

STABILITY ESTIMATE FOR THE DISCRETE CALDERÓN PROBLEM WITH PARTIAL DATA

XIAOMENG ZHAO^{ID} AND GANGHUA YUAN^{*ID}

Abstract. In this paper, we focus on the analysis of discrete versions of the Calderón problem with partial boundary data in dimension $d \geq 3$. In particular, we establish logarithmic stability estimates for the discrete Calderón problem on an arbitrarily small portion of the boundary under suitable *a priori* bounds. For this end, we will use CGO solutions and derive a new discrete Carleman estimate and a key novel unique continuation estimate. Unlike the continuum case, we use a new strategy inspired by [L. Robbiano, *Asymptot. Anal.* **10** (1995) 95–115] to prove the key discrete unique continuation estimate by utilizing the new Carleman estimate with boundary observations for a discrete Laplace operator.

Mathematics Subject Classification. 35R30, 35J25, 65N06.

Received April 10, 2025. Accepted March 13, 2026.

1. INTRODUCTION

In this work, we are interested in a discrete version of the Calderón problem proposed by Calderón [1] in 1980. He asked if it is possible to determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse problem is known as the Calderón problem and has been the origin of many interesting works. An excellent survey of the history of this problem and its developments can be found in [2].

To be more precise, let \mathcal{G} be a smooth bounded domain of \mathbb{R}^d . Let us denote by $n = n(x)$ the outward normal vector at $x \in \partial\mathcal{G}$. Given κ a positive conductivity function, we define the Dirichlet-to-Neumann (DtN) map associated with the conductivity problem

$$\Lambda(\kappa) : H^{\frac{1}{2}}(\partial\mathcal{G}) \rightarrow H^{-\frac{1}{2}}(\partial\mathcal{G}), \quad \Lambda(\kappa) : g \mapsto \kappa \nabla u \cdot n, \quad (1.1)$$

where u is the unique solution to the elliptic problem

$$\operatorname{div}(\kappa \nabla u) = 0 \text{ in } \mathcal{G} \quad \text{and} \quad u = g \text{ on } \partial\mathcal{G}. \quad (1.2)$$

The Calderón problem is the following: given $\Lambda(\kappa)$, can we retrieve the conductivity κ ? Such a question is considered as the archetype of inverse problems and was originally motivated by oil prospecting [1] but has

Keywords and phrases: Discrete Calderón problem, inverse problem, stability estimate, partial boundary data, discrete Carleman estimate.

¹ KLAS, School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin 130024, China.

* Corresponding author: yuangh925@nenu.edu.cn

application in EIT, in the medical imaging method [3] and in several other inverse problems as detection of leaks in buried pipes [4].

On the other hand, it is well known that the Liouville transform allows rewriting (1.2) in the following way:

$$-\kappa^{-\frac{1}{2}} \operatorname{div} \left(\kappa \nabla (\kappa^{-\frac{1}{2}} v) \right) = -\Delta v + qv = 0, \quad \text{where } q = \frac{\Delta \sqrt{\kappa}}{\sqrt{\kappa}}. \quad (1.3)$$

Thus, Calderón's problem can be considered as: given q a regular potential function, we define the DtN map in the following way:

$$\Lambda(q) : H^{\frac{1}{2}}(\partial\mathcal{G}) \rightarrow H^{-\frac{1}{2}}(\partial\mathcal{G}), \quad \Lambda(q) : g \mapsto \nabla u \cdot n, \quad (1.4)$$

for a prescribed voltage g on $\partial\mathcal{G}$, where u is the unique solution of the problem

$$-\Delta u + qu = 0 \text{ in } \mathcal{G} \quad \text{and} \quad u = g \text{ on } \partial\mathcal{G}. \quad (1.5)$$

Hence, the Calderón problem consists in recovering the potential q from the DtN map in (1.3).

Focusing on $d \geq 3$, there are many references in the continuum case. And there are subcases that consider measurements on the whole boundary $\partial\mathcal{G}$ (full data) (see *e.g.*, [5]) or on a nonempty open subset of $\partial\mathcal{G}$ (partial data) (see *e.g.*, [6–13]). We can see the survey [14] for more references.

The discrete version of these inverse problems are interesting themselves [15]. The discrete version of the Calderón problem is related to resistor networks, where we have to determine the conductivities for an electrical network from the knowledge of the electrical potential and current in some nodes. The mathematical model for this is a weighted graph and a function defined on the vertices, where the weights of the edges are related to the conductivity of the edges, and the function is the electrical potential. The discrete version of the Calderón problem has been studied in [16] and [17]. In [16], a discrete stability estimate is obtained by using full boundary data. For the partial boundary data case, [17] prove a stability result for the discrete Calderón problem in a cube in \mathbb{R}^d , $d \geq 3$, when measurements are taken on the boundary except for one side of the cube. In this paper, we prove a stability result for the discrete Calderón problem with measurements taken just on an arbitrarily small portion of one side of the cube in \mathbb{R}^d , $d \geq 3$.

To study the discrete Calderón problems, we first derive a Carleman estimate with boundary observation and a key novel unique continuation result for the discrete equation. Then, we use the discrete CGO solution (constructed in [16]) to bound the Fourier transform of the difference of the potentials $q_1 - q_2$. This produces the logarithm in the dependence of the error of the DtN operator on the stability inequality. Since the Carleman parameter is limited by h , the bound of the Fourier transform is obtained only in a ball limited by h , unlike the continuum case.

One of the main difficulties we encounter to deduce the stability estimate is how to develop the key unique continuation estimate for the discrete Calderón problem with homogeneous boundary condition. Due to the “cusps” of the domain under consideration, we can not use the special weight functions as in [9] or [13] to derive the discrete Carleman estimate with arbitrary boundary data, which is a key tool to prove the unique continuation result for the continuum counterpart (see *e.g.*, [9, 13]). Inspired by [18], we overcome this difficulty by using a new strategy and deriving discrete Carleman estimate with other kind of weight functions. Of course, for the discrete unique continuation estimate again, as expected in view of the above discussion, the parameters in the estimate should be limited in some range depending on the mesh size.

In the study of inverse problems and controllability in the continuum setting, Carleman estimates have been one of the main tools, and the discrete setting has not been the exception. In recent years, the development of discrete or semi-discrete Carleman estimates have been used to obtain different results. For instance, the controllability of spatial discrete parabolic systems [19–24], time discrete parabolic systems [25] and the fully discrete case [26, 27]. Furthermore, for inverse problems we refer to [28, 29] for the semi-discrete wave equations, [30] for the two-dimensional semi-discrete damped wave equation, [31] for the two-dimensional semi-discrete

Schrödinger equation and [32] for the one-dimensional semi-discrete stochastic parabolic equation. For the discrete Laplacian operator, we refer to [16] and [17]. Moreover, in the study of discrete propagation of smallness see [33]. The novelty of the discrete Carleman estimate presented in this paper is that it uses a different weight function and boundary observation.

The novelty of this paper are summarized as follows:

1. Compare with [16] and [17], the inverse problem in this paper just utilizes measurements on an arbitrarily small portion of one side of the cube in \mathbb{R}^d , $d \geq 3$, much less than those in [16] and [17].
2. We use a weight function different from that in [16, 17] to derive a new discrete Carleman estimate with boundary observation.
3. In order to reduce the measurements to an arbitrarily small portions of one side of the cube in \mathbb{R}^d , $d \geq 3$, we need to prove a key novel unique continuation estimate for the discrete elliptic equation. The unique continuation result may have its own interest.
4. In order to prove the key novel unique continuation estimate for the discrete elliptic equation, we use a strategy inspired by [18] with utilizing the new Carleman estimate derived in this paper. This strategy may be used to deal with inverse problems for other kind of discrete equations.

In Section 2, we give some preliminaries of the discrete problem and present the main result. In Section 3, we present discrete calculus results and prove a Carleman estimate which is the basis of the proof of the unique continuation result, given in Section 4. In Section 5, we prove our main result concerning the stability for the discrete Calderón problem with partial data. Finally, in appendix, we focus on the technical steps to obtain the discrete Carleman estimate from Section 3.

2. SOME PRELIMINARIES AND MAIN RESULT

In this section, we introduce notations specific to the discrete problem that will be used throughout this paper.

Let us consider $N \in \mathbb{N}$, and $h := \frac{1}{N+1}$ small enough, which represents the size of our mesh. We also define the Cartesian grid of $[0, 1]^d$ as $\mathcal{K}_h := h[[0, N+1]]^d$, where $[[0, N+1]]^d := \mathbb{Z}^d \cap [0, N+1]^d$. We set $\Omega := (0, 1)^d$, $d \geq 3$, $\Omega_h := \Omega \cap \mathcal{K}_h$ and $\partial\Omega_h := \partial\Omega \cap \mathcal{K}_h$.

2.1. Meshes and operators

For any set of points $\mathcal{W}_h \subset h\mathbb{Z}^d$ or $\mathcal{W}_h \subset \tau_k(h\mathbb{Z}^d)$, we define the mesh in the direction e_k , with $\{e_k\}_{k=1}^d$ the usual base of \mathbb{R}^d , as

$$\mathcal{W}_{h,k}^* := \tau_k(\mathcal{W}_h) \cup \tau_{-k}(\mathcal{W}_h),$$

where $\tau_{\pm k}(\mathcal{W}_h) := \{x \pm \frac{h}{2}e_k | x \in \mathcal{W}_h\}$. Similarly, we define $\mathcal{W}_{h,kj}^* := (\mathcal{W}_{h,k}^*)^* = \{x \pm \frac{h}{2}e_k \pm \frac{h}{2}e_j | x \in \mathcal{W}_h\}$. Obviously, $\mathcal{W}_{h,kj}^* = \mathcal{W}_{h,jk}^*$. We will write briefly $\overline{\mathcal{W}}_{h,k}$ when $k = j$. This enables us to define the boundary points, in the e_k direction, as

$$\partial_k \mathcal{W}_h := \overline{\mathcal{W}}_{h,k} \setminus \mathcal{W}_h.$$

Moreover, for $1 \leq k \leq d$, we define

$$\overline{\mathcal{W}}_h := \bigcup_{\substack{j=1 \\ j \neq k}}^d (\tau_j^2(\overline{\mathcal{W}}_{h,k})) \cup (\tau_{-j}^2(\overline{\mathcal{W}}_{h,k})) \quad \text{and} \quad \partial \mathcal{W}_h := \overline{\mathcal{W}}_h \setminus \mathcal{W}_h.$$

For any set of points $\mathcal{W}_h \subset h\mathbb{Z}^d$ or $\mathcal{W}_h \subset \tau_k(h\mathbb{Z}^d)$, denote as $\mathcal{C}(\mathcal{W}_h)$ the set of discrete functions from \mathcal{W}_h to \mathbb{C} and define the difference and average operators in the direction e_k as

$$(D_k u)(x) := \frac{\tau_k u(x) - \tau_{-k} u(x)}{h}, \quad x \in \mathcal{W}_{h,k}^*,$$

$$(A_k u)(x) := \frac{\tau_k u(x) + \tau_{-k} u(x)}{2}, \quad x \in \mathcal{W}_{h,k}^*,$$

where $\tau_{\pm k} u(x) := u(x \pm \frac{h}{2} e_k)$. We notice that the discrete operators A_k and D_k are defined from $\mathcal{C}(\overline{\mathcal{W}}_h)$ to $\mathcal{C}(\mathcal{W}_{h,k}^*)$.

Moreover, we define the outward normal of the set \mathcal{W}_h in the direction e_k as $n_k \in \mathcal{C}(\partial_k \mathcal{W}_h)$ by

$$\forall x \in \partial_k \mathcal{W}_h, n_k(x) = \begin{cases} 1 & \text{if } \tau_{-k}(x) \in \mathcal{W}_{h,k}^* \text{ and } \tau_k(x) \notin \mathcal{W}_{h,k}^*, \\ -1 & \text{if } \tau_{-k}(x) \notin \mathcal{W}_{h,k}^* \text{ and } \tau_k(x) \in \mathcal{W}_{h,k}^*. \end{cases}$$

Then, we can define $\partial_k^+ \mathcal{W}_h := \{x \in \partial_k \mathcal{W}_h, n_k(x) = 1\}$ and $\partial_k^- \mathcal{W}_h := \{x \in \partial_k \mathcal{W}_h, n_k(x) = -1\}$. Additionally, to introduce the boundary condition we also define the trace operator in the direction k , denoted by t_r^k , for $u \in \mathcal{C}(\mathcal{W}_{h,k}^*)$ as

$$\forall x \in \partial_k \mathcal{W}_h, t_r^k(u)(x) := \begin{cases} \tau_{-k} u(x) & n_k(x) = 1, \\ \tau_k u(x) & n_k(x) = -1. \end{cases}$$

2.2. The discrete Laplace operator

Similar to [16] and [17], we define the operator Δ_h from $\mathcal{C}(\overline{\Omega}_h)$ to $\mathcal{C}(\Omega_h)$ as

$$\Delta_h u := \sum_{k=1}^d D_k(\sigma^k D_k u) \text{ in } \Omega_h, \quad (2.1)$$

where $\sigma^k > 0$ and $\sigma^k \in \mathcal{R}(\cup_{i=1}^d (\overline{\Omega}_h)_i^* \cup \overline{\Omega}_h)$. Here, $\mathcal{R}(\cup_{i=1}^d (\overline{\Omega}_h)_i^* \cup \overline{\Omega}_h)$ denotes the set of discrete functions from $\cup_{i=1}^d (\overline{\Omega}_h)_i^* \cup \overline{\Omega}_h$ to \mathbb{R} . In the sequel, we shall use the symbol σ^k for both the continuous function and its sampling on the discrete sets.

Given a set \mathcal{W}_h , for $u \in \mathcal{C}(\mathcal{W}_h)$, we define its $L_h^\infty(\mathcal{W}_h)$ norm as

$$\|u\|_{L_h^\infty(\mathcal{W}_h)} := \max_{x \in \mathcal{W}_h} \{|u(x)|\}.$$

Then, we introduce the following definition

Definition 2.1.

$$\epsilon_d(h) := \sum_{k,j} \|D_j(\sigma^k)\|_{L_h^\infty(\overline{\Omega}_{h,k})}, \quad \epsilon_a(h) := \sum_{k,j} \|A_j(\sigma^k) - 1\|_{L_h^\infty(\overline{\Omega}_{h,k})},$$

and $M(h) := \sum_k \|\sigma^k\|_{L_h^\infty((\overline{\Omega}_h)_k^*)}$.

Then we consider the following assumption for the mesh parameters.

Assumption 2.2. We suppose that

$$M_0 := \sup_{h \rightarrow 0} M(h) < \infty, \quad \epsilon_d, \epsilon_a < 1.$$

We note that if $\epsilon_a < 1$, there exist $\underline{\sigma}$ and $\bar{\sigma}$, such that,

$$0 < \underline{\sigma} \leq A_j(\sigma^k) \leq \bar{\sigma} \quad \text{in } \bar{\Omega}_{h,k}.$$

Remark 2.3. Assumption 2.2 originates from [16] which is used to assure the existence of CGO solution for the discrete Schrödinger equation (see also [17]).

And for any potential q we consider the following:

Assumption 2.4. The potential $q \in \mathcal{C}(\Omega_h)$ verifies that the system

$$\begin{aligned} -\Delta_h u + qu &= 0, & \text{in } \Omega_h, \\ u &= g, & \text{on } \partial\Omega_h, \end{aligned}$$

has only one solution for any $g \in \mathcal{C}(\partial\Omega_h)$.

Assumption 2.5. The potential $q \in \mathcal{C}(\Omega_h)$ verifies that for the system

$$\begin{aligned} -\Delta_h u + qu &= f, & \text{in } \Omega_h, \\ u &= 0, & \text{on } \partial\Omega_h, \end{aligned}$$

there exists a constant $C > 0$ independent of h such that

$$\|u\|_{H_h^1(\Omega_h)} \leq C \|f\|_{L_h^2(\Omega_h)},$$

where the norms are defined in Section 2.3.

Remark 2.6. If $q \geq 0$, then we can check that Assumption 2.5 holds by using the discrete Poincaré inequality.

Now, let us consider the definition of the discrete normal derivative.

Definition 2.7. We define the normal derivative in $\partial\Omega_h$, for any $u \in \mathcal{C}(\bar{\Omega}_h)$,

$$\partial_n u := \sum_{k=1}^d t_r^k (\sigma^k D_k u) n_k, \quad \text{on } \partial\Omega_h.$$

Moreover, *via* the discrete normal derivative, we consider the DtN operator.

Definition 2.8. We define the DtN map:

$$\Lambda_h[q](g) := \partial_n u, \quad \text{on } \partial\Omega_h,$$

where $u \in \mathcal{C}(\bar{\Omega}_h)$ is the solution of

$$\begin{aligned} -\Delta_h u + qu &= 0, & \text{in } \Omega_h, \\ u &= g, & \text{on } \partial\Omega_h. \end{aligned}$$

2.3. Discrete integral and norm

For a finite set $\mathcal{W}_h \subset h\mathbb{Z}^d$ or $\mathcal{W}_h \subset \tau_k(h\mathbb{Z}^d)$, we define the discrete integral for $u \in \mathcal{C}(\mathcal{W}_h)$ as

$$\int_{\mathcal{W}_h} u := h^d \sum_{x \in \mathcal{W}_h} u(x),$$

and the following L_h^2 inner product in $\mathcal{C}(\mathcal{W}_h)$

$$(u, v)_{\mathcal{W}_h} := \int_{\mathcal{W}_h} u \bar{v}, \quad u, v \in \mathcal{C}(\mathcal{W}_h).$$

We also define the following norms in $\mathcal{C}(\mathcal{W}_h)$:

$$\begin{aligned} \|u\|_{L_h^2(\mathcal{W}_h)}^2 &:= (u, u)_{\mathcal{W}_h}, & \|u\|_{\dot{H}_h^1(\mathcal{W}_h)}^2 &:= \sum_{k=1}^d \int_{\mathcal{W}_{h,k}^*} |D_k u|^2, \\ \|u\|_{H_h^1(\mathcal{W}_h)}^2 &:= \|u\|_{L_h^2(\overline{\mathcal{W}_h})}^2 + \|u\|_{\dot{H}_h^1(\mathcal{W}_h)}^2, & \|u\|_{\dot{H}_h^2(\mathcal{W}_h)}^2 &:= \sum_{k=1}^d \sum_{j=1}^d \int_{\mathcal{W}_{h,kj}^*} |D_k D_j u|^2, \\ \|u\|_{H_h^2(\mathcal{W}_h)}^2 &:= \|u\|_{H_h^1(\mathcal{W}_h)}^2 + \|u\|_{\dot{H}_h^2(\mathcal{W}_h)}^2. \end{aligned}$$

Next, we introduce the discrete integration on the boundary for a subset Γ_h of $\partial\mathcal{W}_h$ as

$$\int_{\Gamma_h} u := h^{d-1} \sum_{x \in \Gamma_h} u(x),$$

where $u \in \mathcal{C}(\Gamma_h)$ and $\Gamma_h \subset \partial\mathcal{W}_h$. Additionally, we introduce a norm on the boundary:

$$|f|_{L_h^2(\Gamma_h)} := \left(h^{d-1} \sum_{x \in \Gamma_h} |f(x)|^2 \right)^{1/2}, \quad \text{for any } \Gamma_h \subset \partial\mathcal{W}_h.$$

Then, as in [16], we also define the Sobolev norms for any trace function g in $\mathcal{C}(\partial\Omega_h)$ as

$$\begin{aligned} |g|_{H_h^{1/2}(\partial\Omega_h)} &:= \min_{u|_{\partial\Omega_h} = g} \|u\|_{H_h^1(\Omega_h)}, \\ |g|_{H_h^{-1/2}(\partial\Omega_h)} &:= \max_{|f|_{H_h^{1/2}(\partial\Omega_h)} = 1} \int_{\partial\Omega_h} f g. \end{aligned}$$

2.4. Main result

Before stating our result precisely, let us introduce the discrete H^r -norms for $r \in \mathbb{R}$. For $h > 0$, set $\hat{\mathcal{K}}_h = [[0, N]]^d := \mathbb{Z}^d \cap [0, N]^d$ and define the discrete Fourier transform of a function $u \in \mathcal{C}(\Omega_h)$, denoted by $\mathcal{F}(u)$ in $\mathcal{C}(\hat{\mathcal{K}}_h)$ as

$$\mathcal{F}(u)(\xi) := h^d \sum_{x \in \Omega_h} u(x) e^{-2\pi i(x \cdot \xi)}, \quad \forall \xi \in \hat{\mathcal{K}}_h.$$

Then, we define the Sobolev-discrete norm H^r , for any $r \in \mathbb{R}$, by

$$|u|_{H_h^r(\Omega_h)} := \left(\sum_{\xi \in \mathcal{K}_h} |\mathcal{F}(u)(\xi)|^2 (1 + |\xi|^2)^r \right)^{1/2}.$$

In [16], the authors mentioned that with this definition, the celebrated Plancherel formula states that the norm $|u|_{H^0(\Omega_h)}$ coincides with $\|u\|_{L_h^2(\Omega_h)}$. Moreover, as in the continuum case, it is classical to show that $\|\cdot\|_{H_h^1(\Omega_h)}$, $\|\cdot\|_{H_h^2(\Omega_h)}$ defined in Section 2.3 and $|\cdot|_{H_h^1(\Omega_h)}$, $|\cdot|_{H_h^2(\Omega_h)}$ are equivalent independently of h respectively.

Now, we present the main result in this paper.

Theorem 2.9. *Let $M > 0$. Let Γ be a non-empty open subset of $\partial\Omega$ satisfying $\Gamma_h := (\Gamma \cap \partial\Omega_h) \subset \partial_i^\pm \Omega_h$, $1 \leq i \leq d$. Suppose that Assumption 2.2 is satisfied. Let $q_1, q_2 \in \mathcal{C}(\Omega_h)$ be two potentials satisfying Assumption 2.4, Assumption 2.5 and $\|q_j\|_{L_h^\infty(\Omega_h)} \leq M$, $j = 1, 2$, $q_1 = q_2$ in $\mathcal{O}_h = \mathcal{O} \cap \Omega_h$, where $\mathcal{O} \subset \Omega$ is a neighborhood of $\partial\Omega$. Set*

$$\|\Lambda_h[q_1] - \Lambda_h[q_2]\|_{\mathcal{L}_h(\Gamma_h)} := \max_{|g|_{H^{1/2}(\partial\Omega_h)}=1} |(\Lambda_h[q_1] - \Lambda_h[q_2])(g)|_{H^{-1/2}(\Gamma_h)}. \quad (2.2)$$

Then there exists a constant $h_0 > 0$ depending on M , such that for $r > 0$ there exists a constant $C > 0$ such that

$$|q_1 - q_2|_{H_h^{-r}(\Omega_h)} \leq C \max \left\{ \epsilon_d^{2\alpha}, \epsilon_a^\alpha, h^\alpha, \left| \ln \left(\|\Lambda_h[q_1] - \Lambda_h[q_2]\|_{\mathcal{L}_h(\Gamma_h)} \right) \right|^{-2\alpha} \right\}, \quad (2.3)$$

for all $0 < h < h_0$, where $\alpha := \frac{r}{2r+d}$.

Remark 2.10. The stability estimate (2.3) is influenced by $\epsilon_d, \epsilon_a, h$ and the error in the DtN maps. The term $|\ln(\text{error})|^{-2\alpha}$ on the right-hand side of (2.3) is consistent with the stability estimate for the continuum case (see e.g., [34]). And the terms $\epsilon_d^{2\alpha}, \epsilon_a^\alpha, h^\alpha$ stem from the discrete setting of the problem.

On the other hand, unlike the stability results for the continuum case, Theorem 2.9 does not yield uniqueness. Indeed, if $\Lambda_h[q_1] = \Lambda_h[q_2]$, then for any $r > 0$, we have

$$|q_1 - q_2|_{H_h^{-r}(\Omega_h)} \leq C \max \left\{ \epsilon_d^{2\alpha}, \epsilon_a^\alpha, h^\alpha \right\}. \quad (2.4)$$

Consequently, given $h > 0$, this does not guarantee uniqueness, but rather some kind of asymptotic uniqueness as $h \rightarrow 0$, since the right-hand side of (2.4) goes to zero as $h \rightarrow 0$.

Remark 2.11. In [16], a discrete stability estimate is obtained by using full boundary data. For the partial boundary data case, [17] prove a stability result for the Calderón problem for a suitable class of potentials in a cube in \mathbb{R}^d , $d \geq 3$, when measurements are taken on the boundary except for one side of the cube. To the best of our knowledge, our result is the first stability estimate for the discrete Calderón problem with measurements taken just on arbitrarily small portions of one side of the cube in \mathbb{R}^d , $d \geq 3$ (see Figure 1).

Remark 2.12. The stability result is in line with the known partial data theory in the continuum case. Moreover, the proof in principle can be viewed as a translation of a typical continuum proof into the discrete case. The typical proof uses first unique continuation to pass to the full data case and then employs the theory in the full data case. For a recent description of the continuum case, with a good overview of the literature, see Section 6 of [35]. The main differences between the discrete case and the continuum case are as follows:

1. The ‘‘Calculus’’ for discrete case and continuum case are different (see Sect. 3.1).
2. The stabilities between the discrete case and the continuum case are different (see Rem. 2.10).

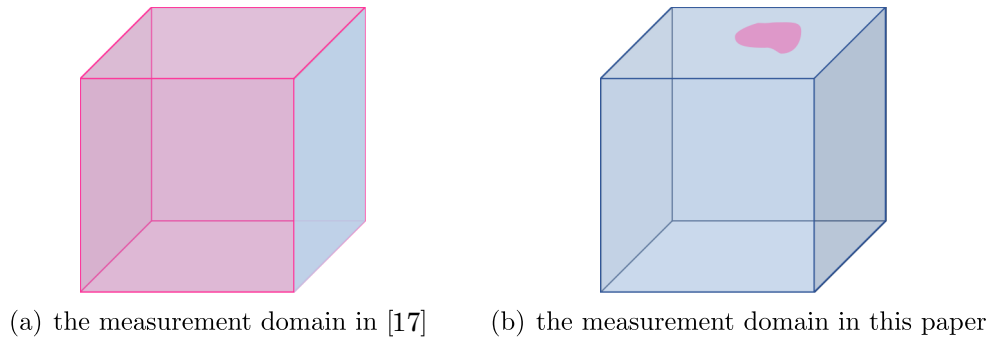


FIGURE 1. The pink area denotes the measurement domain.

3. The discrete domain is in a cube with non-smooth boundary, while the continuum counterpart is often a C^2 -smooth domain (see *e.g.*, [9, 13]). So the method used to prove the unique continuation result in this paper is different from the methods used in [9, 13] (see Sect. 4).

Remark 2.13. There are some limitations of the current result. For example, we need the Dirichlet data (*i.e.* the input data) on the full boundary to obtain the stability. It is very interesting to prove a stability with the Dirichlet data just support on an arbitrary portion of the discrete boundary. Another limitation of the result is that the potential q should be known in a neighborhood of the boundary. To relaxing this assumption will be very challenging. A more challenging and interesting problem is to prove that the sequence q_h (h denotes the size of the discrete mesh) of the discrete potentials converges (in a suitable topology) towards q when the discrete DtN map $\Lambda_h(q_h)$ converges towards $\Lambda(q)$ as $h \rightarrow 0$ (in a suitable topology) (see *e.g.*, [29]).

3. DISCRETE CARLEMAN ESTIMATE

In this section, we establish a discrete Carleman estimate with boundary observation. Before the calculus, we introduce some formulas which will be used in the following article.

3.1. Discrete calculus formulas

First, we provide the product rule for the average and the difference operators.

Lemma 3.1 ([16], Lem. 2.1). *As operators from $\mathcal{C}(\overline{\mathcal{W}}_h)$ to $\mathcal{C}(\mathcal{W}_{h,k_j}^*)$, we have the following identities:*

$$A_k A_j = A_j A_k, \quad A_k D_j = D_j A_k, \quad D_k D_j = D_j D_k.$$

Moreover, for any $u, v \in \mathcal{C}(\overline{\mathcal{W}}_h)$, we have the following identities on $\mathcal{W}_{h,k}^*$:

$$\begin{aligned} D_k(uv) &= D_k u A_k v + A_k u D_k v, \\ A_k(uv) &= A_k u A_k v + \frac{h^2}{4} D_k u D_k v. \end{aligned}$$

As a direct consequence of Lemma 3.1, we have the following Corollary.

Corollary 3.2. *For $u \in \mathcal{C}(\overline{\mathcal{W}}_h)$,*

$$A_k(u^2) = (A_k u)^2 + \frac{h^2}{4} (D_k u)^2, \quad D_k(u^2) = 2D_k u A_k u, \quad \text{on } \mathcal{W}_{h,k}^*.$$

In particular, for all $u \in \mathcal{C}(\overline{\mathcal{W}_h})$,

$$(A_k u)^2 \leq A_k(u^2), \quad (D_k u)^2 \leq \frac{4}{h^2} A_k(u^2), \quad \text{on } \mathcal{W}_{h,k}^*.$$

Then we state the discrete integration by parts for the difference and average operators.

Lemma 3.3 ([17], Lem. 2.2). *Given a mesh \mathcal{W}_h defined as above, for any $v \in \mathcal{C}(\mathcal{W}_{h,k}^*)$, $u \in \mathcal{C}(\overline{\mathcal{W}_{h,k}})$, we have*

$$\int_{\mathcal{W}_h} u D_k v = - \int_{\mathcal{W}_{h,k}^*} v D_k u + \int_{\partial_k \mathcal{W}_h} u t_r^k(v) n_k, \quad (3.1)$$

$$\int_{\mathcal{W}_h} u A_k v = \int_{\mathcal{W}_{h,k}^*} v A_k u - \frac{h}{2} \int_{\partial_k \mathcal{W}_h} u t_r^k(v). \quad (3.2)$$

3.2. Calculus results related to the weight functions

Our Carleman weight function is defined as $e^{s\varphi}$ for $s \geq 1$, with $\varphi = e^{-\lambda\psi}$, $\lambda \geq 1$, where ψ is a continuous function. In this paper, we shall use the same notation for the sample of the continuous function on the discrete sets. We suppose that the function ψ satisfies the following properties.

Assumption 3.4. Let $\tilde{\Omega}$ be an open and connected bounded neighborhood of $\overline{\Omega}$ in \mathbb{R}^d with smooth boundary. The function ψ is in $C^\infty(\tilde{\Omega})$, i.e. infinitely differentiable function, and satisfies

$$|\nabla\psi| > 0 \text{ and } \psi > 0 \text{ in } \tilde{\Omega}.$$

Remark 3.5. We enlarge the open set Ω to a larger open set $\tilde{\Omega}$ as this will guarantee that after performing some discrete operations in Ω , ψ is still well-defined.

In the sequel, C will denote a generic constant independent of h , whose value may change from line to line. As usual, we shall denote by $O(1)$ a bounded function and by $O_\lambda(1)$ a bounded function once λ is fixed. We denote by $O_\lambda(sh)$ the functions that verify $\|O_\lambda(sh)\|_{L_h^\infty(\Omega_h)} \leq C_\lambda sh$ for some constant C_λ depending on λ .

We say that α is a multi-index if $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and for $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ we write:

$$|\alpha| = \alpha_1 + \dots + \alpha_d, \quad y^\alpha = y_1^{\alpha_1} \dots y_d^{\alpha_d}.$$

Moreover, we denote by ∂ the differential operator in the continuum case. For $f = f(x_1, \dots, x_d) \in C^{|\alpha|}(\mathbb{R}^d)$, we write

$$\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} f,$$

where we abbreviate $\partial_{x_i}^{\alpha_i}$ as $\partial_i^{\alpha_i}$, $i = 1, \dots, d$.

We now provide some technical lemmas related to discrete operations performed on the Carleman weight functions that is of the form $e^{s\varphi}$ with $\varphi = e^{-\lambda\psi}$, $\psi \in C^\infty$. For concision, we set $r = e^{s\varphi}$ and $\rho = r^{-1}$. The positive parameters s and h will be large and small respectively and we are particularly interested in the dependence on s , h and λ in the following basic estimates.

We assume $s \geq 1$ and $\lambda \geq 1$. We shall use multi-index of the form $\alpha = (\alpha_1, \dots, \alpha_d)$. The proofs can be found in [20, 22, 36].

Lemma 3.6 ([20], Lem. 2.7). *Let α and β be multi-indices. We have*

$$\begin{aligned} \partial^\beta (r \partial^\alpha \rho) &= |\alpha|^{|\beta|} (-s\varphi)^{|\alpha|} \lambda^{|\alpha+\beta|} (-\nabla\psi)^{\alpha+\beta} \\ &\quad + |\alpha|^{|\beta|} (s\varphi)^{|\alpha|} \lambda^{|\alpha+\beta|-1} O(1) + s^{|\alpha|-1} |\alpha| (|\alpha| - 1) O_\lambda(1) \\ &= O_\lambda(s^{|\alpha|}). \end{aligned}$$

The same expressions hold with r and ρ interchanged and with s changed into $-s$.

Lemma 3.7 ([20], Cor. 2.8). *Let α, β and δ be multi-indices. We have*

$$\begin{aligned} \partial^\delta (r^2 (\partial^\alpha \rho) \partial^\beta \rho) &= |\alpha + \beta|^{\delta} (-s\varphi)^{|\alpha+\beta|} \lambda^{|\alpha+\beta+\delta|} (-\nabla \psi)^{\alpha+\beta+\delta} \\ &\quad + |\delta| |\alpha + \beta| (s\varphi)^{|\alpha+\beta|} \lambda^{|\alpha+\beta+\delta|-1} O(1) \\ &\quad + s^{|\alpha+\beta|-1} (|\alpha|(|\alpha| - 1) + |\beta|(|\beta| - 1)) O_\lambda(1) \\ &= O_\lambda(s^{|\alpha+\beta|}). \end{aligned}$$

Lemma 3.8 ([20], Prop. 2.9). *Let α be a multi-index. Let $k, j = 1, \dots, d$, provided $sh \leq \mathcal{R}$, we have*

$$\begin{aligned} r A_k^p \rho &= 1 + O_{\lambda, \mathcal{R}}((sh)^2) = O_{\lambda, \mathcal{R}}(1), \quad p = 1, 2, \\ r A_k^p D_k \rho &= r \partial_k \rho + s O_{\lambda, \mathcal{R}}((sh)^2) = s O_{\lambda, \mathcal{R}}(1), \quad p = 0, 1, \\ r D_k^{p_k} D_j^{p_j} \rho &= r \partial_k^{p_k} \partial_j^{p_j} \rho + s^2 O_{\lambda, \mathcal{R}}((sh)^2) = s^2 O_{\lambda, \mathcal{R}}(1), \quad p_k + p_j \leq 2. \end{aligned}$$

The same estimates hold with ρ and r interchanged.

Lemma 3.9 ([20], Prop. 2.12). *Let $p \in \mathbb{N}$, $k, j = 1, \dots, d$, provided $sh \leq \mathcal{R}$, we have*

$$\begin{aligned} D_k^p (r D_k^2 \rho) &= s^2 O_{\lambda, \mathcal{R}}(1), \\ D_k^p (r A_k^2 \rho) &= O_{\lambda, \mathcal{R}}((sh)^2). \end{aligned}$$

The same estimates hold with r and ρ interchanged.

Lemma 3.10 ([20], Prop. 2.13). *Let α be a multi-index, $k, j = 1, \dots, d$ and $p_k, p'_k, p_j, p'_j \in \mathbb{N}$. For $p_k + p_j \leq 2$, provided $sh \leq \mathcal{R}$ we have*

$$\begin{aligned} A_k^{p'_k} A_j^{p'_j} D_k^{p_k} D_j^{p_j} (r^2 A_k D_k \rho D_j^2 \rho) &= \partial_k^{p_k} \partial_j^{p_j} (r^2 (\partial_k \rho) \partial_j^2 \rho) + s^3 O_{\lambda, \mathcal{R}}((sh)^2) = s^3 O_{\lambda, \mathcal{R}}(1), \\ A_k^{p'_k} A_j^{p'_j} D_k^{p_k} D_j^{p_j} (r^2 A_j D_j \rho A_k^2 \rho) &= \partial_k^{p_k} \partial_j^{p_j} (r \partial_j \rho) + s O_{\lambda, \mathcal{R}}((sh)^2) = s O_{\lambda, \mathcal{R}}(1). \end{aligned}$$

3.3. Discrete Carleman estimate

We introduce the following notations related to the coefficients $\sigma^k, k = 1, \dots, d$ for any function f ,

$$\nabla_\sigma f = (\sqrt{\sigma^1} \partial_1 f, \dots, \sqrt{\sigma^d} \partial_d f)^T, \quad \Delta_\sigma f = \sum_{k=1}^d \sigma^k \partial_k^2 f.$$

The enlarged neighborhood $\tilde{\Omega}$ of Ω introduced in Assumption 3.4 allows us to apply multiple discrete operators such as D_k and A_k on the weight functions.

We are now in position to state and prove the following discrete Carleman estimate.

Proposition 3.11 (Discrete Carleman estimate). *Suppose that Assumption 2.2 is satisfied. Let ψ be a function verifying Assumption 3.4 and let $q \in C(\Omega_h)$ satisfying $\|q\|_{L_h^\infty(\Omega_h)} \leq M$. Then there exists a constant $\lambda_0 \geq 1$ such that for each $\lambda \geq \lambda_0$, there exist $s_0(\lambda) \geq 1$, $0 < h_0 < 1$, $\varepsilon_0 > 0$ and $C = C(\varepsilon_0, s_0, \lambda, M)$ independent of $h > 0$*

such that

$$\begin{aligned} s^3 \|e^{s\varphi} u\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\Omega_{h,k}^*} e^{2s\varphi} |D_k u|^2 \\ \leq C \left(\|e^{s\varphi} (-\Delta_h + q) u\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k (e^{2s\varphi}) |\partial_n u|^2 \right), \end{aligned} \quad (3.3)$$

for all $s \geq s_0(\lambda)$, $0 < h \leq h_0$, $sh \leq \varepsilon_0$ and $u \in \mathcal{C}(\overline{\Omega})$ satisfying $u|_{\partial\Omega_h} = 0$.

Remark 3.12. The parameter s is limited by the condition $sh \leq \varepsilon_0$. This restriction on the range of the Carleman parameter appears in discrete Carleman estimates (see *e.g.*, [16, 20, 22, 28, 29, 36]). This is related to the fact that the conjugation of discrete operators with the exponential weight behaves as in the continuum case only for sh small enough, since for instance

$$e^{s\varphi} D_k (e^{-s\varphi}) \simeq -s \partial_k \varphi \quad \text{only for } sh \text{ small enough.}$$

Remark 3.13. Several recent works have been concerned with discrete and semi-discrete Carleman estimates for second-order differential operators (see *e.g.*, [16, 17, 20, 22, 28, 29, 36, 37]). The main differences between our estimate and the existing results are that we choose a different weight function and a more general discretization for the Laplacian. At the same time, we consider boundary observation.

Proof of Proposition 3.11. We make the change of variable $v = ru$. Our first task is to obtain an expression for $\Delta_{h,\varphi} v := r \Delta_h(\rho v)$ with the change of variable proposed. By using Lemma 3.1, we have

$$\begin{aligned} \Delta_{h,\varphi} v = \sum_{k=1}^d r (D_k(\sigma^k D_k \rho) A_k^2 v + A_k(\sigma^k D_k \rho) D_k A_k v \\ + D_k A_k \rho A_k(\sigma^k D_k v) + A_k^2 \rho D_k(\sigma^k D_k v)), \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} A_k(\sigma^k D_k v) &= A_k \sigma^k A_k D_k v + \frac{h}{4} D_k \sigma^k (\tau_k D_k v - \tau_{-k} D_k v), \\ A_k(\sigma^k D_k \rho) &= A_k \sigma^k A_k D_k \rho + \frac{h^2}{4} D_k \sigma^k D_k^2 \rho, \\ D_k(\sigma^k D_k \rho) &= D_k \sigma^k A_k D_k \rho + A_k \sigma^k D_k^2 \rho. \end{aligned}$$

Recalling that, in each term, σ^k is the sampling of the given continuous coefficient σ^k on the discrete sets, so that we can deduce from Assumption 2.2 that $\|A_k \sigma^k - \sigma^k\|_\infty \leq Ch$. Then, we set

$$\begin{aligned} A_1 v &= \sum_{k=1}^d r A_k^2 \rho D_k(\sigma^k D_k v), & A_2 v &= \sum_{k=1}^d \sigma^k r D_k^2 \rho A_k^2 v, \\ B_1 v &= 2 \sum_{k=1}^d \sigma^k r A_k D_k \rho A_k D_k v, & B_2 v &= -2s(\Delta_\sigma \varphi) v \end{aligned} \quad (3.5)$$

and

$$\begin{aligned}
g = & \Delta_{h,\varphi} v - \sum_{k=1}^d \frac{h}{4} r D_k A_k \rho D_k \sigma^k (\tau_k D_k v - \tau_{-k} D_k v) \\
& - \sum_{k=1}^d \frac{h^2}{4} r D_k^2 \rho D_k \sigma^k D_k A_k v - h \sum_{k=1}^d O(1) r A_k D_k \rho A_k D_k v \\
& - \sum_{k=1}^d (r D_k A_k \rho D_k \sigma^k + h O(1) r D_k^2 \rho) A_k^2 v - 2s (\Delta_\sigma \varphi) v.
\end{aligned}$$

As in [38], we set

$$Av = A_1 v + A_2 v, \quad Bv = B_1 v + B_2 v.$$

An explanation for the introduction of this additional term $B_2 v$ is provided in [39]. Thus, equation (3.4) reads $Av + Bv = g$ and we have

$$\|Av\|_{L_h^2(\Omega_h)}^2 + \|Bv\|_{L_h^2(\Omega_h)}^2 + 2\operatorname{Re}(Av, Bv)_{\Omega_h} = \|g\|_{L_h^2(\Omega_h)}^2. \quad (3.6)$$

We shall need the following estimation of $\|g\|_{L_h^2(\Omega_h)}^2$. The proof can be found in Appendix A.

Lemma 3.14. *For $sh \leq \mathcal{R}$, we have*

$$\|g\|_{L_h^2(\Omega_h)}^2 \leq C_{\lambda, \mathcal{R}} \left(\|\Delta_{h,\varphi} v\|_{L_h^2(\Omega_h)}^2 + s^2 \|v\|_{L_h^2(\Omega_h)}^2 + (sh)^2 \|v\|_{\tilde{H}_h^1(\Omega_h)}^2 \right). \quad (3.7)$$

Now, we shall estimate the inner-product

$$\operatorname{Re}(Av, Bv)_{\Omega_h} = \sum_{i,j=1}^2 \operatorname{Re}(A_i v, B_j v)_{\Omega_h}. \quad (3.8)$$

To estimate each term of (3.8), we obtain the following results Lemma 3.15-Lemma 3.18. To make the paper more concise, we put the proofs of Lemma 3.15-Lemma 3.18 to Appendix A.

Lemma 3.15. *For $sh \leq \mathcal{R}$, we have*

$$\operatorname{Re}(A_1 v, B_1 v)_{\Omega_h} \geq - \sum_{k=1}^d \int_{\Omega_{h,k}^*} s \lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_k v|^2 + Y_1 + Z_1,$$

where

$$\begin{aligned}
Y_1 = & \sum_{k=1}^d \int_{\partial_k \Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k (|D_k v|^2) n_k - h \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k (D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2) \\
& - \frac{h^2}{2} \sum_{k=1}^d \int_{\partial_k \Omega_h} D_k^2 (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k (|D_k v|^2) n_k,
\end{aligned}$$

with $\gamma_1^k = rA_k^2\rho$, $\beta_1^k = rA_kD_k\rho$ and

$$\begin{aligned} Z_1 = & - \sum_{k=1}^d \int_{\Omega_{h,k}^*} (s\lambda\varphi|O(1)| + s|O_{\lambda,R}(sh)|)|D_kv|^2 \\ & - \sum_{k=1}^d \int_{\Omega_h} h^2(s\lambda\varphi|O(1)| + s|O_{\lambda,R}(sh)|)|D_k^2v|^2 \\ & - \sum_{k=1}^d \int_{\Omega_h} (s\lambda\varphi|O(1)| + s|O_{\lambda,R}(sh)|)|A_kD_kv|^2 \\ & - \sum_{\substack{j,k=1 \\ k \neq j}}^d \int_{\Omega_{h,kj}^*} h^2(s\lambda\varphi|O(1)| + s|O_{\lambda,\mathcal{R}}(sh)|)|D_kD_jv|^2. \end{aligned}$$

Lemma 3.16. For $sh \leq \mathcal{R}$, we have

$$\operatorname{Re}(A_2v, B_1v)_{\Omega_h} \geq 3 \int_{\Omega_h} s^3\lambda^4\varphi^3|\nabla_\sigma\psi|^4|v|^2 + Y_2 + Z_2,$$

where

$$Y_2 = \sum_{k=1}^d \int_{\partial_k\Omega_h} \gamma_2^k\beta_1^k(\sigma^k)^2t_r^k(|A_kv|^2)n_k,$$

with $\gamma_2^k = rD_k^2\rho$, $\beta_1^k = rA_kD_k\rho$ and

$$\begin{aligned} Z_2 = & - \int_{\Omega_h} ((s\lambda\varphi)^3|O(1)| + s^2|O_{\lambda,\mathcal{R}}(1)| + s^3|O_{\lambda,\mathcal{R}}(sh)|)|v|^2 \\ & - \sum_{k=1}^d \int_{\Omega_h} s|O_{\lambda,\mathcal{R}}(sh)||D_kA_kv|^2 - \sum_{k=1}^d \int_{\Omega_{h,k}^*} s|O_{\lambda,\mathcal{R}}((sh)^2)||D_kv|^2. \end{aligned}$$

Lemma 3.17. For $sh \leq \mathcal{R}$, we have

$$\operatorname{Re}(A_1v, B_2v)_{\Omega_h} \geq 2 \sum_{k=1}^d \int_{\Omega_{h,k}^*} s\lambda^2\varphi\sigma^k|\nabla_\sigma\psi|^2|D_kv|^2 + Z_3,$$

where

$$Z_3 = - \int_{\Omega_h} s^2|O_{\lambda,\mathcal{R}}(1)||v|^2 - \sum_{k=1}^d \int_{\Omega_{h,k}^*} (s\lambda\varphi|O(1)| + s|O_{\lambda,\mathcal{R}}(sh)|)|D_kv|^2.$$

Lemma 3.18. For $sh \leq \mathcal{R}$, we have

$$\operatorname{Re}(A_2v, B_2v)_{\Omega_h} \geq -2 \int_{\Omega_h} s^3\lambda^4\varphi^3|\nabla_\sigma\psi|^4|v|^2 + Z_4,$$

where

$$Z_4 = - \int_{\Omega_h} ((s\lambda\varphi)^3 |O(1)| + s^2 |O_\lambda(1)| + s^3 |O_{\lambda, \mathcal{R}}((sh)^2)|) |v|^2 - \sum_{k=1}^d \int_{\Omega_{h,k}^*} s |O_{\lambda, \mathcal{R}}(sh)| |D_k v|^2.$$

Combining the estimates in Lemma 3.14-Lemma 3.18 with (3.6), for $sh \leq \mathcal{R}$, we have

$$\begin{aligned} & \int_{\Omega_h} 2s^3 \lambda^4 \varphi^3 |\nabla_\sigma \psi|^4 |v|^2 + \sum_{k=1}^d \int_{\Omega_{h,k}^*} 2s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_k v|^2 + 2(Y_1 + Y_2 + Z) \\ & \leq C_{\lambda, \mathcal{R}} \left(\|\Delta_{h, \varphi} v\|_{L_h^2(\Omega_h)}^2 + s^2 \|v\|_{L_h^2(\Omega_h)}^2 + (sh)^2 \|v\|_{\dot{H}_h^1(\Omega_h)}^2 \right), \end{aligned} \quad (3.9)$$

where $Z = Z_1 + Z_2 + Z_3 + Z_4$.

Lemma 3.19. *For $sh \leq \mathcal{R}$, we have*

$$\begin{aligned} \sum_{k=1}^d \int_{\Omega_{h,k}^*} s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_k v|^2 & \geq \frac{h^2}{4d} \sum_{\substack{j,k=1 \\ j \neq k}}^d \int_{\Omega_{h,kj}^*} s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_j D_k v|^2 \\ & \quad + \frac{1}{d} \sum_{k=1}^d \int_{\Omega_h} s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |A_k D_k v|^2 \\ & \quad + \frac{h^2}{4d} \sum_{k=1}^d \int_{\Omega_h} s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_k^2 v|^2 + Y_3 + Z_5, \end{aligned}$$

where

$$Y_3 = \frac{h^2}{4} \sum_{k=1}^d \int_{\partial_k \Omega_h} D_k (\varphi \sigma^k |\nabla_\sigma \psi|^2) t_r^k (|D_k v|^2) n_k,$$

and

$$\begin{aligned} Z_5 & = - \sum_{\substack{j,k=1 \\ j \neq k}}^d \int_{\Omega_{h,kj}^*} h^2 |O_{\lambda, \mathcal{R}}(sh)| |D_j D_k v|^2 - \sum_{k=1}^d \int_{\Omega_h} |O_{\lambda, \mathcal{R}}(sh)| |A_k D_k v|^2 \\ & \quad - \sum_{k=1}^d \int_{\Omega_h} h^2 |O_{\lambda, \mathcal{R}}(sh)| |D_k^2 v|^2 - \sum_{k=1}^d \int_{\Omega_{h,k}^*} |O_{\lambda, \mathcal{R}}(sh)| |D_k v|^2. \end{aligned}$$

The proof of Lemma 3.19 can be found in Appendix A.

Combining (3.9) with Lemma 3.19, there exists a constant $\lambda_0 \geq 1$ sufficiently large, for $\lambda \geq \lambda_0$, there exist $s_0(\lambda) \geq 1$ sufficiently large, $\varepsilon_0, 0 < h_0 < 1$ sufficiently small and $\tilde{C}_{\lambda, \varepsilon_0, s_0}, \tilde{C}_{\lambda, \varepsilon_0, s_0} > 0$, such that for $s \geq s_0(\lambda)$, $0 < h \leq h_0$ and $sh \leq \varepsilon_0$, we have

$$\tilde{C}_{\lambda, \varepsilon_0, s_0} \left(s^3 \|v\|_{L_h^2(\Omega_h)}^2 + s \|v\|_{\dot{H}_h^1(\Omega_h)}^2 \right) \leq \tilde{C}_{\lambda, \varepsilon_0, s_0} \|\Delta_{h, \varphi} v\|_{L_h^2(\Omega_h)}^2 - 2(Y_1 + Y_2) - Y_3. \quad (3.10)$$

Now we estimate the boundary terms. Since $v = 0$ on $\partial_k \Omega_h$, we have $t_r^k(A_k v) = -\frac{h}{2} t_r^k(D_k v) n_k$ for all $x \in \partial_k \Omega_h$, which leads to $t_r^k(|A_k v|^2) = \frac{h^2}{4} t_r^k(|D_k v|^2)$ by virtue of $t_r^k(|A_k v|^2) = |t_r^k(A_k v)|^2$. Thus, we have

$$\begin{aligned} -2(Y_1 + Y_2) - Y_3 &= \sum_{k=1}^d \int_{\partial_k \Omega_h} \left(-2\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k(|D_k v|^2) n_k + h t_r^k(D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k)) t_r^k(|D_k v|^2) \right. \\ &\quad \left. - \frac{h^2}{4} D_k(\varphi \sigma^k |\nabla_\sigma \psi|^2) t_r^k(|D_k v|^2) n_k - \frac{h^2}{2} \gamma_2^k \beta_1^k (\sigma^k)^2 t_r^k(|D_k v|^2) n_k \right. \\ &\quad \left. + h^2 D_k^2(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k(|D_k v|^2) n_k \right), \end{aligned}$$

where $\gamma_1^k = r A_k^2 \rho$, $\gamma_2^k = r D_k^2 \rho$, $\beta_1^k = r A_k D_k \rho$. Similar to Lemma A.1, we have following results.

$$\begin{aligned} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k &= s \lambda \varphi(\sigma^k)^2 \partial_k \psi + s O_{\lambda, \varepsilon_0}(sh), \\ D_k^2(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) &= s O_{\lambda, \varepsilon_0}(1), \quad t_r^k(D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k)) = s O_{\lambda, \varepsilon_0}(1), \\ D_k(\varphi \sigma^k |\nabla_\sigma \psi|^2) &= O_\lambda(1), \quad \gamma_2^k \beta_1^k (\sigma^k)^2 = s^3 O_{\lambda, \varepsilon_0}(1). \end{aligned}$$

Thus, we have

$$\begin{aligned} -2(Y_1 + Y_2) - Y_3 &\leq \sum_{k=1}^d \int_{\partial_k \Omega_h} \left(2s \lambda \varphi(\sigma^k)^2 |\partial_k \psi| + s |O_{\lambda, \varepsilon_0}(sh)| \right. \\ &\quad \left. + h^2 |O_\lambda(1)| + h |O_{\lambda, \varepsilon_0}(sh)| \right) t_r^k(|D_k v|^2). \end{aligned} \quad (3.11)$$

Combining (3.10) with (3.11), there exists constant $C > 0$ such that

$$s^3 \|v\|_{L_h^2(\Omega_h)}^2 + s \|v\|_{\tilde{H}_h^1(\Omega_h)}^2 \leq C \left(\|\Delta_{h, \varphi} v\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k(|D_k v|^2) \right), \quad (3.12)$$

for $s \geq s_0(\lambda)$, $0 < h \leq h_0$ and $sh \leq \varepsilon_0$. Finally, we need to express all the terms in the estimate above in terms of the original function u . To this end, we need the following Lemma.

Lemma 3.20. *For $sh \leq \mathcal{R}$, we have*

$$\sum_{k=1}^d \int_{\Omega_{h,k}^*} r^2 |D_k u|^2 \leq C_{\lambda, \mathcal{R}} \left(s^2 \|v\|_{L_h^2(\Omega_h)}^2 + \|v\|_{\tilde{H}_h^1(\Omega_h)}^2 \right), \quad (3.13)$$

$$\sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k(|D_k v|^2) \leq C_{\lambda, \mathcal{R}} \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k(r^2) |\partial_n u|^2. \quad (3.14)$$

The proof of Lemma 3.20 can be found in Appendix A.

Recalling that $q \in C(\Omega_h)$ satisfying $\|q\|_{L_h^\infty(\Omega_h)} \leq M$ and combining (3.12) with Lemma 3.20, we obtain

$$\begin{aligned} s^3 \|ru\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\Omega_{h,k}^*} r^2 |D_k u|^2 \\ \leq C_{\lambda, \varepsilon_0, s_0, M} \left(\|r(-\Delta_h u + qu)\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k(r^2) |\partial_n u|^2 \right), \end{aligned}$$

for $s \geq s_0(\lambda)$, $0 < h \leq h_0$ and $sh \leq \varepsilon_0$.

Thus the proof of Proposition 3.11 is complete. \square

4. A UNIQUE CONTINUATION ESTIMATE

In this section, we prove a unique continuation result for the discrete Schrödinger equation with homogeneous boundary condition which will be a crucial step in the proof of the main result. Our argument inspired by the idea of [18], see also [40] and [41]. By applying the discrete Carleman estimate with full boundary observation, we show a unique continuation result from boundary data on an arbitrary small part $\Gamma_h \subset \partial_i^\pm \Omega_h$, $i=1, \dots, d$.

Before presenting the result, we introduce some notations. Let $\mathcal{O} \subset \Omega$ be an arbitrary neighborhood of $\partial\Omega$. We choose $\varrho, \varrho_1, \varrho_2 > 0$ such that

$$\omega(8\varrho) := \{x \in \Omega, \text{dist}(x, \partial\Omega) < 8\varrho\} \subset \mathcal{O} \quad (4.1)$$

and

$$\omega(\varrho_1, \varrho_2) := \{x \in \Omega, \varrho_1 < \text{dist}(x, \partial\Omega) < \varrho_2\} \subset \mathcal{O}, \quad \varrho_1 < \varrho_2 < 8\varrho. \quad (4.2)$$

We then denote $\mathcal{O}_h := \mathcal{O} \cap \Omega_h$, $\omega_h(8\varrho) := \omega(8\varrho) \cap \Omega_h$, $\omega_h(\varrho_1, \varrho_2) := \omega(\varrho_1, \varrho_2) \cap \Omega_h$ (note that these sets are always non-empty for h small enough).

Let $q \in \mathcal{C}(\Omega_h)$ satisfying $\|q\|_{L^\infty(\Omega_h)} \leq M$, $f \in \mathcal{C}(\Omega_h)$ satisfying $f = 0$ in \mathcal{O}_h and let $u \in \mathcal{C}(\overline{\Omega}_h)$ be such that

$$\begin{aligned} (-\Delta_h + q)u &= f, & \text{in } \Omega_h, \\ u &= 0, & \text{on } \partial\Omega_h. \end{aligned} \quad (4.3)$$

We are now ready to show the result.

Proposition 4.1 (Unique continuation estimate). *Let Γ be a non-empty open subset of $\partial\Omega$ satisfying $\Gamma_h := (\Gamma \cap \partial\Omega_h) \subset \partial_i^\pm \Omega_h$, $1 \leq i \leq d$. Then there exist constants $0 < h_0 < 1$, $\tilde{\varepsilon} > 0$, $C > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for all $0 < h \leq h_0$, $0 < h\tau \leq \tilde{\varepsilon}$ and all $u \in \mathcal{C}(\overline{\Omega}_h)$ satisfying (4.3), we have*

$$\|u\|_{H_h^1(\omega_h(\varrho, 3\varrho))} \leq C \left(e^{-\alpha_1\tau} \|u\|_{H_h^1(\Omega_h)} + e^{\alpha_2\tau} |\partial_n u|_{L_h^2(\Gamma_h)} \right). \quad (4.4)$$

In order to prove Proposition 4.1, we first introduce a cut-off function χ (see Figure 2) satisfying $0 \leq \chi \leq 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi(p) = \begin{cases} 0, & p \leq \frac{1}{2}, \quad p \geq 8, \\ 1, & \frac{3}{4} \leq p \leq 7. \end{cases} \quad (4.5)$$

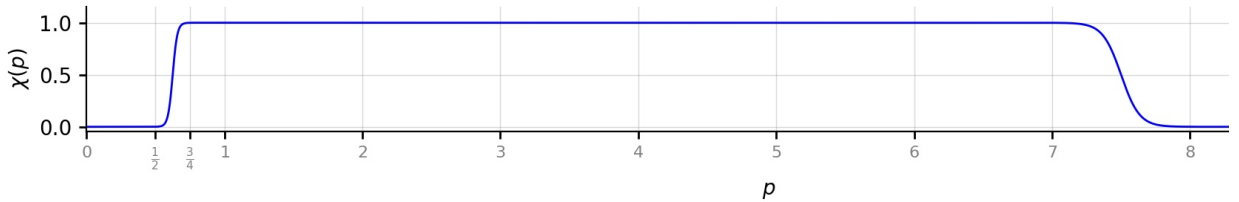


FIGURE 2. cut-off function $\chi(p)$.

In addition, for $x^{(1)} \in \Omega$, we set $B^1 := B(x^{(1)}, r) = \{x \in \mathbb{R}^d, |x - x^{(1)}| < r\}$ and $B_h^1 := B^1 \cap \Omega_h$. We shall begin to estimate u in B_h^1 near the boundary part Γ_h .

Lemma 4.2. *Let u be a solution to (4.3). Then there exist $B_h^1 = B(x^{(1)}, r) \cap \Omega_h$ and $m_0 \in (0, 1)$ such that the following estimate holds:*

$$\|u\|_{H_h^1(B_h^1)} \leq C \left(e^{-\frac{C_2 \varepsilon_0}{m_0 h}} \|u\|_{H_h^1(\Omega_h)} + |\partial_n u|_{L_h^2(\Gamma_h)} \right)^{m_0} \|u\|_{H_h^1(\Omega_h)}^{1-m_0}, \quad (4.6)$$

for some positive constants C and C_2 . Here ε_0 is the constant given in Proposition 3.11.

Remark 4.3. Compared with the continuum case, there is one extra term $e^{-\frac{C_2 \varepsilon_0}{m_0 h}} \|u\|_{H_h^1(\Omega_h)}$ due to the restriction $sh \leq \varepsilon_0$ of the discrete Carleman estimate in Proposition 3.11.

Proof of Lemma 4.2. Let us choose $\delta > 0$ and $x^{(0)} \in \mathbb{R}^d \setminus \bar{\Omega}$ such that

$$\delta < \frac{\varrho}{4}, \quad \overline{B(x^{(0)}, \delta)} \cap \bar{\Omega} = \emptyset, \quad B(x^{(0)}, 2\delta) \cap \Omega \neq \emptyset, \quad B(x^{(0)}, 4\delta) \cap \partial\Omega \subset \Gamma. \quad (4.7)$$

That is, $x^{(0)}$ is an outer point of $\bar{\Omega}$ and is near Γ . We define the functions $\psi_0(x)$ and $\varphi_0(x)$ by:

$$\psi_0(x) = |x - x^{(0)}|^2, \quad \varphi_0(x) = e^{-\frac{\lambda_0}{\delta^2} \psi_0(x)},$$

where λ_0 is the constant given in Proposition 3.11. Denote $w = \chi(\frac{\psi_0}{\delta^2})u$. Taking into account $u = 0$ on $\partial\Omega_h$ and applying the discrete Carleman estimate (3.3), we obtain

$$\begin{aligned} & s^3 \|e^{s\varphi_0} w\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\Omega_{h,k}^*} e^{2s\varphi_0} |D_k w|^2 \\ & \leq C \left(\|e^{s\varphi_0} (-\Delta_h + q)w\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k (e^{2s\varphi_0}) |\partial_n w|^2 \right), \end{aligned} \quad (4.8)$$

for $s_0 \leq s \leq \frac{\varepsilon_0}{h}$. By (4.5) and (4.7), we have

$$\begin{aligned} s^3 \|e^{s\varphi_0} w\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\Omega_{h,k}^*} e^{2s\varphi_0} |D_k w|^2 & \geq s^3 \|e^{s\varphi_0} w\|_{L_h^2(E_h^1)}^2 + s \sum_{k=1}^d \int_{(E_h^1)_k^*} e^{2s\varphi_0} |D_k w|^2 \\ & \geq s e^{2s e^{-4\lambda_0}} \left(s^2 \|u\|_{L_h^2(E_h^1)}^2 + \|u\|_{\dot{H}_h^1(E_h^1)}^2 \right), \end{aligned} \quad (4.9)$$

where $E_h^1 := \{x \in \Omega_h, \delta^2 \leq \psi_0(x) \leq 4\delta^2\}$. In view of $A_k^2 \chi = \chi + \frac{h^2}{4} D_k^2 \chi$ and (4.3), we get

$$\begin{aligned} (-\Delta_h + q)w & = - \sum_{k=1}^d [D_k(\sigma^k D_k \chi) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi) + (D_k A_k \chi) A_k(\sigma^k D_k u)] \\ & \quad - \sum_{k=1}^d \left(\chi + \frac{h^2}{4} D_k^2 \chi \right) D_k(\sigma^k D_k u) + q \chi u \end{aligned}$$

$$\begin{aligned}
&= - \sum_{k=1}^d [D_k(\sigma^k D_k \chi) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi) + (D_k A_k \chi) A_k(\sigma^k D_k u)] \\
&\quad - \frac{h^2}{4} \sum_{k=1}^d (D_k^2 \chi) D_k(\sigma^k D_k u),
\end{aligned} \tag{4.10}$$

for all $x \in \Omega_h$, where we have used $f \chi = 0$ in Ω_h by noting $f = 0$ in \mathcal{O}_h and (4.5). Taking (4.7) into account, we see that $|x - x^{(0)}| > \delta$ for all $x \in \Omega_h$, so that we obtain from (4.5) and (4.10) that

$$\begin{aligned}
\|e^{s\varphi_0}(-\Delta_h + q)w\|_{L_h^2(\Omega_h)}^2 &= \| -e^{s\varphi_0} \sum_{k=1}^d [D_k(\sigma^k D_k \chi) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi) \\
&\quad + (D_k A_k \chi) A_k(\sigma^k D_k u)] - \frac{h^2}{4} e^{s\varphi_0} \sum_{k=1}^d (D_k^2 \chi) D_k(\sigma^k D_k u) \|_{L_h^2(E_h^2)}^2 \\
&\leq C e^{2se^{-6\lambda_0}} \sum_{k=1}^d \left(\|A_k^2 u\|_{L_h^2(\Omega_h)}^2 + \|D_k A_k u\|_{L_h^2(\Omega_h)}^2 + \|A_k(\sigma^k D_k u)\|_{L_h^2(\Omega_h)}^2 \right) \\
&\quad + C e^{2se^{-6\lambda_0}} \left\| \sum_{k=1}^d D_k(\sigma^k D_k u) \right\|_{L_h^2(E_h^2)}^2,
\end{aligned}$$

where $E_h^2 := \{x \in \Omega_h, 6\delta^2 \leq \psi_0(x) \leq 9\delta^2\}$. Recalling that $f = 0$ in $\mathcal{O}_h \supset \omega_h(8\varrho) \supset E_h^2$, we get

$$\left\| \sum_{k=1}^d D_k(\sigma^k D_k u) \right\|_{L_h^2(E_h^2)}^2 = \|qu\|_{L_h^2(E_h^2)}^2 \leq C \|u\|_{L_h^2(\Omega_h)}^2.$$

Furthermore, using Corollary 3.2 and (3.2), we have

$$\begin{aligned}
\|A_k^2 u\|_{L_h^2(\Omega_h)}^2 &\leq \int_{\Omega_h} A_k(|A_k u|^2) = \int_{\Omega_{h,k}^*} |A_k u|^2 - \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k(|A_k u|^2) \\
&\leq \int_{\Omega_{h,k}^*} |A_k u|^2 \leq \int_{\Omega_{h,k}^*} A_k(|u|^2) = \int_{\Omega_h} |u|^2.
\end{aligned}$$

In the last equality, we have used (3.2) by virtue of $u = 0$ on $\partial_k \Omega_h$. Similarly, we have

$$\|D_k A_k u\|_{L_h^2(\Omega_h)}^2 \leq \int_{\Omega_{h,k}^*} |D_k u|^2, \quad \|A_k(\sigma^k D_k u)\|_{L_h^2(\Omega_h)}^2 \leq C \int_{\Omega_{h,k}^*} |D_k u|^2.$$

Then, we get

$$\|e^{s\varphi_0}(-\Delta_h + q)w\|_{L_h^2(\Omega_h)}^2 \leq C e^{2se^{-6\lambda_0}} \|u\|_{H_h^1(\Omega_h)}^2. \tag{4.11}$$

Taking (4.7) into account, we see that $|x - x^{(0)}| > 4\delta$ for all $x \in \partial \Omega_h \setminus \Gamma_h$ and $|x - x^{(0)}| > \delta$ for all $x \in \Gamma_h$, so that we obtain

$$s \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k(e^{2s\varphi_0}) |\partial_n w|^2 = s \int_{\Gamma_h} t_r^k(e^{2s\varphi_0}) |\partial_n w|^2$$

$$\begin{aligned}
&\leq se^{2se^{-\lambda_0}} \int_{\Gamma_h} |t_r^i(\sigma^i D_i(\chi u)) n_i|^2 \\
&\leq Cse^{2se^{-\lambda_0}} \int_{\Gamma_h} (|t_r^i(\sigma^i D_i \chi A_i u)|^2 + |t_r^i(\sigma^i A_i \chi D_i u)|^2) \\
&\leq Cse^{2se^{-\lambda_0}} \int_{\Gamma_h} (|t_r^i(\sigma^i) t_r^i(A_i u)|^2 + |t_r^i(\sigma^i D_i u)|^2) \\
&\leq Cse^{2se^{-\lambda_0}} |\partial_n u|_{L_h^2(\Gamma_h)}^2,
\end{aligned} \tag{4.12}$$

since $\Gamma_h \subset \partial_i^\pm \Omega_h$ and $t_r^i(A_i v) = -\frac{h}{2} t_r^i(D_i v) n_i$ when $v = 0$ on Γ_h .

Combining (4.8) with (4.9), (4.11) and (4.12), we obtain

$$se^{2se^{-4\lambda_0}} \left(s^2 \|u\|_{L_h^2(E_h^1)}^2 + \|u\|_{\dot{H}_h^1(E_h^1)}^2 \right) \leq C \left(se^{2se^{-\lambda_0}} |\partial_n u|_{L_h^2(\Gamma_h)}^2 + e^{2se^{-6\lambda_0}} \|u\|_{H_h^1(\Omega_h)}^2 \right), \tag{4.13}$$

where $E_h^1 := \{x \in \Omega_h, \delta^2 \leq \psi_0(x) \leq 4\delta^2\}$.

We can select $r > 0$ and $x^{(1)} \in \Omega$ such that

$$\text{dist}(x^{(1)}, \Gamma_h) \geq 4r, \quad B_h^1 := B(x^{(1)}, r) \cap \Omega_h \subset E_h^1. \tag{4.14}$$

This is possible because the second condition in (4.7) implies the existence of $x^{(1)} \in \Omega$ such that $|x^{(1)} - x^{(0)}| < 2\delta$. Therefore, for sufficiently small $r > 0$, condition (4.14) is satisfied. From (4.13), we can obtain that for $s_0 \leq s \leq \frac{\varepsilon_0}{h}$, there exist constants $C_1, C_2 > 0$ such that

$$\|u\|_{H_h^1(B_h^1)}^2 \leq C \left(e^{C_1 s} |\partial_n u|_{L_h^2(\Gamma_h)}^2 + e^{-C_2 s} \|u\|_{H_h^1(\Omega_h)}^2 \right). \tag{4.15}$$

If $\frac{1}{C_1+C_2} \log \frac{\|u\|_{H_h^1(\Omega_h)}^2}{|\partial_n u|_{L_h^2(\Gamma_h)}^2} \in [s_0, \frac{\varepsilon_0}{h}]$, then take $s = \frac{1}{C_1+C_2} \log \frac{\|u\|_{H_h^1(\Omega_h)}^2}{|\partial_n u|_{L_h^2(\Gamma_h)}^2}$ and we obtain

$$\|u\|_{H_h^1(B_h^1)}^2 \leq C |\partial_n u|_{L_h^2(\Gamma_h)}^{2m_0} \|u\|_{H_h^1(\Omega_h)}^{2(1-m_0)}, \quad m_0 = \frac{C_2}{C_1+C_2}. \tag{4.16}$$

If $\frac{1}{C_1+C_2} \log \frac{\|u\|_{H_h^1(\Omega_h)}^2}{|\partial_n u|_{L_h^2(\Gamma_h)}^2} \leq s_0$, then

$$\|u\|_{H_h^1(\Omega_h)}^2 \leq C |\partial_n u|_{L_h^2(\Gamma_h)}^2.$$

Therefore,

$$\|u\|_{H_h^1(B_h^1)}^2 \leq \|u\|_{H_h^1(\Omega_h)}^2 = \|u\|_{H_h^1(\Omega_h)}^{2m_0} \|u\|_{H_h^1(\Omega_h)}^{2(1-m_0)} \leq C |\partial_n u|_{L_h^2(\Gamma_h)}^{2m_0} \|u\|_{H_h^1(\Omega_h)}^{2(1-m_0)}. \tag{4.17}$$

If $\frac{1}{C_1+C_2} \log \frac{\|u\|_{H_h^1(\Omega_h)}^2}{|\partial_n u|_{L_h^2(\Gamma_h)}^2} \geq \frac{\varepsilon_0}{h}$, then

$$|\partial_n u|_{L_h^2(\Gamma_h)}^2 \leq e^{-\frac{\varepsilon_0}{h}(C_1+C_2)} \|u\|_{H_h^1(\Omega_h)}^2.$$

In view of (4.15), we have

$$\begin{aligned} \|u\|_{H_h^1(B_h^1)}^2 &\leq C \left(e^{C_1 \frac{\varepsilon_0}{h} - \frac{\varepsilon_0}{h} (C_1 + C_2)} \|u\|_{H_h^1(\Omega_h)}^2 + e^{-C_2 \frac{\varepsilon_0}{h}} \|u\|_{H_h^1(\Omega_h)}^2 \right) \\ &\leq C e^{-C_2 \frac{\varepsilon_0}{h}} \|u\|_{H_h^1(\Omega_h)}^{2m_0} \|u\|_{H_h^1(\Omega_h)}^{2(1-m_0)}. \end{aligned} \quad (4.18)$$

From (4.16)–(4.18), we get (4.6). Thus, we complete the proof of Lemma 4.2. \square

Next we extend the estimate from B_h^1 to $\omega_h(\varrho, 3\varrho)$. To accomplish this, we adapt the techniques developed in [18] to our discrete case. Let $B(x^{(j)}, r)$, $2 \leq j \leq N_0$, be a finite covering of $\omega(\varrho, 3\varrho)$, such that $B_h(x^{(j)}, r) = B(x^{(j)}, r) \cap \Omega_h$, $2 \leq j \leq N_0$ is a finite covering of $\omega_h(\varrho, 3\varrho) = \omega(\varrho, 3\varrho) \cap \Omega_h$. We can assume that $x^{(j)}$ satisfies $\text{dist}(x^{(j)}, \partial\Omega) \geq 4r$. In the sequel, we assume without loss of generality that

$$B(x^{(j+1)}, r) \subset B(x^{(j)}, 2r), \quad (4.19)$$

and we set $B^j = B(x^{(j)}, r)$, $B_h^j = B(x^{(j)}, r) \cap \Omega_h$, $2 \leq j \leq N_0$.

Lemma 4.4. *Let u be a solution to (4.3). Then there exist $m \in (0, 1)$ and $C, C_4 > 0$ such that the following estimate holds:*

$$\|u\|_{H_h^1(B_h^{j+1})} \leq C \left(e^{-\frac{C_4 \varepsilon_0}{m h}} \|u\|_{H_h^1(\Omega_h)} + \|u\|_{H_h^1(B_h^j)} \right)^m \|u\|_{H_h^1(B_h^j)}^{1-m}, \quad 1 \leq j \leq N_0 - 1. \quad (4.20)$$

Here ε_0 is the constant given in Proposition 3.11.

Proof. In order to prove (4.20), we define the functions $\psi_j(x)$ and $\varphi_j(x)$ by:

$$\psi_j(x) = |x - x^{(j)}|^2, \quad \varphi_j(x) = e^{-\frac{\lambda_0}{r^2} \psi_j(x)},$$

where λ_0 is the constant given in Proposition 3.11. Moreover, we set $w = \chi(\frac{\psi_j}{r^2})u$. Taking (4.5) and $\text{dist}(x^{(j)}, \partial\Omega) \geq 4r$ into account, we have $w = \partial_n w = 0$ on $\partial\Omega_h$. By applying the discrete Carleman estimate (3.3), we obtain, for $1 \leq s_0 \leq s \leq \frac{\varepsilon_0}{h}$,

$$s^3 \|e^{s\varphi_j} w\|_{L_h^2(\Omega_h)}^2 + s \sum_{k=1}^d \int_{\Omega_{h,k}^*} e^{2s\varphi_j} |D_k w|^2 \leq C \|e^{s\varphi_j} (-\Delta_h + q)w\|_{L_h^2(\Omega_h)}^2. \quad (4.21)$$

In a similar way as the proof of (4.8) by (4.13) in Lemma 4.2, we can prove

$$s e^{2s e^{-5\lambda_0}} \left(\|u\|_{L_h^2(E_h^4)}^2 + \|u\|_{H_h^1(E_h^4)}^2 \right) \leq C \left(e^{2s e^{-\frac{\lambda_0}{4}}} \|u\|_{H_h^1(E_h^5)}^2 + e^{2s e^{-6\lambda_0}} \|u\|_{H_h^1(E_h^6)}^2 \right),$$

where $E_h^4 := \{x \in \Omega_h, r^2 \leq \psi_j(x) \leq 5r^2\}$, $E_h^5 := \{x \in \Omega_h, \frac{1}{4}r^2 \leq \psi_j(x) \leq r^2\}$ and $E_h^6 := \{x \in \Omega_h, 6r^2 \leq \psi_j(x) \leq 9r^2\}$. Let $E_h^7 = \{x \in \Omega_h, \psi_j(x) \leq 5r^2\}$ and $E_h^8 = \{x \in \Omega_h, \psi_j(x) \leq r^2\}$. Hence, by (4.21) and noting $E_h^7 = E_h^8 \cup E_h^4$, $E_h^5 \subset E_h^8$, we obtain

$$\begin{aligned} s e^{2s e^{-5\lambda_0}} \|u\|_{H_h^1(E_h^7)}^2 &\leq s e^{2s e^{-5\lambda_0}} \left(\|u\|_{H_h^1(E_h^8)}^2 + \|u\|_{H_h^1(E_h^4)}^2 \right) \\ &\leq C \left(s e^{2s e^{-5\lambda_0}} \|u\|_{H_h^1(E_h^8)}^2 + e^{2s e^{-\frac{\lambda_0}{4}}} \|u\|_{H_h^1(E_h^5)}^2 + e^{2s e^{-6\lambda_0}} \|u\|_{H_h^1(E_h^6)}^2 \right) \end{aligned}$$

$$\leq C \left(e^{2se^{-\frac{\lambda_0}{4}}} \|u\|_{H_h^1(E_h^8)}^2 + e^{2se^{-6\lambda_0}} \|u\|_{H_h^1(E_h^6)}^2 \right).$$

Thus we obtain

$$\|u\|_{H_h^1(E_h^7)}^2 \leq C \left(e^{C_3 s} \|u\|_{H_h^1(E_h^8)}^2 + e^{-C_4 s} \|u\|_{H_h^1(\Omega_h)}^2 \right),$$

for $s_0 \leq s \leq \frac{\varepsilon_0}{h}$ and some constants $C_3, C_4 > 0$. Similarly, minimizing the right-hand side with respect to s , for some $m \in (0, 1)$, we have

$$\|u\|_{H_h^1(E_h^7)} \leq C \left(e^{-\frac{C_4 \varepsilon_0}{mh}} \|u\|_{H_h^1(\Omega_h)} + \|u\|_{H_h^1(E_h^8)} \right)^m \|u\|_{H_h^1(\Omega_h)}^{(1-m)}.$$

Since

$$B_h^{j+1} \subset E_h^7, \quad E_h^8 \subset B_h^j,$$

we obtain (4.20). This completes the proof of Lemma 4.4. \square

Lemma 4.5. *Let u be a solution to (4.3). Then there exists constant $C > 0$ such that*

$$\|u\|_{H_h^1(B_h^j)} \leq C \left(e^{-\frac{C_4 \varepsilon_0}{mh}} \|u\|_{H_h^1(\Omega_h)} + \|u\|_{H_h^1(B_h^1)} \right)^{m^j} \|u\|_{H_h^1(\Omega_h)}^{1-m^j}, \quad 2 \leq j \leq N_0. \quad (4.22)$$

Here $m \in (0, 1)$ is the constant given in Lemma 4.4.

Proof. Put

$$\alpha_j = \|u\|_{H_h^1(B_h^j)}, \quad A = e^{-\frac{C_4 \varepsilon_0}{mh}} \|u\|_{H_h^1(\Omega_h)}, \quad B = C^{\frac{1}{1-m}} \|u\|_{H_h^1(\Omega_h)},$$

by (4.20) we have

$$\alpha_{j+1} \leq B^{1-m} (\alpha_j + A)^m, \quad 1 \leq j \leq N_0 - 1.$$

Then, applying Lemma 4 in [42], we obtain, for all $\mu \in (0, m^j]$,

$$\alpha_j \leq 2^{\frac{1}{1-m}} B^{1-\mu} (\alpha_1 + A)^\mu, \quad 2 \leq j \leq N_0.$$

This completes the proof of the lemma. \square

Proof of Proposition 4.1. By Lemma 4.5 and Young's inequality, we easily obtain

$$\|u\|_{H_h^1(B_h^j)} \leq C \left(\varepsilon^p \|u\|_{H_h^1(\Omega_h)} + \varepsilon^{-p'} \left(e^{-\frac{C_4 \varepsilon_0}{mh}} \|u\|_{H_h^1(\Omega_h)} + \|u\|_{H_h^1(B_h^1)} \right) \right),$$

for all $\varepsilon > 0, 2 \leq j \leq N_0$. Here

$$p = \frac{1}{1-m^j}, \quad p' = \frac{1}{m^j}.$$

Selecting $\varepsilon = e^{-\frac{C_5}{p}\tau}$, $C_5 > 0, \tau > 0$, we have

$$\|u\|_{H_h^1(B_h^j)} \leq C \left(e^{-C_5\tau} \|u\|_{H_h^1(\Omega_h)} + e^{C_6(j)\tau - \frac{C_4\varepsilon_0}{mh}} \|u\|_{H_h^1(\Omega_h)} + e^{C_6(j)\tau} \|u\|_{H_h^1(B_h^1)} \right),$$

where $C_6(j) := \frac{C_5 p'}{p}$. Then, for $h\tau \leq \frac{C_4\varepsilon_0}{(C_5+C_6(j))m}$ such that $C_6(j)\tau - \frac{C_4\varepsilon_0}{mh} \leq -C_5\tau$, we get

$$\|u\|_{H_h^1(B_h^j)} \leq C \left(e^{-C_5\tau} \|u\|_{H_h^1(\Omega_h)} + e^{C_6(j)\tau} \|u\|_{H_h^1(B_h^1)} \right). \quad (4.23)$$

Similarly, we obtain from Lemma 4.2 and Young's inequality

$$\|u\|_{H_h^1(B_h^1)} \leq C \left(\varepsilon^{p_0} \|u\|_{H_h^1(\Omega_h)} + \varepsilon^{-p'_0} \left(e^{-\frac{C_2\varepsilon_0}{m_0h}} \|u\|_{H_h^1(\Omega_h)} + |\partial_n u|_{L_h^2(\Gamma_h)} \right) \right),$$

for all $\varepsilon > 0$. Here

$$p_0 = \frac{1}{1-m_0}, \quad p'_0 = \frac{1}{m_0}.$$

Selecting $\varepsilon = e^{-\frac{C_5+C_6(j)}{p_0}\tau}$ and setting $C_7(j) = \frac{(C_5+C_6(j))p'_0}{p_0}$, we obtain

$$\|u\|_{H_h^1(B_h^1)} \leq C \left(e^{-(C_5+C_6(j))\tau} \|u\|_{H_h^1(\Omega_h)} + e^{C_7(j)\tau - \frac{C_2\varepsilon_0}{m_0h}} \|u\|_{H_h^1(\Omega_h)} + e^{C_7(j)\tau} |\partial_n u|_{L_h^2(\Gamma_h)} \right).$$

Choosing $h\tau \leq \frac{C_2\varepsilon_0}{(C_5+C_6(j)+C_7(j))m_0}$ such that $C_7(j)\tau - \frac{C_2\varepsilon_0}{m_0h} \leq -(C_5+C_6(j))\tau$, we have

$$\|u\|_{H_h^1(B_h^1)} \leq C \left(e^{-(C_5+C_6(j))\tau} \|u\|_{H_h^1(\Omega_h)} + e^{C_7(j)\tau} |\partial_n u|_{L_h^2(\Gamma_h)} \right). \quad (4.24)$$

By (4.23) and (4.24), we have

$$\|u\|_{H_h^1(B_h^j)} \leq C e^{-C_5\tau} \|u\|_{H_h^1(\Omega_h)} + e^{C(j)\tau} |\partial_n u|_{L_h^2(\Gamma_h)}, \quad (4.25)$$

for some positive constant $C(j)$ and $\tau > 0$ satisfying $h\tau \leq \min\left\{\frac{C_4\varepsilon_0}{(C_5+C_6(j))m}, \frac{C_2\varepsilon_0}{(C_5+C_6(j)+C_7(j))m_0}\right\} := \varepsilon_j$. Setting $\tilde{\varepsilon} := \min_{j \in \{2, \dots, N_0\}} \varepsilon_j$, addition of inequalities (4.25) for $j \in \{2, \dots, N_0\}$ yields the final inequality (4.4). Thus, the proof of Proposition 4.1 is complete. \square

5. STABILITY ESTIMATE FOR THE DISCRETE CALDERÓN PROBLEM

In this section, we focus on the proof of Theorem 2.9.

Similar to the continuum case, the proof of the stability estimate is based on the existence of CGO solutions. The following results state their existence.

Lemma 5.1 ([16], Thm. 4.4). *Let $M > 0$. For all $q \in C(\Omega_h)$ satisfying $\|q\|_{L_h^\infty(\Omega_h)} \leq M$ and Assumption 2.4, there exist constants $a_0 > 0$ and $c > 0$ that depend on M such that $\forall \eta \in \mathbb{C}^d$ with $\eta \cdot \eta = 0$, if $a := |\operatorname{Im}\eta|$ verifies $a_0 \leq a \leq c \min\{\varepsilon_d^{-1}, h^{-\frac{2}{3}}\}$, there exists $u \in \mathcal{C}(\bar{\Omega}_h)$ a solution of*

$$-\Delta_h u + qu = 0, \quad \text{on } \Omega_h,$$

that satisfies

$$u(x) = e^{i\eta \cdot x}(1 + r(x)), \quad \text{on } \overline{\Omega}_h, \quad (5.1)$$

with

$$\|r\|_{\dot{H}_h^1(\Omega_h)} + a\|r\|_{L_h^2(\overline{\Omega}_h)} \leq C(1 + a^2\epsilon_a + a^4h^2). \quad (5.2)$$

Moreover, the solution u satisfies

$$\|u\|_{H_h^1(\Omega_h)} \leq Ce^a a^2, \quad (5.3)$$

where C only depends on m, a_0 and c .

Remark 5.2. As it mentioned in (4.21) in [16], the solution $u = e^{\eta \cdot x} + e^{\mathcal{R}(\eta) \cdot x} r(x)$ in [16] can be rewritten as (5.1) in our paper.

Proof of Theorem 2.9. Let $q_1 \geq 0$ and $q_2 \geq 0$ in $\mathcal{C}(\Omega_h)$ be two potentials satisfying Assumption 2.4 and $q_1 = q_2$ in $\mathcal{O}_h = \mathcal{O} \cap \Omega_h$, where $\mathcal{O} \subset \Omega$ is a neighborhood of $\partial\Omega$ and $\|q\|_{L_h^\infty(\Omega_h)} \leq M$. For all $\xi \in \hat{\mathcal{K}}_h = \mathbb{Z}^d \cap [0, N]^d$ satisfying $\pi|\xi| \leq a$, we set

$$\eta_1 := -\pi\xi + \sqrt{a^2 - \pi^2|\xi|^2}\zeta_1 + ia\zeta_2, \quad \eta_2 := -\pi\xi - \sqrt{a^2 - \pi^2|\xi|^2}\zeta_1 - ia\zeta_2,$$

where $a_0 \leq a \leq c \min\{\epsilon_d^{-1}, \epsilon_a^{-1}, h^{-\frac{2}{3}}\}$ and $\zeta_1, \zeta_2 \in \mathbb{R}^d$ satisfying $\xi \cdot \zeta_1 = \xi \cdot \zeta_2 = \zeta_1 \cdot \zeta_2 = 0, |\zeta_1| = |\zeta_2| = 1$. It follows from Lemma 5.1 that there exist r_1 and r_2 , such that $u_1 := e^{i\eta_1 \cdot x}(1 + r_1(x))$ and $u_2 := e^{i\eta_2 \cdot x}(1 + r_2(x))$ are solutions of

$$\begin{aligned} -\Delta_h u_1 + q_1 u_1 &= 0 & \text{in } \Omega_h, \\ -\Delta_h u_2 + q_2 u_2 &= 0 & \text{in } \Omega_h, \end{aligned}$$

respectively. Let $v \in \mathcal{C}(\overline{\Omega}_h)$ be the solution to the following problem

$$\begin{aligned} (-\Delta_h + q_1)v &= 0 & \text{in } \Omega_h, \\ v &= u_2 & \text{on } \partial\Omega_h. \end{aligned}$$

Then, by defining $u = v - u_2 \in \mathcal{C}(\overline{\Omega}_h)$, we get

$$\begin{aligned} (-\Delta_h + q_1)u &= (q_2 - q_1)u_2 & \text{in } \Omega_h, \\ u &= 0 & \text{on } \partial\Omega_h. \end{aligned} \quad (5.4)$$

To proceed, we introduce a cut-off function $\chi_1 \in C_0^\infty(\Omega)$ (see Figure 3) satisfying $0 \leq \chi_1 \leq 1$ and

$$\chi_1(x) = \begin{cases} 0, & x \in \omega(\frac{3}{2}\varrho), \\ 1, & x \in \overline{\Omega} \setminus \omega(\frac{5}{2}\varrho), \end{cases} \quad (5.5)$$

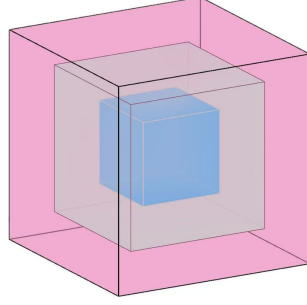


FIGURE 3. Pink region: $\chi_1(x) = 0$; Blue region: $\chi_1(x) = 1$.

where $\omega(\frac{3}{2}\varrho)$ and $\omega(\frac{5}{2}\varrho)$ are defined in Section 4. If we put $\tilde{u} = \chi_1 u$, it follows from $A_k^2 \chi_1 = \chi_1 + \frac{h^2}{4} D_k^2 \chi_1$ and (5.4) that

$$\begin{aligned}
(-\Delta_h + q_1)\tilde{u} &= - \sum_{k=1}^d [D_k(\sigma^k D_k \chi_1) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi_1) + (D_k A_k \chi_1) A_k(\sigma^k D_k u)] \\
&\quad - \sum_{k=1}^d (\chi_1 + \frac{h^2}{4} D_k^2 \chi_1) D_k(\sigma^k D_k u) + q_1 \chi_1 u \\
&= - \sum_{k=1}^d [D_k(\sigma^k D_k \chi_1) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi_1) + (D_k A_k \chi_1) A_k(\sigma^k D_k u)] \\
&\quad - \frac{h^2}{4} \sum_{k=1}^d (D_k^2 \chi_1) D_k(\sigma^k D_k u) + (q_2 - q_1) u_2,
\end{aligned} \tag{5.6}$$

for all $x \in \Omega_h$, where we have used $\chi_1(q_2 - q_1) = q_2 - q_1$ in Ω_h by noting $q_1 = q_2$ in \mathcal{O}_h and (5.5).

Multiplying (5.6) by u_1 and integrating over Ω_h , we obtain

$$\begin{aligned}
&\int_{\Omega_h} (\Delta_h - q_1)\tilde{u} u_1 + \int_{\Omega_h} (q_2 - q_1) u_1 u_2 \\
&= \int_{\Omega_h} u_1 \sum_{k=1}^d [D_k(\sigma^k D_k \chi_1) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi_1) \\
&\quad + (D_k A_k \chi_1) A_k(\sigma^k D_k u) + \frac{h^2}{4} (D_k^2 \chi_1) D_k(\sigma^k D_k u)].
\end{aligned} \tag{5.7}$$

We begin with the first integral in (5.7). Since $t_r^k(\tilde{u})|_{\partial_k \Omega_h} = t_r^k(D_k \tilde{u})|_{\partial_k \Omega_h} = 0, k = 1, \dots, d$, it follows from (3.1) that

$$\begin{aligned}
\int_{\Omega_h} (\Delta_h - q_1)\tilde{u} u_1 &= \sum_{k=1}^d \int_{\Omega_h} D_k(\sigma^k D_k \tilde{u}) u_1 - \int_{\Omega_h} q_1 \tilde{u} u_1 \\
&= - \sum_{k=1}^d \int_{\Omega_{h,k}^*} \sigma^k D_k \tilde{u} D_k u_1 - \int_{\Omega_h} q_1 \tilde{u} u_1 \\
&= \sum_{k=1}^d \int_{\Omega_h} D_k(\sigma^k D_k u_1) \tilde{u} - \int_{\Omega_h} q_1 \tilde{u} u_1 \\
&= \int_{\Omega_h} \Delta_h u_1 \tilde{u} - \int_{\Omega_h} q_1 u_1 \tilde{u} \\
&= 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{\Omega_h} (q_2 - q_1) u_1 u_2 &= \int_{\Omega_h} u_1 \sum_{k=1}^d [D_k(\sigma^k D_k \chi_1) A_k^2 u + (D_k A_k u) A_k(\sigma^k D_k \chi_1) \\ &\quad + (D_k A_k \chi_1) A_k(\sigma^k D_k u) + \frac{h^2}{4} (D_k^2 \chi_1) D_k(\sigma^k D_k u)] . \end{aligned} \quad (5.8)$$

In order to simplify the notations, we denote the right-hand side of (5.8) as F . Noting that u satisfies the conditions in Proposition 4.1, we apply (4.4) and (5.5) to get

$$|F|^2 \leq C \|u_1\|_{L_h^2(\Omega_h)}^2 \sum_{k=1}^d \int_{\omega_h(\varrho, 3\varrho)} \left(|A_k^2 u|^2 + |D_k A_k u|^2 + |A_k(\sigma^k D_k u)|^2 + \frac{h^2}{4} |D_k(\sigma^k D_k u)|^2 \right),$$

for $0 < h < h_0 < 1$. Similar to the discussions in Section 4, we have

$$\begin{aligned} \|A_k^2 u\|_{L_h^2(\omega_h(\varrho, 3\varrho))}^2 &\leq \|u\|_{L_h^2(\omega_h(\varrho, 3\varrho))}^2, \\ \|D_k A_k u\|_{L_h^2(\omega_h(\varrho, 3\varrho))}^2 &\leq \|u\|_{\dot{H}_h^1(\omega_h(\varrho, 3\varrho))}^2, \\ \|A_k(\sigma^k D_k u)\|_{L_h^2(\omega_h(\varrho, 3\varrho))}^2 &\leq C \|u\|_{\dot{H}_h^1(\omega_h(\varrho, 3\varrho))}^2, \\ \frac{h^2}{4} \|D_k(\sigma^k D_k u)\|_{L_h^2(\omega_h(\varrho, 3\varrho))}^2 &\leq C \|u\|_{\dot{H}_h^1(\omega_h(\varrho, 3\varrho))}^2. \end{aligned}$$

In the last inequality, we used $\frac{h^2}{4} |D_k(\sigma^k D_k u)|^2 \leq A_k(|\sigma^k D_k u|^2)$. Thus, we get

$$|F| \leq C \|u_1\|_{L_h^2(\Omega_h)} \|u\|_{\dot{H}_h^1(\omega_h(\varrho, 3\varrho))} \quad (5.9)$$

Combining (5.9) with (5.8) and (4.4), we obtain

$$\left| \int_{\Omega_h} (q_2 - q_1) u_1 u_2 \right| \leq C \|u_1\|_{L_h^2(\Omega_h)} \left(e^{-\alpha_1 \tau} \|u\|_{\dot{H}_h^1(\Omega_h)} + e^{\alpha_2 \tau} |\partial_n u|_{L_h^2(\Gamma_h)} \right), \quad (5.10)$$

where $\alpha_1, \alpha_2 > 0$ and $0 < \tau \leq \frac{\tilde{\epsilon}}{h}$.

Applying Proposition B.1 and noting that u satisfies (5.4), we have

$$\begin{aligned} |\partial_n u|_{L_h^2(\Gamma_h)} &\leq |\partial_n u|_{H_h^{-\frac{1}{2}}(\Gamma_h)}^{\frac{1}{2}} |\partial_n u|_{H_h^{\frac{1}{2}}(\Gamma_h)}^{\frac{1}{2}} = |(\Lambda_h[q_1] - \Lambda_h[q_2])(u_2)|_{H_h^{-\frac{1}{2}}(\Gamma_h)}^{\frac{1}{2}} |\partial_n u|_{H_h^{\frac{1}{2}}(\Gamma_h)}^{\frac{1}{2}} \\ &\leq \delta^{\frac{1}{2}} |u_2|_{H_h^{\frac{1}{2}}(\partial\Omega_h)}^{\frac{1}{2}} |\partial_n u|_{H_h^{\frac{1}{2}}(\Gamma_h)}^{\frac{1}{2}} \leq \delta^{\frac{1}{2}} \|u_2\|_{H_h^1(\Omega_h)}^{\frac{1}{2}} \|u\|_{\dot{H}_h^2(\Omega_h)}^{\frac{1}{2}} \\ &\leq C \delta^{\frac{1}{2}} \|u_2\|_{H_h^1(\Omega_h)}^{\frac{1}{2}} \|u_2\|_{L_h^2(\Omega_h)}^{\frac{1}{2}}, \end{aligned}$$

where we abbreviate $\delta := \|\Lambda_h[q_1] - \Lambda_h[q_2]\|_{\mathcal{L}_h(\Gamma_h)}$. It follows from (5.3) that

$$|\partial_n u|_{L_h^2(\Gamma_h)} \leq C \delta^{\frac{1}{2}} e^a a^2,$$

and

$$\|u_1\|_{L_h^2(\Omega_h)} \leq Ce^a a^2.$$

Similarly, we apply Assumption 2.5 and (5.3) to get

$$\|u\|_{H_h^1(\Omega_h)} \leq C\|(q_2 - q_1)u_2\|_{L_h^2(\Omega_h)} \leq Ce^a a^2.$$

Thus, we have

$$\left| \int_{\Omega_h} (q_2 - q_1)u_1u_2 \right| \leq Ce^{2a} a^4 \left(e^{-\alpha_1\tau} + \delta^{\frac{1}{2}} e^{\alpha_2\tau} \right), \quad (5.11)$$

for all $0 < \tau \leq \frac{\tilde{\varepsilon}}{h}$ and $\alpha_1 > 0, \alpha_2 > 0, a_0 \leq a \leq c \min\{\epsilon_d^{-1}, \epsilon_a^{-1}, h^{-\frac{2}{3}}\}$. Then we choose a constant $\tilde{\gamma} > 0$ sufficiently large and set $\tau = a\tilde{\gamma}$ so that

$$e^{2a} a^4 e^{-\alpha_1\tau} \leq \frac{C}{a}, \quad \text{and} \quad e^{2a} e^{\alpha_2\tau} \leq e^{\alpha_3 a}, \quad (5.12)$$

for some constants $C > 0$ and $\alpha_3 > 0$. It is easy to see that $0 < \tau \leq \frac{\tilde{\varepsilon}}{h}$ implies that $a \leq \frac{\tilde{\varepsilon}}{\tilde{\gamma}} h^{-1}$. Furthermore, by (5.11), (5.12) and noting $u_1 = e^{i\eta_1 \cdot x}(1 + r_1(x))$, $u_2 = e^{i\eta_2 \cdot x}(1 + r_2(x))$, we obtain

$$\begin{aligned} |\mathcal{F}_h(q_2 - q_1)(\xi)| &\leq \frac{C}{a} + Ca^4 e^{\alpha_3 a} \delta^{\frac{1}{2}} + \left| \int_{\Omega_h} (q_2 - q_1) e^{-2\pi i \xi \cdot x} (r_1 + r_2 + r_1 r_2) \right| \\ &\leq C \left(\frac{1}{a} + a^4 e^{\alpha_3 a} \delta^{\frac{1}{2}} + \|r_1\|_{L_h^2(\Omega_h)} + \|r_2\|_{L_h^2(\Omega_h)} + \|r_1\|_{L_h^2(\Omega_h)} \|r_2\|_{L_h^2(\Omega_h)} \right). \end{aligned} \quad (5.13)$$

In addition, it follows from (5.2) that $\|r_1\|_{L_h^2(\Omega_h)}, \|r_2\|_{L_h^2(\Omega_h)}$ are bounded and

$$\|r_1\|_{L_h^2(\Omega_h)} + \|r_2\|_{L_h^2(\Omega_h)} + \|r_1\|_{L_h^2(\Omega_h)} \|r_2\|_{L_h^2(\Omega_h)} \leq C \left(\frac{1}{a} + a\varepsilon_a + a^3 h^2 \right).$$

We conclude from (5.13) that

$$|\mathcal{F}_h(q_2 - q_1)(\xi)| \leq C \left(a^4 e^{\alpha_3 a} \delta^{\frac{1}{2}} + \frac{1}{a} + a\varepsilon_a + a^3 h^2 \right), \quad (5.14)$$

for $\alpha_3 > 0$, $a_0 \leq a \leq \tilde{c} \min\{\epsilon_d^{-1}, \epsilon_a^{-1}, h^{-\frac{2}{3}}\}$, $\tilde{c} := \min\{\frac{\tilde{\varepsilon}}{\tilde{\gamma}}, c\}$ and $\xi \in \hat{\mathcal{K}}_h$ satisfying $\pi|\xi| \leq a$.

Setting $\tilde{\mu} = \tilde{c}^{-1} \max\{\varepsilon_a^{\frac{1}{2}}, h^{\frac{1}{2}}, \varepsilon_d\}$, we always have $\frac{1}{\tilde{\mu}} \leq \tilde{c} \min\{\epsilon_d^{-1}, \epsilon_a^{-1}, h^{-\frac{2}{3}}\}$ and a simple study shows that, setting $a = \frac{1}{\tilde{\mu}}$,

$$\frac{1}{a} + a\varepsilon_a + a^3 h^2 \leq C\tilde{\mu}.$$

Therefore, if $\delta = 0$, we set $a = \frac{1}{\tilde{\mu}}$, then for all $\xi \in \hat{\mathcal{K}}_h$ satisfying $\pi|\xi| \leq \frac{1}{\tilde{\mu}}$, we have

$$|\mathcal{F}_h(q_2 - q_1)(\xi)| \leq C\tilde{\mu}.$$

If δ is small enough, for example, for $-\ln \delta \geq \frac{2(\alpha_3+3)}{\tilde{\mu}}$, such that

$$\delta^{\frac{1}{2}} \frac{1}{\tilde{\mu}^4} e^{\frac{\alpha_3}{\tilde{\mu}}} \leq \tilde{\mu},$$

taking $a = \frac{1}{\tilde{\mu}}$ in (5.14), we obtain, for all $\xi \in \hat{\mathcal{K}}_h$ satisfying $\pi|\xi| \leq \frac{1}{\tilde{\mu}}$,

$$|\mathcal{F}_h(q_2 - q_1)(\xi)| \leq C\tilde{\mu}.$$

If $-\ln \delta \in (2(\alpha_3 + 3)a_0, \frac{2(\alpha_3+3)}{\tilde{\mu}})$, taking $a = \frac{-\ln \delta}{2(\alpha_3+3)}$ in (5.14), we obtain, for all $\xi \in \hat{\mathcal{K}}_h$ satisfying $\pi|\xi| \leq \frac{-\ln \delta}{2(\alpha_3+3)}$,

$$\begin{aligned} |\mathcal{F}_h(q_2 - q_1)(\xi)| &\leq C \left(\delta^{\frac{3}{2(\alpha_3+3)}} |\ln \delta|^4 + \frac{1}{|\ln \delta|} + |\ln \delta| \varepsilon_a + |\ln \delta|^3 h^2 \right) \\ &\leq \frac{C}{|\ln \delta|}, \end{aligned}$$

where we have used $\tilde{\mu} = \tilde{c}^{-1} \max\{\varepsilon_a^{1/2}, h^{1/2}, \varepsilon_d\} \leq \frac{2(\alpha_3+3)}{|\ln \delta|}$.

Combining these cases, we obtain that, if $\delta \leq e^{-2(\alpha_3+3)a_0}$, setting

$$\mu = \max\left\{\tilde{\mu}, \frac{2(\alpha_3 + 3)}{|\ln \delta|}\right\},$$

for all $\xi \in \hat{\mathcal{K}}_h$ satisfying $\pi|\xi| \leq \frac{1}{\mu}$,

$$|\mathcal{F}_h(q_2 - q_1)(\xi)| \leq C\mu. \tag{5.15}$$

In order to estimate $|q_1 - q_2|_{H_h^{-r}(\Omega_h)}$, we take $\rho \in (0, \frac{1}{\pi\mu})$ to be chosen and use (5.15) to get

$$\begin{aligned} |q_1 - q_2|_{H_h^{-r}(\Omega_h)}^2 &= \sum_{\xi \in \hat{\mathcal{K}}_h} |\mathcal{F}_h(q_2 - q_1)(\xi)|^2 (1 + |\xi|^2)^{-r} \\ &= \sum_{|\xi| < \rho, \xi \in \hat{\mathcal{K}}_h} |\mathcal{F}_h(q_2 - q_1)(\xi)|^2 (1 + |\xi|^2)^{-r} \\ &\quad + \sum_{|\xi| > \rho, \xi \in \hat{\mathcal{K}}_h} |\mathcal{F}_h(q_2 - q_1)(\xi)|^2 (1 + |\xi|^2)^{-r} \\ &\leq C(\rho^d \mu^2 + \rho^{-2r}). \end{aligned}$$

Setting $\rho = \pi^{-1} \mu^{\frac{-2}{d+2r}}$, which is indeed smaller than $\frac{1}{\pi\mu}$, we have

$$\begin{aligned} |q_1 - q_2|_{H_h^{-r}(\Omega_h)}^2 &\leq C\mu^{\frac{4r}{d+2r}} \\ &\leq C \max \left\{ \epsilon_d^{\frac{4r}{d+2r}}, \epsilon_a^{\frac{2r}{d+2r}}, h^{\frac{2r}{d+2r}}, |\ln(\|\Lambda_h[q_1] - \Lambda_h[q_2]\|_{\mathcal{L}_h(\Gamma_h)})|^{-\frac{4r}{d+2r}} \right\}, \end{aligned}$$

which completes the proof of Theorem 2.9. \square

ACKNOWLEDGMENTS

The authors sincerely thank the anonymous referees for their thorough reading and invaluable comments. The research of XZ was supported in part by National Key R&D Program of China under grant 2023YFA1009002. The research of GY was supported in part by NSFC (Nos. 11771074 and 12371421) and National Key R&D Program of China (No. 2020YFA0714102).

DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

REFERENCES

- [1] A.P. Calderón, On an inverse boundary value problem. *Comput. Appl. Math.* **25** (2006) 133–138.
- [2] G. Uhlmann, 30 years of Calderón’s problem, Séminaire Laurent Schwartz-Équations aux dérivées partielles et applications. *Année* (2012–2013) 25.
- [3] M. Cheney, D. Isaacson and J.C. Newell, Electrical impedance tomography. *SIAM Rev.* **41** (1999) 85–101.
- [4] J. Jordana, M. Gasulla and R. Pallás-Areny, Electrical resistance tomography to detect leaks from buried pipes. *Meas. Sci. Technol.* **12** (2001) 1061–1068.
- [5] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. Math.* **125** (1987) 153–169.
- [6] A.L. Bukhgeim and G. Uhlmann, Recovering a potential from partial Cauchy data. *Commun. Part. Diff. Equ.* **27** (2002) 653–668.
- [7] V. Isakov, On uniqueness in the inverse conductivity problem with local data. *Inverse Probl. Imaging* **1** (2007) 95–105.
- [8] C. Kenig, J. Sjöstrand and G. Uhlmann, The Calderón problem with partial data. *Ann. Math.* **165** (2007) 567–591.
- [9] K. Krupchyk and G. Uhlmann, Stability estimates for partial data inverse problems for Schrödinger operators in the high frequency limit. *J. Math. Pures Appl.* **126** (2019) 273–291.
- [10] R.-Y. Lai, X. Lu and T. Zhou, Partial data inverse problems for the nonlinear time-dependent Schrödinger equation. *SIAM J. Math. Anal.* **56** (2024) 4712–4741.
- [11] R.-Y. Lai and T. Zhou, An inverse problem for the non-linear fractional magnetic Schrödinger equation. *J. Diff. Equ.* **343** (2023) 64–89.
- [12] R.-Y. Lai and T. Zhou, Partial data inverse problems for nonlinear magnetic Schrödinger equations. *Math. Res. Lett.* **30** (2023) 1535–1563.
- [13] X. Zhao and G. Yuan, Stability estimates for an inverse problem for Schrödinger operators at high frequencies from arbitrary partial boundary measurements. *Inverse Probl.* **39** (2023) 125009, 21.
- [14] C. Kenig and M. Salo, Recent progress in the Calderón problem with partial data, in *Inverse Problems and Applications*. Contemporary Mathematics, vol. 615. American Mathematical Society, Providence, RI (2014) 193–222.
- [15] M. Horváth and Z. Markó, Discrete inverse problems for the Schrödinger operator on the multi-dimensional square lattice with partial Cauchy data. *Inverse Probl.* **32** (2016) 055006, 9.
- [16] S. Ervedoza and F. De Gournay, Uniform stability estimates for the discrete Calderón problems. *Inverse Probl.* **27** (2011) 125012, 37.
- [17] R. Lecaros, J.H. Ortega, A. Pérez and L. De Teresa, Discrete Calderón problem with partial data. *Inverse Probl.* **39** (2023) 035001, 28.
- [18] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptot. Anal.* **10** (1995) 95–115.
- [19] F. Boyer, V. Hernández-Santamaría and L.D. Teresa, Insensitizing controls for a semilinear parabolic equation: a numerical approach. *Math. Control Relat. Fields* **9** (2019) 117–158.
- [20] F. Boyer, F. Hubert and J.L. Rousseau, Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications. *SIAM J. Control Optim.* **48** (2010) 5357–5397.
- [21] F. Boyer, F. Hubert and J.L. Rousseau, Uniform controllability properties for space/time-discretized parabolic equations. *Numer. Math.* **118** (2011) 601–661.
- [22] F. Boyer and J.L. Rousseau, Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **31** (2014) 1035–1078.

- [23] E. Cerpa, R. Lecaros, T.N.T. Nguyen and A. Pérez, Carleman estimates and controllability for a semi-discrete fourth-order parabolic equation. *J. Math. Pures Appl.* **164** (2022) 93–130.
- [24] T.N.T. Nguyen, Carleman estimates for semi-discrete parabolic operators with a discontinuous diffusion coefficient and applications to controllability. *Math. Control Relat. Fields* **4** (2014) 203–259.
- [25] F. Boyer and V. Hernández-Santamaría, Carleman estimates for time-discrete parabolic equations and applications to controllability. *ESAIM Control Optim. Calc. Var.* **26** (2020) 43.
- [26] P.G. Casanova and V. Hernández-Santamaría, Carleman estimates and controllability results for fully discrete approximations of 1D parabolic equations. *Adv. Comput. Math.* **47** (2021) 71.
- [27] R. Lecaros, R. Morales, A. Pérez and S. Zamorano, Discrete Carleman estimates and application to controllability for a fully-discrete parabolic operator with dynamic boundary conditions. *J. Diff. Equ.* **365** (2023) 832–881.
- [28] L. Baudouin and S. Ervedoza, Convergence of an inverse problem for a 1-D discrete wave equation. *SIAM J. Control Optim.* **51** (2013) 556–598.
- [29] L. Baudouin, S. Ervedoza and A. Osses, Stability of an inverse problem for the discrete wave equation and convergence results. *J. Math. Pures Appl.* **103** (2015) 1475–1522.
- [30] W. Zhang and Z. Zhao, Convergence analysis of a coefficient inverse problem for the semi-discrete damped wave equation. *Appl. Anal.* **101** (2022) 1430–1455.
- [31] Z. Zhao and W. Zhang, Stability of a coefficient inverse problem for the discrete Schrödinger equation and a convergence result. *J. Math. Anal. Appl.* **518** (2023) 126665, 25.
- [32] B. Wu, Y. Wang and Z. Wang, Carleman estimates for space semi-discrete approximations of one-dimensional stochastic parabolic equation and its applications. *Inverse Probl.* **40** (2024) 115003, 33.
- [33] A. Fernández-Bertolin, L. Roncal, A. Rüländ and D. Stan, Discrete Carleman estimates and three balls inequalities. *Calc. Var. Part. Diff. Equ.* **60** (2021) 28.
- [34] G. Alessandrini, Stable determination of conductivity by boundary measurements. *Appl. Anal.* **27** (1988) 153–172.
- [35] A. Rüländ and M. Salo, Quantitative Runge approximation and inverse problems. *Int. Math. Res. Not.* **20** (2019) 6216–6234.
- [36] F. Boyer, F. Hubert and J.L. Rousseau, Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations. *J. Math. Pures Appl.* **93** (2010) 240–276.
- [37] R. Lecaros, J.H. Ortega and A. Pérez, Stability estimate for the semi-discrete linearized Benjamin–Bona–Mahony equation. *ESAIM Control Optim. Calc. Var.* **27** (2021) 30.
- [38] A.V. Fursikov and O.Y. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series, vol. 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul (1996).
- [39] J.L. Rousseau and G. Lebeau, On Carleman estimates for elliptic and parabolic operators. *ESAIM Control Optim. Calc. Var.* **18** (2012) 712–747.
- [40] M. Bellassoued, Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation. *Inverse Probl.* **20** (2004) 1033–1052.
- [41] M. Bellassoued and M. Yamamoto, Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation. *J. Math. Pures Appl.* **85** (2006) 193–224.
- [42] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Part. Diff. Equ.* **20** (1995) 335–356.



Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.

APPENDIX A. PROOFS OF INTERMEDIATE RESULTS

A.1 Proof of Lemma 3.14

By Assumption 2.2 and the second equality in Lemma 3.8, we have

$$\begin{aligned} \|rD_k A_k \rho D_k \sigma^k \tau_k D_k v\|_{L_h^2(\Omega_h)}^2 &\leq \int_{\Omega_h} |rD_k A_k \rho|^2 |\tau_k D_k v|^2 \leq C_{\lambda, \mathcal{R}} s^2 \int_{\Omega_h} |\tau_k D_k v|^2 \\ &= C_{\lambda, \mathcal{R}} s^2 \int_{\tau_k(\Omega_h)} |D_k v|^2 \leq C_{\lambda, \mathcal{R}} s^2 \int_{\Omega_{h,k}^*} |D_k v|^2. \end{aligned} \quad (\text{A.1})$$

Similarly, we have

$$\|rD_k A_k \rho D_k \sigma^k \tau_{-k} D_k v\|_{L_h^2(\Omega_h)}^2 \leq C_{\lambda, \mathcal{R}} s^2 \int_{\Omega_{h,k}^*} |D_k v|^2. \quad (\text{A.2})$$

By Assumption 2.2, Corollary 3.2 and (3.2), we observe that

$$\begin{aligned} \|rD_k^2 \rho D_k \sigma^k D_k A_k v\|_{L_h^2(\Omega_h)}^2 &\leq \int_{\Omega_h} |rD_k^2 \rho|^2 A_k (|D_k v|^2) \\ &= \int_{\Omega_{h,k}^*} A_k (|rD_k^2 \rho|^2) |D_k v|^2 - \frac{h}{2} \int_{\partial_k \Omega_h} |rD_k^2 \rho|^2 t_r^k (|D_k v|^2) \\ &\leq \int_{\Omega_{h,k}^*} A_k (|rD_k^2 \rho|^2) |D_k v|^2, \end{aligned}$$

which, by the third equality in Lemma 3.8, yields

$$\|rD_k^2 \rho D_k \sigma^k D_k A_k v\|_{L_h^2(\Omega_h)}^2 \leq C_{\lambda, \mathcal{R}} s^4 \int_{\Omega_{h,k}^*} |D_k v|^2. \quad (\text{A.3})$$

We also find

$$\|rA_k D_k \rho A_k D_k v\|_{L_h^2(\Omega_h)}^2 \leq C_{\lambda, \mathcal{R}} s^2 \int_{\Omega_{h,k}^*} |D_k v|^2. \quad (\text{A.4})$$

We note that

$$\|A_k^2 v\|_{L_h^2(\Omega_h)}^2 \leq \int_{\Omega_h} A_k (|A_k v|^2) = \int_{\Omega_{h,k}^*} |A_k v|^2 - \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k (|A_k v|^2) \leq \int_{\Omega_{h,k}^*} |A_k v|^2 \leq \int_{\Omega_h} |v|^2,$$

by Corollary 3.2, (3.2) and since $v|_{\partial\Omega_h} = 0$. Then by the second and third equality in Lemma 3.8, we have

$$\|(rD_k A_k \rho D_k \sigma^k + hO(1)rD_k^2 \rho) A_k^2 v\|_{L_h^2(\Omega_h)}^2 \leq C_{\lambda, \mathcal{R}} s^2 (1 + (sh)^2) \|v\|_{L_h^2(\Omega_h)}^2. \quad (\text{A.5})$$

Similarly, since $\Delta_\sigma \varphi$ is bounded, estimates (A.1)-(A.5) yield the result.

A.2 Proof of Lemma 3.15

From the definitions of $A_1 v$ and $B_1 v$ in (3.5) we have

$$\text{Re}(A_1 v, B_1 v)_{\Omega_h} = \sum_{j,k=1}^d \text{Re} \int_{\Omega_h} 2r^2 A_k^2 \rho D_k (\sigma^k D_k v) \sigma^j A_j D_j \rho A_j D_j \bar{v}$$

$$\begin{aligned}
&:= \sum_{j,k=1}^d \operatorname{Re} \int_{\Omega_h} 2\gamma_1^k D_k(\sigma^k D_k v) \beta_1^j \sigma^j A_j D_j \bar{v} \\
&:= \sum_{j,k=1}^d I_{jk},
\end{aligned}$$

where $\gamma_1^k = rA_k^2 \rho$ and $\beta_1^j = rA_j D_j \rho$.

A.2.1 Computation of I_{kk}

By Lemma 3.1 and (3.1), we have

$$\begin{aligned}
I_{kk} &= \operatorname{Re} \left(\int_{\Omega_h} 2\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k D_k^2 v D_k A_k \bar{v} + \int_{\Omega_h} 2\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k |A_k D_k v|^2 \right) \\
&= \int_{\Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k D_k (|D_k v|^2) + \int_{\Omega_h} 2\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k |A_k D_k v|^2 \\
&= - \int_{\Omega_{h,k}^*} D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2 + \int_{\partial_k \Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k (|D_k v|^2) n_k \\
&\quad + \int_{\Omega_h} 2\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k |A_k D_k v|^2.
\end{aligned} \tag{A.6}$$

The discrete integration by parts with respect to the average operator A_k (3.2) yields

$$\begin{aligned}
&- \int_{\Omega_{h,k}^*} D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2 \\
&= \int_{\Omega_{h,k}^*} D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2 - 2 \int_{\Omega_h} A_k (D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2) \\
&\quad - \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k (D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2).
\end{aligned} \tag{A.7}$$

A further use of Lemma 3.1 and the discrete integration by parts with respect to the difference operator D_k (3.1) yield

$$\begin{aligned}
&- 2 \int_{\Omega_h} A_k (D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2) \\
&= - 2 \int_{\Omega_h} A_k D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) A_k (|D_k v|^2) - \frac{h^2}{2} \int_{\Omega_h} D_k^2 (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) D_k (|D_k v|^2) \\
&= - 2 \int_{\Omega_h} A_k D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |A_k D_k v|^2 - \frac{h^2}{2} \int_{\Omega_h} A_k D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k^2 v|^2 \\
&\quad + \frac{h^2}{2} \int_{\Omega_{h,k}^*} D_k^3 (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) D_k |D_k v|^2 - \frac{h^2}{2} \int_{\partial_k \Omega_h} D_k^2 (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k (|D_k v|^2) n_k.
\end{aligned} \tag{A.8}$$

Combining (A.6)–(A.8), we have

$$\begin{aligned}
I_{kk} &= \int_{\Omega_{h,k}^*} D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2 - 2 \int_{\Omega_h} A_k D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |A_k D_k v|^2 \\
&\quad + 2 \int_{\Omega_h} \gamma_1^k \beta_1^k \sigma^k D_k \sigma^k |A_k D_k v|^2 - \frac{h^2}{2} \int_{\Omega_h} A_k D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k^2 v|^2 \\
&\quad + \frac{h^2}{2} \int_{\Omega_{h,k}^*} D_k^3(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) D_k |D_k v|^2 + Y_{1k},
\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}
Y_{1k} &= -\frac{h^2}{2} \int_{\partial_k \Omega_h} D_k^2(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k (|D_k v|^2) n_k - \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k (D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2) \\
&\quad + \int_{\partial_k \Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k (|D_k v|^2) n_k.
\end{aligned}$$

Lemma A.1. *For $sh \leq \mathcal{R}$, we have*

$$\begin{aligned}
\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k &= s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \\
D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) &= -s\lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 + s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \\
A_k D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) &= -s\lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 + s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \\
h^2 D_k^3(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) &= s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}(sh),
\end{aligned}$$

where $\gamma_1^k = rA_k^2 \rho$ and $\beta_1^k = rA_k D_k \rho$.

Proof. By the second equality in Lemma 3.10 and Lemma 3.6, we have

$$\begin{aligned}
\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k &= (r\partial_k \rho + sO_{\lambda, \mathcal{R}}((sh)^2)) \sigma^k D_k \sigma^k \\
&= (s\lambda\varphi \partial_k \psi + sO_{\lambda, \mathcal{R}}((sh)^2)) \sigma^k D_k \sigma^k.
\end{aligned}$$

Then with Assumption 2.2 we obtain the estimate of $\gamma_1^k \beta_1^k \sigma^k D_k \sigma^k$ since $\sigma^k D_k \sigma^k = O(1)$.

Applying Lemma 3.1, we get

$$\begin{aligned}
D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) &= D_k(\gamma_1^k \beta_1^k) A_k(\sigma^k A_k \sigma^k) + A_k(\gamma_1^k \beta_1^k) D_k(\sigma^k A_k \sigma^k) \\
&= D_k(\gamma_1^k \beta_1^k) ((\sigma^k)^2 + hO(1)) + A_k(\gamma_1^k \beta_1^k) O(1),
\end{aligned}$$

since $\|A_k \sigma^k - \sigma^k\|_\infty \leq Ch$ and $D_k(\sigma^k A_k \sigma^k) = D_k(\sigma^k) A_k^2 \sigma^k + A_k \sigma^k A_k D_k \sigma^k = O(1)$. Moreover, by the second equality in Lemma 3.10 and Lemma 3.6, we have

$$\begin{aligned}
D_k(\gamma_1^k \beta_1^k) &= \partial_k(r\partial_k \rho) + sO_{\lambda, \mathcal{R}}((sh)^2) = -s\lambda^2 \varphi(\partial_k \psi)^2 + s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}((sh)^2), \\
A_k(\gamma_1^k \beta_1^k) &= r\partial_k \rho + sO_{\lambda, \mathcal{R}}((sh)^2) = s\lambda\varphi \partial_k \psi + sO_{\lambda, \mathcal{R}}((sh)^2).
\end{aligned}$$

Then we get

$$D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) = -s\lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 + s\lambda\varphi O(1) + sO_{\lambda, \mathcal{R}}(sh).$$

Similarly, we obtain the third estimate.

To estimate $h^2 D_k^3(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k)$, introducing $\alpha_1 = \sigma^k A_k \sigma^k$ and $\alpha_2 = \gamma_1^k \beta_1^k$ we first write

$$D_k^3(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) = D_k^3 \alpha_2 A_k^3 \alpha_1 + 3D_k^2 A_k \alpha_2 A_k^2 D_k \alpha_1 + 3A_k^2 D_k \alpha_2 D_k^2 A_k \alpha_1 + A_k^3 \alpha_2 D_k^3 \alpha_1.$$

We note that we have

$$A_k^3 \alpha_1 = O(1), \quad A_k^2 D_k \alpha_1 = O(1), \quad h D_k^2 A_k \alpha_1 = O(1), \quad h^2 D_k^3 \alpha_1 = O(1),$$

and, with the second equality in Lemma 3.10,

$$\begin{aligned} h D_k^3 \alpha_2 &= s O_{\lambda, \mathcal{R}}(1), & D_k^2 A_k \alpha_2 &= s O_{\lambda, \mathcal{R}}(1), \\ A_k^2 D_k \alpha_2 &= s O_{\lambda, \mathcal{R}}(1), & A_k^3 \alpha_2 &= s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}((sh)^2). \end{aligned}$$

The estimate for $h^2 D_k^3(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k)$ then follows. □

By (A.9) and Lemma A.1, we get

$$\begin{aligned} I_{kk} &= - \int_{\Omega_{h,k}^*} s \lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 |D_k v|^2 + \int_{\Omega_{h,k}^*} (s \lambda \varphi O(1) + s O_{\lambda \mathcal{R}}(sh)) |D_k v|^2 \\ &\quad + 2 \int_{\Omega_h} s \lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 |A_k D_k v|^2 + \int_{\Omega_h} (s \lambda \varphi O(1) + s O_{\lambda \mathcal{R}}(sh)) |A_k D_k v|^2 \\ &\quad + \frac{h^2}{2} \int_{\Omega_h} s \lambda^2 \varphi(\sigma^k)^2 (\partial_k \psi)^2 |D_k^2 v|^2 + h^2 \int_{\Omega_h} (s \lambda \varphi O(1) + s O_{\lambda \mathcal{R}}(sh)) |D_k^2 v|^2 + Y_{1k}, \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} Y_{1k} &= - \frac{h^2}{2} \int_{\partial_k \Omega_h} D_k^2(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k (|D_k v|^2) n_k - \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k (D_k(\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k)) |D_k v|^2 \\ &\quad + \int_{\partial_k \Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k (|D_k v|^2) n_k. \end{aligned}$$

A.2.2 Computation of I_{jk} , $k \neq j$

The discrete integration by parts with respect to the difference operator D_k (3.1) gives

$$I_{jk} = -\operatorname{Re} \int_{\Omega_{h,k}^*} D_k(2\gamma_1^k \beta_1^j \sigma^j A_j D_j \bar{v}) \sigma^k D_k v,$$

since $A_j D_j \bar{v} = 0$ in $\partial_k \Omega_h$ when $k \neq j$. Moreover, I_{jk} can be rewritten as

$$\begin{aligned} I_{jk} &= -\operatorname{Re} \int_{\Omega_{h,k}^*} D_k(2\gamma_1^k \beta_1^j \sigma^j) A_k A_j D_j \bar{v} \sigma^k D_k v - \operatorname{Re} \int_{\Omega_{h,k}^*} A_k(2\gamma_1^k \beta_1^j \sigma^j) D_k A_j D_j \bar{v} \sigma^k D_k v \\ &:= I_{jk}^{(1)} + I_{jk}^{(2)}, \end{aligned}$$

due to Lemma 3.1. Integrating by parts with respect to the average operator A_k (3.2) for $I_{jk}^{(1)}$ gives

$$\begin{aligned} I_{jk}^{(1)} &= -\operatorname{Re} \int_{\Omega_h} A_k(D_k(2\gamma_1^k \beta_1^j \sigma^j) \sigma^k D_k v) A_j D_j \bar{v} \\ &= -\operatorname{Re} \left(\int_{\Omega_h} A_k(D_k(2\gamma_1^k \beta_1^j \sigma^j) \sigma^k) A_k D_k v A_j D_j \bar{v} - \frac{h^2}{2} \int_{\Omega_h} D_k(D_k(\gamma_1^k \beta_1^j \sigma^j) \sigma^k) D_k^2 v A_j D_j \bar{v} \right), \end{aligned}$$

where we have used $A_j D_j \bar{v} = 0$ in $\partial_k \Omega_h$ for $k \neq j$ and Lemma 3.1. Similarly, we also have

$$\begin{aligned} I_{jk}^{(2)} &= -\operatorname{Re} \int_{\Omega_{h,jk}^*} A_j (A_k (2\gamma_1^k \beta_1^j \sigma^j) \sigma^k D_k v) D_k D_j \bar{v} \\ &= -\operatorname{Re} \int_{\Omega_{h,jk}^*} A_j (A_k (2\gamma_1^k \beta_1^j \sigma^j) \sigma^k) A_j D_k v D_k D_j \bar{v} - \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) |D_k D_j v|^2 \\ &= \int_{\Omega_{h,k}^*} D_j A_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) |D_k v|^2 - \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) |D_k D_j v|^2, \end{aligned}$$

where we have used $D_k v = 0$ on $\partial_j(\Omega_{h,k}^*)$ for $k \neq j$ and $2A_j D_k v D_k D_j \bar{v} = D_j(|D_k v|^2)$.

We thus have

$$\begin{aligned} I_{jk} &= \int_{\Omega_{h,k}^*} D_j A_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) |D_k v|^2 - 2\operatorname{Re} \int_{\Omega_h} A_k (D_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) A_k D_k v A_j D_j \bar{v} \\ &\quad - \frac{h^2}{2} \operatorname{Re} \int_{\Omega_h} D_k (D_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) D_k^2 v A_j D_j \bar{v} - \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) |D_k D_j v|^2. \end{aligned} \tag{A.11}$$

Lemma A.2. For $sh \leq \mathcal{R}$ and $1 \leq j, k \leq d$, we have

$$\begin{aligned} D_j A_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) &= -s\lambda^2 \varphi \sigma^j \sigma^k (\partial_j \psi)^2 + s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \\ A_k (D_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) &= -s\lambda^2 \varphi \sigma^j \sigma^k (\partial_k \psi) (\partial_j \psi) + s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \\ hD_k (D_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) &= s\lambda \varphi O(1) + O_{\lambda, \mathcal{R}}(sh), \\ D_j (A_k (\gamma_1^k \beta_1^j \sigma^j) \sigma^k) &= -s\lambda^2 \varphi \sigma^j \sigma^k (\partial_j \psi)^2 + s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh), \end{aligned}$$

where $\gamma_1^k = rA_k^2 \rho$ and $\beta_1^j = rA_j D_j \rho$.

Proof. The estimates all follow from Lemma 3.6 and the second equality in Lemma 3.10, arguing as in the proof of Lemma A.1. \square

With (A.11) and Lemma A.2, we obtain

$$\begin{aligned} I_{jk} &= - \int_{\Omega_{h,k}^*} s\lambda^2 \varphi \sigma^j \sigma^k (\partial_j \psi)^2 |D_k v|^2 + \int_{\Omega_{h,k}^*} (s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh)) |D_k v|^2 \\ &\quad + 2\operatorname{Re} \left(\int_{\Omega_h} s\lambda^2 \varphi \sigma^j \sigma^k (\partial_k \psi) (\partial_j \psi) A_k D_k v A_j D_j \bar{v} + \int_{\Omega_h} (s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh)) A_k D_k v A_j D_j \bar{v} \right) \\ &\quad + \frac{h^2}{2} \int_{\Omega_{h,jk}^*} s\lambda^2 \varphi \sigma^j \sigma^k (\partial_j \psi)^2 |D_k D_j v|^2 + h^2 \int_{\Omega_{h,jk}^*} (s\lambda \varphi O(1) + sO_{\lambda, \mathcal{R}}(sh)) |D_k D_j v|^2 \\ &\quad + h\operatorname{Re} \int_{\Omega_h} (s\lambda \varphi O(1) + O_{\lambda, \mathcal{R}}(sh)) D_k^2 v A_j D_j \bar{v}. \end{aligned} \tag{A.12}$$

A.2.3 Estimate of $\operatorname{Re}(A_1 v, B_1 v)_{\Omega_h}$

We now collect the terms (A.10) and (A.12) to write

$$\operatorname{Re}(A_1 v, B_1 v)_{\Omega_h} = - \sum_{k=1}^d \int_{\Omega_{h,k}^*} s\lambda^2 \varphi \sigma^k |\nabla_\sigma \psi|^2 |D_k v|^2 + Y_1 + \sum_{i=1}^5 X_i,$$

where

$$Y_1 = \sum_{k=1}^d Y_{1k} = \sum_{k=1}^d \int_{\partial_k \Omega_h} \gamma_1^k \beta_1^k \sigma^k A_k \sigma^k t_r^k (|D_k v|^2) n_k - \frac{h}{2} \sum_{k=1}^d \int_{\partial_k \Omega_h} t_r^k (D_k (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) |D_k v|^2) \\ - \frac{h^2}{2} \sum_{k=1}^d \int_{\partial_k \Omega_h} D_k^2 (\gamma_1^k \beta_1^k \sigma^k A_k \sigma^k) t_r^k (|D_k v|^2) n_k,$$

$$X_1 = 2 \sum_{k=1}^d \int_{\Omega_h} s \lambda^2 \varphi (\sigma^k)^2 (\partial_k \psi)^2 |A_k D_k v|^2 + 2 \sum_{\substack{j,k=1 \\ j \neq k}}^d \operatorname{Re} \int_{\Omega_h} s \lambda^2 \varphi \sigma^j \sigma^k (\partial_k \psi) (\partial_j \psi) A_k D_k v A_j D_j \bar{v} \\ = 2 \int_{\Omega_h} s \lambda^2 \varphi \left| \sum_{k=1}^d \sigma^k (\partial_k \psi) A_k D_k v \right|^2 \geq 0,$$

$$X_2 = \frac{h^2}{2} \sum_{k=1}^d \int_{\Omega_h} s \lambda^2 \varphi (\sigma^k)^2 (\partial_k \psi)^2 |D_k^2 v|^2 + \frac{h^2}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^d \int_{\Omega_{h,jk}^*} s \lambda^2 \varphi \sigma^j \sigma^k (\partial_j \psi)^2 |D_k D_j v|^2 \geq 0,$$

$$X_3 = \sum_{k=1}^d \int_{\Omega_{h,k}^*} (s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}(sh)) |D_k v|^2 + \sum_{k=1}^d \int_{\Omega_h} (s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}(sh)) |A_k D_k v|^2,$$

$$X_4 = \sum_{\substack{j,k=1 \\ j \neq k}}^d \operatorname{Re} \int_{\Omega_h} (s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}(sh)) A_k D_k v A_j D_j \bar{v}$$

and

$$X_5 = \sum_{\substack{j,k=1 \\ j \neq k}}^d \int_{\Omega_{h,jk}^*} h^2 (s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}(sh)) |D_k D_j v|^2 + \sum_{k=1}^d \int_{\Omega_h} h^2 (s \lambda \varphi O(1) + s O_{\lambda, \mathcal{R}}(sh)) |D_k^2 v|^2 \\ + \sum_{\substack{j,k=1 \\ j \neq k}}^d \operatorname{Re} \int_{\Omega_h} h (s \lambda \varphi O(1) + O_{\lambda, \mathcal{R}}(sh)) D_k^2 v A_j D_j \bar{v}.$$

We conclude with Cauchy-Schwarz inequalities that yields

$$|X_4| \leq \sum_{k=1}^d \int_{\Omega_h} (s \lambda \varphi |O(1)| + s |O_{\lambda, \mathcal{R}}(sh)|) |A_k D_k v|^2,$$

and

$$|X_5| \leq \sum_{k=1}^d \int_{\Omega_h} h^2 (s \lambda \varphi |O(1)| + s |O_{\lambda, \mathcal{R}}(sh)|) |D_k^2 v|^2 + \sum_{k=1}^d \int_{\Omega_h} (s \lambda \varphi |O(1)| + |O_{\lambda, \mathcal{R}}(sh)|) |A_k D_k v|^2 \\ + \sum_{\substack{j,k=1 \\ j \neq k}}^d \int_{\Omega_{h,jk}^*} h^2 (s \lambda \varphi |O(1)| + s |O_{\lambda, \mathcal{R}}(sh)|) |D_k D_j v|^2.$$

Then we complete the proof of Lemma 3.15.

A.3 Proof of Lemma 3.16

From the definitions of A_2v and B_1v in (3.5) we have

$$\begin{aligned} \operatorname{Re}(A_2v, B_1v)_{\Omega_h} &= \sum_{j,k=1}^d \operatorname{Re} \int_{\Omega_h} 2\sigma^j r^2 D_j^2 \rho A_j^2 v \sigma^k A_k D_k \rho A_k D_k \bar{v} \\ &:= \sum_{j,k=1}^d \operatorname{Re} \int_{\Omega_h} 2\gamma_2^j \beta_1^k \sigma^j \sigma^k A_j^2 v A_k D_k \bar{v} \\ &:= \sum_{j,k=1}^d J_{jk}, \end{aligned}$$

where $\gamma_2^j = rD_j^2 \rho$ and $\beta_1^k = rA_k D_k \rho$.

A.3.1 Computation of J_{kk}

Since $2A_k^2 v D_k A_k \bar{v} = D_k(|A_k v|^2)$, we have

$$\begin{aligned} J_{kk} &= \int_{\Omega_h} \gamma_2^k \beta_1^k (\sigma^k)^2 D_k(|A_k v|^2) \\ &= - \int_{\Omega_{h,k}^*} D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) |A_k v|^2 + \int_{\partial_k \Omega_h} \gamma_2^k \beta_1^k (\sigma^k)^2 t_r^k (|A_k v|^2) n_k \\ &= - \int_{\Omega_{h,k}^*} D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) A_k(|v|^2) + \frac{h^2}{4} \int_{\Omega_{h,k}^*} D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) |D_k v|^2 \\ &\quad + \int_{\partial_k \Omega_h} \gamma_2^k \beta_1^k (\sigma^k)^2 t_r^k (|A_k v|^2) n_k \\ &= - \int_{\Omega_h} A_k D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) |v|^2 + \frac{h^2}{4} \int_{\Omega_{h,k}^*} D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) |D_k v|^2 \\ &\quad + \int_{\partial_k \Omega_h} \gamma_2^k \beta_1^k (\sigma^k)^2 t_r^k (|A_k v|^2) n_k, \end{aligned}$$

where we have used (3.1), (3.2) and the fact that $v = 0$ on $\partial_k \Omega_h$.

Lemma A.3. *For $sh \leq \mathcal{R}$, we have*

$$\begin{aligned} A_k D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) &= -3s^3 \lambda^4 \varphi^3 (\sigma^k)^2 (\partial_k \psi)^4 + (s\lambda\varphi)^3 O(1) + s^2 O_{\lambda, \mathcal{R}}(1) + s^3 O_{\lambda, \mathcal{R}}((sh)^2), \\ D_k(\gamma_2^k \beta_1^k (\sigma^k)^2) &= s^3 O_{\lambda, \mathcal{R}}(1), \end{aligned}$$

where $\gamma_2^k = rD_k^2 \rho$ and $\beta_1^k = rA_k D_k \rho$.

Proof. The estimates follow from the first equality in Lemma 3.10 and Lemma 3.7 arguing as in the proof of Lemma A.1. \square

Then we get

$$\begin{aligned} J_{kk} &= 3 \int_{\Omega_h} s^3 \lambda^4 \varphi^3 (\sigma^k)^2 (\partial_k \psi)^4 |v|^2 + \int_{\Omega_h} ((s\lambda\varphi)^3 O(1) + s^2 O_{\lambda, \mathcal{R}}(1) + s^3 O_{\lambda, \mathcal{R}}((sh)^2)) |v|^2 \\ &\quad + \int_{\Omega_{h,k}^*} s O_{\lambda, \mathcal{R}}((sh)^2) |D_k v|^2 + Y_{2k}, \end{aligned} \tag{A.13}$$

where

$$Y_{2k} = \int_{\partial_k \Omega_h} \gamma_2^k \beta_1^k (\sigma^k)^2 t_r^k (|A_k v|^2) n_k.$$

A.3.2 Computation of J_{jk} , $j \neq k$

The discrete integration by parts with respect to the average operator A_j (3.2) gives

$$J_{jk} = \operatorname{Re} \int_{\Omega_{h,j}^*} A_j (2\gamma_2^j \beta_1^k \sigma^j \sigma^k A_k D_k \bar{v}) A_j v,$$

since $A_k D_k \bar{v} = 0$ on $\partial_j \Omega_h$ when $k \neq j$. Moreover, J_{jk} can be rewritten as

$$\begin{aligned} J_{jk} &= \operatorname{Re} \int_{\Omega_{h,j}^*} A_j (2\gamma_2^j \beta_1^k \sigma^j \sigma^k) A_j A_k D_k \bar{v} A_j v + \frac{h^2}{2} \operatorname{Re} \int_{\Omega_{h,j}^*} D_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) D_j A_k D_k \bar{v} A_j v \\ &:= J_{jk}^{(1)} + \frac{h^2}{2} \operatorname{Re} \int_{\Omega_{h,j}^*} D_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) D_j A_k D_k \bar{v} A_j v. \end{aligned} \quad (\text{A.14})$$

due to Lemma 3.1. Further, by (3.2) and Lemma 3.1, we have

$$\begin{aligned} J_{jk}^{(1)} &= \operatorname{Re} \int_{\Omega_{h,jk}^*} A_k (A_j (2\gamma_2^j \beta_1^k \sigma^j \sigma^k) A_j v) A_j D_k \bar{v} \\ &= \operatorname{Re} \int_{\Omega_{h,jk}^*} A_k A_j (2\gamma_2^j \beta_1^k \sigma^j \sigma^k) A_k A_j v A_j D_k \bar{v} + \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j D_k v|^2 \\ &= \int_{\Omega_{h,jk}^*} A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) D_k (|A_j v|^2) + \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j D_k v|^2 \\ &:= J_{jk}^{(11)} + \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j D_k v|^2, \end{aligned} \quad (\text{A.15})$$

where we have used $A_j v = 0$ on $\partial_k(\Omega_{h,j}^*)$ for $k \neq j$. Using (3.1), Lemma 3.1 and (3.2), we obtain

$$\begin{aligned} J_{jk}^{(11)} &= - \int_{\Omega_{h,j}^*} D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j v|^2 \\ &= - \int_{\Omega_{h,j}^*} D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) A_j (|v|^2) + \frac{h^2}{4} \int_{\Omega_{h,j}^*} D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |D_j v|^2 \\ &= - \int_{\Omega_h} D_k A_k A_j^2 (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |v|^2 + \frac{h^2}{4} \int_{\Omega_{h,j}^*} D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |D_j v|^2, \end{aligned} \quad (\text{A.16})$$

by virtue of $A_j v = 0$ on $\partial_k(\Omega_{h,j}^*)$ for $k \neq j$ and $v = 0$ on $\partial_j \Omega_h$.

Combing (A.14)-(A.16), we have

$$\begin{aligned} J_{jk} &= - \int_{\Omega_h} D_k A_k A_j^2 (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |v|^2 + \frac{h^2}{2} \int_{\Omega_{h,jk}^*} D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j D_k v|^2 \\ &\quad + \frac{h^2}{4} \int_{\Omega_{h,j}^*} D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |D_j v|^2 + \frac{h^2}{2} \operatorname{Re} \int_{\Omega_{h,j}^*} D_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) D_j A_k D_k \bar{v} A_j v. \end{aligned} \quad (\text{A.17})$$

Lemma A.4. For $sh \leq \mathcal{R}$ and $1 \leq j, k \leq d$, we have

$$\begin{aligned} D_k A_k A_j^2 (\gamma_2^j \beta_1^k \sigma^j \sigma^k) &= -3s^3 \lambda^4 \varphi^3 \sigma_j \sigma^k (\partial_k \psi)^2 (\partial_j \psi)^2 \\ &\quad + (s\lambda\varphi)^3 O(1) + s^2 O_{\lambda, \mathcal{R}}(1) + s^3 O_{\lambda, \mathcal{R}}((sh)^2), \\ D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) &= s^3 O_{\lambda, \mathcal{R}}(1), \\ D_k A_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) &= s^3 O_{\lambda, \mathcal{R}}(1), \\ D_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) &= s^3 O_{\lambda, \mathcal{R}}(1), \end{aligned}$$

where $\gamma_2^j = rD_j^2 \rho$ and $\beta_1^k = rA_k D_k \rho$.

Proof. The estimates follow from the first equality in Lemma 3.10 and Lemma 3.7 arguing as in the proof of Lemma A.1. \square

Using Lemma A.4 and the facts that $|A_j D_k v|^2 \leq A_j (|D_k v|^2)$ and $D_k v = 0$ on $\partial_j(\Omega_{h,k}^*)$, we get

$$\begin{aligned} \frac{h^2}{2} \left| \int_{\Omega_{h,jk}^*} D_k A_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) |A_j D_k v|^2 \right| &\leq \int_{\Omega_{h,jk}^*} s |O_{\lambda, \mathcal{R}}((sh)^2)| A_j (|D_k v|^2) \\ &= \int_{\Omega_{h,k}^*} s |O_{\lambda, \mathcal{R}}((sh)^2)| |D_k v|^2. \end{aligned} \quad (\text{A.18})$$

By Young's inequality and Lemma A.4 we note that

$$\begin{aligned} &\frac{h^2}{2} \left| \operatorname{Re} \int_{\Omega_{h,j}^*} D_j (\gamma_2^j \beta_1^k \sigma^j \sigma^k) D_j A_k D_k \bar{v} A_j v \right| \\ &\leq s^3 (sh) \int_{\Omega_{h,j}^*} |O_{\lambda, \mathcal{R}}(1)| |A_j v|^2 + sh^2 (sh) \int_{\Omega_{h,j}^*} |O_{\lambda, \mathcal{R}}(1)| |D_j D_k A_k v|^2 \\ &\leq s^3 (sh) \int_{\Omega_{h,j}^*} |O_{\lambda, \mathcal{R}}(1)| A_j (|v|^2) + s (sh) \int_{\Omega_{h,j}^*} |O_{\lambda, \mathcal{R}}(1)| A_j (|D_k A_k v|^2) \\ &= s^3 \int_{\Omega_h} |O_{\lambda, \mathcal{R}}(sh)| |v|^2 + s \int_{\Omega_h} |O_{\lambda, \mathcal{R}}(sh)| |D_k A_k v|^2, \end{aligned} \quad (\text{A.19})$$

since $v = 0$ on $\partial_j \Omega_h$ and $D_k A_k v = 0$ on $\partial_j \Omega_h$ for $k \neq j$ and using Corollary 3.2.

In terms of (A.17)-(A.19) and Lemma A.4, we obtain

$$\begin{aligned} J_{jk} &\geq 3 \int_{\Omega_h} s^3 \lambda^4 \varphi^3 \sigma_j \sigma^k (\partial_k \psi)^2 (\partial_j \psi)^2 |v|^2 - \int_{\Omega_h} s |O_{\lambda, \mathcal{R}}(sh)| |D_k A_k v|^2 \\ &\quad - \int_{\Omega_h} ((s\lambda\varphi)^3 |O(1)| + s^2 |O_{\lambda, \mathcal{R}}(1)| + s^3 |O_{\lambda, \mathcal{R}}(sh)|) |v|^2 \\ &\quad - \int_{\Omega_{h,k}^*} s |O_{\lambda, \mathcal{R}}((sh)^2)| |D_k v|^2 - \int_{\Omega_{h,j}^*} s |O_{\lambda, \mathcal{R}}((sh)^2)| |D_j v|^2 \end{aligned} \quad (\text{A.20})$$

A.3.3 Estimate of $\operatorname{Re}(A_2 v, B_1 v)_{\Omega_h}$

Combining (A.13) and (A.20), the Lemma follows.

A.4 Proof of Lemma 3.17

From the definitions of $A_1 v$ and $B_2 v$ in (3.5) we have

$$\operatorname{Re}(A_1 v, B_2 v)_{\Omega_h} = - \sum_{k=1}^d \operatorname{Re} \int_{\Omega_h} 2sr A_k^2 \rho D_k(\sigma^k D_k v)(\Delta_\sigma \varphi) \bar{v} \quad (\text{A.21})$$

For concision we now set $\omega_1^k = r A_k^2 \rho(\Delta_\sigma \varphi)$. The discrete integration by parts (3.1) yields

$$\begin{aligned} -\operatorname{Re} \int_{\Omega_h} 2s \omega_1^k D_k(\sigma^k D_k v) \bar{v} &= \operatorname{Re} \int_{\Omega_{h,k}^*} 2s D_k(\omega_1^k \bar{v}) \sigma^k D_k v \\ &= \int_{\Omega_{h,k}^*} 2s A_k(\omega_1^k) \sigma^k |D_k v|^2 + \operatorname{Re} \int_{\Omega_{h,k}^*} 2s D_k(\omega_1^k) \sigma^k A_k \bar{v} D_k v, \end{aligned} \quad (\text{A.22})$$

where we have used $\bar{v} = 0$ on $\partial_k \Omega_h$ and Lemma 3.1.

Lemma A.5. *For $sh \leq \mathcal{R}$, we have*

$$\begin{aligned} A_k(\omega_1^k) &= \lambda^2 \varphi |\nabla_\sigma \psi|^2 + \lambda \varphi O(1) + O_{\lambda, \mathcal{R}}(h + (sh)^2), \\ D_k(\omega_1^k) &= O_{\lambda, \mathcal{R}}(1), \end{aligned}$$

where $\omega_1^k = r A_k^2 \rho(\Delta_\sigma \varphi)$.

Proof. Observing that

$$\Delta_\sigma \varphi = \sum_{k=1}^d \sigma^k \partial_k^2 \varphi = - \sum_{k=1}^d \sigma^k \partial_k (\lambda \varphi \partial_k \psi) = \lambda^2 \varphi |\nabla_\sigma \psi|^2 + \lambda \varphi O(1),$$

by the first equality in Lemma 3.8 and the second equality in Lemma 3.9, we obtain the desired results. \square

By the discrete integration by parts (3.2) and Young's inequality, we obtain

$$\begin{aligned} \left| \int_{\Omega_{h,k}^*} s O_{\lambda, \mathcal{R}}(1) \sigma^k A_k \bar{v} D_k v \right| &\leq \int_{\Omega_{h,k}^*} s^2 |O_{\lambda, \mathcal{R}}(1)| |A_k v|^2 + \int_{\Omega_{h,k}^*} |O_{\lambda, \mathcal{R}}(1)| |D_k v|^2 \\ &\leq \int_{\Omega_{h,k}^*} s^2 |O_{\lambda, \mathcal{R}}(1)| A_k(|v|^2) + \int_{\Omega_{h,k}^*} |O_{\lambda, \mathcal{R}}(1)| |D_k v|^2 \\ &= \int_{\Omega_h} s^2 |O_{\lambda, \mathcal{R}}(1)| |v|^2 + \int_{\Omega_k^*} |O_{\lambda, \mathcal{R}}(1)| |D_k v|^2, \end{aligned} \quad (\text{A.23})$$

since $|A_k v|^2 \leq A_k(|v|^2)$ and $v = 0$ on $\partial_k \Omega_h$.

Consequently, we obtain the estimate of $\operatorname{Re}(A_1 v, B_2 v)_{\Omega_h}$ from (A.21), (A.22), Lemma A.5 and (A.23).

Thus the proof of Lemma 3.17 is complete.

A.5 Proof of Lemma 3.18

From the definitions of $A_2 v$ and $B_2 v$ in (3.5) we have

$$\operatorname{Re}(A_2 v, B_2 v)_{\Omega_h} = - \sum_{k=1}^d \operatorname{Re} \int_{\Omega_h} 2s \sigma^k r D_k^2 \rho A_k^2 v(\Delta_\sigma \varphi) \bar{v}$$

$$\begin{aligned}
&= \sum_{k=1}^d \left(- \int_{\Omega_h} 2s\sigma^k r D_k^2 \rho(\Delta_\sigma \varphi) |v|^2 - \frac{h^2}{2} \operatorname{Re} \int_{\Omega_h} s\sigma^k r D_k^2 \rho(\Delta_\sigma \varphi) \bar{v} D_k^2 v \right) \\
&:= \sum_{k=1}^d \left(- \int_{\Omega_h} 2s\omega_2^k |v|^2 - \frac{h^2}{2} \operatorname{Re} \int_{\Omega_h} s\omega_2^k \bar{v} D_k^2 v \right) \\
&:= Q_{1k} + Q_{2k},
\end{aligned} \tag{A.24}$$

where $\omega_2^k = \sigma^k r D_k^2 \rho(\Delta_\sigma \varphi)$ and we have used $A_k^2 v = v + \frac{h^2}{4} D_k^2 v$. By using integration by parts (3.1) twice, we can prove

$$\begin{aligned}
Q_{2k} &= \frac{sh^2}{2} \operatorname{Re} \int_{\Omega_{h,k}^*} D_k(\omega_2^k \bar{v}) D_k v \\
&= \frac{sh^2}{2} \int_{\Omega_{h,k}^*} A_k(\omega_2^k) |D_k v|^2 + \frac{sh^2}{4} \int_{\Omega_{h,k}^*} D_k(\omega_2^k) D_k(|v|^2) \\
&= \frac{sh^2}{2} \int_{\Omega_{h,k}^*} A_k(\omega_2^k) |D_k v|^2 - \frac{sh^2}{4} \int_{\Omega_h} D_k^2(\omega_2^k) |v|^2,
\end{aligned} \tag{A.25}$$

where we have used $v = 0$ on $\partial_k \Omega_h$ and $A_k \bar{v} D_k v = \frac{1}{2} D_k(|v|^2)$.

Similar to Lemma A.5 and Lemma A.1, we have the following Lemma.

Lemma A.6. *For $sh \leq \mathcal{R}$, we have*

$$\begin{aligned}
\omega_2^k &= s^2 \lambda^4 \varphi^3 |\nabla_\sigma \psi|^2 \sigma^k (\partial_k \psi)^2 + s^2 \lambda^3 \varphi^3 O(1) + s O_\lambda(1) + s^2 O_{\lambda, \mathcal{R}}((sh)^2), \\
A_k(\omega_2^k) &= s^2 O_{\lambda, \mathcal{R}}(1), \\
h^2 D_k^2(\omega_2^k) &= s O_{\lambda, \mathcal{R}}(sh),
\end{aligned}$$

where $\omega_2^k = \sigma^k r D_k^2 \rho(\Delta_\sigma \varphi)$.

Then we get the estimate of $\operatorname{Re}(A_2 v, B_2 v)_{\Omega_h}$ from (A.24), (A.25) and Lemma A.6. Thus the proof of Lemma 3.18 is complete.

A.6 Proof of Lemma 3.19

We set $\omega_3^k = \varphi \sigma^k |\nabla_\sigma \psi|^2$. For $k, j = 1, \dots, d$ with $k \neq j$, by (3.2), (3.1) and Lemma 3.1, we have

$$\begin{aligned}
\int_{\Omega_{h,k}^*} \omega_3^k |D_k v|^2 &= \int_{\Omega_{h,k,j}^*} A_j(\omega_3^k) |D_k v|^2 \\
&= \int_{\Omega_{h,k,j}^*} A_j(\omega_3^k) A_j(|D_k v|^2) + \frac{h^2}{4} \int_{\Omega_{h,k,j}^*} D_j(\omega_3^k) D_j(|D_k v|^2) \\
&= \int_{\Omega_{h,k,j}^*} A_j(\omega_3^k) |A_j D_k v|^2 + \frac{h^2}{4} \int_{\Omega_{h,k,j}^*} A_j(\omega_3^k) |D_j D_k v|^2 - \frac{h^2}{4} \int_{\Omega_{h,k}^*} D_j^2(\omega_3^k) |D_k v|^2 \\
&\geq \frac{h^2}{4} \int_{\Omega_{h,k,j}^*} A_j(\omega_3^k) |D_j D_k v|^2 - \frac{h}{4} \int_{\Omega_{h,k}^*} ((\tau_j - \tau_{-j}) D_j(\omega_3^k)) |D_k v|^2,
\end{aligned}$$

since $D_k v = 0$ on $\partial_j(\Omega_{h,k}^*)$.

On the other hand, for $k = 1, \dots, d$, by (3.2), (3.1) and Lemma 3.1, we have

$$\int_{\Omega_{h,k}^*} \omega_3^k |D_k v|^2 = \int_{\Omega_h} A_k(\omega_3^k) |D_k v|^2 + \frac{h}{2} \int_{\partial_k \Omega_h} t_r^k(\omega_3^k) |D_k v|^2$$

$$\begin{aligned}
&\geq \int_{\Omega_h} A_k(\omega_3^k) A_k(|D_k v|^2) + \frac{h^2}{4} \int_{\Omega_h} D_k(\omega_3^k) D_k(|D_k v|^2) \\
&= \int_{\Omega_h} A_k(\omega_3^k) |A_k D_k v|^2 + \frac{h^2}{4} \int_{\Omega_h} A_k(\omega_3^k) |D_k^2 v|^2 \\
&\quad - \frac{h}{4} \int_{\Omega_{h,k}^*} ((\tau_k - \tau_{-k}) D_k(\omega_3^k)) |D_k v|^2 + \frac{h^2}{4} \int_{\partial_k \Omega_h} D_k(\omega_3^k) t_r^k (|D_k v|^2) n_k.
\end{aligned}$$

We thus have

$$\begin{aligned}
\sum_{j,k=1}^d \int_{\Omega_{h,k}^*} \omega_3^k |D_k v|^2 &\geq \sum_{\substack{j,k=1 \\ j \neq k}}^d \left(\frac{h^2}{4} \int_{\Omega_{h,kj}^*} A_j(\omega_3^k) |D_j D_k v|^2 - \frac{h}{4} \int_{\Omega_{h,k}^*} ((\tau_j - \tau_{-j}) D_j(\omega_3^k)) |D_k v|^2 \right) \\
&\quad + \sum_{k=1}^d \left(\frac{h^2}{4} \int_{\Omega_h} A_k(\omega_3^k) |D_k^2 v|^2 - \frac{h}{4} \int_{\Omega_{h,k}^*} ((\tau_k - \tau_{-k}) D_k(\omega_3^k)) |D_k v|^2 \right) \\
&\quad + \sum_{k=1}^d \int_{\Omega_h} A_k(\omega_3^k) |A_k D_k v|^2 + Y_3,
\end{aligned} \tag{A.26}$$

where

$$Y_3 = \frac{h^2}{4} \sum_{k=1}^d \int_{\partial_k \Omega_h} D_k(\omega_3^k) t_r^k (|D_k v|^2) n_k.$$

Moreover, for $j, k = 1, \dots, d$, we have

$$A_j(\omega_3^k) = \omega_3^k + hO_{\lambda, \mathcal{R}}(1), \quad (\tau_j - \tau_{-j}) D_k(\omega_3^k) = O_{\lambda, \mathcal{R}}(1), \tag{A.27}$$

where $\omega_3^k = \varphi \sigma^k |\nabla_\sigma \psi|^2$. From (A.26) and (A.27), the result of Lemma 3.19 follows.

A.7 Proof of Lemma 3.20

We begin proving the first inequality (3.13) of our Lemma. Recalling that $v = ru$, $u = \rho v$, thanks to Lemma 3.1 and Young's inequality, we have

$$\begin{aligned}
\int_{\Omega_{h,k}^*} r^2 |D_k u|^2 &\leq 2 \left(\int_{\Omega_{h,k}^*} r^2 |D_k v A_k \rho|^2 + \int_{\Omega_{h,k}^*} r^2 |A_k v D_k \rho|^2 \right) \\
&:= 2(H_1 + H_2).
\end{aligned}$$

Let us first estimate H_2 . Using Lemma 3.1, (3.2) and the second equality in Lemma 3.8 we obtain

$$H_2 \leq \int_{\Omega_{h,k}^*} |r D_k \rho|^2 A_k(|v|^2) = \int_{\Omega_h} A_k(|r D_k \rho|^2) |v|^2 \leq s^2 \int_{\Omega_h} |O_{\lambda, \mathcal{R}}(1)| |v|^2,$$

since $v = 0$ on $\partial_k \Omega_h$. It remains to prove that

$$H_1 \leq \int_{\Omega_{h,k}^*} |O_{\lambda, \mathcal{R}}(1)| |D_k v|^2,$$

which follows from the first equality in Lemma 3.8, and inequality (3.13) is proved.

To prove the inequality (3.14), we note that

$$|\rho D_k v|^2 = |\rho A_k r D_k u + \rho D_k r A_k u|^2 \leq C_{\lambda, \mathcal{R}} (|D_k u|^2 + s^2 |A_k u|^2),$$

due to Lemma 3.1 and the first and second equality Lemma 3.8. Then, by virtue of $t_r^k(|A_k u|^2) = \frac{h^2}{4} t_r^k(|D_k u|^2)$ and $sh \leq \mathcal{R}$, we have

$$t_r^k(|D_k v|^2) = t_r^k(r^2 |\rho D_k v|^2) \leq C_{\lambda, \mathcal{R}} t_r^k(r^2) t_r^k(|D_k u|^2) \quad (\text{A.28})$$

Moreover, by Definition 2.7, we have

$$|\partial_n u|^2 = |t_r^k(\sigma^k D_k u) n_k|^2 = |t_r^k(\sigma^k)|^2 |t_r^k(D_k u)|^2 \geq C t_r^k(|D_k u|^2), \quad (\text{A.29})$$

for $x \in \partial_k \Omega_h$, $k = 1, \dots, d$. Combining (A.28) with (A.29), we get (3.14). Thus the proof of Lemma 3.20 is complete.

APPENDIX B.

Proposition B.1. *Let Assumption 2.4 and Assumption 2.5 hold. Let $M > 0$, $q \in C(\Omega_h)$ satisfying $\|q\|_{L_h^\infty(\Omega_h)} \leq M$ and let $u \in C(\bar{\Omega}_h)$ satisfying*

$$\begin{aligned} -\Delta_h u + qu &= f, & \text{in } \Omega_h, \\ u &= 0, & \text{on } \partial\Omega_h, \end{aligned} \quad (\text{B.1})$$

where $f \in C(\Omega_h)$ and $f = 0$ on $\partial\Omega_h$. Then there exists constant $C > 0$ independent of h such that

$$\|u\|_{H_h^2(\Omega_h)} \leq C \|f\|_{L_h^2(\Omega_h)}. \quad (\text{B.2})$$

Proof. By Assumption 2.5, we have

$$\|u\|_{H_h^1(\Omega_h)} \leq C \|f\|_{L_h^2(\Omega_h)}, \quad (\text{B.3})$$

for some constant $C > 0$ independent of h . Accordingly, replacing f by $f - qu$, we are reduced to the case $q = 0$, that we assume from now.

Since $\Omega_h = h\mathbb{Z}^d \cap (0, 1)^d$, we first propose to extend u a priori defined on the discrete domain Ω_h to $\Omega_{\text{ext}, h} = h\mathbb{Z}^d \cap (-1, 2)^d$ as follows. For $x = (x_1, \dots, x_d) \in ([0, 1]^{d-1} \times (-1, 2)) \cap \Omega_{\text{ext}, h}$, we set $\tilde{u}(x) = -u(x_1, \dots, x_{d-1}, -x_d)$ for $x_d \in (-1, 0)$ and $\tilde{u}(x) = -u(x_1, \dots, x_{d-1}, 1 - (x_d - 1))$ for $x_d \in (1, 2)$. This defines \tilde{u} on $([0, 1]^{d-1} \times (-1, 2)) \cap \Omega_{\text{ext}, h}$. Then, for $x = (x_1, \dots, x_d) \in ([0, 1]^{d-2} \times (-1, 2) \times [0, 1]) \cap \Omega_{\text{ext}, h}$, we set $\tilde{u}(x) = -u(x_1, \dots, x_{d-2}, -x_{d-1}, x_d)$ for $x_{d-1} \in (-1, 0)$ and $\tilde{u}(x) = -u(x_1, \dots, 1 - (x_{d-1} - 1), x_d)$ for $x_{d-1} \in (1, 2)$. This defines \tilde{u} on $([0, 1]^{d-2} \times (-1, 2) \times [0, 1]) \cap \Omega_{\text{ext}, h}$. By induction, we then extend it for $x = (x_1, \dots, x_d) \in \Omega_{\text{ext}, h}$ by setting $\tilde{u} = -u(-x_1, \dots, x_d)$ for $x_1 \in (-1, 0)$ and $\tilde{u} = -u(1 - (x_1 - 1), \dots, x_d)$ for $x_1 \in (1, 2)$. We do a similar extension \tilde{f} of f on $\Omega_{\text{ext}, h}$ taking care of choosing $\tilde{f} = 0$ on $\partial\Omega_h$.

We thus have constructed a solution \tilde{u} of

$$\begin{aligned} -\Delta_h \tilde{u} &= \tilde{f}, & \text{in } \Omega_{\text{ext}, h}, \\ \tilde{u} &= 0, & \text{on } \partial\Omega_{\text{ext}, h}. \end{aligned} \quad (\text{B.4})$$

We then choose a function $\chi \in C_c^\infty((-1, 2)^d)$ such that $\chi = 1$ on $[0, 1]^d$ and we multiply (B.4) by $-\chi D_1^2 \tilde{u}$, after some integrations by parts where all the boundary terms vanish due to the choice of χ , we obtain

$$\begin{aligned} & \int_{\Omega_{ext,h}} \chi A_1 \sigma^1 |D_1^2 \tilde{u}|^2 + \sum_{k=2}^d \int_{(\Omega_{ext,h})_1^*} A_1 (A_k \chi \sigma^k) |D_1 D_k \tilde{u}|^2 \\ &= - \int_{\Omega_{ext,h}} \chi D_1^2 \tilde{u} \tilde{f} - \int_{\Omega_{ext,h}} \chi D_1 \sigma^1 D_1^2 \tilde{u} A_1 D_1 \tilde{u} + \sum_{k=2}^d \int_{\Omega_{ext,h}} A_k (D_k \chi \sigma^k D_k \tilde{u}) D_1^2 \tilde{u} \\ & \quad - \sum_{k=2}^d \int_{(\Omega_{ext,h})_1^*} D_1 (A_k \chi \sigma^k) A_1 D_k \tilde{u} D_k D_1 \tilde{u}. \end{aligned} \quad (\text{B.5})$$

Of course, since $\chi = 1$ on $[0, 1]^d$, the left hand-side of (B.5) is bounded from below by

$$C \left(\int_{\Omega_h} |D_1^2 u|^2 + \sum_{k=2}^d \int_{\Omega_{h,k1}^*} |D_1 D_k u|^2 \right).$$

On the other hand, noting that \tilde{u} and \tilde{f} are symmetric extensions of u and f , the right hand-side of (B.5) is bounded from above by

$$C \left(\left(\int_{\Omega_h} |D_1^2 u|^2 \right)^{\frac{1}{2}} + \sum_{k=2}^d \left(\int_{\Omega_{h,k1}^*} |D_1 D_k u|^2 \right)^{\frac{1}{2}} \right) \left(\|f\|_{L_h^2(\Omega_h)}^2 + \|u\|_{H_h^1(\Omega_h)} \right).$$

We thus obtain

$$\left(\int_{\Omega_h} |D_1^2 u|^2 \right)^{\frac{1}{2}} + \sum_{k=2}^d \left(\int_{\Omega_{h,k1}^*} |D_1 D_k u|^2 \right)^{\frac{1}{2}} \leq C \left(\|f\|_{L_h^2(\Omega_h)}^2 + \|u\|_{H_h^1(\Omega_h)} \right). \quad (\text{B.6})$$

Similarly, we have

$$\left(\int_{\Omega_h} |D_i^2 u|^2 \right)^{\frac{1}{2}} + \sum_{k=2}^d \left(\int_{\Omega_{h,ki}^*} |D_i D_k u|^2 \right)^{\frac{1}{2}} \leq C \left(\|f\|_{L_h^2(\Omega_h)}^2 + \|u\|_{H_h^1(\Omega_h)} \right), \quad (\text{B.7})$$

for $i = 2, \dots, d$. Combining (B.6), (B.7) and (B.3), we obtain the inequality (B.2). \square