(\frac{1}{M}\|g\|_{L^\infty(B_{2r_0}(x_1))}\right)^{\theta_1},$$

then we have

$$\|G\|_{L^\infty(D_{2r_0}(0))} \le C_2 C_1^{\theta_1} \mu_\nu(B_{r_0}(x_1))^{-\theta\theta_1/2} \left(\frac{1}{M}\|f\|_{L^2(B)}\right)^{\theta\theta_1}. \tag{3.26}$$

Meanwhile, we can apply the Hadamard three-circle theorem to deduce that for each $m \in \mathbb{N}^+$,

$$\|G\|_{L^\infty(D_{2r_0}(mr_0))} \le \|G\|^{1/2}_{L^\infty(D_{r_0}(mr_0))} \|G\|^{1/2}_{L^\infty(D_{4r_0}(mr_0))} \le \|G\|^{1/2}_{L^\infty(D_{r_0}(mr_0))}.$$

We see that for each $m \in \mathbb{N}^+$,

$$\|G\|_{L^\infty(D_{r_0}((m+1)r_0))} \le \|G\|_{L^\infty(D_{2r_0}(mr_0))} \le \|G\|^{1/2}_{L^\infty(D_{r_0}(mr_0))},$$

which implies that for each $m \in \mathbb{N}^+$,

$$\|G\|_{L^\infty(D_{r_0}((m+1)r_0))} \le \|G\|^{1/2}_{L^\infty(D_{r_0}(mr_0))} \le \cdots \le \|G\|^{(1/2)^m}_{L^\infty(D_{r_0}(r_0))}.$$

This yields

$$\begin{aligned} \|G\|_{L^\infty(\cup_{1\le m\le n} D_{r_0}(mr_0))} &= \sup_{1\le m\le n} \|G\|_{L^\infty(D_{r_0}(mr_0))} \le \sup_{1\le m\le n} \|G\|^{(1/2)^{m-1}}_{L^\infty(D_{r_0}(r_0))} \\ &\le \sup_{1\le m\le n} \|G\|^{(1/2)^{n-1}}_{L^\infty(D_{r_0}(r_0))} \le \|G\|^{(1/2)^K}_{L^\infty(D_{r_0}(r_0))}, \end{aligned} \tag{3.27}$$

where n is the integer such that

$$nr_0 \ge |x_0 - x_1| + \frac{a}{2} + \frac{b}{2} > (n-1)r_0, \tag{3.28}$$

and

$$K = \frac{|x_0 - x_1| + \frac{a}{2} + \frac{b}{2}}{r_0}.$$

Because it follows from (3.28) that

$$\left[0, |x_0 - x_1| + \frac{a}{2} + \frac{b}{2}\right] \subset \bigcup_{1\le m\le n} D_{r_0}(mr_0)) \;\; and \;\; D_{r_0}(r_0) \subset D_{2r_0}(0),$$

we see from (3.27) that for all $s \in [0, |x_0 - x_1| + \frac{a}{2} + \frac{b}{2}]$,

$$|G(s)| \le \|G\|_{L^\infty(\cup_{1\le m\le n} D_{r_0}(mr_0))} \le \|G\|^{(1/2)^K}_{L^\infty(D_{r_0}(r_0))} \le \|G\|^{(1/2)^K}_{L^\infty(D_{2r_0}(0))}. \tag{3.29}$$

From (3.26) and (3.29), we find that for all $s \in [0, |x_0 - x_1| + \frac{a}{2} + \frac{b}{2}]$,

$$\begin{aligned} |g(x_1 + s)| &= M|G(s)| \le M\|G\|^{(1/2)^K}_{L^\infty(D_{2r_0}(0))} \\ &\le M\left(C_2 C_1^{\theta_1}\mu_\nu(B_{r_0}(x_1))^{-\theta\theta_1/2}\left(\frac{1}{M}\|f\|_{L^2(B)}\right)^{\theta\theta_1}\right)^{(1/2)^K} \\ &= \left(C_2 C_1^{\theta_1}\mu_\nu(B_{r_0}(x_1))^{-\theta\theta_1/2}\right)^{2^{-K}} M^{1-\theta\theta_1/2^K}\|f\|^{\theta\theta_1/2^K}_{L^2(B)}. \end{aligned}$$

One can easily find that the above inequality holds for $g(x_1 - s)$, for all $s \in [0, |x_0 - x_1| + \frac{a}{2} + \frac{b}{2}]$, too. We see that,

$$\sup_{|x-x_1|\le|x_0-x_1|+\frac{a}{2}+\frac{b}{2}} |g(x)| \le \left(C_2 C_1^{\theta_1}\mu_\nu(B_{r_0}(x_1))^{-\theta\theta_1/2}\right)^{2^{-K}} M^{1-\theta\theta_1/2^K}\|f\|^{\theta\theta_1/2^K}_{L^2(B)}.$$

Since $A \subset \{x : |x - x_1| \le |x_0 - x_1| + \frac{a}{2} + \frac{b}{2}\} \cap \overline{\mathbb{R}^+}$ and

$$\sup_{\{x:|x-x_1|\le|x_0-x_1|+\frac{a}{2}+\frac{b}{2}\}\cap\overline{\mathbb{R}^+}} |g(x)| \le \sup_{|x-x_1|\le|x_0-x_1|+\frac{a}{2}+\frac{b}{2}} |g(x)|,$$

the above estimate yields

$$\begin{aligned} \int_A |f(x)|^2 \mathrm{d}x = \int_A |g(x)|^2 \mathrm{d}\mu_\nu(x) &\le \mu_\nu(A) \sup_{\{x:|x-x_1|\le|x_0-x_1|+\frac{a}{2}+\frac{b}{2}\}\cap\overline{\mathbb{R}^+}} |g(x)|^2 \\ &\le \mu_\nu(A)\left(C_2 C_1^{\theta_1}\mu_\nu(B_{r_0}(x_1))^{-\theta\theta_1/2}\right)^{2^{-(K-1)}} M^{2(1-\theta\theta_1/2^K)}\|f\|^{2\theta\theta_1/2^K}_{L^2(B)}. \end{aligned}$$

We observe that for $I = [c, d] \subset \overline{\mathbb{R}^+}$,

$$\mu_\nu([c, d]) = \int_c^d x^{2\nu+1}\mathrm{d}x = \frac{d^{2(\nu+1)} - c^{2(\nu+1)}}{2(\nu + 1)}.$$

By inequality $(a^p + b^p) \le (a + b)^p$, $b \ge 0$, $a \ge 0$, $p \ge 1$, we get

$$\frac{d^{2(\nu+1)} - c^{2(\nu+1)}}{2(\nu + 1)} \ge \frac{(d - c)^{2(\nu+1)}}{2(\nu + 1)} \qquad i.e. \quad \mu_\nu(I) \ge \frac{|I|^{2\nu+2}}{2(\nu + 1)}.$$

Thus, we know that

$$\begin{aligned}\int_A |f(x)|^2 \mathrm{d}x &\le \mu_\nu(A)\left(C_2 C_1^{\theta_1}\left(\frac{(2r_0)^{2\nu+2}}{2(\nu+1)}\right)^{-\theta\theta_1/2}\right)^{2^{-(K-1)}} M^{2(1-\theta\theta_1/2^K)}\|f\|_{L^2(B)}^{2\theta\theta_1/2^K}\\ &\le \mu_\nu(A)(C_2C_1^{\theta_1}r_0^{-(\nu+1)}+1)^2 M^{2(1-\theta\theta_1/2^K)}\|f\|_{L^2(B)}^{2\theta\theta_1/2^K}\\ &= \frac{(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})}{2(\nu+1)}(C_2C_1^{\theta_1}r_0^{-(\nu+1)}+1)^2 M^{2(1-\theta\theta_1/2^K)}\|f\|_{L^2(B)}^{2\theta\theta_1/2^K}.\end{aligned}$$

Finally, noting that $M \ge \|f\|_{L^2(B)}$ and $r_0 \ge \frac{C^{-1}(1\wedge\frac{b}{2})}{5}$, we obtain

$$\begin{aligned}\int_A |f(x)|^2 \mathrm{d}x &\le \frac{(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})}{2(\nu+1)}(1+C_3)^2(5C)^{2\nu+2}\left(\left(1\wedge\frac{b}{2}\right)^{-(\nu+1)}+1\right)^2 M^2\left(\frac{\|f\|_{L^2(B)}^2}{M^2}\right)^{\alpha_1}\\ &\le 4\frac{(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})}{2(\nu+1)}(1+C_3)^2(5C)^{2\nu+2}2^{2\nu+2}(b^{-2(\nu+1)}+1)M^2\left(\frac{\|f\|_{L^2(B)}^2}{M^2}\right)^{\alpha_1}\\ &\le C_4^{2(\nu+1)}(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})(b^{-2(\nu+1)}+1)M^2\left(\frac{\|f\|_{L^2(B)}^2}{M^2}\right)^{\alpha_2},\end{aligned}$$

where $\theta_2 := \theta\theta_1$, $\alpha_1 := \theta_2(\frac{1}{2})^{\frac{|x_0-x_1|+\frac{a}{2}+\frac{b}{2}}{r_0}}$ and $\alpha_2 := \min\left\{\theta_2, (\frac{1}{2})^{5C}\right\}^{1+\frac{|x_0-x_1|+\frac{a}{2}+\frac{b}{2}}{1\wedge\frac{b}{2}}}$.

Step 2. We show that (3.24) holds for $\lambda > 0$.
Define $h(x) = \lambda^{1/2} f(\lambda x)$, $\lambda > 0$, $x \in \mathbb{R}^+$. It is clear that

$$h \in L^2(\mathbb{R}^+) \ \ \textit{and} \ \ F_\nu(h)(x) = \lambda^{-\frac{1}{2}}F_\nu(f)(x/\lambda).$$

Since $F_\nu(f) \in C_0^\infty(\mathbb{R}^+)$, the above implies $F_\nu(h)(x) \in C_0^\infty(\mathbb{R}^+)$. Thus, by Step 1, there exist constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$ such that

$$\begin{aligned}\int_{A/\lambda} |h(x)|^2 \mathrm{d}x \le C &\left(\left(\frac{a_2}{\lambda}\right)^{2(\nu+1)} - \left(\frac{a_1}{\lambda}\right)^{2(\nu+1)}\right)\left(\left(\frac{b}{\lambda}\right)^{-2(\nu+1)}+1\right)\\ &\times\left(\int_{B/\lambda}|h(x)|^2\mathrm{d}x\right)^{\theta^{p_1}}\left(\int_0^\infty |F_\nu(h)(y)|^2 e^y dy\right)^{1-\theta^{p_1}},\end{aligned} \tag{3.30}$$

where

$$p_1 = 1 + \frac{|\frac{x_0}{\lambda}-\frac{x_1}{\lambda}|+\frac{a}{2\lambda}+\frac{b}{2\lambda}}{1\wedge\frac{b}{2\lambda}} = 1 + \frac{|x_0-x_1|+\frac{a}{2}+\frac{b}{2}}{\lambda\wedge\frac{b}{2}}.$$

From (3.30), we find that

$$\begin{aligned}\int_A |f(x)|^2\mathrm{d}x &= \int_{A/\lambda} |h(x)|^2\mathrm{d}x \\ &\le C(a_2^{2(\nu+1)} - a_1^{2(\nu+1)})(\lambda^{-2(\nu+1)} + b^{-2(\nu+1)}) \left(\int_B |f(x)|^2\mathrm{d}x\right)^{\theta^{p_1}} \left(\int_{\mathbb{R}^+} |F_\nu(f)(y)|^2 \mathrm{e}^{\lambda y}\mathrm{d}y\right)^{1-\theta^{p_1}}.\end{aligned}$$

Thus, we complete the proof of the lemma. □

We omit the proofs of the following two corollaries, as they follow from the above lemma using the same arguments as in [23], Corollaries 2.6 and 2.7.

Corollary 3.12. *There exist constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0, 1)$ such that for any b, $\lambda > 0$,*

$$\int_{\mathbb{R}^+} |f(x)|^2\mathrm{d}x \le C\left(1 + \frac{b^{2\nu+2}}{\lambda^{2\nu+2}}\right)\left(\int_{(0,b]^c} |f(x)|^2\mathrm{d}x\right)^{\theta^{1+\frac{b}{\lambda}}} \left(\int_{\mathbb{R}^+} |F_\nu(f)(y)|^2\mathrm{e}^{\lambda y}\mathrm{d}y\right)^{1-\theta^{1+\frac{b}{\lambda}}} \tag{3.31}$$

for each $f \in L^2(\mathbb{R}^+)$ with $F_\nu(f) \in C_0^\infty(\mathbb{R}^+)$.

Corollary 3.13. *There exists a positive constant $C = C(\nu)$ such that for each $b > 0$ and $N > 0$ and all $f \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} F_\nu f \subset [0, N]$,*

$$\int_{\mathbb{R}^+} |f(x)|^2\mathrm{d}x \le \mathrm{e}^{C(1+bN)} \int_{(0,b]^c} |f(x)|^2\mathrm{d}x. \tag{3.32}$$

With the above lemmas and corollaries in hand, we can now prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. We fix λ, b, $T > 0$ and $u_0 \in C_0^\infty(\mathbb{R}^+)$. Define

$$f(x) := \mathrm{e}^{-\frac{ix^2}{4T}} u(x, T; u_0), \qquad x \in \mathbb{R}^+. \tag{3.33}$$

From (2.5), we know that

$$(2T)^{\frac{1}{2}}\mathrm{e}^{\frac{i(\nu+1)\pi}{2}} f(x) = F_\nu(\mathrm{e}^{\frac{iy^2}{4T}} u_0(y))(x/2T), \qquad x \in \mathbb{R}^+. \tag{3.34}$$

This implies that for a.e. $x \in \mathbb{R}^+$,

$$\begin{aligned}F_\nu(f)(x) &= \int_0^\infty \sqrt{xy}J_\nu(xy)f(y)\mathrm{d}y \\ &= (2T)^{-\frac{1}{2}}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}} \int_0^\infty \sqrt{xy}J_\nu(xy)(2T)^{\frac{1}{2}}\mathrm{e}^{\frac{i(\nu+1)\pi}{2}} f(y)\mathrm{d}y \\ &= (2T)^{\frac{1}{2}}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}} \int_0^\infty \sqrt{2Txz}J_\nu(2Txz)(2T)^{\frac{1}{2}}\mathrm{e}^{\frac{i(\nu+1)\pi}{2}} f(2Tz)\mathrm{d}z \\ &= (2T)^{\frac{1}{2}}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}} \int_0^\infty \sqrt{2Txz}J_\nu(2Txz)F_\nu(\mathrm{e}^{\frac{iy^2}{4T}} u_0(y))(z)\mathrm{d}z \\ &= (2T)^{\frac{1}{2}}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}}\mathrm{e}^{\frac{iy^2}{4T}} u_0(y)|_{y=2Tx} \\ &= (2T)^{\frac{1}{2}}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}}\mathrm{e}^{iTx^2} u_0(2Tx).\end{aligned} \tag{3.35}$$

We are going to prove the conclusions (i) and (ii) of the theorem one by one. The proof of (iii) is given in Section 5; see the proof of Theorem 5.3(ii).

(i) By Corollary 3.12, with λ replaced by $2T\lambda$, we obtain that there exist constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$ such that

$$\begin{aligned}\int_{\mathbb{R}^+} |u(x,T;u_0)|^2 \mathrm{d}x &= \int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x \\ &\le C\left(1+\frac{b^{2\nu+2}}{(2\lambda T)^{2\nu+2}}\right)\left(\int_{(0,b]^c} |f(x)|^2 \mathrm{d}x\right)^{\theta^{1+b/(2\lambda T)}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{2T\lambda x}|F_\nu(f)(x)|^2 \mathrm{d}x\right)^{1-\theta^{1+b/(2\lambda T)}} \\ &= C\left(1+\frac{b^{2\nu+2}}{(2\lambda T)^{2\nu+2}}\right)\left(\frac{\int_{(0,b]^c} |f(x)|^2 \mathrm{d}x}{\int_{\mathbb{R}^+} \mathrm{e}^{2T\lambda x}|F_\nu(f)(x)|^2 \mathrm{d}x}\right)^{\theta^{1+b/(2\lambda T)}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{2T\lambda x}|F_\nu(f)(x)|^2 \mathrm{d}x\right) \\ &\le C\left(1+\frac{b^{2\nu+2}}{(\lambda T)^{2\nu+2}}\right)\left(\frac{\int_{(0,b]^c} |f(x)|^2 \mathrm{d}x}{\int_{\mathbb{R}^+} \mathrm{e}^{2T\lambda x}|F_\nu(f)(x)|^2 \mathrm{d}x}\right)^{\theta^{1+b/(\lambda T)}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{2T\lambda x}|F_\nu(f)(x)|^2 \mathrm{d}x\right).\end{aligned}$$

By (3.33), (3.35), and after some computations, we find that

$$\int_{\mathbb{R}^+} |u(x,T;u_0)|^2 \mathrm{d}x \le C\left(1+\frac{b^{2\nu+2}}{(\lambda T)^{2\nu+2}}\right)\left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x\right)^{\theta^{1+b/(\lambda T)}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2 \mathrm{d}x\right)^{1-\theta^{1+b/(\lambda T)}}.$$

The above inequality, along with the conservation law for the Schrödinger equation, leads to (1.13). Hence (i) is true.

(ii) Fix $\beta > 1$ and $\gamma \in (0,1)$. For the function $f(x)$ defined above, we claim there exists $C = C(\nu)$ such that

$$\int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x \le C\mathrm{e}^{\left(\frac{C^\beta b^\beta}{\lambda(1-\gamma)T^\beta}\right)^{1/(\beta-1)}} \left(\int_{(0,b]^c} |f(x)|^2 \mathrm{d}x\right)^{\gamma} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda(2Tx)^\beta}|F_\nu(f)(x)|^2 \mathrm{d}x\right)^{1-\gamma}. \tag{3.36}$$

In fact, for any fixed $N > 0$, we make the following decomposition: $f = g_1 + g_2$ in $L^2(\mathbb{R}^+)$ where

$$F_\nu(g_1) := \chi_{(0,N]}F_\nu(f), \quad F_\nu(g_2) := \chi_{(0,N]^c}F_\nu(f).$$

Then by applying Corollary 3.13 to g_1, we find that

$$\begin{aligned}\int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x &\le 2\int_{\mathbb{R}^+} |g_1(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^+} |g_2(x)|^2 \mathrm{d}x \\ &\le 2\mathrm{e}^{C(1+bN)}\int_{(0,b]^c} |g_1(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^+} |g_2(x)|^2 \mathrm{d}x \\ &\le 4\mathrm{e}^{C(1+bN)}\int_{(0,b]^c} (|f(x)|^2 + |g_2(x)|^2) \mathrm{d}x + 2\int_{\mathbb{R}^+} |g_2(x)|^2 \mathrm{d}x \\ &\le 4\mathrm{e}^{C(1+bN)}\int_{(0,b]^c} |f(x)|^2 \mathrm{d}x + 6\mathrm{e}^{C(1+bN)}\int_{\mathbb{R}^+} |g_2(x)|^2 \mathrm{d}x\end{aligned} \tag{3.37}$$

for some $C>0$ depending only on ν. Meanwhile, since the Hankel transform F_ν is an isometry, we get

$$\begin{aligned}\int_{\mathbb{R}^+}|g_2(x)|^2\mathrm{d}x&=\int_{\mathbb{R}^+}|F_\nu(g_2)(x)|^2\mathrm{d}x=\int_{\mathbb{R}^+}|\chi_{(0,N]^c}(x)F_\nu(f)(x)|^2\mathrm{d}x\\&=\mathrm{e}^{-\lambda(2TN)^\beta}\int_{\mathbb{R}^+}|\chi_{(0,N]^c}(x)F_\nu(f)(x)|^2\mathrm{e}^{\lambda(2TN)^\beta}\mathrm{d}x.\end{aligned}$$

This, together with (3.37), yields

$$\int_{\mathbb{R}^+}|f(x)|^2\mathrm{d}x\le 4\mathrm{e}^{C(1+bN)}\int_{(0,b]^c}|f(x)|^2\mathrm{d}x+6\mathrm{e}^{C(1+bN)-\lambda(2TN)^\beta}\int_{\mathbb{R}^+}|F_\nu(f)(x)|^2\mathrm{e}^{\lambda(2Tx)^\beta}\mathrm{d}x. \tag{3.38}$$

Since it follows from the Young inequality that

$$\begin{aligned}CbN&=[Cb((1-\gamma)\lambda(2T)^\beta)^{-1/\beta}][((1-\gamma)\lambda(2T)^\beta)^{1/\beta}N]\\&\le(1-\frac{1}{\beta})[Cb((1-\gamma)\lambda(2T)^\beta)^{-1/\beta}]^{\frac{\beta}{\beta-1}}+\frac{1}{\beta}[((1-\gamma)\lambda(2T)^\beta)^{1/\beta}N]^\beta\\&\le[(Cb)^\beta/((1-\gamma)\lambda(2T)^\beta)]^{\frac{1}{\beta-1}}+(1-\gamma)\lambda(2TN)^\beta,\end{aligned}$$

we deduce from (3.38) that

$$\int_{\mathbb{R}^+}|f(x)|^2\mathrm{d}x\le 6\mathrm{e}^{C+(\frac{(Cb)^\beta}{(1-\gamma)\lambda(2T)^\beta})^{\frac{1}{\beta-1}}}\left(\mathrm{e}^{(1-\gamma)\lambda(2TN)^\beta}\int_{(0,b]^c}|f(x)|^2\mathrm{d}x+\mathrm{e}^{-\gamma\lambda(2TN)^\beta}\int_{\mathbb{R}^+}|F_\nu(f)(x)|^2\mathrm{e}^{\lambda(2Tx)^\beta}\mathrm{d}x\right).$$

As N was arbitrary from $(0,\infty)$, the above indicates that for all $\varepsilon\in(0,1)$,

$$\int_{\mathbb{R}^+}|f(x)|^2\mathrm{d}x\le 6\mathrm{e}^{C+\left(\frac{(Cb)^\beta}{(1-\gamma)\lambda(2T)^\beta}\right)^{\frac{1}{\beta-1}}}\left(\varepsilon^{-(1-\gamma)}\int_{(0,b]^c}|f(x)|^2\mathrm{d}x+\varepsilon^\gamma\int_{\mathbb{R}^+}|F_\nu(f)(x)|^2\mathrm{e}^{\lambda(2Tx)^\beta}\mathrm{d}x\right).$$

It is easy to check that the above inequality holds in fact for all $\varepsilon>0$. Minimizing it with respect to $\varepsilon>0$ leads to (3.36). Here, we use the inequality

$$\inf_{\varepsilon>0}(\varepsilon^{-(1-\gamma)}A+\varepsilon^\gamma B)\le 2A^\gamma B^{1-\gamma}\quad\text{ for all } A,B\ge 0.$$

This proves (3.36).
Finally from (3.33), (3.35) and (3.36), after some computations, we know that (ii) is true. □

Proof of Theorem 1.3. Let f be as defined in the proof of Theorem 1.2. By (3.33), (3.35) and Lemma 3.11 (with λ replaced by $2\lambda T$), we find that

$$\begin{aligned}\int_A|u(x,T;u_0)|^2\,\mathrm{d}x&=\int_A|f(x)|^2\,\mathrm{d}x\\&\le C(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})\big((2\lambda T)^{-2(\nu+1)}+b^{-2(\nu+1)}\big)\\&\quad\times\left(\int_B|f(x)|^2\,\mathrm{d}x\right)^{\theta^{\alpha_1}}\left(\int_{\mathbb{R}^+}|F_\nu(f)(y)|^2\mathrm{e}^{2\lambda Ty}\,\mathrm{d}y\right)^{1-\theta^{\alpha_1}}\\&\le C(a_2^{2(\nu+1)}-a_1^{2(\nu+1)})\big((\lambda T)^{-2(\nu+1)}+b^{-2(\nu+1)}\big)\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_B |u(x,T;u_0)|^2\,\mathrm{d}x\right)^{\theta^{\alpha_1}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2\,\mathrm{d}x\right)^{1-\theta^{\alpha_1}} \\
&\leq C(a_2^{2(\nu+1)} - a_1^{2(\nu+1)})\big((\lambda T)^{-1} + b^{-1}\big)^{2\nu+2} \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2\,\mathrm{d}x \left(\frac{\int_B |u(x,T;u_0)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2\,\mathrm{d}x}\right)^{\theta^{\alpha_1}} \\
&\leq 2C(a_2^{2(\nu+1)} - a_1^{2(\nu+1)})\big((\lambda T) \wedge b\big)^{-(2\nu+2)} \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2\,\mathrm{d}x \left(\frac{\int_B |u(x,T;u_0)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|u_0(x)|^2\,\mathrm{d}x}\right)^{\theta^{\alpha_2}}
\end{aligned}
$$

for some constants $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$, where

$$
\alpha_1 = 1 + \frac{|x_0 - x_1| + \frac{a}{2} + \frac{b}{2}}{(2\lambda T) \wedge \frac{b}{2}}, \quad \alpha_2 = 1 + \frac{|x_0 - x_1| + \frac{a}{2} + \frac{b}{2}}{(\lambda T) \wedge \frac{b}{2}}.
$$

This proves our theorem. □

4. Proofs of Theorems 1.4–1.6

Theorem 1.4 is mainly based on Theorems 1.1 and 1.2. Theorem 1.5 is a consequence of Theorem 1.3. Theorem 1.6 is based on Theorem 1.3, along with an additional property of the Schrödinger equation, which is presented in Lemma 4.2 of this paper.

Proof of Theorem 1.4. (i) The proof presented here is inspired by the approach used in [18], Theorem 1.3. Fix $b, T > 0$. According to (iii) of Theorem 1.1(with $a = b$), there exists $C = C(\nu)$ so that when $0 \leq s < t \leq T$,

$$
\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq C\mathrm{e}^{C(1+\frac{b^2}{t-s})} \left(\int_{(0,b]^c} |u(x,s;u_0)|^2 \mathrm{d}x + \int_{(0,b]^c} |u(x,t;u_0)|^2 \mathrm{d}x\right). \tag{4.1}
$$

Since $0 \leq s < t \leq T$, and if $(s,t) \in [0, T/3] \times [2T/3, T]$, we have

$$
(t-s) \geq T/3. \tag{4.2}
$$

Integrating (4.1) with s over $s \in [0, T/3]$ and t over $t \in [2T/3, T]$, using (4.2), we obtain that

$$
\begin{aligned}
\left(\frac{T}{3}\right)^2 \int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x &\leq C \int_{\frac{2T}{3}}^{T} \int_0^{\frac{T}{3}} \mathrm{e}^{C(1+\frac{b^2}{T/3})} \left(\int_{(0,b]^c} |u(x,s;u_0)|^2 \mathrm{d}x + \int_{(0,b]^c} |u(x,t;u_0)|^2 \mathrm{d}x\right) dsdt \\
&\leq \frac{CT}{3}\mathrm{e}^{C(1+\frac{3b^2}{T})} \left(\int_0^{\frac{T}{3}} \int_{(0,b]^c} |u(x,s;u_0)|^2 dxds + \int_{\frac{2T}{3}}^{T} \int_{(0,b]^c} |u(x,t;u_0)|^2 dxdt\right) \\
&\leq \frac{CT}{3}\mathrm{e}^{C(1+\frac{3b^2}{T})} \int_0^T \int_{(0,b]^c} |u(x,t;u_0)|^2 dxdt.
\end{aligned}
$$

From the above, we obtain

$$
\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq \frac{3C}{T}\mathrm{e}^{C(1+\frac{3b^2}{T})} \int_0^T \int_{(0,b]^c} |u(x,t;u_0)|^2 dxdt,
$$

which implies (1.17).

We now establish the sharpness of the growth rate $C\mathrm{e}^{C\frac{b^2}{T}}$ as $T \to 0$. To this end, recall the Hankel transform identity from [44], p. 29, (10):

$$F_\nu\left(x^{\nu+\frac{1}{2}}\mathrm{e}^{-ax^2}\right)(y)=\frac{y^{\nu+\frac{1}{2}}}{(2a)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{4a}},\quad \operatorname{Re}a>0,\quad \operatorname{Re}\nu>-1. \tag{4.3}$$

For every $T>0$, consider the initial datum $u_T(x)=x^{\nu+\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{4T}}$. Its L^2-norm is given by $\|u_T\|^2_{L^2(\mathbb{R}^+)}=2^\nu T^{\nu+1}\Gamma(\nu+1)$. Using (1.7) and (4.3), we obtain

$$u(x,t;u_T)=\left(\frac{t}{T}-i\right)^{-(\nu+1)}\mathrm{e}^{-\frac{i(\nu+1)\pi}{2}}\mathrm{e}^{\frac{itx^2}{4(t^2+T^2)}}x^{\nu+\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{4T(t^2/T^2+1)}},\quad |u(x,t;u_T)|=\frac{T^{\nu+1}}{(t^2+T^2)^{\frac{\nu+1}{2}}}x^{\nu+\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{4T(t^2/T^2+1)}}.$$

We now estimate both the upper and lower bounds of the double integral $\int_0^T\int_b^\infty|u(x,t;u_T)|^2dxdt$. For the upper bound, a straightforward calculation shows that

$$\begin{aligned}\int_0^T\int_b^\infty|u(x,t;u_T)|^2dxdt&=\int_0^T\int_b^\infty\frac{T^{2(\nu+1)}}{(t^2+T^2)^{\nu+1}}x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{2T(t^2/T^2+1)}}dxdt\\&\le\int_0^T\int_b^\infty x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{4T}}dxdt\\&\le T\mathrm{e}^{-\frac{b^2}{8T}}\int_0^\infty x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{8T}}\mathrm{d}x\\&=2^{3\nu+2}T^{\nu+2}\Gamma(\nu+1)\mathrm{e}^{-\frac{b^2}{8T}}.\end{aligned}\tag{4.4}$$

For the lower bound, assume $b\ge\sqrt{2T}$, which is a natural assumption in the regime $T\to0$. Then we have

$$\begin{aligned}\int_0^T\int_b^\infty|u(x,t;u_T)|^2dxdt&=\int_0^T\int_b^\infty\frac{T^{2(\nu+1)}}{(t^2+T^2)^{\nu+1}}x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{2T(t^2/T^2+1)}}dxdt\\&\ge\int_0^T\int_b^\infty\frac{T^{2(\nu+1)}}{(2T^2)^{\nu+1}}x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{2T}}dxdt\\&=T\cdot2^{-(\nu+1)}\int_b^\infty x^{2\nu+1}\mathrm{e}^{-\frac{x^2}{2T}}\mathrm{d}x\\&=T^{\nu+2}\int_{\frac{b}{\sqrt{2T}}}^\infty\mathrm{e}^{-z^2}z^{2\nu+1}du\\&\ge T^{\nu+2}\cdot\frac{b}{\sqrt{2T}}\int_{\frac{b}{\sqrt{2T}}}^\infty\mathrm{e}^{-z^2}\mathrm{d}z.\end{aligned}$$

Set $z_0=\frac{b}{\sqrt{2T}}$. By applying integration by parts to the Gaussian tail integral, we obtain

$$\int_{z_0}^\infty\mathrm{e}^{-z^2}\mathrm{d}z=\frac{\mathrm{e}^{-z_0^2}}{2z_0}-\frac{1}{2}\int_{z_0}^\infty\mathrm{e}^{-z^2}z^{-2}\mathrm{d}z\ge\frac{\mathrm{e}^{-z_0^2}}{2z_0}-\frac{1}{2z_0^2}\int_{z_0}^\infty\mathrm{e}^{-z^2}\mathrm{d}z.$$

It follows that

$$\int_{z_0}^\infty\mathrm{e}^{-z^2}\mathrm{d}z\ge\frac{z_0\mathrm{e}^{-z_0^2}}{1+2z_0^2}.$$

Thus, when $b \geq \sqrt{2T}$, we find that

$$\int_0^T \int_b^\infty |u(x,t;u_T)|^2 dxdt \geq T^{\nu+2} \cdot \frac{b^2 \mathrm{e}^{-\frac{b^2}{2T}}}{2(b^2+T)} \geq T^{\nu+2} \cdot \frac{\mathrm{e}^{-\frac{b^2}{2T}}}{3}. \tag{4.5}$$

Combining the upper bound (4.4) and the lower bound (4.5) with the initial energy $\|u_T\|^2_{L^2(\mathbb{R}^+)} = 2^\nu T^{\nu+1}\Gamma(\nu+1)$, we conclude that

$$\frac{T}{2^\nu \Gamma(\nu+1)} \cdot \frac{\mathrm{e}^{-\frac{b^2}{2T}}}{3} \leq \frac{\int_0^T \int_b^\infty |u(x,t;u_T)|^2 dxdt}{\int_0^\infty |u_T(x)|^2 dx} \leq 2^{2(\nu+1)} T \mathrm{e}^{-\frac{b^2}{8T}}.$$

As $T \to 0$, this ratio exhibits exponential decay of the form $\sim \mathrm{e}^{-\frac{b^2}{T}}$, thereby confirming the sharpness of the rate in the cost constant $C\mathrm{e}^{C\frac{b^2}{T}}$. This completes the proof of (i).

(ii) Given $u_0 \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} u_0 \subset [0,N]$. By a standard density argument, we can apply Theorem 1.2 (i) (with $\lambda = b/T$) to get that for some $C = C(\nu) > 0$ and $\theta = \theta(\nu) \in (0,1)$,

$$\begin{aligned} \int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x &\leq 2C \left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{\theta^2} \left(\int_{\mathbb{R}^+} \mathrm{e}^{bx/T} |u_0(x)|^2 \mathrm{d}x \right)^{1-\theta^2} \\ &\leq 2C\mathrm{e}^{bN(1-\theta^2)/T} \left(\int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{\theta^2} \left(\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \right)^{1-\theta^2}. \end{aligned} \tag{4.6}$$

This implies that

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq (2C)^{\frac{1}{\theta^2}} \mathrm{e}^{bN(1-\theta^2)/T\theta^2} \int_{(0,b]^c} |u(x,T;u_0)|^2 \mathrm{d}x.$$

Hence, we end the proof of (ii). □

We recall Lemma 3.1 from [23], which will be used in the proofs of Theorems 1.5 and 1.6.

Lemma 4.1. *Let $x, \theta \in (0,1)$.*
(i) For each $a > 0$,

$$\sum_{k=1}^\infty x^{\theta^k} \mathrm{e}^{-ak} \leq \frac{\mathrm{e}^a}{|\ln\theta|} \Gamma\left(\frac{a}{|\ln\theta|}\right) |\ln x|^{-a/|\ln\theta|}. \tag{4.7}$$

(ii) For each $\varepsilon > 0$ and $\alpha > 0$,

$$\sum_{k=1}^\infty x^{\theta^k} k^{-1-\varepsilon} \leq \frac{4}{\varepsilon} \alpha^\varepsilon \mathrm{e}^{\varepsilon \ln\varepsilon + \varepsilon + e\alpha^{-1}\theta^{-1}} (\ln(\alpha|\ln x| + e))^{-\varepsilon}. \tag{4.8}$$

Proof of Theorem 1.5. When $u_0 = 0$, (1.20) holds clearly for all $\varepsilon \in (0,1)$. We now fix $u_0 \in C_0^\infty(\mathbb{R}^+)\backslash\{0\}$. For convenience, we define

$$A_1 := \int_{\mathbb{R}^+} \mathrm{e}^{\lambda_1 x} |u_0(x)|^2 \,\mathrm{d}x, \qquad B_1 := \int_B |u(x,T;u_0)|^2 \,\mathrm{d}x,$$
$$R_{\lambda_2} := \int_{\mathbb{R}^+} \mathrm{e}^{-\lambda_2 x} |u(x,T;u_0)|^2 \,\mathrm{d}x.$$

The proof of (1.20) is divided into several steps.

Step 1. There exist positive constants $C_1(\nu)$ and $C_2(\nu)$ such that

$$R_{\lambda_2} \le C_3(x_0,b,\lambda_1,\lambda_2,T) \left(\ln \frac{A_1}{B_1}\right)^{-C_2\lambda_2((\lambda_1 T)\wedge \frac{b}{2})} A_1, \tag{4.9}$$

where

$$C_3(x_0,b,\lambda_1,\lambda_2,T) := 1 + C_1 \Gamma\left(C_2\lambda_2\left((\lambda_1 T)\wedge \frac{b}{2}\right)\right) \times \exp\left(\lambda_2^{-1}\left((\lambda_1 T)\wedge \frac{b}{2}\right)^{-1} + \lambda_2\left(x_0 + \frac{b}{2}\right)\right). \tag{4.10}$$

According to Theorem 1.3 (with $(x_0, x_1, \frac{a}{2}, \frac{b}{2})$ replaced by $(2(k-1)\lambda_2^{-1} + \lambda_2^{-1}, x_0, \lambda_2^{-1}, \frac{b}{2})$ with $k \in \mathbb{N}^+$), we notice that

$$\begin{aligned}
&\int_{\mathbb{R}^+} \mathrm{e}^{-\lambda_2 x}|u(x,T;u_0)|^2 \mathrm{d}x \\
&\le \sum_{k=1}^{\infty} \int_{2(k-1)\lambda_2^{-1} < x < 2k\lambda_2^{-1}} \mathrm{e}^{-2(k-1)} |u(x,T;u_0)|^2 \mathrm{d}x \\
&\le C((\lambda_1 T)\wedge b)^{-(2\nu+2)} \left(\sum_{k=1}^{\infty} \mathrm{e}^{-2k+2}\left((2k\lambda_2^{-1})^{2\nu+2} - (2(k-1)\lambda_2^{-1})^{2\nu+2}\right)\left(\frac{B_1}{A_1}\right)^{\theta^{1+\frac{|x_0-(2k-1)\lambda_2^{-1}|+\lambda_2^{-1}+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \right) A_1 \\
&\le C((\lambda_1 T)\wedge b)^{-(2\nu+2)} \left(\sum_{k=1}^{\infty} \mathrm{e}^{-2k+2}(2k\lambda_2^{-1})^{2\nu+2}\left(\frac{B_1}{A_1}\right)^{\theta^{1+\frac{|x_0-(2k-1)\lambda_2^{-1}|+\lambda_2^{-1}+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \right) A_1 \\
&\le C(2\lambda_2^{-1})^{2\nu+2}(2\nu+2)^{2\nu+2}((\lambda_1 T)\wedge b)^{-(2\nu+2)}\mathrm{e}^2 \left(\sum_{k=1}^{\infty} \mathrm{e}^{-k}\left(\frac{B_1}{A_1}\right)^{\theta^{1+\frac{|x_0-(2k-1)\lambda_2^{-1}|+\lambda_2^{-1}+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \right) A_1 \\
&\le C(2\lambda_2^{-1})^{2\nu+2}(2\nu+2)^{2\nu+2}((\lambda_1 T)\wedge b)^{-(2\nu+2)}\mathrm{e}^2 \left(\sum_{k=1}^{\infty} \mathrm{e}^{-k}\left(\frac{B_1}{A_1}\right)^{\theta^{1+\frac{x_0+2k\lambda_2^{-1}+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \right) A_1
\end{aligned} \tag{4.11}$$

for some $\theta \in (0,1)$ and $C > 0$ depending only on ν. In the fourth inequality, we used the fact that $k \le n e^{k/n}$ for all $k \in \mathbb{N}^+$. The last inequality is due to $B_1 < A_1$ (which follows from the definitions of A_1 and B_1, the conservation law for the Schrödinger equation and the fact that $u_0 \neq 0$).
Now, we apply (4.7) in Lemma 4.1 with

$$(a, x, \theta) = (1, (B_1/A_1)^{\theta^{1+\frac{x_0+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}}, \theta^{\frac{2}{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}})$$

to get

$$\sum_{k=1}^{\infty} \mathrm{e}^{-k} \left(\frac{B_1}{A_1}\right)^{\theta^{1+\frac{x_0+2k\lambda_2^{-1}+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}} \le \frac{e\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}\Gamma\left(\frac{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}\right)\left(\theta^{1+\frac{x_0+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}\left|\ln\frac{B_1}{A_1}\right|\right)^{\frac{-\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}}.$$

This, along with (4.11) and the facts that $x^a \le ([a]+1)!\mathrm{e}^x$ for all $x > 0$, $a > 0$ where $[a]$ is the integral part of a, imply that

$$\begin{aligned}
&\int_{\mathbb{R}^+} \mathrm{e}^{-\lambda_2 x}|u(x,T;u_0)|^2 \mathrm{d}x \\
&\le C(2\lambda_2^{-1})^{2\nu+2}\left(\frac{2\nu+2}{(\lambda_1 T)\wedge b}\right)^{2\nu+2} \mathrm{e}^3 \frac{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{|\ln\theta|}\Gamma\left(\frac{\lambda_2\left((\lambda_1 T)\wedge\frac{b}{2}\right)}{2|\ln\theta|}\right)\left(\theta^{1+\frac{x_0+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}}\left|\ln\frac{B_1}{A_1}\right|\right)^{\frac{-\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}} A_1 \\
&\le \frac{C2^{2\nu+2}\mathrm{e}^3}{|\ln\theta|}(2\nu+2)^{2\nu+2}(\lambda_2((\lambda_1 T)\wedge\frac{b}{2}))^{-(2\nu+2)+1}\mathrm{e}^{\frac{\lambda_2}{2}((\lambda_1 T)\wedge\frac{b}{2}+x_0+\frac{b}{2})}\Gamma\left(\frac{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}\right)\left(\ln\frac{A_1}{B_1}\right)^{\frac{-\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}} A_1 \\
&\le \frac{C2^{2\nu+2}\mathrm{e}^3}{|\ln\theta|}(2\nu+2)^{2\nu+2}([2\nu+1]+1)!\mathrm{e}^{\lambda_2^{-1}((\lambda_1 T)\wedge\frac{b}{2})^{-1}+\lambda_2(x_0+\frac{b}{2})}\Gamma\left(\frac{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}\right)\left(\ln\frac{A_1}{B_1}\right)^{\frac{-\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}{2|\ln\theta|}} A_1.
\end{aligned}$$

This leads to (4.9).

Step 2. (1.20) holds for $\lambda_2 \le \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$.

First, we claim that for each $\varepsilon \in (0,1)$,

$$R_{\lambda_2} \le C_3(\varepsilon A_1 + \varepsilon \mathrm{e}^{\varepsilon^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}} B_1), \tag{4.12}$$

where $C_3 = C_3(x_0, b, \lambda_1, \lambda_2, T)$ is given by (4.10). Indeed, if $R_{\lambda_2} \le C_3\varepsilon A_1$, (4.12) is obvious. So we only consider the case: $R_{\lambda_2} > C_3\varepsilon A_1$. In this case, we have the following observation:

$$0 < \varepsilon < \frac{R_{\lambda_2}}{C_3 A_1} < 1. \tag{4.13}$$

Besides, two facts are given in order. First, since $\lambda_2 \le \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$, the function $x \to x\mathrm{e}^{x^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}$ is decreasing on $(0,1)$. This, along with (4.13), indicates that

$$\frac{R_{\lambda_2}}{C_3A_1}\mathrm{e}^{\left(\frac{R_{\lambda_2}}{C_3A_1}\right)^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}} \le \varepsilon\mathrm{e}^{\varepsilon^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}. \tag{4.14}$$

Second, since the function $f(x) = \mathrm{e}^{x^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}$ is decreasing on $(0,\infty)$ and its inverse is the function $g(x) := (\ln x)^{-C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}$, we deduce from (4.9) that

$$\frac{A_1}{B_1} = f\left(g\left(\frac{A_1}{B_1}\right)\right) \le f\left(\frac{R_{\lambda_2}}{C_3A_1}\right) = \mathrm{e}^{\left(\frac{R_{\lambda_2}}{C_3A_1}\right)^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}. \tag{4.15}$$

According to (4.14) and (4.15), we have

$$R_{\lambda_2} = C_3\frac{R_{\lambda_2}}{C_3A_1}\frac{A_1}{B_1}B_1 \le C_3\left(\frac{R_{\lambda_2}}{C_3A_1}\mathrm{e}^{\left(\frac{R_{\lambda_2}}{C_3A_1}\right)^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}\right)B_1 \le C_3\varepsilon\mathrm{e}^{\varepsilon^{-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}B_1.$$

Since $\varepsilon \in (0,1)$ was arbitrary, the above leads to (4.12) for $R_{\lambda_2} > C_3\varepsilon A_1$. Hence, (4.12) is true.

Next, we claim that

$$C_3(x_0, b, \lambda_1, \lambda_2, T) \le \exp\left\{2(C_1 + C_2^{-1} + 1)\left(1 + \frac{\lambda_2^{-1} + x_0 + \frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}\right)\right\}. \tag{4.16}$$

In fact, by some computations, we first observe that for each $s \in (0,1]$,

$$\Gamma(s) \le \mathrm{e}^{2s^{-1}}. \tag{4.17}$$

Since $\lambda_2 \le \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$, it follows from (4.10) and (4.17) with $s = C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})$ that

$$\begin{aligned} C_3(x_0, b, \lambda_1, \lambda_2, T) &\le 1 + \mathrm{e}^{C_1}\mathrm{e}^{2C_2^{-1}\lambda_2^{-1}((\lambda_1 T)\wedge\frac{b}{2})^{-1}} \times \exp(\lambda_2^{-1}((\lambda_1 T)\wedge\frac{b}{2})^{-1} + \lambda_2(x_0 + \frac{b}{2})) \\ &\le e\cdot\exp\left(C_1 + (2C_2^{-1}+1)\lambda_2^{-1}((\lambda_1 T)\wedge\frac{b}{2})^{-1} + C_2^{-1}\frac{x_0 + \frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}\right). \end{aligned}$$

This leads to (4.16).

Now, by (4.12) and (4.16), we obtain

$$R_{\lambda_2} \le C_4(x_0, b, \lambda_1, \lambda_2, T)(\varepsilon A_1 + \varepsilon\mathrm{e}^{\varepsilon^{-1-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}B_1), \tag{4.18}$$

where

$$C_4(x_0, b, \lambda_1, \lambda_2, T) := C_1\exp\left\{2(C_1 + C_2^{-1} + 1)(C_2+1)\left(1 + \frac{\lambda_2^{-1} + x_0 + \frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}\right)\right\}.$$

Since

$$\varepsilon e^{\varepsilon^{-1-\frac{\alpha}{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}} \leq \varepsilon e^{\varepsilon^{-1-\frac{\beta}{\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}} \text{ when } 0<\alpha<\beta \text{ and } \varepsilon\in(0,1), \tag{4.19}$$

it follows from the estimate (4.19) and (4.18) that (1.20) holds for $\lambda_2 \leq \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$.

Step 3. (1.20) holds for $\lambda_2 > \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$.

First, by the definition of R_{λ_2}, we obtain that $R_{\lambda_2} \leq R_{\frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}}$. Then combining (4.12) and (4.16) (with λ_2 replaced by $\frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$), we notice that for each $\varepsilon\in(0,1)$,

$$\begin{aligned} R_{\lambda_2} &\leq \exp\left\{2(C_1+C_2^{-1}+1)\left(1+\frac{C_2((\lambda_1 T)\wedge\frac{b}{2})+x_0+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}\right)\right\}(\varepsilon A_1+\varepsilon e^{\varepsilon^{-1}}B_1) \\ &\leq \exp\left\{2(C_1+C_2^{-1}+1)\left(1+C_2+\frac{\lambda_2^{-1}+x_0+\frac{b}{2}}{(\lambda_1 T)\wedge\frac{b}{2}}\right)\right\}(\varepsilon A_1+\varepsilon e^{\varepsilon^{-1-\frac{1}{C_2\lambda_2((\lambda_1 T)\wedge\frac{b}{2})}}}B_1), \end{aligned}$$

which, together with (4.19), yields (1.20) for $\lambda_2 > \frac{1}{C_2((\lambda_1 T)\wedge\frac{b}{2})}$.

Hence, this ends the proof of Theorem 1.5. □

We first prove a lemma on the regularity propagation property of the Schrödinger equation, and then use this result to prove Theorem 1.6.

Lemma 4.2. *For every $\nu\geq 0$, given $k\in\mathbb{N}^+$, there exists a constant $C(k,\nu)$ such that for any $T>0$ and $u_0\in C_0^\infty(\mathbb{R}^+)$,*

$$\int_{\mathbb{R}^+} x^{2k}|u(x,T;u_0)|^2 dx \leq C(k,\nu)\left(T+\frac{1}{T}\right)^{2k}\left(\|u_0\|^2_{H^{4k}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}x^{8k}|u_0|^2 dx+\int_{\mathbb{R}^+}\frac{1}{x^{4k}}|u_0|^2 dx\right). \tag{4.20}$$

Proof. Fix $k\in\mathbb{N}^+$, $T>0$ and $u_0\in C_0^\infty(\mathbb{R}^+)$. Based on the identity (1.7), the unitary property of the Hankel transform, and the formula (2.4), we conclude that

$$\begin{aligned} \|x^k u(x,T;u_0)\|^2_{L^2(\mathbb{R}^+)} &= \|x^k(2T)^{-\frac{1}{2}}F_\nu(f)(x/2T)\|^2_{L^2(\mathbb{R}^+)} \\ &= \int_{\mathbb{R}^+} x^{2k}(2T)^{-1}|F_\nu(f)(x/2T)|^2 dx \\ &= (2T)^{2k}\int_{\mathbb{R}^+} x^{2k}|F_\nu(f)(x)|^2 dx \\ &= (2T)^{2k}\|F_\nu x^k F_\nu(f)\|^2_{L^2(\mathbb{R}^+)} \\ &= (2T)^{2k}\left\|H_\nu^{\frac{k}{2}}f\right\|^2_{L^2(\mathbb{R}^+)}, \end{aligned} \tag{4.21}$$

where $f := e^{\frac{ix^2}{4T}}u_0$.

Since the operator H_ν^k with domain $C_0^\infty(\mathbb{R}^+)$ is a polynomial in $\frac{1}{x}$ and ∂_x, of degree $2k$, and after some computations, we can find that the polynomial H_ν^k is a linear combination of the monomials

$$\left\{\frac{1}{x^r}\partial_x^s : r+s=2k,\ r,\ s\in\mathbb{N},\ r\neq 1\right\}.$$

From this, we see that

$$\begin{aligned}\int_{\mathbb{R}^+}\left|H_\nu^{\frac{k}{2}}f\right|^2\mathrm{d}x &= \int_{\mathbb{R}^+}\left\langle H_\nu^k f, f\right\rangle_{\mathbb{C}}\mathrm{d}x\\ &\leq C(k,\nu)\sum_{r+s=2k, r\neq 1}\int_{\mathbb{R}^+}\left|\left\langle \partial_x^s f, \frac{1}{x^r}f\right\rangle_{\mathbb{C}}\right|\mathrm{d}x,\end{aligned}\tag{4.22}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes the pointwise Hermitian inner product on $\mathbb{C}$, that is, $\langle f(x), g(x)\rangle_{\mathbb{C}} := f(x)\overline{g(x)}$. Then by some computations, we find that

$$\begin{aligned}\|\partial_x^s f\|^2_{L^2(\mathbb{R}^+)} = \int_{\mathbb{R}^+}\left|\left\langle \partial_x^{2s}f, f\right\rangle_{\mathbb{C}}\right|\mathrm{d}x &\leq C_1(s)\max\{T^{-2s},1\}\sum_{0\leq m+n\leq 2s}\int_{\mathbb{R}^+}|\langle\partial_x^m u_0, x^n u_0\rangle_{\mathbb{C}}|\,\mathrm{d}x\\ &\leq C_2(s)\max\{T^{-2s},1\}\left(\|u_0\|^2_{H^{2s}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}x^{4s}|u_0|^2\mathrm{d}x\right).\end{aligned}\tag{4.23}$$

From (4.21), (4.22) and (4.23), we get that

$$\begin{aligned}\int_{\mathbb{R}^+}x^{2k}|u(x,T;u_0)|^2\mathrm{d}x &= (2T)^{2k}\int_{\mathbb{R}^+}\left|H_\nu^{\frac{k}{2}}f\right|^2\mathrm{d}x\\ &\leq C(k,\nu)(2T)^{2k}\sum_{r+s=2k, r\neq 1}\int_{\mathbb{R}^+}\left|\left\langle\partial_x^s f, \frac{1}{x^r}f\right\rangle_{\mathbb{C}}\right|\mathrm{d}x\\ &\leq C(k,\nu)(2T)^{2k}\sum_{r+s=2k, r\neq 1}\left(\int_{\mathbb{R}^+}|\partial_x^s f|^2\mathrm{d}x+\int_{\mathbb{R}^+}\left|\frac{1}{x^r}f\right|^2\mathrm{d}x\right) \qquad (4.24)\\ &\leq C(k,\nu)\left(T+\frac{1}{T}\right)^{2k}\left(\|u_0\|^2_{H^{4k}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}x^{8k}|u_0|^2\mathrm{d}x+\int_{\mathbb{R}^+}\frac{1}{x^{4k}}|u_0|^2\mathrm{d}x\right). \qquad (4.25)\end{aligned}$$

This completes the proof. □

Remark 4.3. Lemma 4.2 provides a quantitative estimate for the solutions of equation (1.1) for all $\nu\geq 0$. In the special case $\nu=\frac{1}{2}$, which corresponds to the free Schrödinger equation on the half-line, we can derive the following quantitative estimate from the previous proof, by combining (4.21) and (4.23): Given $k\in\mathbb{N}^+$, for any $u_0\in C_0^\infty(\mathbb{R}^+)$, let $u(x,t;u_0)$ denote the solution to equation (1.1) with $\nu=\frac{1}{2}$, then there exists a constant $C(k)>0$ such that for any $T>0$,

$$\int_{\mathbb{R}^+}x^{2k}|u(x,T;u_0)|^2\mathrm{d}x\leq C(k)(1+T)^{2k}\left(\|u_0\|^2_{H^{2k}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}x^{4k}|u_0|^2\mathrm{d}x\right).$$

This estimate is consistent with the quantitative estimate established in Lemma 3.2 of [23] for the free Schrödinger equation in $\mathbb{R}^n$. For general $\nu\geq 0$, however, the additional requirements on the regularity and decay of the function u_0 stem from inequality (4.24). It is natural to wonder if the additional requirements for the regularity and decay of u_0, even the last term $\int_{\mathbb{R}^+} \frac{1}{x^{4k}}|u_0|^2 \mathrm{d}x$ can be removed. We point out that for general $\nu \geq 0$, if $k=1$, the estimate

$$\int_{\mathbb{R}^+} x^2 |u(x,T;u_0)|^2 \, \mathrm{d}x \leq C(\nu)(1+T)^2 \left(\|u_0\|^2_{H^2(\mathbb{R}^+)} + \int_{\mathbb{R}^+} x^4 |u_0|^2 \, \mathrm{d}x \right)$$

holds true. This is because the Sobolev norms $\|H_\nu^{\frac{1}{2}} f\|^2_{L^2(\mathbb{R}^+)}$ and $\|(-\Delta)^{\frac{1}{2}} f\|^2_{L^2(\mathbb{R}^+)}$ are equivalent if $k=1$. However, for $k \geq 2$, the equivalence of these two norms remains an open question. For more information on this topic, we refer readers to [52–55].

Remark 4.4. The proof of this lemma borrows ideas from the proof of Lemma 3.2 in [23], with some notable differences. Their proof relies on the commutativity of the operators $(x_j + 2i(t-T)\partial_{x_j})^k$ and $i\partial_t + \Delta$. However, in the presence of a potential term, it remains unclear how to construct similar commutators to obtain an estimate like (4.20). Using the identity (1.7), we can deduce the estimate (4.20), and this method can also be applied to prove Lemma 3.2 in [23]. It would be interesting to determine whether this quantitative estimate holds for a general potential V. However, this is a challenging question, as both approaches may fail in this setting. To the best of our knowledge, no such quantitative estimates exist for a general potential V.

Proof of Theorem 1.6. When $u_0 = 0$, (1.21) holds clearly for all $\varepsilon \in (0,1)$. We now fix $u_0 \in C_0^\infty(\mathbb{R}^+)\backslash\{0\}$. For convenience, we define

$$A_2 := \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x} |u_0(x)|^2 \mathrm{d}x + \|u_0\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)} + \int_{\mathbb{R}^+} \frac{1}{x^{4([\nu]+3)}} |u_0|^2 \mathrm{d}x, \quad B_2 := \int_B |u(x,T;u_0)|^2 \mathrm{d}x,$$
$$A_3 := \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x} |u_0(x)|^2 \mathrm{d}x.$$

The proof of (1.21) is divided into several steps.

Step 1. There exists $C_0 = C_0(\nu) > 1$ such that

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq C_1(x_0, b, \lambda, T) \frac{A_2}{\sqrt{\ln\left(\ln \frac{A_2}{B_2} + e\right)}}, \tag{4.26}$$

where

$$C_1(x_0,b,\lambda,T) := \left(T + \frac{1}{T}\right)^{[\nu]+3} (1+T)^{4([\nu]+3)} \mathrm{e}^{C_0^{1+\frac{x_0+\frac{b}{2}+1}{(\lambda T)\wedge \frac{b}{2}}}}. \tag{4.27}$$

By the conservation law for the Schrödinger equation, and the Hölder inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x &= \int_{\mathbb{R}^+} |u(x,T;u_0)|^2 \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^+} (1+x)^{2\nu+2+2} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^+} (1+x)^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}^+} (1+x)^{2([\nu]+3)} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^+} (1+x)^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x \right)^{1/2}. \end{aligned} \tag{4.28}$$

Next, the proof of (4.26) is organized in two parts.

Part 1.1. There exists a constant $C_2(\nu) > 1$ such that

$$\int_{\mathbb{R}^+} (1+x)^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x \le C_3(x_0,b,\lambda,T) \frac{1}{\ln(\ln \frac{A_2}{B_2} + e)} A_2, \tag{4.29}$$

where

$$C_3(x_0,b,\lambda,T) := \mathrm{e}^{C_2^{1+\frac{x_0+\frac{b}{2}+1}{(\lambda T)\wedge \frac{b}{2}}}}. \tag{4.30}$$

By Theorem 1.3 (with $(x_0,\, x_1,\, \frac{a}{2},\, \frac{b}{2})$ replaced by $(k-\frac{1}{2},\, x_0,\, \frac{1}{2},\, \frac{b}{2})$ for $k \in \mathbb{N}^+$), we find that

$$\begin{aligned}\int_{\mathbb{R}^+} (1+x)^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x &\le \sum_{k=1}^{\infty} \int_{k-1\le x<k} k^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x \\ &\le C((\lambda T)\wedge b)^{-(2\nu+2)} \sum_{k=1}^{\infty} k^{-2} B_2^{\theta^{1+\frac{x_0+k+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}} A_3^{1-\theta^{1+\frac{x_0+k+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}} \\ &\le C((\lambda T)\wedge b)^{-(2\nu+2)} \left(\sum_{k=1}^{\infty} k^{-2} \left(\frac{B_2}{A_2}\right)^{\theta^{1+\frac{x_0+k+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}} \right) A_2\end{aligned} \tag{4.31}$$

for some $C > 0$ and $\theta \in (0,1)$ depending only on ν. Since $u_0 \ne 0$, by the definitions of A_2 and B_2, and by the conservation law for the Schrödinger equation, we obtain $B_2 < A_2$. Then by (4.8) in Lemma 4.1 with

$$(x,\theta,\varepsilon,\alpha) = ((B_2/A_2)^{\theta^{1+\frac{x_0+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}}, \theta^{\frac{1}{(\lambda T)\wedge \frac{b}{2}}}, 1, \theta^{-1-\frac{x_0+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}),$$

to get

$$\sum_{k=1}^{\infty} k^{-2} \left(\frac{B_2}{A_2}\right)^{\theta^{1+\frac{x_0+k+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}} \le 4\theta^{-1-\frac{x_0+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}} \mathrm{e}^{1+e\theta^{1+\frac{x_0+\frac{b}{2}-1}{(\lambda T)\wedge \frac{b}{2}}}} \frac{1}{\ln\left(\left|\ln \frac{B_2}{A_2}\right| + e\right)}. \tag{4.32}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^+} (1+x)^{-(2\nu+2)-2} |u(x,T;u_0)|^2 \mathrm{d}x &\le 4C((\lambda T)\wedge \frac{b}{2})^{-(2\nu+2)} \theta^{-1-\frac{x_0+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}} \mathrm{e}^{1+e\theta^{1+\frac{x_0+\frac{b}{2}-1}{(\lambda T)\wedge \frac{b}{2}}}} \frac{A_2}{\ln\left(\left|\ln \frac{B_2}{A_2}\right| + e\right)} \\ &\le 4C([2\nu+2]+1)! \mathrm{e}^{\frac{1}{(\lambda T)\wedge \frac{b}{2}}} \mathrm{e}^{\theta^{-1-\frac{x_0+\frac{b}{2}}{(\lambda T)\wedge \frac{b}{2}}}} \mathrm{e}^{1+e\theta^{1+\frac{x_0+\frac{b}{2}-1}{(\lambda T)\wedge \frac{b}{2}}}} \frac{A_2}{\ln\left(\left|\ln \frac{B_2}{A_2}\right| + e\right)} \\ &\le 4C([2\nu+2]+1)! e \cdot \mathrm{e}^{(\theta^{-1}+e+1)\theta^{-2\frac{x_0+\frac{b}{2}+1}{(\lambda T)\wedge \frac{b}{2}}}} \frac{A_2}{\ln\left(\left|\ln \frac{B_2}{A_2}\right| + e\right)}.\end{aligned} \tag{4.33}$$

The first inequality in (4.33) follows from (4.31) and (4.32), while the last two inequalities rely on the facts that

$$\theta \in (0,1) \textit{ and } \left((\lambda T)\wedge\frac{b}{2}\right)^{-(2\nu+2)} \leq ([2\nu+2]+1)!\mathrm{e}^{\frac{1}{(\lambda T)\wedge\frac{b}{2}}} \leq ([2\nu+2]+1)!\mathrm{e}^{\theta^{-2\frac{1}{(\lambda T)\wedge\frac{b}{2}}}}.$$

Since $\theta \in (0,1)$, (4.29) follows from (4.33), as also do (4.30), so we get Part 1.1.

Part 1.2. There exists a positive constant $C_4(\nu)$ such that

$$\int_{\mathbb{R}^+}(1+x)^{2([\nu]+3)}|u(x,T;u_0)|^2\mathrm{d}x \leq C_4\left(T+\frac{1}{T}\right)^{2([\nu]+3)}(1+T)^{8([\nu]+3)}\left(1+\left((\lambda T)\wedge\frac{b}{2}\right)^{-1}\right)^{8([\nu]+3)} A_2. \tag{4.34}$$

By Lemma 4.2 (with $k=[\nu]+3$), we find that

$$\int_{\mathbb{R}^+}x^{2([\nu]+3)}|u(x,T;u_0)|^2\mathrm{d}x \leq C_{41}\left(T+\frac{1}{T}\right)^{2([\nu]+3)}\left(\|u_0\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}x^{8([\nu]+3)}|u_0|^2\mathrm{d}x+\int_{\mathbb{R}^+}\frac{1}{x^{4([\nu]+3)}}|u_0|^2\mathrm{d}x\right)$$

for some $C_{41}>0$ depending only on ν. It follows that

$$\begin{aligned}\int_{\mathbb{R}^+}(1+x)^{2([\nu]+3)}|u(x,T;u_0)|^2\mathrm{d}x &\leq \int_{\mathbb{R}^+}2^{2([\nu]+3)}\left(1+x^{2([\nu]+3)}\right)|u(x,T;u_0)|^2\mathrm{d}x\\ &\leq C_{42}\left(T+\frac{1}{T}\right)^{2([\nu]+3)}\left(\int_{\mathbb{R}^+}|u(x,T;u_0)|^2\mathrm{d}x+\|u_0\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)}\right.\\ &\qquad\left.+\int_{\mathbb{R}^+}x^{8([\nu]+3)}|u_0|^2\mathrm{d}x+\int_{\mathbb{R}^+}\frac{1}{x^{4([\nu]+3)}}|u_0|^2\mathrm{d}x\right),\end{aligned} \tag{4.35}$$

for some constant $C_{42}>0$ depending only on ν. Since

$$(\lambda x)^{8([\nu]+3)} \leq (8([\nu]+3))!\mathrm{e}^{\lambda x}, \quad x\in\mathbb{R}^+,$$

and

$$\begin{aligned}\max\{1,\lambda^{-8([\nu]+3)}\} &= \max\{1,(\lambda T)^{-8([\nu]+3)}T^{8([\nu]+3)}\}\\ &\leq (1+T)^{8([\nu]+3)}\max\left\{1,\left((\lambda T)\wedge\frac{b}{2}\right)^{-8([\nu]+3)}\right\}\\ &\leq (1+T)^{8([\nu]+3)}\left(1+\left((\lambda T)\wedge\frac{b}{2}\right)^{-1}\right)^{8([\nu]+3)},\end{aligned}$$

this, along with (4.35), yields that

$$\int_{\mathbb{R}^+}(1+x)^{2([\nu]+3)}|u(x,T;u_0)|^2\mathrm{d}x$$

$$\begin{aligned}&\le C_{43}\left(T+\frac{1}{T}\right)^{2([\nu]+3)}\left(\|u_0\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)}+\int_{\mathbb{R}^+}\lambda^{-8([\nu]+3)}\mathrm{e}^{\lambda x}|u_0|^2\mathrm{d}x+\int_{\mathbb{R}^+}\frac{1}{x^{4([\nu]+3)}}|u_0|^2\mathrm{d}x\right)\\&\le C_{43}\left(T+\frac{1}{T}\right)^{2([\nu]+3)}\max\{1,\lambda^{-8([\nu]+3)}\}A_2\\&\le C_{43}\left(T+\frac{1}{T}\right)^{2([\nu]+3)}(1+T)^{8([\nu]+3)}\left(1+\left((\lambda T)\wedge\frac{b}{2}\right)^{-1}\right)^{8([\nu]+3)}A_2\end{aligned}\tag{4.36}$$

for some $C_{43}>0$ depending only on ν. Hence, we get Part 1.2.

Now, by (4.29) and (4.34), we get

$$\begin{aligned}&\int_{\mathbb{R}^+}|u_0(x)|^2\mathrm{d}x\\&\le\left(\int_{\mathbb{R}^+}(1+x)^{2([\nu]+3)}|u(x,T;u_0)|^2\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^+}(1+x)^{-(2\nu+2)-2}|u(x,T;u_0)|^2\mathrm{d}x\right)^{1/2}\\&\le\sqrt{C_4}\left(T+\frac{1}{T}\right)^{[\nu]+3}(1+T)^{4([\nu]+3)}\left(1+\left((\lambda T)\wedge\frac{b}{2}\right)^{-1}\right)^{4([\nu]+3)}\sqrt{C_3(x_0,b,\lambda,T)}\frac{A_2}{\sqrt{\ln\left(\ln\frac{A_2}{B_2}+e\right)}}\\&\le\sqrt{C_4}\left(T+\frac{1}{T}\right)^{[\nu]+3}(1+T)^{4([\nu]+3)}(4([\nu]+3))!\mathrm{e}^{1+((\lambda T)\wedge\frac{b}{2})^{-1}}\sqrt{C_3(x_0,b,\lambda,T)}\frac{A_2}{\sqrt{\ln\left(\ln\frac{A_2}{B_2}+e\right)}}.\end{aligned}\tag{4.37}$$

The last inequality in (4.37) follows from the fact that $x^{4([\nu]+3)}\le(4([\nu]+3))!\mathrm{e}^x$ for all $x>0$. Consequently, (4.26) follows directly from (4.37) and (4.30).

Step 2. (1.21) holds for the above-mentioned u_0 and each $\varepsilon\in(0,1)$.
It suffices to show that for each $\varepsilon\in(0,1)$,

$$A:=\int_{\mathbb{R}^+}|u_0(x)|^2\mathrm{d}x\le C_1(\varepsilon A_2+\varepsilon\mathrm{e}^{\mathrm{e}^{\varepsilon^{-2}}}B_2),\tag{4.38}$$

where $C_1=C_1(x_0,b,\lambda,T)$ is given by (4.27). In fact, if $A\le C_1\varepsilon A_2$, (4.38) is obvious. So we only consider the case: $A>C_1\varepsilon A_2$. In this case, two observations are given in order: First, since $C_1>1$(see (4.27)), we deduce from the definitions of A and A_2 that

$$0<\varepsilon<\frac{A}{C_1A_2}<1.\tag{4.39}$$

Since the function $x\mapsto x\mathrm{e}^{\mathrm{e}^{x^{-2}}}$ is decreasing on (0,1), we see from (4.39) that

$$\frac{A}{C_1A_2}\mathrm{e}^{\mathrm{e}^{\left(\frac{A}{C_1A_2}\right)^{-2}}}\le\varepsilon\mathrm{e}^{\mathrm{e}^{\varepsilon^{-2}}}.\tag{4.40}$$

Second, since $x \mapsto \mathrm{e}^{-e}\mathrm{e}^{\mathrm{e}^{x^{-2}}}$ is decreasing on (0,1) and because its inverse is: $x \mapsto \frac{1}{\sqrt{\ln(\ln x+e)}}$ on $(1,\infty)$, it follows from (4.26) that

$$\frac{A_2}{B_2} \leq \mathrm{e}^{-e}\mathrm{e}^{\mathrm{e}^{\left(\frac{A}{C_1 A_2}\right)^{-2}}}. \tag{4.41}$$

Now we infer from (4.40) and (4.41) that

$$A = C_1 \frac{A}{C_1 A_2}\frac{A_2}{B_2} B_2 \leq C_1 \left(\frac{A}{C_1 A_2}\mathrm{e}^{-e}\mathrm{e}^{\mathrm{e}^{(\frac{A}{C_1 A_2})^{-2}}}\right) B_2 \leq C_1 \varepsilon \mathrm{e}^{-e}\mathrm{e}^{\mathrm{e}^{\varepsilon^{-2}}} B_2 \leq C_1 \varepsilon \mathrm{e}^{\mathrm{e}^{\varepsilon^{-2}}} B_2.$$

As $\varepsilon \in (0,1)$ was arbitrary, we obtain (4.38). This ends the proof of (1.21). □

5. The sharpness of the main results

The purpose of this section is to show the optimality of the inequalities established in Theorem 1.1 and Theorem 1.2.

5.1. The sharpness of Theorem 1.1

To show the sharpness of Theorem 1.1, we establish the following theorem.

Theorem 5.1. *(i) For every $\nu \geq 0$, let $A = [a_1, a_2] \subset \mathbb{R}^+$, $B = [b_1, b_2] \subset \mathbb{R}^+$, $a = a_2 - a_1$, $b = b_2 - b_1$, and $a, b, T > 0$. Then one can find a sequence $\{u_k\}_{k\in\mathbb{N}^+} \subset L^2(\mathbb{R}^+)$ with*

$$\int_{\mathbb{R}^+} |u_k(x)|^2 \mathrm{d}x = 1 \tag{5.1}$$

such that

$$\lim_{k\to\infty}\int_{A^c} |u_k(x)|^2 \mathrm{d}x = \lim_{k\to\infty}\int_B |u(x,T;u_k)|^2 \mathrm{d}x = 0. \tag{5.2}$$

(ii) For every $\nu \geq 0$, let $A = [a_1, a_2] \subset \mathbb{R}^+$, $B = [b_1, b_2] \subset \mathbb{R}^+$, $a = a_2 - a_1$, $b = b_2 - b_1$, and $a, b, S_1, S_2 > 0$. Then one can find a sequence $\{u_k\}_{k\in\mathbb{N}^+} \subset L^2(\mathbb{R}^+)$ with

$$\int_{\mathbb{R}^+} |u_k(x)|^2 \mathrm{d}x = 1 \tag{5.3}$$

such that

$$\lim_{k\to\infty}\int_{A^c} |u(x,S_1;u_k)|^2 \mathrm{d}x = \lim_{k\to\infty}\int_0^{S_2}\int_B |u(x,t;u_k)|^2 dxdt = 0. \tag{5.4}$$

(iii) For every $\nu \geq 0$, each $A \subset \mathbb{R}^+$ with $|A^c| > 0$, and each $T > 0$, there is no constant $C > 0$ such that for all $u_0 \in L^2(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq C \int_A |u(x,T;u_0)|^2 \mathrm{d}x. \tag{5.5}$$

Proof. (i) Let x_0 and x_1 be the centers of A and B, respectively. Choose

$$g \in C_0^\infty(\mathbb{R}^+) \ \text{ such that } \ \|g\|_{L^2(\mathbb{R}^+)} = 1. \tag{5.6}$$

For each $k \in \mathbb{N}^+$, we set

$$g_k(x) := k^{\frac{1}{2}} g(k(x - x_0)), \quad \operatorname{supp} g_k \subset [x_0, \infty). \tag{5.7}$$

We define a sequence $\{u_k\} \subset L^2(\mathbb{R}^+)$ as follows:

$$u_k(x) := \mathrm{e}^{-ix^2/4T} g_k(x), \quad x \in \mathbb{R}^+, \ k \in \mathbb{N}^+. \tag{5.8}$$

Two observations are in order: First, by (5.7) and (5.8),

$$\lim_{k\to\infty} \int_{A^c} |u_k(x)|^2 \mathrm{d}x = \lim_{k\to\infty} \int_{\frac{ak}{2}}^{\infty} |g(x)|^2 \mathrm{d}x = 0;$$

second, from (5.6)–(5.8), we observe that

$$\int_{\mathbb{R}^+} |u_k(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^+} |g_k(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^+} |g(x)|^2 \,\mathrm{d}x = 1, \quad \text{for all } k \in \mathbb{N}^+.$$

Next, it suffices to prove $\int_B |u(x, T; u_k)|^2 \mathrm{d}x$ goes to zero as $k \to \infty$. By (2.5), we see that for each $k \in \mathbb{N}^+$,

$$u(x, T; u_k) := (2T)^{-\frac{1}{2}} \mathrm{e}^{-\frac{i}{2}(\nu+1)\pi} \mathrm{e}^{ix^2/4T} F_\nu(g_k)(x/2T), \quad x \in \mathbb{R}^+. \tag{5.9}$$

Meanwhile, from (5.7), it follows that a.e. $x \in \mathbb{R}^+$,

$$\begin{aligned} F_\nu(g_k(y))(x) &= \int_0^\infty \sqrt{xy} J_\nu(xy) g_k(y) \mathrm{d}y \\ &= \int_{x_0}^\infty \sqrt{xy} J_\nu(xy) k^{\frac{1}{2}} g(k(y - x_0)) \mathrm{d}y \\ &= \int_0^\infty \sqrt{x(x_0 + y')} J_\nu(x(x_0 + y')) k^{\frac{1}{2}} g(ky') \mathrm{d}y' \\ &= k^{-\frac{1}{2}} \int_0^\infty \sqrt{x\left(x_0 + \frac{y}{k}\right)} J_\nu\left(x\left(x_0 + \frac{y}{k}\right)\right) g(y) \mathrm{d}y. \end{aligned}$$

This, together with (5.9) and (5.6), as well as the fact that $|J_\nu(x)| \le C_\nu$ for all $x \in \mathbb{R}^+$, where C_ν is a constant depending only on ν (see [47], pp. 578–580), shows that

$$\begin{aligned}
\int_B |u(x,T;u_k)|^2 \mathrm{d}x &\le |B| \sup_{x\in B} |u(x,T;u_k)|^2 \\
&\le |B| C_\nu^2 \left((2T)^{-\frac{1}{2}} k^{-\frac{1}{2}} \int_0^\infty \sqrt{\frac{x_1+\frac{b}{2}}{2T}\left(x_0+\frac{y}{k}\right)} |g(y)| \mathrm{d}y \right)^2 \\
&\le |B| C_\nu^2 \left((2T)^{-\frac{1}{2}} k^{-\frac{1}{2}} \left(\int_0^\infty \sqrt{\frac{x_1+\frac{b}{2}}{2T} x_0} |g(y)| \mathrm{d}y + \int_0^\infty \sqrt{\frac{x_1+\frac{b}{2}}{2T}\frac{y}{k}} |g(y)| \mathrm{d}y \right) \right)^2 \\
&\le 2|B| C_\nu^2 (2T)^{-1} \left(k^{-1} \left(\int_0^\infty \sqrt{\frac{x_1+\frac{b}{2}}{2T} x_0} |g(y)| \mathrm{d}y \right)^2 + k^{-2} \left(\int_0^\infty \sqrt{\frac{x_1+\frac{b}{2}}{2T}} y |g(y)| \mathrm{d}y \right)^2 \right),
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \int_B |u(x,T;u_k)|^2 \mathrm{d}x = 0.$$

Now, from the above proof, we get (5.1) and (5.2).

(ii) Let g and g_k, with $k \in \mathbb{N}^+$, satisfy (5.6) and (5.7), respectively. Since the Schrödinger equation is time-reversible, we can find a sequence $\{u_k\} \subset L^2(\mathbb{R}^+)$ such that

$$v_k(x) := u(x, S_1; u_k) = g_k(x), \quad x \in \mathbb{R}^+,\ k \in \mathbb{N}^+. \tag{5.10}$$

By (5.10), (5.6) and (5.7), we have the following two observations: First, we notice that

$$\lim_{k\to\infty} \int_{A^c} |v_k(x)|^2 \mathrm{d}x = \lim_{k\to\infty} \int_{[0,\frac{ak}{2}]^c} |g(x)|^2 \mathrm{d}x = 0. \tag{5.11}$$

Second, we have

$$\int_{\mathbb{R}^+} |v_k(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^+} |g_k(x)|^2 \mathrm{d}x = 1 \quad \text{for all } k \in \mathbb{N}^+. \tag{5.12}$$

Then by (2.5) and (5.10), for each $k \in \mathbb{N}^+$,

$$u(x,t;v_k) = \frac{1}{2it} \int_0^\infty \sqrt{xy} J_\nu\left(\frac{xy}{2t}\right) \mathrm{e}^{-\frac{x^2+y^2}{4it}} \mathrm{e}^{\frac{-i\nu\pi}{2}} g_k(y) \mathrm{d}y, \quad (x,t) \in \mathbb{R}^+ \times (\mathbb{R}\backslash\{0\}). \tag{5.13}$$

In a similar way to (i), we can get

$$\lim_{k\to\infty} \int_B |u(x,t;v_k)|^2 \mathrm{d}x = 0. \tag{5.14}$$

At the same time, by the conservation law for the Schrödinger equation and (5.12), we obtain that for all $k \in \mathbb{N}^+$ and $t \in \mathbb{R}\backslash\{0\}$,

$$\int_B |u(x,t;v_k)|^2 \mathrm{d}x \leq \int_{\mathbb{R}^+} |u(x,t;v_k)|^2 \mathrm{d}x = \int_{\mathbb{R}^+} |v_k(x)|^2 \mathrm{d}x = 1.$$

Hence by (5.14), we can apply the Lebesgue dominated convergence theorem to get

$$\lim_{k\to\infty} \int_{-S_1}^{S_2-S_1} \int_B |u(x,t;v_k)|^2 dxdt = 0. \tag{5.15}$$

Recalling from (5.10) that $v_k(x) = u(x, S_1; u_k)$, $x \in \mathbb{R}^+$, and combining (5.11), (5.12), and (5.15), we conclude that the sequence $\{u_k\}$ satisfies both (5.3) and (5.4). This completes the proof of (ii).

(iii) For a contradiction, suppose that (iii) is not true. Then there exist $A_0 \subset \mathbb{R}^+$ with $|A_0^c| > 0$ and $C_1, T > 0$ such that

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \leq C_1 \int_{A_0} |u(x,T;u_0)|^2 \mathrm{d}x \quad \text{for all } u_0 \in L^2(\mathbb{R}^+). \tag{5.16}$$

we define

$$u_{T,f}(x) := \mathrm{e}^{-ix^2/4T} f(x), \qquad x \in \mathbb{R}^+, \tag{5.17}$$

for each $f \in L^2(\mathbb{R}^+)$.

By (5.17), (5.16) and (2.5), it follows that

$$\begin{aligned}\int_{\mathbb{R}^+} |F_\nu(f)(x)|^2 \mathrm{d}x &= \int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^+} |u_{T,f}(x)|^2 \mathrm{d}x \\ &\leq C_1 \int_{A_0} |u(x,T;u_{T,f})|^2 \mathrm{d}x = C_1 \int_{A_0/2T} |F_\nu(f)(x)|^2 \mathrm{d}x,\end{aligned}$$

for every $f \in L^2(\mathbb{R}^+)$. Since $|A_0^c| > 0$, by taking $f \in L^2(\mathbb{R}^+)\backslash\{0\}$ with $\operatorname{supp} F_\nu(f) \subset A_0^c/2T$ in the above inequality, we are led to a contradiction.

□

5.2. The sharpness of Theorem 1.2

Next, we establish Theorem 5.3 below, which demonstrates the sharpness of Theorem 1.2. Before that, we need a lemma, whose proof borrows some ideas from [56], Proposition 3.4.

Lemma 5.2. *Let $A \subset \mathbb{R}^+$ be a measurable set such that* $\operatorname{dist}(0, A) > 0$. *For each $\nu \geq 0$, there exist constants $C_0 > 0$ and $N_0 > 0$ such that for all $N \geq N_0$, there exists a function $f \in L^2(\mathbb{R}^+)$ with* $\operatorname{supp} F_\nu(f) \subset [0, N]$ *satisfying*

$$\|f\|_{L^2(\mathbb{R}^+)} \geq \mathrm{e}^{C_0 N} \|f\|_{L^2(A)}.$$

Proof. Let

$$d_0 := \operatorname{dist}(0, A) > 0. \tag{5.18}$$

We consider the function $\phi_s(x) = x^{\nu+\frac{1}{2}}\mathrm{e}^{-sx^2}$ defined for $s > 0$ and $x \in \mathbb{R}^+$, whose Hankel transform is given by (see [44], p. 29)

$$F_\nu(\phi_s)(y) = \frac{y^{\nu+\frac{1}{2}}}{(2s)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{4s}}, \quad s > 0, \ y \in \mathbb{R}^+. \tag{5.19}$$

We define $f \in L^2(\mathbb{R}^+)$ by its Hankel transform as follows:

$$F_\nu(f)(y) = \chi_{(0,N]}(y)\frac{y^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{N}}, \quad y \in \mathbb{R}^+, \ N > 0.$$

We first give an estimate of the L^2-norm of f over the whole domain $\mathbb{R}^+$; we have

$$\|f\|_{L^2(\mathbb{R}^+)} \geq 2^{\nu-1}\Gamma(\nu+1)N^{-(\nu+1)}, \tag{5.20}$$

In fact, by using the unitarity of F_ν and performing the following change of variables $t = \frac{2y^2}{N}$, we obtain that

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^+)} = \|F_\nu(f)\|^2_{L^2(\mathbb{R}^+)} &= \int_0^N \left(\frac{y^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}}\right)^2 \mathrm{e}^{-\frac{2y^2}{N}}\mathrm{d}y = \frac{2^\nu}{N^{\nu+1}}\int_0^{2N} t^\nu \mathrm{e}^{-t}\mathrm{d}t \\
&= \frac{2^\nu}{N^{\nu+1}}\left(\int_0^\infty t^\nu \mathrm{e}^{-t}\mathrm{d}t - \int_{2N}^\infty t^\nu \mathrm{e}^{-t}\mathrm{d}t\right) \\
&\geq \frac{2^\nu}{N^{\nu+1}}\left(\int_0^\infty t^\nu \mathrm{e}^{-t}\mathrm{d}t - \mathrm{e}^{-N}\int_{2N}^\infty t^\nu \mathrm{e}^{-t/2}\mathrm{d}t\right) \\
&\geq \frac{2^\nu}{N^{\nu+1}}\left(\int_0^\infty t^\nu \mathrm{e}^{-t}\mathrm{d}t - \mathrm{e}^{-N}\int_0^\infty t^\nu \mathrm{e}^{-t/2}\mathrm{d}t\right) \\
&= \frac{2^\nu\Gamma(\nu+1)}{N^{\nu+1}}(1 - 2^{\nu+1}\mathrm{e}^{-N}).
\end{aligned}$$

By choosing N sufficiently large such that $2^{\nu+1}\mathrm{e}^{-N} < 1/2$, we obtain (5.20).

We now aim to estimate the L^2-norm of f over the subset A. To this end, we use the inverse Hankel transform, which yields

$$\begin{aligned}
f(x) &= \int_0^\infty \sqrt{xy}J_\nu(xy)F_\nu f(y)\mathrm{d}y \\
&= \int_0^N \sqrt{xy}J_\nu(xy)\frac{y^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{N}}\mathrm{d}y \\
&= \int_0^\infty \sqrt{xy}J_\nu(xy)\frac{y^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{N}}\mathrm{d}y - \int_N^\infty \sqrt{xy}J_\nu(xy)\frac{y^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}}\mathrm{e}^{-\frac{y^2}{N}}\mathrm{d}y \\
&= \phi_{s=\frac{N}{4}}(x) - R(x).
\end{aligned} \tag{5.21}$$

For the first term in (5.21), by (5.18) and some computations similar to the estimate of $\|f\|^2_{L^2(\mathbb{R}^+)}$, we get

$$\|\phi_{s=\frac{N}{4}}\|^2_{L^2(A)} \leq \int_{x>d_0}\left(\phi_{s=\frac{N}{4}}(x)\right)^2\mathrm{d}x = \int_{x>d_0}(x^{\nu+\frac{1}{2}}\mathrm{e}^{-\frac{N}{4}x^2})^2\mathrm{d}x \leq 2^{2\nu+1}\Gamma(\nu+1)\mathrm{e}^{-\frac{N}{4}d_0^2}N^{-(\nu+1)}.$$

For the second term in (5.21), we apply the unitarity of the Hankel transform together with some computations to obtain

$$\|R\|_{L^2(A)}^2 \le \|R\|_{L^2(\mathbb{R}^+)}^2 = \int_N^\infty \left(\frac{x^{\nu+\frac{1}{2}}}{\left(\frac{N}{2}\right)^{\nu+1}} \right)^2 \mathrm{e}^{-\frac{2x^2}{N}} \mathrm{d}x \le 2^{2\nu+1}\Gamma(\nu+1)\mathrm{e}^{-N} N^{-(\nu+1)}.$$

Setting $C_1 = \min\{1, d_0^2/4\}$, we thus obtain

$$\|f\|_{L^2(A)} \le 2^{2\nu+1}\Gamma(\nu+1)\mathrm{e}^{-C_1 N} N^{-(\nu+1)}. \tag{5.22}$$

We conclude the proof with (5.20), (5.22) and by choosing C_0 such that $0 < C_0 < C_1$. □

Theorem 5.3. *(i) For every $\nu \ge 0$, let $A = [a_1, a_2] \subset \mathbb{R}^+$, $a = a_2 - a_1$, and $a, \lambda, T > 0$, $\beta \ge 1$. Then one can find a sequence $\{u_k\}_{k\in\mathbb{N}^+} \subset C_0^\infty(\mathbb{R}^+)$ and $M > 0$ such that*

$$\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x^\beta} |u_k(x)|^2 \mathrm{d}x \le M \text{ and } \int_{\mathbb{R}^+} |u_k(x)|^2 \mathrm{d}x = 1 \tag{5.23}$$

and

$$\lim_{k\to\infty} \int_A |u(x, T; u_k)|^2 \mathrm{d}x = 0. \tag{5.24}$$

(ii) For every $\nu \ge 0$, let $\lambda, b, T > 0$ and $\alpha(s)$, $s \in \mathbb{R}^+$, be an increasing function with $\lim_{s\to\infty} \alpha(s)/s = 0$. Then for each $\gamma \in (0,1)$, there is no positive constant C such that for all $u_0 \in C_0^\infty(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |u_0(x)|^2 \mathrm{d}x \le C \left(\int_{(0,b]^c} |u(x, T; u_0)|^2 \mathrm{d}x \right)^\gamma \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda\alpha(x)} |u_0(x)|^2 \mathrm{d}x \right)^{1-\gamma}. \tag{5.25}$$

Proof. (i) Suppose x_0 be the center of A. Let g and g_k, with $k \in \mathbb{N}^+$, satisfy (5.6) and (5.7), respectively. As a same way to the proof of Theorem 5.1(i), we define a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^+)$ as follows:

$$u_k(x) := \mathrm{e}^{-ix^2/4T} g_k(x), \quad x \in \mathbb{R}^+, \ k \in \mathbb{N}^+. \tag{5.26}$$

We can get

$$\int_{\mathbb{R}^+} |u_k(x)|^2 \mathrm{d}x = 1 \quad \text{for all } k \in \mathbb{N}^+$$

and

$$\lim_{k\to\infty} \int_A |u(x, T; u_k)|^2 \mathrm{d}x = 0.$$

Meanwhile, from (5.6), (5.7) and (5.26), we find that for each $k \in \mathbb{N}^+$,

$$\begin{aligned}\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x^\beta} |u_k(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x^\beta} |g_k(x)|^2 \mathrm{d}x = \int_{x_0}^\infty \mathrm{e}^{\lambda x^\beta} |k^{\frac{1}{2}} g(k(x - x_0))|^2 \mathrm{d}x \\ = \int_{\mathbb{R}^+} \mathrm{e}^{\lambda(\frac{x}{k}+x_0)^\beta} |g(x)|^2 \mathrm{d}x \le \int_{\mathbb{R}^+} \mathrm{e}^{\lambda(x+x_0)^\beta} |g(x)|^2 \mathrm{d}x < \infty.\end{aligned}$$

Hence, this ends the proof of (i).
(ii) To derive a contradiction, we assume that the conclusion is not true. Then there exist $\bar{b}$, $\bar{\lambda}$, $\bar{T}$, $\bar{C} > 0$, $\bar{\gamma} \in (0,1)$, and an increasing function $\bar{\alpha}(s)$ on $(0,\infty)$ with $\lim_{s\to\infty} \bar{\alpha}(s)/s = 0$ such that for each $v_0 \in C_0^\infty(\mathbb{R}^+)$,

$$\int_{\mathbb{R}^+} |v_0(x)|^2 \mathrm{d}x \leq \bar{C} \left(\int_{(0,\bar{b}]^c} |u(x,\bar{T};v_0)|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\bar{\lambda}\bar{\alpha}(x)} |v_0(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}}. \tag{5.27}$$

For any fixed $g \in L^2(\mathbb{R}^+)$ with $F_\nu(g) \in C_0^\infty(\mathbb{R}^+)$, we define $v_{0,g} \in C_0^\infty(\mathbb{R}^+)$ *via*

$$F_\nu(g)(x) = (2\bar{T})^{\frac{1}{2}} \mathrm{e}^{-\frac{i}{2}(\nu+1)\pi} \mathrm{e}^{i\bar{T}x^2} v_{0,g}(2\bar{T}x), \quad x \in \mathbb{R}^+. \tag{5.28}$$

From (3.33), (3.35) (with $(T,u_0) = (\bar{T}, v_{0,g})$) and (5.28), we find that

$$g(x) = \mathrm{e}^{-ix^2/4\bar{T}} u(x,\bar{T};v_{0,g}), \quad x \in \mathbb{R}^+. \tag{5.29}$$

By (5.29), the conservation law (for the Schrödinger equation), (5.27) and (5.28), we get that

$$\begin{aligned}
\int_{\mathbb{R}^+} |g(x)|^2 \mathrm{d}x &= \int_{\mathbb{R}^+} |u(x,\bar{T};v_{0,g})|^2 \mathrm{d}x = \int_{\mathbb{R}^+} |v_{0,g}(x)|^2 \mathrm{d}x \\
&\leq \bar{C} \left(\int_{(0,\bar{b}]^c} |u(x,\bar{T};v_{0,g})|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\bar{\lambda}\bar{\alpha}(x)} |v_{0,g}(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}} \\
&= \bar{C} \left(\int_{(0,\bar{b}]^c} |g(x)|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\bar{\lambda}\bar{\alpha}(2\bar{T}x)} |F_\nu g(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}}.
\end{aligned}$$

From this, using a standard density argument, we obtain that for each $g \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} F_\nu(g)$ compact,

$$\int_{\mathbb{R}^+} |g(x)|^2 \mathrm{d}x \leq \bar{C} \left(\int_{(0,\bar{b}]^c} |g(x)|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\bar{\lambda}\bar{\alpha}(2\bar{T}x)} |F_\nu g(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}}.$$

Since $\bar{\alpha}(\cdot)$ is increasing and because the Hankel transform is an isometry, the above implies that for each $N \geq 1$ and each $g \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} F_\nu(g) \subset [0,N]$,

$$\begin{aligned}
\int_{\mathbb{R}^+} |g(x)|^2 \mathrm{d}x &\leq \bar{C} \left(\int_{(0,\bar{b}]^c} |g(x)|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} \mathrm{e}^{\bar{\lambda}\bar{\alpha}(2\bar{T}x)} |F_\nu g(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}} \\
&= \bar{C} \mathrm{e}^{(1-\bar{\gamma})\bar{\lambda}\bar{\alpha}(2\bar{T}N)} \left(\int_{(0,\bar{b}]^c} |g(x)|^2 \mathrm{d}x \right)^{\bar{\gamma}} \left(\int_{\mathbb{R}^+} |F_\nu g(x)|^2 \mathrm{d}x \right)^{1-\bar{\gamma}}.
\end{aligned} \tag{5.30}$$

So (5.30) implies that for $N \geq 1$ and each $g \in L^2(\mathbb{R}^+)$ with $\operatorname{supp} F_\nu(g) \subset [0,N]$,

$$\int_{\mathbb{R}^+} |g(x)|^2 \mathrm{d}x \leq \bar{C}^{\frac{1}{\bar{\gamma}}} \mathrm{e}^{\frac{1-\bar{\gamma}}{\bar{\gamma}} \bar{\lambda}\bar{\alpha}(2\bar{T}N)} \int_{(0,\bar{b}]^c} |g(x)|^2 \mathrm{d}x. \tag{5.31}$$

In addition, according to Lemma 5.2, there are C_0, $N_0 > 0$ such that for each $N \geq N_0$ there is $f_N \in L^2(\mathbb{R}^+)\backslash\{0\}$ with $\operatorname{supp} F_\nu(f_N) \subset [0, N]$ such that

$$\mathrm{e}^{C_0 N} \int_{(0,\bar{b}]^c} |f_N(x)|^2 \mathrm{d}x \leq \int_{\mathbb{R}^+} |f_N(x)|^2 \mathrm{d}x. \tag{5.32}$$

By (5.31) and (5.32), we get that for each $N \geq N_0$,

$$\mathrm{e}^{C_0 N} \leq \bar{C}^{\frac{1}{\bar{\gamma}}} \mathrm{e}^{\frac{1-\bar{\gamma}}{\bar{\gamma}} \bar{\lambda} \bar{\alpha}(2\bar{T}N)},$$

from which it follows that

$$0 < \frac{\bar{\gamma} C_0}{2(1-\bar{\gamma})\bar{\lambda}\bar{T}} \leq \lim_{N\to\infty} \frac{\bar{\alpha}(2\bar{T}N)}{2\bar{T}N}.$$

This contradicts $\lim_{s\to\infty} \frac{\bar{\alpha}(s)}{s} = 0$. Hence, the conclusion (ii) is true. □

6. Applications

We now turn to the applications. We mainly consider the applications of Theorem 1.1–1.6 to different kinds of controllability for the Schrödinger equations. Based on the abstract result [23], Lemma 5.1, which establishes the equivalence between observability and controllability, we derive the following results. From the perspective of controllability type, Theorem 6.1 and Theorem 6.7 concern exact controllability (see Rem. 6.2 for the detailed explanations); Theorem 6.8 concerns nonstandard exact controllability (see Rem. 6.9); Theorems 6.3 and 6.12 give two kinds of nonstandard approximate controllability (see Rem. 6.4 and Rem. 6.13); Thm. 6.5 and Theorem 6.10 yield two kinds of nonstandard approximate null controllability (see Rem. 6.6 and Rem. 6.11).

6.1. Applications of Theorems 1.1–1.3 to controllability

We first combine Theorem 1.1 with Lemma 5.1 in [23] to establish the following exact controllability result.

Theorem 6.1. *Let A, B be two measurable sets in $\mathbb{R}^+$ with finite measure. Let $0 \leq t_1 < t_2 \leq T$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=t_1\}}\chi_{A^c}(x) f_1(x) + \delta_{\{t=t_2\}}\chi_{B^c}(x) f_2(x),\ x \in \mathbb{R}^+,\ t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.1}$$

Denote by $u_1(\cdot,\cdot,u_0,f_1,f_2)$ the solution to the equation (6.1). Then for any u_0, $u_T \in L^2(\mathbb{R}^+)$, there exists a pair of controls $(f_1, f_2) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ such that

$$u_1(x,T,u_0,f_1,f_2) = u_T(x),\ x \in \mathbb{R}^+, \tag{6.2}$$

and

$$\|f_1\|^2_{L^2(\mathbb{R}^+)} + \|f_2\|^2_{L^2(\mathbb{R}^+)} \leq C\|u_T - \mathrm{e}^{-iH_\nu T}u_0\|^2_{L^2(\mathbb{R}^+)}, \tag{6.3}$$

where the constant $C = C(\nu, A, B, t_2 - t_1)$ is given by Theorem 1.1.

Proof. We sketch the proof here. Consider the following dual equation

$$\begin{cases} i\partial_t\psi(x,t) - H_\nu\psi(x,t) = 0, \ x \in \mathbb{R}^+, \ t \in (0,T), \\ \psi(x,T) = g(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.4}$$

Write $\psi(\cdot,\cdot,T,g)$ for the solution to (6.4). Then Theorem 1.1 implies that

$$\int_{\mathbb{R}^+} |g|^2 \mathrm{d}x \le C\left(\int_{A^c} |\psi(x,t_1,T,g)|^2 \mathrm{d}x + \int_{B^c} |\psi(x,t_2,T,g)|^2 \mathrm{d}x\right). \tag{6.5}$$

Next, we use [23], Lemma 5.1 and (6.5) to prove (6.2) and (6.3). For this purpose, we let

$$X := L^2(\mathbb{R}^+) = X^*, \quad Y := L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) = Y^*, \quad Z := L^2(\mathbb{R}^+) = Z^*$$

and define the state transformation operator $R : Z \to X$ and the observation operator $O : Z \to Y$ as follows:

$$Rg := g, \quad Og := (\chi_{A^c}(\cdot)\psi(\cdot,t_1,T,g), \ \chi_{B^c}(\cdot)\psi(\cdot,t_2,T,g)) \quad \text{for } g \in Z. \tag{6.6}$$

From (6.5) and (6.6), we find that for each $g \in L^2(\mathbb{R}^+)$,

$$\|Rg\|^2_{L^2(\mathbb{R}^+)} \le C\|Og\|^2_{L^2(\mathbb{R}^+)\times L^2(\mathbb{R}^+)} + \frac{1}{k}\|g\|^2_{L^2(\mathbb{R}^+)}, \quad k \in \mathbb{N}^+. \tag{6.7}$$

By Lemma 5.1 in [23], there is a pair $(f_{1k}, f_{2k}) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, $k \in \mathbb{N}^+$ such that the following dual inequality holds

$$\frac{1}{C}\|(f_{1k}, f_{2k})\|^2_{L^2(\mathbb{R}^+)\times L^2(\mathbb{R}^+)} + k\|R^*g - O^*(f_{1k}, f_{2k})\|^2_{L^2(\mathbb{R}^+)} \le \|g\|^2_{L^2(\mathbb{R}^+)}, \quad k \in \mathbb{N}^+, \tag{6.8}$$

where

$$R^*g = g; \quad O^*(f_{1k}, f_{2k}) = u_1(\cdot, T, 0, f_{1k}, f_{2k}). \tag{6.9}$$

Define

$$g(x) = u_T(x) - \mathrm{e}^{-iH_\nu T}u_0(x), \quad x \in \mathbb{R}^+. \tag{6.10}$$

Then (6.2) and (6.3) are followed by choosing a weakly convergent subsequence in (6.8) and a limiting procedure. □

Remark 6.2. The above theorem can be understood as follows: For any u_0, $u_T \in L^2(\mathbb{R}^+)$, there exists a pair of controls $(f_1, f_2) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$ steering the solution of (6.1) from u_0 at time 0 to u_T at time T. Moreover, a bound of the norm of this pair of controls is explicitly given.

Similarly, by combining Theorems 1.2–1.3 with Lemma 5.1 in [23], we obtain the following controllability results. Before stating the main results, we define, for each $\lambda > 0$,

$$X_\lambda := \left\{ f \in L^2(\mathbb{R}^+) : \int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|f(x)|^2 \,\mathrm{d}x < \infty \right\},$$

with the norm $\|\cdot\|_{X_\lambda}$ given by

$$\|f\|_{X_\lambda} := \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|f(x)|^2 \,\mathrm{d}x\right)^{1/2}, \quad f \in X_\lambda.$$

One can directly check that the dual space of X_λ is

$$X_\lambda^* = \overline{C_0^\infty(\mathbb{R}^+)}^{\|\cdot\|_{X_\lambda^*}},$$

with the norm $\|\cdot\|_{X_\lambda^*}$ given by

$$\|g\|_{X_\lambda^*} := \left(\int_{\mathbb{R}^+} \mathrm{e}^{-\lambda x}|g(x)|^2 \,\mathrm{d}x\right)^{1/2}, \quad g \in X_\lambda^*.$$

Theorem 6.3. *Let $b, \lambda > 0$ and $T > s \geq 0$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=s\}}\chi_{(0,b]^c}(x) f(x,t), \ x \in \mathbb{R}^+, \ t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.11}$$

Denote by $u_2(\cdot,\cdot,u_0,f)$ the solution to the equation (6.11). Then for any $\varepsilon > 0$ and u_0, $u_T \in L^2(\mathbb{R}^+)$, there exists a control $f \in L^2(\mathbb{R}^+)$ such that

$$\varepsilon^{\frac{1-q}{q}} \int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x + \varepsilon^{-1}\|u_2(\cdot,T,u_0,f) - u_T(\cdot)\|_{X_\lambda^*}^2 \leq C\left(1 + \frac{b^{2\nu+2}}{(\lambda(T-s))^{2\nu+2}}\right)\int_{\mathbb{R}^+} |u_T(x) - \mathrm{e}^{-iH_\nu T}u_0(x)|^2 \mathrm{d}x, \tag{6.12}$$

where $q := \theta^{1+\frac{b}{\lambda(T-s)}} \in (0,1)$, C and θ are given by Theorem 1.2 (i).

Remark 6.4. The above theorem can be understood as follows: For any u_0, $u_T \in L^2(\mathbb{R}^+)$ and $\varepsilon > 0$, there exists a control $f \in L^2(\mathbb{R}^+)$ steering the solution of (6.11) from u_0 at time 0 to the target $B_\varepsilon^{X_\lambda^*}(u_T)$ at time T. (Here, $B_\varepsilon^{X_\lambda^*}(u_T)$ denotes the closed ball in X_λ^*, centered at u_T and of radius ε.) Moreover, a bound of the norm of this control is explicitly given.

Theorem 6.5. *Given any interval $A = [a_1, a_2] \subset \mathbb{R}^+$, $B = [b_1, b_2] \subset \mathbb{R}^+$, $a = a_2 - a_1$, $b = b_2 - b_1$, and $a, b, \lambda, T > 0$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=0\}}\chi_B(x) f(x,t), \ x \in \mathbb{R}^+, \ t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.13}$$

Denote by $u_3(\cdot,\cdot,u_0,f)$ the solution to the equation (6.13). Then for any $\varepsilon > 0$ and $u_0 \in \tilde{L}^2(A;\mathbb{C})$, there exists a control $f \in L^2(\mathbb{R}^+)$ such that

$$\varepsilon^{\frac{1-\theta^p}{\theta^p}} \int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x + \varepsilon^{-1}\|u_3(\cdot,T,u_0,f)\|_{X_\lambda^*}^2 \leq C(a_2^{2\nu+2} - a_1^{2\nu+2})((\lambda T) \wedge b)^{-(2\nu+2)} \int_A |u_0(x)|^2 \mathrm{d}x, \tag{6.14}$$

where C, θ and p are given by Theorem 1.3, $\tilde{L}^2(A) := \{g \in L^2(\mathbb{R}^+) : g = 0 \text{ on } A^c\}$.

Remark 6.6. The above theorem can be understood as follows: For any $u_0 \in \tilde{L}^2(A;\mathbb{C})$ and $\varepsilon > 0$, there exists a control $f \in L^2(\mathbb{R}^+)$ steering the solution of (6.13) from u_0 at time 0 to the target $B_\varepsilon^{X_\lambda^*}(0)$ at time T. Moreover, a bound of the norm of this control is explicitly given.

6.2. Applications of Theorems 1.4–1.6 to controllability

First, as a consequence of the classical HUM method (see, for example, [17]), the inequality (1.17) is equivalent to the following standard exact controllability result for the Schrödinger equation.

Theorem 6.7. *Let $b > 0$. Then for any $T > 0$ and any $(u_0, u_T) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, there exists a control $f \in L^2(\mathbb{R}^+ \times (0,T))$ such that the unique solution of the Schrödinger equation*

$$i\partial_t u(x,t) - H_\nu u(x,t) = \chi_{(0,b]^c}(x) f(x,t) \tag{6.15}$$

with initial data $u(x,0) = u_0(x)$ satisfies $u(x,T) = u_T(x)$.

This theorem states that for any u_0, $u_T \in L^2(\mathbb{R}^+)$, there exists a control $f \in L^2(\mathbb{R}^+ \times (0,T))$ steering the solution of (6.15) from u_0 at time 0 to u_T at time T. Next, as presented in Subsection 6.1, we will combine Lemma 5.1 from [23] with the results of Theorem 1.4 (ii), Theorem 1.5, and Theorem 1.6, respectively, to obtain various non-standard controllability results.

Theorem 6.8. *Let $b, N > 0$ and $0 \le s < T$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=s\}} \chi_{(0,b]^c}(x) f(x,t), \ x \in \mathbb{R}^+, \ t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.16}$$

Denote by $u_4(\cdot,\cdot,u_0,f)$ the solution to the equation (6.16). Then for any $u_0, u_T \in L^2(\mathbb{R}^+)$, there exists a control $f \in L^2(\mathbb{R}^+)$ such that

$$u_4(x,T,u_0,f) = u_T(x), \quad x \in (0,N] \tag{6.17}$$

and

$$\|f\|^2_{L^2(\mathbb{R}^+)} \le \mathrm{e}^{C(1+\frac{bN}{T-s})} \|u_T - \mathrm{e}^{-iH_\nu T} u_0\|^2_{L^2(\mathbb{R}^+)}, \tag{6.18}$$

where the constant $C = C(\nu)$ is given by Theorem 1.4(ii).

Remark 6.9. The above theorem can be understood as follows: For any u_0, $u_T \in L^2(\mathbb{R}^+)$ and $N > 0$, there exists a control $f \in L^2(\mathbb{R}^+)$ steering the solution of (6.16) from u_0 at time 0 to u_T at time T over the interval $(0,N]$. Moreover, a bound of the norm of this control is explicitly given.

Theorem 6.10. *Given any interval $B = [b_1, b_2] \subset \mathbb{R}^+$, $b = b_2 - b_1$, $b, \lambda_1, \lambda_2, T > 0$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=0\}} \chi_B(x) f(x,t), \ x \in \mathbb{R}^+, \ t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.19}$$

Denote by $u_5(\cdot,\cdot,u_0,f)$ the solution to the equation (6.19). Then for any $\varepsilon \in (0,1)$ and $u_0 \in X_{\lambda_2}$, there exists a control $f \in L^2(\mathbb{R}^+)$ such that

$$\frac{1}{\varepsilon} \mathrm{e}^{-(\frac{1}{\varepsilon})^{1+\frac{C}{\lambda_2((\lambda_1 T)\wedge \frac{b}{2})}}} \int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}x + \frac{1}{\varepsilon} \|u_5(\cdot,T,u_0,f)\|^2_{X^*_{\lambda_1}} \le C(x_0, b, \lambda_1, \lambda_2, T) \|u_0\|^2_{X_{\lambda_2}}, \tag{6.20}$$

where $C(x_0, b, \lambda_1, \lambda_2, T)$ and C are given by Theorem 1.5.

Remark 6.11. The above theorem can be understood as follows: For each $u_0 \in X_{\lambda_2}$ and $\varepsilon > 0$, there exists a control $f \in L^2(\mathbb{R}^+)$ steering the solution of (6.19) from u_0 at time 0 to the target $B_\varepsilon^{X^*_{\lambda_1}}(0)$ at time T. Moreover, a bound of the norm of this control is explicitly given.

Finally, we are going to show the last theorem. Before stating the main result, we introduce a functional space. For each $\lambda > 0$, we denote R_λ for the completion of $C_0^\infty(\mathbb{R}^+)$ in the norm

$$\|f\|_{R_\lambda} := \left(\int_{\mathbb{R}^+} \mathrm{e}^{\lambda x}|f(x)|^2 \mathrm{d}x + \|f\|^2_{H^{4([\nu]+3)}(\mathbb{R}^+)} + \int_{\mathbb{R}^+} \frac{1}{x^{4([\nu]+3)}}|f(x)|^2\mathrm{d}x\right)^{1/2}, \quad f \in C_0^\infty(\mathbb{R}^+).$$

Write R^*_λ for the dual space of R_λ with respect to the pivot space $L^2(\mathbb{R}^+)$.

Theorem 6.12. *Given any interval $B = [b_1, b_2] \subset \mathbb{R}^+$, $b = b_2 - b_1$, $b, \lambda > 0$ and $T > s \geq 0$. Consider the following impulse controlled Schrödinger equation:*

$$\begin{cases} i\partial_t u(x,t) - H_\nu u(x,t) = \delta_{\{t=s\}}\chi_B(x) f(x,t), \; x \in \mathbb{R}^+, \; t \in (0,T), \\ u(x,0) = u_0(x) \in L^2(\mathbb{R}^+). \end{cases} \tag{6.21}$$

Denote by $u_6(\cdot,\cdot,u_0,f)$ the solution to the equation (6.21). Then for any $\varepsilon \in (0,1)$ and $u_0, u_T \in L^2(\mathbb{R}^+)$, there exists a control $f \in L^2(\mathbb{R}^+)$ such that

$$\varepsilon^{-1}\mathrm{e}^{-\mathrm{e}^{\varepsilon^{-2}}} \int_{\mathbb{R}^+} |f(x)|^2\mathrm{d}x + \varepsilon^{-1}\|u_6(\cdot,T,u_0,f) - u_T(\cdot)\|^2_{R^*_\lambda} \leq C(x_0,b,\lambda,T-s)\|u_T - \mathrm{e}^{-iH_\nu T}u_0\|^2_{L^2(\mathbb{R}^+)}, \tag{6.22}$$

where $C(x_0,b,\lambda,T-s)$ is given by Theorem 1.6.

Remark 6.13. The above theorem can be understood as follows: For any u_0, $u_T \in L^2(\mathbb{R}^+)$ and $\varepsilon > 0$, there exists a control $f \in L^2(\mathbb{R}^+)$ steering the solution of (6.21) from u_0 at time 0 to the target $B_\varepsilon^{R^*_\lambda}(u_T)$ at time T. (Here, $B_\varepsilon^{R^*_\lambda}(u_T)$ denotes the closed ball in R^*_λ, centered at u_T and of radius ε.) Moreover, a bound of the norm of this control is explicitly given.

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No new data/codes were created or analyzed in this study.

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