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GLOBAL NULL CONTROLLABILITY OF STOCHASTIC SEMILINEAR COMPLEX GINZBURG–LANDAU EQUATIONS

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Abstract. In this paper, we study the null controllability of forward and backward stochastic semilinear complex Ginzburg–Landau equations with global Lipschitz nonlinear terms. For this purpose, by deriving an improved global Carleman estimates for linear systems, we obtain the controllability results for the stochastic linear systems with a L^2-valued source term. Based on it, together with a Banach fixed point argument, the desired null controllability of semilinear systems is derived.

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1. Introduction and main results

The Ginzburg–Landau equation is an important nonlinear mathematical physics equation, which was proposed initially by Ginzburg and Landau in [1] to serve as a phenomenological description of superconductivity. Now, it is used to describe many other physical phenomena, such as superfluidity, Bose–Einstein condensation, liquid crystals and strings in field theory, nonlinear waves, and second-order phase transition (see *e.g.*, [2, 3]). For more physical background and applications, we refer to [2, 4, 5] and the references therein. In practice, systems are usually affected by uncertainty. When considering the random influence, stochastic complex Ginzburg–Landau equations arise naturally. In many situations, stochastic partial differential equations (SPDEs) are more realistic than deterministic ones. Recently, stochastic complex Ginzburg–Landau equations have been extensively studied (see *e.g.*, [6–8]). In this paper, we will study the controllability of stochastic semilinear complex Ginzburg–Landau equations.

Let G be a nonempty bounded domain in $\mathbb{R}^n$ (n is a positive integer) with a boundary Γ of class C^4 and $T > 0$. Put $Q := (0,T) \times G$ and $\Sigma := (0,T) \times \Gamma$. Assume G_0 to be a given nonempty open subset of G and denote by χ_{G_0} the characteristic function of the set G_0. For simplicity, we will use the notation $y_j := \frac{\partial y}{\partial x_j}$, where x_j is the jth coordinate of a generic point $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$. Similarly, we use the notation z_j, v_j, *etc.* for the partial derivatives of z, v with respect to x_j. Moreover, for any complex number c, we denote by $\overline{c}$, Rec, and Imc its complex conjugate, real part, and imaginary part, respectively.

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $B(\cdot)$, augmented

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by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Let H be a Banach space. Denote by $L^2_{\mathcal{F}_t}(\Omega;H)$ the space of all $\mathcal{F}_t$-measurable random variables ζ such that $\mathbb{E}|\zeta|^2_H<\infty$; denote by $L^2_{\mathbb{F}}(0,T;H)$ the space consisting of all H-valued $\mathbf{F}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)})<\infty$; by $L^\infty_{\mathbb{F}}(0,T;H)$ the space consisting of all H-valued $\mathbf{F}$-adapted bounded processes $X(\cdot)$; and by $L^2_{\mathbb{F}}(\Omega;C([0,T];H))$ the space consisting of all H-valued $\mathbf{F}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C(0,T;H)})<\infty$. Similarly, one can define $L^2_{\mathbb{F}}(\Omega;C^k([0,T];H))$ for any positive integer k. All of these spaces are Banach spaces with canonical norms.

Consider the following controlled forward stochastic complex Ginzburg–Landau equation:

$$\begin{cases} \mathrm{d}y-(a+\mathrm{i}b)\sum\limits_{j,k=1}^{n}(a^{jk}y_j)_k\mathrm{d}t=[f(\omega,t,x,y)+\chi_{G_0}h]\mathrm{d}t+[g(\omega,t,x,y)+H]\mathrm{d}B(t) & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(0)=y_0 & \text{in } G, \end{cases} \tag{1.1}$$

and controlled backward stochastic complex Ginzburg–Landau equation:

$$\begin{cases} \mathrm{d}y+(a-\mathrm{i}b)\sum\limits_{j,k=1}^{n}(a^{jk}y_j)_k\mathrm{d}t=[\Upsilon(\omega,t,x,y,Y)+\chi_{G_0}h]\mathrm{d}t+Y\mathrm{d}B(t) & \text{in } Q,\\ y=0 & \text{on } \Sigma,\\ y(T)=y_T & \text{in } G, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit, and $a>0$, $b\in\mathbb{R}$. In (1.1) (resp., (1.2)), y_0 (y_T) is a given initial value (resp., terminal value). In both cases, y is the state variable. In (1.1), the control variable consists of the pair (h,H); while in (1.2), the control variable is only h. Unless otherwise stated, we assume that all functions mentioned in this paper are complex-valued. We also assume that the functions a^{jk}, f, g and Υ satisfy the following assumptions:

($\mathbf{H_1}$). $a^{jk}(x)\in C^2(\overline{G};\mathbb{R})$ and $a^{jk}=a^{kj}$ $(j,k=1,2,\dots,n)$. There is some constant $s_0>0$ such that $\sum\limits_{j,k=1}^{n}a^{jk}(x)\xi^j\overline{\xi^k}\geq s_0|\xi|^2$ for any $(x,\xi)=(x,\xi^1,\dots,\xi^n)\in G\times\mathbb{C}^n$.

($\mathbf{H_2}$). $f(\cdot,\cdot,\cdot,y)$ and $g(\cdot,\cdot,\cdot,y)$ are $\mathbf{F}$-adapted, L^2-valued stochastic processes for each $y\in L^2(G;\mathbb{C})$.

($\mathbf{H_3}$). $\forall\ (\omega,t,x)\in\Omega\times Q,\ f(\omega,t,x,0)=0$.

($\mathbf{H_4}$). $\exists\ \kappa>0,\ \forall\ (\omega,t,x,a_1,a_2)\in\Omega\times Q\times\mathbb{R}^2,\ |f(\omega,t,x,a_1)-f(\omega,t,x,a_2)|\leq\kappa|a_1-a_2|$.

($\mathbf{H_5}$). $\exists\ \kappa_1>0,\ \forall\ (\omega,t,x,a_1,a_2)\in\Omega\times Q\times\mathbb{R}^2,\ |g(\omega,t,x,a_1)-g(\omega,t,x,a_2)|\leq\kappa_1|a_1-a_2|$.

($\mathbf{H_6}$). $\Upsilon(\cdot,\cdot,\cdot,y,Y)$ are $\mathbf{F}$-adapted, L^2-valued stochastic processes for each $y,Y\in L^2(G)$.

($\mathbf{H_7}$). $\forall\ (\omega,t,x)\in\Omega\times Q,\ \Upsilon(\omega,t,x,0,0)=0$.

($\mathbf{H_8}$). $\exists\ \kappa_2>0,\ \forall\ (\omega,t,x,a_1,a_2,b_1,b_2)\in\Omega\times Q\times\mathbb{R}^4$, $|\Upsilon(\omega,t,x,a_1,b_1)-\Upsilon(\omega,t,x,a_2,b_2)|\leq\kappa_2(|a_1-a_2|+|b_1-b_2|)$.

Remark 1.1. With the assumptions (H_1)-(H_5) and $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G;\mathbb{C}))$, $(h, H) \in L^2_{\mathbb{F}}(0,T; L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T; L^2(G;\mathbb{C}))$, it is known (see *e.g.*, [9],Thm. 3.24) that system (1.1) admits a unique solution

$$y \in \mathcal{H}_T := L^2_{\mathbb{F}}(\Omega; C([0,T]; L^2(G;\mathbb{C}))) \cap L^2_{\mathbb{F}}(0,T; H^1_0(G;\mathbb{C})).$$

With the assumptions (H_1), (H_6)-(H_8) and $y_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G;\mathbb{C}))$, $h \in L^2_{\mathbb{F}}(0,T; L^2(G_0;\mathbb{C}))$, it is known (see *e.g.*, [9],Thm. 4.11) that system (1.2) admits a unique solution

$$(y, Y) \in \mathcal{H}_T \times L^2_{\mathbb{F}}(0,T; L^2(G;\mathbb{C})).$$

The main purpose of this paper is to study the null controllability of the semilinear stochastic complex Ginzburg–Landau equations (1.1) and (1.2), respectively. System (1.1) (resp., (1.2)) is said to be *globally null controllable* if for any $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G;\mathbb{C}))$ (resp., $y_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G;\mathbb{C}))$) , there exists a control $(h, H) \in L^2_{\mathbb{F}}(0,T; L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T; L^2(G;\mathbb{C}))$ (resp., $h \in L^2_{\mathbb{F}}(0,T; L^2(G_0;\mathbb{C}))$) such that the corresponding solution y of (1.1) (resp., (1.2)) satisfies $y(T,\cdot) = 0$ in G, $\mathbb{P}$-a.s. (resp., $y(0,\cdot) = 0$ in G, $\mathbb{P}$-a.s.)

The controllability of deterministic nonlinear partial differential equations (PDEs) has been studied by many authors and the results are relatively rich, such as nonlinear parabolic equation (see *e.g.*, [10–14]), Ginzburg–Landau equation (see *e.g.*, [15, 16]). It can be seen from these results that the authors usually use the following strategies to study controllability problems. First, linearize the nonlinear system and study the controllability of the linearized system. Then the controllability problem of nonlinear systems is solved by using appropriate fixed point methods, usually Schauder or Kakutani fixed point methods. At this point, the property of compactness plays a crucial role, for example, the compact embedding result of the Aubin–Lions lemma is used commonly.

Compared with deterministic PDEs, although the controllability problems of SPDEs have not been widely studied, some progress has been made in recent years. We refer to [8, 17–25] for some known results in this respect. Reference [17, 21, 24] showed the null controllability and approximate controllability for the stochastic parabolic equations. In [8, 18], the null controllability for the stochastic complex Ginzburg–Landau equation was obtained. In [22], the exact controllability for stochastic Schrödinger equation was established. References [20] and [25] showed the exact controllability for a refined stochastic wave equation and a refined stochastic beam equation, respectively. In [19] and [23], the null controllability for the stochastic fourth order parabolic type equations were discussed.

It is worth mentioning that the above works mainly focus on the controllability of linear SPDEs systems, while there is little literature on the controllability of nonlinear stochastic systems. This is due to the lack of compactness embedding for the function spaces related to SPDEs (see [24], Rem. 2.5 or [9]), which makes some classical strategies in deterministic setting (see *e.g.*, [13]) fail to establish null controllability for semilinear systems at the stochastic level. As far as we know, the known results in this direction seem to be only [26–29], in which the author established the null controllability of semilinear stochastic parabolic equations. However, to our best knowledge, nothing is known about the controllability of nonlinear stochastic complex Ginzburg–Landau equations. In this paper, we study the null controllability problems for stochastic semilinear complex Ginzburg–Landau equations (1.1) and (1.2).

The main results in this paper read as follows.

Theorem 1.2. *Let assumptions (H_1)-(H_5) be satisfied. Then system (1.1) is globally null controllable.*

Theorem 1.3. *Let assumptions (H_1), (H_6)-(H_8) be satisfied. Then system (1.2) is globally null controllable.*

Our strategy for proving Theorem 1.2 is as follows. To overcome the lack of compactness mentioned above, we borrow some ideas from [27, 30]. First, using the weighted identity method, we obtain a new global Carleman estimate for the backward stochastic complex Ginzburg–Landau equation with a L^2-valued source by introducing a suitable singular weighted function. Next, by combining the new Carleman estimate and a duality technique

introduced in [30], we establish the null controllability for a linear system of the form

$$\begin{cases} \mathrm{d}y - (a+\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}y_j)_k\mathrm{d}t = (F+\chi_{G_0}h)\mathrm{d}t + H\mathrm{d}B(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases}$$

where the source F is in some suitable functional space. At last, we need to prove that a nonlinear mapping $F \to f(\omega,t,x,y)$ is strictly contractive in a suitable functional space. Through a Banach fixed point method which does not rely on any compactness argument, we can obtain the null controllability for (1.1). The strategy to prove Theorem 1.3 is very close to that of Theorem 1.2, but one major difference can be found. For this case, it is not necessary to prove a Carleman estimate for the forward stochastic Ginzburg–Landau equation. Actually, it suffices to use the deterministic Carleman estimate and also employ the duality technique introduced by [30].

In this paper, the dual technique is mainly used to establish the null controllability of a linear system with a source term (see Thm. 2.6 for linear forward stochastic system and Thm. 3.3 for linear backward stochastic system). Moreover, the duality method is also used to derive Carleman estimates. Compared with the weighted identity method, there are some advantages for using the duality method to establish Carleman estimate. First, when some assumption on the regularity for coefficients in the equation are relaxed, a Carleman estimate can still be obtained by using the duality method. Second, a Carleman estimate can be obtained with the drift term taking values in negative order Sobolev space by using the duality method. Therefore, one may be able to use these new Carleman estimates to consider new controllability problems (see *e.g.*, [8, 28–31]).

Now, we give some remarks in order.

Remark 1.4. The reason why assumption (H_5) is added to Theorem 1.2 is to ensure the well-posedness of system (1.1). We do not use assumption (H_5) in the proof of Theorem 1.2. In fact, we claim that the null controllability of equation (1.1) can be reduced to the null controllability of the following system without the g in the diffusion term

$$\begin{cases} \mathrm{d}y - (a+\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}y_j)_k\mathrm{d}t = [f(\omega,t,x,y)+\chi_{G_0}h]\mathrm{d}t + H\mathrm{d}B(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G. \end{cases} \tag{1.3}$$

Indeed, assume that one can find two controls $h \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C}))$ and $H \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ such that the corresponding solution y of (1.3) satisfies $y(T,\cdot) = 0$ in G, $\mathbb{P}$-a.s. Since the control H is distributed in the whole domain G, the state y still satisfies equation (1.1) with the controls $h^* = h$ and

$$H^* = H - g(\omega,t,x,y) \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C})),$$

which is well defined by (H_2). Moreover, we still have the controllability property for equation (1.1), *i.e.* $y(T,\cdot) = 0$ in G, $\mathbb{P}$-a.s. This is why we can drop the Lipschitz condition (H_5) on g in the proof of Theorem 1.2.

Remark 1.5. From the following, it can be found that since $0 < T < 1$ is assumed, we actually obtain that the system (1.1) and (1.2) are globally null controllable at any small time. Although we focus on the proof of null controllability at small time for simplicity, the null controllability in the case of $T \geq 1$ can also be obtained by the same method.

Remark 1.6. From Theorem 1.2 and Theorem 1.3, we can obtain the null controllability for semilinear stochastic parabolic equations, which implies that the results in this paper include the results in [27].

Remark 1.7. Theorem 1.2 requires an extra control $H \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ on the diffusion term. It would be quite interesting to establish the null controllability for (1.1) by only one control force or the control H acting only on a sub-domain of G. As far as we know, the results in this direction seem to be only [21] and [26]. However, the random coefficients in the system considered in [21] and [26] have some restrictive conditions. For forward SPDEs with general coefficients, it still seems to be a challenging open problem to establish null controllability with only one control, even for general linear SPDEs (see *e.g.*, [8, 19, 24]).

Remark 1.8. Theorem 1.2 and Theorem 1.3 provide a partial positive answer to the open question provided in [8], Remark 1.10. However, the global controllability for system (1.1) and (1.2) with more general nonlinearities $f(\cdot)$, $g(\cdot)$ and $\Upsilon(\cdot)$, such as the super-linear nonlinearity considered for deterministic Ginzburg–Landau equation (see *e.g.*, [15]) or parabolic-type PDEs (see *e.g.*, [12, 13]), is still an interesting but challenging problem.

As mentioned above, we prove Theorem 1.2 and Theorem 1.3 by Carleman estimates. In the past few years, the Carleman estimates for SPDEs have received much attention. Although there are numerous results for the global Carleman estimate for deterministic Ginzburg–Landau equation and parabolic-type equation (*e.g.*, [14, 15, 32]), the Carleman estimates for the stochastic counterpart are much less. In this respect, we refer to [8, 18, 19, 21, 23, 24, 30, 31, 33–38] for some known results. In [8, 18], authors studied Carleman estimats for stochastic complex Ginzburg–Landau equations. The references [21, 24, 30, 31, 33–37] are devoted to stochastic second order parabolic equations. The references [19, 23, 38] are concerned with stochastic fourth order parabolic equations.

The remainder of the paper is organized as follows. In Section 2, we present the proof of Theorem 1.2. In particular, we give a new Carleman estimate for the backward stochastic complex Ginzburg–Landau equation. In Section 3, we present the proof of Theorem 1.3.

2. Controllability of a semilinear forward stochastic complex Ginzburg–Landau equation

2.1. A new Carleman estimate for a backward stochastic complex Ginzburg–Landau equation

In this part, we first establish the Carleman estimate for the following backward linear stochastic complex Ginzburg–Landau equation with a source Ξ in drift term:

$$\begin{cases} \mathrm{d}z + (a-\mathrm{i}b)\sum\limits_{j,k=1}^{n}(a^{jk}z_j)_k\mathrm{d}t = \Xi\mathrm{d}t + Z\mathrm{d}B(t) & \text{in } Q,\\ z = 0 & \text{on } \Sigma,\\ z(T) = z_T & \text{in } G, \end{cases} \tag{2.1}$$

where $a>0$, $b\in\mathbb{R}$ and coefficients a^{jk} satisfy the assumption (H_1). With the above assumptions for the coefficients and $z_T \in L^2_{\mathcal{F}_T}(\Omega;L^2(G;\mathbb{C}))$, $\Xi \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C}))$, the system (2.1) admits a unique solution $(z,Z)\in\mathcal{H}_T\times L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ (see *e.g.*, [9], Thm. 4.11). We refer to [8] for some known Carleman estimates for the equation (2.1).

At first, we introduce an identity for a stochastic parabolic operator with complex principal parts, which plays a key role in proving the Carleman estimates for complex Ginzburg–Landau equations. Set

$$\mathcal{L}z := \mathrm{d}z + (a-\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}z_j)_k\mathrm{d}t.$$

Lemma 2.1 ([8], Lem. 2.1)**.** *Suppose that $\hat{\ell} \in C^3(Q;\mathbb{R})$ and $\Phi \in C^1(Q;\mathbb{C})$. Let z be an $H^2(G;\mathbb{C})$-valued continuous semimartingale. Set $\hat{\theta} = \mathrm{e}^{\hat{\ell}}$ and $v = \hat{\theta} z$. Then for a.e. $(x,t) \in Q$, it holds that*

$$\begin{aligned}
2\mathrm{Re}(\hat{\theta}\overline{I_1}\mathcal{L}z) =& 2|I_1|^2\mathrm{d}t + \mathrm{d}M + \sum_{k=1}^{n} V_k^k + B|v|^2\mathrm{d}t + \sum_{j,k=1}^{n} D^{jk}v_j\overline{v}_k\mathrm{d}t \\
& - 2\sum_{j=1}^{n}[\mathrm{Re}(aE^j\overline{v}v_j) - \mathrm{Im}(bF^jv\overline{v}_j)]\mathrm{d}t + \sum_{j,k=1}^{n} aa^{jk}\mathrm{d}v_j\mathrm{d}\overline{v}_k \\
& + (\hat{\ell}_t - aA)\mathrm{d}v\mathrm{d}\overline{v} - 2b\sum_{j,k=1}^{n} a^{jk}\hat{\ell}_k\mathrm{Im}(\mathrm{d}v\mathrm{d}\overline{v}_j) \\
& + 2\Big[b\sum_{j,k=1}^{n}(a^{jk}\hat{\ell}_k)_j\mathrm{Im}(v\mathrm{d}\overline{v}) + \mathrm{Re}(\overline{\Phi}\overline{v}\mathrm{d}v)\Big],
\end{aligned} \tag{2.2}$$

where

$$\begin{cases}
A = \sum_{j,k=1}^{n}[a^{jk}\hat{\ell}_j\hat{\ell}_k - (a^{jk}\hat{\ell}_j)_k], \quad \Lambda = \sum_{j,k=1}^{n}(a^{jk}v_j)_k + Av, \\
I_1 = a\Lambda + 2\mathrm{i}b\sum_{j,k=1}^{n} a^{jk}\hat{\ell}_jv_k + (\Phi - \hat{\ell}_t)v,
\end{cases} \tag{2.3}$$

$$\begin{aligned}
B =& 2(a^2+b^2)\sum_{j,k=1}^{n}(Aa^{jk}\hat{\ell}_j)_k - aA_t - 2aA\mathrm{Re}\Phi - 2bA\mathrm{Im}\Phi \\
& + 2\mathrm{Re}(\Phi\hat{\ell}_t) - 2|\Phi|^2 + \hat{\ell}_{tt}, \\
M =& aA|v|^2 + \sum_{j,k=1}^{n} a^{jk}[-av_j\overline{v}_k + 2b\hat{\ell}_j\mathrm{Im}(\overline{v}_kv)] - \hat{\ell}_t|v|^2, \\
V^k =& 2a\sum_{j=1}^{n} a^{jk}\mathrm{Re}(v_j\mathrm{d}\overline{v}) - 2b\sum_{j=1}^{n} a^{jk}\hat{\ell}_j\mathrm{Im}(v\mathrm{d}\overline{v}) - 2A(a^2+b^2)\sum_{j=1}^{n} a^{jk}\hat{\ell}_j|v|^2\mathrm{d}t \\
& - 2a\sum_{j=1}^{n} a^{jk}\mathrm{Re}(\overline{v}_j\Phi v)\mathrm{d}t + 2b\sum_{j=1}^{n} a^{jk}\mathrm{Im}[v_j(\overline{\Phi} - \hat{\ell}_t)\overline{v}]\mathrm{d}t \\
& + 2(a^2+b^2)\sum_{j,j',k'=1}^{n}[a^{jk}a^{j'k'}\hat{\ell}_jv_{j'}\overline{v}_{k'} - a^{jk'}a^{j'k}\hat{\ell}_j(v_{j'}\overline{v}_{k'} + \overline{v}_{j'}v_{k'})]\mathrm{d}t, \\
D^{jk} =& aa_t^{jk} + 2b\mathrm{Im}\Phi a^{jk} + 2a\mathrm{Re}\Phi a^{jk} \\
& + 2(a^2+b^2)\sum_{j',k'=1}^{n}[a^{jk'}(a^{j'k}\hat{\ell}_{j'})_{k'} + a^{k'k}(a^{j'j}\hat{\ell}_{j'})_{k'} - (a^{jk}a^{j'k'}\hat{\ell}_{j'})_{k'}], \\
E^j =& \sum_{k=1}^{n} a^{jk}(2\hat{\ell}_k\overline{\Phi} - 2\hat{\ell}_k\hat{\ell}_t - \overline{\Phi}_k), \\
\textit{and} \quad F^j =& \sum_{k=1}^{n}(a^{jk}\Phi_k - 2a^{jk}\hat{\ell}_{tk} - a_t^{jk}\hat{\ell}_k - 2a^{jk}\hat{\ell}_k\Phi).
\end{aligned}$$

To state our Carleman estimates for (2.1), we first introduce the weight function. To begin with, we introduce some auxiliary functions. Let G' be any given nonempty open subset of G satisfying $G' \subset\subset G_0$. It can be seen from [14] that there exists a function $\beta \in C^4(\overline{G})$ such that

$$\begin{cases} 0 < \beta(x) \leq 1 \quad \forall x \in G, \\ \beta(x) = 0 \quad \forall x \in \partial G, \\ \inf\limits_{G \setminus \overline{G'}} \{|\nabla \beta(x)|\} \geq \alpha_0 > 0. \end{cases} \tag{2.4}$$

Without loss of generality, in what follows we assume that $0 < T < 1$. For some constant $m \geq 1$ and $\sigma \geq 2$, we define the following weight function depending on the time variable:

$$\begin{cases} \gamma(t) = 1 + \left(1 - \dfrac{4t}{T}\right)^{\sigma}, & t \in [0, T/4), \\ \gamma(t) = 1, & t \in [T/4, T/2), \\ \gamma(t) \text{ is increasing on } [T/2, 3T/4), & \\ \gamma(t) = \dfrac{1}{(T-t)^m}, & t \in [3T/4, T], \\ \gamma(t) \in C^2([0, T)). \end{cases} \tag{2.5}$$

Set

$$\alpha(x) := \mathrm{e}^{\mu(6m+\beta(x))} - \mu \mathrm{e}^{\mu(6m+6)}, \quad \varphi(t,x) := \gamma(t)\alpha(x), \quad \xi(t,x) := \gamma(t)\mathrm{e}^{\mu(6m+\beta(x))}, \tag{2.6}$$

where μ is a positive parameter with $\mu \geq 1$ and σ is chosen as

$$\sigma = \lambda \mu^2 \mathrm{e}^{\mu(6m-4)} \tag{2.7}$$

for some parameter $\lambda \geq 1$. We finally set the weight

$$\theta := \mathrm{e}^{\ell}, \text{ where } \ell(t,x) := \lambda \varphi(t,x). \tag{2.8}$$

It can be seen from the (2.8) that, compared with the classical weight function (see, *e.g.* [24]), the main difference here is that the weight does not degenerate as $t \to 0^+$.

In the following, for any $k \in \mathbb{N}$, we denote by $\mathcal{O}(\mu^k)$ a function of order μ^k for sufficiently large μ, and by $\mathcal{O}_\mu(\lambda^k)$ a function of order λ^k for fixed μ and sufficiently large λ. Moreover, we use C to denote a generic positive constant that depends only on G, G_0, T, m, s_0 and α_0, but is independent of a and b. It may change from line to line. If it is necessary, the dependence of a constant C on some parameters, say "$\tau_1, \tau_2, \cdots, \tau_k$", will be written by $C(\tau_1, \tau_2, \cdots, \tau_k)$.

Now, we state the results of the Carleman estimate.

Theorem 2.2. *There exist three positive constants $\lambda_1 = \lambda_1(a, b, s_0, \alpha_0)$, $\mu_1 = \mu_1(a, b, s_0, \alpha_0)$ and C, such that for any $\Xi \in L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{C}))$ and $r_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G; \mathbb{C}))$, any solution $(z, Z) \in \mathcal{H}_T \times L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{C}))$ of*

(2.1) *satisfies*

$$\begin{aligned}
&\mathbb{E}\int_G \theta^2(0)\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}|z(0)|^2\mathrm{d}x + \mathbb{E}\int_0^{T/4}\int_G \theta^2\lambda^2\mu^2\xi|\varphi||\gamma_t||z|^2\mathrm{d}x\mathrm{d}t \\
&+\mathbb{E}\int_Q \theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2\lambda\mu^2\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t \\
\leq& C_1\left[\mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t\right],
\end{aligned} \tag{2.9}$$

for all $\lambda \geq \lambda_1$ *and* $\mu \geq \mu_1$, *where* $C_1 = C_1(a,b)$ *is given by*

$$C_1 = \frac{C(a^2+b^2+1)}{\min\{a^2,1\}}. \tag{2.10}$$

Remark 2.3. This type of Carleman estimate was first considered in [39] to deal with the local trajectory controllability for the incompressible Navier–Stokes equations. Later, references [27] and [28] developed the ideas in [39] to study the global null controllability of stochastic semilinear parabolic equations.

Remark 2.4. Since the weight function $\gamma(t)$ does not blow up as $t \to 0^+$, it prevents θ from vanishing at $t = 0$. Compared with the classical proof of the Carleman estimate for the linear stochastic complex Ginzburg–Landau equation (see *e.g.*, [18], Thm. 3.1 and [8], Thm. 2.2), this change brings additional difficulties and enable us to include the term $\mathbb{E}\int_G \theta^2(0)\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}|z(0)|^2\mathrm{d}x$ on the left-hand side of (2.9).

Remark 2.5. At first glance, Carleman estimate (2.9) looks very similar to the global Carleman estimate for backward stochastic parabolic equations in [27]. Furthermore, one can find that the idea behind the proofs in this paper and [27] are analogous. Nevertheless, since the diffusion coefficients a^{jk} and complex-valued functions are involved in this paper, the specific proofs have some differences and additional difficulties. We have to use an identity (2.2) of linear stochastic complex Ginzburg–Landau operators instead of the identity introduced in [27]. This means that we need more calculations and estimates than [27] to derive the desired Carleman estimate (2.9). So, one cannot simply mimic the proofs in [27] to obtain Theorem 2.2. Indeed, even in the deterministic setting, the proof of the global Carleman estimate for parabolic operator with complex principal parts is much more complicated than real-valued cases (see *e.g.*, [15]). When (2.1) degenerates to a real linear backward stochastic parabolic equation ($a = 1$ and $b = 0$) and the coefficient matrix $(a^{jk})_{n\times n}$ degenerates to the identity matrix, the Carleman estimate in Theorem 2.2 is the same as the one in [27].

Proof of Theorem 2.2: By a density argument, it suffices to prove Theorem 2.2 with $(z, Z) \in L^2_{\mathbb{F}}(\Omega; C([0,T]; H^2(G;\mathbb{C}))) \times L^2_{\mathbb{F}}(0,T; H^1(G;\mathbb{C}))$ (see *e.g.*, [19, 23]). For readability, we divide the proof into several steps.

Step 1. We apply Lemma 2.1 with $\hat{\theta} = \theta$, $\hat{\ell} = \ell$ and set

$$\Phi = 2(a+\mathrm{i}b)\sum_{j,k=1}^n (a^{jk}\ell_j)_k. \tag{2.11}$$

Integrating (2.2) in Q, taking mathematical expectation on both sides, we conclude that

$$\begin{aligned}
&2\mathbb{E}\int_Q \mathrm{Re}(\theta\overline{I_1}\mathcal{L}z)\mathrm{d}x\\
&=\mathbb{E}\int_Q \mathrm{d}M\mathrm{d}x+\mathbb{E}\int_Q B|v|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\sum_{j,k=1}^n D^{jk}v_j\overline{v}_k\mathrm{d}x\mathrm{d}t-2\mathbb{E}\int_Q\sum_{j=1}^n[\mathrm{Re}(aE^j\overline{v}v_j)\\
&\quad-\mathrm{Im}(bF^jv\overline{v}_j)]\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\sum_{k=1}^n V_k^k\mathrm{d}x+\mathbb{E}\int_Q\sum_{j,k=1}^n aa^{jk}\mathrm{d}v_j\mathrm{d}\overline{v}_k\mathrm{d}x\\
&\quad+\mathbb{E}\int_Q(\ell_t-aA)\mathrm{d}v\mathrm{d}\overline{v}\mathrm{d}x-2b\mathbb{E}\int_Q\sum_{j,k=1}^n a^{jk}\ell_k\mathrm{Im}(\mathrm{d}v\mathrm{d}\overline{v}_j)\mathrm{d}x\\
&\quad+2\mathbb{E}\int_Q\Big[b\sum_{j,k=1}^n(a^{jk}\ell_k)_j\mathrm{Im}(v\mathrm{d}\overline{v})+\mathrm{Re}(\overline{\Phi}\overline{v}\mathrm{d}v)\Big]\mathrm{d}x+2\mathbb{E}\int_Q|I_1|^2\mathrm{d}x\mathrm{d}t=:\sum_{i=1}^{10}\mathcal{J}_i.
\end{aligned}\tag{2.12}$$

In addition, it is easy to check that for any $j,k=1,\cdots,n$,

$$\ell_j=\lambda\mu\xi\beta_j,\quad \ell_{jk}=\lambda\mu^2\xi\beta_j\beta_k+\lambda\mu\xi\beta_{jk},\quad \ell_t=\lambda\gamma_t\alpha,$$

which are useful in the remainder of the proof.

Step 2. In this step, let us estimate every term on the right side of (2.12).

Estimate for $\mathcal{J}_1$. From the definitions of $v=\theta z$ and ℓ, we easily see that $\lim_{t\to T-}\ell(t,\cdot)=-\infty$ and thus the term at $t=T$ vanishes. By expression of M, we have

$$M|_0^T=-M(0)=-aA(0)|v(0)|^2-\sum_{j,k=1}^n a^{jk}[-av_j\overline{v}_k+2b\ell_j\mathrm{Im}(\overline{v}_kv)]\Big|_{t=0}+\ell_t(0)|v(0)|^2.\tag{2.13}$$

For the first term on the right-hand side of (2.13), using the explicit expression of A in (2.3) and the fact $\gamma(0)=2$, we can obtain

$$\begin{aligned}
A=&\sum_{j,k=1}^n(\lambda^2\mu^2\xi^2a^{jk}\beta_j\beta_k-\lambda\mu^2\xi a^{jk}\beta_j\beta_k-\lambda\mu\xi a_k^{jk}\beta_j-\lambda\mu\xi a^{jk}\beta_{jk})\\
=&\lambda^2\mu^2\xi^2\sum_{j,k=1}^n a^{jk}\beta_j\beta_k+\mathcal{O}(\lambda)\mu^2\xi,
\end{aligned}\tag{2.14}$$

which imply that

$$\begin{aligned}
-aA(0)=&-a[4\lambda^2\mu^2\mathrm{e}^{2\mu(6m+\beta)}\sum_{j,k=1}^n a^{jk}(0)\beta_j\beta_k+\mathcal{O}(\lambda)\mu^2\mathrm{e}^{\mu(6m+\beta)}]\\
\geq&-Ca\lambda^2\mu^2\mathrm{e}^{2\mu(6m+\beta)}.
\end{aligned}\tag{2.15}$$

For the second one, by assumption (H_1) on the functions a^{jk}, we have

$$a\sum_{j,k=1}^n a^{jk}v_j\overline{v}_k\Big|_{t=0}\geq as_0|\nabla v(0)|^2.\tag{2.16}$$

And, for any $\epsilon_1 > 0$, using Young inequality, we obtain

$$\left|2b\sum_{j,k=1}^{n} a^{jk}\ell_j \mathrm{Im}(\overline{v}_k v)\right| = \left|2b\lambda\mu\xi\sum_{j,k=1}^{n} a^{jk}\beta_j \mathrm{Im}(\overline{v}_k v)\right| \leq \epsilon_1|\nabla v|^2 + \frac{Cb^2}{\epsilon_1}\lambda^2\mu^2\xi^2|v|^2. \tag{2.17}$$

Choosing $\epsilon_1 = \frac{as_0}{2}$ in (2.17), we get

$$\left|2b\sum_{j,k=1}^{n} a^{jk}\ell_j \mathrm{Im}(\overline{v}_k v)\Big|_{t=0}\right| \leq \frac{as_0}{2}|\nabla v(0)|^2 + \frac{Cb^2}{a}\lambda^2\mu^2 \mathrm{e}^{2\mu(6m+\beta)}|v(0)|^2, \tag{2.18}$$

From (2.16) and (2.18), we get

$$-\sum_{j,k=1}^{n} a^{jk}[-av_j\overline{v}_k + 2b\ell_j \mathrm{Im}(\overline{v}_k v)]\Big|_{t=0} \geq \frac{as_0}{2}|\nabla v(0)|^2 - \frac{Cb^2}{a}\lambda^2\mu^2 \mathrm{e}^{2\mu(6m+\beta)}|v(0)|^2. \tag{2.19}$$

For the third one, from the explicit expression of the function $\gamma(t)$ in (2.5), we get

$$\gamma_t(t) = -\frac{4\sigma}{T}\left(1 - \frac{4t}{T}\right)^{\sigma-1} \quad \forall t \in [0, T/4]. \tag{2.20}$$

Using (2.7) and the above expression, we have

$$\begin{aligned} \ell_t(0,x) &= -\frac{4}{T}\lambda^2\mu^2 \mathrm{e}^{\mu(6m-4)}\alpha(x) \\ &= \frac{4}{T}\lambda^2\mu^2 \mathrm{e}^{\mu(6m-4)}[\mu \mathrm{e}^{\mu(6m+6)} - \mathrm{e}^{\mu(6m+\beta(x))}] \\ &\geq c_0\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)} \end{aligned} \tag{2.21}$$

for all $\mu > 1$ and some constant $c_0 > 0$. Set

$$K_0 \equiv K_0(a,b) := \max\left\{\frac{2C(a^2+b^2)}{c_0 a}, 1\right\}.$$

By (2.15), (2.19), (2.21) and (2.13), there exists $\mu_* \equiv \mu_*(a,b) = K_0$, such that for all $\mu \geq \mu_*$, it holds that

$$M|_0^T \geq \frac{as_0}{2}|\nabla v(0)|^2 + c_1\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}|v(0)|^2 \geq c_1\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}|v(0)|^2 \tag{2.22}$$

for some constant $c_1 > 0$. From (2.22), we can easily get

$$\mathcal{J}_1 := \mathbb{E}\int_G M|_0^T \mathrm{d}x \geq c_1\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}\mathbb{E}\int_G |v(0)|^2 \mathrm{d}x. \tag{2.23}$$

Estimate for $\mathcal{J}_2$. To abridge the notation, in what follows, we set

$$\Psi = \sum_{j,k=1}^{n} a^{jk}\beta_j\beta_k.$$

By the definition of B in Lemma 2.1 and (2.11), we have that

$$\begin{aligned}
B &= 2(a^2+b^2)\sum_{j,k=1}^{n}(Aa^{jk}\ell_j)_k - aA_t - 2aA\mathrm{Re}\Phi - 2bA\mathrm{Im}\Phi \\
&\quad + 2\mathrm{Re}(\Phi\ell_t) - 2|\Phi|^2 + \ell_{tt} \\
&= 2(a^2+b^2)\sum_{j,k=1}^{n}A_k a^{jk}\ell_j + 2(a^2+b^2)A\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k \\
&\quad - 4a^2A\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k - 4b^2A\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k \\
&\quad - 8(a^2+b^2)\Big[\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k\Big]^2 - aA_t + 4a\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k\ell_t + \ell_{tt} \\
&= 2(a^2+b^2)\Big[\sum_{j,k=1}^{n}A_k a^{jk}\ell_j - A\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k\Big] \\
&\quad - 8(a^2+b^2)\Big[\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k\Big]^2 - aA_t + 4a\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k\ell_t + \ell_{tt} =: \sum_{j=1}^{5} I_j
\end{aligned}$$

Next, we estimate I_j $(j=1,\dots,5)$. Using (2.14), we have

$$A_k = 2\lambda^2\mu^3\xi^2\beta_k\Psi + \mathcal{O}(\lambda^2)\mu^2\xi^2 + \mathcal{O}(\lambda)\mu^3\xi. \tag{2.24}$$

Further, using (2.24) and (2.14), we obtain respectively that

$$\sum_{j,k=1}^{n}A_k a^{jk}\ell_j = 2\lambda^3\mu^4\xi^3\Psi^2 + \mathcal{O}(\lambda^3)\mu^3\xi^3 + \mathcal{O}(\lambda^2)\mu^4\xi^2 \tag{2.25}$$

and

$$A\sum_{j,k=1}^{n}(a^{jk}\ell_j)_k = \lambda^3\mu^4\xi^3\Psi^2 + \mathcal{O}(\lambda^3)\mu^3\xi^3 + \mathcal{O}(\lambda^2)\mu^4\xi^2. \tag{2.26}$$

By (2.25) and (2.26), I_1 simplifies to

$$I_1 = 2(a^2+b^2)\lambda^3\mu^4\xi^3\Psi^2 + (a^2+b^2)[\mathcal{O}(\lambda^3)\mu^3\xi^3 + \mathcal{O}(\lambda^2)\mu^4\xi^2],$$

which imply that

$$\begin{aligned}
\mathbb{E}\int_Q I_1|v|^2\mathrm{d}x\mathrm{d}t \geq{}& 2(a^2+b^2)\mathbb{E}\int_Q \lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t \\
&- C(a^2+b^2)\mathbb{E}\int_Q(\lambda^3\mu^3\xi^3 + \lambda^2\mu^4\xi^2)|v|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.27}$$

For I_2, we easily have

$$I_2 = (a^2+b^2)\mathcal{O}(\lambda^2)\mu^4\xi^2,$$

which imply that

$$\mathbb{E}\int_Q I_2|v|^2\mathrm{d}x\mathrm{d}t \geq -C(a^2+b^2)\mathbb{E}\int_Q \lambda^2\mu^4\xi^2|v|^2\mathrm{d}x\mathrm{d}t. \tag{2.28}$$

Further, we estimate the third and the fourth terms together. Using (2.14), we have

$$\begin{aligned} A_t =& \sum_{j,k=1}^n (\lambda^2\mu^2\xi^2 a^{jk}\beta_j\beta_k - \lambda\mu^2\xi a^{jk}\beta_j\beta_k - \lambda\mu\xi a_k^{jk}\beta_j - \lambda\mu\xi a^{jk}\beta_{jk})_t \\ =&2\lambda^2\mu^2\xi\xi_t\Psi - \lambda\mu^2\xi_t\Psi + \mathcal{O}(\lambda)\mu\xi_t \\ =&\frac{\gamma_t}{\gamma}\left[2\lambda^2\mu^2\xi^2\Psi + \mathcal{O}(\lambda)\mu^2\xi\right]. \end{aligned} \tag{2.29}$$

In addition, we get

$$\sum_{j,k=1}^n (a^{jk}\ell_j)_k\ell_t = \lambda^2\mu^2\xi\gamma_t\alpha\Psi + \mathcal{O}(\lambda^2)\mu\xi\gamma_t\alpha = \frac{\gamma_t}{\gamma}\left[\lambda^2\mu^2\xi\varphi\Psi + \mathcal{O}(\lambda^2)\mu\xi\varphi\right]. \tag{2.30}$$

By (2.29) and (2.30), we obtain that

$$I_3 + I_4 = W, \tag{2.31}$$

where

$$W = -\frac{\gamma_t}{\gamma}\left[2a\lambda^2\mu^2\xi^2\Psi + a\mathcal{O}(\lambda)\mu^2\xi\right] - \frac{\gamma_t}{\gamma}\left[4a\lambda^2\mu^2\xi(-\varphi)\Psi + a\mathcal{O}(\lambda^2)\mu\xi(-\varphi)\right].$$

Further, we estimate the W. From the definition of γ, it is clear that W vanishes on $(T/4, T/2)$. On $(T/2, T)$, we use the fact that there exists $C>0$ such that $|\gamma_t| \leq C|\gamma|^2$. Therefore, there exists a constant $C>0$ only depending on G, G' such that

$$|W| \leq Ca\lambda^2\mu^3\xi^3, \quad (t,x) \in (T/2,T)\times G, \tag{2.32}$$

where we have used that $|\varphi\gamma| \leq \mu\xi^2$. On $(0, T/4)$, noting the facts that $\mu\xi \leq |\varphi|$, $\gamma_t \leq 0$, $\varphi \leq 0$ and $\gamma \in [1,2]$, it holds that

$$W \geq 4a\frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi|\varphi|\Psi - Ca\frac{|\gamma_t|}{\gamma}\lambda^2\mu\xi|\varphi|, \quad (t,x) \in (0,T/4)\times G. \tag{2.33}$$

By (2.31), (2.32) and (2.33), we get

$$\begin{aligned} \mathbb{E}\int_Q (I_3+I_4)|v|^2\mathrm{d}x\mathrm{d}t \geq& 4a\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi||v|^2\mathrm{d}x\mathrm{d}t \\ -& Ca\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}(\lambda\mu^2+\lambda^2\mu)\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t - Ca\mathbb{E}\int_Q \lambda^2\mu^3\xi^3|v|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.34}$$

Further, we estimate I_5. Using the definition of $\gamma(t)$, it is not difficult to see that $|\gamma_{tt}| \leq C\lambda^2\mu^4 \mathrm{e}^{2\mu(6m-4)}$, thus

$$|I_5| = |\ell_{tt}| = |\lambda\gamma_{tt}\alpha| \leq C\lambda^3\mu^5 \mathrm{e}^{2\mu(6m-4)}\mathrm{e}^{\mu(6m+6)} \leq C\lambda^3\mu^2\xi^3, \quad t \in (0, T/4), \tag{2.35}$$

where we have used that $\mu^3\mathrm{e}^{-2\mu} < \frac{1}{2}$ for all $\mu > 1$. Noting that estimates $|\gamma_{tt}| \leq C\gamma^3$ and $|\varphi\gamma| \leq \mu\xi^2$, we obtain

$$|I_5| = |\ell_{tt}| = \left|\lambda\frac{\gamma_{tt}}{\gamma}\varphi\right| \leq C\lambda\mu\xi^3, \quad t \in (T/2, T). \tag{2.36}$$

Since ℓ_{tt} vanishes for $t \in (T/4, T/2)$, from (2.35) and (2.36), we deduce that

$$\mathbb{E}\int_Q I_5|v|^2\mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_Q \lambda^3\mu^2\xi^3|v|^2\mathrm{d}x\mathrm{d}t. \tag{2.37}$$

By (2.27), (2.28), (2.34) and (2.37), we obtain that

$$\begin{aligned}
\mathcal{J}_2 :=& \mathbb{E}\int_Q B|v|^2\mathrm{d}x\mathrm{d}t = \sum_{j=1}^5 \mathbb{E}\int_Q I_j|v|^2\mathrm{d}x\mathrm{d}t \\
\geq& 4a\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi||v|^2\mathrm{d}x\mathrm{d}t \\
&- Ca\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}(\lambda\mu^2 + \lambda^2\mu)\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t \\
&+ 2(a^2+b^2)\mathbb{E}\int_Q \lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t \\
&- C(a^2+b^2)\mathbb{E}\int_Q (\lambda^3\mu^3\xi^3 + \lambda^2\mu^4\xi^2)|v|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.38}$$

Estimate for $\mathcal{J}_3$. By the definition of D^{jk} in Lemma 2.1 and (2.11), we find that

$$\begin{aligned}
D^{jk} =& aa_t^{jk} + 2b\mathrm{Im}\Phi a^{jk} + 2a\mathrm{Re}\Phi a^{jk} \\
&+ 2(a^2+b^2)\sum_{j',k'=1}^n [a^{jk'}(a^{j'k}\ell_{j'})_{k'} + a^{k'k}(a^{j'j}\ell_{j'})_{k'} - (a^{jk}a^{j'k'}\ell_{j'})_{k'}] \\
=& 2(a^2+b^2)a^{jk}\lambda\mu^2\xi\Psi + 4(a^2+b^2)\lambda\mu^2\xi\sum_{j',k'=1}^n a^{jk'}a^{j'k}\beta_{j'}\beta_{k'} + (a^2+b^2)\mathcal{O}(\lambda)\mu\xi.
\end{aligned}$$

Hence, it holds that

$$\begin{aligned}
\mathcal{J}_3 :=& \mathbb{E}\int_Q \sum_{j,k=1}^n D^{jk}v_j\overline{v}_k\mathrm{d}x\mathrm{d}t \geq 2(a^2+b^2)\mathbb{E}\int_Q \lambda\mu^2\xi\Psi\sum_{j,k=1}^n a^{jk}v_j\overline{v}_k\mathrm{d}x\mathrm{d}t \\
&+ 4(a^2+b^2)\mathbb{E}\int_Q \lambda\mu^2\xi\left|\sum_{j,k=1}^n a^{jk}\beta_j v_k\right|^2\mathrm{d}x\mathrm{d}t - C(a^2+b^2)\mathbb{E}\int_Q \lambda\mu\xi|\nabla v|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.39}$$

Estimate for $\mathcal{J}_4$. First, notice that for any $j = 1, 2, \dots, n$,

$$\mathrm{Im}(bF^j v\overline{v}_j) = \mathrm{Re}(\mathrm{i}b\overline{F^j}\overline{v}v_j).$$

By the definitions of E^j and F^j in Lemma 2.1 and (2.11), we find that

$$\begin{aligned}
&-2\sum_{j=1}^{n}[\mathrm{Re}(aE^j\overline{v}v_j) - \mathrm{Im}(bF^j v\overline{v}_j)] = -2\sum_{j=1}^{n}\mathrm{Re}[(aE^j - \mathrm{i}b\overline{F^j})\overline{v}v_j]\\
=&-4\sum_{j,k=1}^{n} a^{jk}\ell_k\mathrm{Re}[(a+\mathrm{i}b)\overline{\Phi}\overline{v}v_j] + 2\sum_{j,k=1}^{n} a^{jk}\mathrm{Re}[(a+\mathrm{i}b)\overline{\Phi}_k\overline{v}v_j]\\
&+4\sum_{j,k=1}^{n} aa^{jk}\ell_k\ell_t\mathrm{Re}(\overline{v}v_j) - 2\sum_{j,k=1}^{n}\mathrm{Re}[\mathrm{i}b(2a^{jk}\ell_{kt} + a_t^{jk}\ell_k)\overline{v}v_j]\\
=&-8(a^2+b^2)\sum_{j,k,j',k'=1}^{n} a^{jk}\ell_k(a^{j'k'}\ell_{j'})_{k'}\mathrm{Re}(\overline{v}v_j) + 4\sum_{j,k=1}^{n} aa^{jk}\ell_k\ell_t\mathrm{Re}(\overline{v}v_j)\\
&-4b\sum_{j,k=1}^{n} a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j) + (a^2+b^2)\mathcal{O}(\lambda)\mu^3\xi|v||\nabla v| + |b|\mathcal{O}(\lambda)\mu\xi|v||\nabla v|\\
=&-4(a^2+b^2)\sum_{j,k,j',k'=1}^{n}\Big\{[a^{jk}\ell_k(a^{j'k'}\ell_{j'})_{k'}|v|^2]_j - [a^{jk}\ell_k(a^{j'k'}\ell_{j'})_{k'}]_j|v|^2\Big\}\\
&+2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t|v|^2)_j - 2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j|v|^2 - 4b\sum_{j,k=1}^{n} a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j)\\
&+(a^2+b^2)\mathcal{O}(\lambda)\mu^3\xi|v||\nabla v| + |b|\mathcal{O}(\lambda)\mu\xi|v||\nabla v|\\
=&-2\sum_{j,k=1}^{n}\left[2(a^2+b^2)\sum_{j',k'=1}^{n} a^{jk}\ell_k(a^{j'k'}\ell_{j'})_{k'}|v|^2 - aa^{jk}\ell_k\ell_t|v|^2\right]_j\\
&+(a^2+b^2)\mathcal{O}(\lambda^2)\mu^4\xi^2|v|^2 + (a^2+b^2)\mathcal{O}(\lambda)\mu^3\xi|v||\nabla v| + |b|\mathcal{O}(\lambda)\mu\xi|v||\nabla v|\\
&-2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j|v|^2 - 4b\sum_{j,k=1}^{n} a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j).
\end{aligned} \tag{2.40}$$

Further, we estimate the last two terms of the above expression. First, we have

$$-2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j = -2a\frac{\gamma_t}{\gamma}\lambda^2\mu^2\xi\Psi\varphi - 2a\frac{\gamma_t}{\gamma}\lambda^2\mu^2\xi^2\Psi + a\mathcal{O}(\lambda^2)\mu\xi\frac{\gamma_t}{\gamma}\varphi.$$

On $(0, T/4)$, since the second term of the above expression is positive, it holds that

$$-2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j \geq -2a\frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi| + a\mathcal{O}(\lambda^2)\mu\xi\frac{\gamma_t}{\gamma}\varphi, \quad (t,x)\in(0,T/4)\times G. \tag{2.41}$$

On $(T/2,T)$, noting $|\gamma_t|\leq C|\gamma|^2$ and $|\gamma\varphi|\leq\mu\xi^2$, we have

$$\left|-2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j\right|\leq Ca\lambda^2\mu^3\xi^3,\quad (t,x)\in(T/2,T)\times G. \tag{2.42}$$

Since obviously it vanishes for $t\in(T/4,T/2)$, we can put estimates (2.41) and (2.42) together to deduce that

$$\begin{aligned}\int_0^T -2\sum_{j,k=1}^{n}(aa^{jk}\ell_k\ell_t)_j|v|^2\mathrm{d}t\geq\int_0^{T/4}\left[-2a\frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi|+a\mathcal{O}(\lambda^2)\mu\xi\frac{|\gamma_t|}{\gamma}|\varphi|\right]|v|^2\mathrm{d}t\\ -Ca\int_0^T\lambda^2\mu^3\xi^3|v|^2\mathrm{d}t.\end{aligned} \tag{2.43}$$

Next, we have

$$-4b\sum_{j,k=1}^{n}a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j)=-4b\frac{|\gamma_t|}{\gamma}\lambda\mu\xi\mathrm{Re}\left[\mathrm{i}\Big(\sum_{j,k=1}^{n}a^{jk}\beta_kv_j\Big)\overline{v}\right].$$

On $(T/2,T)$, due to $|\gamma_t|\leq C\gamma^2$, we obtain

$$\left|-4b\sum_{j,k=1}^{n}a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j)\right|\leq C|b|\lambda\mu\xi^2\left|\sum_{j,k=1}^{n}a^{jk}\beta_kv_j\right||v|. \tag{2.44}$$

On $(0,T/4)$, by (2.20), it holds that

$$\begin{aligned}\left|-4b\sum_{j,k=1}^{n}a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j)\right|\leq&C|b|\lambda^2\mu^3\xi\mathrm{e}^{\mu(6m-4)}\left|\sum_{j,k=1}^{n}a^{jk}\beta_kv_j\right||v|\\ \leq&C|b|\lambda^2\xi^2\left|\sum_{j,k=1}^{n}a^{jk}\beta_kv_j\right||v|,\end{aligned} \tag{2.45}$$

where we also have used $\mu^3\mathrm{e}^{-2\mu}\leq\frac{1}{2}$ for all $\mu>1$. Similarly, it also vanishes for $t\in(T/4,T/2)$, using (2.44) and (2.45), we have

$$\int_0^T-4b\sum_{j,k=1}^{n}a^{jk}\ell_{kt}\mathrm{Re}(\mathrm{i}\overline{v}v_j)\mathrm{d}t\geq-C|b|\int_0^T\lambda^2\mu\xi^2\left|\sum_{j,k=1}^{n}a^{jk}\beta_kv_j\right||v|\mathrm{d}t. \tag{2.46}$$

By (2.40), (2.43), (2.46) and $v=0$ on Σ, we have

$$\begin{aligned}\mathcal{J}_4 :=& \mathbb{E}\int_Q -2\sum_{j=1}^n [\mathrm{Re}(aE^j\overline{v}v_j)-\mathrm{Im}(bF^jv\overline{v}_j)]\mathrm{d}x\mathrm{d}t\\ \geq& -2a\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi||v|^2\mathrm{d}x\mathrm{d}t - Ca\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}\lambda^2\mu\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t\\ &- C(a^2+b^2+a+|b|)\mathbb{E}\int_Q(\lambda^2\mu^4\xi^2+\lambda^2\mu^3\xi^3+\lambda^3\mu^2\xi^3)|v|^2\mathrm{d}x\mathrm{d}t\\ &- C(a^2+b^2+|b|)\mathbb{E}\int_Q\mu^2|\nabla v|^2\mathrm{d}x\mathrm{d}t - C|b|\mathbb{E}\int_Q\lambda\xi\left|\sum_{j,k=1}^n a^{jk}\beta_k v_j\right|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.47}$$

Estimate for $\mathcal{J}_5$. Noting that $v=0$ on Σ and $\beta=0$ on Σ, for any $j=1,\dots,n$, we obtain

$$v_j=\frac{\partial v}{\partial\nu}\nu_j,\quad \beta_j=\frac{\partial\beta}{\partial\nu}\nu_j,$$

where $\nu=(\nu_1,\dots,\nu_n)$ denotes the unit outer normal vector on Γ. By the definition of V^k in Lemma 2.1, for any $k=1,\dots,n$, it holds that

$$\begin{aligned}V^k|_\Sigma =& 2(a^2+b^2)\sum_{j,j',k'=1}^n [a^{jk}a^{j'k'}\ell_j v_{j'}\overline{v}_{k'} - a^{jk'}a^{j'k}\ell_j(v_{j'}\overline{v}_{k'}+\overline{v}_{j'}v_{k'})]\mathrm{d}t\Big|_\Sigma\\ =& -2(a^2+b^2)\sum_{j,j',k'=1}^n \lambda\mu\xi\frac{\partial\beta}{\partial\nu}\left|\frac{\partial v}{\partial\nu}\right|^2 a^{jk}a^{j'k'}\nu_j\nu_{j'}\nu_{k'}\mathrm{d}t\Big|_\Sigma.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}\mathcal{J}_5 :=& \mathbb{E}\int_Q\sum_{k=1}^n V_k^k\mathrm{d}x = \mathbb{E}\int_\Sigma\sum_{k=1}^n V^k\nu_k\mathrm{d}\sigma\\ =& -2(a^2+b^2)\mathbb{E}\int_\Sigma\lambda\mu\xi\frac{\partial\beta}{\partial\nu}\left|\frac{\partial v}{\partial\nu}\right|^2\sum_{j,j',k',k=1}^n a^{jk}a^{j'k'}\nu_j\nu_{j'}\nu_{k'}\nu_k\mathrm{d}\sigma\geq 0.\end{aligned} \tag{2.48}$$

Estimate for $\mathcal{J}_6,\mathcal{J}_7$ *and* $\mathcal{J}_8$. By the first equation of (2.1), condition (H_1) and the fact $v=\theta z$, we have

$$\begin{aligned}\mathcal{J}_6 :=& \mathbb{E}\int_Q\sum_{j,k=1}^n aa^{jk}\mathrm{d}v_j\mathrm{d}\overline{v}_k\mathrm{d}x\\ =& \mathbb{E}\int_Q\sum_{j,k=1}^n aa^{jk}\theta^2\mathrm{d}(\ell_j z+z_j)\mathrm{d}(\ell_k\overline{z}+\overline{z}_k)\mathrm{d}x\\ \geq& as_0\mathbb{E}\int_Q\theta^2|\nabla Z+\nabla\ell Z|^2\mathrm{d}x\mathrm{d}t\\ \geq& \frac{1}{2}as_0\mathbb{E}\int_Q\theta^2|\nabla Z|^2\mathrm{d}x\mathrm{d}t - Ca\mathbb{E}\int_Q\lambda^2\mu^2\xi^2\theta^2|Z|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Similar to the previous analysis of ℓ_t and (2.14), we have

$$\begin{aligned}|\mathcal{J}_7| &= \left|\mathbb{E}\int_Q (\ell_t - aA)\mathrm{d}v\mathrm{d}\overline{v}\mathrm{d}x\right| \le \mathbb{E}\int_Q \left|\frac{\gamma_t}{\gamma}\lambda\varphi - aA\right|\theta^2|Z|^2\mathrm{d}x\mathrm{d}t\\ &\le C\mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + Ca\mathbb{E}\int_Q \lambda^2\mu^2\xi^2\theta^2|Z|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathcal{J}_8 :=& -2b\mathbb{E}\int_Q \sum_{j,k=1}^n a^{jk}\ell_k \mathrm{Im}(\mathrm{d}v\mathrm{d}\overline{v}_j)\mathrm{d}x\\ =& -2b\mathbb{E}\int_Q \sum_{j,k=1}^n a^{jk}\beta_k\lambda\mu\xi\theta^2\mathrm{Im}[\mathrm{d}z\mathrm{d}(\ell_j\overline{z}+\overline{z}_j)]\mathrm{d}x\\ \ge& -C|b|\mathbb{E}\int_Q \theta^2(\lambda^2\mu^2\xi^2|Z|^2 + \lambda\mu\xi|Z||\nabla Z|)\mathrm{d}x\mathrm{d}t\\ \ge& -\frac{1}{2}as_0\mathbb{E}\int_Q \theta^2|\nabla Z|^2\mathrm{d}x\mathrm{d}t - \frac{Cb^2}{a}\mathbb{E}\int_Q \theta^2\lambda^2\mu^2\xi^2|Z|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore, it holds that

$$\mathcal{J}_6 + \mathcal{J}_7 + \mathcal{J}_8 \ge -C\mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t - \frac{C(a^2+b^2)}{a}\mathbb{E}\int_Q \lambda^2\mu^2\xi^2\theta^2|Z|^2\mathrm{d}x\mathrm{d}t. \tag{2.49}$$

Estimate for $\mathcal{J}_9$. First, noting that the fact $\mathrm{Im}(c) = \mathrm{Re}(\mathrm{i}\overline{c})$ for any $c \in \mathbb{C}$, we obtain that

$$\begin{aligned}&2b\sum_{j,k=1}^n (a^{jk}\ell_k)_j \mathrm{Im}(v\mathrm{d}\overline{v}) + 2\mathrm{Re}(\overline{\Phi}\overline{v}\mathrm{d}v)\\ =& \,2\sum_{j,k=1}^n (a^{jk}\ell_k)_j \mathrm{Re}[(2a-\mathrm{i}b)\overline{v}\mathrm{d}v]\\ =& \,2a\sum_{j,k=1}^n (a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}+\overline{v}\mathrm{d}v) + \mathrm{i}b\sum_{j,k=1}^n (a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}-\overline{v}\mathrm{d}v)\\ =& \,2a\mathrm{d}\Big[\sum_{j,k=1}^n (a^{jk}\ell_k)_j|v|^2\Big] - 2a\sum_{j,k=1}^n (a^{jk}\ell_k)_{jt}|v|^2\mathrm{d}t - 2a\sum_{j,k=1}^n (a^{jk}\ell_k)_j\mathrm{d}v\mathrm{d}\overline{v}\\ &+\mathrm{i}b\sum_{j,k=1}^n (a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}-\overline{v}\mathrm{d}v).\end{aligned} \tag{2.50}$$

It is easy to check that

$$\begin{aligned}2a\sum_{j,k=1}^n (a^{jk}\ell_k)_j|v|^2\Big|_0^T =& \Big[-4a\lambda\mu^2\mathrm{e}^{\mu(6m+\beta)}\sum_{j,k=1}^n a^{jk}\beta_j\beta_k + a\mathcal{O}(\lambda)\mu\mathrm{e}^{\mu(6m+\beta)}\Big]|v(0)|^2\\ \ge& -Ca\lambda\mu^2\mathrm{e}^{\mu(6m+\beta)}|v(0)|^2.\end{aligned} \tag{2.51}$$

Further, we have

$$-2a\sum_{j,k=1}^{n}(a^{jk}\ell_k)_{jt}=-2a\frac{\gamma_t}{\gamma}[\Psi\lambda\mu^2\xi+\mathcal{O}(\lambda)\mu\xi]. \tag{2.52}$$

Combining with (2.52), similar to the analysis for W in Estimate for $\mathcal{J}_2$, we obtain

$$\begin{aligned}&-2a\mathbb{E}\int_Q\sum_{j,k=1}^{n}(a^{jk}\ell_k)_{jt}|v|^2\mathrm{d}x\mathrm{d}t\\ \geq&-Ca\mathbb{E}\int_0^{T/4}\int_G\frac{|\gamma_t|}{\gamma}\lambda\mu^2\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t-Ca\mathbb{E}\int_Q\lambda\mu^2\xi^2|v|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.53}$$

Further, we have

$$-2a\sum_{j,k=1}^{n}(a^{jk}\ell_k)_j\mathrm{d}v\mathrm{d}\overline{v}=a\mathcal{O}(\lambda)\mu^2\xi\theta^2|Z|^2\mathrm{d}t. \tag{2.54}$$

Integtating (2.50) on Q, taking mathematical expectation on both sides and noting (2.51), (2.53) and (2.54), we conclude that

$$\begin{aligned}\mathcal{J}_9:=&\,\mathbb{E}\int_Q 2b\sum_{j,k=1}^{n}(a^{jk}\ell_k)_j\mathrm{Im}(v\mathrm{d}\overline{v})+2\mathrm{Re}(\overline{\Phi}\overline{v}\mathrm{d}v)\mathrm{d}x\\ \geq&-Ca\mathbb{E}\int_G\lambda\mu^2\mathrm{e}^{\mu(6m+\beta)}|v(0)|^2\mathrm{d}x-Ca\mathbb{E}\int_0^{T/4}\int_G\frac{|\gamma_t|}{\gamma}\lambda\mu^2\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t\\ &-Ca\mathbb{E}\int_Q\lambda\mu^2\xi^2|v|^2\mathrm{d}x\mathrm{d}t-Ca\mathbb{E}\int_Q\lambda\mu^2\xi\theta^2|Z|^2\mathrm{d}x\mathrm{d}t\\ &+\mathbb{E}\int_Q\mathrm{i}b\sum_{j,k=1}^{n}(a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}-\overline{v}\mathrm{d}v)\mathrm{d}x.\end{aligned} \tag{2.55}$$

Next, we estimate the last term of (2.55). By the same analysis as (2.23) and (2.24) in [8], replacing ψ, φ, f, z, y and Y in [8] by β, ξ, Ξ, v, z and Z respectively, we obtain that

$$\begin{aligned}&\mathbb{E}\int_Q\mathrm{i}b\sum_{j,k=1}^{n}(a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}-\overline{v}\mathrm{d}v)\mathrm{d}x\\ \geq&-C\mathbb{E}\int_Q\theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t-C|ab|\mathbb{E}\int_Q\lambda\mu^3\xi|v||\nabla v|\mathrm{d}x\mathrm{d}t\\ &+(-a^2+b^2)\mathbb{E}\int_Q\lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t-Cb^2\mathbb{E}\int_Q(\lambda^2\mu^4\xi^2+\lambda^3\mu^3\xi^3)|v|^2\mathrm{d}x\mathrm{d}t\\ &-2b^2\mathbb{E}\int_Q\lambda\mu^2\xi\Psi\sum_{j,k=1}^{n}a^{jk}v_k\overline{v}_j\mathrm{d}x\mathrm{d}t-(a^2+b^2)\mathbb{E}\int_Q\lambda\mu^2\xi\left|\sum_{j,k=1}^{n}a^{jk}\beta_jv_k\right|^2\mathrm{d}x\mathrm{d}t\\ &-C(a^2+b^2)\mathbb{E}\int_Q[\lambda\mu\xi|\nabla v|^2+\lambda^3\mu^3\xi^3|v|^2]\mathrm{d}x\mathrm{d}t,\end{aligned}$$

which imply that

$$\begin{aligned}
&\mathbb{E}\int_Q \mathrm{i}b\sum_{j,k=1}^n (a^{jk}\ell_k)_j(v\mathrm{d}\overline{v}-\overline{v}\mathrm{d}v)\mathrm{d}x\\
\geq&-C\mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t+(b^2-a^2)\mathbb{E}\int_Q \lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t\\
&-2b^2\mathbb{E}\int_Q \lambda\mu^2\xi\Psi\sum_{j,k=1}^n a^{jk}v_k\overline{v}_j\mathrm{d}x\mathrm{d}t-(a^2+b^2)\mathbb{E}\int_Q \lambda\mu^2\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t\\
&-C(a^2+b^2)\bigg[\mathbb{E}\int_Q(\lambda^2\mu^4\xi^2+\lambda^3\mu^3\xi^3)|v|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q(\lambda\mu\xi+\mu^2)|\nabla v|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}\tag{2.56}$$

By (2.55) and (2.56), we have

$$\begin{aligned}
\mathcal{J}_9\geq&-C\bigg[a\mathbb{E}\int_G \lambda\mu^2\mathrm{e}^{\mu(6m+\beta)}|v(0)|^2\mathrm{d}x+a\mathbb{E}\int_0^{T/4}\int_G\frac{|\gamma_t|}{\gamma}\lambda\mu^2\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t\\
&+(a^2+b^2+a)\mathbb{E}\int_Q(\lambda^2\mu^4\xi^2+\lambda^3\mu^3\xi^3)|v|^2\mathrm{d}x\mathrm{d}t+a\mathbb{E}\int_Q\lambda\mu^2\xi\theta^2|Z|^2\mathrm{d}x\mathrm{d}t\\
&+(a^2+b^2)\mathbb{E}\int_Q(\lambda\mu\xi+\mu^2)|\nabla v|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t\bigg]\\
&+(b^2-a^2)\mathbb{E}\int_Q\lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t-2b^2\mathbb{E}\int_Q\lambda\mu^2\xi\Psi\sum_{j,k=1}^n a^{jk}v_k\overline{v}_j\mathrm{d}x\mathrm{d}t\\
&-(a^2+b^2)\mathbb{E}\int_Q\lambda\mu^2\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{2.57}$$

Step 3: Towards the Carleman estimate. Combining (2.23), (2.38), (2.39), (2.47), (2.48), (2.49), (2.57) with (2.12), we obtain that

$$\begin{aligned}
2\mathbb{E}\int_Q\mathrm{Re}(\theta\overline{I_1}\mathcal{L}z)\mathrm{d}x\geq&\,2\mathbb{E}\int_Q|I_1|^2\mathrm{d}x\mathrm{d}t+c_1\mathbb{E}\int_G\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}|v(0)|^2\mathrm{d}x\\
&+2a\mathbb{E}\int_0^{T/4}\int_G\frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi\Psi|\varphi||v|^2\mathrm{d}x\mathrm{d}t+(a^2+3b^2)\mathbb{E}\int_Q\lambda^3\mu^4\xi^3|\Psi|^2|v|^2\mathrm{d}x\mathrm{d}t\\
&+2a^2\mathbb{E}\int_Q\lambda\mu^2\xi\Psi\sum_{j,k=1}^n a^{jk}v_j\overline{v}_k\mathrm{d}x\mathrm{d}t+3(a^2+b^2)\mathbb{E}\int_Q\lambda\mu^2\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t\\
&-C\bigg[a\mathbb{E}\int_G\lambda\mu^2\mathrm{e}^{\mu(6m+1)}|v(0)|^2\mathrm{d}x+a\mathbb{E}\int_0^{T/4}\int_G\frac{|\gamma_t|}{\gamma}(\lambda\mu^2+\lambda^2\mu)\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t\\
&+(a^2+b^2+a+|b|)\mathbb{E}\int_Q(\lambda^3\mu^3\xi^3+\lambda^2\mu^4\xi^2)|v|^2\mathrm{d}x\mathrm{d}t\\
&+(a^2+b^2+|b|)\mathbb{E}\int_Q(\lambda\mu\xi+\mu^2)|\nabla v|^2\mathrm{d}x\mathrm{d}t+|b|\mathbb{E}\int_Q\lambda\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_Q\lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t+\frac{(a^2+b^2)}{a}\mathbb{E}\int_Q\lambda^2\mu^2\xi^2\theta^2|Z|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}\tag{2.58}$$

By (2.1), we easily have

$$2\mathbb{E}\int_Q \mathrm{Re}(\theta\overline{I_1}\mathcal{L}z)\mathrm{d}x \le \mathbb{E}\int_Q |I_1|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t. \tag{2.59}$$

And, if $\mu > \mu_{**}(a,b) := \ln\frac{a^2+b^2}{a}$, it holds that

$$\frac{(a^2+b^2)}{a}\mathbb{E}\int_Q \lambda^2\mu^2\xi^2\theta^2|Z|^2\mathrm{d}x\mathrm{d}t \le \mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t. \tag{2.60}$$

By (H_1), (2.4), (2.59) and (2.60), we can easily show that (2.58) imply that

$$\begin{aligned}
&\mathbb{E}\int_Q |I_1|^2\mathrm{d}x\mathrm{d}t + c_1\mathbb{E}\int_G \lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}|v(0)|^2\mathrm{d}x\\
&+ 2as_0\alpha_0^2\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t + (a^2+3b^2)s_0^2\alpha_0^4\mathbb{E}\int_Q \lambda^3\mu^4\xi^3|v|^2\mathrm{d}x\mathrm{d}t\\
&+ 2a^2s_0^2\alpha_0^2\mathbb{E}\int_Q \lambda\mu^2\xi|\nabla v|^2\mathrm{d}x\mathrm{d}t + 3(a^2+b^2)\mathbb{E}\int_Q \lambda\mu^2\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t\\
\le\ & C\Big[a\mathbb{E}\int_G \lambda\mu^2\mathrm{e}^{\mu(6m+1)}|v(0)|^2\mathrm{d}x + a\mathbb{E}\int_0^{T/4}\int_G \frac{|\gamma_t|}{\gamma}(\lambda\mu^2+\lambda^2\mu)\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t\\
&+ (a^2+b^2+a+|b|)\mathbb{E}\int_Q (\lambda^3\mu^3\xi^3+\lambda^2\mu^4\xi^2)|v|^2\mathrm{d}x\mathrm{d}t\\
&+ (a^2+b^2+|b|)\mathbb{E}\int_Q (\lambda\mu\xi+\mu^2)|\nabla v|^2\mathrm{d}x\mathrm{d}t + |b|\mathbb{E}\int_Q \lambda\xi\Big|\sum_{j,k=1}^n a^{jk}\beta_j v_k\Big|^2\mathrm{d}x\mathrm{d}t\\
&+ \mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_0^T\int_{G'} \lambda\mu^2\xi|\nabla v|^2\mathrm{d}x\mathrm{d}t\\
&+ a\mathbb{E}\int_0^{T/4}\int_{G'} \frac{|\gamma_t|}{\gamma}\lambda^2\mu^2\xi|\varphi||v|^2\mathrm{d}x\mathrm{d}t + (a^2+b^2)\mathbb{E}\int_0^T\int_{G'} \lambda^3\mu^4\xi^3|v|^2\mathrm{d}x\mathrm{d}t\Big].
\end{aligned}$$

Hence, we set

$$\begin{aligned}
K_1 \equiv K_1(a,b,s_0,\alpha_0) &:= \max\left\{\frac{2C}{s_0\alpha_0^2}, \frac{4C(a^2+b^2+a+|b|)}{(a^2+b^2)s_0^2\alpha_0^4}, \frac{2C(a^2+b^2+|b|)}{a^2s_0^2\alpha_0^2}, 1\right\},\\
K_2 \equiv K_2(a,b,s_0,\alpha_0) &:= \max\left\{\frac{2Ca}{c_1}, K_1(a,b), \frac{C|b|}{a^2+b^2}\right\},
\end{aligned}$$

and obtain that there exist constants $\mu_0 \equiv \mu_0(a,b,s_0,\alpha_0) = \max\{K_2, \mu_*, \mu_{**}\}$ and $\lambda_0 \equiv \lambda_0(a,b,s_0,\alpha_0) = K_1$, such that for all $\mu \geq \mu_0$ and $\lambda \geq \lambda_0$, it holds that

$$\begin{aligned}
&\mathbb{E}\int_G \lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}|v(0)|^2\mathrm{d}x + a\mathbb{E}\int_0^{T/4}\int_G \lambda^2\mu^2\xi|\varphi||\gamma_t||v|^2\mathrm{d}x\mathrm{d}t \\
&+ (a^2+b^2)\mathbb{E}\int_Q \lambda^3\mu^4\xi^3|v|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_Q \lambda\mu^2\xi|\nabla v|^2\mathrm{d}x\mathrm{d}t \\
\leq\ & C\bigg[\mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t + a\mathbb{E}\int_0^{T/4}\int_{G'} \lambda^2\mu^2\xi|\varphi||\gamma_t||v|^2\mathrm{d}x\mathrm{d}t \\
&+ (a^2+b^2)\mathbb{E}\int_0^T\int_{G'} \lambda^3\mu^4\xi^3|v|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_0^T\int_{G'} \lambda\mu^2\xi|\nabla v|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}$$

By the definition of v, this implies that

$$\begin{aligned}
&\mathbb{E}\int_G \theta^2(0)\lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}|z(0)|^2\mathrm{d}x + a\mathbb{E}\int_0^{T/4}\int_G \theta^2\lambda^2\mu^2\xi|\varphi||\gamma_t||z|^2\mathrm{d}x\mathrm{d}t \\
&+ (a^2+b^2)\mathbb{E}\int_Q \theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_Q \theta^2\lambda\mu^2\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t \\
\leq\ & C\bigg[\mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t \\
&+ (a^2+b^2)\mathbb{E}\int_0^T\int_{G'} \theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_0^T\int_{G'} \theta^2\lambda\mu^2\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t \\
&+ a\mathbb{E}\int_0^{T/4}\int_{G'} \theta^2\lambda^2\mu^2\xi|\varphi||\gamma_t||z|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned} \tag{2.61}$$

Step 4: Last arrangements and conclusion. We choose a real-valued function $\zeta \in C_0^\infty(G_0)$ such that $\zeta = 1$ in G' and $\frac{|\nabla\zeta|}{\zeta^{1/2}} \in L^\infty(G_0)$. By the Itô formula, we calculate

$$\mathrm{d}(\xi\theta^2\zeta|z|^2) = \xi\theta^2\zeta z\mathrm{d}\overline{z} + \xi\theta^2\zeta\overline{z}\mathrm{d}z + (\xi\theta^2)_t\zeta|z|^2\mathrm{d}t + \xi\theta^2\zeta|\mathrm{d}z|^2.$$

and thus, using the equation (2.1) and $\zeta \in C_0^\infty(G_0)$, integrating it over Q and taking the mathematical expectation, we have

$$\begin{aligned}
&2a\mathbb{E}\int_Q \xi\theta^2\zeta\sum_{j,k=1}^n a^{jk}z_j\overline{z}_k\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q (\xi\theta^2)_t\zeta|z|^2\mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_G \xi(0,x)\theta^2(0,x)\zeta(x)|z(0,x)|^2\mathrm{d}x + \mathbb{E}\int_Q \xi\theta^2\zeta|Z|^2\mathrm{d}x\mathrm{d}t \\
=\ & -2\mathbb{E}\int_Q \mathrm{Re}[(a-\mathrm{i}b)\overline{z}\sum_{j,k=1}^n a^{jk}z_j(\xi\theta^2\zeta)_k]\mathrm{d}x\mathrm{d}t - 2\mathbb{E}\int_Q \xi\theta^2\zeta\mathrm{Re}(z\overline{\Xi})\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.62}$$

We readily see that the last two terms on the left-hand side of (2.62) are positive, so they can be dropped. Also, notice that using the properties of ζ and (H_1), the first term gives the local terms containing $|\nabla z|^2$. For the second term on the left-hand side of (2.62), similar to step 2 above, we can analyze it on different time intervals. In summary, multiplying both sides of (2.62) by $a\lambda\mu^2$, by the properties of ζ, Young inequality and

simple calculation, we have

$$\begin{aligned} &a\mathbb{E}\int_0^{T/4}\int_{G'}\theta^2\lambda^2\mu^2\xi|\varphi||\gamma_t||z|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_0^T\int_{G'}\theta^2\lambda\mu^2\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t \\ \le\ & C(a^2+b^2)\mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_Q\theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t \\ &+ C(a^2+b^2)\mathbb{E}\int_Q\theta^2\lambda^2\mu^4\xi^2|z|^2\mathrm{d}x\mathrm{d}t + Ca^2\mathbb{E}\int_Q\theta^2\mu^2|\nabla z|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.63}$$

Combining (2.61) with (2.63), we obtain

$$\begin{aligned} &\mathbb{E}\int_G\theta^2(0)\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}|z(0)|^2\mathrm{d}x + a\mathbb{E}\int_0^{T/4}\int_G\theta^2\lambda^2\mu^2\xi|\varphi||\gamma_t||z|^2\mathrm{d}x\mathrm{d}t \\ &+ (a^2+b^2)\mathbb{E}\int_Q\theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_Q\theta^2\lambda\mu^2\xi|\nabla z|^2\mathrm{d}x\mathrm{d}t \\ \le& C\bigg[\mathbb{E}\int_Q\lambda^2\mu^2\xi^3\theta^2|Z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q\theta^2|\Xi|^2\mathrm{d}x\mathrm{d}t + a^2\mathbb{E}\int_Q\theta^2\mu^2|\nabla z|^2\mathrm{d}x\mathrm{d}t \\ &+ (a^2+b^2)\mathbb{E}\int_Q\theta^2\lambda^2\mu^4\xi^2|z|^2\mathrm{d}x\mathrm{d}t + (a^2+b^2)\mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|z|^2\mathrm{d}x\mathrm{d}t\bigg]. \end{aligned} \tag{2.64}$$

Finally, from (2.64), we can see that there exist constants $\mu_1 \equiv \mu_1(a,b,s_0,\alpha_0) = \mu_0$ and $\lambda_1 \equiv \lambda_1(a,b,s_0,\alpha_0) = \max\{\lambda_0, 2C\}$ such that (2.9) holds for all $\mu \ge \mu_1$, $\lambda \ge \lambda_1$. Thus, the proof of Theorem 2.2 is complete. □

2.2. The proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. Applying Theorem 2.2, we first establish a controllability result for a linear forward stochastic complex Ginzburg–Landau equation with one source term and two controls, that is,

$$\begin{cases} \mathrm{d}y - (a+\mathrm{i}b)\sum_{j,k=1}^n (a^{jk}y_j)_k\mathrm{d}t = (F+\chi_{G_0}h)\mathrm{d}t + H\mathrm{d}B(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \tag{2.65}$$

where $y = y(t,x)$ denotes the state variable associated to the initial state $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G;\mathbb{C}))$, the control pair $(h,H) \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$. Observe that given the aforementioned regularity on the controls and source term, one can easily show that system (2.65) admits a unique solution $y \in \mathcal{H}_T$ (see *e.g.*, [9], Thm. 3.24).

We define the space

$$\mathscr{S}_{\lambda,\mu} = \left\{F \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C})) \middle| \left(\mathbb{E}\int_Q\theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|F|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} < +\infty\right\},$$

which is a Banach space equipped with the canonical norm denoted by $\|\cdot\|_{\mathscr{S}_{\lambda,\mu}}$.

Theorem 2.6. *Assume that $F \in \mathscr{S}_{\lambda,\mu}$. For any $y_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(G;\mathbb{C}))$, there is a control pair $(\hat{h},\hat{H})$ such that the associated solution $\hat{y}$ to the controlled system (2.65) satisfies $\hat{y}(T) = 0$ in G, $\mathbb{P}$-a.s. Moreover, one can find*

three positive constants $C = C(G, G_0, T, m, s_0, \alpha_0, a, b)$, $\lambda_2 = \lambda_2(a, b)$ *and* $\mu_2 = \mu_2(a, b)$, *such that*

$$\begin{aligned}&\mathbb{E}\int_Q \theta^{-2}|\hat{y}|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{G_0}\theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|\hat{h}|^2\mathrm{d}x\mathrm{d}t\\&+\mathbb{E}\int_Q\theta^{-2}\lambda^{-2}\mu^{-2}\xi^{-3}|\hat{H}|^2\mathrm{d}x\mathrm{d}t\\\leq& C\bigg(\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\theta^{-2}(0)|y_0|^2\mathrm{d}x + \|F\|^2_{\mathscr{I}_{\lambda,\mu}}\bigg),\end{aligned}\tag{2.66}$$

for all $\lambda \geq \lambda_2$ *and* $\mu \geq \mu_2$.

Proof of Theorem 2.6: We borrow some ideas from [30]. The main steps are as follows. First, we construct a family of optimal control problems for equation (2.65). Next, we establish a uniform estimate for these optimal solutions. Finally, by taking the limit, one can obtain the estimate (2.66) and the desired null controllability result. We also divide the proof into three parts.

Step 1. For any $\varepsilon > 0$, consider the following weight function

$$\gamma_\varepsilon(t) = \begin{cases}1 + \left(1 - \dfrac{4t}{T}\right)^\sigma, & t \in [0, T/4),\\ 1, & t \in [T/4, T/2 + \varepsilon),\\ \gamma(t - \varepsilon), & t \in [T/2 + \varepsilon, T],\end{cases}$$

where σ is the same as in (2.7). Using the new weight function $\gamma_\varepsilon(t)$, we set

$$\varphi_\varepsilon(t, x) := \gamma_\varepsilon(t)\alpha(x), \quad \theta_\varepsilon := \mathrm{e}^{\lambda\varphi_\varepsilon}.$$

With this notation, we introduce the functional

$$\begin{aligned}J_\varepsilon(h, H) :=& \frac{1}{2}\mathbb{E}\int_Q\theta_\varepsilon^{-2}|y|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^T\int_{G_0}\theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|h|^2\mathrm{d}x\mathrm{d}t\\&+\frac{1}{2}\mathbb{E}\int_Q\theta^{-2}\lambda^{-2}\mu^{-2}\xi^{-3}|H|^2\mathrm{d}x\mathrm{d}t + \frac{1}{2\varepsilon}\mathbb{E}\int_G|y(T)|^2\mathrm{d}x\end{aligned}$$

and consider the following optimal control problem:

$$\begin{cases}\min\limits_{(h,H)\in\mathscr{H}} J_\varepsilon(h, H)\\ \text{subject to equation (2.65)},\end{cases}\tag{2.67}$$

where

$$\begin{aligned}\mathscr{H} = \bigg\{&(h, H) \in L^2_{\mathbb{F}}(0, T; L^2(G_0; \mathbb{C})) \times L^2_{\mathbb{F}}(0, T; L^2(G; \mathbb{C}))\bigg|\\&\mathbb{E}\int_0^T\int_{G_0}\theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|h|^2\mathrm{d}x\mathrm{d}t < +\infty, \quad \mathbb{E}\int_Q\theta^{-2}\lambda^{-2}\mu^{-2}\xi^{-3}|H|^2\mathrm{d}x\mathrm{d}t < +\infty\bigg\}.\end{aligned}$$

Similar to [40], it is easy to check that for any $\varepsilon > 0$, (2.67) admits a unique optimal pair solution that we denote by $(h_\varepsilon, H_\varepsilon)$. Moreover, by the standard variational method (see [40, 41]), it follows that

$$h_\varepsilon = -\chi_{G_0}\theta^2\lambda^3\mu^4\xi^3 r_\varepsilon, \quad H_\varepsilon = -\theta^2\lambda^2\mu^2\xi^3 R_\varepsilon \quad \text{in } Q,\ \mathbb{P}\text{-}a.s., \tag{2.68}$$

where the pair $(r_\varepsilon, R_\varepsilon)$ verifies the backward stochastic equation

$$\begin{cases} \mathrm{d}r_\varepsilon + (a - \mathrm{i}b)\sum\limits_{j,k=1}^{n}(a^{jk}r_{\varepsilon j})_k \mathrm{d}t = -\theta_\varepsilon^{-2} y_\varepsilon \mathrm{d}t + R_\varepsilon \mathrm{d}B(t) & \text{in } Q, \\ r_\varepsilon = 0 & \text{on } \Sigma, \\ r_\varepsilon(T) = \dfrac{1}{\varepsilon} y_\varepsilon(T) & \text{in } G, \end{cases} \tag{2.69}$$

where y_ε is the solution of (2.65) with the controls $h = h_\varepsilon$ and $H = H_\varepsilon$.

Step 2. We now establish a uniform estimate for the optimal solutions $\{(y_\varepsilon, h_\varepsilon, H_\varepsilon)\}_{\varepsilon>0}$. By Itô's formula, (2.65) and (2.69), it follows that

$$\begin{aligned} &\mathbb{E}\int_G y_\varepsilon(T)\overline{r_\varepsilon}(T)\mathrm{d}x \\ &= \mathbb{E}\int_G y_\varepsilon(0)\overline{r_\varepsilon}(0)\mathrm{d}x + \mathbb{E}\int_Q \Big[(a + \mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}y_{\varepsilon j})_k + (F + \chi_{G_0}h_\varepsilon)\Big]\overline{r_\varepsilon}\mathrm{d}x\mathrm{d}t \\ &\quad + \mathbb{E}\int_Q \Big[-(a + \mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}\overline{r_\varepsilon}_j)_k - \theta_\varepsilon^{-2}\overline{y_\varepsilon}\Big] y_\varepsilon \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q H_\varepsilon\overline{R_\varepsilon}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

This, together with (2.68) and the last equality of (2.69), imply that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2\lambda^2\mu^2\xi^3|R_\varepsilon|^2\mathrm{d}x\mathrm{d}t \\ &+ \mathbb{E}\int_Q \theta_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_G |y_\varepsilon(T)|^2\mathrm{d}x = \mathbb{E}\int_G y_0\overline{r_\varepsilon}(0)\mathrm{d}x + \mathbb{E}\int_Q F\overline{r_\varepsilon}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.70}$$

Applying the Carleman estimate in Theorem 2.2 to equation (2.69) with $\Xi = -\theta_\varepsilon^{-2}y_\varepsilon$, $z = r_\varepsilon$ and $Z = R_\varepsilon$, we get

$$\begin{aligned} &\mathbb{E}\int_G \lambda^2\mu^3 \mathrm{e}^{2\mu(6m+1)}\theta^2(0)|r_\varepsilon(0)|^2\mathrm{d}x + \mathbb{E}\int_Q \lambda^3\mu^4\xi^3\theta^2|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t \\ \le& C_1\Big(\mathbb{E}\int_0^T\int_{G_0}\lambda^3\mu^4\xi^3\theta^2|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2|\theta_\varepsilon^{-2}y_\varepsilon|^2\mathrm{d}x\mathrm{d}t \\ &+ \mathbb{E}\int_Q \lambda^2\mu^2\xi^3\theta^2|R_\varepsilon|^2\mathrm{d}x\mathrm{d}t\Big). \end{aligned} \tag{2.71}$$

In view of (2.71), we use the Young inequality on the right-hand side of (2.70) to obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^2\lambda^2\mu^2\xi^3|R_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_Q\theta_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(T)|^2\mathrm{d}x\\
\le&\,\rho\bigg[\mathbb{E}\int_G\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}\theta^2(0)|r_\varepsilon(0)|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^3\mu^4\xi^3\theta^2|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t\bigg]\\
&+C_\rho\bigg[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\theta^{-2}(0)|y_0|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\xi^{-3}\theta^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg]
\end{aligned}\tag{2.72}$$

for any $\rho>0$. Noting that $\theta^2\theta_\varepsilon^{-2}\le 1$ for any $(t,x)\in Q$ and using (2.71) and (2.72) with sufficiently small $\rho>0$, we obtain that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_0}\theta^2\lambda^3\mu^4\xi^3|r_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^2\lambda^2\mu^2\xi^3|R_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_Q\theta_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(T)|^2\mathrm{d}x\\
\le&\,C\bigg[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\theta^{-2}(0)|y_0|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\xi^{-3}\theta^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}$$

Noting that (2.68), we get

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_0}\theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|h_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^{-2}\lambda^{-2}\mu^{-2}\xi^{-3}|H_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\
&+\mathbb{E}\int_Q\theta_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(T)|^2\mathrm{d}x\\
\le&\,C\bigg[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\theta^{-2}(0)|y_0|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\xi^{-3}\theta^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}\tag{2.73}$$

Step 3. By (2.73), it is easy to check that there exists $(\hat{h},\hat{H},\hat{y})$ such that

$$\begin{cases}
h_\varepsilon\rightharpoonup\hat{h} & \text{weakly in } L^2(\Omega\times(0,T);L^2(G_0;\mathbb{C})),\\
H_\varepsilon\rightharpoonup\hat{H} & \text{weakly in } L^2(\Omega\times(0,T);L^2(G;\mathbb{C})),\\
y_\varepsilon\rightharpoonup\hat{y} & \text{weakly in } L^2(\Omega\times(0,T);L^2(G;\mathbb{C})).
\end{cases}\tag{2.74}$$

It is easy to show that $\hat{y}$ is the solution to (2.65) associated to $(\hat{h},\hat{H})$. Moreover, by (2.73), we get that $\hat{y}(T)=0$ in G, $\mathbb{P}$-*a.s.*. Also, from the weak convergence (2.74), Fatou's lemma and the uniform estimate (2.73), we obtain (2.66). Thus, the proof of Theorem 2.6 is complete. □

Based on Theorem 2.6, let us prove Theorem 1.2.

Proof of Theorem 1.2: According to Theorem 2.6, for any given $F\in\mathscr{S}_{\lambda,\mu}$, we know that there exists a control pair $(h,H)\in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C}))\times L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$, such that the corresponding solution $y\in\mathcal{H}_T$ to the controlled system (2.65) satisfies $y(T)=0$ in G, $\mathbb{P}$-a.s. Hence, let us consider a nonlinearity f satisfying assumptions

(H_2)-(H_4) and define the nonlinear map,

$$\mathscr{E} : F \in \mathscr{S}_{\lambda,\mu} \mapsto f(w,t,x,y) \in \mathscr{S}_{\lambda,\mu},$$

where y is the trajectory of (2.65) associated to the data y_0 and F. In the following, to simplify the notation, we simply write $f(y)$.

Next, we will show that $\mathscr{E}$ is a contraction mapping from $\mathscr{S}_{\lambda,\mu}$ into $\mathscr{S}_{\lambda,\mu}$. First, we check that the mapping $\mathscr{E}$ is well defined. In fact, for any $F \in \mathscr{S}_{\lambda,\mu}$, using (H_2)-(H_4) and (2.66), we have

$$\begin{aligned}
\|\mathscr{E}F\|^2_{\mathscr{S}_{\lambda,\mu}} &= \mathbb{E}\int_Q \theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|f(y)|^2\mathrm{d}x\mathrm{d}t \\
&\le \kappa^2\mathbb{E}\int_Q \theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|y|^2\mathrm{d}x\mathrm{d}t \\
&\le \kappa^2\lambda^{-3}\mu^{-4}\mathbb{E}\int_Q \theta^{-2}|y|^2\mathrm{d}x\mathrm{d}t \\
&\le C\lambda^{-3}\mu^{-4}\left(\mathbb{E}\int_G \lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\theta^{-2}(0)|y_0|^2\mathrm{d}x + \|F\|^2_{\mathscr{S}_{\lambda,\mu}}\right) \\
&\le \lambda^{-3}\mu^{-4}\left(C_2\mathbb{E}\|y_0\|^2_{L^2(G)} + C\|F\|^2_{\mathscr{S}_{\lambda,\mu}}\right) < \infty,
\end{aligned}$$

for any sufficiently large parameters $\lambda \ge \lambda_2$ and $\mu \ge \mu_2$, where $C_2 = C_2(G, G_0, T, m, s_0, \alpha_0, a, b, \lambda, \mu) > 0$, and $C = C(G, G_0, T, m, s_0, \alpha_0, a, b) > 0$. This proves that $\mathscr{E}$ is well defined.

Next, we check that the mapping $\mathscr{E}$ is strictly contractive. Let y_1, y_2 be the solutions of the controlled system (2.65) with respect to the source terms F_1, $F_2 \in \mathscr{S}_{\lambda,\mu}$, respectively. Then applying (2.66) in Theorem 2.6 for the equation associated to $F = F_1 - F_2$, $y(0) = y_0 - y_0 = 0$, and using assumption (H_4), we have

$$\begin{aligned}
\|\mathscr{E}F_1 - \mathscr{E}F_2\|_{\mathscr{S}_{\lambda,\mu}} =& \mathbb{E}\int_Q \theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|f(y_1) - f(y_2)|^2\mathrm{d}x\mathrm{d}t \\
\le& \kappa^2\mathbb{E}\int_Q \theta^{-2}\lambda^{-3}\mu^{-4}\xi^{-3}|y_1 - y_2|^2\mathrm{d}x\mathrm{d}t \\
\le& C\kappa^2\lambda^{-3}\mu^{-4}\|F_1 - F_2\|^2_{\mathscr{S}_{\lambda,\mu}},
\end{aligned}$$

for any sufficiently large parameters $\lambda \ge \lambda_2$ and $\mu \ge \mu_2$, where $C = C(G, G_0, T, m, s_0, \alpha_0, a, b) > 0$. Thus, if necessary, we can increase the value of λ and μ such that $C\kappa^2\lambda^{-3}\mu^{-4} < 1$, which implies that the mapping $\mathscr{E}$ is strictly contractive.

Further, by the Banach fixed point theorem, it follows that $\mathscr{E}$ has a unique fixed point $\tilde{F} \in \mathscr{S}_{\lambda,\mu}$. Moreover, it holds that $\tilde{F} = f(\omega, t, x, y)$, where y is the solution for the equation (2.65) associated to $F = \tilde{F}$, which means that y is the solution to equation (1.3). Applying Theorem 2.6, we know that there exists a control pair $(h, H) \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$, such that y to the controlled system (1.3) satisfies $y(T) = 0$ in G, $\mathbb{P}$-a.s. According to Remark 1.4, the proof of Theorem 1.2 is complete. □

3. Controllability of a semilinear backward stochastic complex Ginzburg–Landau equation

Similar to the proof of Theorem 1.2, the key to the proof of Theorem 1.3 is to obtain the controllability results of the following linear backward systems with source terms. However, unlike the previous section, we will not need to prove the Carleman estimate for the corresponding adjoint system (*i.e.*, the forward equation).

Although this is possible, we can simplify the problem by establishing a Carleman estimate for a random Ginzburg–Landau equation, which can be obtained by the deterministic Carleman estimate.

3.1. A new Carleman estimate for a random complex Ginzburg–Landau equation

In this section, we first derive a new Carleman estimate for the following deterministic Ginzburg–Landau equation with source term ϖ:

$$\begin{cases} q_t - (a+\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}q_j)_k = \varpi & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(0) = q_0 & \text{in } G, \end{cases} \tag{3.1}$$

where $a > 0$, $b \in \mathbb{R}$, coefficients a^{jk} satisfy the assumption (H_1) and $\varpi \in L^2(0,T;L^2(G;\mathbb{C}))$. For this, we need to introduce some new auxiliary functions, which are the mirrored version of (2.5) and (2.6). In more detail, also set $0 < T < 1$, we define the function $\tilde{\gamma}(t)$ as

$$\begin{cases} \tilde{\gamma}(t) = \dfrac{1}{t^m}, & t \in [0, T/4), \\ \tilde{\gamma}(t) \text{ is decreasing on } [T/4, T/2), & \\ \tilde{\gamma}(t) = 1, & t \in [T/2, 3T/4), \\ \tilde{\gamma}(t) = 1 + \left(1 - \dfrac{4(T-t)}{T}\right)^{\sigma}, & t \in [3T/4, T], \\ \tilde{\gamma}(t) \in C^2([0,T)), & \end{cases}$$

where $m \geq 1$ and $\sigma \geq 2$ is defined in (2.7). Observe that $\tilde{\gamma}(t)$ is the mirrored version of $\gamma(t)$ in (2.7) with respect to $T/2$. With this new function, we set

$$\tilde{\varphi}(t,x) := \tilde{\gamma}(t)\alpha(x), \quad \tilde{\xi}(t,x) := \tilde{\gamma}(t)\mathrm{e}^{\mu(6m+\beta(x))},$$

where α is defined in (2.6) and μ is also a positive parameter with $\mu \geq 1$. We finally set the weight function

$$\tilde{\theta} := \mathrm{e}^{\tilde{\ell}}, \text{ where } \tilde{\ell}(t,x) := \lambda\tilde{\varphi}(t,x).$$

We have the Carleman estimate for the deterministic Ginzburg–Landau equation (3.1).

Theorem 3.1. *There exist three positive constants $\lambda_3 = \lambda_3(a,b,s_0,\alpha_0)$, $\mu_3 = \mu_3(a,b,s_0,\alpha_0)$ and C, such that for any $\varpi \in L^2(0,T;L^2(G;\mathbb{C}))$ and $q_0 \in L^2(G)$, any solution q of (3.1) satisfies*

$$\begin{aligned} &\int_G \lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}\tilde{\theta}^2(T)|q(T)|^2\mathrm{d}x + \int_Q \lambda\mu^2\tilde{\xi}\tilde{\theta}^2|\nabla q|^2\mathrm{d}x\mathrm{d}t \\ &+ \int_Q \lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|q|^2\mathrm{d}x\mathrm{d}t + \int_{3T/4}^{T}\int_G \lambda^2\mu^2\tilde{\xi}\tilde{\theta}^2|\tilde{\varphi}||\tilde{\gamma}_t||q|^2\mathrm{d}x\mathrm{d}t \\ &\leq C_1\left(\int_0^T\int_{G_0}\lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|q|^2\mathrm{d}x\mathrm{d}t + \int_Q \tilde{\theta}^2|\varpi|^2\mathrm{d}x\mathrm{d}t\right), \end{aligned} \tag{3.2}$$

for all $\lambda \geq \lambda_3$ and $\mu \geq \mu_3$, where C_1 is defined in (2.10).

By using an argument similar to Theorem 2.2, one can establish the Carleman estimate (3.2) in Theorem 3.1. We omit the proof here.

Next, let us consider the forward complex Ginzburg–Landau equation given by

$$\begin{cases} \mathrm{d}q - (a+\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}q_j)_k\mathrm{d}t = \varpi_1\mathrm{d}t + \varpi_2\mathrm{d}B(t) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(0) = q_0 & \text{in } G, \end{cases} \tag{3.3}$$

where $\varpi_j \in L^2(0,T;L^2(G;\mathbb{C}))$, $j=1,2$, and $q_0 \in L^2_{\mathcal{F}_0}(\Omega;L^2(G;\mathbb{C}))$. Notice that when $\varpi_2 = 0$, equation (3.3) becomes a random Ginzburg–Landau equation. Therefore, using Theorem 3.1, we conclude the following global Carleman estimate for random Ginzburg–Landau equations.

Theorem 3.2. *Assume that $\varpi_2 \equiv 0$, then there exist three positive constants $\lambda_4 = \lambda_4(a,b,s_0,\alpha_0)$, $\mu_4 = \mu_4(a,b,s_0,\alpha_0)$ and C, such that for any $\varpi_1 \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ and $q_0 \in L^2_{\mathcal{F}_0}(\Omega;L^2(G;\mathbb{C}))$, any solution $q \in \mathcal{H}_T$ of (3.3) satisfies*

$$\begin{aligned} &\mathbb{E}\int_G \lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}\tilde{\theta}^2(T)|q(T)|^2\mathrm{d}x + \mathbb{E}\int_Q \lambda\mu^2\tilde{\xi}\tilde{\theta}^2|\nabla q|^2\mathrm{d}x\mathrm{d}t \\ &+ \mathbb{E}\int_Q \lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|q|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_{3T/4}^T\int_G \lambda^2\mu^2\tilde{\xi}\tilde{\theta}^2|\tilde{\varphi}||\tilde{\gamma}_t||q|^2\mathrm{d}x\mathrm{d}t \\ \le& C_1\left(\mathbb{E}\int_0^T\int_{G_0}\lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|q|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q\tilde{\theta}^2|\varpi|^2\mathrm{d}x\mathrm{d}t\right), \end{aligned} \tag{3.4}$$

for all $\lambda \ge \lambda_4$ and $\mu \ge \mu_4$, where C_1 is defined in (2.10).

3.2. The proof of Theorem 1.3

In this section, we will give the proof of Theorem 1.3. Inspired by the duality technique in [30], by applying Theorem 3.2, we first establish a controllability result for the following linear backward stochastic complex Ginzburg–Landau equation.

$$\begin{cases} \mathrm{d}y + (a-\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}y_j)_k\mathrm{d}t = (F+\chi_{G_0}h)\mathrm{d}t + Y\mathrm{d}B(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(T) = y_T & \text{in } G, \end{cases} \tag{3.5}$$

where $y_T \in L^2_{\mathcal{F}_T}(\Omega;L^2(G;\mathbb{C}))$ and $F \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ are given and $h \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C})$ is a control. Observing that given the aforementioned regularity on the controls and source term, one can easily show that system (3.5) admits a unique solution $(y,Y) \in \mathcal{H}_T \times L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))$ (see *e.g.*, [9], Thm. 4.11).

We define the space

$$\mathscr{Q}_{\lambda,\mu} = \left\{F \in L^2_{\mathbb{F}}(0,T;L^2(G;\mathbb{C}))\middle|\left(\mathbb{E}\int_Q\tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|F|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} < +\infty\right\},$$

which is a Banach space equipped with the canonical norm denoted by $\|\cdot\|_{\mathscr{Q}_{\lambda,\mu}}$.

Theorem 3.3. *Assume that $F \in \mathscr{Q}_{\lambda,\mu}$. For any $y_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(G;\mathbb{C}))$, there is a control $\hat{h}$ such that the associated solution $(\hat{y}, \hat{Y}) \in \mathcal{H}_T \times L^2_{\mathbb{F}}(0,T; L^2(G;\mathbb{C}))$ to the controlled system (3.5) satisfies $\hat{y}(0) = 0$ in G, $\mathbb{P}$-a.s. Moreover, one can find three positive constants $C = C(G, G_0, T, m, s_0, \alpha_0, a, b)$, $\lambda_5 = \lambda_5(a,b)$ and $\mu_5 = \mu_5(a,b)$, such that*

$$\begin{aligned}
&\mathbb{E}\int_Q \tilde{\theta}^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|\hat{Y}|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \tilde{\theta}^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|\nabla\hat{y}|^2 \mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_Q \tilde{\theta}^{-2}|\hat{y}|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|\hat{h}|^2 \mathrm{d}x\mathrm{d}t \\
&\leq C\bigg(\mathbb{E}\int_G \tilde{\theta}^{-2}(T)\lambda^{-2}\mu^{-2}|y_T|^2 \mathrm{d}x + \|F\|^2_{\mathscr{Q}_{\lambda,\mu}}\bigg),
\end{aligned} \tag{3.6}$$

for all $\lambda \geq \lambda_5$ and $\mu \geq \mu_5$.

Proof of Theorem 3.3. We divide the proof into three parts.
Step 1. For any $\varepsilon > 0$, consider the following weight function

$$\tilde{\gamma}_\varepsilon(t) = \begin{cases} \gamma(t+\varepsilon), & t \in [0, T/2 - \varepsilon), \\ 1, & t \in [T/2 - \varepsilon, 3T/4), \\ 1 + \left(1 - \dfrac{4(T-t)}{T}\right)^\sigma, & t \in [3T/4, T]. \end{cases} \tag{3.7}$$

where σ is the same as in (2.7). In this way, $\tilde{\gamma}_\varepsilon(t) \leq \gamma(t)$ for $t \in [0,T]$. Using the new weight function $\tilde{\gamma}_\varepsilon(t)$, we set

$$\tilde{\varphi}_\varepsilon(t,x) := \tilde{\gamma}_\varepsilon(t)\alpha(x), \quad \tilde{\theta}_\varepsilon := \mathrm{e}^{\lambda\tilde{\varphi}_\varepsilon}. \tag{3.8}$$

With this notation, we introduce the functional

$$J_\varepsilon(h) := \frac{1}{2}\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}|y|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|h|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2\varepsilon}\mathbb{E}\int_G |y(0)|^2 \mathrm{d}x$$

and consider the following optimal control problem:

$$\begin{cases} \min\limits_{h\in\mathscr{H}} J_\varepsilon(h) \\ \text{subject to equation (3.5)}, \end{cases} \tag{3.9}$$

where

$$\mathscr{H} = \bigg\{h \in L^2_{\mathbb{F}}(0,T; L^2(G_0;\mathbb{C})) \bigg| \mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|h|^2 \mathrm{d}x\mathrm{d}t < +\infty\bigg\}.$$

Similar to [40], it is easy to check that for any $\varepsilon > 0$, (3.9) admits a unique optimal solution that we denote by h_ε. Moreover, by the standard variational method (see [40, 41]), it follows that

$$h_\varepsilon = \chi_{G_0}\tilde{\theta}^2\lambda^3\mu^4\tilde{\xi}^3 p_\varepsilon \quad \text{in } Q,\ \mathbb{P}\text{-a.s.}, \tag{3.10}$$

where the p_ε verifies the forward random equation

$$\begin{cases} \mathrm{d}p_\varepsilon - (a+\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}p_{\varepsilon j})_k\mathrm{d}t = \tilde{\theta}_\varepsilon^{-2}y_\varepsilon\mathrm{d}t & \text{in } Q,\\ p_\varepsilon = 0 & \text{on } \Sigma,\\ p_\varepsilon(0) = \dfrac{1}{\varepsilon}y_\varepsilon(0) & \text{in } G, \end{cases} \tag{3.11}$$

where y_ε is the solution of (3.5) with the controls $h = h_\varepsilon$.

Step 2. We now establish a uniform estimate for the optimal solutions $\{(y_\varepsilon, Y_\varepsilon, h_\varepsilon)\}_{\varepsilon>0}$. By Itô's formula, (3.5) and (3.11), it follows that

$$\begin{aligned} &\mathbb{E}\int_G y_\varepsilon(T)\overline{p_\varepsilon}(T)\mathrm{d}x\\ =&\,\mathbb{E}\int_G y_\varepsilon(0)\overline{p_\varepsilon}(0)\mathrm{d}x + \mathbb{E}\int_Q\Big[-(a-\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}y_{\varepsilon j})_k + (F+\chi_{G_0}h_\varepsilon)\Big]\overline{p_\varepsilon}\mathrm{d}x\mathrm{d}t\\ &+\mathbb{E}\int_Q\Big[(a-\mathrm{i}b)\sum_{j,k=1}^{n}(a^{jk}\overline{p_\varepsilon}_j)_k + \tilde{\theta}_\varepsilon^{-2}\overline{y_\varepsilon}\Big]y_\varepsilon\mathrm{d}x\mathrm{d}t. \end{aligned}$$

This, together with (3.10) and the last equality of (3.11), imply that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_{G_0}\tilde{\theta}^2\lambda^3\mu^4\tilde{\xi}^3|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(0)|^2\mathrm{d}x\\ =&\,\mathbb{E}\int_G y_T\overline{p_\varepsilon}(T)\mathrm{d}x - \mathbb{E}\int_Q F\overline{p_\varepsilon}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.12}$$

Now, applying the Carleman estimate (3.4) in Theorem 3.2 to equation (3.11) with $\varpi = \tilde{\theta}_\varepsilon^{-2}y_\varepsilon$ and $q = p_\varepsilon$, we have

$$\begin{aligned} &\mathbb{E}\int_G\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}\tilde{\theta}^2(T)|p_\varepsilon(T)|^2\mathrm{d}x + \mathbb{E}\int_Q\lambda\mu^2\tilde{\xi}\tilde{\theta}^2|\nabla p_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\ &+\mathbb{E}\int_Q\lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\ \le&\,C_1\Big(\mathbb{E}\int_0^T\int_{G_0}\lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q\tilde{\theta}^2|\tilde{\theta}_\varepsilon^{-2}y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\Big). \end{aligned} \tag{3.13}$$

In view of (3.13), we use the Young inequality on the right-hand side of (3.12) to obtain

$$\begin{aligned} &\mathbb{E}\int_0^T\int_{G_0}\tilde{\theta}^2\lambda^3\mu^4\tilde{\xi}^3|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(0)|^2\mathrm{d}x\\ \le&\,\rho\Big[\mathbb{E}\int_G\lambda^2\mu^3\mathrm{e}^{2\mu(6m+1)}\tilde{\theta}^2(T)|p_\varepsilon(T)|^2\mathrm{d}x + \mathbb{E}\int_Q\lambda^3\mu^4\tilde{\xi}^3\tilde{\theta}^2|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t\Big]\\ &+C_\rho\Big[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\tilde{\theta}^{-2}(T)|y_T|^2\mathrm{d}x + \mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}\tilde{\theta}^{-2}|F|^2\mathrm{d}x\mathrm{d}t\Big], \end{aligned} \tag{3.14}$$

for any $\rho>0$. Noting that $\tilde{\theta}^2\tilde{\theta}_\varepsilon^{-2}\leq 1$ for any $(t,x)\in Q$ and using (3.13) and (3.14) with sufficiently small $\rho>0$, we obtain that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{G_0}\tilde{\theta}^2\lambda^3\mu^4\tilde{\xi}^3|p_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(0)|^2\mathrm{d}x\\ \leq& C\bigg[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\tilde{\theta}^{-2}(T)|y_T|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}\tilde{\theta}^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg].\end{aligned}$$

Noting that (3.10), we get

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{G_0}\tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|h_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\mathbb{E}\int_G|y_\varepsilon(0)|^2\mathrm{d}x\\ \leq& C\bigg[\mathbb{E}\int_G\lambda^{-2}\mu^{-3}\mathrm{e}^{-2\mu(6m+1)}\tilde{\theta}^{-2}(T)|y_T|^2\mathrm{d}x+\mathbb{E}\int_Q\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}\tilde{\theta}^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg].\end{aligned}\tag{3.15}$$

Now, we will add a weighted integral of the process Y on the left-hand side of (3.15). To do that, using Itô's formula and equation (3.5) with $h=h_\varepsilon$, we have

$$\begin{aligned}\mathrm{d}(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|y_\varepsilon|^2)=&(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_t|y_\varepsilon|^2\mathrm{d}t+\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|Y_\varepsilon|^2\mathrm{d}t\\&+\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}y_\varepsilon\mathrm{d}\overline{y_\varepsilon}+\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\overline{y_\varepsilon}\mathrm{d}y_\varepsilon.\end{aligned}$$

Integrating the above equality in Q, taking mathematical expectation on both sides, we conclude that

$$\begin{aligned}&\mathbb{E}\int_G\tilde{\theta}_\varepsilon^{-2}(T)\lambda^{-2}\tilde{\xi}^{-2}(T)|y_T|^2\mathrm{d}x-\mathbb{E}\int_G\tilde{\theta}_\varepsilon^{-2}(0)\lambda^{-2}\tilde{\xi}^{-2}(0)|y_\varepsilon(0)|^2\mathrm{d}x\\ =&\,\mathbb{E}\int_Q(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_t|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|Y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\&+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}y_\varepsilon[-(a+\mathrm{i}b)\sum_{j,k=1}^n(a^{jk}\overline{y_\varepsilon}_j)_k+(\overline{F}+\chi_{G_0}\overline{h_\varepsilon})]\mathrm{d}x\mathrm{d}t\\&+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\overline{y_\varepsilon}[-(a-\mathrm{i}b)\sum_{j,k=1}^n(a^{jk}y_{\varepsilon j})_k+(F+\chi_{G_0}h_\varepsilon)]\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.16}$$

Using $\tilde{\xi}^{-2}(0)=0$, the weight $\tilde{\theta}_\varepsilon^{-1}$ does not blow up at $t=0$, we obtain from (3.16) that

$$\begin{aligned}&\mathbb{E}\int_Q(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_t|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|Y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\&+2a\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\sum_{j,k=1}^n a^{jk}y_{\varepsilon k}\overline{y_\varepsilon}_j\mathrm{d}x\mathrm{d}t\\=&\,\mathbb{E}\int_G\tilde{\theta}_\varepsilon^{-2}(T)\lambda^{-2}\tilde{\xi}^{-2}(T)|y_T|^2\mathrm{d}x\\&-2\mathbb{E}\int_Q\mathrm{Re}\bigg[(a+\mathrm{i}b)y_\varepsilon\sum_{j,k=1}^n a^{jk}(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_k\overline{y_\varepsilon}_j\bigg]\mathrm{d}x\mathrm{d}t\\&-2\mathbb{E}\int_Q\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\mathrm{Re}(y_\varepsilon\overline{F})\mathrm{d}x\mathrm{d}t-2\mathbb{E}\int_0^T\int_{G_0}\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\mathrm{Re}(y_\varepsilon\overline{h_\varepsilon})\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.17}$$

Let us analyze some terms in (3.17). First, using (H_1), we have

$$2a\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\sum_{j,k=1}^{n} a^{jk}y_{\varepsilon k}\overline{y_\varepsilon}_j \mathrm{d}x\mathrm{d}t \geq 2as_0\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|\nabla y_\varepsilon|^2\mathrm{d}x\mathrm{d}t. \tag{3.18}$$

Then, by the definition of $\tilde{\gamma}_\varepsilon$, $\tilde{\theta}_\varepsilon$ in (3.7) and (3.8), one can easily obtain that

$$\mathbb{E}\int_Q (\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_t|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t \geq -C\mathbb{E}\int_0^{3T/4}\int_G \tilde{\theta}_\varepsilon^{-2}\lambda^{-1}\mu|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t. \tag{3.19}$$

Further, using Young inequality and Cauchy inequality, we estimate the last three terms of (3.17). For the first one, we have

$$\begin{aligned}&\left|2\mathbb{E}\int_Q \mathrm{Re}\left[(a+\mathrm{i}b)y_\varepsilon\sum_{j,k=1}^{n} a^{jk}(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_k\overline{y_\varepsilon}_j\right]\mathrm{d}x\mathrm{d}t\right|\\ \leq&\rho\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}|\nabla y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + C_\rho\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\mu^2|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\end{aligned} \tag{3.20}$$

for any $\rho>0$, where we have used $|(\tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2})_k| \leq C\tilde{\theta}_\varepsilon^{-2}\lambda^{-1}\mu\tilde{\xi}^{-1}$. For the second one, we obtain

$$\begin{aligned}&\left|2\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\mathrm{Re}(y_\varepsilon\overline{F})\mathrm{d}x\mathrm{d}t\right|\\ \leq&\ \mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-1}\mu^2\tilde{\xi}^{-1}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \tilde{\theta}^{-2}\lambda^{-3}\mu^{-2}\tilde{\xi}^{-3}|F|^2\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.21}$$

where we have used $\tilde{\theta}_\varepsilon^{-2} \leq \tilde{\theta}^{-2}$. For the third one, we get

$$\begin{aligned}&\left|2\mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\tilde{\xi}^{-2}\mathrm{Re}(y_\varepsilon\overline{h_\varepsilon})\mathrm{d}x\mathrm{d}t\right|\\ \leq&\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\mu^2|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}_\varepsilon^{-2}\lambda^{-4}\mu^{-2}\tilde{\xi}^{-4}|h_\varepsilon|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.22}$$

Combining (3.18)–(3.22) with (3.17) and taking $\rho>0$ small enough, we conclude that

$$\begin{aligned}&\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|Y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|\nabla y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\ \leq& C\mathbb{E}\int_G \tilde{\theta}^{-2}(T)\lambda^{-2}\mu^{-2}|y_T|^2\mathrm{d}x + C\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\ &+ C\mathbb{E}\int_Q \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|F|^2\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|h_\varepsilon|^2\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.23}$$

where we have used $\tilde{\theta}_\varepsilon^{-2} \leq \tilde{\theta}^{-2}$, $\tilde{\xi}^{-2}(T) < 1$ and $\lambda^{-1}\tilde{\xi}^{-1} < 1$. Using (3.15) and (3.23), we have

$$\begin{aligned}
&\mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|Y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}\lambda^{-2}\mu^{-2}\tilde{\xi}^{-2}|\nabla y_\varepsilon|^2\mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}\int_Q \tilde{\theta}_\varepsilon^{-2}|y_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{G_0} \tilde{\theta}^{-2}\lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}|h_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_G |y_\varepsilon(0)|^2\mathrm{d}x \qquad (3.24)\\
&\leq C\bigg[\mathbb{E}\int_G \tilde{\theta}^{-2}(T)\lambda^{-2}\mu^{-2}|y_T|^2\mathrm{d}x + \mathbb{E}\int_Q \lambda^{-3}\mu^{-4}\tilde{\xi}^{-3}\tilde{\theta}^{-2}|F|^2\mathrm{d}x\mathrm{d}t\bigg].
\end{aligned}$$

Step 3. By (3.24), it is easy to check that there exists $(\hat{h}, \hat{y}, \hat{Y})$ such that

$$\begin{cases} h_\varepsilon \rightharpoonup \hat{h} & \text{weakly in } L^2(\Omega\times(0,T); L^2(G_0;\mathbb{C})),\\ y_\varepsilon \rightharpoonup \hat{y} & \text{weakly in } L^2(\Omega\times(0,T); H_0^1(G;\mathbb{C})),\\ Y_\varepsilon \rightharpoonup \hat{Y} & \text{weakly in } L^2(\Omega\times(0,T); L^2(G;\mathbb{C})). \end{cases} \qquad (3.25)$$

We can easily check that $(\hat{y}, \hat{Y})$ is the solution to (3.5) associated to $\hat{h}$. Moreover, by (3.24), we get that $\hat{y}(0) = 0$ in G, $\mathbb{P}$-*a.s.*. Also, from the weak convergence (3.25), Fatou's lemma and the uniform estimate (3.24), we obtain (3.6). Thus, the proof of Theorem 3.3 is complete. □

Proof of Theorem 1.3: The proof of Theorem 1.3 is similar to the proof of Theorem 1.2. The main difference is that we should apply the Theorem 3.3 instead of the Theorem 2.6. So we omit the details of the proof of Theorem 1.3. □

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Data availability statement

The research data associated with this article are included in the article.

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