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PRESCRIBED STABILIZATION OF CONSERVATIVE MECHANICAL SYSTEMS *VIA* DELAYED STATE FEEDBACK: SYSTEM STRUCTURE, LAGUERRE POLYNOMIALS AND MID PROPERTY

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Abstract. In recent years, a novel pole placement paradigm has emerged for linear time-invariant systems of functional differential equations, addressing the complexities posed by their infinite spectrum and intricate dynamics. This paradigm is built upon the multiplicity-induced-dominancy (MID) property, which asserts that a characteristic root with sufficiently high multiplicity can dominate the spectral behavior of the system, thereby influencing its dynamic response. While the MID property has proven to be a powerful tool in control design, its applicability often depends on specific system configurations and parameter constraints. In this study, we propose new sufficient conditions ensuring the validity of the MID property in the context of the lowest intermediate over-order multiplicity. Our approach establishes a connection between the MID property and Laguerre polynomials by exploiting their inherent properties. Thanks to their structural properties, these conditions are tailored to the prescribed stabilization of conservative mechanical systems, which are characterized in the Laplace domain by even polynomials. By leveraging these conditions, we present a systematic framework for the precise assignment of dominant roots in the spectrum of conservative mechanical systems stabilized *via* delayed state feedback, enabling accurate control over both the system's solution's long-time behavior as well as its exact exponential decay.

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1. Introduction

Mechanical systems are distinguished by their inherent structural characteristics, which arise from the fundamental physical laws governing their dynamic behavior, such as Newton's laws of motion, principles of energy conservation, and material properties; see, for instance, [1]. These structural attributes encompass key aspects such as mass distribution, stiffness, damping properties, and connectivity between components, which collectively determine the system's response to external forces and internal interactions. The intrinsic structure of

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mechanical systems not only defines their operational behavior but also serves as a foundation for advanced analysis and mathematical modeling. Understanding these characteristics enables developing accurate predictive models that capture the system's dynamics under various operating conditions, including linear and nonlinear regimes [2, 3]. In particular, the study of conservative mechanical systems offers several advantages since they provide simplified mathematical models defining standard benchmarks for advanced control purposes and provide a stepping stone for exploring new concepts; see [4] and the references therein.

This work presents a state feedback control design methodology that addresses the presence of input and/or output delays in the control loop: a common challenge encountered in long transmission lines, communication networks, or due to inherent technological limitations. By leveraging the intrinsic structure of conservative mechanical systems, the proposed approach enables the precise prescription of exponential stabilization for the closed-loop system, ensuring improved performance and stability despite the presence of delays.

Actually, over the past decades there has been a growing focus on advancing the classical pole placement technique, extending and adapting it to infinite-dimensional systems; see, for instance, [3, 5–10]. One of these advanced approaches, known as the *partial pole placement* (PPP) method, has gained significant attention for its effectiveness in addressing complex control challenges such as stabilizing infinite-dimensional systems and optimizing their performance [11], offering new possibilities in applications such as the control of transonic flow in a wind tunnel [12] and modeling the central nervous system action on the human balance with biomechanical and biomedical scope [13, 14]. Unlike its classical counterpart, which deals with finite-dimensional systems, the PPP has been first introduced in the control of delay-differential equations, then it was extended to some reduced-order hyperbolic partial differential equations as in [15, 16] where the boundary control of the transport equation is considered. The PPP addresses one main challenge posed by infinite-dimensional systems, where the spectral properties are far more intricate; see, for instance, [3, 8, 17].

The PPP method is based on two key spectral properties called respectively *multiplicity-induced dominancy* (MID) [12, 16, 18, 19] and *coexisting-real-root-induced dominancy* (CRRID); see, for instance, [20]. While the MID states that under specific conditions, a characteristic root (or eigenvalue) with sufficiently high multiplicity becomes the dominant root in the system, the CRRID, on the other hand, extends this concept by considering a platoon of real characteristic roots and their influence on the system's dynamic response. These properties significantly simplify the control design process by focusing on the rightmost spectral values rather than the entire infinite set of eigenvalues. Together, these properties enable PPP to design control strategies that prescribe a dominant eigenvalue or, equivalently, the closed-loop system's solution's exponential decay.

While the MID property for generic quasipolynomials—referred to as GMID—has been fully characterized in [12, 16], meaning that a spectral value with maximal multiplicity is necessarily dominant due to its established connection with confluent hypergeometric functions and their zero distributions, only sufficient conditions are available to determine when the MID holds for spectral values of *intermediate multiplicities*, known as the IMID property; see, for instance, [18, 19, 21, 22]. Despite the analytical significance of the GMID property, implementing control strategies based on it lacks robustness, as highlighted in [23]. In fact, to enhance robustness against parametric uncertainties in the model, it is more appropriate to relax constraints on the selection of the closed-loop spectral abscissa. This can be achieved by leveraging the IMID property, which allows for the assignment of a root with an over-order intermediate multiplicity, providing greater flexibility in control design.

In this work, we further contribute to the study of the over-order IMID property by explicitly determining novel conditions under which a given multiple root is dominant, which defines the closed-loop solution's exponential decay rate.

The remaining of the paper is organized as follows. Section 2 provides the problem statement, and the motivation of this contribution is presented in Section 3. Since our approach exploits algebraic zeros of quasipolynomials, for the sake of completeness, some preliminary results on polynomials and quasipolynomials are presented in Section 4. The main result of the paper is presented in Section 5. As an illustrative application of the obtained result, the prescribed stabilization of the double inverted pendulum is considered in Section 6. Some open questions limiting the proposed approach are also discussed.

Notation. Throughout the paper, $\mathbb{Z}$, $\mathbb{Z}_+ = \mathbb{N}$, and $\mathbb{Z}_+^* = \mathbb{N}^*$ denote the sets of integers, nonnegative integers, and positive integers, respectively, while $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_+^*$ denote the sets of real numbers, nonnegative real numbers, and positive real numbers, and $\mathbb{C}$ denotes the set of complex numbers. For a complex number λ, $\Re(\lambda)$ and $\Im(\lambda)$ denote its real and imaginary parts, respectively. Given $k, n \in \mathbb{N}$ with $k \leq n$, the binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For a function f of two variables, $f(\cdot, c)$ and $f(c, \cdot)$ denote the single-variable functions that we obtain by setting, respectively, the second and first variables of f to the constant value c. Moreover, for a function f of n variables and $i, k, n \in \mathbb{N}^*$, $\partial_i^k f$ denotes the kth partial derivative of f in its ith variable.

2. Problem statement

Consider the generic delay-differential equation (DDE)

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + ... + a_0 x(t) + b_{n-1} x^{(n-1)}(t-\tau) + ... + b_0 x(t-\tau) = 0 \,, \tag{2.1}$$

where the unknown function x is real-valued, n is a positive integer, $a_n > 0$, $a_i, b_i \in \mathbb{R}$ for $i = 0, 1, ..., n-1$, and $\tau > 0$ is a delay. One approach to analyzing the asymptotic behavior of the solutions of (2.1) is to examine the roots of the associated characteristic function; see, for instance, [3, 8, 24, 25]. For equation (2.1), the characteristic function $D\colon \mathbb{C} \to \mathbb{C}$ is given by

$$D(s) = P(s) + Q(s)\, \mathrm{e}^{-s\tau} \,, \tag{2.2}$$

where

$$\begin{aligned} P(s) &= a_n s^n + a_{n-1} s^{n-1} + ... + a_0 \,, \\ Q(s) &= b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + ... + b_0 \,. \end{aligned} \tag{2.3}$$

More precisely, the exponential behavior of the solutions of (2.1) is given by the real number $\gamma_0 = \sup\{\Re(s) \mid s \in \mathbb{C},\ D(s) = 0\}$, called the *spectral abscissa* of D, in the sense that, for every $\varepsilon > 0$, there exists $C > 0$ such that, for every solution y of (2.1), one has $|y(t)| \leq C \mathrm{e}^{(\gamma_0 + \varepsilon)t} \max_{\theta \in [-\tau, 0]} |y(\theta)|$ [25]. Moreover, all solutions of (2.1) converge exponentially to 0 if, and only if, $\gamma_0 < 0$. The main challenge in the analysis of the asymptotic behavior of (2.1) is that, contrarily to the delay-free case, the corresponding characteristic function D has infinitely many roots.

A single-variable polynomial or quasipolynomial is called *γ-stable* if the real parts of all the roots of that polynomial or quasipolynomial are less than $\gamma \in \mathbb{R}$. Assume that the coefficients a_i in P are known and fixed. Furthermore, assume that the coefficients b_i in Q are independently adjustable control parameters. Then, the quasipolynomial D is said to be *γ-stabilizable* for a given τ if there exists a set of control parameters $b_i(a_0, a_1, ..., a_n, \tau)$, $i = 0, 1, ..., n-1$ for which D is γ-stable.

A root s_0 of a single-variable polynomial or quasipolynomial is called *dominant* if any other root $s_1 \neq s_0$ of that polynomial or quasipolynomial satisfies $\Re(s_1) \leq \Re(s_0)$. Moreover, if the strict inequality $\Re(s_1) < \Re(s_0)$ holds, then s_0 is called *strictly dominant*. In order to determine the admissible values of the multiple real roots of the quasipolynomial D and to prove the dominance of these multiple roots with multiplicity greater than n (over-order), we will investigate the location of the roots of the family of two-variable polynomials $R_k\colon \mathbb{C}^2 \to \mathbb{C}$ given by

$$R_k(s, \tau) = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(s) \tau^{k-i}, \quad k \in \mathbb{N} \,. \tag{2.4}$$

In [18], we gave a sufficient condition for the dominance of a multiple real root of D and a necessary condition for the γ-stabilizability of D in terms of the polynomials R_k. These conditions will be recalled at the beginning of Section 4. If the coefficients a_i are interpreted as model parameters and b_i as control parameters, it is important to note that the polynomials R_k depend solely on the open-loop characteristic polynomial P. Furthermore, in [21, 22] the specific polynomial R_n was referred to as the *elimination-produced polynomial*.

The goal of this paper is to assess the γ-stabilizable intervals of τ by investigating the location of the roots of the polynomials R_k. In particular, we explore the case where the polynomial P is of the form $P(s) = \widehat{P}(s^2)$ such that $\widehat{P}$ is a real-rooted polynomial. Such a structure of P occurs naturally in the case of conservative mechanical systems. This will be illustrated in the next section, emphasizing the added value of the present study.

3. Motivating example

Let P be of the form $P(s) = \widehat{P}(s^2)$ such that $\widehat{P}$ is a real-rooted polynomial, $n = 4$, and $a_4 = 1$. Then, the quasipolynomial D in (2.2) takes the form

$$D(s) = s^4 + a_2 s^2 + a_0 + \mathrm{e}^{-s\tau}(b_3 s^3 + b_2 s^2 + b_1 s + b_0)\,, \tag{3.1}$$

where $\tau > 0$ and $a_2^2 - 4a_0 \geq 0$. Such a quasipolynomial may occur as the characteristic function of controlled mechanical systems, such as a double inverted pendulum with delayed proportional-derivative feedback [26] or a model of human stance on a rolling balance board [14]. Note that the structure of the specific mechanical system may impose further restrictions on the parameters a_2 and a_0. In this example, we assume that the only restriction on a_2 and a_0 is the inequality $a_2^2 - 4a_0 \geq 0$.

Let us define a partition of the domain

$$\mathcal{D} = \{(a_2, a_0) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0\} \tag{3.2}$$

of parameters a_2 and a_0 as

$$\begin{aligned}
\mathcal{D}_1 &= \{(a_2, a_0) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0, a_0 \geq 0, a_2 \leq 0\}\,,\\
\mathcal{D}_2 &= \{(a_2, a_0) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0, a_0 < 0, a_2 \leq 0\}\,,\\
\mathcal{D}_3 &= \{(a_2, a_0) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0, a_0 \leq 0, a_2 > 0\}\,,\\
\mathcal{D}_4 &= \{(a_2, a_0) \in \mathbb{R}^2 \mid a_2^2 - 4a_0 \geq 0, a_0 > 0, a_2 > 0\}\,.
\end{aligned} \tag{3.3}$$

For a visualization of this partition, see Figure 1a. The polynomial $\widehat{P}$, which is given by $\widehat{P}(x) = x^2 + a_2 x + a_0$, has two nonnegative roots if $(a_2, a_0) \in \mathcal{D}_1$, a nonnegative root and a negative root if $(a_2, a_0) \in \mathcal{D}_2 \cup \mathcal{D}_3$, and two negative roots if $(a_2, a_0) \in \mathcal{D}_4$.

The polynomial P has only real roots if, and only if, $(a_2, a_0) \in \mathcal{D}_1$. In the subdomain $\mathcal{D}_1$, the results of [18] apply: if $\gamma \leq 0$, then the quasipolynomial D in (3.1) is γ-stabilizable if, and only if, $0 < \tau < \tau_\gamma$, where τ_γ is the smallest positive real root of $R_4(\gamma, \cdot)$, and the polynomial $R_4(\gamma, \cdot)$ is given by

$$R_4(\gamma, \theta) = (\gamma^4 + a_2\gamma^2 + a_0)\theta^4 + 8\gamma(2\gamma^2 + a_2)\theta^3 + 12(6\gamma^2 + a_2)\theta^2 + 96\gamma\theta + 24\,. \tag{3.4}$$

In this paper, we will show that the same conclusion holds if $(a_2, a_0) \in \mathcal{D}_2$, supplementing the IMID validity domain established in [18]. Figure 1b shows the branches of the real algebraic curves $R_4(s, \tau) = 0$ and $R_3(s, \tau) = 0$ for different values of a_2 and a_0. Figure 1b suggests that the polynomials $R_4(\cdot, \tau)$ and $R_3(\cdot, \tau)$ have strictly dominant real roots for every $\tau > 0$ in the subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$. This property proves to be important in the proofs of Section 5.

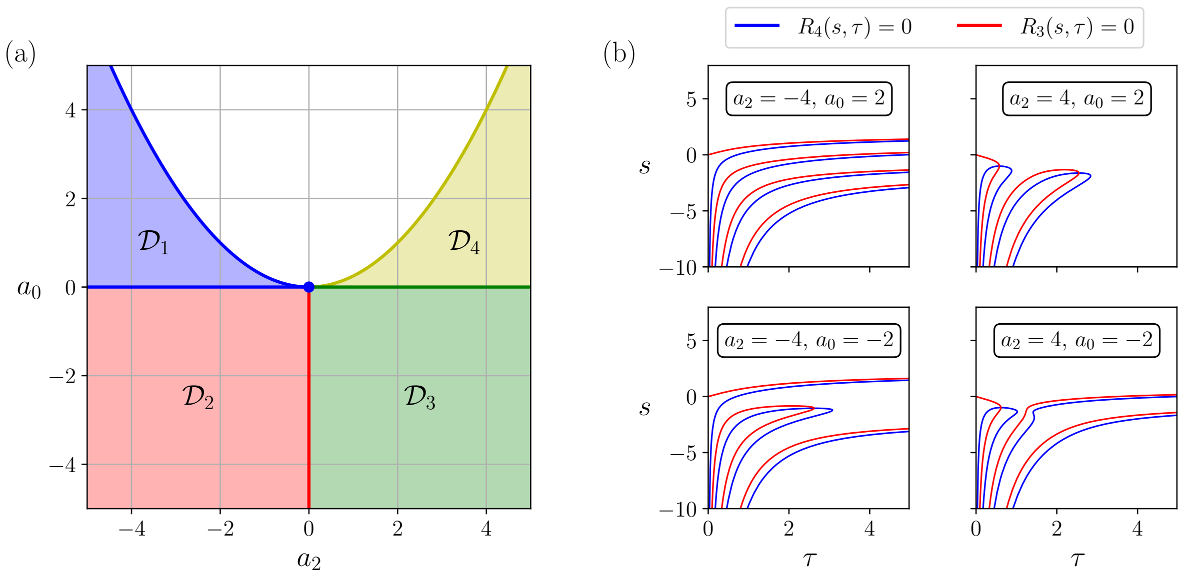

FIGURE 1. Partition of the domain $\mathcal{D}$ of parameters a_2 and a_0 (a). Branches of the real algebraic curves $R_4(s,\tau)=0$ and $R_3(s,\tau)=0$ for different values of a_2 and a_0 (b).

4. Prerequisites on the roots of polynomials and quasipolynomials

The polynomials R_k, $k \in \mathbb{N}^*$, satisfy the properties

$$R_k(s,\tau) = \tau R_{k-1}(s,\tau) + \partial_1 R_{k-1}(s,\tau)\,, \tag{4.1}$$

$$\partial_2 R_k(s,\tau) = k R_{k-1}(s,\tau)\,. \tag{4.2}$$

The following three propositions describe the relationship between the polynomials R_k in (2.4) and the quasipolynomial D in (2.2) based on [18], Proposition 1 and its proof; Remark 1; Proposition 2; Proposition 4.

Proposition 4.1. *The real number s_0 is a root of the quasipolynomial D of multiplicity at least $n+1$ if, and only if, $R_n(s_0,\tau)=0$ and*

$$b_i = -\frac{\mathrm{e}^{s_0\tau}}{i!}\sum_{j=i}^{n-1}\frac{(-s_0)^{j-i}}{(j-i)!}R_j(s_0,\tau)\,,\quad i=0,1,...,n-1. \tag{4.3}$$

Proposition 4.2. *Let s_0 be a real root of the quasipolynomial D of multiplicity at least $n+1$. If $R_{n-1}(s_0,\vartheta)\le 0$ for any $0<\vartheta\le\tau$, then s_0 is a dominant root of D.*

Proposition 4.3. *If the quasipolynomial D is γ-stable, then the polynomial $R_n(\cdot,\tau)$ is γ-stable.*

In view of the preceding propositions, we need to characterize the location of the roots of R_n and R_{n-1}. Since R_k is a linear combination of successive derivatives of P, we can rewrite it as a linear combination of the coefficients of P. For more details on this approach, see, for example, [27], Section 5.3.

Proposition 4.4. *The polynomial R_k can be rewritten as*

$$R_k(s,\tau)=\sum_{i=0}^{k}\binom{k}{i}P^{(i)}(s)\tau^{k-i}=\tau^{k-n}\sum_{j=0}^{n}a_{n-j}(n-j)!\tau^{j}F_k^{(j)}(s\tau)\,, \tag{4.4}$$

where

$$F_k(x)=\begin{cases}\displaystyle\sum_{i=0}^{k}\binom{k}{i}\frac{x^{i+n-k}}{(i+n-k)!} & \text{if } 0\le k<n\,,\\ \displaystyle\sum_{i=0}^{n}\binom{k}{i+k-n}\frac{x^{i}}{i!} & \text{if } k\ge n\,.\end{cases} \tag{4.5}$$

Proof. It is sufficient to show that

$$R_k(s,\tau)=\sum_{i=0}^{k}\binom{k}{i}P^{(i)}(s)\tau^{k-i}=\sum_{l=0}^{n}a_l l!G_k^{(n-l)}(s)=\sum_{j=0}^{n}a_{n-j}(n-j)!G_k^{(j)}(s)\,, \tag{4.6}$$

where

$$G_k(s)=\begin{cases}\displaystyle\sum_{i=0}^{k}\binom{k}{i}\frac{\tau^{i}s^{i+n-k}}{(i+n-k)!} & \text{if } 0\le k<n\,,\\ \displaystyle\sum_{i=0}^{n}\binom{k}{i+k-n}\frac{\tau^{i+k-n}s^{i}}{i!} & \text{if } k\ge n\,.\end{cases} \tag{4.7}$$

After (4.6)–(4.7) are verified, we can complete the proof by noticing that $G_k(s)=\tau^{k-n}F_k(s\tau)$.

If $0\le k<n$, then we have

$$\begin{aligned}R_k(s,\tau)&=\sum_{i=0}^{k}\binom{k}{i}P^{(i)}(s)\tau^{k-i}=\sum_{i=0}^{k}\sum_{l=i}^{n}\binom{k}{i}\tau^{k-i}a_l\frac{l!}{(l-i)!}s^{l-i}\\&=\sum_{l=0}^{k}a_l l!\sum_{i=0}^{l}\binom{k}{i}\frac{\tau^{k-i}s^{l-i}}{(l-i)!}+\sum_{l=k+1}^{n}a_l l!\sum_{i=0}^{k}\binom{k}{i}\frac{\tau^{k-i}s^{l-i}}{(l-i)!}\end{aligned} \tag{4.8}$$

and, for $l\in\mathbb{N}$,

$$G_k^{(n-l)}(s)=\begin{cases}\displaystyle\sum_{i=k-l}^{k}\binom{k}{i}\frac{\tau^{i}s^{i-k+l}}{(i-k+l)!}=\sum_{j=0}^{l}\binom{k}{j}\frac{\tau^{k-j}s^{l-j}}{(l-j)!} & \text{if } 0\le l\le k\,,\\ \displaystyle\sum_{i=0}^{k}\binom{k}{i}\frac{\tau^{i}s^{i-k+l}}{(i-k+l)!}=\sum_{j=0}^{k}\binom{k}{j}\frac{\tau^{k-j}s^{l-j}}{(l-j)!} & \text{if } k<l\le n\,.\end{cases} \tag{4.9}$$

Moreover, if $k\ge n$, then we have

$$R_k(s,\tau)=\sum_{i=0}^{k}\binom{k}{i}P^{(i)}(s)\tau^{k-i}=\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{k}{i}\tau^{k-i}a_l\frac{l!}{(l-i)!}s^{l-i}=\sum_{l=0}^{n}a_l l!\sum_{i=0}^{l}\binom{k}{i}\frac{\tau^{k-i}s^{l-i}}{(l-i)!} \tag{4.10}$$

and, for $l \in \mathbb{N}$,

$$G_k^{(n-l)}(s) = \sum_{i=n-l}^{n} \binom{k}{i+k-n} \frac{\tau^{i+k-n} s^{i-n+l}}{(i-n+l)!} = \sum_{j=0}^{l} \binom{k}{j} \frac{\tau^{k-j} s^{l-j}}{(l-j)!} \quad \text{if } 0 \le l \le n\,. \tag{4.11}$$

Therefore, for both $0 \le k < n$ and $k \ge n$, (4.6)–(4.7) hold. □

Proposition 4.5. *The polynomial F_k has k simple negative real roots and the zero root of multiplicity $n-k$ if $0 \le k < n$, and it has exactly n simple negative real roots if $k \ge n$.*

To prove the above proposition, one needs first to recall the following family of recurrent polynomials called the generalized *Laguerre polynomials* of degree $\nu \in \mathbb{N}$ and parameter $\lambda > -1$, which are defined by

$$L_{\nu,\lambda}(x) = \frac{1}{\nu!} \mathrm{e}^{x} x^{-\lambda} \frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}} (\mathrm{e}^{-x} x^{\nu+\lambda})\,. \tag{4.12}$$

They can also be written in the explicit form

$$L_{\nu,\lambda}(x) = \sum_{i=0}^{\nu} (-1)^i \binom{\nu+\lambda}{\nu-i} \frac{x^i}{i!}\,. \tag{4.13}$$

For a given λ, the polynomials $L_{\nu,\lambda}$ are orthogonal on the interval $(0,\infty)$ with respect to the weight function w given by $w(x) = x^{\lambda}\mathrm{e}^{-x}$; therefore, they have only simple positive real roots if $\nu \in \mathbb{N}^*$ [27].

The derivatives of the generalized Laguerre polynomials satisfy the relation

$$L_{\nu,\lambda}^{(\kappa)}(x) = \sum_{i=\kappa}^{\nu} (-1)^i \binom{\nu+\lambda}{\nu-i} \frac{x^{i-\kappa}}{(i-\kappa)!} = (-1)^{\kappa} L_{\nu-\kappa,\lambda+\kappa}(x) \tag{4.14}$$

for $0 \le \kappa \le \nu$, $\kappa \in \mathbb{N}$.

Proof of Proposition 4.5. Note that $F_k(-x) = L_{n,k-n}(x)$ if $k \ge n$, and $n!/k! F_k(-x)/(-x)^{n-k} = L_{k,n-k}(x)$ if $0 \le k < n$. □

As we will detail later in Section 5, the representation in (4.4) is closely related to the Schur-Szegő composition of polynomials.

Definition 4.6. Let f and g be (complex) polynomials of degree $l \ge 1$ given by

$$f(x) = \sum_{i=0}^{l} \binom{l}{i} c_i x^i\,, \qquad g(x) = \sum_{i=0}^{l} \binom{l}{i} d_i x^i\,. \tag{4.15}$$

Then, the Schur-Szegő composition of f and g is given by

$$(f * g)(x) = \sum_{i=0}^{l} \binom{l}{i} c_i d_i x^i\,. \tag{4.16}$$

In general, the location of the roots of $f * g$ cannot be completely characterized parametrically. However, we have the following result from [28]; see also [29, 30].

Proposition 4.7. *Let f and g be (complex) polynomials of degree l such that x_f and x_g are nonzero roots, respectively, of f and g of respective multiplicities m_f and m_g. Furthermore, define $\mu := m_f + m_g - l$. If $\mu > 0$, then $-x_f x_g$ is a root of $f * g$ of multiplicity μ. Moreover, if $\mu = 0$, then $-x_f x_g$ is not a root of $f * g$.*

If the polynomials f and g are real-rooted, and g has only negative real roots, then more can be said about the location of the roots. Let us call a root of $f * g$ an A-root if it is either zero or of the form $-x_f x_g$, where x_f is a nonzero root of f and x_g is a nonzero root of g such that the sum of their multiplicities is greater than $l = \deg f = \deg g$. All the remaining roots are called B-roots. With these notations, the following proposition holds [28].

Proposition 4.8. *Let f have only real roots, and let g have only negative real roots. Then, $f * g$ is real-rooted, and it has the same number of negative, positive and zero roots as f. Furthermore, every B-root of $f * g$ is simple.*

As an immediate consequence of Proposition 4.7 and Proposition 4.8, we can conclude the following.

Proposition 4.9. *Let f and g be polynomials of degree l. If f has only real roots of multiplicity at most M_f, and g has only negative real roots of multiplicity at most M_g such that $M_f + M_g \leq l + 1$, then the nonzero roots of $f * g$ are simple.*

The fact that the roots of $f * g$ in Proposition 4.8 are real can be deduced from the following proposition [27, 31], which applies to a more general location of the roots of g, but it does not provide information about the multiplicities of the roots.

Proposition 4.10. *If the roots of f are real, and the roots of g lie in a sector $\mathcal{S}(\alpha, \beta) := \{r\mathrm{e}^{\mathrm{i}\varphi} \mid r \geq 0, \alpha \leq \varphi \leq \beta\}$, where $(\alpha, \beta) \in \mathbb{R}^2$ and $0 \leq \beta - \alpha < \pi$, then the roots of $f * g$ lie in the double sector $\mathcal{S}(\alpha, \beta) \cup \mathcal{S}(\alpha + \pi, \beta + \pi)$.*

Remark 4.11. The double sector in Proposition 4.10 contains the complex numbers in $\mathcal{S}(\alpha, \beta)$ and their opposites.

5. Main results

In this section, we assume that the polynomial P is of the form $P(s) = \widehat{P}(s^2)$, where $\widehat{P}$ is a real-rooted polynomial given by

$$\widehat{P}(x) = \widehat{a}_m x^m + \widehat{a}_{m-1} x^{m-1} + \ldots + \widehat{a}_0 \tag{5.1}$$

with $\widehat{a}_m > 0$ and $m \in \mathbb{N}^*$. Hence, the degree of P is $n = 2m$, $a_{2i} = \widehat{a}_i$, $i = 0, 1, \ldots, m$, and $a_{2i-1} = 0$, $i = 1, 2, \ldots, m$. In this case, equation (4.4) gives us the representation

$$R_k(s, \tau) = \tau^{k-2m} \sum_{i=0}^{m} \widehat{a}_{m-i} (2(m-i))! \tau^{2i} F_k^{(2i)}(s\tau)\,, \tag{5.2}$$

where F_k is given by (4.5) with $n = 2m$. Note that, in this section, F_k is a polynomial of degree $2m$ according to (4.5). Furthermore, F_k has k simple negative real roots and the zero root of multiplicity $2m - k$ if $0 \leq k < 2m$, and it has exactly $2m$ simple negative real roots if $k \geq 2m$ by Proposition 4.5. Throughout this section, we denote the roots of F_k by $\zeta_{i,k}$, $i = 1, 2, \ldots, 2m$ in non-increasing order.

Let us define the family of two-variable polynomials $S_k \colon \mathbb{C}^2 \to \mathbb{C}$ by

$$S_k(z, y) = \sum_{i=0}^{m} \widehat{a}_{m-i} (2(m-i))! y^i F_k^{(2i)}(z), \quad k \in \mathbb{N}\,. \tag{5.3}$$

Note that $R_k(s,\tau) = \tau^{k-2m} S_k(s\tau, \tau^2)$. Therefore, by equations (4.1)–(4.2), the polynomials S_k, $k \in \mathbb{N}^*$ satisfy the properties

$$S_k(z,y) = S_{k-1}(z,y) + \partial_1 S_{k-1}(z,y)\,, \tag{5.4}$$

$$(k-2m)S_k(z,y) + z\,\partial_1 S_k(z,y) + 2y\,\partial_2 S_k(z,y) = kS_{k-1}(z,y)\,. \tag{5.5}$$

5.1. Schur–Szegő composition and Laguerre polynomials

The following theorem is the key to describing the location of the roots of S_k.

Theorem 5.1. *Let $\widehat{P}$ be real-rooted, and assume that $\zeta \in \mathbb{R}$ and $\zeta > \zeta_{1,k}$, where $\zeta_{1,k}$ is the largest (real) root of F_k. Then, $S_k(\zeta,\cdot)$ is real-rooted, it has the same number of positive and negative roots as $\widehat{P}$, and it does not have a zero root. If, furthermore, $k \geq m-1$, then the roots of $S_k(\zeta,\cdot)$ are simple.*

Remark 5.2. If $\widehat{P}$ has a zero root of multiplicity m, and $\zeta > \zeta_{1,k}$, then $S_k(\zeta,\cdot)$ is a non-vanishing constant polynomial. In this case, $S_k(\zeta,\cdot)$ does not have any roots.

Proof of Theorem 5.1. In the proof, we will apply the results of Propositions 4.8 and 4.9 twice. First, we consider the polynomials f_1 and g_1 given by

$$f_1(x) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} x^{2i} = (x^2-1)^m\,, \qquad g_1(x) = F_k(x+\zeta)\,. \tag{5.6}$$

By noticing that

$$g_1(x) = F_k(x+\zeta) = \sum_{j=0}^{2m} \frac{F_k^{(j)}(\zeta)}{j!} x^j\,, \tag{5.7}$$

we can write the Schur-Szegő composition of f_1 and g_1 as

$$\begin{aligned} (f_1 * g_1)(x) &= \sum_{i=0}^{m} (-1)^{m-i} \frac{\binom{m}{i}}{\binom{2m}{2i}} \frac{F_k^{(2i)}(\zeta)}{(2i)!} x^{2i} \\ &= \frac{1}{(2m)!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (2(m-i))! F_k^{(2i)}(\zeta) x^{2i}\,. \end{aligned} \tag{5.8}$$

The roots of g_1 are real since F_k has only real roots, and they are negative since $\zeta_{1,k}$ is the largest root of F_k and $\zeta > \zeta_{1,k}$ by assumption. If $0 \leq k < 2m$, then $-\zeta$ is a root of g_1 of multiplicity $2m-k$, and the remaining k roots are simple. If $k \geq 2m$, then all the roots of g_1 are simple. The roots of f_1 are 1 of multiplicity m and -1 of multiplicity m. Hence, $f_1 * g_1$ has only real roots and does not have a zero root by Proposition 4.8. If $m + (2m-k) \leq 2m+1$, that is, if $m-1 \leq k$, then Proposition 4.9 applies. In this case, the roots of $f_1 * g_1$ are simple.

Next, we consider the polynomials f_2 and g_2 given by

$$f_2(y) = \widehat{P}(y)\,, \qquad g_2(y) = \sum_{i=0}^{m} \binom{m}{i} (2i)! F_k^{(2(m-i))}(\zeta) y^i\,. \tag{5.9}$$

The Schur-Szegő composition of f_2 and g_2 can be written as

$$(f_2 * g_2)(y) = \sum_{i=0}^{m} \widehat{a}_i (2i)! F_k^{(2(m-i))}(\zeta) y^i \,. \tag{5.10}$$

Note that g_2 and $f_1 * g_1$ are related by the transformations $h_1(y) = y^m g_2(1/y)$, $h_2(y) = (-1)^m h_1(-y)$, and $(f_1 * g_1)(x) = h_2(x^2)/(2m)!$. Therefore, h_2 has only positive real roots, while h_1 and g_2 have only negative real roots. If $k \geq m-1$, then all the roots of h_2, h_1 and g_2 are simple. By Proposition 4.8, $f_2 * g_2$ is real-rooted, and it has the same number of negative, positive and zero roots as f_2. If $k \geq m-1$, then Proposition 4.9 applies, and the nonzero roots of $f_2 * g_2$ are simple. Now, the statement follows by noticing that $S_k(\zeta, y) = y^m (f_2 * g_2)(1/y)$. □

Theorem 5.1 characterizes the roots of $S_k(\zeta, \cdot)$ for real values of ζ if $\zeta > \zeta_{1,k}$. The next theorem provides important information about the roots of $S_k(\zeta, \cdot)$ for non-real values of ζ if $\Re(\zeta) > \zeta_{1,k}$.

Theorem 5.3. *Let $\widehat{P}$ be real-rooted, and assume that $\eta \in \mathbb{R}$, $\zeta \in \mathbb{C}$, $\Im(\zeta) \neq 0$, and $\Re(\zeta) > \zeta_{1,k}$, where $\zeta_{1,k}$ is the largest (real) root of F_k. Then, $S_k(\zeta, \eta) \neq 0$.*

Proof. The proof follows the same lines as the proof of Theorem 5.1; however, now we rely on Proposition 4.10. We can consider only the case where $\Im(\zeta) < 0$, because $S_k(\cdot, \eta)$ is a polynomial with real coefficients if η is real and the coefficients of $\widehat{P}$ are real.

First, we take f_1 and g_1 as in (5.6). Since $\Re(\zeta) > \zeta_{1,k}$ and $\Im(\zeta) < 0$, the roots of g_1 have negative real parts and positive imaginary parts. For a given ζ, we can find some $\alpha > \pi/2$ and $\beta < \pi$ such that all the roots of g_1 lie in the sector $\mathcal{S}(\alpha, \beta)$, where $0 \leq \beta - \alpha < \pi/2$. Hence, Proposition 4.10 applies to f_1 and g_1, and all the roots of $f_1 * g_1$ lie in $\mathcal{S}(\alpha, \beta) \cup \mathcal{S}(\alpha + \pi, \beta + \pi)$.

Next, we take f_2 and g_2 as in (5.9). The polynomials g_2 and $f_1 * g_1$ are related by $h_1(y) = y^m g_2(1/y)$, $h_2(y) = (-1)^m h_1(-y)$, and $(f_1 * g_1)(x) = h_2(x^2)/(2m)!$. Therefore, all the roots of h_2 lie in $\mathcal{S}(2\alpha, 2\beta) \cup \mathcal{S}(2\alpha + 2\pi, 2\beta + 2\pi)$, which reduces to $\mathcal{S}(2\alpha, 2\beta)$. The roots of h_1 are in $\mathcal{S}(2\alpha + \pi, 2\beta + \pi)$; thus, the roots g_2 lie in $\mathcal{S}(-2\beta - \pi, -2\alpha - \pi)$. Hence, Proposition 4.10 applies to f_2 and g_2, and all the roots of $f_2 * g_2$ lie in $\mathcal{S}(-2\beta - \pi, -2\alpha - \pi) \cup \mathcal{S}(-2\beta, -2\alpha)$.

Since $S_k(\zeta, y) = y^m (f_2 * g_2)(1/y)$, the roots of $S_k(\zeta, \cdot)$ lie in the double sector $\mathcal{S}(2\alpha + \pi, 2\beta + \pi) \cup \mathcal{S}(2\alpha, 2\beta)$, and $S_k(\zeta, \cdot)$ does not have a zero root. Since $\pi/2 < \alpha$ and $\beta < \pi$, we have $2\pi < 2\alpha + \pi$ and $2\beta + \pi < 3\pi$. Therefore, the nonzero complex numbers in $\mathcal{S}(2\alpha + \pi, 2\beta + \pi)$ lie in the open upper half-plane. Similarly, $\pi < 2\alpha$ and $2\beta < 2\pi$, and the nonzero complex numbers in $\mathcal{S}(2\alpha, 2\beta)$ lie in the open lower half-plane. Consequently, $S_k(\zeta, \cdot)$ does not have a real root. □

Remark 5.4. The proof of Theorem 5.3 assumed a fixed complex ζ satisfying $\Im(\zeta) \neq 0$ and $\Re(\zeta) > \zeta_{1,k}$, and it was concluded that $S_k(\zeta, \cdot)$ does not have a real root. However, the statement of Theorem 5.3 can also be interpreted the other way around: for a fixed real η, $S_k(\cdot, \eta)$ does not have a root in the region $\{z \in \mathbb{C} \mid \Im(z) \neq 0, \Re(z) > \zeta_{1,k}\}$.

5.2. Roots as values of real analytic functions

In the following, let p denote the number of positive real roots of the real-rooted polynomial $\widehat{P}$, and let us assume that $p \geq 1$. Then, according to Theorem 5.1, we can define the functions $y_{i,k} \colon (\zeta_{1,k}, \infty) \to \mathbb{R}_+^*$, $i = 1, 2, ..., p$ such that $y_{i,k}(\zeta)$, $i = 1, 2, ..., p$ give the p positive roots of $S_k(\zeta, \cdot)$ in non-decreasing order. The next proposition describes some properties of the functions $y_{i,k}$ using the results of Theorem 5.1 and 5.3.

Proposition 5.5. *If $k \geq m-1$ and $p \geq 1$, then the functions $y_{i,k}$, $i = 1, 2, ..., p$ are real analytic functions, and they satisfy $y_{i,k}(\zeta_0) \neq y_{j,k}(\zeta_0)$ and $y'_{i,k}(\zeta_0) \neq 0$ for any $i \neq j$ and $\zeta_0 > \zeta_{1,k}$.*

Proof. For a given $k \geq m-1$ and $\zeta_0 > \zeta_{1,k}$, the polynomial $S_k(\zeta_0, \cdot)$ has simple roots by Theorem 5.1. This directly gives $y_{i,k}(\zeta_0) \neq y_{j,k}(\zeta_0)$ for $i \neq j$.

By the definition of $y_{i,k}$, the equation $S_k(\zeta, y_{i,k}(\zeta)) = 0$ holds for any $\zeta > \zeta_{1,k}$. Moreover, since $y_{i,k}(\zeta)$ is a simple root of $S_k(\zeta, \cdot)$, we have $\partial_2 S_k(\zeta, y_{i,k}(\zeta)) \neq 0$. Therefore, by the implicit function theorem, the functions $y_{i,k}$ are real analytic functions [32], Theorem 1.8.3.

By taking the derivative of the identity $S_k(\zeta, y_{i,k}(\zeta)) = 0$ with respect to ζ, one gets

$$\partial_1 S_k(\zeta, y_{i,k}(\zeta)) + y'_{i,k}(\zeta)\, \partial_2 S_k(\zeta, y_{i,k}(\zeta)) = 0\,. \tag{5.11}$$

Hence, $y'_{i,k}(\zeta) = 0$ holds if, and only if, $\partial_1 S_k(\zeta, y_{i,k}(\zeta)) = 0$. In other words, $y'_{i,k}(\zeta) \neq 0$ if, and only if, ζ is a simple root of $S_k(\cdot, y_{i,k}(\zeta))$. It can be shown by induction that the lth-order derivative of the identity $S_k(\zeta, y_{i,k}(\zeta)) = 0$ with respect to ζ is of the form

$$\partial_1^l S_k(\zeta, y_{i,k}(\zeta)) + \sum_{j=1}^{l-1} y_{i,k}^{(j)}(\zeta) H_{l,k,j}\big(\zeta, y_{i,k}(\zeta), ..., y_{i,k}^{(j)}(\zeta)\big) + y_{i,k}^{(l)}(\zeta)\, \partial_2 S_k(\zeta, y_{i,k}(\zeta)) = 0\,, \tag{5.12}$$

where $l \geq 2$, and $H_{l,k,j} \colon \mathbb{C}^{j+2} \to \mathbb{C}$ is a polynomial in $j+2$ variables. Hence, if $y_{i,k}^{(j)}(\zeta) = 0$, $j = 1, 2, ..., l-1$, then $y_{i,k}^{(l)}(\zeta) = 0$ if, and only if, $\partial_1^l S_k(\zeta, y_{i,k}(\zeta)) = 0$. This allows us to conclude that $y_{i,k}^{(j)}(\zeta) = 0$, $j = 1, 2, ..., l-1$ and $y_{i,k}^{(l)}(\zeta) \neq 0$ hold if, and only if, ζ is a root of $S_k(\cdot, y_{i,k}(\zeta))$ of multiplicity l with $l \geq 2$. Note that $S_k(\cdot, y_{i,k}(\zeta))$ is a polynomial of degree $2m$; therefore, the multiplicity of ζ as a root of $S_k(\cdot, y_{i,k}(\zeta))$ is at most $2m$.

Assume, for a proof by contradiction, that $\zeta_0 > \zeta_{1,k}$ is a root of $S_k(\cdot, y_{i,k}(\zeta_0))$ of multiplicity $M \geq 2$. Then, for a small real δ, the perturbation $y_{i,k}(\zeta_0) + \delta$ results in exactly M roots of $S_k(\cdot, y_{i,k}(\zeta_0) + \delta)$ in a small complex neighborhood of ζ_0 by the continuity of the roots with respect to the coefficients. By Theorem 5.3, $S_k(\cdot, \eta)$ does not have a root in the region $\{z \in \mathbb{C} \mid \Im(z) \neq 0,\, \Re(z) > \zeta_{1,k}\}$ for any real η. Therefore, all the M roots in the small complex neighborhood of ζ_0 have to be real. Let us now count the possible number of real roots. If M is odd, then $y_{i,k}$ is strictly increasing or decreasing at ζ_0. If M is even, then $y_{i,k}$ has a strict local minimum or maximum at ζ_0. Hence, the perturbation $y_{i,k}(\zeta_0) + \delta$ results in only simple real roots. Moreover, the number of real roots is 1 for both positive and negative δ if M is odd, and the number of real roots is 2 for positive (resp., negative) δ and 0 for negative (resp., positive) δ if M is even and $y_{i,k}$ has a strict local minimum (resp., maximum) at ζ_0. In all cases, there is a perturbation that results in less than M roots in the complex neighborhood of ζ_0, and we reach a contradiction. Therefore, ζ_0 is a simple root of $S_k(\cdot, y_{i,k}(\zeta_0))$, and $y'_{i,k}(\zeta_0) \neq 0$. □

The next proposition shows that $y_{i,k}(\zeta)$ approaches infinity as ζ approaches infinity for any $i = 1, 2, ..., p$ and $k \geq 0$.

Proposition 5.6. *If $p \geq 1$, then $\lim_{\zeta\to\infty} y_{i,k}(\zeta) = \infty$.*

Proof. The statement can be proved using the Cauchy bound on the moduli of the roots of polynomials [27], Theorem 8.1.3. Let us consider the polynomial f given by

$$f(y) = y^m S_k(\zeta, 1/y) = \sum_{i=0}^{m} \widehat{a}_i (2i)! F_k^{(2(m-i))}(\zeta) y^i \tag{5.13}$$

with $\zeta > \zeta_{1,k}$. The moduli of the roots of f in (5.13) are smaller than or equal to the unique positive root $\rho(\zeta)$ of the polynomial g given by

$$g(y) = |\widehat{a}_m (2m)! F_k(\zeta)|\, y^m - \sum_{i=0}^{m-1} \left|\widehat{a}_i (2i)! F_k^{(2(m-i))}(\zeta)\right| y^i\,. \tag{5.14}$$

Since $F_k^{(2(m-i))}(\zeta) > 0$ for $i = 0, 1, ..., m$ and $\zeta > \zeta_{1,k}$, the polynomial g in (5.14) has the same roots as the polynomial h given by

$$h(y) = y^m - \sum_{i=0}^{m-1} \frac{|\widehat{a}_i|\,(2i)!F_k^{(2(m-i))}(\zeta)}{\widehat{a}_m(2m)!F_k(\zeta)} y^i \,. \tag{5.15}$$

Note that $0 \leq \deg\big(F_k^{(2(m-i))}\big) < \deg\big(F_k\big) = 2m$, $i = 0, 1, ..., m-1$; therefore, $h(y)$ approaches y^m as ζ approaches infinity. Hence, by continuity, the unique positive root of g in (5.14) approaches 0, that is, $\lim_{\zeta\to\infty} \rho(\zeta) = 0$. The proof is completed by noticing that $1/\rho(\zeta)$ is a lower bound on the moduli of the roots of $S_k(\zeta, \cdot)$. ☐

Using Proposition 5.6, we can establish that the sign of $y'_{i,k}(\zeta)$ is positive independently of i and k if $k \geq m-1$.

Proposition 5.7. *If $k \geq m-1$ and $p \geq 1$, then $y'_{i,k}(\zeta_0) > 0$ for any $\zeta_0 > \zeta_{1,k}$.*

Proof. Since $y'_{i,k}$ does not change sign by Proposition 5.5, $y'_{i,k}(\zeta)$ is either positive for any $\zeta > \zeta_{1,k}$ or negative for any $\zeta > \zeta_{1,k}$. If $y'_{i,k}(\zeta) < 0$ for $\zeta > \zeta_{1,k}$, then $y_{i,k}(\zeta) > 0$ is bounded from above; hence, we arrive at a contradiction by Proposition 5.6. ☐

In the following, we consider the limiting case $\zeta = \zeta_{1,k}$. Note that $\zeta_{1,k}$ is the largest root of the real-rooted polynomial F_k, and F_k has a positive leading coefficient. If $k \geq 2(m-1)$, then the multiplicity of $\zeta_{1,k}$ as a root of F_k is at most 2 by Proposition 4.5. Hence, all the roots of the polynomials $F_k^{(2i)}$, $i = 1, 2, ..., m$ are smaller than $\zeta_{1,k}$, and we have $F_k^{(2i)}(\zeta_{1,k}) > 0$, $i = 1, 2, ..., m$ and $F_k(\zeta_{1,k}) = 0$. If we also assume, as earlier, that $\widehat{P}$ has at least one positive real root, then at least one of the coefficients $\widehat{a}_j$, $j = 0, 1, ..., m-1$ has to be nonzero. Thus, under these assumptions, $S_k(\zeta, \cdot)$ cannot be a constant polynomial for $\zeta \geq \zeta_{1,k}$, and the leading coefficient of $S_k(\zeta, \cdot)$ cannot approach zero as ζ approaches $\zeta_{1,k}$. Therefore, because of the continuity of the roots of $S_k(\zeta, \cdot)$, the limits

$$y_{i,k}(\zeta_{1,k}) \coloneqq \lim_{\zeta\to\zeta_{1,k}+} y_{i,k}(\zeta)\,, \quad i = 1, 2, ..., p\,, \quad k \geq 2(m-1) \tag{5.16}$$

exist, and they are nonnegative real values such that $S_k(\zeta_{1,k}, y_{i,k}(\zeta_{1,k})) = 0$.

Proposition 5.8. *Let $k \geq 2(m-1)$, $p \geq 1$, and $\widehat{a}_{m-1} \leq 0$. Then, $y_{1,k}(\zeta_{1,k}) = 0$, and $y_{i,k}(\zeta_{1,k}) > 0$, $i = 2, 3, ..., p$.*

Proof. The polynomial $S_k(\zeta_{1,k}, \cdot)$ takes the form

$$S_k(\zeta_{1,k}, y) = \sum_{i=0}^{m} \widehat{a}_{m-i}(2(m-i))!y^i F_k^{(2i)}(\zeta_{1,k})\,. \tag{5.17}$$

By Theorem 5.1 and continuity, $S_k(\zeta_{1,k}, \cdot)$ has only real roots.

Let zero be a root of $\widehat{P}$ of multiplicity $m-l$ with $l \in \mathbb{N}$, $0 \leq l \leq m$ such that $l = m$ if zero is not a root of $\widehat{P}$. Then, we have $\widehat{a}_j = 0$, $j = 0, 1, ..., m-l-1$ and $\widehat{a}_{m-l} \neq 0$ if $0 \leq l < m$ and $\widehat{a}_0 \neq 0$ if $l = m$. Since $\widehat{P}$ has p positive real roots, the multiplicity of zero as a root of $\widehat{P}$ is at most $m-p$. Therefore, we have $1 \leq p \leq l \leq m$. Since $\widehat{P}$ has only real roots, the polynomial f given by

$$f(x) = x^m \widehat{P}(1/x) = \sum_{i=0}^{l} \widehat{a}_{m-i} x^i \tag{5.18}$$

has only real roots such that $\widehat{a}_m > 0$ and $\widehat{a}_{m-l} \neq 0$. Therefore, if $\widehat{a}_{m-1} = 0$, then $\widehat{a}_{m-2} < 0$ by [27], Lemma 5.4.4 and Remark 5.4.5.

Note that we have $F_k(\zeta_{1,k}) = 0$, $F_k^{(2i)}(\zeta_{1,k}) > 0$, $i = 1, 2, ..., m$, and $F_k^{(2i)}(\zeta) > 0$, $i = 0, 1, ..., m$, $\zeta > \zeta_{1,k}$. Thus, for both $\widehat{a}_{m-1} = 0$ and $\widehat{a}_{m-1} < 0$, the number of sign changes in the sequence of the coefficients of $S_k(\zeta_{1,k}, \cdot)$ decreases by one compared to $S_k(\zeta, \cdot)$, $\zeta > \zeta_{1,k}$. Hence, by Descartes' rule of signs [27], Corollary 10.1.10 and Corollary 10.1.12, $S_k(\zeta_{1,k}, \cdot)$ has exactly $p - 1$ positive roots. Since $F_k(\zeta_{1,k}) = 0$, zero is a root of $S_k(\zeta_{1,k}, \cdot)$. If $\widehat{a}_{m-1} < 0$, then zero is a simple root. If $\widehat{a}_{m-1} = 0$, then zero is a double root. □

The previous four propositions completely characterize the functions $y_{1,k}$ under some conditions on k, p, and $\widehat{a}_{m-1}$.

Proposition 5.9. *Let $k \geq 2(m-1)$, $p \geq 1$, and $\widehat{a}_{m-1} \leq 0$. Then, the functions $y_{1,k} \colon (\zeta_{1,k}, \infty) \to \mathbb{R}_+^*$ are bijective and real analytic such that $y'_{1,k}(\zeta) > 0$ for any $\zeta > \zeta_{1,k}$.*

Proof. The statement follows directly from Proposition 5.5, Proposition 5.6, Proposition 5.7, and Proposition 5.8. □

In the following, we focus our attention on $y_{1,2m-1}$ and $y_{1,2m}$.

Proposition 5.10. *Let $p \geq 1$ and $\widehat{a}_{m-1} \leq 0$. Furthermore, assume that $\zeta_{1,2m} < \zeta \leq 0$ and $0 < \eta \leq y_{1,2m}(\zeta)$. Then, $S_{2m-1}(\zeta, \eta) < 0$.*

Proof. Assume that there are some ζ and η satisfying $\zeta_{1,2m} < \zeta \leq 0$ and $0 < \eta \leq y_{1,2m}(\zeta)$ such that $S_{2m-1}(\zeta, \eta) = 0$. Note that $\zeta_{1,2m-1} = 0$ by Proposition 4.5. Therefore, by Proposition 5.9, there exists some $\zeta^* > 0$ such that $y_{1,2m-1}(\zeta^*) = \eta$ and $S_{2m-1}(\zeta^*, \eta) = 0$.

By equation (5.4), the polynomials S_{2m-1} and S_{2m} satisfy

$$S_{2m}(z, y) = S_{2m-1}(z, y) + \partial_1 S_{2m-1}(z, y)\,, \tag{5.19}$$

which can be written as

$$S_{2m}(z, y) = \mathrm{e}^{-z} \frac{\partial}{\partial z} \left(\mathrm{e}^{z} S_{2m-1}(z, y) \right) . \tag{5.20}$$

By (5.20) and Rolle's theorem, $S_{2m}(\cdot, \eta)$ has a root ζ_* such that $\zeta < \zeta_* < \zeta^*$. By Proposition 5.9, we have $y_{1,2m}(\zeta_*) > y_{1,2m}(\zeta) \geq \eta$ because $\zeta_* > \zeta$. That is, $S_{2m}(\zeta_*, \eta) = 0$ and $S_{2m}(\zeta_*, y_{1,2m}(\zeta_*)) = 0$ hold such that $0 < \eta < y_{1,2m}(\zeta_*)$ and $\zeta_* > \zeta_{1,2m}$. Since $y_{1,2m}(\zeta_*)$ is the smallest positive root of $S_{2m}(\zeta_*, \cdot)$, this is a contradiction.

Now, it remains to show that $S_{2m-1}(\zeta_0, \eta_0) < 0$ for some specific ζ_0 and η_0 satisfying $\zeta_{1,2m} < \zeta_0 \leq 0$ and $0 < \eta_0 \leq y_{1,2m}(\zeta_0)$. Note that $S_{2m-1}(\zeta, 0) = \widehat{a}_m (2m)! F_{2m-1}(\zeta)$, and F_{2m-1} has $2m-1$ simple negative real roots and a simple zero root by Proposition 4.5. Since F_{2m-1} has a positive leading coefficient, $F_{2m-1}(-\varepsilon)$ is negative for a sufficiently small $\varepsilon > 0$; therefore, the same holds true for $S_{2m-1}(-\varepsilon, 0)$. Furthermore, the function $S_{2m-1}(-\varepsilon, \cdot)$ is continuous; hence, $S_{2m-1}(-\varepsilon, \delta)$ remains negative for a sufficiently small $\delta > 0$. For sufficiently small $\varepsilon > 0$ and $\delta > 0$, the values $\zeta_0 = -\varepsilon$ and $\eta_0 = \delta$ also satisfy the prescribed inequalities. □

Since $R_k(s, \tau) = \tau^{k-2m} S_k(s\tau, \tau^2)$, we can reformulate the results on the location of the roots of S_{2m} and S_{2m-1} in terms of R_{2m} and R_{2m-1}. To this end, let us define the functions $\tau_{1,k} \colon (\zeta_{1,k}, \infty) \to \mathbb{R}_+^*$ given by

$$\tau_{1,k}(\zeta) \coloneqq \sqrt{y_{1,k}(\zeta)} \tag{5.21}$$

and the functions $z_{1,k} \colon \mathbb{R}_+^* \to (\zeta_{1,k}, \infty)$ given by

$$z_{1,k}(\tau) \coloneqq \tau_{1,k}^{-1}(\tau)\,, \tag{5.22}$$

where $\tau_{1,k}^{-1}$ is the inverse function of $\tau_{1,k}$. If $k \geq 2(m-1)$, $p \geq 1$, and $\widehat{a}_{m-1} \leq 0$, then the functions $y_{1,k}\colon (\zeta_{1,k}, \infty) \to \mathbb{R}_+^*$ are bijective and real analytic such that $y_{1,k}'(\zeta) > 0$ for any $\zeta > \zeta_{1,k}$ by Proposition 5.9. Under the same conditions, $\tau_{1,k}$ and $z_{1,k}$ are well defined and have the same properties as $y_{1,k}$ [32], Proposition 1.3.3 and Theorem 1.4.3.

Proposition 5.11. *Let $k \geq 2(m-1)$, $p \geq 1$, and $\widehat{a}_{m-1} \leq 0$. Then, the functions $\tau_{1,k}\colon (\zeta_{1,k}, \infty) \to \mathbb{R}_+^*$ are bijective and real analytic such that $\tau_{1,k}'(\zeta) > 0$ for any $\zeta > \zeta_{1,k}$. Moreover, the functions $z_{1,k}\colon \mathbb{R}_+^* \to (\zeta_{1,k}, \infty)$ are bijective and real analytic such that $z_{1,k}'(\tau) > 0$ for any $\tau > 0$.*

Now, let us define the functions $s_{1,k}\colon \mathbb{R}_+^* \to \mathbb{R}$ by

$$s_{1,k}(\tau) \coloneqq \frac{z_{1,k}(\tau)}{\tau}\,. \tag{5.23}$$

The next proposition states that $s_{1,2m}(\tau)$ is a strictly dominant root of the polynomial $R_{2m}(\cdot, \tau)$.

Proposition 5.12. *Let $p \geq 1$ and $\widehat{a}_{m-1} \leq 0$, and assume that $\tau > 0$. Then,*

$$R_{2m}\big(s_{1,2m}(\tau), \tau\big) = 0\,. \tag{5.24}$$

Moreover, for $s \in \mathbb{C}$,

$$R_{2m}(s, \tau) \neq 0 \tag{5.25}$$

if $\Re(s) \geq s_{1,2m}(\tau)$ and $\Im(s) \neq 0$, or if $\Re(s) > s_{1,2m}(\tau)$ and $\Im(s) = 0$.

Proof. For a given $\tau > 0$, we can find a unique $\zeta^* > \zeta_{1,2m}$ such that $\tau_{1,2m}(\zeta^*) = \tau$. Therefore, we have $\tau^2 = y_{1,2m}(\zeta^*)$ and $z_{1,2m}(\tau) = \zeta^*$. Hence,

$$R_{2m}\big(s_{1,2m}(\tau), \tau\big) = S_{2m}\big(z_{1,2m}(\tau), \tau^2\big) = S_{2m}\big(\zeta^*, y_{1,2m}(\zeta^*)\big) = 0\,, \tag{5.26}$$

and equation (5.24) holds.

Assume that $R_{2m}(\sigma, \tau) = 0$ such that $\tau > 0$ and $\sigma > s_{1,2m}(\tau)$. Then,

$$R_{2m}(\sigma, \tau) = S_{2m}(\sigma\tau, \tau^2) = S_{2m}\big(\sigma\tau, y_{1,2m}(\zeta^*)\big) = 0 \tag{5.27}$$

such that $\sigma\tau > z_{1,2m}(\tau) = \zeta^*$. By Proposition 5.9, $y_{1,2m}(\sigma\tau) > y_{1,2m}(\zeta^*) > 0$. Since $y_{1,2m}(\sigma\tau)$ is the smallest positive root of $S_{2m}(\sigma\tau, \cdot)$, we arrive at a contradiction.

Finally, assume that $R_{2m}(s, \tau) = 0$ such that $\tau > 0$, $\Re(s) \geq s_{1,2m}(\tau)$ and $\Im(s) \neq 0$. Then, $R_{2m}(s, \tau) = S_{2m}(s\tau, \tau^2) = 0$ such that $\tau^2 > 0$, $\Re(s\tau) \geq z_{1,2m}(\tau) > \zeta_{1,2m}$ and $\Im(s\tau) \neq 0$, which is a contradiction by Theorem 5.3. □

Proposition 5.13. *Let $p \geq 1$ and $\widehat{a}_{m-1} \leq 0$. If $0 < \tau \leq \tau_{1,2m}(0)$, then $s_{1,2m}'(\tau) > 0$.*

Proof. Note that $z_{1,2m}(\tau_{1,2m}(0)) = 0$. Therefore, if $0 < \tau \leq \tau_{1,2m}(0)$, then $z_{1,2m}(\tau) \leq 0$. Consequently, we can take the derivative of $s_{1,2m}$ as

$$s_{1,2m}'(\tau) = \frac{z_{1,2m}'(\tau)\tau - z_{1,2m}(\tau)}{\tau^2} > 0\,. \tag{5.28}$$

□

Proposition 5.14. *Let $p \geq 1$ and $\widehat{a}_{m-1} \leq 0$. Furthermore, assume that $0 < \tau \leq \tau_{1,2m}(0)$, $0 < \vartheta \leq \tau$, and $s_{1,2m}(\tau) \leq \sigma \leq 0$. Then, $R_{2m-1}(\sigma, \vartheta) < 0$.*

Proof. Note that $R_{2m-1}(\sigma,\vartheta) = \vartheta^{-1} S_{2m-1}(\sigma\vartheta, \vartheta^2)$. Furthermore, by Proposition 5.13, we have $s_{1,2m}(\vartheta) \leq s_{1,2m}(\tau)$; hence, $z_{1,2m}(\vartheta) \leq s_{1,2m}(\tau)\vartheta \leq \sigma\vartheta$. Therefore, we obtain the inequalities

$$\zeta_{1,2m} < z_{1,2m}(\tau) \leq \sigma\tau \leq \sigma\vartheta \leq 0 \tag{5.29}$$

and

$$0 < \vartheta^2 = \tau_{1,2m}^2(z_{1,2m}(\vartheta)) \leq \tau_{1,2m}^2(\sigma\vartheta) = y_{1,2m}(\sigma\vartheta)\,. \tag{5.30}$$

Hence, Proposition 5.10 with $\zeta = \sigma\vartheta$ and $\eta = \vartheta^2$ entails that $S_{2m-1}(\sigma\vartheta, \vartheta^2) < 0$. ☐

5.3. Consequences on the distribution of quasipolynomials' roots

Propositions 5.12 and 5.14 allow us to apply Propositions 4.1–4.3 to the quasipolynomial D in (2.2) under some conditions on P and γ. In this way, we can prove the main result of this paper.

Theorem 5.15. *Assume that the polynomial P is of the form $P(s) = \widehat{P}(s^2)$, where $\widehat{P}$ is a real-rooted polynomial given by*

$$\widehat{P}(x) = \widehat{a}_m x^m + \widehat{a}_{m-1} x^{m-1} + \ldots + \widehat{a}_0 \tag{5.31}$$

with $m \in \mathbb{N}^*$. Furthermore, assume that $\widehat{a}_m > 0$, $\widehat{a}_{m-1} \leq 0$, $\tau > 0$, and $\gamma \leq 0$. Let τ_γ denote the smallest positive real root of $R_{2m}(\gamma, \cdot)$, and let s_τ denote the largest real root of $R_{2m}(\cdot, \tau)$. Then, the following statements hold.

(i) *The quasipolynomial D is γ-stabilizable if, and only if, $0 < \tau < \tau_\gamma$.*
(ii) *If $0 < \tau < \tau_\gamma$ and*

$$b_i = -\frac{e^{s_\tau \tau}}{i!} \sum_{j=i}^{2m-1} \frac{(-s_\tau)^{j-i}}{(j-i)!} R_j(s_\tau, \tau)\,, \quad i = 0, 1, \ldots, 2m-1\,, \tag{5.32}$$

then s_τ is a dominant root of D such that $s_\tau < \gamma$.

Remark 5.16. If $\widehat{P}$ has a zero root of multiplicity m and $\gamma = 0$, then $R_{2m}(\gamma, \cdot)$ does not have a positive real root. In this case, set $\tau_\gamma = \infty$. In any other case, τ_γ and s_τ are well defined under the assumptions of Theorem 5.15, as detailed below in the proof of the theorem.

Proof of Theorem 5.15. Since $\widehat{a}_m > 0$ and $\widehat{a}_{m-1} \leq 0$, the sum of the roots of $\widehat{P}$ is nonnegative by Vieta's formulas. This means that either $\widehat{P}$ has at least one positive root, that is $p \geq 1$, or $\widehat{P}$ has a zero root of multiplicity m.

First, let us consider the case where $p \geq 1$. By Proposition 5.13, the function $s_{1,2m}$ is strictly increasing on the interval $(0, \tau_{1,2m}(0)]$, and it maps $(0, \tau_{1,2m}(0)]$ onto $(-\infty, 0]$. Therefore, we can find a unique τ^* such that $s_{1,2m}(\tau^*) = \gamma \leq 0$ and $0 < \tau^* \leq \tau_{1,2m}(0)$. By Proposition 5.12, we have

$$R_{2m}(\gamma, \tau^*) = R_{2m}\big(s_{1,2m}(\tau^*), \tau^*\big) = 0\,. \tag{5.33}$$

Let us assume that τ^* is not the smallest positive real root of $R_{2m}(\gamma, \cdot)$, that is, $R_{2m}(\gamma, \tau_*) = 0$ for some $0 < \tau_* < \tau^*$. Since $\gamma = s_{1,2m}(\tau^*) > s_{1,2m}(\tau_*)$, this is a contradiction by Proposition 5.12. Hence, $\tau^* = \tau_\gamma$.

For any $\tau > 0$, Proposition 5.12 gives $R_{2m}(s_{1,2m}(\tau), \tau) = 0$ such that $s_{1,2m}(\tau)$ is the largest real root of $R_{2m}(\cdot, \tau)$. Hence, $s_{1,2m}(\tau) = s_\tau$. Therefore, by Proposition 4.1, $s_{1,2m}(\tau)$ is a root of D of multiplicity at least $2m + 1$ if (5.32) holds.

If $0 < \tau < \tau_\gamma$, then Proposition 5.14 with $\sigma = s_{1,2m}(\tau)$ gives $R_{2m-1}(s_{1,2m}(\tau), \vartheta) < 0$ for any $0 < \vartheta \le \tau$. Therefore, if $0 < \tau < \tau_\gamma$, and (5.32) holds, then $s_{1,2m}(\tau)$ is a dominant root of D by Proposition 4.2. Moreover, if $0 < \tau < \tau_\gamma$, then $s_{1,2m}(\tau) < s_{1,2m}(\tau_\gamma) = \gamma$. Hence, the quasipolynomial D is γ-stabilizable if $0 < \tau < \tau_\gamma$.

If $\tau \ge \tau_\gamma$, then either $\tau_\gamma \le \tau \le \tau_{1,2m}(0)$ or $\tau > \tau_{1,2m}(0)$. If $\tau_\gamma \le \tau \le \tau_{1,2m}(0)$, then we have $s_{1,2m}(\tau) \ge s_{1,2m}(\tau_\gamma) = \gamma$. If $\tau > \tau_{1,2m}(0)$, then $z_{1,2m}(\tau) > z_{1,2m}(\tau_{1,2m}(0)) = 0$ by Proposition 5.11; hence, $s_{1,2m}(\tau) = z_{1,2m}(\tau)/\tau > 0 \ge \gamma$. In both cases, $s_{1,2m}(\tau) \ge \gamma$, and $R_{2m}(s_{1,2m}(\tau), \tau) = 0$ by Proposition 5.12. Therefore, $R_{2m}(\cdot, \tau)$ is not γ-stable; hence, the quasipolynomial D is not γ-stabilizable for $\tau \ge \tau_\gamma$ by Proposition 4.3.

In the following, we consider the case where $\widehat{P}$ has a zero root of multiplicity m. Then, $\widehat{a}_i = 0$, $i = 0, 1, ..., m-1$ hold, and R_k takes the form

$$R_k(s, \theta) = \theta^{k-2m} \widehat{a}_m (2m)! F_k(s\theta) \,. \tag{5.34}$$

Hence, for $\gamma < 0$, $R_{2m}(\gamma, \theta) = 0$ if, and only if, $\theta = \zeta_{i,2m}/\gamma$, $i = 1, 2, ..., 2m$. Note that $\zeta_{i,k}$, $i = 1, 2, ..., 2m$ are the roots of F_k in non-increasing order, and F_{2m} has only simple negative roots by Proposition 4.5. Therefore, $\tau_\gamma = \zeta_{1,2m}/\gamma$. Since $\lim_{\gamma \to 0-} \zeta_{1,2m}/\gamma = \infty$, we set $\tau_\gamma = \infty$ for $\gamma = 0$.

According to the proof of Proposition 4.5 and equation (4.14), we have $F_{2m-1}(x) = x F'_{2m}(x)/(2m)$. Moreover, F_{2m-1} has a positive leading coefficient. As a result, $F_{2m-1}(x) < 0$ holds for any $\zeta_{1,2m} \le x < 0$. Hence, for a given $\tau > 0$ and any $0 < \vartheta \le \tau$, we obtain

$$R_{2m-1}(\zeta_{1,2m}/\tau, \vartheta) = \vartheta^{-1} \widehat{a}_m (2m)! F_{2m-1}(\zeta_{1,2m}\vartheta/\tau) < 0 \,. \tag{5.35}$$

Moreover, we have

$$R_{2m}(\zeta_{1,2m}/\tau, \tau) = \widehat{a}_m (2m)! F_{2m}(\zeta_{1,2m}) = 0 \,. \tag{5.36}$$

Note that $\zeta_{1,2m}/\tau$ is the largest real root of $R_{2m}(\cdot, \tau)$. Consequently, $\zeta_{1,2m}/\tau = s_\tau$. If $0 < \tau < \tau_\gamma$, and (5.32) holds, then $\zeta_{1,2m}/\tau$ is a dominant root of D by equations (5.35)–(5.36), Propositions 4.1 and 4.2. Next, if $0 < \tau < \tau_\gamma$, then $\zeta_{1,2m}/\tau < \gamma$. Hence, the quasipolynomial D is γ-stabilizable if $0 < \tau < \tau_\gamma$.

If $\gamma < 0$ and $\tau \ge \tau_\gamma$, then $\zeta_{1,2m}/\tau \ge \gamma$. Therefore, the quasipolynomial D is not γ-stabilizable for $\tau \ge \tau_\gamma$ by equation (5.36) and Proposition 4.3. □

Remark 5.17. The case where $\widehat{P}$ has a zero root of multiplicity m was included in Theorem 5.15 for the sake of completeness. For a similar argument in this case, see [18, 22].

6. Applications: prescribed stabilization of conservative mechanical systems

Consider the linear delay differential equation of the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{C}(\mathbf{K}_\mathrm{p}\mathbf{q}(t-\tau) + \mathbf{K}_\mathrm{d}\dot{\mathbf{q}}(t-\tau)) \tag{6.1}$$

under appropriate initial conditions, where $\mathbf{q}$ is the unknown function, $\mathbf{q}(t) \in \mathbb{R}^{m\times 1}$, $t \in \mathbb{R}$, $\mathbf{M} \in \mathbb{R}^{m\times m}$ is a symmetric and positive definite matrix, $\mathbf{S} \in \mathbb{R}^{m\times m}$ is a symmetric matrix, $\mathbf{C} \in \mathbb{R}^{m\times 1}$, $\mathbf{K}_\mathrm{p}$, $\mathbf{K}_\mathrm{d} \in \mathbb{R}^{1\times m}$, and $\tau > 0$. In equation (6.1), $\dot{\mathbf{q}}$ denotes the derivative of $\mathbf{q}$.

If $\mathbf{K}_\mathrm{p} = \mathbf{0}$ and $\mathbf{K}_\mathrm{d} = \mathbf{0}$, then (6.1) takes the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{S}\mathbf{q}(t) = \mathbf{0} \,. \tag{6.2}$$

If there are only potential forces in a holonomic scleronomic mechanical system of finitely many degrees of freedom such that the potential energy does not explicitly depend on time (*i.e.*, if the mechanical system is *conservative*), then we obtain an equation of motion of the form (6.2) in the neighborhood of a position of equilibrium [33]. Note that in such mechanical systems, there are no dissipative forces, gyroscopic forces, or any other nonconservative forces.

The right-hand side of equation (6.1) represents the control action given by delayed full-state feedback. In order to see this, we can write (6.1) in the first-order form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{x}(t-\tau)\,, \tag{6.3}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{S} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{\mathrm{p}} & \mathbf{K}_{\mathrm{d}} \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix}. \tag{6.4}$$

Since $\mathbf{B} \in \mathbb{R}^{2m\times 1}$ and $\mathbf{K} \in \mathbb{R}^{1\times 2m}$, the characteristic function D of (6.3) can be given using the matrix determinant lemma [34, 35] as

$$\begin{aligned} D(s) &= \det\Big(s\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\mathrm{e}^{-s\tau}\Big) \\ &= \det(s\mathbf{I} - \mathbf{A}) - \mathbf{K}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}\mathrm{e}^{-s\tau} \\ &= P(s) + Q(s)\mathrm{e}^{-s\tau}\,, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} P(s) &= a_{2m}s^{2m} + a_{2m-1}s^{2m-1} + ... + a_0\,, \\ Q(s) &= b_{2m-1}s^{2m-1} + b_{2m-2}s^{2m-2} + ... + b_0\,, \end{aligned} \tag{6.6}$$

and $a_{2m} = 1$. In Section 2, the exponential bound on the solutions of the delay differential equation (2.1) was given in terms of the spectral abscissa of its characteristic function D in (2.2). Note that there exist analogous results for the system of delay differential equations (6.3) and its characteristic function D in (6.5); see, for example, [24].

By expanding the adjugate of $s\mathbf{I} - \mathbf{A}$ [34], pp. 124–125, the coefficients b_i, $i = 0, 1, 2m-1$ in the polynomial Q in (6.5) can be expressed as

$$\begin{bmatrix} b_0 & b_1 & \cdots & b_{2m-1} \end{bmatrix} = -\mathbf{K}\mathbf{M}_{\mathrm{c}}\mathbf{E}\,, \tag{6.7}$$

where

$$\mathbf{M}_{\mathrm{c}} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{2m-1}\mathbf{B} \end{bmatrix} \tag{6.8}$$

and

$$\mathbf{E} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{2m-1} & a_{2m} \\ a_2 & a_3 & a_4 & \cdots & a_{2m} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2m-1} & a_{2m} & 0 & \cdots & 0 & 0 \\ a_{2m} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{6.9}$$

Since $\det(\mathbf{E}) = (-1)^m a_{2m}^{2m} = \pm 1 \neq 0$, the coefficients b_i, $i = 0, 1, 2m-1$ can be adjusted independently using $\mathbf{K}$ if, and only if, the matrix $\mathbf{M}_\mathrm{c}$ is invertible, that is, if, and only if, the pair $(\mathbf{A}, \mathbf{B})$ is controllable [36]. In this way, we may be able to stabilize (6.3) by an appropriate choice of $\mathbf{K}$.

Using (6.4), we can write the polynomial P in (6.5) in the form

$$P(s) = \det\left(s\mathbf{I} - \mathbf{A}\right) = \det\left(s^2\mathbf{I} + \mathbf{M}^{-1}\mathbf{S}\right) = \det\left(s^2\mathbf{I} + \mathbf{M}^{-\frac{1}{2}}\mathbf{S}\mathbf{M}^{-\frac{1}{2}}\right), \tag{6.10}$$

where $\mathbf{M}^{\frac{1}{2}}$ is the symmetric and positive definite square root of $\mathbf{M}$, and $\mathbf{M}^{-\frac{1}{2}}$ is the inverse of $\mathbf{M}^{\frac{1}{2}}$. Since the matrix $\mathbf{M}^{-\frac{1}{2}}\mathbf{S}\mathbf{M}^{-\frac{1}{2}}$ is symmetric, P is of the form $P(s) = \widehat{P}(s^2)$, where $\widehat{P}$ is a real-rooted polynomial. Therefore, P has the structure that we assumed in Section 5 invariably. However, it should be noted that most of the results in Section 5 apply only if $\widehat{P}$ has at least one positive real root and $a_{2m-2} \leq 0$. In particular, Theorem 5.15 applies under the condition $a_{2m-2} \leq 0$.

The polynomial P has only real roots if, and only if, every root of $\widehat{P}$ is nonnegative, that is, if, and only if, $\mathbf{S}$ is negative semidefinite. In this case, the γ-stabilizability of D in (6.5) can be assessed using the results of [18]. If every root of $\widehat{P}$ is nonnegative, then the sum of the roots of $\widehat{P}$ is also nonnegative; therefore, we always have $a_{2m-2} \leq 0$ by Vieta's formulas. In this sense, Theorem 5.15 extends the results of [18].

If $\widehat{P}$ has only negative real roots, that is, if $\mathbf{S}$ is positive definite, then we always have $a_{2m-2} > 0$. In this case, the γ-stabilizability of D in (6.5) cannot be completely characterized using the techniques of Sections 4 and 5; hence, further investigation is needed; it will be carried out in Subsection 6.2 in the case where $m = 1$.

The next two subsections demonstrate the results of this paper through specific examples of controlled mechanical systems. In Subsection 6.1, we consider the double inverted pendulum with a torsion spring. Then, in Subsection 6.2, the limitations of the results will be discussed in the case of a single-degree-of-freedom mass-spring system.

6.1. Double inverted pendulum with a torsion spring

Consider the pinned inverted pendulum system shown in Figure 2. The mechanical system moves in the vertical plane, and it consists of two rods of equal mass M and length L and a torsion spring of stiffness S_t. The control torque

$$T_\mathrm{c}(t) = K_{\mathrm{p},1}\varphi_1(t-\tau) + K_{\mathrm{d},1}\dot{\varphi}_1(t-\tau) + K_{\mathrm{p},2}\varphi_2(t-\tau) + K_{\mathrm{d},2}\dot{\varphi}_2(t-\tau) \tag{6.11}$$

is applied at the first (lowest) rod, where $\tau > 0$ is the feedback delay.

The equation of motion is of the form (6.1) with

$$\mathbf{M} = \frac{ML^2}{6}\begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{S} = -\frac{MgL}{2}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + S_\mathrm{t}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{q}(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}, \tag{6.12}$$

and

$$\mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{K}_\mathrm{p} = \begin{bmatrix} -K_{\mathrm{p},1} & -K_{\mathrm{p},2} \end{bmatrix}, \quad \mathbf{K}_\mathrm{d} = \begin{bmatrix} -K_{\mathrm{d},1} & -K_{\mathrm{d},2} \end{bmatrix}, \tag{6.13}$$

where $M > 0$, $L > 0$, $g > 0$, and $S_\mathrm{t} > 0$. With $p_1 = S_\mathrm{t}/(MgL) > 0$, $p_2 = g/L > 0$, and $p_3 = 1/(ML^2) > 0$, the corresponding characteristic function D takes the form

$$D(s) = s^4 + \frac{6}{7}(2p_1 - 7)p_2 s^2 - \frac{9}{7}(2p_1 - 3)p_2^2 + \mathrm{e}^{-s\tau}(b_3 s^3 + b_2 s^2 + b_1 s + b_0), \tag{6.14}$$

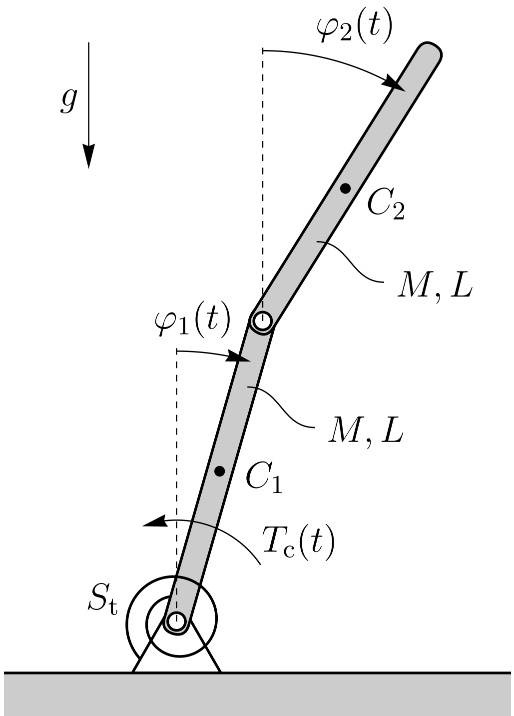

FIGURE 2. Mechanical model of the double inverted pendulum with a torsion spring.

where

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = -\frac{18}{7} p_3 \begin{bmatrix} p_2 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} K_{\mathrm{p},1} \\ K_{\mathrm{p},2} \\ K_{\mathrm{d},1} \\ K_{\mathrm{d},2} \end{bmatrix} . \tag{6.15}$$

Since $p_2 > 0$ and $p_3 > 0$, the matrix of the linear map (6.15) is invertible. In other words, the matrices $\mathbf{A}$ and $\mathbf{B}$ in (6.4) form a controllable pair. Moreover, in accordance with (6.10), we have $P(s) = \widehat{P}(s^2)$, where $\widehat{P}$ is a real-rooted polynomial given by

$$\widehat{P}(x) = x^2 + \frac{6}{7}(2p_1 - 7)p_2 x - \frac{9}{7}(2p_1 - 3)p_2^2 . \tag{6.16}$$

Since $p_1 > 0$ and $p_2 > 0$, the γ-stabilizability of D in (6.14) can be characterized using Theorem 5.15 if $0 < p_1 \leq 7/2$. Note that if $0 < p_1 < 3/2$, then the polynomial $\widehat{P}$ in (6.16) has positive real roots. On the other hand, if $3/2 < p_1 \leq 7/2$, then $\widehat{P}$ has a positive real root and a negative real root such that the sum of the roots is non-negative. If $p_1 = 3/2$, then $\widehat{P}$ has a zero root and a positive real root.

Let us now consider γ-stabilizability with $\gamma = 0$. The polynomial $R_4(0, \cdot)$ takes the form

$$R_4(0, \theta) = -\frac{9}{7}(2p_1 - 3)p_2^2\theta^4 + \frac{72}{7}(2p_1 - 7)p_2\theta^2 + 24 . \tag{6.17}$$

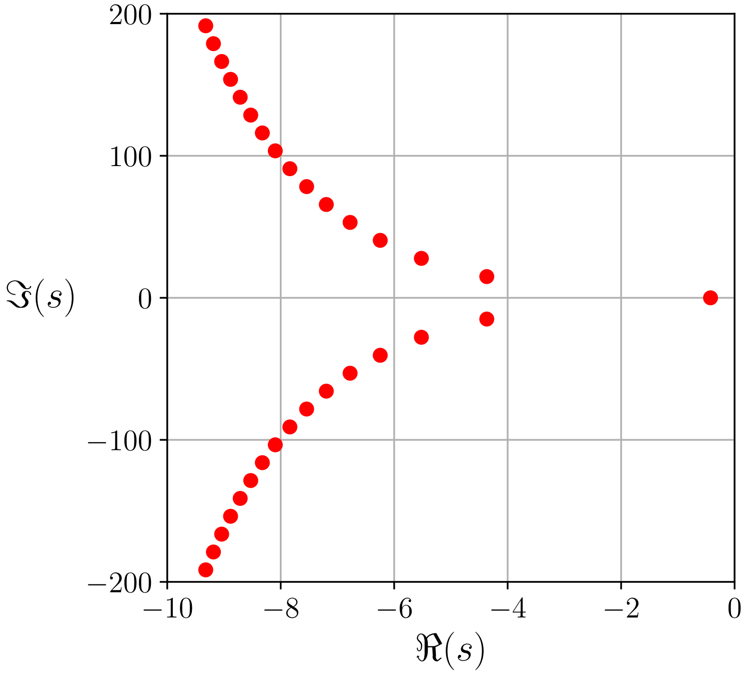

FIGURE 3. The spectrum distribution of the closed-loop characteristic function D in (6.14) if $\tau = 1/2$, $p_1 = 2$, $p_2 = 1$, and the control parameters b_0, b_1, b_2, and b_3 are chosen according to (5.32). The spectral abscissa is achieved at the quintuple root $s \approx -0.424$. Note that τ is smaller than $\tau_0(2;1) \approx 0.868$ given by (6.18).

If $p_2 > 0$, then the smallest positive real root of $R_4(0,\cdot)$ in (6.17) can be given as

$$\tau_0(p_1;p_2) = \sqrt{\frac{12(7-2p_1) - 2\sqrt{6}\sqrt{273-154p_1+24p_1^2}}{3p_2(3-2p_1)}}\,. \tag{6.18}$$

Note that (6.18) is also valid at $p_1 = 3/2$ in the sense that

$$\lim_{p_1\to 3/2} \tau_0(p_1;p_2) = \sqrt{\frac{7}{12p_2}}\,. \tag{6.19}$$

According to Theorem 5.15, if $0 < p_1 \leq 7/2$, and $p_2 > 0$, then D in (6.14) is γ-stabilizable with $\gamma = 0$ if, and only if, $0 < \tau < \tau_0(p_1;p_2)$. Figure 3 shows the spectrum of D for specific values of τ, p_1, and p_2 if the control parameters b_0, b_1, b_2, and b_3 are given by (5.32). Figure 4 illustrates the effective exponential stability of the solutions of the closed-loop system for specific initial conditions. Note that if the parameters $K_{\mathrm{p},1}$, $K_{\mathrm{p},2}$, $K_{\mathrm{d},1}$, and $K_{\mathrm{d},2}$ are given by (5.32) and (6.15), then the solutions do not depend on the value of $p_3 > 0$.

6.2. Limitations of the design method and open questions

Consider a single-degree-of-freedom mass-spring system with delayed proportional-derivative feedback given by

$$M\ddot{q}(t) + Sq(t) = -K_{\mathrm{p}}q(t-\tau) - K_{\mathrm{d}}\dot{q}(t-\tau) \tag{6.20}$$

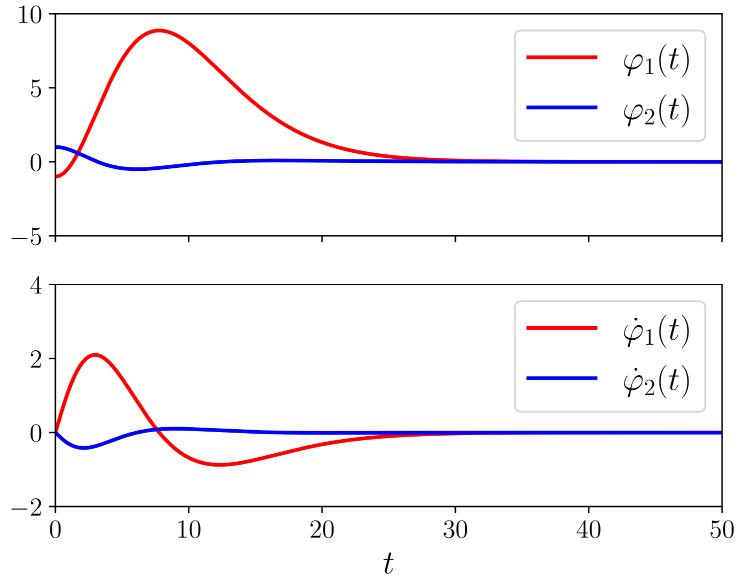

FIGURE 4. The solution of the closed-loop system (6.1) with (6.12)–(6.13) if $\tau = 1/2$, $p_1 = 2$, $p_2 = 1$, and the parameters $K_{\mathrm{p},1}$, $K_{\mathrm{p},2}$, $K_{\mathrm{d},1}$, and $K_{\mathrm{d},2}$ are chosen according to (5.32) and (6.15). The initial conditions are taken such that for all $t \in [-\tau, 0]$ one has $\varphi_1(t) = -1$ and $\varphi_2(t) = 1$. The components φ_1 and φ_2 of the solution $\mathbf{q}$ are shown along with their derivatives.

where $M > 0$, $S > 0$, and $\tau > 0$. The characteristic function D of (6.20) is of the form

$$D(s) = s^2 + \omega^2 + \mathrm{e}^{-s\tau}(b_1 s + b_0)\,, \tag{6.21}$$

where $\omega = \sqrt{S/M} > 0$, $b_1 = K_\mathrm{d}/M$, and $b_0 = K_\mathrm{p}/M$.

Since $P(s) = s^2 + \omega^2$ and $m = 1$, the polynomials R_{2m-1} and R_{2m} take the form

$$R_1(s,\tau) = \tau s^2 + 2s + \omega^2\tau\,, \tag{6.22}$$

$$R_2(s,\tau) = \tau^2 s^2 + 4\tau s + 2 + \omega^2\tau^2\,. \tag{6.23}$$

For $\tau > 0$, the roots of $R_1(\cdot,\tau)$ and $R_2(\cdot,\tau)$ can be given as

$$s_{\pm,1}(\tau) = \frac{-1 \pm \sqrt{1-\omega^2\tau^2}}{\tau} \quad \text{and} \quad s_{\pm,2}(\tau) = \frac{-2 \pm \sqrt{2-\omega^2\tau^2}}{\tau}\,, \tag{6.24}$$

respectively. The roots $s_{\pm,1}(\tau)$ are real if $0 < \tau \leq 1/\omega$, and $\Re(s_{\pm,1}(\tau)) = -1/\tau$ if $\tau > 1/\omega$. Similarly, the roots $s_{\pm,2}(\tau)$ are real if $0 < \tau \leq \sqrt{2}/\omega$, and $\Re(s_{\pm,2}(\tau)) = -2/\tau$ if $\tau > \sqrt{2}/\omega$. The location of the roots of $R_1(\cdot,\tau)$ and $R_2(\cdot,\tau)$ is shown in Figure 5 for $\omega = 1$.

The properties of the roots of $R_1(\cdot,\tau)$ and $R_2(\cdot,\tau)$ allow us to show, using Propositions 4.1–4.3, that D in (6.21) is γ-stabilizable for a given $\gamma \leq -\sqrt{2}/\omega$ if, and only if, $0 < \tau < \tau_\gamma$, where τ_γ is the smallest positive root of $R_2(\gamma,\cdot)$. However, as explained below, this statement does not hold for $-\sqrt{2}/\omega < \gamma \leq 0$.

In [11], the minimum of the spectral abscissa of D in (6.21) with respect to b_0 and b_1 was characterized as follows. The minimum of the spectral abscissa corresponds to a real characteristic root of multiplicity 3 if $0 < \tau < \sqrt{2}/\omega$, and it corresponds to a real characteristic root of multiplicity 4 if $\tau = \sqrt{2}/\omega$. In both cases, the multiple real characteristic root is given by $s_{+,2}(\tau)$; see also [19]. On the other hand, the minimum of the

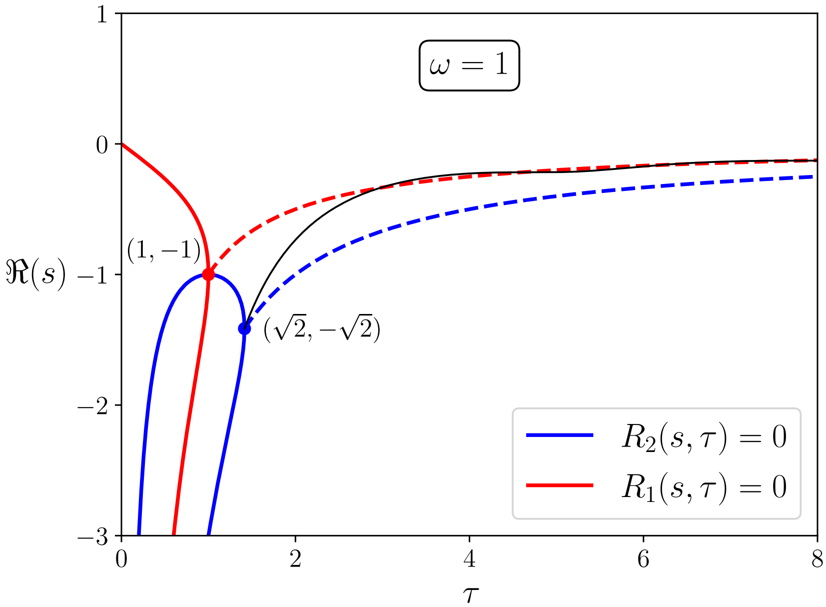

FIGURE 5. Location of the roots of $R_1(\cdot,\tau)$ and $R_2(\cdot,\tau)$ in (6.22)–(6.23) and the minimum of the spectral abscissa of D in (6.21) with respect to b_0 and b_1 for $\omega = 1$ and $\tau > 0$. Solid blue and red lines: real roots of $R_2(\cdot,\tau)$ and $R_1(\cdot,\tau)$; dashed blue and red lines: real parts of the non-real roots of $R_2(\cdot,\tau)$ and $R_1(\cdot,\tau)$; solid black line: the minimum of the spectral abscissa of D for $\tau > \sqrt{2}$. For $0 < \tau \leq \sqrt{2}$, the minimum of the spectral abscissa of D coincides with the largest real root of $R_2(\cdot,\tau)$.

spectral abscissa corresponds to the real part of a pair of complex conjugate characteristic roots of multiplicity 2 if $\tau > \sqrt{2}/\omega$. In the latter case, the minimum of the spectral abscissa can be given by solving a system of trigonometric-polynomial equations as shown in [37]. The numerically obtained solution is represented by a black line in Figure 5.

7. Conclusion

Motivated by the structure of conservative mechanical systems, we derived new sufficient conditions for the validity of the lowest over-order MID property for even plants. These results were obtained by exploiting the properties of Schur-Szegő composition and Laguerre polynomials, which represents a novel approach in the analysis and control of delay systems. We also note that analogous results for odd plants can be established with minor modifications of the arguments presented in Section 5. Such a structure may arise in controlled conservative mechanical systems due to the integrating behavior of the controller [38].

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Data availability statement

No new data/codes were created or analyzed in this study.

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