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REFLECTED STOCHASTIC RECURSIVE CONTROL PROBLEMS WITH JUMPS: DYNAMIC PROGRAMMING AND STOCHASTIC VERIFICATION THEOREMS

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Abstract. This paper mainly investigates reflected stochastic recursive control problems governed by jump-diffusion dynamics. The system's state evolution is described by a stochastic differential equation driven by both Brownian motion and Poisson random measures, while the recursive cost functional is formulated *via* the solution process Y of a reflected backward stochastic differential equation driven by the same dual stochastic sources. By establishing the dynamic programming principle, we provide the probabilistic interpretation of an obstacle problem for partial integro-differential equations of Hamilton-Jacobi-Bellman type in the viscosity solution sense through our control problem's value function. Furthermore, the value function is proved to inherit the semi-concavity and joint Lipschitz continuity in state and time coordinates, which play key roles in deriving stochastic verification theorems of control problem within the framework of viscosity solutions. We remark that some restrictions in previous study are eliminated, such as the frozen of the reflected processes in time and state, and the independence of the driver from diffusion variables.

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1. Introduction

Reflected backward stochastic differential equations (RBSDEs), as a pivotal extension of classical backward stochastic differential equations (BSDEs), were first systematically studied by El Karoui *et al.* [1]. Distinguished from standard BSDEs, RBSDE incorporates an additional non-decreasing process that enforces the solution path Y to maintain above (or below) a prescribed barrier in a kind of minimal way. This structure innovation has promoted RBSDEs as a powerful mathematical tool in various fields, such as options pricing [2, 3], mixed stochastic game problems [3, 4], obstacle problems for partial differential equations (PDEs) [2, 5–7] and so on.

In response to the inherent discontinuities observed in financial markets (such as abrupt price jumps, liquidity shocks), the theoretical framework of RBSDEs has been extended to incorporate both Brownian motion and Poisson random measures, termed as RBSDEs with jumps. Such an advancement is particularly critical not only in mathematical theory but also in applications. The work by Hamadène, Ouknine [8] established the

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existence and uniqueness of solutions for such systems when the obstacle process has only inaccessible jumps. Essaky [9] further generalized these results to the case when the obstacle process has predictable jumps through a penalization method. In the aspect of application, some researches has been done to link these stochastic systems to obstacle problems for partial integro-differential equations (PIDEs). Matoussi *et al.* [10] presented a probabilistic interpretation for weak Sobolev solutions for semilinear parabolic PIDEs with obstacles. In the viscosity solution sense, Sylla [11] gave a probabilistic interpretation for PIDEs *via* the solution of RBSDEs with jumps. Zhang, Liu [12] employed the solutions to reflected forward-backward stochastic differential equations (FBSDEs) to establish a nonlinear Feynman-Kac representation for mild supersolutions of PIDEs. However, despite these contributions, the study of obstacle problems specifically for PIDEs of HJB type rising from stochastic control problems remains largely unexplored.

Motivated by this, we investigate a class of stochastic recursive optimal control problems with jump-diffusion dynamics. Precisely, the system's state evolution is governed by a stochastic differential equation driven by both Brownian motion and Poisson random measures (SDE with jumps), while the recursive cost functional is defined through the solution process Y of a reflected backward stochastic differential equation driven by the same dual stochastic sources (RBSDE with jumps). The approach of dynamic programming principle is employed here to link the value function with the unique viscosity solution to an obstacle problem for PIDE of HJB type. Throughout the study, a sequence of penalized control problems is introduced to approximate the original control problem. Subsequently, more properties of the value function are explored. We demonstrate the value function inherits the semi-concavity and joint Lipschitz continuity in state and time coordinates under some additional conditions. These properties are indispensable and key, which is revealed in the research of stochastic verification theorems of our control problem, especially in the framework of viscosity solutions. The following is about the main contributions and novelties of this work.

The first is the study of the regular properties (including the semi-concavity and joint Lipschitz continuity in state and time coordinates) of the value function. The study of the semi-concavity is very complex and needs the meticulous deduction. Seeing more clearly from the study of semi-concavity, we are involved in the comparison and estimates of three different time-state configurations of the value function, while encountered with the difficulties caused by jump diffusion. The conventional transformation technique from [13, 14], effective in pure Brownian motion settings, is insufficient when we confront with the jump diffusion. Our resolution lies in an ingenious adaptation of a kind of time-stretching transformations of the jump noise introduced by [15], which we call as the Kulik's transformation, refer to [16, 17], *etc.* However, this powerful tool comes with a price, which necessitates a technical assumption $\nu(E) < \infty$ in Section 4. We point out such condition is not necessary in other parts.

Furthermore, this work makes a significant breakthrough in relaxing the critical constraints imposed by the existing studies for RBSDEs. While prior studies like [13] required either the reflected process to be frozen in time and state or the driver coefficient to be independent from diffusion variables (see Conditions (H6), (H7) therein). By recasting the problem into a sequence of penalized control problems, we focus on the study of the semi-concavity of the penalized value functions, which allows the residual terms in the inequality characterization of the semi-concavity. Crucially, these residual terms exhibit asymptotic decay as the penalized system converges to the original control problem, thereby preserving essential regularity properties without relying on prior restrictive assumptions. The appearance and rigorous characterization of the residual terms within the semi-concavity inequalities are the key innovations in the research.

The paper is organized as follows. Section 2 introduces the Wiener-Poisson space and presents the preliminaries about RBSDEs with jumps. Section 3 formulates stochastic control problems involving reflected FBSDEs with jumps. Moreover, by employing a dynamic programming approach, the value function is connected with the corresponding obstacle problem of PIDE of HJB type. Section 4 investigates more regularity properties of the value functions, focusing on the semi-concavity and joint Lipschitz continuity in (t, x). The concluding Section 5 develops the stochastic verification theorems in both classical solution and viscosity solution frameworks.

2. Preliminaries

Let the triple $(\Omega, \mathcal{F}, \mathbb{P})$ be the completed product of the two probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, *i.e.*, $\Omega := \Omega_1 \times \Omega_2$, $\mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2$, $\mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_2$ with $\mathcal{F}$ being completed with respect to $\mathbb{P}$. The detailed information of spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ are as follows.

- $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ is a classical Wiener space, *i.e.*, $\Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$, $\mathcal{F}_1$ is the completed Borel σ-field on Ω_1, $\mathbb{P}_1$ is the Wiener measure. Under $\mathbb{P}_1$, the canonical processes $B_s(\omega) = \omega_1(s)$ and $B_{-s}(\omega) = \omega_1(-s)$, $s \in \mathbb{R}_+$, $\omega_1 \in \Omega_1$ are two independent d-dimensional Brownian motions. Denote $\mathbb{F}^B = \{\mathcal{F}^B_s\}_{s \geqslant 0}$ is the natural filtration generated by the Brownian motion $B(\cdot)$, *i.e.*, $\mathcal{F}^B_s := \sigma\big\{B_r, r \leqslant s\big\} \vee \mathcal{N}_{\mathbb{P}_1}$, with $\mathcal{N}_{\mathbb{P}_1}$ being the collection of $\mathbb{P}_1$-null sets.
- $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ is a Poisson space. Precisely, Ω_2 is the set of all point functions $p : D_p \to E$, with D_p being a countable subset of $\mathbb{R}$ and $E := \mathbb{R}^l \backslash \{0\}$ being equipped with its Borel σ-field $\mathcal{B}(E)$. We identify the point function p with $N(p, \cdot)$, where N is the counting measure defined on $\mathbb{R} \times E$, *i.e.*,

  $$N(p, (s,t] \times A) := \sharp\big\{r \in D_p \cap (s,t] \mid p(r) \in A\big\},\ A \in \mathcal{B}(E),\ s, t \in \mathbb{R},\ s < t.$$

  Here $\sharp$ represents the cardinal number of the set. $\mathcal{F}_2$ denotes the smallest σ-field on Ω_2 such that the coordinate mapping $p \to N\big(p, (s,t] \times A\big)$, $A \in \mathcal{B}(E)$, $s, t \in \mathbb{R}$, $s < t$ is measurable. $\mathbb{P}_2$ is the probability on $(\Omega_2, \mathcal{F}_2)$ such that the coordinate measure $N(p, \mathrm{d}t\mathrm{d}e)$ is a Poisson random measure with the compensator $\hat{N}(\mathrm{d}t\mathrm{d}e) := \mathrm{d}t\nu(\mathrm{d}e)$, where ν is supposed to be a σ-finite measure on $(E, \mathcal{B}(E))$ satisfying $\int_E (1 \wedge |e|^2)\nu(\mathrm{d}e) < \infty$. Then, for any $A \in \mathcal{B}(E)$ with $\nu(A) < \infty$, the process $\big\{\tilde{N}\big((s,t] \times A\big)\big\}_{t \geqslant s} := \big\{(N - \hat{N})\big((s,t] \times A\big)\big\}_{t \geqslant s}$ is a martingale. By setting

  $$\dot{\mathcal{F}}^N_t := \sigma\Big\{N\big((s,r] \times A\big), -\infty < s \leqslant r \leqslant t, A \in \mathcal{B}(E)\Big\},\ t \geqslant 0,$$

  we get the filtration $\mathbb{F}^N := (\mathcal{F}^N_t)_{t \geqslant 0}$ with $\mathcal{F}^N_t := \Big(\bigcap_{r > t} \dot{\mathcal{F}}^N_r\Big) \vee \mathcal{N}_{\mathbb{P}_2}$.

Based on the above, the filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is introduced as $\mathbb{F} := \{\mathcal{F}_t\}_{t \geqslant 0}$ with $\mathcal{F}_t := \mathcal{F}^B_t \otimes \mathcal{F}^N_t$ augmented by all $\mathbb{P}$-null sets.

For any $t \in [0, T]$, $p \geqslant 1$ and Euclidean space $\mathbb{R}^k$ $(k \geqslant 1)$, we introduce the following spaces,

$$L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^k) := \Big\{\xi : \Omega \to \mathbb{R}^k \mid \xi \text{ is } \mathcal{F}_t\text{-measurable},\ \mathbb{E}|\xi|^2 < \infty\Big\};$$

$$\mathcal{S}^2_{\mathbb{F}}(t, T; \mathbb{R}^k) := \Big\{\phi : \Omega \times [t,T] \to \mathbb{R}^k \mid \phi(\cdot) \text{ is } \mathbb{F}\text{-adapted, càdlàg, and } \mathbb{E}\Big[\sup_{s \in [t,T]} |\phi_s|^2\Big] < \infty\Big\};$$

$$\mathcal{M}^2_{\mathbb{F}}(t, T; \mathbb{R}^k) := \Big\{\phi : \Omega \times [t,T] \to \mathbb{R}^k \mid \phi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable, and } \mathbb{E}\Big[\int_t^T |\phi(s)|^2 \mathrm{d}s\Big] < \infty\Big\};$$

$$\mathcal{K}^2_{\mathbb{F}}(t, T; \mathbb{R}) := \Big\{K : \Omega \times [t,T] \times E \to \mathbb{R} \mid K(\cdot) \text{ is } \mathcal{P}_{t,T}{}^{1} \otimes \mathcal{B}(E)\text{-measurable,}$$
$$\text{and } \|K(\cdot)\|^2_{\mathcal{K}^2_{\mathbb{F}}(t,T;\mathbb{R})} = \mathbb{E}\Big[\int_t^T \int_E |K_s(e)|^2 \nu(\mathrm{d}e)\mathrm{d}s\Big] < \infty\Big\};$$

$$\mathcal{L}^p_\nu(E; \mathbb{R}) := \Big\{K : E \to \mathbb{R} \mid K(\cdot) \text{ is } \mathcal{B}(E)\text{-measurable, and } \|K(\cdot)\|^p_{\nu,p} = \int_E |K(e)|^p \nu(\mathrm{d}e) < \infty\Big\};$$

$$\mathcal{A}^2_{\mathbb{F}}(t, T; \mathbb{R}) := \Big\{\phi : \Omega \times [t,T] \to \mathbb{R} \mid \phi(\cdot) \text{ is } \mathbb{F}\text{-adapted, càdlàg and increasing, } \phi(t) = 0,\ \mathbb{E}\big[|\phi(T)|^2\big] < \infty\Big\}.$$

For any $t \in [0,T]$, we put

$$\mathscr{S}^2_{\mathbb{F}}[t,T] := \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{R}^d) \times \mathcal{K}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times \mathcal{A}^2_{\mathbb{F}}(t,T;\mathbb{R}).$$

2.1. Reflected backward stochastic differential equations with jumps

In this part, we recall some known results about RBSDEs with jumps. Consider

$$\begin{cases} \text{(i) } (Y,Z,V,A) \in \mathscr{S}^2_{\mathbb{F}}[0,T]; \\ \text{(ii) } Y_s = \xi + \displaystyle\int_s^T f\big(r, Y_r, Z_r, \int_E l(e)V_r(e)\nu(\mathrm{d}e)\big)\mathrm{d}r - (A_T - A_s) \\ \qquad\qquad - \displaystyle\int_s^T Z_r \mathrm{d}B_r - \int_s^T\int_E V_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s \in [0,T]; \\ \text{(iii) } Y_s \leqslant S_s, \quad \text{a.e. } s \in [0,T]; \\ \text{(iv) } \displaystyle\int_0^T (S_{s-} - Y_{s-})\mathrm{d}A_s = 0. \end{cases} \tag{2.1}$$

In the above, f, ξ and S is the driver coefficient, the terminal condition and the obstacle term of (2.1), resp. We sometimes use the triple (f,ξ,S) to represent (2.1). The condition on (f,ξ,S) is given as follows.

(A$_1$) (i) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P}_{0,T}$-measurable for every fixed $(y,z,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, and $f(\cdot,0,0,0) \in \mathcal{M}^2_{\mathbb{F}}(0,T;\mathbb{R})$;
(ii) f is Lipschitz continuous in $(y,z,v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, uniformly with respect to $(\omega,s) \in \Omega \times [0,T]$, $\mathbb{P}$-a.s.;
(iii) $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$, S is a real-valued $\mathbb{F}$-progressively measurable, càdlàg stochastic process satisfying $\mathbb{E}\Big[\sup\limits_{s\in[0,T]} |S_s|^2\Big] < \infty$, and $\xi \leqslant S_T$, $\mathbb{P}$-a.s.;
(iv) there exists some constant $\kappa > 0$ such that $0 \leqslant l(e) \leqslant \kappa(1 \wedge |e|)$;

(A$_2$) $v \to f(t,y,z,v)$ is non-decreasing, for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

Now we present some results about the wellposedness of RBSDE with jumps, which can be refer to [9, 18]. Notice that the obstacle in (2.1) is the upper one, so we need to adapt the lower obstacle case studied in [9, 18] to our case. Such a transformation between the two cases is natural and easy, so that we shall not repeat it here and present the results directly.

First, for each $n \in \mathbb{N}$, we introduce the following BSDE with jumps,

$$\begin{aligned} Y^n_s = \xi + \int_s^T f\big(r, Y^n_r, Z^n_r, \int_E l(e)V^n_r(e)\nu(\mathrm{d}e)\big)\mathrm{d}r - (A^n_T - A^n_s) - \int_s^T Z^n_r \mathrm{d}B_r \\ - \int_s^T\int_E V^n_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s \in [0,T], \end{aligned} \tag{2.2}$$

where $A^n_s := \displaystyle\int_0^s n\big(S_r - Y^n_r\big)^- \mathrm{d}r$. From [9], we know (2.2) is the penalized equation of RBSDE with jumps (2.1). Clearly, for each $n \in \mathbb{N}$, the condition **(A$_1$)** guarantees the uniquely existence of $(Y^n,Z^n,V^n) \in \mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times \mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{R}^d) \times \mathcal{K}^2_{\mathbb{F}}(t,T;\mathbb{R})$ satisfying (2.2), refer to [19, 20]. With the help of Theorems 4.2 and 5.1 in [9], we have the following approximation result from (2.2) to (2.1), and thereby the wellposedness of (2.1).

Lemma 2.1. *Assume* **(A$_1$)** *and* **(A$_2$)** *hold. Then the following two results hold.*

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[1] $\mathcal{P}_{t,T}$ denotes the σ-field of $\mathbb{F}$-predictable subsets of $\Omega \times [t,T]$.

(i) *The sequence* $\{(Y^n,Z^n,V^n,A^n)\}_{n\geqslant 0}$ *has a limit* (Y,Z,V,A) *such that* Y^n *converges decreasingly to* $Y\in\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R})$, *and* (Z,V,A) *is the weak limit of* (Z^n,V^n,A^n) *in* $\mathcal{M}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)\times\mathcal{K}^2_{\mathbb{F}}(0,T;\mathbb{R})\times\mathcal{A}^2_{\mathbb{F}}(0,T;\mathbb{R})$.

(ii) *The limit* $(Y,Z,V,A)\in\mathscr{S}^2_{\mathbb{F}}[0,T]$ *is the unique solution of RBSDE with jumps* (2.1).

Next, we present the estimate and the continuous dependence of the solutions of RBSDEs with jumps.

Lemma 2.2. *Assume* $\mathbf{(A_1)}$ *holds.*

(i) *Let* $(Y,Z,V,K)\in\mathscr{S}^2_{\mathbb{F}}[0,T]$ *be the solution of RBSDE with jumps* (2.1), *then there exists a constant* $C>0$ *such that*

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{s\in[t,T]}|Y_s|^2+\int_t^T(|Z_s|^2+\|V_s(\cdot)\|^2_{\nu,2})\mathrm{d}s+\sup_{s\in[t,T]}|A_s|^2\Big]\\&\leqslant C\mathbb{E}^{\mathcal{F}_t}\Big[|\xi|^2+\Big(\int_t^T|f(s,0,0,0)|^2\mathrm{d}s\Big)+\sup_{s\in[t,T]}|S_s|^2\Big],\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

(ii) *For any given data* (g',ξ',S') *satisfying* $\mathbf{(A_1)}$, *denote* (Y',Z',V',A') *by the solution of* (2.1) *with* (g',ξ',S'). *Then there exists some constant* $C>0$ *such that*

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\big[\sup_{s\in[t,T]}|\widehat{Y}_s|^2+\int_t^T(|\widehat{Z}_s|^2+\|\widehat{V}_s(\cdot)\|^2_{\nu,2})\mathrm{d}s+|\widehat{A}_T-\widehat{A}_t|^2\big]\\&\leqslant C\mathbb{E}^{\mathcal{F}_t}\Big[|\widehat{\xi}|^2+\Big(\int_t^T\big|\widehat{f}\big(s,Y_s,Z_s,\int_E l(e)V_s(e)\nu(\mathrm{d}e)\big)\big|^2\mathrm{d}s\Big)\Big]+C\Big(\mathbb{E}^{\mathcal{F}_t}\Big[\sup_{s\in[t,T]}|\widehat{S}_s|^2\Big]\Big)^{\frac{1}{2}}\Psi^{\frac{1}{2}}_{t,T},\quad\mathbb{P}\text{-a.s.,}\end{aligned}$$

where $\widehat{\xi}:=\xi-\xi'$, $(\widehat{Y},\widehat{Z},\widehat{V},\widehat{A}):=(Y-Y',Z-Z',V-V',A-A')$, $\widehat{S}:=S-S'$, $\widehat{f}:=f-f'$ *and*

$$\Psi_{t,T}:=\mathbb{E}^{\mathcal{F}_t}\Big[|\xi|^2+|\xi'|^2+\int_t^T|f(s,0,0,0)|^2\mathrm{d}s+\int_t^T|f'(s,0,0,0)|^2\mathrm{d}s+\sup_{s\in[t,T]}|S_s|^2+\sup_{s\in[t,T]}|S'_s|^2\Big].$$

Note that, (i) can be referred to Proposition 2.2 in [9]. Moreover, the detail of (ii) is similar to (i), so we omit here.

For the later study, we also need the comparison theorem of (2.1).

Lemma 2.3. *(Comparison Theorem) Let* (f_i,ξ_i,S^i), $i=1,2$ *satisfy* $\mathbf{(A_1)}$ *and* $\mathbf{(A_2)}$, *and* (Y^i,Z^i,V^i,A^i), $i=1,2$ *be the unique solution of RBSDE with jumps* (2.1) *associated with* (f_i,ξ_i,S^i), $i=1,2$, *resp. Then* $Y^1_s\leqslant Y^2_s$, $\mathbb{P}$*-a.s.,* $s\in[0,T]$, *whenever* $\xi_1\leqslant\xi_2$, $f_1(s,y,z,v)\leqslant f_2(s,y,z,v)$, $S^1_s\leqslant S^2_s$, $s\in[0,T]$, $(y,z,v)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, $\mathbb{P}$*-a.s.*

Note that the comparison theorem in [9] (Thm. 5.2) is with the same obstacle process. Our Lemma 2.3 generalizes it to the case with different barriers by observing

$$f^1_n(s,y,z,v):=f_1(s,y,z,v)-n(S^1_s-y)^-,\qquad f^2_n(s,y,z,v):=f_2(s,y,z,v)-n(S^2_s-y)^-.$$

3. Reflected stochastic recursive control problems with jumps

3.1. Formulation of the control problem

Let $U\subset\mathbb{R}^m$ be compact. For $t\in[0,T]$, the admissible control $u(\cdot)$ on $[t,T]$ is introduced as the U-valued, $\mathbb{F}$-predictable stochastic process. We denote by $\mathcal{U}_{t,T}$ all the admissible controls on $[t,T]$.

The state process is described by the following controlled SDE with jumps,

$$\begin{cases}\mathrm{d}X_s = b(s,X_s,u_s)\mathrm{d}s + \sigma(s,X_s,u_s)\mathrm{d}B_s + \displaystyle\int_E \gamma(s,X_{s-},u_s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e), \quad s\in[t,T],\\ X_t = x_t,\end{cases} \tag{3.1}$$

where $(t,x_t)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is the initial pair, $u(\cdot)\in\mathcal{U}_{t,T}$, and the above involved coefficients $b:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}^{n\times d}$, $\gamma:[0,T]\times\mathbb{R}^n\times U\times E\to\mathbb{R}^n$ are assumed to satisfy the following condition.

($\mathbf{H}_1$) (i) For all $x\in\mathbb{R}^n$, $e\in E$, $b(\cdot,x,\cdot)$, $\sigma(\cdot,x,\cdot)$ and $\gamma(\cdot,x,\cdot,e)$ are continuous in $(r,u)\in[0,T]\times U$;
(ii) b,σ are Lipschitz continuous in $x\in\mathbb{R}^n$, uniformly with respect to $(r,u)\in[0,T]\times U$. And there exists a map $\ell(\cdot)\in\mathcal{L}^2_\nu(E;\mathbb{R})$ such that for all $(r,u)\in[0,T]\times U$, $x_0,x_1\in\mathbb{R}^n$, $e\in E$,

$$|\gamma(r,x_0,u,e)-\gamma(r,x_1,u,e)|\leqslant \ell(e)|x_0-x_1|.$$

From the classical theory of SDE with jumps, we get the following results (refer to [20–23]).

Lemma 3.1. *Under ($\mathbf{H}_1$), for any $(t,x_t)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $u(\cdot)\in\mathcal{U}_{t,T}$, the SDE with jumps (3.1) admits the unique solution $X\equiv X^{t,x_t;u}\in\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$. Moreover, for any $p\geqslant 2$, there exists some constant $C>0$ such that, for $t\in[0,T]$, $x_t,x_t'\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot)\in\mathcal{U}_{t,T}$, the following estimates hold, $\mathbb{P}$-a.s.,*

$$\begin{aligned}&\text{(i) } \mathbb{E}^{\mathcal{F}_t}\Big[\sup_{r\in[t,T]}|X_r^{t,x_t;u}|^p\Big]\leqslant C(1+|x_t|^p),\\ &\text{(ii) } \mathbb{E}^{\mathcal{F}_t}\Big[|X_s^{t,x_t;u}-x_t|^p\Big]\leqslant C(s-t)(1+|x_t|^p),\\ &\text{(iii) } \mathbb{E}^{\mathcal{F}_t}\Big[\sup_{r\in[t,T]}|X_r^{t,x_t;u}-X_r^{t,x_t';u}|^2\Big]\leqslant C|x_t-x_t'|^2.\end{aligned}$$

Next, with the state $X^{t,x_t;u}$ from (3.1), we consider the following RBSDE with jumps,

$$\begin{cases}\text{(i) } (Y,Z,V,A)\in\mathscr{S}^2_{\mathbb{F}}[t,T];\\ \text{(ii) } Y_s = \Phi(X_T^{t,x_t;u}) + \displaystyle\int_s^T f\big(r,X_r^{t,x_t;u},Y_r,Z_r,\int_E l(e)V_r(e)\nu(\mathrm{d}e),u_r\big)\mathrm{d}r - (A_T-A_s)\\ \qquad\qquad - \displaystyle\int_s^T Z_r\mathrm{d}B_r - \int_s^T\int_E V_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s\in[t,T];\\ \text{(iii) } Y_s\leqslant h(s,X_s^{t,x_t;u}), \quad \text{a.e. } s\in[t,T];\\ \text{(iv) } \displaystyle\int_t^T \big(h(s,X_s^{t,x_t;u})-Y_s\big)\mathrm{d}A_s = 0,\end{cases} \tag{3.2}$$

where the driver $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times U\to\mathbb{R}$, the terminal $\Phi:\mathbb{R}^n\to\mathbb{R}$ and the obstacle $h:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are assumed to satisfy

($\mathbf{H}_2$) (i) for all $(x,y,z,v)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, $f(\cdot,x,y,z,v,\cdot)$ is continuous in $(r,u)\in[0,T]\times U$, $h(\cdot,x)$ is continuous in $r\in[0,T]$;
(ii) f,h,Φ is Lipschitz continuous in $(x,y,z,v)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, uniformly with respect to $(r,u)\in[0,T]\times U$;
(iii) there exists some constant $\kappa>0$ such that $0\leqslant l(e)\leqslant\kappa(1\wedge|e|)$;
(iv) for all $x\in\mathbb{R}^n$, $\Phi(x)\leqslant h(T,x)$.

(C) $v \to f(t,x,y,z,v)$ is non-decreasing, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$.

We note that the obstacle process in (3.2) is $h(\cdot, X_\cdot^{t,x_t;u})$, which obviously satisfies **$(\mathbf{A}_1)$-(iii)**. Hence, under **$(\mathbf{H}_1)$** and **$(\mathbf{H}_2)$**, for any $(t,x_t) \in [0,T] \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}_{t,T}$, Lemma 2.1 can be applied here to guarantee (3.2) to admit the unique solution $(Y,Z,V,A) \equiv (Y^{t,x_t;u}, Z^{t,x_t;u}, V^{t,x_t;u}, A^{t,x_t;u}) \in \mathscr{S}^2_{\mathbb{F}}[t,T]$. Moreover, we observe that the increasing process A is not only càdlàg but in fact continuous. This follows from the fact that the solution X of (3.1) exhibits only inaccessible jumps and the function h is continuous. As a result, the obstacle process $h(\cdot, X_\cdot)$ inherits only inaccessible jumps, which implies the continuity of A as stated. Therefore, the jumping times of Y are also inaccessible.

Under **$(\mathbf{H}_1)$**, **$(\mathbf{H}_2)$** and **(C)**, by adopting the technique of Proposition 6.1 in [7] and using Lemma 3.1, the following estimates hold, for $t \in [0,T]$, $x_t, x'_t \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}_{t,T}$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\text{(i) } \mathbb{E}^{\mathcal{F}_t}\Big[\sup_{r\in[t,T]} |Y_r^{t,x_t;u}|^2 + \int_t^T \big(|Z_r^{t,x_t;u}|^2 + \|V_r^{t,x_t;u}(\cdot)\|^2_{\nu,2}\big)dr + \sup_{r\in[t,T]} |A_r^{t,x_t;u}|^2\Big] \leqslant C(1+|x_t|^2);\\
&\text{(ii) } \mathbb{E}^{\mathcal{F}_t}\Big[\sup_{r\in[t,T]} |Y_r^{t,x_t;u} - Y_r^{t,x'_t;u}|^2\Big] \leqslant C|x_t - x'_t|^2.
\end{aligned} \tag{3.3}$$

With the above preparation, for any initial pair $(t,x) \in [0,T] \times \mathbb{R}^n$ and the admissible control $u(\cdot) \in \mathcal{U}_{t,T}$, we define the cost functional of the control problem as follows,

$$J(t,x;u(\cdot)) := Y_t^{t,x;u},$$

which is of recursive form.
Notice that it is classical in the theory of BSDEs that, for $(t,x_t) \in [0,T] \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}_{t,T}$, $J(t,x_t;u(\cdot)) = J(t,x;u(\cdot))|_{x=x_t} = Y_t^{t,x_t;u}$ holds, $\mathbb{P}$-a.s., for which we refer to [5–7], *etc.*

Now, we formulate the following reflected stochastic recursive control problem with jumps which is parameterized by the initial pair $(t,x) \in [0,T] \times \mathbb{R}^n$.

Problem $(\mathbf{C})_{t,x}$ *For any $(t,x) \in [0,T] \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}_{t,T}$ such that*

$$J(t,x;\bar{u}(\cdot)) = \operatorname*{essinf}_{u(\cdot)\in\mathcal{U}_{t,T}} J(t,x;u(\cdot)) := W(t,x). \tag{3.4}$$

The control $\bar{u}(\cdot)$ satisfying (3.4) is called as an optimal control of Problem $(\mathrm{C})_{t,x}$, the corresponding $\bar{X} = X^{t,x;\bar{u}}$ is the corresponding optimal state process, and $W : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to be the value function of Problem $(\mathrm{C})_{t,x}$.

Note that, as the essential infimum of the $\mathcal{F}_t$-measurable cost functional $J(t,x;u(\cdot))$ over a family of control processes, for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $W(t,x)$ is an $\mathcal{F}_t$-measurable random variable. However, we can prove it to be deterministic under our conditions, similar to Proposition 3.3 in [24], and Proposition 3.1 in [7], *etc.*

Proposition 3.2. *For all $(t,x) \in [0,T] \times \mathbb{R}^n$, the value function $W(t,x)$ is a deterministic function, i.e., $\mathbb{E}[W(t,x)] = W(t,x)$, $\mathbb{P}$-a.s.*

From the definition of the value function and the estimates in (3.3), we get the following result directly.

Proposition 3.3. *Under the conditions* **$(\mathbf{H}_1)$**, **$(\mathbf{H}_2)$** *and* **(C)**, *$W(\cdot,\cdot)$ is of linear growth and Lipschitz continuous in $x \in \mathbb{R}^n$, i.e., for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$,*

$$|W(t,x)| \leqslant C(1+|x|), \qquad |W(t,x) - W(t,x')| \leqslant C|x-x'|.$$

To study the dynamic programming principle (DPP, for short) of Problem $(\mathrm{C})_{t,x}$, we need to generalize the notation of the stochastic backward semigroup introduced initially by Peng [25] to our framework.

Definition 3.4. *Given the initial data* $(t,x)\in[0,T]\times\mathbb{R}^n$, $\delta\in[0,T-t]$, $u(\cdot)\in\mathcal{U}_{t,t+\delta}$ *and a real-valued random variable* $\eta\in L^2_{\mathcal{F}_{t+\delta}}(\Omega;\mathbb{R}^n)$, *we define the backward stochastic semigroup* $G^{t,x;u}_{s,t+\delta}[\cdot]$ *as*

$$G^{t,x;u}_{s,t+\delta}[\eta]:=\tilde{Y}^{t,x;u}_s,\quad s\in[t,t+\delta],$$

where $(\tilde{Y}^{t,x;u},\tilde{Z}^{t,x;u},\tilde{V}^{t,x;u},\tilde{A}^{t,x;u})$ *is the solution of the following RBSDE with jumps on* $[t,t+\delta]$:

$$\begin{cases}\text{(i) } (\tilde{Y}^{t,x;u},\tilde{Z}^{t,x;u},\tilde{V}^{t,x;u},\tilde{A}^{t,x;u})\in\mathscr{S}^2_{\mathbb{F}}[t,t+\delta];\\ \text{(ii) } \tilde{Y}^{t,x;u}_s=\eta+\displaystyle\int_s^{t+\delta}f\big(r,X^{t,x;u}_r,\tilde{Y}^{t,x;u}_r,\tilde{Z}^{t,x;u}_r,\int_E l(e)\tilde{V}^{t,x;u}_r(e)\nu(\mathrm{d}e),u_r\big)\mathrm{d}r-(\tilde{A}^{t,x;u}_{t+\delta}-\tilde{A}^{t,x;u}_s)\\ \qquad\qquad -\displaystyle\int_s^{t+\delta}\tilde{Z}^{t,x;u}_r\mathrm{d}B_r-\int_s^{t+\delta}\int_E\tilde{V}^{t,x;u}_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e),\quad s\in[t,t+\delta];\\ \text{(iii) } \tilde{Y}^{t,x;u}_s\leqslant h(s,X^{t,x;u}_s),\quad a.e.\ s\in[t,t+\delta];\\ \text{(iv) } \displaystyle\int_t^{t+\delta}\big(h(r,X^{t,x;u}_r)-\tilde{Y}^{t,x;u}_r\big)\mathrm{d}\tilde{A}^{t,x;u}_r=0.\end{cases}$$

Note that for RBSDE with jumps (3.2), we have the following

$$J(t,x;u(\cdot))=Y^{t,x;u}_t=G^{t,x;u}_{t,T}\big[\Phi(X^{t,x;u}_T)\big]=G^{t,x;u}_{t,t+\delta}\big[Y^{t,x;u}_{t+\delta}\big]=G^{t,x;u}_{t,t+\delta}\big[J\big(t+\delta,X^{t,x;u}_{t+\delta};u(\cdot)\big)\big].$$

With the help of backward stochastic semigroup, we obtain the dynamic programming principle of Problem $(\mathrm{C})_{t,x}$ as follows.

Proposition 3.5. *(DPP) Under* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{C})$, *for any* $0\leqslant t<t+\delta\leqslant T,\ x\in\mathbb{R}^n$,

$$W(t,x)=\operatorname*{essinf}_{u\in\mathcal{U}_{t,t+\delta}}G^{t,x;u}_{t,t+\delta}\big[W(t+\delta,X^{t,x;u}_{t+\delta})\big].\tag{3.5}$$

Due to the lack of the continuity of the coefficients in control variable, the method in [25] can not be used here. However, we can adopt the approach introduced in [5, 24] to prove (3.5). The detail is similar to Theorem 3.1 in [5], so that we skip it here.

Thanks to the DPP, we can also get the continuity of $W(\cdot,\cdot)$ in $t\in[0,T]$ as follows.

Proposition 3.6. *Under* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{C})$, *the value function* $W(\cdot,\cdot)$ *is continuous in* $t\in[0,T]$.

For the proof, we may refer to Theorem 3.2 in [7].

3.2. The Obstacle problems of partial integral-differential equations of HJB type

In this subsection, we aim to associate Problem $(\mathrm{C})_{t,x}$ with a kind of partial differential equations. To simplified the notations, we first put

$$\begin{aligned}&\mathcal{L}^u\Psi(t,x):=\Psi_x(t,x).b(t,x,u)+\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^\top(t,x,u)\Psi_{xx}(t,x)\right),\\&\mathcal{B}^u\Psi(t,x):=\int_E\big[\Psi\big(t,x+\gamma(t,x,u,e)\big)-\Psi(t,x)-\Psi_x(t,x).\gamma(t,x,u,e)\big]\nu(\mathrm{d}e),\\&\mathcal{C}^u\Psi(t,x):=\int_E l(e)\big[\Psi\big(t,x+\gamma(t,x,u,e)\big)-\Psi(t,x)\big]\nu(\mathrm{d}e),\\&\mathbb{H}(t,x,(\Psi,\Psi_x,\Psi_{xx})(t,x),u):=\mathcal{L}^u\Psi(t,x)+\mathcal{B}^u\Psi(t,x)\\&\qquad\qquad+f\big(t,x,\Psi(t,x),\Psi_x(t,x).\sigma(t,x,u),\mathcal{C}^u\Psi(t,x),u\big),\end{aligned}$$

where $(t,x,u)\in[0,T]\times\mathbb{R}^n\times U$ and $\Psi\in C^{1,2}([0,T]\times\mathbb{R}^n;\mathbb{R})$. Then we consider the following obstacle problem for PIDE of HJB type,

$$\begin{cases}\max\Big\{W(t,x)-h(t,x),-\dfrac{\partial W}{\partial t}(t,x)-\inf\limits_{u\in U}\mathbb{H}\big(t,x,(W,W_x,W_{xx})(t,x),u\big)\Big\}=0,\ (t,x)\in[0,T]\times\mathbb{R}^n,\\ W(T,x)=\Phi(x),\quad x\in\mathbb{R}^n.\end{cases}\tag{3.6}$$

For convenience, we call (3.6) as PIDE. The aim is to associate the value function $W(\cdot,\cdot)$ (in (3.4)) with (3.6). Under our condition, $W(\cdot,\cdot)$ is not necessarily smooth, which pushes us to resort to a kind of weak solution, *i.e.*, viscosity solution, introduced firstly by Crandall, Lions [26]; we also refer to Crandall, Ishii, and Lions [27]. We first generalize the notion of the viscosity solution to adapt to PIDE (3.6).

Definition 3.7. *Let $W\in C\big([0,T]\times\mathbb{R}^n;\mathbb{R}\big)$.*

(i) *We call $W(\cdot,\cdot)$ as a viscosity subsolution of (3.6) if $W(T,x)\leqslant\Phi(x)$, for all $x\in\mathbb{R}^n$, and if for all functions $\varphi\in C^3_{l,b}\big([0,T]\times\mathbb{R}^n\big)$ and any sufficiently small $\delta>0$,*

$$\begin{aligned}\max\Big\{W(t,x)-h(t,x),-\frac{\partial\varphi}{\partial t}(t,x)-\inf_{u\in U}\big\{\mathcal{L}^u\varphi(t,x)+\mathcal{B}^{\delta,u}(W,\varphi)(t,x)\\ +f\big(t,x,W(t,x),\varphi_x(t,x).\sigma(t,x,u),\mathcal{C}^{\delta,u}(W,\varphi)(t,x),u\big)\big\}\Big\}\leqslant 0,\end{aligned}$$

holds at any local maximum point (t,x) of $W-\varphi$, where

$$\begin{aligned}\mathcal{B}^{\delta,u}(W,\varphi)(t,x):=&\int_{E_\delta}\big[\varphi(t,x+\gamma(t,x,u,e))-\varphi(t,x)-\varphi_x(t,x).\gamma(t,x,u,e)\big]\nu(\mathrm{de})\\ &+\int_{E_\delta^c}\big[W(t,x+\gamma(t,x,u,e))-W(t,x)-\varphi_x(t,x).\gamma(t,x,u,e)\big]\nu(\mathrm{de}),\\ \mathcal{C}^{\delta,u}(W,\varphi)(t,x):=&\int_{E_\delta}l(e)\big[\varphi(t,x+\gamma(t,x,u,e))-\varphi(t,x)\big]\nu(\mathrm{de})\\ &+\int_{E_\delta^c}l(e)\big[W(t,x+\gamma(t,x,u,e))-W(t,x)\big]\nu(\mathrm{de}),\end{aligned}$$

with $E_\delta:=\{e\in E\mid |e|<\delta\}$.

(ii) *We call $W(\cdot,\cdot)$ as a viscosity supersolution of equation (3.6) if $W(T,x)\geqslant\Phi(x)$, for all $x\in\mathbb{R}^n$, and if for all functions $\varphi\in C^3_{l,b}\big([0,T]\times\mathbb{R}^n\big)$, sufficiently small $\delta>0$ and the local minimizer $(t,x)\in[0,T)\times\mathbb{R}^n$ of $W-\varphi$,*

$$\begin{aligned}\max\Big\{W(t,x)-h(t,x),-\frac{\partial\varphi}{\partial t}(t,x)-\inf_{u\in U}\big\{\mathcal{L}^u\varphi(t,x)+\mathcal{B}^{\delta,u}(W,\varphi)(t,x)\\ +f\big(t,x,W(t,x),\varphi_x(t,x).\sigma(t,x,u),\mathcal{C}^{\delta,u}(W,\varphi)(t,x),u\big)\big\}\Big\}\geqslant 0.\end{aligned}\tag{3.7}$$

(iii) *If $W(\cdot,\cdot)$ is both a viscosity subsolution and a viscosity supersolution of (3.6), it is called as a viscosity solution of PIDE (3.6).*

Remark 3.8. *We point out that, due to the linear growth of the value function $W(\cdot,\cdot)$ in Proposition 3.3, the local maximum (resp. minimum) of $W-\varphi$ in* Definition 3.7-(i) *(resp.* Def. 3.7-(ii)) can be replaced by a global one in the proof of Theorem 3.12. Moreover, $\mathcal{B}^{\delta,u}(W,\varphi)(t,x)$ and $\mathcal{C}^{\delta,u}(W,\varphi)(t,x)$ in Definition 3.7 can be replaced by $\mathcal{B}^u\varphi(t,x)$ and $\mathcal{C}^u\varphi(t,x)$, resp. Such an observation can be referred to [21, 22], *etc.*

To prove the value function $W(\cdot,\cdot)$ in (3.4) to be a viscosity solution of (3.6), we introduce the penalized equations of RBSDE with jumps (3.2). Precisely, for each $n\in\mathbb{N}$,

$$\begin{aligned} Y^{n,t,x;u}_s &= \Phi(X^{t,x;u}_T)+\int_s^T f\big(r,X^{t,x;u}_r,Y^{n,t,x;u}_r,Z^{n,t,x;u}_r,\int_E l(e)V^{n,t,x;u}_r(e)\nu(\mathrm{d}e),u_r\big)\mathrm{d}r\\ &\quad -(A^{n,t,x;u}_T-A^{n,t,x;u}_s)-\int_s^T Z^{n,t,x;u}_r\mathrm{d}B_r-\int_s^T\int_E V^{n,t,x;u}_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e),\quad s\in[t,T], \end{aligned} \tag{3.8}$$

where $A^{n,t,x;u}_s:=\displaystyle\int_t^s n\big(h(r,X^{t,x;u}_r)-Y^{n,t,x;u}_r\big)^-\mathrm{d}r$ and $X^{t,x;u}$ satisfies (3.1) with $x_t=x\in\mathbb{R}^n$. It is obvious that, for each $n\in\mathbb{N}$, (3.8) admits the unique solution $(Y^{n,t,x;u},Z^{n,t,x;u},V^{n,t,x;u})\in\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R})\times\mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{R}^d)\times\mathcal{K}^2_\nu(t,T;\mathbb{R})$. Moreover, we have the following approximation result from Lemma 2.1.

Lemma 3.9. *Assume* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* **(C)** *hold. The sequence* $\{(Y^{n,t,x;u},Z^{n,t,x;u},V^{n,t,x;u},A^{n,t,x;u})\}_{n\in\mathbb{N}}$ *has a limit* $(Y^{t,x;u},Z^{t,x;u},V^{t,x;u},A^{t,x;u})$*, which is indeed the solution of RBSDE with jumps* (3.2)*. Moreover, the convergence of* $Y^{n,t,x;u}$ *to* $Y^{t,x;u}$ *in* $\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R})$ *is decreasingly.*

With (3.1) and (3.8), we formulate a sequence of control problems as follows. For each $n\in\mathbb{N}$, we introduce the cost functional

$$J_n(t,x;u(\cdot)):=Y^{n,t,x;u}_t,\quad (t,x)\in[0,T]\times\mathbb{R}^n,\ u(\cdot)\in\mathcal{U}_{t,T},$$

and the value function

$$W^n(t,x)=\operatorname*{essinf}_{u(\cdot)\in\mathcal{U}_{t,T}} J_n(t,x;u(\cdot)),\quad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{3.9}$$

According to Theorems 4.1 and 5.1 in [22], for each $n\in\mathbb{N}$, $W^n(\cdot,\cdot)$ is the unique viscosity solution of the following HJB equation,

$$\begin{cases} -\dfrac{\partial W^n}{\partial t}(t,x)-\inf\limits_{u\in U}\mathbb{H}^n\big(t,x,(W^n,W^n_x,W^n_{xx})(t,x),u\big)=0,\quad (t,x)\in[0,T]\times\mathbb{R}^n,\\ W^n(T,x)=\Phi(x),\quad x\in\mathbb{R}^n, \end{cases} \tag{3.10}$$

where

$$\mathbb{H}^n(t,x,(W^n,W^n_x,W^n_{xx})(t,x),u)=\mathbb{H}(t,x,(W^n,W^n_x,W^n_{xx})(t,x),u)-n\big(h(t,x)-W^n(t,x)\big)^-.$$

Notice that the uniqueness holds in the following space:

$$\begin{aligned}\Theta=\Big\{&\varphi\in C\big([0,T]\times\mathbb{R}^n;\mathbb{R}\big)\mid\exists\widetilde{A}>0\text{ such that}\\ &\lim_{|x|\to\infty}\varphi(t,x)\exp\Big\{-\widetilde{A}\big[\log((|x|^2+1)^{\frac12})\big]^2\Big\}=0,\ \text{uniformly in } t\in[0,T]\Big\}.\end{aligned}$$

The above space is characterized by a growth condition, which is weaker than polynomial growth. Actually, this growth condition, introduced by Barles *et al.* [21] to prove the uniqueness of the viscosity solution for a class of PIDEs associated with FBSDEs with jumps, was shown to be optimal, meaning it generally cannot be relaxed.

From Lemma 3.9 and the definition of $W^n(\cdot,\cdot)$, we give the following result without the proof. The readers may refer to Lemma 4.3 and Remark 4.4 in [6].

Lemma 3.10. *For all $(t,x)\in[0,T]\times\mathbb{R}^n$, $n\in\mathbb{N}$,*

$$W^1(t,x)\geqslant\cdots\geqslant W^n(t,x)\geqslant W^{n+1}(t,x)\geqslant\cdots\geqslant W(t,x).$$

Moreover, for all $(t,x)\in[0,T]\times\mathbb{R}^n$, $\lim_{n\to\infty}W^n(t,x)=W(t,x)$, which is also uniform on compact sets.

Going further, the following result is true, referring to Lemma 4.6 in [6].

Lemma 3.11. *For all $(t,x)\in[0,T]\times\mathbb{R}^n$ with $(t_n,x_n)\to(t,x)$ as $n\to\infty$, and $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R}^n)$, we have*

$$\lim_{n\to\infty}\inf_{u\in U}\mathbb{H}(t_n,x_n,(W^n,\varphi_x,\varphi_{xx})(t_n,x_n),u)=\inf_{u\in U}\mathbb{H}(t,x,(W,\varphi_x,\varphi_{xx})(t,x),u).$$

With the above approximation results, we get the first main result of this work.

Theorem 3.12. *Under* $\mathbf{(H_1)}$, $\mathbf{(H_2)}$ *and* $\mathbf{(C)}$, *the value function $W(\cdot,\cdot)$ introduced in (3.4) is the unique viscosity solution of (3.6) in Θ.*

The above results can be established using standard techniques found in [6, 24, 28], with appropriate modifications to account for the integral operator and the obstacle term in PIDE (3.6). However, due to space constraints, we omit the detailed proofs here. Interested readers may refer to the complete proofs provided in the arXiv version of this paper.

4. More properties of the value function

Based on the previous study, we further explore more properties of the value function $W(\cdot,\cdot)$ under some additional conditions, such as the semi-concavity and the joint Lipschitz continuity. The importance of these issues will be revealed in the study of stochastic verification theorem in Section 5.

The research of these issues will involve the comparison between stochastic systems with differential initial times and initial states, which is very complex. To make the study more directly, we first try to transform these systems to operate on the same time interval. The appearance of Brownian motion and Poisson random measure at the same time increases the difficulty. A kind of time-stretching transformations of the jump noise, called as Kulik's transformation, turns out to be a key tool in our research. For the initial literatures about the Kulik's transformation, we can refer to [15–17]. We first recall some known results about the Kulik's transformation in the following.

For any $t_0,t_1\in[0,T)$, $\lambda\in[0,1]$, we introduce $t_\lambda:=(1-\lambda)t_0+\lambda t_1$ and the following transformation

$$\tau_i^\lambda(s):=t_\lambda+\frac{T-t_\lambda}{T-t_i}(s-t_i),\quad s\in[t_i,T],\ i=0,1. \tag{4.1}$$

Obviously, for $i=0,1$, τ_i^λ maps $[t_i,T]$ to $[t_\lambda,T]$, and $\dot\tau_i^\lambda=\frac{\mathrm{d}}{\mathrm{d}s}\tau_i^\lambda(s)=\frac{T-t_\lambda}{T-t_i}$.

In the following, we also use the inverse time changes ϱ_i^λ of τ_i^λ, that is,

$$\varrho_i^\lambda(s)=t_i+\frac{T-t_i}{T-t_\lambda}(s-t_\lambda),$$

which maps $[t_\lambda,T]$ to $[t_i,T]$, $i=0,1$.

For the above time changes, we have the following results by the directly computation. It can also be referred to [13, 17].

Lemma 4.1. *For any $\delta \in (0,T)$, there exists a constant $C_\delta > 0$ only depending on δ and T, such that, for all $s \in [t_\lambda, T-\delta]$,*

$$\begin{aligned}
&|\varrho_1^\lambda(s)-\varrho_0^\lambda(s)|+\Big|\frac{1}{\dot\tau_1^\lambda}-\frac{1}{\dot\tau_0^\lambda}\Big|+\Big|\frac{1}{\sqrt{\dot\tau_1^\lambda}}-\frac{1}{\sqrt{\dot\tau_0^\lambda}}\Big|\leqslant C_\delta|t_1-t_0|,\\
&\lambda\Big|1-\frac{1}{\sqrt{\dot\tau_1^\lambda}}\Big|+(1-\lambda)\Big|1-\frac{1}{\sqrt{\dot\tau_0^\lambda}}\Big|\leqslant\frac{1}{2\delta}\lambda(1-\lambda)|t_1-t_0|,\\
&\Big|\lambda\Big(1-\frac{1}{\sqrt{\dot\tau_1^\lambda}}\Big)+(1-\lambda)\Big(1-\frac{1}{\sqrt{\dot\tau_0^\lambda}}\Big)\Big|\leqslant\frac{1}{8\delta^2}\lambda(1-\lambda)|t_1-t_0|^2,\\
&\lambda\Big(1-\frac{1}{\dot\tau_1^\lambda}\Big)=-(1-\lambda)\Big(1-\frac{1}{\dot\tau_0^\lambda}\Big)=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_1-t_0),\\
&\lambda\varrho_1^\lambda(s)+(1-\lambda)\varrho_0^\lambda(s)=s,\quad s\in[t_\lambda,T].
\end{aligned}$$

Denote $B_s^\lambda := B_s - B_{t_\lambda}$, $s \in [t_\lambda, T]$ and N^λ by the restriction of N from $[0,T]\times E$ to $[t_\lambda,T]\times E$. The filtration generated by B^λ and N^λ is denoted by $\mathbb{F}^\lambda = \{\mathcal{F}_s^\lambda\}_{s\in[t_\lambda,T]}$. Then, for $i = 0,1$, $s \in [t_i,T]$, $A \in \mathcal{B}(E)$, we introduce

$$\begin{aligned}
&\mathbb{B}^i_s := \frac{1}{\sqrt{\dot\tau_i^\lambda}}B^\lambda_{\tau_i^\lambda(s)},\quad \mathbb{N}^i(p,(t_i,s]\times A) := \tau_i^\lambda(N^\lambda)(p,(t_i,s]\times A) = N^\lambda\big(p,(t_\lambda,\tau_i^\lambda(s)]\times A\big),\\
&g_{\tau_i} := \exp\Big\{-\ln\Big(\frac{T-t_\lambda}{T-t_i}\Big)\tau_i^\lambda(N^\lambda)([t_i,T]\times E)+(t_i-t_\lambda)\nu(E)\Big\},\quad \mathbb{Q}_{\tau_i^\lambda}:=g_{\tau_i^\lambda}\mathbb{P}.
\end{aligned}\tag{4.2}$$

Note that, for $i = 0,1$, $\mathbb{Q}_{\tau_i^\lambda}$ is a new probability measure, which depends strongly on the following additional condition on the measure ν.

(H$_3$) $\nu(E) < \infty$.

Thus, from now on, **(H$_3$)** is assumed.

According to [15–17], for $i = 0,1$, $\{\mathbb{B}^i_s\}_{s\in[t_i,T]}$ is a Brownian motion under the probability measures $\mathbb{P}$ and $\mathbb{Q}_{\tau_i}$, and the point process $\mathbb{N}^i$ defined on $[t_i,T]\times E$ is a Poisson random measure under $\mathbb{Q}_{\tau_i}$, which has the same distribution as N^λ under $\mathbb{P}$. Moreover, its compensator is still $\nu(\mathrm{d}e)\mathrm{d}s$. The compensated Poisson random measure for $\mathbb{N}^i$ under $\mathbb{Q}_{\tau_i}$ is introduced as follows,

$$\widetilde{\mathbb{N}}^i(\mathrm{d}s,\mathrm{d}e) := \mathbb{N}^i(\mathrm{d}s,\mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}s.$$

Moreover, for $i = 0,1$, $\mathbb{B}^i$ and $\mathbb{N}^i$ are independent under $\mathbb{P}$ and $\mathbb{Q}_{\tau_i}$. In this case, we denote $\mathbf{F}^i = \{\mathscr{F}^i_s\}_{s\in[t_i,T]}$ by the filtration generated by $\mathbb{B}^i$ and $\mathbb{N}^i$, *i.e.*,

$$\mathscr{F}^i_s = \sigma\Big\{\mathbb{B}^i_r, \mathbb{N}^i([t_i,r]\times A):\ r\in[t_i,s],\ A\in\mathcal{B}(E)\Big\}\vee\mathcal{N}_{\mathbb{Q}_{\tau_i}},\quad s\in[t_i,T].$$

For convenience, for $\lambda \in [0,1]$ and $i = 0,1$, we set $\mathcal{U}^\lambda_{t_\lambda,T}$ (resp., $\mathcal{U}^i_{t_i,T}$) as the set of all U-valued, $\mathbb{F}^\lambda$ (resp., $\mathbf{F}^i$)-predictable stochastic processes on $[t_\lambda,T]$ (resp., $[t_i,T]$). Then, for any $u^\lambda(\cdot) \in \mathcal{U}^\lambda_{t_\lambda,T}$, we have $u^i(\cdot) := u^\lambda\big(\tau_i^\lambda(\cdot)\big) \in \mathcal{U}^i_{t_i,T}$, $i = 0,1$.

4.1. The semi-concavity of $W(\cdot,\cdot)$ in (t,x)

Let us recall the definition of semi-concavity as follows.

Definition 4.2. *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be semi-concave, if there exists a constant $C \geqslant 0$ such that $\varphi(x) - C|x|^2$ is concave in $x \in \mathbb{R}^n$.*

According to Definition 4.3 in [14] or Definition 1.1 in [13], the semi-concavity of φ is equivalent to the following inequality, for some constant $C>0$,

$$\lambda\varphi(x)+(1-\lambda)\varphi(x')\leqslant\varphi\big(\lambda x+(1-\lambda)x'\big)+C\lambda(1-\lambda)|x-x'|^2,\quad \forall x,x'\in\mathbb{R}^n,\ \lambda\in[0,1].$$

Now we study the semi-concavity of the value function. For this, some additional assumptions are needed.

$(\mathbf{H}_4)$ (i) $b,\ \sigma,\ f$ and h are Lipschitz continuous with respect to $t\in[0,T]$; for all $t,t'\in[0,T]$ and $(x,u,e)\in\mathbb{R}^n\times U\times E$, $|\gamma(t,x,u,e)-\gamma(t',x,u,e)|\leqslant C(1\wedge|e|)|t-t'|$;
(ii) $b,\ \sigma,\ f,\ h$ and Φ are bounded in $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$; for all $x\in\mathbb{R}^n$ and $e\in E$, $|\gamma(t,x,u,e)|\leqslant C(1\wedge|e|)$;
(iii) $b,\ \sigma,\ \gamma$ are differentiable in t and x, and

$$\begin{cases}|b_x(t,x,u)-b_x(t',x',u)|+|\sigma_x(t,x,u)-\sigma_x(t',x',u)|\leqslant C\big(|t-t'|+|x-x'|\big),\\ |b_t(t,x,u)-b_t(t',x',u)|+|\sigma_t(t,x,u)-\sigma_t(t',x',u)|\leqslant C\big(|t-t'|+|x-x'|\big),\\ |\gamma_x(t,x,u,e)-\gamma_x(t',x',u,e)|+|\gamma_t(t,x,u,e)-\gamma_t(t',x',u,e)|\leqslant C(1\wedge|e|)\big(|t-t'|+|x-x'|\big),\end{cases}$$

where C is a nonnegative constant;

$(\mathbf{H}_5)$ f is semi-concave in $(t,x,y,z,v)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, uniformly with respect to $u\in U$; Φ is semi-concave in $x\in\mathbb{R}^n$.

Remark 4.3. *In the following, for $p\geqslant 2$, $\int_E(1\wedge|e|)^p\nu(\mathrm{d}e)<\infty$ is needed, which holds naturally under* $(\mathbf{H}_3)$. For convenience, we always denote $\ell(e)=C(1\wedge|e|),\ e\in E$ in the following, even with different C.

Theorem 4.4. *Assume that the conditions $(\mathbf{H}_1)$–$(\mathbf{H}_5)$ and $(\mathbf{C})$ hold. For all $\delta>0$, the value function $W(\cdot,\cdot)$ is semi-concave on $[0,T-\delta]\times\mathbb{R}^n$, that is, there exists some constant $C_\delta>0$ such that for any (t_0,x_0), $(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^n$, and $\lambda\in[0,1]$,*

$$\lambda W(t_1,x_1)+(1-\lambda)W(t_0,x_0)\leqslant W(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)\big(|t_0-t_1|^2+|x_0-x_1|^2\big),$$

where $(t_\lambda,x_\lambda):=\lambda(t_1,x_1)+(1-\lambda)(t_0,x_0)$.

Indeed, combined with the property of $W(\cdot,\cdot)$ in Lemma 3.10, Theorem 4.4 can be proved clearly if the following result is true.

Proposition 4.5. *Let $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ and $(\mathbf{C})$ hold. Then, for all $\delta>0$, there exists some constant $C_\delta>0$ such that for any (t_0,x_0), $(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^n$ and $\lambda\in[0,1]$,*

$$\lambda W^n(t_1,x_1)+(1-\lambda)W^n(t_0,x_0)\leqslant W^n(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)\big(|t_0-t_1|^2+|x_0-x_1|^2\big)+\mathscr{A}^n,$$

with $\lim\limits_{n\to\infty}\mathscr{A}^n=0$.

The proof of Proposition 4.5 is very complex and occupies an extensive length. Therefore, we first make the following abbreviations,

$$\varphi^{\lambda,u}:=\varphi^{t_\lambda,x_\lambda;u},\quad \varphi^{i,u}:=\varphi^{t_i,x_i;u},\ \text{where }\varphi=X,Y,Z,V,A,\ i=0,1,\ \text{resp.};\quad \mathcal{X}^u:=\lambda X^{1,u}+(1-\lambda)X^{0,u};$$
$$(\mathcal{Y}^u,\mathcal{Z}^u,\mathcal{V}^u,\mathcal{A}^u):=(\lambda Y^{1,u}+(1-\lambda)Y^{0,u},\lambda Z^{1,u}+(1-\lambda)Z^{0,u},\lambda V^{1,u}+(1-\lambda)V^{0,u},\lambda A^{1,u}+(1-\lambda)A^{0,u}).$$

For the above processes, we can recognize the equations satisfied by them as follows. For $\lambda \in [0,1]$ and $u^\lambda(\cdot) \in \mathcal{U}^\lambda_{t_\lambda,T}$,

$$\begin{cases} \mathrm{d}X^{\lambda,u^\lambda}_s = b(s, X^{\lambda,u^\lambda}_s, u^\lambda_s)\mathrm{d}s + \sigma(s, X^{\lambda,u^\lambda}_s, u^\lambda_s)\mathrm{d}B^\lambda_s + \displaystyle\int_E \gamma(s, X^{\lambda,u^\lambda}_{s-}, u^\lambda_s, e)\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_\lambda, T], \\ X^{\lambda,u^\lambda}_{t_\lambda} = x_\lambda, \quad (t_\lambda, x_\lambda) \in [0, T-\delta] \times \mathbb{R}^n, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \text{(i) } (Y^{\lambda,u^\lambda}, Z^{\lambda,u^\lambda}, V^{\lambda,u^\lambda}, A^{\lambda,u^\lambda}) \in \mathscr{S}^2_{\mathbb{F}^\lambda}[t_\lambda, T]; \\ \text{(ii) } Y^{\lambda,u^\lambda}_s = \Phi(X^{\lambda,u^\lambda}_T) + \displaystyle\int_s^T f\big(r, X^{\lambda,u^\lambda}_r, Y^{\lambda,u^\lambda}_r, Z^{\lambda,u^\lambda}_r, \int_E l(e)V^{\lambda,u^\lambda}_r(e)\nu(\mathrm{d}e), u^\lambda_r\big)\mathrm{d}r \\ \qquad\qquad -(A^{\lambda,u^\lambda}_T - A^{\lambda,u^\lambda}_s) - \displaystyle\int_s^T Z^{\lambda,u^\lambda}_r \mathrm{d}B^\lambda_r - \int_s^T \int_E V^{\lambda,u^\lambda}_r(e)\tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_\lambda, T]; \\ \text{(iii) } Y^{\lambda,u^\lambda}_s \leqslant h(s, X^{\lambda,u^\lambda}_s), \quad \text{a.e. } s \in [t_\lambda, T]; \\ \text{(iv) } \displaystyle\int_t^T \big(h(s, X^{\lambda,u^\lambda}_s) - Y^{\lambda,u^\lambda}_s\big)\mathrm{d}A^{\lambda,u^\lambda}_s = 0. \end{cases} \tag{4.4}$$

For $i = 0, 1$ and $u^i(\cdot) \in \mathcal{U}^i_{t_i,T}$,

$$\begin{cases} \mathrm{d}X^{i,u^i}_s = b(s, X^{i,u^i}_s, u^i_s)\mathrm{d}s + \sigma(s, X^{i,u^i}_s, u^i_s)\mathrm{d}\mathbb{B}^i_s + \displaystyle\int_E \gamma(s, X^{i,u^i}_{s-}, u^i_s, e)\tilde{\mathbb{N}}^i(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_i, T], \\ X^{i,u^i}_{t_i} = x_i, \quad (t_i, x_i) \in [0, T-\delta] \times \mathbb{R}^n, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \text{(i) } (Y^{i,u^i}, Z^{i,u^i}, V^{i,u^i}, A^{i,u^i}) \in \mathscr{S}^2_{\mathbf{F}^i}[t_i, T]; \\ \text{(ii) } Y^{i,u^i}_s = \Phi(X^{i,u^i}_T) + \displaystyle\int_s^T f\big(r, X^{i,u^i}_r, Y^{i,u^i}_r, Z^{i,u^i}_r, \int_E l(e)V^{i,u^i}_r(e)\nu(\mathrm{d}e), u^i_r\big)\mathrm{d}r \\ \qquad\qquad -(A^{i,u^i}_T - A^{i,u^i}_s) - \displaystyle\int_s^T Z^{i,u^i}_r \mathrm{d}\mathbb{B}^i_r - \int_s^T \int_E V^{i,u^i}_r(e)\tilde{\mathbb{N}}^i(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_i, T]; \\ \text{(iii) } Y^{i,u^i}_s \leqslant h(s, X^{i,u^i}_s), \quad \text{a.e. } s \in [t_i, T]; \\ \text{(iv) } \displaystyle\int_t^T \big(h(s, X^{i,u^i}_s) - Y^{i,u^i}_s\big)\mathrm{d}A^{i,u^i}_s = 0. \end{cases} \tag{4.6}$$

Under our conditions, the above equations are all well-posed. Moreover, for RBSDE with jumps (4.4) and (4.6), their penalized equations are as follows, for $n \in \mathbb{N}$, $\lambda \in [0,1]$,

$$\begin{cases} \mathrm{d}Y^{n,\lambda,u^\lambda}_s = -f\big(s, X^{\lambda,u^\lambda}_s, Y^{n,\lambda,u^\lambda}_s, Z^{n,\lambda,u^\lambda}_s, \displaystyle\int_E l(e)V^{n,\lambda,u^\lambda}_s(e)\nu(\mathrm{d}e), u^\lambda_s\big)\mathrm{d}s + \mathrm{d}A^{n,\lambda,u^\lambda}_s + Z^{n,\lambda,u^\lambda}_s \mathrm{d}B^\lambda_s \\ \qquad\qquad + \displaystyle\int_E V^{n,\lambda,u^\lambda}_s(e)\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_\lambda, T], \\ Y^{n,\lambda,u^\lambda}_T = \Phi(X^{\lambda,u^\lambda}_T); \end{cases} \tag{4.7}$$

and for $i = 0, 1$,

$$\begin{cases} \mathrm{d}Y^{n,i,u^i}_s = -f\big(s, X^{i,u^i}_s, Y^{n,i,u^i}_s, Z^{n,i,u^i}_s, \int_E l(e) V^{n,i,u^i}_s(e)\nu(\mathrm{d}e), u^i_s\big)\mathrm{d}s + \mathrm{d}A^{n,i,u^i}_s + Z^{n,i,u^i}_s \mathrm{d}\mathbb{B}^i_s \\ \qquad\qquad + \int_E V^{n,i,u^i}_s(e)\tilde{\mathbb{N}}^i(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_i, T], \\ Y^{n,i,u^i}_T = \Phi(X^{i,u^i}_T), \end{cases} \tag{4.8}$$

where $A^{n,\lambda,u^\lambda}_\cdot := n\int_{t_\lambda}^{\cdot} \big(h(r, X^{\lambda,u^\lambda}_r) - Y^{n,\lambda,u^\lambda}_r\big)^- \mathrm{d}r$ and $A^{n,i,u^i}_\cdot := n\int_{t_i}^{\cdot} \big(h(r, X^{i,u^i}_r) - Y^{n,i,u^i}_r\big)^- \mathrm{d}r$. Moreover, the wellposedness of the above equations is obvious.

Recall the definition of $W^n(\cdot,\cdot)$ (in (3.9)), to prove Proposition 4.5, we need to prove formally

$$\lambda Y^{n,1,u^1}_{t_1} + (1-\lambda)Y^{n,0,u^0}_{t_0} - Y^{n,\lambda,u^\lambda}_{t_\lambda} \leqslant C_\delta \lambda(1-\lambda)\big(|t_1 - t_0|^2 + |x_1 - x_0|^2\big) + \mathscr{A}^n, \tag{4.9}$$

with $\lim\limits_{n\to\infty} \mathscr{A}^n = 0$. Note that, the processes involved in (4.9) are fixed with three different times, so that the direct proof of (4.9) is infeasible. Therefore, for $i = 0, 1$, by introducing

$$\tilde{X}^{i,u^\lambda}_s := X^{i,u^\lambda}_{\varrho^\lambda_i(s)},\ \tilde{\varphi}^{n,i,u^\lambda}_s := \varphi^{n,i,u^i}_{\varrho^\lambda_i(s)},\ \varphi = Y, V, A, \text{ resp., and } \tilde{Z}^{n,i,u^\lambda}_s := \frac{1}{\sqrt{\dot{\tau}^\lambda_i}} Z^{n,i,u^i}_{\varrho^\lambda_i(s)},\ s \in [t_\lambda, T],\ \mathbb{P}\text{-a.s.},$$

and applying the inverse time change to SDE (4.5) and BSDE (4.8), we get (note (4.2))

$$\begin{cases} \mathrm{d}\tilde{X}^{i,u^\lambda}_s = \frac{1}{\dot{\tau}^\lambda_i} b\big(\varrho^\lambda_i(s), \tilde{X}^{i,u^\lambda}_s, u^\lambda_s\big)\mathrm{d}s + \frac{1}{\sqrt{\dot{\tau}^\lambda_i}}\sigma\big(\varrho^\lambda_i(s), \tilde{X}^{i,u^\lambda}_s, u^\lambda_s\big)\mathrm{d}B^\lambda_s \\ \qquad\qquad + \int_E \gamma\big(\varrho^\lambda_i(s), \tilde{X}^{i,u^\lambda}_{s-}, u^\lambda_s, e\big)\Big(\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e) + (1 - \frac{1}{\dot{\tau}^\lambda_i})\nu(\mathrm{d}e)\mathrm{d}s\Big), \quad s \in [t_\lambda, T], \\ \tilde{X}^{i,u^\lambda}_{t_\lambda} = x_i, \quad (t_\lambda, x_i) \in [0, T-\delta] \times \mathbb{R}^n, \end{cases} \tag{4.10}$$

and

$$\begin{cases} \mathrm{d}\tilde{Y}^{n,i,u^\lambda}_s = -\frac{1}{\dot{\tau}^\lambda_i} f\big(\varrho^\lambda_i(s), \tilde{X}^{i,u^\lambda}_s, \tilde{Y}^{n,i,u^\lambda}_s, \sqrt{\dot{\tau}^\lambda_i}\tilde{Z}^{n,i,u^\lambda}_s, \int_E l(e)\tilde{V}^{n,i,u^\lambda}_s(e)\nu(\mathrm{d}e), u^\lambda_s\big)\mathrm{d}s + \mathrm{d}\tilde{A}^{n,i,u^\lambda}_s \\ \qquad\qquad + \tilde{Z}^{n,i,u^\lambda}_s \mathrm{d}B^\lambda_s + \int_E \tilde{V}^{n,i,u^\lambda}_s(e)\Big(\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e) + (1 - \frac{1}{\dot{\tau}^\lambda_i})\nu(\mathrm{d}e)\mathrm{d}s\Big), \quad s \in [t_\lambda, T], \\ \tilde{Y}^{n,i,u^\lambda}_T = \Phi(\tilde{X}^{i,u^\lambda}_T), \end{cases} \tag{4.11}$$

where $u^\lambda_s = u^i_{\varrho^i_i(s)}$ and $\tilde{A}^{n,i,u^\lambda}_s = \frac{n}{\dot{\tau}^\lambda_i}\int_{t_\lambda}^s \big(h(\varrho^\lambda_i(r), \tilde{X}^{i,u^\lambda}_r) - \tilde{Y}^{n,i,u^\lambda}_r\big)^- \mathrm{d}r$.

By the above transformation, (4.9) turns out to be equivalent to

$$\lambda \tilde{Y}^{n,1,u^\lambda}_{t_\lambda} + (1-\lambda)\tilde{Y}^{n,0,u^\lambda}_{t_\lambda} - Y^{n,\lambda,u^\lambda}_{t_\lambda} \leqslant C_\delta \lambda(1-\lambda)\big(|t_1 - t_0|^2 + |x_1 - x_0|^2\big) + \mathscr{A}^n. \tag{4.12}$$

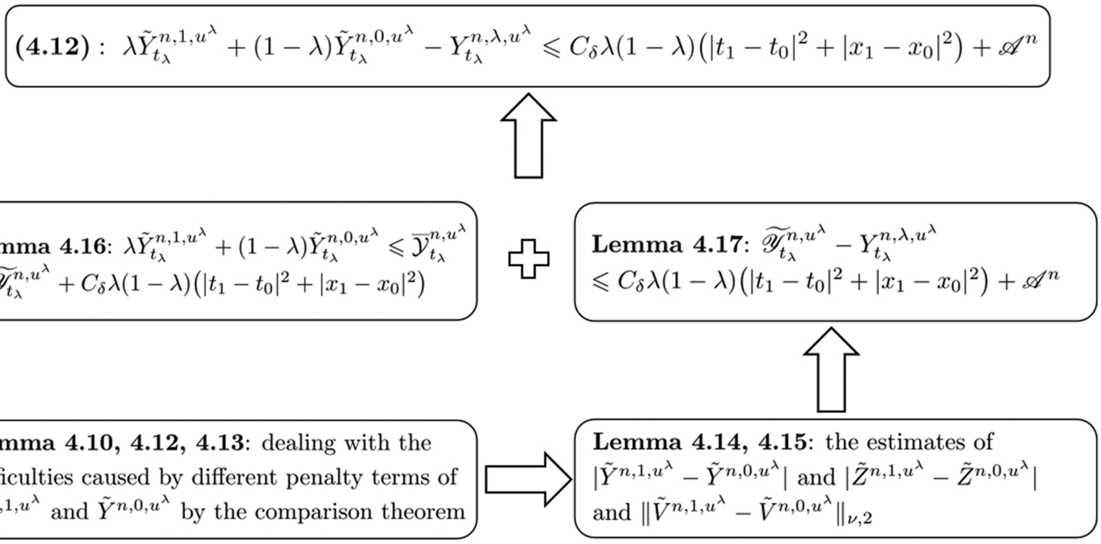

FIGURE 1. The auxiliary Lemmas.

Once (4.12) is true, Proposition 4.5 can be proved as follows. Note that, for any $n \geqslant 1$, $(t_0,x_0),(t_1,x_1),(t_\lambda,x_\lambda) \in [0,T-\delta]\times\mathbb{R}^n$,

$$
\begin{aligned}
W^n(t_i,x_i) &= \operatorname*{essinf}_{u^i(\cdot)\in\mathcal{U}^i_{t_i,T}} J_n(t_i,x_i;u^i(\cdot)) = \operatorname*{essinf}_{u^\lambda(\cdot)\in\mathcal{U}^\lambda_{t_\lambda,T}} \tilde{Y}^{n,i,u^\lambda}_{t_\lambda}, \quad i=0,1,\\
W^n(t_\lambda,x_\lambda) &= \operatorname*{essinf}_{u^\lambda(\cdot)\in\mathcal{U}^\lambda_{t_\lambda,T}} J_n(t_\lambda,x_\lambda;u^\lambda(\cdot)) = \operatorname*{essinf}_{u^\lambda(\cdot)\in\mathcal{U}^\lambda_{t_\lambda,T}} Y^{n,\lambda,u^\lambda}_{t_\lambda}.
\end{aligned}
\tag{4.13}
$$

Then, for any $\varepsilon>0$, there exists some $u^{\lambda,\varepsilon}(\cdot)\in\mathcal{U}^\lambda_{t_\lambda,T}$ such that

$$
\begin{aligned}
&\lambda W^n(t_1,x_1)+(1-\lambda)W^n(t_0,x_0) \leqslant \lambda\tilde{Y}^{n,1,u^{\lambda,\varepsilon}}_{t_\lambda} + (1-\lambda)\tilde{Y}^{n,0,u^{\lambda,\varepsilon}}_{t_\lambda}\\
&\leqslant Y^{n,\lambda,u^{\lambda,\varepsilon}}_{t_\lambda} + C_\delta\lambda(1-\lambda)\big(|t_1-t_0|^2+|x_1-x_0|^2\big)+\mathscr{A}^n\\
&\leqslant \varepsilon + W^n(t_\lambda,x_\lambda) + C_\delta\lambda(1-\lambda)\big(|t_1-t_0|^2+|x_1-x_0|^2\big)+\mathscr{A}^n.
\end{aligned}
\tag{4.14}
$$

Due to the arbitrariness of $\varepsilon>0$, we complete the proof of Proposition 4.5.

From now on, we focus on the study of (4.12). The proof of (4.12) is complicated and needs some techniques. To make it clear, we make a flow chart to present the logical relationship of the auxiliary Lemmas.

Besides the above, we point out that the study of $|\tilde{X}^{1,u^\lambda}-\tilde{X}^{0,u^\lambda}|$, $|\lambda\tilde{X}^{1,u^\lambda}+(1-\lambda)\tilde{X}^{0,u^\lambda}-X^{\lambda,u^\lambda}|$ in Lemmas 4.8, 4.9 and the penalized equations (4.8) in Lemmas 4.6, 4.7 are all the basis of the above inferences.

Now we study some properties of the penalized equations (4.8).

Lemma 4.6. *Under the conditions* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_4)$-(ii), *for all* $n\geqslant 1$ *and* $i=1,2$,

$$|Y^{n,i,u^i}_s| \leqslant C, \quad \forall s\in[t_i,T],\ \mathbb{P}\text{-a.s.}$$

Furthermore, for any $p \geqslant 1$, there exists some constant $C_p > 0$ such that

$$\mathbb{E}^{\mathcal{F}^i_s}\Big[\Big(\int_s^T |Z^{n,i,u^i}_r|^2\mathrm{d}r\Big)^p + \Big(\int_s^T \|V^{n,i,u^i}_r(\cdot)\|^2_{\nu,2}\mathrm{d}r\Big)^p + \Big(A^{n,i,u^i}_T - A^{n,i,u^i}_s\Big)^{2p}\Big] \leqslant C_p, \quad \forall s \in [t_i, T],\ \mathbb{P}\text{-a.s.},$$

where the constant C_p depends on p as well as the bounds of f, h and Φ.

Proof. Firstly, similar to the proof of Theorem 4.2 in [9], we know the existence of some constant $C > 0$ independent of n such that

$$\mathbb{E}^{\mathcal{F}^i_s}\Big[\sup_{s\in[t_i,T]}|Y^{n,i,u^i}_s|^2 + \int_s^T \Big(|Z^{n,i,u^i}_r|^2 + \|V^{n,i,u^i}_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r + \Big(A^{n,i,u^i}_T - A^{n,i,u^i}_s\Big)^2\Big] \leqslant C, \quad \mathbb{P}\text{-a.s.} \tag{4.15}$$

Especially, we get, for all $n \geqslant 1$, $i = 1, 2$ and $s \in [t_i, T]$,

$$|Y^{n,i,u^i}_s| \leqslant C, \quad \mathbb{P}\text{-a.s.}$$

Next, according to (4.15) and the boundedness of f, for (4.8), we obtain

$$\begin{aligned}
A^{n,i,u^i}_T - A^{n,i,u^i}_s &= \Phi(X^{i,u^i}_T) - Y^{n,i,u^i}_s + \int_s^T f\Big(r, X^{i,u^i}_r, Y^{n,i,u^i}_r, Z^{n,i,u^i}_r, \int_E l(e)V^{n,i,u^i}_r(e)\nu(\mathrm{d}e), u^i_r\Big)\mathrm{d}r\\
&\quad - \int_s^T Z^{n,i,u^i}_r \mathrm{d}\mathbb{B}^i_r - \int_s^T\int_E V^{n,i,u^i}_r(e)\tilde{\mathbb{N}}^i(\mathrm{d}r, \mathrm{d}e)\\
&\leqslant C + \Big|\int_s^T Z^{n,i,u^i}_r \mathrm{d}\mathbb{B}^i_r\Big| + \Big|\int_s^T\int_E V^{n,i,u^i}_r(e)\tilde{\mathbb{N}}^i(\mathrm{d}r, \mathrm{d}e)\Big|.
\end{aligned}$$

Thereby, for all $p \geqslant 1$, from the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}^i_s}\Big[\Big(A^{n,i,u^i}_T - A^{n,i,u^i}_s\Big)^{2p}\Big]\\
&\leqslant C_p + C_p\mathbb{E}^{\mathcal{F}^i_s}\Big[\Big(\int_s^T |Z^{n,i,u^i}_r|^2\mathrm{d}r\Big)^p + \Big(\int_s^T\int_E |V^{n,i,u^i}_r(e)|^2\mathbb{N}^i(\mathrm{d}r, \mathrm{d}e)\Big)^p\Big], \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{4.16}$$

Applying Itô's formula to $|Y^{n,i,u^i}_s|^2$, we have

$$\begin{aligned}
&|Y^{n,i,u^i}_s|^2 + \int_s^T |Z^{n,i,u^i}_r|^2\mathrm{d}r + \int_s^T\int_E |V^{n,i,u^i}_r(e)|^2\mathbb{N}^i(\mathrm{d}r, \mathrm{d}e)\\
&= |\Phi(X^{i,u^i}_T)|^2 + 2\int_s^T Y^{n,i,u^i}_r f\Big(r, X^{i,u^i}_r, Y^{n,i,u^i}_r, Z^{n,i,u^i}_r, \int_E l(e)V^{n,i,u^i}_r(e)\nu(\mathrm{d}e), u^i_r\Big)\mathrm{d}r\\
&\quad -2\int_s^T Y^{n,i,u^i}_r \mathrm{d}A^{n,i,u^i}_r - 2\int_s^T Y^{n,i,u^i}_r Z^{n,i,u^i}_r \mathrm{d}\mathbb{B}^i_r - 2\int_s^T\int_E Y^{n,i,u^i}_r V^{n,i,u^i}_r(e)\tilde{\mathbb{N}}^i(\mathrm{d}r, \mathrm{d}e), \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{4.17}$$

Then, from the inequalities (4.15) and (4.17), for all $p \geqslant 1$,

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{2p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{2p}\Big] \\
&\leqslant \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r + \int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{2p}\Big] \\
&\leqslant C_p + C_p \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{2p} + \Big|\int_s^T Y_r^{n,i,u^i} Z_r^{n,i,u^i} \mathrm{d}\mathbb{B}_r^i\Big|^{2p} + \Big|\int_s^T \int_E Y_r^{n,i,u^i} V_r^{n,i,u^i}(e) \tilde{\mathbb{N}}^i(\mathrm{d}r,\mathrm{d}e)\Big|^{2p}\Big] \\
&\leqslant C_p + C_p \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{2p} + \Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{p}\Big], \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Further, combined with (4.16), we have

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{4p} + \Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{2p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{2p}\Big] \\
&\leqslant C_p + C_p \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{2p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{2p}\Big] \\
&\leqslant C_p + C_p \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{2p} + \Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{p}\Big], \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Consequently, due to (4.15), for all integers of the form $p = 2^k$ with $k = 0, 1, 2, \cdots$,

$$
\mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{p} + \Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{2p}\Big] \leqslant C_p, \quad \mathbb{P}\text{-a.s.}
$$

Hence, for any $p \geqslant 1$, the above conclusion is valid.

Finally, using Lemma 3.1 in [23], for all $p \geqslant 1$,

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{p} + \Big(\int_s^T \|V_r^{n,i,u^i}(\cdot)\|_{\nu,2}^2 \mathrm{d}r\Big)^{p} + \big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\big)^{2p}\Big] \\
&\leqslant C \mathbb{E}^{\mathcal{F}_s^i}\Big[\Big(\int_s^T |Z_r^{n,i,u^i}|^2 \mathrm{d}r\Big)^{p} + \Big(\int_s^T \int_E |V_r^{n,i,u^i}(e)|^2 \mathbb{N}^i(\mathrm{d}r,\mathrm{d}e)\Big)^{p} + \Big(A_T^{n,i,u^i} - A_s^{n,i,u^i}\Big)^{2p}\Big] \leqslant C_p, \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

∎

Similar to Lemma 6.1 in [1], the following result holds. For the readers' convenience, we sketch the proof.

Lemma 4.7. *Under* **(H$_1$)**, **(H$_2$)**, **(H$_4$)**-(ii) *and* **(C)**, *for any* $p \geqslant 2$, $\mathbb{P}$-*a.s.*,

$$
\text{(i)}\ \lim_{n\to\infty} \mathbb{E}^{\mathcal{F}_s^i}\Big[\sup_{r\in[s,T]} |(h(r, X_r^{i,u^i}) - Y_r^{n,i,u^i})^-|^p\Big] = 0, \quad s \in [t_i, T],
$$

$$
\text{(ii)}\ \lim_{n\to\infty} \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big|\int_s^T (h(\rho_i^\lambda(r), \tilde{X}_r^{i,u^\lambda}) - \tilde{Y}_r^{n,i,u^\lambda})^- \mathrm{d}\tilde{A}_r^{n,j,u^\lambda}\Big|^{\frac{p}{2}}\Big] = 0, \quad i, j = 0, 1,\ s \in [t_\lambda, T].
$$

Proof. It is obvious that, for all $s \in [t_i, T]$, $\{Y_s^{n,i,u^i}\}$ is decreasing in n, thereby $Y_s^{n,i,u^i} \leqslant Y_s^{0,i,u^i}$, $\mathbb{P}$-a.s. Next we shall prove $Y_s^{n,i,u^i} \leqslant h(s, X_s^{i,u^i})$, $s \in [t_i, T]$, $\mathbb{P}$-a.s. By the comparison theorem of BSDE with jumps, we know

$Y^{n,i,u^i}_s \leqslant \mathbb{Y}^{n,i,u^i}_s$, where

$$\begin{cases} \mathrm{d}\mathbb{Y}^{n,i,u^i}_s = -\Big[f\big(s, X^{i,u^i}_s, \mathbb{Y}^{n,i,u^i}_s, \mathbb{Z}^{n,i,u^i}_s, \displaystyle\int_E l(e)\mathbb{V}^{n,i,u^i}_s(e)\nu(\mathrm{d}e), u^i_s\big) - n\big(\mathbb{Y}^{n,i,u^i}_s - h(s, X^{i,u^i}_s)\big)\Big]\mathrm{d}s \\ \qquad\qquad +\mathbb{Z}^{n,i,u^i}_s\mathrm{d}\mathbb{B}^i_s + \displaystyle\int_E \mathbb{V}^{n,i,u^i}_s(e)\tilde{\mathbb{N}}^i(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_i, T], \\ \mathbb{Y}^{n,i,u^i}_T = \Phi(X^{i,u^i}_T). \end{cases}$$

Let ϑ be a stopping time such that $s \leqslant \vartheta \leqslant T$, then

$$\begin{aligned} \mathbb{Y}^{n,i,u^i}_\vartheta = \mathbb{E}^{\mathcal{F}^i_\vartheta}\Big[&\mathrm{e}^{-n(T-\vartheta)}\Phi(X^{i,u^i}_T) + \int_\vartheta^T \mathrm{e}^{-n(r-\vartheta)} f\big(r, X^{i,u^i}_r, \mathbb{Y}^{n,i,u^i}_r, \mathbb{Z}^{n,i,u^i}_r, \int_E l(e)\mathbb{V}^{n,i,u^i}_r(e)\nu(\mathrm{d}e), u^i_r\big)\mathrm{d}r \\ &+ \int_\vartheta^T n\mathrm{e}^{-n(r-\vartheta)} h(r, X^{i,u^i}_r)\mathrm{d}r\Big]. \end{aligned}$$

As $n \to \infty$, it is easy to check that,

$$\mathrm{e}^{-n(T-\vartheta)}\Phi(X^{i,u^i}_T) + \int_\vartheta^T n\mathrm{e}^{-n(r-\vartheta)} h(r, X^{i,u^i}_r)\mathrm{d}r \to \Phi(X^{i,u^i}_T)\mathbf{I}_{\{\vartheta = T\}} + h(\vartheta, X^{i,u^i}_\vartheta)\mathbf{I}_{\{\vartheta < T\}},$$

$\mathbb{P}$-a.s. and in L^2 sense. Furthermore, due to f is bounded, we also have

$$\mathbb{E}^{\mathcal{F}^i_\vartheta}\Big[\int_\vartheta^T \mathrm{e}^{-n(r-\vartheta)} f\big(r, X^{i,u^i}_r, \mathbb{Y}^{n,i,u^i}_r, \mathbb{Z}^{n,i,u^i}_r, \int_E l(e)\mathbb{V}^{n,i,u^i}_r(e)\nu(\mathrm{d}e), u^i_r\big)\mathrm{d}r\Big] \to 0,$$

in L^2 sense. Consequently, we get $\mathbb{Y}^{n,i,u^i}_\vartheta \to \Phi(X^{i,u^i}_T)\mathbf{I}_{\{\vartheta=T\}} + h(\vartheta, X^{i,u^i}_\vartheta)\mathbf{I}_{\{\vartheta<T\}}$ in mean square and $Y^{n,i,u^i}_\cdot \leqslant h(\cdot, X^{i,u^i}_\cdot)$. Combined with the section theorem in Dellacherie and Meyer [29], we have $Y^{n,i,u^i}_s \leqslant h(s, X^{i,u^i}_s)$, $s \in [t_i, T]$, $\mathbb{P}$-a.s. Therefore, $(h(s, X^{i,u^i}_s) - Y^{n,i,u^i}_s)^- \downarrow 0$, $s \in [t_i, T]$, $\mathbb{P}$-a.s., as $n \to \infty$.

Denote ${}^p\Lambda$ be the predictable projection of any process Λ. Based on the aforementioned results, the predictable projection of $Y^{n,i,u^i}_\cdot$ possesses the following properties:

$$ {}^pY^{n,i,u^i} \downarrow {}^pY^{i,u^i} \text{ in } \mathcal{S}^2_{\mathbb{F}}(t_i, T; \mathbb{R}), \qquad {}^pY^{n,i,u^i}_s \leqslant {}^ph(s, X^{i,u^i}_s), \quad s \in [t_i, T], \ \mathbb{P}\text{-a.s.} \tag{4.18}$$

Note that the jumping times of both $Y^{n,i,u^i}_\cdot$ and $h(\cdot, X^{i,u^i}_\cdot)$ are inaccessible. Then, for all $s \in [t_i, T]$, we obtain ${}^pY^{n,i,u^i}_s = Y^{n,i,u^i}_{s-}$ and ${}^ph(s, X^{i,u^i}_s) = h(s, X^{i,u^i}_{s-})$, $\mathbb{P}$-a.s. Combined with (4.18), $\big(h(s, X^{i,u^i}_{s-}) - Y^{n,i,u^i}_{s-}\big)^- \downarrow 0$, $s \in [t_i, T]$, $\mathbb{P}$-a.s., as $n \to \infty$.

Furthermore, based on the continuity and monotonicity of power functions, for any $p \geqslant 2$, we have

$$\big|\big(h(s, X^{i,u^i}_s) - Y^{n,i,u^i}_s\big)^-\big|^p \downarrow 0, \qquad \big|\big(h(s, X^{i,u^i}_{s-}) - Y^{n,i,u^i}_{s-}\big)^-\big|^p \downarrow 0, \quad s \in [t_i, T], \ \mathbb{P}\text{-a.s., as } n \to \infty.$$

From the Dini's Theorem for càdlàg processes (see p. 202 in [30]), we know such a convergence is uniform in t. Further, using the dominated convergence theorem, we conclude that (i) holds.

Now, let's prove the second result. For any $p \geqslant 2$ and $i, j = 0, 1$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big|\int_s^T (h(\rho_i^\lambda(r),\tilde X_r^{i,u^\lambda})-\tilde Y_r^{n,i,u^\lambda})^- \mathrm{d}\tilde A_r^{n,j,u^\lambda}\Big|^{\frac p2}\Big]\\
&\leqslant \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{r\in[s,T]}\Big|(h(\rho_i^\lambda(r),\tilde X_r^{i,u^\lambda})-\tilde Y_r^{n,i,u^\lambda})^-\Big|^{\frac p2}\cdot\Big|\tilde A_T^{n,j,u^\lambda}-\tilde A_s^{n,j,u^\lambda}\Big|^{\frac p2}\Big]\\
&\leqslant \Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{r\in[s,T]}\Big|(h(\rho_i^\lambda(r),\tilde X_r^{i,u^\lambda})-\tilde Y_r^{n,i,u^\lambda})^-\Big|^p\Big]\Big)^{\frac12}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\tilde A_T^{n,j,u^\lambda}-\tilde A_s^{n,j,u^\lambda}\Big|^p\Big]\Big)^{\frac12}\\
&\to 0,\quad \text{as } n\to\infty,
\end{aligned}$$

where the last inequality have used (i) and Lemma 4.6. ∎

Now we estimate the difference of $\tilde X_\cdot^{0,u^\lambda}$ and $\tilde X_\cdot^{1,u^\lambda}$.

Lemma 4.8. *Suppose* **(H**$_1$**)**–**(H**$_3$**)** *and* **(H**$_4$**)**-(i), (ii) *hold. Then, for all $p \geqslant 1$ and $s \in [t_\lambda, T]$, we have*

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}|\tilde X_\tau^{1,u^\lambda}-\tilde X_\tau^{0,u^\lambda}|^p+\int_s^T|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^p\mathrm{d}r\Big]\leqslant C_{T,p,\delta}\Big(|t_1-t_0|^p+|\tilde X_s^{1,u^\lambda}-\tilde X_s^{0,u^\lambda}|^p\Big),\quad \mathbb{P}\text{-a.s.}$$

Proof. From (4.10), we know

$$\begin{cases}
\mathrm{d}(\tilde X_s^{1,u^\lambda}-\tilde X_s^{0,u^\lambda})=\Big(\dfrac{1}{\dot\tau_1^\lambda}b\big(\varrho_1^\lambda(s),\tilde X_s^{1,u^\lambda},u_s^\lambda\big)-\dfrac{1}{\dot\tau_0^\lambda}b\big(\varrho_0^\lambda(s),\tilde X_s^{0,u^\lambda},u_s^\lambda\big)\Big)\mathrm{d}s\\
\quad+\Big(\dfrac{1}{\sqrt{\dot\tau_1^\lambda}}\sigma\big(\varrho_1^\lambda(s),\tilde X_s^{1,u^\lambda},u_s^\lambda\big)-\dfrac{1}{\sqrt{\dot\tau_0^\lambda}}\sigma\big(\varrho_0^\lambda(s),\tilde X_s^{0,u^\lambda},u_s^\lambda\big)\Big)\mathrm{d}B_s^\lambda\\
\quad+\displaystyle\int_E\Big(\gamma\big(\varrho_1^\lambda(s),\tilde X_{s-}^{1,u^\lambda},u_s^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(s),\tilde X_{s-}^{0,u^\lambda},u_s^\lambda,e\big)\Big)\tilde N^\lambda(\mathrm{d}s,\mathrm{d}e)\\
\quad+\displaystyle\int_E\Big(\gamma\big(\varrho_1^\lambda(s),\tilde X_{s-}^{1,u^\lambda},u_s^\lambda,e\big)(1-\frac{1}{\dot\tau_1^\lambda})-\gamma\big(\varrho_0^\lambda(s),\tilde X_{s-}^{0,u^\lambda},u_s^\lambda,e\big)(1-\frac{1}{\dot\tau_0^\lambda})\Big)\nu(\mathrm{d}e)\mathrm{d}s,\quad s\in[t_\lambda,T],\\
\tilde X_{t_\lambda}^{1,u^\lambda}-\tilde X_{t_\lambda}^{0,u^\lambda}=x_1-x_0.
\end{cases}$$

From Lemma 4.1, the boundedness and Lipschitz continuity of b, σ, γ, for any $p \geqslant 1$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\Big(\frac{1}{\dot\tau_1^\lambda}b\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda\big)-\frac{1}{\dot\tau_0^\lambda}b\big(\varrho_0^\lambda(r),\tilde X_r^{0,u^\lambda},u_r^\lambda\big)\Big)\mathrm{d}r\Big|^p\Big]\\
&\leqslant C_{T,p}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big|\Big(\frac{1}{\dot\tau_1^\lambda}-\frac{1}{\dot\tau_0^\lambda}\Big)b\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda\big)+\frac{1}{\dot\tau_0^\lambda}\Big(b\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda\big)-b\big(\varrho_0^\lambda(r),\tilde X_r^{0,u^\lambda},u_r^\lambda\big)\Big)\Big|^p\mathrm{d}r\Big]\\
&\leqslant C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|t_1-t_0|^p+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^p\Big)\mathrm{d}r\Big].
\end{aligned}\tag{4.19}$$

Using the Burkholder-Davis-Gundy inequality, we get

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\Big(\frac{1}{\sqrt{\dot\tau_1^\lambda}}\sigma\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda\big)-\frac{1}{\sqrt{\dot\tau_0^\lambda}}\sigma\big(\varrho_0^\lambda(r),\tilde X_r^{0,u^\lambda},u_r^\lambda\big)\Big)\mathrm{d}B_r^\lambda\Big|^p\Big]\\
&\leqslant\begin{cases}C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\Big(\displaystyle\int_s^T\big(|t_1-t_0|^2+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^2\big)\mathrm{d}r\Big)^{\frac{p}{2}}\Big], & 1\leqslant p<2,\\ C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_t}\Big[\displaystyle\int_s^T\big(|t_1-t_0|^p+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^p\big)\mathrm{d}r\Big], & p\geqslant 2,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{t\in[s,T]}\Big|\int_s^t\int_E\Big(\gamma\big(\varrho_1^\lambda(r),\tilde X_{r-}^{1,u^\lambda},u_r^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(r),\tilde X_{r-}^{0,u^\lambda},u_r^\lambda,e\big)\Big)\tilde N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big|^p\Big]\\
&\leqslant\begin{cases}C_{T,p}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\displaystyle\int_s^T\int_E|\gamma\big(\varrho_1^\lambda(r),\tilde X_{r-}^{1,u^\lambda},u_r^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(r),\tilde X_{r-}^{0,u^\lambda},u_r^\lambda,e\big)|^2\nu(\mathrm{d}e)\mathrm{d}r\Big)^{\frac{p}{2}}\Big], & 1\leqslant p<2,\\ C_{T,p}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\displaystyle\int_s^T\int_E|\gamma\big(\varrho_1^\lambda(r),\tilde X_{r-}^{1,u^\lambda},u_r^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(r),\tilde X_{r-}^{0,u^\lambda},u_r^\lambda,e\big)|^2\nu(\mathrm{d}e)\mathrm{d}r\Big)^{\frac{p}{2}} & \\ \qquad+\displaystyle\int_s^T\int_E|\gamma\big(\varrho_1^\lambda(r),\tilde X_{r-}^{1,u^\lambda},u_r^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(r),\tilde X_{r-}^{0,u^\lambda},u_r^\lambda,e\big)|^p\nu(\mathrm{d}e)\mathrm{d}r\Big], & p\geqslant 2,\end{cases}\\
&\leqslant\begin{cases}C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\displaystyle\int_s^T\Big(|t_1-t_0|^2+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^2\Big)\mathrm{d}r\Big]\Big)^{\frac{p}{2}}, & 1\leqslant p<2,\\ C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\displaystyle\int_s^T\Big(|t_1-t_0|^p+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^p\Big)\mathrm{d}r\Big], & p\geqslant 2,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\int_E\Big(\gamma\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda,e\big)(1-\frac{1}{\dot\tau_1^\lambda})-\gamma\big(\varrho_0^\lambda(r),\tilde X_r^{0,u^\lambda},u_r^\lambda,e\big)(1-\frac{1}{\dot\tau_0^\lambda})\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big|^p\Big]\\
&\leqslant\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\int_E\Big|\Big(\frac{1}{\dot\tau_0^\lambda}-\frac{1}{\dot\tau_1^\lambda}\Big)\gamma\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda,e\big)\\
&\qquad\qquad+(1-\frac{1}{\dot\tau_0^\lambda})\Big(\gamma\big(\varrho_1^\lambda(r),\tilde X_r^{1,u^\lambda},u_r^\lambda,e\big)-\gamma\big(\varrho_0^\lambda(r),\tilde X_r^{0,u^\lambda},u_r^\lambda,e\big)\Big)\Big|\nu(\mathrm{d}e)\mathrm{d}r\Big)^p\Big]\\
&\leqslant C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|t_1-t_0|^p+|\tilde X_r^{1,u^\lambda}-\tilde X_r^{0,u^\lambda}|^p\Big)\mathrm{d}r\Big].
\end{aligned}
\tag{4.20}
$$

Then, for all $\tau \in [s,T]$ and $p \geqslant 1$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\tilde{X}_\tau^{1,u^\lambda}-\tilde{X}_\tau^{0,u^\lambda}|^p\Big]\\
&\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\mathrm{d}r\Big]\Big)\\
&\quad+C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^2\mathrm{d}r\Big)^{\frac{p}{2}}\mathbf{I}_{\{1\leqslant p<2\}}\Big]\\
&\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\mathrm{d}r\Big]\Big)\\
&\quad+C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^2\mathrm{d}r\Big]\Big)^{\frac{p}{2}}\mathbf{I}_{\{1\leqslant p<2\}}.
\end{aligned}$$

Therefore, for $p \geqslant 2$, using Gronwall's inequality, we get

$$\sup_{\tau\in[s,T]}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\tilde{X}_\tau^{1,u^\lambda}-\tilde{X}_\tau^{0,u^\lambda}|^p\Big]\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p\Big). \tag{4.21}$$

For $1 \leqslant p < 2$, $\tau \in [s,T]$ using (4.21) with $p = 2$, we have

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\tilde{X}_\tau^{1,u^\lambda}-\tilde{X}_\tau^{0,u^\lambda}|^p\Big]\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\mathrm{d}r\Big]\Big).$$

Using Gronwall's inequality again, (4.21) also holds for $1 \leqslant p < 2$.

Finally, from (4.19)–(4.20) and using (4.21) with $p \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}|\tilde{X}_\tau^{1,u^\lambda}-\tilde{X}_\tau^{0,u^\lambda}|^p\Big]\\
&\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\mathrm{d}r\Big]\Big)\\
&\quad+C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^2\mathrm{d}r\Big]\Big)^{\frac{p}{2}}\mathbf{I}_{\{1\leqslant p<2\}}\\
&\leqslant C_{T,p,\delta}\Big(|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p+|t_1-t_0|^p\Big),
\end{aligned}$$

holds for $p \geqslant 1$. The desired result is proved. ∎

Next, we denote by $\tilde{\mathcal{X}}_\cdot^{u^\lambda}$ the convex combination of $\tilde{X}_\cdot^{0,u^\lambda}$ and $\tilde{X}_\cdot^{1,u^\lambda}$, *i.e.*, $\tilde{\mathcal{X}}_s^{u^\lambda} := \lambda\tilde{X}_s^{1,u^\lambda}+(1-\lambda)\tilde{X}_s^{0,u^\lambda}$, $s \in [t_\lambda, T]$. The estimate of $|\tilde{\mathcal{X}}_\cdot^{u^\lambda} - X_\cdot^{\lambda,u^\lambda}|$ is given as follows.

Lemma 4.9. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$ *hold, for all* $p \geqslant 1$ *and* $s \in [t_\lambda, T]$, *we have,* $\mathbb{P}$*-a.s.,*

$$\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\sup_{\tau\in[s,T]}|\tilde{\mathcal{X}}^{u^\lambda}_\tau - X^{\lambda,u^\lambda}_\tau|^p\Big] \leqslant C_{T,p,\delta}\Big(|t_1-t_0|^p + |\tilde{\mathcal{X}}^{u^\lambda}_s - X^{\lambda,u^\lambda}_s|^p\Big) + C_{T,p,\delta}\lambda^p(1-\lambda)^p\Big(|t_1-t_0|^{2p} + |\tilde{X}^{1,u^\lambda}_s - \tilde{X}^{0,u^\lambda}_s|^{2p}\Big).$$

Proof. First, $\tilde{\mathcal{X}}^{u^\lambda}_\cdot - X^{\lambda,u^\lambda}_\cdot$ satisfies the following equation

$$\begin{aligned}
&\tilde{\mathcal{X}}^{u^\lambda}_s - X^{\lambda,u^\lambda}_s = \lambda\tilde{X}^{1,u^\lambda}_s + (1-\lambda)\tilde{X}^{0,u^\lambda}_s - X^{\lambda,u^\lambda}_s\\
&= \int_{t_\lambda}^s \Big(J_b(r) + b(r,\tilde{\mathcal{X}}^{u^\lambda}_r,u^\lambda_r) - b(r,X^{\lambda,u^\lambda}_r,u^\lambda_r)\Big)\mathrm{d}r + \int_{t_\lambda}^s \Big(J_\sigma(r) + \sigma(r,\tilde{\mathcal{X}}^{u^\lambda}_r,u^\lambda_r) - \sigma(r,X^{\lambda,u^\lambda}_r,u^\lambda_r)\Big)\mathrm{d}B^\lambda_r\\
&\quad + \int_{t_\lambda}^s\int_E \Big(J_\gamma(r,e) + \gamma(r,\tilde{\mathcal{X}}^{u^\lambda}_{r-},u^\lambda_r,e) - \gamma(r,X^{\lambda,u^\lambda}_{r-},u^\lambda_r,e)\Big)\tilde{N}^\lambda(\mathrm{d}r,\mathrm{d}e)\\
&\quad + \int_{t_\lambda}^s\int_E \Big(\lambda(1-\frac{1}{\dot{\tau}^\lambda_1})\gamma\big(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_r,u^\lambda_r,e\big) + (1-\lambda)(1-\frac{1}{\dot{\tau}^\lambda_0})\gamma\big(\varrho^\lambda_0(r),\tilde{X}^{0,u^\lambda}_r,u^\lambda_r,e\big)\Big)\nu(\mathrm{d}e)\mathrm{d}r,
\end{aligned}$$

where $s \in [t_\lambda, T]$ and

$$\begin{aligned}
&J_b(r) := \frac{\lambda}{\dot{\tau}^\lambda_1}b\big(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_r,u^\lambda_r\big) + \frac{1-\lambda}{\dot{\tau}^\lambda_0}b\big(\varrho^\lambda_0(r),\tilde{X}^{0,u^\lambda}_r,u^\lambda_r\big) - b(r,\tilde{\mathcal{X}}^{u^\lambda}_r,u^\lambda_r),\\
&J_\sigma(r) := \frac{\lambda}{\sqrt{\dot{\tau}^\lambda_1}}\sigma\big(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_r,u^\lambda_r\big) + \frac{1-\lambda}{\sqrt{\dot{\tau}^\lambda_0}}\sigma\big(\varrho^\lambda_0(r),\tilde{X}^{0,u^\lambda}_r,u^\lambda_r\big) - \sigma(r,\tilde{\mathcal{X}}^{u^\lambda}_r,u^\lambda_r),\\
&J_\gamma(r,e) := \lambda\gamma\big(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_{r-},u^\lambda_r,e\big) + (1-\lambda)\gamma\big(\varrho^\lambda_0(r),\tilde{X}^{0}_{r-},u^\lambda_r,e\big) - \gamma(r,\tilde{\mathcal{X}}^{\lambda,u^\lambda}_{r-},u^\lambda_r,e), \quad r\in[t_\lambda,T],\ e\in E.
\end{aligned}$$

From Lemma 4.1, for any $X_1, X_0 \in \mathbb{R}^n$ and $u \in U$, we have (set $X_\lambda := \lambda X_1 + (1-\lambda)X_0$)

$$\begin{aligned}
&\Big|\frac{\lambda}{\dot{\tau}^\lambda_1}b\big(\varrho^\lambda_1(r),X_1,u\big) + \frac{1-\lambda}{\dot{\tau}^\lambda_0}b\big(\varrho^\lambda_0(r),X_0,u\big) - b(r,X_\lambda,u)\Big|\\
&\leqslant \Big|\lambda b\big(\varrho^\lambda_1(r),X_1,u\big) + (1-\lambda)b\big(\varrho^\lambda_0(r),X_0,u\big) - b(r,X_\lambda,u)\Big|\\
&\quad + \Big|(\frac{1}{\dot{\tau}^\lambda_1}-1)\lambda b\big(\varrho^\lambda_1(r),X_1,u\big) + (\frac{1}{\dot{\tau}^\lambda_0}-1)(1-\lambda)b\big(\varrho^\lambda_0(r),X_0,u\big)\Big|\\
&\leqslant C\lambda(1-\lambda)|\varrho^\lambda_1(r)-\varrho^\lambda_0(r)|\cdot\int_0^1 \big|b_s\big(r+\eta(1-\lambda)(\varrho^\lambda_1(r)-\varrho^\lambda_0(r)),X_\lambda+\eta(1-\lambda)(X_1-X_0),u\big)\\
&\qquad\qquad -b_s\big(r-\eta\lambda(\varrho^\lambda_1(r)-\varrho^\lambda_0(r)),X_\lambda-\eta\lambda(X_1-X_0),u\big)\big|\mathrm{d}\eta\\
&\quad + C\lambda(1-\lambda)|X_1-X_0|\cdot\int_0^1 \big|b_x\big(r+\eta(1-\lambda)(\varrho^\lambda_1(r)-\varrho^\lambda_0(r)),X_\lambda+\eta(1-\lambda)(X_1-X_0),u\big)\\
&\qquad\qquad -b_x\big(r-\eta\lambda(\varrho^\lambda_1(r)-\varrho^\lambda_0(r)),X_\lambda-\eta\lambda(X_1-X_0),u\big)\big|\mathrm{d}\eta\\
&\quad + \frac{\lambda(1-\lambda)}{T-t_\lambda}|t_1-t_0|\big(|\varrho^\lambda_1(r)-\varrho^\lambda_0(r)| + |X_1-X_0|\big)\\
&\leqslant C\lambda(1-\lambda)\big(|t_1-t_0|^2 + |X_1-X_0|^2\big).
\end{aligned}$$

Therefore,

$$|J_b(r)| \leqslant C\lambda(1-\lambda)\Big(|t_1-t_0|^2 + |\tilde{X}^{1,u^\lambda}_r - \tilde{X}^{0,u^\lambda}_r|^2\Big), \quad r\in[t_\lambda,T]. \tag{4.22}$$

Similarly, using Lemma 4.1-(i), (ii) and (iii), we also obtain

$$\begin{aligned}
&|J_\sigma(r)|\leqslant C\lambda(1-\lambda)\Big(|t_1-t_0|^2+|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^2\Big),\\
&|J_\gamma(r,e)|\leqslant C\ell(e)\lambda(1-\lambda)\Big(|t_1-t_0|^2+|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^2\Big),\quad r\in[t_\lambda,T],\ e\in E.
\end{aligned}\tag{4.23}$$

Based on the above estimates, using the methods used in (4.19)–(4.20), for $p\geqslant 1$, we get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\Big(J_b(r)+b(r,\tilde{\mathcal{X}}_r^{u^\lambda},u_r^\lambda)-b(r,X_r^{\lambda,u^\lambda},u_r^\lambda)\Big)\mathrm{d}r\Big|^p\Big]\\
&\leqslant C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|J_b(r)|^p+|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^p\Big)\mathrm{d}r\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\Big(J_\sigma(r)+\sigma(r,\tilde{\mathcal{X}}_r^{u^\lambda},u_r^\lambda)-\sigma(r,X_r^{\lambda,u^\lambda},u_r^\lambda)\Big)\mathrm{d}B_r^\lambda\Big|^p\Big]\\
&\leqslant\begin{cases}
C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|J_\sigma(r)|^2+|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^2\Big)\mathrm{d}r\Big]\Big)^{\frac{p}{2}}, & 1\leqslant p<2,\\
C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|J_\sigma(r)|^p+|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^p\Big)\mathrm{d}r\Big], & p\geqslant 2,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\int_E\Big(J_\gamma(r,e)+\gamma(r,\tilde{\mathcal{X}}_{r-}^{u^\lambda},u_r^\lambda,e)-\gamma(r,X_{r-}^{\lambda,u^\lambda},u_r^\lambda,e)\Big)\tilde{N}^\lambda(\mathrm{d}r,\mathrm{d}e)\Big|^p\Big]\\
&\leqslant\begin{cases}
C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\int_E\Big(|J_\gamma(r,e)|^2+\ell^2(e)|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^2\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big]\Big)^{\frac{p}{2}}, & 1\leqslant p<2,\\
C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\int_E\Big(|J_\gamma(r,e)|^2+\ell^2(e)|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^2\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big)^{\frac{p}{2}} & \\
\qquad+\int_s^T\int_E\Big(|J_\gamma(r,e)|^p+\ell^p(e)|\tilde{\mathcal{X}}_r^{u^\lambda}-X_r^{\lambda,u^\lambda}|^p\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big], & p\geqslant 2,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}\Big|\int_s^\tau\int_E\Big(\lambda(1-\frac{1}{\dot{\tau}_1^\lambda})\gamma\big(\varrho_1^\lambda(r),\tilde{X}_r^{1,u^\lambda},u_r^\lambda,e\big)+(1-\lambda)(1-\frac{1}{\dot{\tau}_0^\lambda})\gamma\big(\varrho_0^\lambda(r),\tilde{X}_r^{0,u^\lambda},u_r^\lambda,e\big)\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big|^p\Big]\\
&\leqslant C_{T,p,\delta}\lambda^p(1-\lambda)^p\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big|\int_t^s\int_E\ell(e)\Big(|t_1-t_0|+|\tilde{X}_{r-}^{1,u^\lambda}-\tilde{X}_{r-}^{0,u^\lambda}|\Big)\nu(\mathrm{d}e)\mathrm{d}r\Big|^p\Big]\\
&\leqslant C_{T,p,\delta}\lambda^p(1-\lambda)^p\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_t^s\Big(|t_1-t_0|^p+|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\Big)\mathrm{d}r\Big].
\end{aligned}$$

Similar to the proof of Lemma 4.8, for $p \geqslant 1$, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{\tau\in[s,T]}|\tilde{\mathcal{X}}_\tau^{u^\lambda}-X_\tau^{\lambda,u^\lambda}|^p\Big]\\
&\leqslant C_p|\tilde{\mathcal{X}}_s^{u^\lambda}-X_s^{\lambda,u^\lambda}|^p+C_{T,p,\delta}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|J_b(r)|^p+|J_\sigma(r)|^p+\int_E|J_\gamma(r,e)|^p\nu(\mathrm{d}e)\Big)\mathrm{d}r\Big]\\
&\quad+C_{T,p,\delta}\Big(\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|J_\sigma(r)|^2+\int_E|J_\gamma(r,e)|^2\nu(\mathrm{d}e)\Big)\mathrm{d}r\Big]\Big)^{\frac{p}{2}}\\
&\quad+C_{T,p,\delta}\lambda^p(1-\lambda)^p\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_t^s\Big(|t_1-t_0|^p+|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|^p\Big)\mathrm{d}r\Big]\\
&\leqslant C_{T,p,\delta}\big(|t_1-t_0|^p+|\tilde{\mathcal{X}}_s^{u^\lambda}-X_s^{\lambda,u^\lambda}|^p\big)+C_{T,p,\delta}\lambda^p(1-\lambda)^p\Big(|t_1-t_0|^{2p}+|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^{2p}\Big),
\end{aligned}
$$

where we have used (4.22), (4.23) and Lemma 4.8. ▮

For convenience, we denote

$$
\Delta_s:=C_\delta\sup_{r\in[t_\lambda,s]}\Big(|t_1-t_0|+|\tilde{X}_r^{1,u^\lambda}-\tilde{X}_r^{0,u^\lambda}|\Big). \tag{4.24}
$$

Without loss of generality, we also assume $t_0\leqslant t_1$. Otherwise, the following study is made for $\tilde{Y}_\cdot^{n,1,u^\lambda}$.

Lemma 4.10. *Suppose* $\mathbf{(H_1)}$, $\mathbf{(H_2)}$, $\mathbf{(H_4)}$-(i), (ii) *and* $\mathbf{(C)}$ *hold, for any* $s\in[t_\lambda,T]$, *we have*

$$
\tilde{Y}_s^{n,0,u^\lambda}\leqslant\overline{Y}_s^{n,0,u^\lambda},\quad \mathbb{P}\text{-a.s.},
$$

where

$$
\begin{cases}
\mathrm{d}\overline{Y}_s^{n,0,u^\lambda}=-\Big[\dfrac{1}{\dot{\tau}_1^\lambda}f\Big(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda},\overline{Y}_s^{n,0,u^\lambda}-\Delta_s,\sqrt{\dot{\tau}_1^\lambda}\,\overline{Z}_s^{n,0,u^\lambda},\displaystyle\int_E l(e)\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e),u_s^\lambda\Big)\\
\qquad\qquad-\dfrac{n}{\dot{\tau}_1^\lambda}\Big(h(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda})+\Delta_s-\overline{Y}_s^{n,0,u^\lambda}\Big)^- -(1-\dfrac{1}{\dot{\tau}_1^\lambda})\displaystyle\int_E\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\\
\qquad\qquad+C_\delta|t_1-t_0|\cdot\Big(\Big|\overline{Z}_s^{n,0,u^\lambda}\Big|+\Big|\displaystyle\int_E\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big)+\Delta_s\Big]\mathrm{d}s\\
\qquad\quad+\overline{Z}_s^{n,0,u^\lambda}\mathrm{d}B_s^\lambda+\displaystyle\int_E\overline{V}_s^{n,0,u^\lambda}(e)\tilde{N}^\lambda(\mathrm{d}s,\mathrm{d}e),\quad s\in[t_\lambda,T],\\
\overline{Y}_T^{n,0,u^\lambda}=\Phi(\tilde{X}_T^{1,u^\lambda})+\Delta_T.
\end{cases} \tag{4.25}
$$

Proof. For all $s\in[t_\lambda,T]$, $(y,z,v)\in\mathbb{R}\times\mathbb{R}^d\times\mathcal{L}_\nu^2(E;\mathbb{R})$, we have

$$
\begin{aligned}
&\frac{1}{\dot{\tau}_0^\lambda}f\Big(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda},y,\sqrt{\dot{\tau}_0^\lambda}z,\int_E l(e)v(e)\nu(\mathrm{d}e),u_s^\lambda\Big)-(1-\frac{1}{\dot{\tau}_0^\lambda})\int_E v(e)\nu(\mathrm{d}e)\\
&\leqslant\frac{1}{\dot{\tau}_1^\lambda}f\Big(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda},y-\Delta_s,\sqrt{\dot{\tau}_1^\lambda}z,\int_E l(e)v(e)\nu(\mathrm{d}e),u_s^\lambda\Big)-(1-\frac{1}{\dot{\tau}_1^\lambda})\int_E v(e)\nu(\mathrm{d}e)\\
&\quad+C_\delta|t_1-t_0|\cdot\Big(|z|+\Big|\int_E v(e)\nu(\mathrm{d}e)\Big|\Big)+\Delta_s,
\end{aligned}
$$

where we have used the Lipschitz continuity of f and Lemma 4.1. Furthermore,

$$\begin{aligned}&\frac{1}{\dot{\tau}_0^\lambda}\big(h(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda})-y\big)-\frac{1}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda})-y\big)\\&=\Big(\frac{1}{\dot{\tau}_0^\lambda}-\frac{1}{\dot{\tau}_1^\lambda}\Big)\big(h(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda})-y\big)+\frac{1}{\dot{\tau}_1^\lambda}\big(h(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda})-h(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda})\big)\leqslant\Delta_s(1+|y|).\end{aligned}$$

Note the boundedness of $\tilde{Y}_s^{n,0,u^\lambda}$ obtained in Lemma 4.6, we know

$$\frac{n}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda})+\Delta_s-\tilde{Y}_s^{n,0,u^\lambda}\big)^-\leqslant\frac{n}{\dot{\tau}_0^\lambda}\big(h(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda})-\tilde{Y}_s^{n,0,u^\lambda}\big)^-.$$

Moreover, $\Phi(\tilde{X}_T^{0,u^\lambda})\leqslant\Phi(\tilde{X}_T^{1,u^\lambda})+C|\tilde{X}_T^{1,u^\lambda}-\tilde{X}_T^{0,u^\lambda}|\leqslant\Phi(\tilde{X}_T^{1,u^\lambda})+\Delta_T$.

Finally, $t_0\leqslant t_1$ guarantees us to apply the comparison theorem of BSDE with jumps to (4.25) and (4.11) with $i=0$, we get

$$\tilde{Y}_s^{n,0,u^\lambda}\leqslant\overline{Y}_s^{n,0,u^\lambda},\quad\forall s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.}$$

∎

Remark 4.11. *The process* $\{\Delta_s\}_{s\in[t_0,T]}$ *defined in* (4.24) *is an* $\mathbb{F}^\lambda$*-adapted, continuous increasing process, and from Lemma* 4.8*, for all* $p\geqslant1$*,*

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^p-\Delta_s^p\Big]\leqslant C_{p,\delta}\Big(|t_0-t_1|^p+|\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}|^p\Big),\quad\mathbb{P}\text{-a.s.}\tag{4.26}$$

Lemma 4.12. *Let* $\mathbf{(H_1)}$*–*$\mathbf{(H_3)}$, $\mathbf{(H_4)}$-(i), (ii) *and* **(C)** *hold. Then*

$$-C\leqslant\tilde{Y}_s^{n,0,u^\lambda}\leqslant\overline{Y}_s^{n,0,u^\lambda}\leqslant C_\delta+C_\delta\Delta_s,\quad s\in[t_\lambda,T],\ n\geqslant1,\ \mathbb{P}\text{-a.s.}$$

Moreover,

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|\overline{Z}_r^{n,0,u^\lambda}|^2+\|\overline{V}_r^{n,0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\leqslant C_\delta(1+\Delta_s^2),\quad s\in[t_\lambda,T],\ n\geqslant1,\ \mathbb{P}\text{-a.s.}$$

Proof. By Lemmas 4.6 and 4.10, for all $s\in[t_\lambda,T]$, $n\geqslant1$, $\mathbb{P}$-a.s., we have

$$-C\leqslant\tilde{Y}_s^{n,0,u^\lambda}\leqslant\overline{Y}_s^{n,0,u^\lambda}.$$

Now, let us consider another BSDE with jumps as follows,

$$\begin{cases}\mathrm{d}\mathscr{Y}_s^{0,u^\lambda}=-\Big[C_1+C_\delta|t_1-t_0|\cdot\Big(\Big|\mathscr{Z}_s^{0,u^\lambda}\Big|+\Big|\displaystyle\int_E\mathscr{V}_s^{0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big)+\Delta_s\Big]\mathrm{d}s+\mathscr{Z}_s^{0,u^\lambda}\mathrm{d}B_s^\lambda\\\qquad\qquad+\displaystyle\int_E\mathscr{V}_s^{0,u^\lambda}(e)\tilde{N}^\lambda(\mathrm{d}s,\mathrm{d}e),\quad s\in[t_\lambda,T],\\\mathscr{Y}_T^{0,u^\lambda}=\Phi(\tilde{X}_T^{1,u^\lambda})+C_1\Delta_T.\end{cases}$$

From the boundedness of f as well as Lemma 4.1, we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}&\frac{1}{\dot{\tau}_1^\lambda} f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \overline{Y}_s^{n,0,u^\lambda} - \Delta_s, \sqrt{\dot{\tau}_1^\lambda}\,\overline{Z}_s^{n,0,u^\lambda}, \int_E l(e)\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big) - (1-\frac{1}{\dot{\tau}_1^\lambda})\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\\&-\frac{n}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) + \Delta_s - \overline{Y}_s^{n,0,u^\lambda}\big)^- + C_\delta|t_1-t_0|\cdot\Big(\Big|\overline{Z}_s^{n,0,u^\lambda}\Big| + \Big|\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big) + \Delta_s\\&\leqslant C_1 + C_\delta|t_1-t_0|\cdot\Big(\Big|\overline{Z}_s^{n,0,u^\lambda}\Big| + \Big|\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big) + \Delta_s.\end{aligned}$$

Therefore, using the comparison theorem, we get

$$\overline{Y}_s^{n,0,u^\lambda} \leqslant \mathscr{Y}_s^{0,u^\lambda}, \quad s\in[t_\lambda, T],\ n\geqslant 1,\ \mathbb{P}\text{-a.s.}$$

Next, we want to prove $\mathscr{Y}_s^{0,u^\lambda} \leqslant C_\delta + C_\delta\Delta_s,\ s\in[t_\lambda,T],\ n\geqslant 1,\ \mathbb{P}$-a.s. For some constant β, by applying Itô's formula to $\mathrm{e}^{\beta s}|\mathscr{Y}_s^{0,u^\lambda}|^2$, we obtain, for all $s\in[t_\lambda,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}&\mathrm{e}^{\beta s}|\mathscr{Y}_s^{0,u^\lambda}|^2 + \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\beta r}\Big(\beta|\mathscr{Y}_r^{0,u^\lambda}|^2 + |\mathscr{Z}_r^{0,u^\lambda}|^2 + \|\mathscr{V}_r^{0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\\&= \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\mathrm{e}^{\beta T}|\Phi(\tilde{X}_T^{1,u^\lambda}) + C_1\Delta_T|^2\Big] + 2C_1\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\beta r}\mathscr{Y}_r^{0,u^\lambda}\mathrm{d}r\Big] + 2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\beta r}\mathscr{Y}_r^{0,u^\lambda}\Delta_r\mathrm{d}r\Big]\\&\quad +2C_\delta|t_1-t_0|\cdot\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\beta r}\mathscr{Y}_r^{0,u^\lambda}\Big(|\mathscr{Z}_s^{0,u^\lambda}| + \Big|\int_E \mathscr{V}_s^{0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big)\mathrm{d}r\Big].\\&\leqslant C_\delta\big(1+\Delta_s^2\big) + C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\beta r}|\mathscr{Y}_r^{0,u^\lambda}|^2\mathrm{d}r\Big] + \frac{1}{2}\mathbb{E}\Big[\int_s^T \mathrm{e}^{\beta r}\Big(|\mathscr{Z}_r^{0,u^\lambda}|^2 + \|\mathscr{V}_r^{0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big].\end{aligned}$$

By choosing a sufficiently large β, we obtain

$$|\mathscr{Y}_s^{0,u^\lambda}|^2 + \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \Big(|\mathscr{Y}_r^{0,u^\lambda}|^2 + |\mathscr{Z}_r^{0,u^\lambda}|^2 + \|\mathscr{V}_r^{0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big] \leqslant C_\delta(1+\Delta_s^2), \quad \mathbb{P}\text{-a.s.}$$

In particular, $|\mathscr{Y}_s^{0,u^\lambda}| \leqslant C_\delta(1+\Delta_s),\ s\in[t_\lambda,T],\ \mathbb{P}$-a.s. Therefore,

$$-C \leqslant \overline{Y}_s^{n,0,u^\lambda} \leqslant \mathscr{Y}_s^{0,u^\lambda} \leqslant C_\delta(1+\Delta_s), \quad \mathbb{P}\text{-a.s.}$$

Now, we proceed to establish the second result. Let C_h be the bound of h, applying Itô's formula to $(\overline{Y}^{n,0,u^\lambda}_s+C_h-\Delta_s)^2$, we obtain

$$\begin{aligned}
&\big(\overline{Y}^{n,0,u^\lambda}_s+C_h-\Delta_s\big)^2+\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T\Big(|\overline{Z}^{n,0,u^\lambda}_r|^2+\|\overline{V}^{n,0,u^\lambda}_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r\Big]\\
&=\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[(\Phi(\tilde{X}^{1,u^\lambda}_T)+C_h)^2\Big]-\frac{2n}{\dot{\tau}^\lambda_1}\mathbb{E}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\big(h(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_r)+\Delta_r-\overline{Y}^{n,0,u^\lambda}_r\big)^-\mathrm{d}r\Big]\\
&\quad+\frac{2}{\dot{\tau}^\lambda_1}\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\ f\Big(\varrho^\lambda_1(r),\tilde{X}^{1,u^\lambda}_r,\overline{Y}^{n,0,u^\lambda}_r-\Delta_r,\sqrt{\dot{\tau}^\lambda_1}\overline{Z}^{n,0,u^\lambda}_r,\int_E l(e)\overline{V}^{n,0,u^\lambda}_r(e)\nu(\mathrm{d}e),u^\lambda_r\Big)\mathrm{d}r\Big]\\
&\quad+2C_\delta|t_1-t_0|\cdot\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\Big(|\overline{Z}^{n,0,u^\lambda}_r|+\Big|\int_E\overline{V}^{n,0,u^\lambda}_r(e)\nu(\mathrm{d}e)\Big|\Big)\mathrm{d}r\Big]\\
&\quad+2\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\Delta_r\mathrm{d}r\Big]+2\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\mathrm{d}\Delta_r\Big]\\
&\quad+2(1-\frac{1}{\dot{\tau}^\lambda_1})\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T\int_E(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\overline{V}^{n,0,u^\lambda}_r(e)\nu(\mathrm{d}e)\mathrm{d}r\Big].
\end{aligned}$$

Further, according to the results of (i), (4.26), Cauchy inequality and the fact

$$(\overline{Y}^{n,0,u^\lambda}_s+C_h-\Delta_s)\big(h(\varrho^\lambda_1(s),\tilde{X}^{1,u^\lambda}_s)+\Delta_s-\overline{Y}^{n,0,u^\lambda}_s\big)^-\geqslant 0,\quad n\geqslant 1,\ s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.},$$

we have, for all $s\in[t_\lambda,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\big(\overline{Y}^{n,0,u^\lambda}_s+C_h-\Delta_s\big)^2+\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T\Big(|\overline{Z}^{n,0,u^\lambda}_r|^2+\|\overline{V}^{n,0,u^\lambda}_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r\Big]\\
&\leqslant C_\delta(1+\Delta_s^2)+2\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\mathrm{d}\Delta_r\Big]+\frac{1}{2}\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T\Big(|\overline{Z}^{n,0,u^\lambda}_r|^2+\|\overline{V}^{n,0,u^\lambda}_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big(\overline{Y}^{n,0,u^\lambda}_s+C_h-\Delta_s\big)^2+\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T\Big(|\overline{Z}^{n,0,u^\lambda}_r|^2+\|\overline{V}^{n,0,u^\lambda}_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r\Big]\\
&\leqslant C_\delta(1+\Delta_s^2)+2\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\mathrm{d}\Delta_r\Big],\quad s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.}
\end{aligned}\tag{4.27}$$

Notice that, for any $1<p<2$ and $q>2$ satisfying $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\int_s^T(\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r)\mathrm{d}\Delta_r\Big]\leqslant\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\sup_{r\in[s,T]}|\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r|(\Delta_T-\Delta_s)\Big]\\
&\leqslant\Big(\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\sup_{r\in[s,T]}|\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r|^p\Big]\Big)^{\frac{1}{p}}\Big(\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\Delta_T^q\Big]\Big)^{\frac{1}{q}}\\
&\leqslant\varepsilon\Big(\mathbb{E}^{\mathcal{F}^\lambda_s}\Big[\sup_{r\in[s,T]}|\overline{Y}^{n,0,u^\lambda}_r+C_h-\Delta_r|^p\Big]\Big)^{\frac{2}{p}}+C_\varepsilon\Delta_s^2\\
&\leqslant\varepsilon M_{s,t}^{\frac{2}{p}}+C_\varepsilon\Delta_s^2,
\end{aligned}\tag{4.28}$$

where ε will be given in detail later, and $M_{s,t} := \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup\limits_{r\in[t,T]} |\overline{Y}_r^{n,0,u^\lambda} + C_h - \Delta_r|^p\Big]$, $t_\lambda \leqslant t \leqslant s \leqslant T$. We remark that $(M_{s,t})_{s\geqslant t}$ is an $\mathbb{F}^\lambda$-martingale, and $\dfrac{2}{p} > 1$. So from Doob's martingale inequality,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\sup_{s\in[t,T]} M_{s,t}^{\frac{2}{p}}\Big] \leqslant \Big(\frac{2}{2-p}\Big)^{\frac{2}{p}}\Big(\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[M_{T,t}^{\frac{2}{2-p}}\Big]\Big)^{\frac{2-p}{p}} \leqslant \Big(\frac{2}{2-p}\Big)^{\frac{2}{p}}\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[M_{T,t}^{\frac{2}{p}}\Big] \\
&\leqslant \Big(\frac{2}{2-p}\Big)^{\frac{2}{p}}\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\sup_{s\in[t,T]} |\overline{Y}_s^{n,0,u^\lambda} + C_h - \Delta_s|^2\Big].
\end{aligned} \tag{4.29}$$

Now, we go back to the inequality (4.27). Combined with (4.28) and (4.29), we obtain

$$\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\sup_{s\in[t,T]}\big(\overline{Y}_s^{n,0,u^\lambda} + C_h - \Delta_s\big)^2\Big] \leqslant C_\delta(1+\Delta_t^2) + 2\varepsilon\Big(\frac{2}{2-p}\Big)^{\frac{2}{p}}\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\sup_{s\in[t,T]}|\overline{Y}_s^{n,0,u^\lambda} + C_h - \Delta_s|^2\Big].$$

By choosing a suitable $\varepsilon > 0$ such that $2\varepsilon\Big(\dfrac{2}{2-p}\Big)^{\frac{2}{p}} < 1$, we get

$$\mathbb{E}^{\mathcal{F}_t^\lambda}\Big[\sup_{s\in[t,T]}\big(\overline{Y}_s^{n,0,u^\lambda} + C_h - \Delta_s\big)^2\Big] \leqslant C_\delta(1+\Delta_t^2).$$

Thereby,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \Big(|\overline{Z}_r^{n,0,u^\lambda}|^2 + \|\overline{V}_r^{n,0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big] \\
&\leqslant C_\delta(1+\Delta_s^2) + 2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{r\in[s,T]}|\overline{Y}_r^{n,0,u^\lambda} + C_h - \Delta_r|^2\Big] \leqslant C_\delta(1+\Delta_s^2), \quad s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.}
\end{aligned}$$

∎

For all $s\in[t_\lambda,T]$ and $n\geqslant 1$, we define $\mathcal{Y}_s^{n,0,u^\lambda} := \overline{Y}_s^{n,0,u^\lambda} - \Delta_s$. Then, from (4.25), we get

$$\begin{cases}
\mathrm{d}\mathcal{Y}_s^{n,0,u^\lambda} = -\Big[\dfrac{1}{\dot{\tau}_1^\lambda} f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \mathcal{Y}_s^{n,0,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\,\overline{Z}_s^{n,0,u^\lambda}, \displaystyle\int_E l(e)\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big) \\
\qquad -\dfrac{n}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - \mathcal{Y}_s^{n,0,u^\lambda}\big)^- - \big(1-\dfrac{1}{\dot{\tau}_1^\lambda}\big)\displaystyle\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e) \\
\qquad + C_\delta|t_1 - t_0|\cdot\Big(\Big|\overline{Z}_s^{n,0,u^\lambda}\Big| + \Big|\displaystyle\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big) + \Delta_s\Big]\mathrm{d}s - \mathrm{d}\Delta_s \\
\qquad + \overline{Z}_s^{n,0,u^\lambda}\mathrm{d}B_s^\lambda + \displaystyle\int_E \overline{V}_s^{n,0,u^\lambda}(e)\tilde{N}^\lambda(\mathrm{d}s,\mathrm{d}e), \quad s\in[t_\lambda,T], \\
\mathcal{Y}_T^{n,0,u^\lambda} = \Phi(\tilde{X}_T^{1,u^\lambda}).
\end{cases} \tag{4.30}$$

Notice that the penalization term of the above BSDE (4.30) is the same as BSDE (4.11) with $i = 1$, and they are driven by the same Brownian motion and Poisson random measure in $[t_\lambda, T]$. Thereafter, according to the standard estimates and Doob's martingale inequality, we give an estimate between them as follows.

Lemma 4.13. *Assume that* $\mathbf{(H_1)}$–$\mathbf{(H_3)}$, $\mathbf{(H_4)}$-(i), (ii) *hold. Then, there exists some constant* C_δ, *such that, for all* $n \geqslant 1$ *and* $s \in [t_\lambda, T]$, $\mathbb{P}$-*a.s.*,

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{r\in[s,T]} |\mathcal{Y}_r^{n,0,u^\lambda} - \tilde{Y}_r^{n,1,u^\lambda}|^2 + \int_s^T \Big(|\overline{Z}_r^{n,0,u^\lambda} - \tilde{Z}_r^{n,1,u^\lambda}|^2 + \|\overline{V}_r^{n,0,u^\lambda}(\cdot) - \tilde{V}_r^{n,1,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big] \leqslant C_\delta \Delta_s^2.$$

Proof. Setting $\big(\Delta Y_\cdot, \Delta Z_\cdot, \Delta V_\cdot(\cdot)\big) := \big(\mathcal{Y}_\cdot^{n,0,u^\lambda} - \tilde{Y}_\cdot^{n,1,u^\lambda}, \overline{Z}_\cdot^{n,0,u^\lambda} - \tilde{Z}_\cdot^{n,1,u^\lambda}, \overline{V}_\cdot^{n,0,u^\lambda}(\cdot) - \tilde{V}_\cdot^{n,1,u^\lambda}(\cdot)\big)$, by (4.11) and (4.30), we have

$$\begin{cases} \mathrm{d}\Delta Y_s = -\Big[\frac{1}{\dot{\tau}_1^\lambda} f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \mathcal{Y}_s^{n,0,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\,\overline{Z}_s^{n,0,u^\lambda}, \int_E l(e)\overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big) \\ \qquad -\frac{1}{\dot{\tau}_1^\lambda} f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \tilde{Y}_s^{n,1,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\,\tilde{Z}_s^{n,1,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,1,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big) \\ \qquad +\frac{n}{\dot{\tau}_i^\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - \tilde{Y}_s^{n,1,u^\lambda}\big)^- - \frac{n}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - \mathcal{Y}_s^{n,0,u^\lambda}\big)^- \\ \qquad +C_\delta|t_1 - t_0|\cdot\Big(\Big|\overline{Z}_s^{n,0,u^\lambda}\Big| + \Big|\int_E \overline{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big) + \Delta_s - (1 - \frac{1}{\dot{\tau}_1^\lambda})\int_E \Delta V_s(e)\nu(\mathrm{d}e)\Big]\mathrm{d}s \\ \qquad -\mathrm{d}\Delta_s + \Delta Z_s \mathrm{d}B_s^\lambda + \int_E \Delta V_s(e)\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_\lambda, T], \\ \Delta Y_T = 0. \end{cases} \tag{4.31}$$

For some constant $\alpha > 0$, applying Itô's formula to $\mathrm{e}^{\alpha s}|\Delta Y_s|^2$, we get

$$\begin{aligned} &\mathrm{e}^{\alpha s}|\Delta Y_s|^2 + \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \Big(\alpha \mathrm{e}^{\alpha r}|\Delta Y_r|^2 + \mathrm{e}^{\alpha r}|\Delta Z_r|^2\Big)\mathrm{d}r\Big] + \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \int_E \mathrm{e}^{\alpha r}|\Delta V_r(e)|^2 N^\lambda(\mathrm{d}r, \mathrm{d}e)\Big] \\ &= \frac{2}{\dot{\tau}_1^\lambda}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r\Big(f\big(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}, \mathcal{Y}_r^{n,0,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\,\overline{Z}_r^{n,0,u^\lambda}, \int_E l(e)\overline{V}_r^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_r^\lambda\big) \\ &\qquad\qquad -f\big(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}, \tilde{Y}_r^{n,1,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\,\tilde{Z}_r^{n,1,u^\lambda}, \int_E l(e)\tilde{V}_r^{n,1,u^\lambda}(e)\nu(\mathrm{d}e), u_r^\lambda\big)\Big)\mathrm{d}r\Big] \\ &\quad +\frac{2n}{\dot{\tau}_1^\lambda}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r\Big(\big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - \tilde{Y}_r^{n,1,u^\lambda}\big)^- - \big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - \mathcal{Y}_r^{n,0,u^\lambda}\big)^-\Big)\mathrm{d}r\Big] \\ &\quad +2C_\delta|t_1 - t_0|\cdot\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r\Big(\Big|\overline{Z}_r^{n,0,u^\lambda}\Big| + \Big|\int_E \overline{V}_r^{n,0,u^\lambda}(e)\nu(\mathrm{d}e)\Big|\Big)\mathrm{d}r\Big] + 2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r \Delta_r \mathrm{d}r\Big] \\ &\quad -2(1 - \frac{1}{\dot{\tau}_1^\lambda})\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \int_E \mathrm{e}^{\alpha r}\Delta Y_r \Delta V_r(e)\nu(\mathrm{d}e)\mathrm{d}r\Big] + 2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r \mathrm{d}\Delta_r\Big] \\ &\qquad \leqslant \big(11C_\delta^2 + 2\big)\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}|\Delta Y_r|^2\mathrm{d}r\Big] + \frac{1}{2}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Big(|\Delta Z_r|^2 + \|\Delta V_r(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big] \\ &\qquad\quad +C_\delta|t_1 - t_0|^2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \Big(|\overline{Z}_r^{n,0,u^\lambda}|^2 + \|\,\overline{V}_r^{n,0,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big] + 2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r \mathrm{d}\Delta_r\Big], \end{aligned}$$

where we have used $\Delta Y_r\Big(\big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - \tilde{Y}_r^{n,1,u^\lambda}\big)^- - \big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - \mathcal{Y}_r^{n,0,u^\lambda}\big)^-\Big) \leqslant 0,\ r \in [t_\lambda, T],\ \mathbb{P}$-a.s.

For α large enough, using Lemma 4.12 and Lemma 3.1 in [23], we get

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\Delta Y_s|^2+\int_s^T\Big(|\Delta Y_r|^2+|\Delta Z_s|^2+\|\Delta V_r(\cdot)\|^2_{\nu,2}\Big)\mathrm{d}r\ \Big]\leqslant C_\delta\Delta_s^2+2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r\mathrm{d}\Delta_r\Big].$$

The term $\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \mathrm{e}^{\alpha r}\Delta Y_r\mathrm{d}\Delta_r\Big]$ can be dealt with by employing the technique in the proof of Lemma 4.12-(ii) (refer to (4.28), (4.29)). Then the desired result can be proved. ∎

Based on the above preparations, we have the following estimates.

Lemma 4.14. *Assume the conditions of Lemma 4.12 hold, then*

$$|\tilde{Y}_s^{n,1,u^\lambda}-\tilde{Y}_s^{n,0,u^\lambda}|\leqslant C_\delta\Delta_s,\quad \mathbb{P}\text{-a.s.}$$

Proof. For all $n\geqslant 1$ and $s\in[t_\lambda,T]$, based on Lemma 4.12 and Lemma 4.13, we have

$$\tilde{Y}_s^{n,0,u^\lambda}-\tilde{Y}_s^{n,1,u^\lambda}\leqslant \overline{Y}_s^{n,0,u^\lambda}-\tilde{Y}_s^{n,1,u^\lambda}=\mathcal{Y}_s^{n,0,u^\lambda}+\Delta_s-\tilde{Y}_s^{n,1,u^\lambda}\leqslant C_\delta\Delta_s,\quad n\geqslant 1,\ s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.}$$

Moreover, due to the symmetry, by the same argument, we also get

$$\tilde{Y}_s^{n,0,u^\lambda}-\tilde{Y}_s^{n,1,u^\lambda}\geqslant -C_\delta\Delta_s,\quad n\geqslant 1,\ s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.}$$

Thus, $|\tilde{Y}_s^{n,1,u^\lambda}-\tilde{Y}_s^{n,0,u^\lambda}|\leqslant C_\delta\Delta_s$, for all $n\geqslant 1,\ s\in[t_\lambda,T],\ \mathbb{P}$-a.s. ∎

Lemma 4.15. *Assume* $\mathbf{(H_1)}$–$\mathbf{(H_3)}$, $\mathbf{(H_4)}$-(i), (ii) *and* $\mathbf{(C)}$.
(i) Setting

$$\mathscr{D}_s^n:=\int_s^T\big(h(\varrho_0^\lambda(r),\tilde{X}_r^{0,u^\lambda})-\tilde{Y}_r^{n,0,u^\lambda}\big)^-\mathrm{d}\tilde{A}_r^{n,1,u^\lambda}+\int_s^T\big(h(\varrho_1^\lambda(r),\tilde{X}_r^{1,u^\lambda})-\tilde{Y}_r^{n,1,u^\lambda}\big)^-\mathrm{d}\tilde{A}_r^{n,0,u^\lambda},\tag{4.32}$$

for any $p\geqslant 2$, we have, $\lim\limits_{n\to\infty}\mathbb{E}^{\mathcal{F}_s^\lambda}[(\mathscr{D}_s^n)^{\frac{p}{2}}]=0$, $s\in[t_\lambda,T]$, $\mathbb{P}$-a.s.
(ii) For any $p\geqslant 2$ and $s\in[t_\lambda,T]$, $\mathbb{P}$-a.s.,

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\big(|\tilde{Z}_r^{n,1,u^\lambda}-\tilde{Z}_r^{n,0,u^\lambda}|^2+\|\tilde{V}_r^{n,1,u^\lambda}(\cdot)-\tilde{V}_r^{n,0,u^\lambda}(\cdot)\|^2_{\nu,2}\big)\mathrm{d}r\Big)^{\frac{p}{2}}+|\widehat{A}_T-\widehat{A}_s|^p\Big]\leqslant C_\delta\Delta_s^{\frac{p}{2}}+C\mathbb{E}^{\mathcal{F}_s^\lambda}[(\mathscr{D}_s^n)^{\frac{p}{2}}].$$

Proof. The first result can be directly proved by using Lemma 4.7-(ii). Now we focus on the second one. For all $n\geqslant 1$, $s\in[t_\lambda,T]$, setting $\widehat{X}_s:=\tilde{X}_s^{1,u^\lambda}-\tilde{X}_s^{0,u^\lambda}$ and $\widehat{\varphi}_s:=\tilde{\varphi}_s^{n,1,u^\lambda}-\tilde{\varphi}_s^{n,0,u^\lambda}$ with $\varphi=Y,Z,V,A$. From equation

(4.11), we have

$$\begin{cases}\mathrm{d}\widehat{Y}_s=-\Big\{\dfrac{1}{\dot{\tau}_1^\lambda}f\Big(\varrho_1^\lambda(s),\tilde{X}_s^{1,u^\lambda},\tilde{Y}_s^{n,1,u^\lambda},\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda},\displaystyle\int_E l(e)\tilde{V}_s^{n,1,u^\lambda}(e)\nu(\mathrm{d}e),u_s^\lambda\Big)\\ \qquad\qquad-\dfrac{1}{\dot{\tau}_0^\lambda}f\Big(\varrho_0^\lambda(s),\tilde{X}_s^{0,u^\lambda},\tilde{Y}_s^{n,0,u^\lambda},\sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda},\displaystyle\int_E l(e)\tilde{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e),u_s^\lambda\Big)\\ \qquad\qquad-\displaystyle\int_E\Big(\widehat{V}_s(e)-\frac{1}{\dot{\tau}_1^\lambda}\tilde{V}_s^{n,1,u^\lambda}(e)+\frac{1}{\dot{\tau}_0^\lambda}\tilde{V}_s^{n,0,u^\lambda}(e)\Big)\nu(\mathrm{d}e)\Big\}\mathrm{d}s+\mathrm{d}\widehat{A}_s\\ \qquad\quad+\widehat{Z}_s\mathrm{d}B_s^\lambda+\displaystyle\int_E\widehat{V}_s(e)\tilde{N}^\lambda(\mathrm{d}s,\mathrm{d}e),\quad s\in[t_\lambda,T],\\ \widehat{Y}_T=\Phi(\tilde{X}_T^{1,u^\lambda})-\Phi(\tilde{X}_T^{0,u^\lambda}).\end{cases}\tag{4.33}$$

Step 1. We prove the stated result holds with $p=2$. For this, applying Itô's formula to $|\widehat{Y}_s|^2$, we have

$$\begin{aligned}&|\widehat{Y}_s|^2+\int_s^T|\widehat{Z}_r|^2\mathrm{d}r+\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\\&=|\Phi(\tilde{X}_T^{1,u^\lambda})-\Phi(\tilde{X}_T^{0,u^\lambda})|^2+2\int_s^T\widehat{Y}_r\big(I_r^1-I_r^2\big)\mathrm{d}r-2\int_s^T\widehat{Y}_r\mathrm{d}\widehat{A}_r-\int_s^T2\widehat{Y}_r\widehat{Z}_r\mathrm{d}B_r^\lambda\\&\quad-\int_s^T\int_E2\widehat{Y}_r\widehat{V}_r(e)\tilde{N}^\lambda(\mathrm{d}r,\mathrm{d}e),\end{aligned}\tag{4.34}$$

where

$$\begin{aligned}I_r^1&:=\frac{1}{\dot{\tau}_1^\lambda}f\Big(\varrho_1^\lambda(r),\tilde{X}_r^{1,u^\lambda},\tilde{Y}_r^{n,1,u^\lambda},\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_r^{n,1,u^\lambda},\int_E l(e)\tilde{V}_r^{n,1,u^\lambda}(e)\nu(\mathrm{d}e),u_r^\lambda\Big)\\&\quad-\frac{1}{\dot{\tau}_0^\lambda}f\Big(\varrho_0^\lambda(r),\tilde{X}_r^{0,u^\lambda},\tilde{Y}_r^{n,0,u^\lambda},\sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_r^{n,0,u^\lambda},\int_E l(e)\tilde{V}_r^{n,0,u^\lambda}(e)\nu(\mathrm{d}e),u_r^\lambda\Big),\\I_r^2&:=\int_E\Big(\widehat{V}_r(e)-\frac{1}{\dot{\tau}_1^\lambda}\tilde{V}_r^{n,1,u^\lambda}(e)+\frac{1}{\dot{\tau}_0^\lambda}\tilde{V}_r^{n,0,u^\lambda}(e)\Big)\nu(\mathrm{d}e).\end{aligned}$$

Then, according to Lemma 4.14, we have

$$\begin{aligned}|I_r^1|&\leqslant|t_1-t_0|+|\varrho_1^\lambda(r)-\varrho_0^\lambda(r)|+|\widehat{X}_r|+|\widehat{Y}_r|+\Big|\sqrt{\dot{\tau}_1^\lambda}-\sqrt{\dot{\tau}_0^\lambda}\Big|\cdot|\tilde{Z}_r^{n,1,u^\lambda}|+\Big|\sqrt{\dot{\tau}_0^\lambda}\Big|\cdot|\widehat{Z}_r|\\&\qquad+\|l(\cdot)\|_{\nu,2}\cdot\|\widehat{V}_r(\cdot)\|_{\nu,2}\\&\leqslant C\Big(\Delta_r+|t_1-t_0|\cdot|\tilde{Z}_r^{n,1,u^\lambda}|+|\widehat{Z}_r|+\|\widehat{V}_r(\cdot)\|_{\nu,2}\Big),\end{aligned}\tag{4.35}$$

and

$$|I_r^2|\leqslant C\int_E\Big(\Big|1-\frac{1}{\dot{\tau}_0^\lambda}\Big|\cdot|\widehat{V}_r(e)|+|t_1-t_0|\cdot|\tilde{V}_r^{n,1,u^\lambda}(e)|\Big)\nu(\mathrm{d}e).\tag{4.36}$$

Moreover,

$$\begin{aligned}
&-\int_s^T \widehat{Y}_r \mathrm{d}\widehat{A}_r = -\int_s^T (\tilde{Y}_r^{n,1,u^\lambda} - \tilde{Y}_r^{n,0,u^\lambda}) \mathrm{d}(\tilde{A}_r^{n,1,u^\lambda} - \tilde{A}_r^{n,0,u^\lambda}) \\
&\leqslant \int_s^T \big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - \tilde{Y}_r^{n,1,u^\lambda}\big)^- \mathrm{d}\tilde{A}_r^{n,0,u^\lambda} + \int_s^T (h(\varrho_0^\lambda(r), \tilde{X}_r^{0,u^\lambda}) - \tilde{Y}_r^{n,0,u^\lambda})^- \mathrm{d}\tilde{A}_r^{n,1,u^\lambda} \\
&\quad -\int_s^T \big(h(\varrho_1^\lambda(r), \tilde{X}_r^{1,u^\lambda}) - h(\varrho_0^\lambda(r), \tilde{X}_r^{0,u^\lambda})\big) \mathrm{d}(\tilde{A}_r^{n,1,u^\lambda} - \tilde{A}_r^{n,0,u^\lambda}) \\
&\leqslant \mathscr{D}_s^n + C\Delta_T |\widehat{A}_T - \widehat{A}_s|.
\end{aligned}$$

Based on the above estimates, we return to (4.34) and obtain

$$\begin{aligned}
&|\widehat{Y}_s|^2 + \int_s^T |\widehat{Z}_r|^2 \mathrm{d}r + \int_s^T \int_E |\widehat{V}_r(e)|^2 N^\lambda(\mathrm{d}r, \mathrm{d}e) \\
&\leqslant C|\widehat{X}_T|^2 + C_\delta \int_s^T \Delta_r \Big(\Delta_r + |t_1 - t_0| \cdot |\tilde{Z}_r^{n,1,u^\lambda}| + |\widehat{Z}_r| + \|\widehat{V}_r(\cdot)\|_{\nu,2}\Big) \mathrm{d}r \\
&\quad + C_\delta \int_s^T \int_E \Delta_r \Big(|1 - \frac{1}{\dot{\tau}_0^\lambda}| \cdot |\widehat{V}_r(e)| + |t_1 - t_0| \cdot |\tilde{V}_r^{n,1,u^\lambda}(e)|\Big) \nu(\mathrm{d}e)\mathrm{d}r + C\Delta_T|\widehat{A}_T - \widehat{A}_s| + 2\mathscr{D}_s^n \\
&\quad - \int_s^T 2\widehat{Y}_r \widehat{Z}_r \mathrm{d}B_r^\lambda - \int_s^T \int_E 2\widehat{Y}_r \widehat{V}_r(e) \tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e) \\
&\leqslant C_{\delta,\varepsilon} \Delta_T^2 + C_\delta |t_1 - t_0|^2 \int_s^T \Big(|\tilde{Z}_r^{n,1,u^\lambda}|^2 + \|\tilde{V}_r^{n,1,u^\lambda}(\cdot)\|_{\nu,2}^2\Big) \mathrm{d}r + \varepsilon \int_s^T \Big(|\widehat{Z}_r|^2 + \|\widehat{V}_r(\cdot)\|_{\nu,2}^2\Big) \mathrm{d}r \\
&\quad + C\Delta_T |\widehat{A}_T - \widehat{A}_s| + 2\mathscr{D}_s^n - \int_s^T 2\widehat{Y}_r \widehat{Z}_r \mathrm{d}B_r^\lambda - \int_s^T \int_E 2\widehat{Y}_r \widehat{V}_r(e) \tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e).
\end{aligned} \tag{4.37}$$

Then, choosing $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
&|\widehat{Y}_s|^2 + \int_s^T |\widehat{Z}_r|^2 \mathrm{d}r + \int_s^T \|\widehat{V}_r(\cdot)\|_{\nu,2}^2 \mathrm{d}r \\
&\leqslant C_\delta \Delta_T^2 + C_\delta |t_1 - t_0|^2 \int_s^T \Big(|\tilde{Z}_r^{n,1,u^\lambda}|^2 + \|\tilde{V}_r^{n,1,u^\lambda}(\cdot)\|_{\nu,2}^2\Big) \mathrm{d}r + C\Delta_T |\widehat{A}_T - \widehat{A}_s| + C\mathscr{D}_s^n \\
&\quad - C\int_s^T \widehat{Y}_r \widehat{Z}_r \mathrm{d}B_r^\lambda - C\int_s^T \int_E \widehat{Y}_r \widehat{V}_r(e) \tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e) - C\int_s^T \int_E |\widehat{V}_r(e)|^2 \tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e).
\end{aligned}$$

Therefore, according to Lemma 4.6, for any $\varepsilon > 0$,

$$|\widehat{Y}_s|^2 + \mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T \Big(|\widehat{Z}_r|^2 + \|\widehat{V}_r(\cdot)\|_{\nu,2}^2\Big) \mathrm{d}r\Big] \leqslant C_{\delta,\varepsilon} \Delta_s^2 + C\mathbb{E}^{\mathcal{F}_s^\lambda}[\mathscr{D}_s^n] + \varepsilon \mathbb{E}^{\mathcal{F}_s^\lambda}[|\widehat{A}_T - \widehat{A}_s|^2], \quad \mathbb{P}\text{-a.s.} \tag{4.38}$$

Now we deal with $\mathbb{E}^{\mathcal{F}_s^\lambda}[|\widehat{A}_T-\widehat{A}_s|^2]$, from equation (4.33), (4.35) and (4.36),

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\big|\widehat{A}_T-\widehat{A}_s\big|^2\Big]\\
&\leqslant C|\widehat{Y}_s|^2+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\Phi(\tilde{X}_T^{1,u^\lambda})-\Phi(\tilde{X}_T^{0,u^\lambda})|^2\Big]+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T(|\widehat{Z}_r|^2+\|\widehat{V}_r(\cdot)\|_{\nu,2}^2)\mathrm{d}r\Big]\\
&\quad+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T(I_r^1-I_r^2)\mathrm{d}r\Big)^2\Big]\\
&\leqslant C\Delta_s^2+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|\widehat{Z}_r|^2+\|\widehat{V}_r(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big],\quad \mathbb{P}\text{-a.s.}
\end{aligned}\tag{4.39}$$

Combined with (4.38), choosing ε small enough,

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\big|\widehat{A}_T-\widehat{A}_s\big|^2\Big]\leqslant C_\delta\Delta_s^2,\quad \mathbb{P}\text{-a.s.}$$

Thus,

$$|\widehat{Y}_s|^2+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|\widehat{Z}_r|^2+\|\widehat{V}_r(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\leqslant C_\delta\Delta_s^2+C\mathbb{E}^{\mathcal{F}_s^\lambda}[\mathscr{D}_s^n],\quad \mathbb{P}\text{-a.s.}$$

Step 2. Now we focus on the case $p>2$. From (4.37), for $p>2$,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\widehat{Y}_s|^2+\Big(\int_s^T|\widehat{Z}_r|^2\mathrm{d}r\Big)^{\frac{p}{2}}+\Big(\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{2}}\Big]\\
&\leqslant\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(|\widehat{Y}_s|^2+\int_s^T|\widehat{Z}_r|^2\mathrm{d}r+\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{2}}\Big]\\
&\leqslant C_\delta\Delta_s^p+C_\delta|t_1-t_0|^p\cdot\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T(|\tilde{Z}_r^{n,1,u^\lambda}|^2+\|\tilde{V}_r^{n,1,u^\lambda}(\cdot)\|_{\nu,2}^2)\mathrm{d}r\Big)^{\frac{p}{2}}\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}|\widehat{A}_T-\widehat{A}_s|^{\frac{p}{2}}\Big]\\
&\quad+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T|\widehat{Y}_r|^2|\widehat{Z}_r|^2\mathrm{d}r\Big)^{\frac{p}{4}}\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\int_E|\widehat{Y}_r|^2|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{4}}\Big]\\
&\quad+C\epsilon^{\frac{p}{2}}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T(|\widehat{Z}_r|^2+\|\widehat{V}_r(\cdot)\|_{\nu,2}^2)\mathrm{d}r\Big)^{\frac{p}{2}}\Big]\\
&\leqslant C_\delta\Delta_s^p+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}|\widehat{A}_T-\widehat{A}_s|^{\frac{p}{2}}\Big]+C\epsilon^{\frac{p}{2}}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\Big(|\widehat{Z}_r|^2+\|\widehat{V}_r(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big)^{\frac{p}{2}}\Big]\\
&\quad+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}\Big(\int_s^T|\widehat{Z}_r|^2\mathrm{d}r\Big)^{\frac{p}{4}}\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}\Big(\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{4}}\Big]\\
&\leqslant C_\delta\Delta_s^p+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}|\widehat{A}_T-\widehat{A}_s|^{\frac{p}{2}}\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}]+C\epsilon^{\frac{p}{2}}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T|\widehat{Z}_r|^2\mathrm{d}r\Big)^{\frac{p}{2}}\Big]\\
&\quad+C\epsilon^{\frac{p}{2}}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{2}}\Big],
\end{aligned}$$

where we have used Lemma 3.1 in [23], Lemma 4.6, Lemma 4.14 and (4.26).

Thus, by choosing ϵ small enough, we get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\widehat{Y}_s|^2+\Big(\int_s^T|\widehat{Z}_r|^2\mathrm{d}r\Big)^{\frac{p}{2}}+\Big(\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}r,\mathrm{d}e)\Big)^{\frac{p}{2}}\Big]\\
&\leqslant C_\delta\Delta_s^p+C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Delta_T^{\frac{p}{2}}|\widehat{A}_T-\widehat{A}_s|^{\frac{p}{2}}\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}]\\
&\leqslant C_\delta\Delta_s^p+C\varepsilon\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[|\widehat{A}_T-\widehat{A}_s|^p\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}].
\end{aligned}$$

Further, similar to the proof of (4.39), we get

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\Big(\int_s^T|\widehat{Z}_r|^2\mathrm{d}s\Big)^{\frac{p}{2}}+\Big(\int_s^T\int_E|\widehat{V}_r(e)|^2N^\lambda(\mathrm{d}s,\mathrm{d}e)\Big)^{\frac{p}{2}}+\big|\widehat{A}_T-\widehat{A}_s\big|^p\Big]\leqslant C_\delta\Delta_s^p+C\mathbb{E}^{\mathcal{F}_s^\lambda}[|\mathscr{D}_s^n|^{\frac{p}{2}}].$$

Finally, combined with Lemma 3.1 in [23], the desired result is proved. ∎

Recalling the notation $\tilde{\mathcal{X}}_\cdot^{u^\lambda}=\lambda\tilde{X}_\cdot^{1,u^\lambda}+(1-\lambda)\tilde{X}_\cdot^{0,u^\lambda}$, we also put

$$\big(\tilde{\mathcal{Y}}_\cdot^{n,u^\lambda},\tilde{\mathcal{Z}}_\cdot^{n,u^\lambda},\tilde{\mathcal{V}}_\cdot^{n,u^\lambda}(\cdot)\big)=\big(\lambda\tilde{Y}_\cdot^{n,1,u^\lambda}+(1-\lambda)\tilde{Y}_\cdot^{n,0,u^\lambda},\lambda\tilde{Z}_\cdot^{n,1,u^\lambda}+(1-\lambda)\tilde{Z}_\cdot^{n,0,u^\lambda},\lambda\tilde{V}_\cdot^{n,1,u^\lambda}(\cdot)+(1-\lambda)\tilde{V}_\cdot^{n,0,u^\lambda}(\cdot)\big).$$

Lemma 4.16. *Suppose* **(H**$_1$**)–(H**$_5$**)** *and* **(C)** *hold, for all $n\geqslant 1$, we have*

$$\tilde{\mathcal{Y}}_s^{n,u^\lambda}\leqslant\overline{\mathcal{Y}}_s^{n,u^\lambda},\qquad s\in[t_\lambda,T],\ \mathbb{P}\text{-a.s.},$$

where

$$\begin{cases}
\mathrm{d}\overline{\mathcal{Y}}_s^{n,u^\lambda}=-\Big[f\big(s,X_s^{\lambda,u^\lambda},\overline{\mathcal{Y}}_s^{n,u^\lambda}-\overline{\Delta}_s,\overline{\mathcal{Z}}_s^{n,u^\lambda},\int_E l(e)\overline{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e),u_s^\lambda\big)-n\big(h(s,X_s^{\lambda,u^\lambda})-\overline{\mathcal{Y}}_s^{n,u^\lambda}+\overline{\Delta}_s\big)^-\\
\qquad\qquad+C\overline{\Delta}_s+C_\delta\lambda(1-\lambda)\Big(|\widehat{Z}_s|^2+\|\widehat{V}_s(\cdot)\|_{\nu,2}^2+|t_0-t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big)\Big]\mathrm{d}s\\
\qquad\quad+\overline{\mathcal{Z}}_s^{n,u^\lambda}\mathrm{d}B_s^\lambda+\int_E\overline{\mathcal{V}}_s^{n,u^\lambda}(e)\tilde{N}^\lambda(\mathrm{d}s,\mathrm{d}e),\quad s\in[t_\lambda,T],\\
\overline{\mathcal{Y}}_T^{n,u^\lambda}=\Phi(X_T^{\lambda,u^\lambda})+\overline{\Delta}_T,
\end{cases}$$

with $\overline{\Delta}_s = C\tilde{\Delta}_s + C_\delta \lambda(1-\lambda)\Delta_s^2$, $\tilde{\Delta}_s := \sup\limits_{r\in[t_\lambda,s]} |\tilde{\mathcal{X}}_r^{u^\lambda} - X_r^{\lambda,u^\lambda}|$, $\forall s \in [t_\lambda, T]$.

Proof. Using the semi-concavity and Lipschitz continuity of f,

$$\begin{aligned}
&\lambda f\big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \tilde{Y}_s^{n,1,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,1,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big)\\
&+(1-\lambda) f\big(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda}, \tilde{Y}_s^{n,0,u^\lambda}, \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big)\\
&\leqslant f\big(s, \tilde{\mathcal{X}}_s^{u^\lambda}, \tilde{\mathcal{Y}}_s^{n,u^\lambda}, \lambda\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda} + (1-\lambda)\sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}, \int_E l(e)\tilde{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big)\\
&\quad +C_\delta\lambda(1-\lambda)\Big(|\varrho_1^\lambda(s) - \varrho_0^\lambda(s)|^2 + |\widehat{X}_s|^2 + |\widehat{Y}_s|^2 + \Big|\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda} - \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}\Big|^2 + \Big|\int_E l(e)\widehat{V}_s(e)\nu(\mathrm{d}e)\Big|^2\Big)\\
&\leqslant f\big(s, X_s^{\lambda,u^\lambda}, \tilde{\mathcal{Y}}_s^{n,u^\lambda}, \tilde{\mathcal{Z}}_s^{n,u^\lambda}, \int_E l(e)\tilde{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big) + C\Big|\lambda\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda} + (1-\lambda)\sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda} - \tilde{\mathcal{Z}}_s^{n,u^\lambda}\Big|\\
&\quad +C_\delta\lambda(1-\lambda)\Big(|\varrho_1^\lambda(s) - \varrho_0^\lambda(s)|^2 + |\widehat{X}_s|^2 + |\widehat{Y}_s|^2 + \Big|\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda} - \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}\Big|^2 + \Big|\int_E l(e)\widehat{V}_s(e)\nu(\mathrm{d}e)\Big|^2\Big)\\
&\quad +C|\tilde{\mathcal{X}}_s^{u^\lambda} - X_s^{\lambda,u^\lambda}|\\
&\leqslant f(s, X_s^{\lambda,u^\lambda}, \tilde{\mathcal{Y}}_s^{n,u^\lambda}, \tilde{\mathcal{Z}}_s^{n,u^\lambda}, \int_E l(e)\tilde{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda) + C\overline{\Delta}_s\\
&\quad +C_\delta\lambda(1-\lambda)\Big(|\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0 - t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big),
\end{aligned}$$

where $\big(\widehat{X}_\cdot, \widehat{Y}_\cdot, \widehat{Z}_\cdot, \widehat{V}_\cdot(\cdot)\big)$ are the ones introduced in the proof of Lemma 4.15.

From the Lipschitz continuity of f, we obtain

$$\begin{aligned}
&(\frac{1}{\dot{\tau}_1^\lambda} - 1)\lambda f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \tilde{Y}_s^{n,1,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,1,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big)\\
&+(\frac{1}{\dot{\tau}_0^\lambda} - 1)(1-\lambda) f\Big(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda}, \tilde{Y}_s^{n,0,u^\lambda}, \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big)\\
&\leqslant C_\delta\lambda(1-\lambda)|t_0 - t_1|\Big(|\varrho_1^\lambda(s) - \varrho_0^\lambda(s)| + |\widehat{X}_s| + |\widehat{Y}_s| + |\sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda} - \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}| + |\int_E \widehat{V}_s(e)l(e)\nu(\mathrm{d}e)|\Big)\\
&\leqslant C_\delta\lambda(1-\lambda)\Big(\Delta_s^2 + |\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0 - t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big),
\end{aligned}$$

Moreover,

$$-\int_E \Big(\lambda(1 - \frac{1}{\dot{\tau}_1^\lambda})\tilde{V}_s^{n,1,u^\lambda}(e) + (1-\lambda)(1 - \frac{1}{\dot{\tau}_0^\lambda})\tilde{V}_s^{n,0,u^\lambda}(e)\Big)\nu(\mathrm{d}e) \leqslant C_\delta\lambda(1-\lambda)\big(\Delta_s^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2\big).$$

Thus, by adding the terms in the left sides of the above inequalities, we get

$$\begin{aligned}
&\frac{\lambda}{\dot{\tau}_1^\lambda} f\Big(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}, \tilde{Y}_s^{n,1,u^\lambda}, \sqrt{\dot{\tau}_1^\lambda}\tilde{Z}_s^{n,1,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,1,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big)\\
&+\frac{1-\lambda}{\dot{\tau}_0^\lambda} f\Big(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda}, \tilde{Y}_s^{n,0,u^\lambda}, \sqrt{\dot{\tau}_0^\lambda}\tilde{Z}_s^{n,0,u^\lambda}, \int_E l(e)\tilde{V}_s^{n,0,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\Big)\\
&-\int_E \Big(\lambda(1-\frac{1}{\dot{\tau}_1^\lambda})\tilde{V}_s^{n,1,u^\lambda}(e) + (1-\lambda)(1-\frac{1}{\dot{\tau}_0^\lambda})\tilde{V}_s^{n,0,u^\lambda}(e)\Big)\nu(\mathrm{d}e)\\
&\leqslant f(s, X_s^{\lambda,u^\lambda}, \tilde{\mathcal{Y}}_s^{n,u^\lambda}, \tilde{\mathcal{Z}}_s^{n,u^\lambda}, \int_E l(e)\tilde{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda) + C\overline{\Delta}_s\\
&\quad +C_\delta\lambda(1-\lambda)\Big(\Delta_s^2 + |\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0-t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big)\\
&\leqslant f(s, X_s^{\lambda,u^\lambda}, \tilde{\mathcal{Y}}_s^{n,u^\lambda} - \overline{\Delta}_s, \tilde{\mathcal{Z}}_s^{n,u^\lambda}, \int_E l(e)\tilde{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda) + C\overline{\Delta}_s\\
&\quad +C_\delta\lambda(1-\lambda)\Big(|\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0-t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big).
\end{aligned} \tag{4.40}$$

Similarly, based on the semi-concavity and Lipschitz continuity of h and Φ, we obtain

$$\begin{aligned}
&-\frac{\lambda}{\dot{\tau}_1^\lambda} n\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - \tilde{Y}_s^{n,1,u^\lambda}\big)^- - \frac{1-\lambda}{\dot{\tau}_0^\lambda} n\big(h(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda}) - \tilde{Y}_s^{n,0,u^\lambda}\big)^-\\
&\leqslant -n\Big(\frac{\lambda}{\dot{\tau}_1^\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - \tilde{Y}_s^{n,1,u^\lambda}\big) + \frac{1-\lambda}{\dot{\tau}_0^\lambda}\big(h(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda}) - \tilde{Y}_s^{n,0,u^\lambda}\big)\Big)^-\\
&= -n\Big(-\tilde{\mathcal{Y}}_s^{n,u^\lambda} - \lambda(1-\lambda)\frac{t_0-t_1}{T-t_\lambda}\widehat{Y}_s + \lambda(1-\lambda)\frac{t_0-t_1}{T-t_\lambda}\big(h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) - h(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda})\big)\\
&\quad +\lambda h(\varrho_1^\lambda(s), \tilde{X}_s^{1,u^\lambda}) + (1-\lambda)h(\varrho_0^\lambda(s), \tilde{X}_s^{0,u^\lambda})\Big)^-\\
&\leqslant -n\big(h(s, X_s^{\lambda,u^\lambda}) - \tilde{\mathcal{Y}}_s^{n,u^\lambda} + \overline{\Delta}_s\big)^-,
\end{aligned} \tag{4.41}$$

and

$$\lambda\Phi(\tilde{X}_T^{1,u^\lambda}) + (1-\lambda)\Phi(\tilde{X}_T^{0,u^\lambda}) \leqslant \Phi(\tilde{\mathcal{X}}_T^{u^\lambda}) + C_\delta\lambda(1-\lambda)|\tilde{X}_T^{1,u^\lambda} - \tilde{X}_T^{0,u^\lambda}|^2 \leqslant \Phi(X_T^{\lambda,u^\lambda}) + C\overline{\Delta}_T. \tag{4.42}$$

Finally, based on the above inequalities (4.40)-(4.42), we can apply the comparison theorem to get the desired result. ∎

The last auxiliary process is introduced as follows.

$$\widetilde{\mathscr{Y}}_s^{n,u^\lambda} := \overline{\mathcal{Y}}_s^{n,u^\lambda} - \overline{\Delta}_s, \quad s \in [t_\lambda, T],\ \mathbb{P}\text{-a.s.}$$

Then,

$$\begin{cases}
\mathrm{d}\widetilde{\mathscr{Y}}_s^{n,u^\lambda} = -\Big[f\big(s, X_s^{\lambda,u^\lambda}, \widetilde{\mathscr{Y}}_s^{n,u^\lambda}, \overline{\mathcal{Z}}_s^{n,u^\lambda}, \int_E l(e)\overline{\mathcal{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big) - n\big(h(s, X_s^{\lambda,u^\lambda}) - \widetilde{\mathscr{Y}}_s^{n,u^\lambda}\big)^-\\
\qquad\qquad +C\overline{\Delta}_s + C_\delta\lambda(1-\lambda)\Big(|\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0-t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big)\Big]\mathrm{d}s - \mathrm{d}\overline{\Delta}_s\\
\qquad\quad +\overline{\mathcal{Z}}_s^{n,u^\lambda}\mathrm{d}B_s^\lambda + \int_E \overline{\mathcal{V}}_s^{n,u^\lambda}(e)\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_\lambda, T],\\
\overline{\mathcal{Y}}_T^{n,u^\lambda} = \Phi(X_T^{\lambda,u^\lambda}).
\end{cases} \tag{4.43}$$

Note that the penalization term of the above BSDE (4.43) is the same as BSDE (4.7), and they are driven by the same Brownian motion and Poisson random measure in $[t_\lambda, T]$. Similar to the proof of Lemmas 4.12 and 4.13, we give an estimate between them as follows.

Lemma 4.17. *Assume the conditions* $(\mathbf{H}_1)$*-*$(\mathbf{H}_4)$ *and* $(\mathbf{C})$ *hold. Then, there exists some constant* $C_\delta \geqslant 0$ *such that, for all* $n \geqslant 1$ *and* $s \in [t_\lambda, T]$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{r\in[s,T]}|\widetilde{\mathscr{Y}}_r^{n,u^\lambda} - Y_r^{n,\lambda,u^\lambda}|^2 + \int_s^T \Big(|\overline{\mathscr{Z}}_r^{n,u^\lambda} - Z_r^{n,\lambda,u^\lambda}|^2 + \|\overline{\mathscr{V}}_r^{n,u^\lambda}(\cdot) - V_r^{n,\lambda,u^\lambda}(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\\
&\leqslant C\overline{\Delta}_s^2 + \mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\mathscr{D}_s^n|^2\big],
\end{aligned}$$

where $\mathscr{D}_\cdot^n$ *is defined in* (4.32).

Proof. For all $s \in [t_\lambda, T]$, denote $\big(\mathscr{Y}_s^n, \mathscr{Z}_s^n, \mathscr{V}_s^n\big) := \big(\widetilde{\mathscr{Y}}_s^{n,u^\lambda} - Y_s^{n,\lambda,u^\lambda}, \overline{\mathscr{Z}}_s^{n,u^\lambda} - Z_s^{n,\lambda,u^\lambda}, \overline{\mathscr{V}}_s^{n,u^\lambda} - V_s^{n,\lambda,u^\lambda}\big)$. Then,

$$\begin{cases}
\mathrm{d}\mathscr{Y}_s^n = -\Big[f\big(s, X_s^{\lambda,u^\lambda}, \widetilde{\mathscr{Y}}_s^{n,u^\lambda}, \overline{\mathscr{Z}}_s^{n,u^\lambda}, \displaystyle\int_E l(e)\overline{\mathscr{V}}_s^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big) - n\big(h(s, \mathcal{X}_s^{u^\lambda}) - \widetilde{\mathscr{Y}}_s^{n,u^\lambda}\big)^- + C\overline{\Delta}_s + D_s^n\\
\qquad\qquad - f\big(s, X_s^{\lambda,u^\lambda}, Y_s^{n,\lambda,u^\lambda}, Z_s^{n,\lambda,u^\lambda}, \displaystyle\int_E l(e)V_s^{n,\lambda,u^\lambda}(e)\nu(\mathrm{d}e), u_s^\lambda\big) + n\big(h(s, X_s^{\lambda,u^\lambda}) - Y_s^{n,\lambda,u^\lambda}\big)^-\Big]\mathrm{d}s\\
\qquad\quad - \mathrm{d}\overline{\Delta}_s + \mathscr{Z}_s^n\mathrm{d}B_s^\lambda + \displaystyle\int_E \mathscr{V}_s(e)\tilde{N}^\lambda(\mathrm{d}s, \mathrm{d}e), \quad s \in [t_\lambda, T],\\
\mathscr{Y}_T^n = 0,
\end{cases}$$

where $D_s^n := C_\delta\lambda(1-\lambda)\Big(|\widehat{Z}_s|^2 + \|\widehat{V}_s(\cdot)\|_{\nu,2}^2 + |t_0 - t_1|^2|\tilde{Z}_s^{n,0,u^\lambda}|^2\Big)$, $s \in [t_\lambda, T]$.

For some constant $\alpha > 0$, we apply Itô's formula to $\mathrm{e}^{\alpha s}|\mathscr{Y}_s^n|^2$,

$$\begin{aligned}
&\mathrm{e}^{\alpha s}|\mathscr{Y}_s^n|^2 + \int_s^T \alpha\mathrm{e}^{\alpha r}|\mathscr{Y}_r^n|^2\mathrm{d}r + \int_s^T \mathrm{e}^{\alpha r}|\mathscr{Z}_r^n|^2\mathrm{d}r + \int_s^T\int_E \mathrm{e}^{\alpha r}|\mathscr{V}_r^n(e)|^2N^\lambda(\mathrm{d}r, \mathrm{d}e)\\
&= 2\int_s^T \mathrm{e}^{\alpha r}\mathscr{Y}_r^n\Big[I_r^3 + n\big(h(r, X_r^{\lambda,u^\lambda}) - Y_r^{n,\lambda,u^\lambda}\big)^- - n\big(h(r, \mathcal{X}_r^{u^\lambda}) - \widetilde{\mathscr{Y}}_r^{n,u^\lambda}\big)^- + C\overline{\Delta}_r + D_r^n\Big]\mathrm{d}r\\
&\quad + 2\int_s^T \mathrm{e}^{\alpha r}\mathscr{Y}_r^n\mathrm{d}\overline{\Delta}_r - 2\int_s^T \mathrm{e}^{\alpha r}\mathscr{Y}_r^n\mathscr{Z}_r^n\mathrm{d}B_r^\lambda - 2\int_s^T\int_E \mathrm{e}^{\alpha r}\mathscr{Y}_r^n\mathscr{V}_r^n(e)\tilde{N}^\lambda(\mathrm{d}r, \mathrm{d}e),
\end{aligned}$$

with

$$I_r^3 := f\big(r, X_r^{\lambda,u^\lambda}, \widetilde{\mathscr{Y}}_r^{n,u^\lambda}, \overline{\mathscr{Z}}_r^{n,u^\lambda}, \int_E l(e)\overline{\mathscr{V}}_r^{n,u^\lambda}(e)\nu(\mathrm{d}e), u_r^\lambda\big) - f\big(r, X_r^{\lambda,u^\lambda}, Y_r^{n,\lambda,u^\lambda}, Z_r^{n,\lambda,u^\lambda}, \int_E l(e)V_r^{n,\lambda,u^\lambda}(e)\nu(\mathrm{d}e), u_r^\lambda\big).$$

Then, based on the continuity of f,

$$\begin{aligned}
&\mathrm{e}^{\alpha s}|\mathscr{Y}_s^n|^2+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(\alpha\mathrm{e}^{\alpha r}|\mathscr{Y}_r^n|^2+\mathrm{e}^{\alpha r}|\mathscr{Z}_r^n|^2+\int_E\mathrm{e}^{\alpha r}|\mathscr{V}_r^n(e)|^2\nu(\mathrm{d}e)\Big)\mathrm{d}r\Big]\\
&\leqslant C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathrm{e}^{\alpha r}\mathscr{Y}_r^n\Big(|\mathscr{Y}_r^n|+|\mathscr{Z}_r^n|+\Big|\int_E l(e)\mathscr{V}_r^n(e)\nu(\mathrm{d}e)\Big|\Big)\mathrm{d}r\Big]+2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathrm{e}^{\alpha r}\mathscr{Y}_r^n\big(C\overline{\Delta}_r+D_r^n\big)\mathrm{d}r\Big]\\
&\quad+2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathrm{e}^{\alpha r}\mathscr{Y}_r^n\mathrm{d}\overline{\Delta}_r\Big]\\
&\leqslant C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathrm{e}^{\alpha r}|\mathscr{Y}_r^n|^2\mathrm{d}r\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\overline{\Delta}_T|^2\big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\mathscr{D}_s^n|^2\big]+2\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathrm{e}^{\alpha r}\mathscr{Y}_r^n\mathrm{d}\overline{\Delta}_r\Big]\\
&\quad+\frac{1}{2}\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(\mathrm{e}^{\alpha r}|\mathscr{Z}_r^n|^2+\int_E\mathrm{e}^{\alpha r}|\mathscr{V}_r^n(e)|^2\nu(\mathrm{d}e)\Big)\mathrm{d}r\Big],
\end{aligned}$$

where we haved used Lemmas 4.6 and 4.15.

Taking α large enough, we get

$$|\mathscr{Y}_s^n|^2+\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|\mathscr{Z}_r^n|^2+\|\mathscr{V}_r^n(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\leqslant C\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathscr{Y}_r^n\mathrm{d}\overline{\Delta}_r\Big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\overline{\Delta}_T|^2\big]+C\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\mathscr{D}_s^n|^2\big].$$

Following the techniques used in (4.28) and (4.29), we can tackle with $\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\mathscr{Y}_r^n\mathrm{d}\overline{\Delta}_r\Big]$ similarty to get

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\sup_{s\in[t,T]}|\mathscr{Y}_s^n|^2\Big]\leqslant C\overline{\Delta}_s^2+C\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\mathscr{D}_s^n|^2\big],$$

and

$$\mathbb{E}^{\mathcal{F}_s^\lambda}\Big[\int_s^T\Big(|\mathscr{Z}_r^n|^2+\|\mathscr{V}_r^n(\cdot)\|_{\nu,2}^2\Big)\mathrm{d}r\Big]\leqslant C\overline{\Delta}_s^2+\mathbb{E}^{\mathcal{F}_s^\lambda}\big[|\mathscr{D}_s^n|^2\big].$$

$\blacksquare$

Based on the above preparations, we give the proof of (4.12).

For any $(t_0,x_0),\ (t_1,x_1)\in[0,T-\delta]\times\mathbb{R}$, and $\lambda\in[0,1]$, according to the Lemma 4.16, we have

$$\lambda\tilde{Y}_{t_\lambda}^{n,1,u^\lambda}+(1-\lambda)\tilde{Y}_{t_\lambda}^{n,0,u^\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}=\tilde{\mathfrak{Y}}_{t_\lambda}^{n,u^\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}\leqslant\overline{\mathfrak{Y}}_{t_\lambda}^{n,u^\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}.$$

Furthermore, from the definition of $\overline{\mathfrak{Y}}_\cdot^{n,u^\lambda}$ and Lemma 4.17,

$$\overline{\mathfrak{Y}}_{t_\lambda}^{n,u^\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}\leqslant\widetilde{\mathscr{Y}}_{t_\lambda}^{n,u^\lambda}+\overline{\Delta}_{t_\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}\leqslant C\overline{\Delta}_{t_\lambda}+\Big(\mathbb{E}\big[|\mathscr{D}_{t_\lambda}^n|^2\big]\Big)^{\frac{1}{2}},$$

that is,

$$\lambda\tilde{Y}_{t_\lambda}^{n,1,u^\lambda}+(1-\lambda)\tilde{Y}_{t_\lambda}^{n,0,u^\lambda}-Y_{t_\lambda}^{n,\lambda,u^\lambda}\leqslant C_\delta\lambda(1-\lambda)\big(|t_1-t_0|^2+|x_1-x_0|^2\big)+\Big(\mathbb{E}\big[|\mathscr{D}_{t_\lambda}^n|^2\big]\Big)^{\frac{1}{2}}.\tag{4.44}$$

Then the desired (4.12) is proved.

According to (4.13) and (4.14), Proposition 4.5 is proved. For Theorem 4.4, we only need to take the limit $n\to\infty$ in Proposition 4.5.

4.2. The Lipschitz continuity of $W(\cdot,\cdot)$ in (t,x)

In this part, we study the Lipschitz continuity of $W(\cdot,\cdot)$ with respect to (t,x). For this, the transformation introduced in the previous part will be used again.

Taking $\lambda=0$, $i=1$ in Subsection 4.1, we get a transformation $\tau_1^0(s)$ from (4.1), which is the only one used in the following. So we denote $\tau(\cdot):=\tau_1^\lambda(\cdot)$. Obviously, τ maps $[t_1,T]$ to $[t_0,T]$, and $\dot\tau=\frac{\mathrm{d}}{\mathrm{d}s}\tau(s)=\frac{T-t_0}{T-t_1}$. The rest settings are similar to Subsection 4.1 (keeping in mind that $\lambda=0$). Then, the meanings of $\mathbb{B}_s^1$, $\mathbb{N}_s^1$, $\tilde{\mathbb{N}}_s^1$, g_τ, $\mathbf{F}^1$, $\mathscr{F}_s^1$, $\mathcal{U}_{t_0,T}$, $\mathcal{U}_{t_1,T}$ are recognized.

In this framework, some results can degenerate into the following ones. The first one is from Lemma 4.1. We also refer to [17, 22].

Lemma 4.18. *There exists the constant $C_\delta>0$ only depending on δ, such that*

$$|\varrho(s)-s|+|\frac{1}{\dot\tau}-1|+|\frac{1}{\sqrt{\dot\tau}}-1|\leqslant C_\delta|t_0-t_1|,\quad s\in[t_0,T].$$

According to Lemma 4.9, we have

Lemma 4.19. *Suppose* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$-(i),(ii) *hold, for all $p\geqslant 1$ and $s\in[t_0,T]$, we have, $\mathbb{P}$-a.s.,*

$$\mathbb{E}^{\mathcal{F}_s^0}\Big[\sup_{\tau\in[s,T]}|X_\tau^{0,u^0}-\tilde{X}_\tau^{1,u^0}|^p\Big]\leqslant C_{T,p,\delta}\Big(|t_0-t_1|^p+|X_s^{0,u^0}-\tilde{X}_s^{1,u^0}|^p\Big),\quad \mathbb{P}\text{-a.s.}$$

From Lemma 4.14, we have

Lemma 4.20. *Under the conditions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$, $(\mathbf{H}_4)$-(i), (ii) *and* $(\mathbf{C})$,

$$|\tilde{Y}_s^{n,1,u^0}-Y_s^{n,0,u^0}|\leqslant C_\delta\Delta_s,\quad \forall s\in[t_0,T],\ \mathbb{P}\text{-a.s.}$$

Proposition 4.21. *Assume that* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$-(i), (ii) *hold. Then, for all $\delta>0$, there exists a constant $C_{T,p,\delta}>0$ only depending on δ, the bounds and Lipschitz constants of σ, b and γ, such that for any $t_0,t_1\in[0,T-\delta]$, $x_0,x_1\in\mathbb{R}^n$,*

$$|W^n(t_0,x_0)-W^n(t_1,x_1)|\leqslant C_\delta\big(|t_0-t_1|+|x_0-x_1|\big).$$

Proof. For all $n\geqslant 1$ and $(t_0,x_0),(t_1,x_1)\in[0,T-\delta]\times\mathbb{R}^n$, we recall

$$\begin{aligned}W^n(t_0,x_0)&=\operatorname*{essinf}_{u^0(\cdot)\in\mathcal{U}^0_{t_0,T}}J_n(t_0,x_0;u^0(\cdot))=\operatorname*{essinf}_{u^0(\cdot)\in\mathcal{U}^0_{t_0,T}}Y_{t_0}^{n,0,u^0},\\ W^n(t_1,x_1)&=\operatorname*{essinf}_{u^1(\cdot)\in\mathcal{U}^1_{t_1,T}}J_n(t_1,x_1;u^1(\cdot))=\operatorname*{essinf}_{u^0(\cdot)\in\mathcal{U}^0_{t_0,T}}\widetilde{Y}_{t_0}^{n,1,u^0}.\end{aligned}$$

Then, for any $\varepsilon>0$, there exists some $u^{1,\varepsilon}(\cdot)\in\mathcal{U}^1_{t_1,T}$ such that

$$W^n(t_1,x_1)>Y_{t_1}^{n,1,u_s^{1,\varepsilon}}-\varepsilon=\widetilde{Y}_{t_0}^{n,1,u_s^{0,\varepsilon}}-\varepsilon,$$

where $u_s^{0,\varepsilon}=u^{1,\varepsilon}_{\varrho(s)}\in\mathcal{U}^0_{t_0,T}$.

Therefore, using Lemma 4.20,

$$W^n(t_0,x_0)-W^n(t_1,x_1)\leqslant Y^{n,0,u^{0,\varepsilon}}_{t_0}-\widetilde{Y}^{n,1,u^{0,\varepsilon}}_{t_0}+\varepsilon\leqslant C_\delta\Delta_{t_0}+\varepsilon=C_\delta(|t_0-t_1|+|x_0-x_1|)+\varepsilon.$$

Due to the symmetry, for any $\varepsilon>0$,

$$|W^n(t_0,x_0)-W^n(t_1,x_1)|\leqslant C_\delta(|t_0-t_1|+|x_0-x_1|)+\varepsilon.$$

By the arbitrariness of ε, the desired result holds. ∎

Letting $n\to\infty$ in Proposition 4.21 and using Lemma 3.10, we get the following result.

Theorem 4.22. *Assume* **(H$_1$)–(H$_3$)**, **(H$_4$)**-(i), (ii) *and* **(C)** *hold. Then, for all $\delta>0$, there exists a constant $C_\delta>0$, such that for any $t_0,t_1\in[0,T-\delta]$, $x_0,x_1\in\mathbb{R}^n$,*

$$|W(t_0,x_0)-W(t_1,x_1)|\leqslant C_\delta\big(|t_0-t_1|+|x_0-x_1|\big).$$

That is to say, the value function $W(\cdot,\cdot)$ is Lipschitz continuous on $[0,T-\delta]\times\mathbb{R}^n$, for all $\delta>0$.

5. Stochastic verification theorems

In this section, we focus on the research of stochastic verification theorems of Problem $(\mathrm{C})_{t,x}$. The study will be carried out in two cases: classical solutions and viscosity solutions.

5.1. The classical solution case

We begin with the case when PIDE (3.6) admits the classical solution. In our framework, we try to construct an optimal feedback control of Problem $(\mathrm{C})_{t,x}$ from the classical solution of PIDE (3.6). To begin with, we introduce the following definition of admissible feedback control laws.

Definition 5.1. *Let $t\in[0,T]$. A measurable mapping $\mathbb{u}:[t,T]\times\mathbb{R}^n\to U$ is said to be an admissible feedback control law, if for all $x\in\mathbb{R}^n$, the following equation*

$$\begin{cases}\mathrm{d}X^{t,x;\mathbb{u}}_s=b\big(s,X^{t,x;\mathbb{u}}_s,\mathbb{u}(s,X^{t,x;\mathbb{u}}_s)\big)\mathrm{d}s+\sigma\big(s,X^{t,x;\mathbb{u}}_s,\mathbb{u}(s,X^{t,x;\mathbb{u}}_s)\big)\mathrm{d}B_s\\ \qquad\qquad+\displaystyle\int_E\gamma\big(s,X^{t,x;\mathbb{u}}_{s-},\mathbb{u}(s,X^{t,x;\mathbb{u}}_{s-}),e\big)\tilde{N}(\mathrm{d}s,\mathrm{d}e),\\ X^{t,x;\mathbb{u}}_t=x,\quad(t,x)\in[0,T]\times\mathbb{R}^n,\end{cases}\tag{5.1}$$

admit a unique strong solution $X^{t,x;\mathbb{u}}_\cdot\in\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$, and $\mathbb{u}(\cdot,X^{t,x;\mathbb{u}}_\cdot)\in\mathcal{U}_{t,T}$. The set of all such admissible feedback control laws on $[t,T]$ is denoted by $\mathscr{U}_{t,T}$.

For convenience, we introduce the mapping $\psi:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\to U$ such that

$$\psi(r,x,y,p,P)\in\operatorname{argmin}\mathbb{H}(r,x,y,p,P,\cdot)\equiv\Big\{\bar u\in U\mid\mathbb{H}(r,x,y,p,P,\bar u)=\min_{u\in U}\mathbb{H}(r,x,y,p,P,u)\Big\}.$$

Now we present the first main result of this subsection.

Theorem 5.2. *Assume* **(H$_1$)**, **(H$_2$)** *and* **(C)**. *Let $\mathbb{W}(\cdot,\cdot)\in C^{1,2}([0,T]\times\mathbb{R}^n)$ be the classical solution of PIDE (3.6). Then*

(i) *for any $(t,x)\in[0,T]\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}_{t,T}$, we have $\mathbb{W}(t,x)\leqslant J(t,x;u(\cdot))$;*

(ii) *for any* $(t,x) \in [0,T] \times \mathbb{R}^n$, *defining* $\bar{\mathrm{u}} : [t,T] \times \mathbb{R}^n \to U$ *as*

$$\bar{\mathrm{u}}(s,y) = \psi\big(s,y,(W,W_x,W_{xx})(s,y)\big), \quad (s,y) \in [t,T] \times \mathbb{R}^n, \tag{5.2}$$

if $\bar{\mathrm{u}}(\cdot,\cdot) \in \mathscr{U}_{t,T}$, *then* $\bar{\mathrm{u}}\big(\cdot,\mathbb{X}.\big)$ *is an optimal control of Problem* $(C)_{t,x}$, *where* $\mathbb{X}. = X.^{t,x;\bar{\mathrm{u}}}$ *satisfies* (5.1) *with* $\bar{\mathrm{u}}(\cdot,\cdot)$. *Moreover,* $\mathbb{W}(\cdot,\cdot)$ *is in fact the value function* $W(\cdot,\cdot)$, *i.e.*,

$$\mathbb{W}(t,x) = J\big(t,x;\bar{\mathrm{u}}(\cdot,\mathbb{X}.)\big) = W(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Proof. (i) For any $(t,x) \in [0,T] \times \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}_{t,T}$, denote $X. = X.^{t,x;u}$. Applying Itô's formula to $\mathbb{W}(\cdot,X.)$, we have, for any $s \in [t,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\mathbb{W}\big(s,X_s\big) = {}& \Phi(X_T) - \int_s^T \Big[\frac{\partial W}{\partial r}(r,X_r) + \mathbb{H}\big(r,X_r,\big(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx}\big)(r,X_r),u_r\big)\Big]\mathrm{d}r \\ & + \int_s^T \Big[f\big(r,X_r,\mathbb{W}(r,X_r),\mathbb{W}_x(r,X_r).\sigma(r,X_r,u_r),\mathcal{C}^u\mathbb{W}(r,X_r),u_r\big)\Big]\mathrm{d}r \\ & - \int_s^T \mathbb{W}_x(r,X_r).\sigma\big(r,X_r,u_r\big)\mathrm{d}B_r - \int_s^T\int_E \big(\mathbb{W}(r,X_{r-}+\gamma(r,X_r,u_r,e)) - \mathbb{W}(r,X_{r-})\big)\tilde{N}(\mathrm{d}r,\mathrm{d}e).\end{aligned} \tag{5.3}$$

On the other hand, for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $n \in \mathbb{N}$, we consider the following penalized BSDE with jumps,

$$\begin{aligned}{}^nY_s = {}& \Phi\big(X_T\big) + \int_s^T f\big(r,X_r,{}^nY_r,{}^nZ_r,\int_E l(e)\,{}^nV_r(e)\nu(\mathrm{d}e),u_r\big)\mathrm{d}r - n\int_s^T \big(h(r,X_r) - {}^nY_r\big)^-\mathrm{d}r \\ & - \int_s^T {}^nZ_r\mathrm{d}B_r - \int_s^T\int_E {}^nV_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s \in [t,T].\end{aligned} \tag{5.4}$$

By the fact that $\mathbb{W}(\cdot,\cdot)$ being the classical solution of (3.6), we get the following two cases,

Case (a). at any point $(t,x) \in [0,T] \times \mathbb{R}^n$ such that $(\mathbb{W}-h)(t,x) = 0$,

$$-\frac{\partial W}{\partial t}(t,x) - \inf_{u\in U} \mathbb{H}\big(t,x,(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx})(t,x),u\big) \leqslant 0;$$

Case (b). at any point $(t,x) \in [0,T] \times \mathbb{R}^n$ such that $(\mathbb{W}-h)(t,x) < 0$,

$$-\frac{\partial W}{\partial t}(t,x) - \inf_{u\in U} \mathbb{H}\big(t,x,(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx})(t,x),u\big) = 0.$$

No matter (a) or (b), for any $r \in [t,T]$ and $u(\cdot) \in \mathcal{U}_{t,T}$, $n \in \mathbb{N}$, we have

$$\begin{aligned}& f\big(r,X_r,\mathbb{W}\big(r,X_r\big),\mathbb{W}_x\big(r,X_r\big).\sigma(r,X_r,u_r),\mathcal{C}^u\mathbb{W}(r,X_r),u_r\big) \\ & -\frac{\partial \mathbb{W}}{\partial r}\big(r,X_r\big) - \mathbb{H}\big(r,X_r,(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx})(r,X_r),u_r\big) \\ & \leqslant f\big(r,X_r,\mathbb{W}\big(r,X_r\big),\mathbb{W}_x\big(r,X_r\big).\sigma(r,X_r,u_r),\mathcal{C}^u\mathbb{W}(r,X_r),u_r\big) \\ & \quad -\frac{\partial \mathbb{W}}{\partial r}\big(r,X_r\big) - \inf_{u\in U} \mathbb{H}\big(r,X_r,(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx})(r,X_r),u\big) \\ & \leqslant f\big(r,X_r,\mathbb{W}\big(r,X_r\big),\mathbb{W}_x\big(r,X_r\big).\sigma(r,X_r,u_r),\mathcal{C}^u\mathbb{W}(r,X_r),u_r\big) - n\big(h(r,X_r) - \mathbb{W}(r,X_r)\big)^-.\end{aligned} \tag{5.5}$$

Therefore, by using the comparison theorem to them, for all $n \in \mathbb{N}$, we get

$$\mathbb{W}\big(s, X_s\big) \leqslant {}^nY_s, \quad s \in [t,T], \ \mathbb{P}\text{-a.s.} \tag{5.6}$$

Notice that Lemma 2.1, letting $n \to \infty$ in (5.6), we get

$$\mathbb{W}(s, X_s) \leqslant Y_s^{t,x,u}, \quad s \in [t,T], \ \mathbb{P}\text{-a.s.}$$

Here $Y_{\cdot}^{t,x,u}$ is the first component of the solution of RBSDE with jumps (3.2).

Especially, when $s = t$,

$$\mathbb{W}(t,x) \leqslant Y_t^{t,x,u} = J(t,x;u(\cdot)), \quad \text{for any } u(\cdot) \in \mathcal{U}_{t,T}. \tag{5.7}$$

(ii) Let $\mathbb{X} \in \mathcal{S}_{\mathbb{F}}^2(t,T;\mathbb{R}^n)$ be the solutions of SDE with jumps (5.1) with $\mathbbm{u}(\cdot,\cdot)$ replaced by $\bar{\mathbbm{u}}(\cdot,\cdot) \in \mathscr{U}_{t,T}$ introduced in (5.2). And the following RBSDE with jumps

$$\begin{cases}\text{(i) } (\mathbb{Y},\mathbb{Z},\mathbb{V},\mathbb{A}) \in \mathscr{S}_{\mathbb{F}}^2[t,T];\\ \text{(ii) } \mathbb{Y}_s = \Phi(\mathbb{X}_T) + \displaystyle\int_s^T f\Big(r,\mathbb{X}_r,\mathbb{Y}_r,\mathbb{Z}_r,\int_E l(e)\mathbb{V}_r(e)\nu(\mathrm{d}e),\bar{\mathbbm{u}}(r,\mathbb{X}_r)\Big)\mathrm{d}r - (\mathbb{A}_T - \mathbb{A}_s) - \int_s^T \mathbb{Z}_r \mathrm{d}B_r\\ \qquad\qquad - \displaystyle\int_s^T\int_E \mathbb{V}_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s \in [t,T];\\ \text{(iii) } \mathbb{Y}_s \leqslant h(s,\mathbb{X}_s), \ \text{a.e. } s \in [t,T];\\ \text{(iv) } \displaystyle\int_t^T \big(h(s,\mathbb{X}_s) - \mathbb{Y}_s\big)\mathrm{d}\mathbb{A}_s = 0,\end{cases} \tag{5.8}$$

admits the unique $\mathbb{F}$-adapted solution $(\mathbb{Y},\mathbb{Z},\mathbb{V},\mathbb{A})$. Note that $\bar{\mathbbm{u}}(\cdot,\cdot) \in \mathscr{U}_{t,T}$ implies $\bar{\mathbbm{u}}(\cdot,\mathbb{X}_\cdot) \in \mathcal{U}_{t,T}$.

Based on Lemma 3.9, (5.8) also has the penalized equation as follows.

$$\begin{aligned}{}^n\mathbb{Y}_s &= \Phi(\mathbb{X}_T) + \int_s^T f\Big(r,\mathbb{X}_r,{}^n\mathbb{Y}_r,{}^n\mathbb{Z}_r,\int_E l(e)\,{}^n\mathbb{V}_r(e)\nu(\mathrm{d}e),\bar{\mathbbm{u}}(r,\mathbb{X}_r)\Big)\mathrm{d}r - \int_s^T n\big(h(r,\mathbb{X}_r) - {}^n\mathbb{Y}_r\big)^-\mathrm{d}r\\ &\quad - \int_s^T {}^n\mathbb{Z}_r\mathrm{d}B_r - \int_s^T\int_E {}^n\mathbb{V}_r(e)\tilde{N}(\mathrm{d}r,\mathrm{d}e), \quad s \in [t,T], \ n \in \mathbb{N}.\end{aligned}$$

In Case (a), for $(t,x) \in [0,T] \times \mathbb{R}^n$ such that $\mathbb{W}(t,x) = h(t,x)$, combined with the obstacle condition in RBSDE with jumps (5.8), we get

$$\mathbb{W}(t,x) = h(t,x) \geqslant \mathbb{Y}_t = J\big(t,x;\bar{\mathbbm{u}}(\cdot,\mathbb{X}_\cdot)\big), \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

In Case (b), applying Itô's formula to $\mathbb{W}(s,\mathbb{X}_s)$ on $[t,T]$, we have

$$\begin{aligned}&\mathbb{W}\big(s,\mathbb{X}_s\big) = \Phi\big(\mathbb{X}_T\big) - \int_s^T \Big(\frac{\partial\mathbb{W}}{\partial r}\big(r,\mathbb{X}_r\big) + \mathbb{H}\big(r,\mathbb{X}_r,(\mathbb{W},\mathbb{W}_x,\mathbb{W}_{xx})(r,\mathbb{X}_r),\bar{\mathbbm{u}}(r,\mathbb{X}_r)\big)\Big)\mathrm{d}r\\ &+ \int_s^T f\big(r,\mathbb{X}_r,\mathbb{W}\big(r,\mathbb{X}_r\big),\mathbb{W}_x\big(r,\mathbb{X}_r\big).\sigma\big(r,\mathbb{X}_r,\bar{\mathbbm{u}}(r,\mathbb{X}_r)\big),\mathcal{C}^{\bar{\mathbbm{u}}}\mathbb{W}(r,\mathbb{X}_r),\bar{\mathbbm{u}}(r,\mathbb{X}_r)\big)\mathrm{d}r\\ &- \int_s^T \mathbb{W}_x\big(r,\mathbb{X}_r\big).\sigma\big(r,\mathbb{X}_r,\bar{\mathbbm{u}}(r,\mathbb{X}_r)\big)\mathrm{d}B_r - \int_s^T\int_E \big(\mathbb{W}\big(r,\mathbb{X}_{r-} + \gamma(r,\mathbb{X}_{r-},u_r,e)\big) - \mathbb{W}(r,\mathbb{X}_{r-})\big)\tilde{N}(\mathrm{d}r,\mathrm{d}e).\end{aligned}$$

Under (5.2) and Case (b), we follow a procedure analogous to part (i). Then, employing the comparison theorem for BSDE with jumps, we get, for all $n \in \mathbb{N}$, $\mathbb{W}\big(t,x\big) \geqslant {}^n\mathbb{Y}_t$. Similarly to (i), letting $n \to \infty$ and $s = t$, we have $\mathbb{W}\big(t,x\big) \geqslant \mathbb{Y}_t$.

Finally, combined with (5.7), for any $(t,x) \in [0,T] \times \mathbb{R}^n$, we get

$$\mathbb{W}(t,x) = J\big(t,x;\bar{\mathbb{u}}(\cdot,\mathbb{X}.)\big) = \operatorname*{essinf}_{u(\cdot)\in\mathcal{U}_{t,T}} J\big(t,x;u(\cdot)\big) = W(t,x).$$

That is, the classical solution $\mathbb{W}(\cdot,\cdot)$ of PIDE (3.6) is indeed the value function $W(\cdot,\cdot)$ of Problem $(\mathrm{C})_{t,x}$, and $\bar{\mathbb{u}}(\cdot,\mathbb{X}.)$ is the optimal control of Problem $(\mathrm{C})_{t,x}$. ∎

Remark 5.3. *We make some explanations on the collection $\mathscr{U}_{t,T}$ of the admissible feedback control laws. Let us first give a relatively direct condition*

(H$_7$) (i) *For every $(r,x,y,z,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $b(r,x,\cdot)$, $\sigma(r,x,\cdot)$ are Lipschitz continuous in $u \in U$. And, for any $u_1, u_2 \in U$, $|\gamma(r,x,u_1,e) - \gamma(r,x,u_2,e)| \leqslant \ell(e)|u_1 - u_2|$.*
(ii) *$\mathbb{u}(\cdot,\cdot) \in \mathcal{L}$, where $\mathcal{L}$ is the class of measurable mappings $\mathbb{u} : [0,T] \times \mathbb{R}^n \to U$ satisfying the two properties:* (a) *for every fixed $x \in \mathbb{R}^n$, $u(\cdot,x)$ is continuous in $r \in [0,T]$;* (b) *for every $r \in [0,T]$, $\mathbb{u}(r,\cdot)$ is Lipschitz continuous in $x \in \mathbb{R}^n$.*

If **(H$_1$)**, **(H$_2$)** *and* **(H$_7$)** *hold, it is easy to check that $\mathbb{u}(\cdot,\cdot) \in \mathscr{U}_{t,T}$. These are essentially the Lipschitz conditions on the coefficients, which may be hard for us to seek such $\mathbb{u}(\cdot,\cdot) \in \mathcal{L}$. However, as the development of the theory of SDEs with discontinuous coefficients (such as the singular or the irregular coefficients), there are many researches devoted to seeking a strong solution of SDE with jumps under weak assumptions, such as [31–33], etc. Therefore, with the help of these studies, some appropriately mild conditions may be posed to ensure $\mathscr{U}_{t,T}$ to be non-empty. We just keep in mind that the constructed $\mathbb{u}(\cdot,\cdot)$ must lie in $\mathscr{U}_{t,T}$ when dealing with the specific (or computable) control problems.*

5.2. The viscosity solution case

We now look at the case when the solution of PIDE is not necessarily smooth, *i.e.*, viscosity solution. In order to give a clear statement of the main result, we rewrite (3.6) as follows.

$$\begin{cases} \max\Big\{W(t,x) - h(t,x), -\dfrac{\partial W}{\partial t}(t,x) - \inf\limits_{u\in U} \mathcal{H}\big(t,x,(W,W_x,W_{xx})(t,x),\mathcal{B}^u W(t,x),\mathcal{C}^u W(t,x),u\big)\Big\} = 0, \\ \hfill (t,x) \in [0,T] \times \mathbb{R}^n, \\ W(T,x) = \Phi(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Here, for all $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$ and $\Psi \in C^{1,2}([0,T] \times \mathbb{R}^n;\mathbb{R})$,

$$\begin{aligned} &\mathcal{H}\big(t,x,\Psi(t,x),\Psi_x(t,x),\Psi_{xx}(t,x),\mathcal{B}^u\Psi(t,x),\mathcal{C}^u\Psi(t,x),u\big) = \mathbb{H}\big(t,x,(\Psi,\Psi_x,\Psi_{xx})(t,x),u\big) \\ &:= \mathcal{L}^u\Psi(t,x) + \mathcal{B}^u\Psi(t,x) + f\big(t,x,\Psi(t,x),\Psi_x(t,x).\sigma(t,x,u),\mathcal{C}^u\Psi(t,x),u\big). \end{aligned}$$

First, it is necessary to introduce the definition of second-order right parabolic superdifferentials (refer to [34]).

Definition 5.4. Let $(t,x)\in[0,T]\times\mathbb{R}^n$ and $w\in C([0,T]\times\mathbb{R}^n;\mathbb{R})$, the *second-order parabolic superdifferential* of w at (t,x) is defined as

$$D^{1,2,+}_{t+,x}w(t,x):=\Big\{(q,p,P)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^n\Big|\ \varlimsup_{s\to t^+,y\to x}\frac{1}{|s-t|+|y-x|^2}\big[w(s,y)-w(t,x)$$
$$-q(s-t)-\langle\, p,y-x\,\rangle-\frac{1}{2}(y-x)^\top P(y-x)\big]\leqslant 0\Big\}.$$

The following is about the characterization of $D^{1,2,+}_{t+,x}w(t,x)$.

Lemma 5.5. *Let $w\in C([0,T]\times\mathbb{R}^n)$ and $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ be given. Then, $(q,p,P)\in D^{1,2,+}_{t+,x}w(t_0,x_0)$ if and only if there exists a function $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that, for any $(t,x)\in[t_0,T]\times\mathbb{R}^n$, $(t,x)\neq(t_0,x_0)$, $\varphi(t,x)>w(t,x)$, and*

$$\big(\varphi(t_0,x_0),\varphi_t(t_0,x_0),\varphi_x(t_0,x_0),\varphi_{xx}(t_0,x_0)\big)=\big(w(t_0,x_0),q,p,P\big).$$

Moreover, if for some $k\geqslant 1$, $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$|w(t,x)|\leqslant C(1+|x|^k),\tag{5.9}$$

then we can choose φ such that φ, φ_t, φ_x, φ_{xx} also satisfy (5.9) with different constants C.

Next, similar to [35], we gather the functions φ in Lemma 5.5 as follows.

$$\mathfrak{I}(q,p,P;w)(t,x)=\Big\{\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)\mid w-\varphi\text{ attains a strict maximum over }[0,T]\times\mathbb{R}^n\text{ at }(t,x)$$
$$\text{and }\big(\varphi(t,x),\varphi_t(t,x),\varphi_x(t,x),\varphi_{xx}(t,x)\big)=\big(w(t,x),q,p,P\big)\Big\}.$$

The following result can be found in [14, 36].

Lemma 5.6. *Given $g\in C([0,T])$. Let's extend g to $(-\infty,+\infty)$ by setting $g(t)=\begin{cases}g(0),\ t<0\\ g(t),\ t\in[0,T].\\ g(T),\ t>T\end{cases}$ Suppose that for all $\delta\in(0,T)$, there is a function $\rho(\cdot)\in L^1(0,T-\delta;\mathbb{R})$ and some $h_2>0$, such that*

$$\frac{g(t+h)-g(t)}{h}\leqslant\rho(t),\ \text{a.e.},\ t\in[0,T-\delta),\ h\leqslant h_2,$$

then,

$$g(\beta)-g(\alpha)\leqslant\int_\alpha^\beta\varlimsup_{h\to0^+}\frac{g(t+h)-g(t)}{h}dt,\qquad 0\leqslant\alpha<\beta\leqslant T-\delta.$$

Now, we give the stochastic verification theorem within the framework of viscosity solutions.

Theorem 5.7. *Assume that the conditions* **(H$_1$)-(H$_3$)** and **(C)** *hold. Let $W(\cdot,\cdot)\in C_1([0,T]\times\mathbb{R}^n)$ be the viscosity solution of the PIDE (3.6). For any $\delta>0$, assume that $W(\cdot,\cdot)$ is semi-concave in $x\in\mathbb{R}^n$ and Lipschitz continuous in $t\in[0,T-\delta]$, uniformly in x. For an initial pair $(t,x)\in[0,T]\times\mathbb{R}^n$, let $(\bar u(\cdot),X^{t,x,\bar u}_\cdot)$ be an admissible pair and $(Y^{t,x,\bar u}_\cdot,Z^{t,x,\bar u}_\cdot,V^{t,x,\bar u}_\cdot,A^{t,x,\bar u}_\cdot)$ solve RBSDE with jumps (3.2) under the control process*

$\bar{u}(\cdot)\in\mathcal{U}_{t,T}$. Suppose there exists a triple $\big(\bar{q},\bar{p},\bar{P}\big)\in\mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{R})\times\mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{R}^n)\times\mathcal{M}^2_{\mathbb{F}}(t,T;\mathbb{S}^n)$ such that the following conditions hold,

$$\begin{cases}\text{(i) } \big(\bar{q}(s),\bar{p}(s),\bar{P}(s)\big)\in D^{1,2,+}_{t+,x}W\big(s,X^{t,x,\bar{u}}_s\big),\ \text{a.e. } s\in[t,T],\ \mathbb{P}\text{-a.s.};\\ \text{(ii) } \bar{p}(s).\sigma\big(s,X^{t,x,\bar{u}}_s,\bar{u}_s\big)=Z^{t,x,\bar{u}}_s,\ \text{a.e. } s\in[t,T],\ \mathbb{P}\text{-a.s.};\\ \text{(iii) } W\big(s,X^{t,x,\bar{u}}_{s-}+\gamma(s,X^{t,x,\bar{u}}_{s-},\bar{u}_s,e)\big)-W\big(s,X^{t,x,\bar{u}}_{s-}\big)=V^{t,x,\bar{u}}_s(e),\ e\in E,\ \text{a.e., } s\in[t,T],\ \mathbb{P}\text{-a.s.};\\ \text{(iv) } \textit{For some } \varphi(\cdot,\cdot)\in\mathfrak{I}(\bar{q},\bar{p},\bar{P};W)\big(s,X^{t,x,\bar{u}}_s\big),\\ \qquad \mathbb{E}\Big[\displaystyle\int_t^T\Big(\bar{q}(s)+\mathcal{H}\big(s,X^{t,x,\bar{u}}_s,Y^{t,x,\bar{u}}_s,\bar{p}(s),\bar{P}(s),\mathcal{B}^{\bar{u}}\varphi(s,X^{t,x,\bar{u}}_s),\mathcal{C}^{\bar{u}}\varphi(s,X^{t,x,\bar{u}}_s),\bar{u}_s\big)\Big)\mathrm{d}s\Big]\leqslant 0.\end{cases}$$

Then, $\bar{u}(\cdot)$ is an optimal control of Problem $(C)_{t,x}$.

Proof. Firstly, from the uniqueness of the viscosity solution of (3.6) (Thm. 3.12), we know, for any $(t,x)\in[0,T]\times\mathbb{R}^n$ and $u(\cdot)\in\mathcal{U}_{t,T}$

$$W(t,x)\leqslant J(t,x;u(\cdot)). \tag{5.10}$$

For any fixed $t_0\in[t,T]$ satisfied (i) and given $\omega_0=(\omega_0',p_0')\in\Omega$, we construct a new probability space $\big(\Omega,\mathcal{F},\mathbb{P}(\cdot\mid\mathcal{F}^t_{t_0})(\omega_0)\big)$ equipped with a new filtration $\{\mathcal{F}^{t_0}_s\}_{t_0\leqslant s\leqslant T}$. Here, for any $\varsigma,\varsigma'\in[t,T]$ satisfied $\varsigma\leqslant\varsigma'$, define $\mathcal{F}^{\varsigma}_{\varsigma'}$ as $\mathcal{F}^{B,\varsigma}_{\varsigma'}\otimes\mathcal{F}^{\varsigma}_{\varsigma'}$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$, where

$$\mathcal{F}^{B,\varsigma}_{\varsigma'}:=\sigma\{B_r:\varsigma\leqslant r\leqslant\varsigma'\}\vee\mathcal{N}_{\mathbb{P}_1},\quad \mathcal{F}^{\mu,\varsigma}_{\varsigma'}:=\Big(\bigcap_{s>\varsigma'}\sigma\{N((\tau,r]\times\Delta),\varsigma\leqslant\tau\leqslant r\leqslant s,\Delta\in\mathcal{B}(E)\}\Big)\vee\mathcal{N}_{\mathbb{P}_2}.$$

We define, for any $A\in\mathcal{F}$ and $\omega\in\Omega$, $\mathbb{P}(A\mid\mathcal{F}^t_{t_0})(\omega):=\mathbb{E}[\mathbf{1}_A\mid\mathcal{F}^t_{t_0}](\omega)$. Since the underlying probability space is standard Borel, this yields a regular conditional probability. That is, for each ω, $\mathbb{P}(\cdot\mid\mathcal{F}^t_{t_0})(\omega)$ is a probability measure, and for each $A\in\mathcal{F}$, $\mathbb{P}(A\mid\mathcal{F}^t_{t_0})(\cdot)$ is a version of the conditional expectation.

In this new probability space, for the fixed ω_0, the regular conditional probability $\mathbb{P}(\cdot\mid\mathcal{F}^t_{t_0})(\omega_0)$ serves as the probability measure on the new space. The corresponding expectation $\mathbb{E}_{t,t_0}$ is related to the expectation $\mathbb{E}$ of the initial probability space as follows,

$$\mathbb{E}_{t,t_0}[\cdot]=\mathbb{E}\big[\cdot\mid\mathcal{F}^t_{t_0}(\omega_0)\big](\omega_0).$$

Furthermore, it is easy to check that $\{B_s\}_{s\geqslant t_0}$ is still a standard Brownian motion with $B_{t_0}=B_{t_0}(\omega_0')$ almost surely, and μ is a Poisson random measure restricted to $[t_0,T]$ with $N((t,t_0],\Delta)=N(p_0',(t,t_0]\times\Delta)$, $\Delta\in\mathcal{B}(E)$. And, the control process $\bar{u}(\cdot)$ is adapted to the new filtration. For ω_0, the process $X^{t,x,\bar{u}}$ is a solution of (3.1) on $[t_0,T]$ in $\big(\Omega,\mathcal{F},\mathbb{P}(\cdot\mid\mathcal{F}^t_{t_0})(\omega_0)\big)$ with the initial condition $X^{t,x,\bar{u}}_{t_0}=X^{t,x,\bar{u}}_{t_0}(\omega_0)$.

From now on, we keep in mind that (t_0,ω_0) is fixed. From $\big(\bar{q}(t_0),\bar{p}(t_0),\bar{P}(t_0)\big)\in D^{1,2,+}_{t+,x}W\big(t_0,X^{t,x,\bar{u}}_{t_0}\big)$ and Lemma 5.5, we know there exists a function $\bar{\varphi}(t_0,\omega_0,\cdot,\cdot)\in\mathfrak{I}(\bar{q},\bar{p},\bar{P};W)\big(t_0,X^{t,x,\bar{u}}_{t_0}\big)$. Furthermore, the linear growth of $W(\cdot,\cdot)$ in Proposition 3.3 implies us $\bar{\varphi}$, $\bar{\varphi}_t$, $\bar{\varphi}_x$, $\bar{\varphi}_{xx}$ are also linear growth in x, *i.e.*,

$$|\bar{\varphi}(t_0,\omega_0,t,x)|+|\bar{\varphi}_t(t_0,\omega_0,t,x)|+|\bar{\varphi}_x(t_0,\omega_0,t,x)|+|\bar{\varphi}_{xx}(t_0,\omega_0,t,x)|\leqslant C(1+|x|),\quad (t,x)\in[0,T]\times\mathbb{R}^n. \tag{5.11}$$

Obviously, $\bar{\varphi}$ is deterministic on the space $\big(\Omega, \mathcal{F}, \mathbb{P}(\cdot \mid \mathcal{F}^t_{t_0})(\omega_0)\big)$. Then, for any $h > 0$, applying Itô's formula to $\bar{\varphi}\big(\cdot, X^{t,x,\bar{u}}_{\cdot}\big)$ on $[t_0, t_0 + h]$, we obtain

$$\begin{aligned}
&\bar{\varphi}\big(t_0 + h, X^{t,x,\bar{u}}_{t_0+h}\big) - \bar{\varphi}\big(t_0, X^{t,x,\bar{u}}_{t_0}\big)\\
&= \int_{t_0}^{t_0+h} \Big(\bar{\varphi}_t\big(r, X^{t,x,\bar{u}}_r\big) + \mathcal{L}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r) + \mathcal{B}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r)\Big)\mathrm{d}r + \int_{t_0}^{t_0+h} \bar{\varphi}_x\big(r, X^{t,x,\bar{u}}_r\big).\sigma\big(r, X^{t,x,\bar{u}}_r, \bar{u}_r\big)\mathrm{d}B_r\\
&\quad + \int_{t_0}^{t_0+h} \int_E \Big(\bar{\varphi}\big(r, X^{t,x,\bar{u}}_{r-} + \gamma(r, X^{t,x,\bar{u}}_{r-}, \bar{u}_r, e)\big) - \bar{\varphi}(r, X^{t,x,\bar{u}}_{r-})\Big)\tilde{N}(\mathrm{d}r, \mathrm{d}e).
\end{aligned}$$

Taking the expectation $\mathbb{E}_{t,t_0}[\cdot]$, we get

$$\begin{aligned}
&\mathbb{E}_{t,t_0}\Big[W\big(t_0 + h, X^{t,x,\bar{u}}_{t_0+h}\big) - W\big(t_0, X^{t,x,\bar{u}}_{t_0}\big)\Big] \leqslant \mathbb{E}_{t,t_0}\Big[\bar{\varphi}\big(t_0 + h, X^{t,x,\bar{u}}_{t_0+h}\big) - \bar{\varphi}\big(t_0, X^{t,x,\bar{u}}_{t_0+h}\big)\Big]\\
&= \mathbb{E}_{t,t_0}\Big[\int_{t_0}^{t_0+h} \Big(\bar{\varphi}_t\big(r, X^{t,x,\bar{u}}_r\big) + \mathcal{L}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r) + \mathcal{B}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r)\Big)\mathrm{d}r\Big].
\end{aligned} \tag{5.12}$$

Next, we will estimate $\overline{\lim_{h\to 0^+}} \frac{1}{h}\mathbb{E}_{t,t_0}\Big[W\big(t_0 + h, \bar{X}_{t_0+h}\big) - W\big(t_0, \bar{X}_{t_0}\big)\Big]$. Before that, we make the following preparations. For all $h \in (0, T - t_0)$, according to (5.11) and Lemma 3.1,

$$\begin{aligned}
&\int_{t_0}^{t_0+h} \Big|\mathbb{E}_{t,t_0}[\mathcal{L}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r)]\Big|\mathrm{d}r\\
&\leqslant \mathbb{E}_{t,t_0}\Big[\int_{t_0}^{t_0+h} \Big|\bar{\varphi}_x\big(r, X^{t,x,\bar{u}}_r\big).b\big(r, X^{t,x,\bar{u}}_r, \bar{u}_r\big) + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^\top(r, X^{t,x,\bar{u}}_r, \bar{u}_r)\bar{\varphi}_{xx}(r, X^{t,x,\bar{u}}_r)\right|\mathrm{d}r\Big]\\
&\leqslant C\mathbb{E}_{t,t_0}\Big[\int_{t_0}^{t_0+h} \big(1 + |X^{t,x,\bar{u}}_r| + |X^{t,x,\bar{u}}_r|^2 + |X^{t,x,\bar{u}}_r|^3\big)\mathrm{d}r\Big] < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t_0}^{t_0+h} \Big|\mathbb{E}_{t,t_0}[\mathcal{B}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r)]\Big|\mathrm{d}r\\
&\leqslant \mathbb{E}_{t,t_0}\Big[\int_{t_0}^{t_0+h}\Big| \int_E\int_0^1 \big\langle(1-\theta)\bar{\varphi}_{xx}\big(r, X^{t,x,\bar{u}}_r + \theta\gamma(r, X^{t,x,\bar{u}}_r, \bar{u}_r, e)\big)\gamma(r, X^{t,x,\bar{u}}_r, \bar{u}_r, e), \gamma(r, X^{t,x,\bar{u}}_r, \bar{u}_r, e)\big\rangle\mathrm{d}\theta\nu(\mathrm{d}e)\Big|\mathrm{d}r\Big]\\
&\leqslant C\int_{t_0}^{t_0+h} \Big|\mathbb{E}_{t,t_0}\Big[\int_E \ell^2(e)(\ell(e) + 1)(1 + |X^{t,x,\bar{u}}_r|^2 + |X^{t,x,\bar{u}}_r|^3)\nu(\mathrm{d}e)\Big]\Big|\mathrm{d}r < +\infty.
\end{aligned}$$

Then, from the Lebesgue differentiation theorem, we get

$$\begin{aligned}
&\overline{\lim_{h\to 0^+}} \frac{1}{h}\mathbb{E}_{t,t_0}\Big[W\big(t_0 + h, X^{t,x,\bar{u}}_{t_0+h}\big) - W\big(t_0, X^{t,x,\bar{u}}_{t_0}\big)\Big]\\
&\leqslant \overline{\lim_{h\to 0^+}} \frac{1}{h}\mathbb{E}_{t,t_0}\Big[\int_{t_0}^{t_0+h} \Big(\bar{\varphi}_t\big(r, X^{t,x,\bar{u}}_r\big) + \mathcal{L}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r) + \mathcal{B}^{\bar{u}}\bar{\varphi}(r, X^{t,x,\bar{u}}_r)\Big)\mathrm{d}r\Big]\\
&= \bar{\varphi}_t\big(t_0, X^{t,x,\bar{u}}_{t_0}\big) + \mathcal{L}^{\bar{u}}\bar{\varphi}(t_0, X^{t,x,\bar{u}}_{t_0}) + \mathcal{B}^{\bar{u}}\bar{\varphi}(t_0, X^{t,x,\bar{u}}_{t_0})\\
&= \bar{q}(t_0) + \bar{p}(t_0).b\big(t_0, X^{t,x,\bar{u}}_{t_0}, \bar{u}_{t_0}\big) + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^\top(t_0, X^{t,x,\bar{u}}_{t_0}, \bar{u}_{t_0})\bar{P}(t_0)\right) + \mathcal{B}^{\bar{u}}\bar{\varphi}(t_0, X^{t,x,\bar{u}}_{t_0}).
\end{aligned} \tag{5.13}$$

Next, we claim that, for all $\delta\in(0,T)$ and the previous t_0 lying in $[t,T-\delta)$, for any $h>0$ with $t_0+h\leqslant T-\delta$,

$$\begin{aligned}&\text{(a) } \frac{1}{h}\mathbb{E}_{t,t_0}\Big[W\big(t_0+h,X^{t,x,\bar u}_{t_0+h}\big)-W\big(t_0,X^{t,x,\bar u}_{t_0}\big)\Big]\leqslant C(1+|X^{t,x,\bar u}_{t_0}|^2),\quad \mathbb{P}\text{-a.s.};\\ &\text{(b) } \frac{1}{h}\mathbb{E}\Big[W\big(t_0+h,X^{t,x,\bar u}_{t_0+h}\big)-W\big(t_0,X^{t,x,\bar u}_{t_0}\big)\Big]\leqslant C(1+|x|^2).\end{aligned}\tag{5.14}$$

In fact, from Theorems 4.22 and 4.4 as well as $\big(\bar q(t_0),\bar p(t_0),\bar P(t_0)\big)\in D^{1,2,+}_{t+,x}W(t_0,X^{t,x;\bar u}_{t_0})$, we know, for all $\delta\in(0,T)$, for any $h\in(0,T-t_0-\delta]$,

$$W\big(t_0+h,X^{t,x,\bar u}_{t_0+h}\big)-W\big(t_0,X^{t,x,\bar u}_{t_0}\big)\leqslant \mathrm{I}+\mathrm{II}+\mathrm{III},$$

with

$$\begin{cases}\mathrm{I}:=C\big(1+|X^{t,x,\bar u}_{t_0+h}|\big)h,\\ \mathrm{II}:=\big\langle \bar p(t_0),X^{t,x,\bar u}_{t_0+h}-X^{t,x,\bar u}_{t_0}\big\rangle,\\ \mathrm{III}:=C|X^{t,x,\bar u}_{t_0+h}-X^{t,x,\bar u}_{t_0}|^2.\end{cases}$$

Using (5.11), we have $|\bar q(t_0)|+|\bar p(t_0)|+|\bar P(t_0)|\leqslant C(1+|X^{t,x,\bar u}_{t_0}|)$. Furthermore, combine with Lemma 3.1, we get

$$\begin{aligned}&\mathbb{E}_{t,t_0}\big[\mathrm{I}\big]\leqslant Ch+Ch\Big(\mathbb{E}_{t,t_0}\Big[\sup_{r\in[t_0,t_0+h]}|X^{t,x,\bar u}_r|^2\Big]\Big)^{\frac12}\leqslant Ch(1+|X^{t,x,\bar u}_{t_0}|),\\ &\mathbb{E}_{t,t_0}\big[\mathrm{II}\big]\leqslant \mathbb{E}_{t,t_0}\Big[\Big\langle\bar p(t_0),\int_{t_0}^{t_0+h}b\big(r,X^{t,x,\bar u}_r,\bar u_r\big)\mathrm{d}r\Big\rangle\Big]\leqslant\Big(\mathbb{E}_{t,t_0}\big[|\bar p(t_0)|^2\big]\Big)^{\frac12}\Big(\mathbb{E}_{t,t_0}\Big[\Big|\int_{t_0}^{t_0+h}b\big(r,X^{t,x,\bar u}_r,\bar u_r\big)\mathrm{d}r\Big|^2\Big]\Big)^{\frac12}\\ &\qquad\qquad\leqslant Ch(1+|X^{t,x,\bar u}_{t_0}|^2),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_{t,t_0}\big[\mathrm{III}\big]&=C\mathbb{E}_{t,t_0}\big[|X^{t,x,\bar u}_{t_0+h}-X^{t,x,\bar u}_{t_0}|^2\big]\\ &\leqslant C\mathbb{E}_{t,t_0}\Big[\Big(\int_{t_0}^{t_0+h}b\big(r,X^{t,x,\bar u}_r,\bar u_r\big)\mathrm{d}r\Big)^2\Big]+C\mathbb{E}_{t,t_0}\Big[\Big(\int_{t_0}^{t_0+h}\sigma\big(r,X^{t,x,\bar u}_r,\bar u_r\big)\mathrm{d}B_r\Big)^2\Big]\\ &\quad+C\mathbb{E}_{t,t_0}\Big[\Big(\int_{t_0}^{t_0+h}\int_E\gamma\big(r,X^{t,x,\bar u}_r,\bar u_r,e\big)\tilde N(\mathrm{d}r,\mathrm{d}e)\Big)^2\Big]\leqslant Ch(1+|X^{t,x,\bar u}_{t_0}|^2).\end{aligned}$$

Therefore,

$$\frac1h\mathbb{E}_{t,t_0}\Big[W\big(t_0+h,X^{t,x,\bar u}_{t_0+h}\big)-W\big(t_0,X^{t,x,\bar u}_{t_0}\big)\Big]\leqslant\frac1h\mathbb{E}_{t,t_0}\Big[\mathrm{I}+\mathrm{II}+\mathrm{III}\Big]\leqslant C(1+|X^{t,x,\bar u}_{t_0}|+|X^{t,x,\bar u}_{t_0}|^2).$$

All the above constants C can be different and do not depend on t_0. Furthermore, by taking the expectation on the both sides of the above inequality, we get (5.14)-(b).

Taking expectation on the both sides of (5.13), and applying Fatou's Lemma (needing (5.14)-(a)), we have

$$
\begin{aligned}
&\overline{\lim_{h\to 0^+}}\,\frac{1}{h}\mathbb{E}\Big[W\big(t_0+h, X^{t,x,\bar u}_{t_0+h}\big) - W\big(t_0, X^{t,x,\bar u}_{t_0}\big)\Big]\\
&=\overline{\lim_{h\to 0^+}}\,\frac{1}{h}\mathbb{E}\Big[\mathbb{E}_{t,t_0}\big[W\big(t_0+h, X^{t,x,\bar u}_{t_0+h}\big) - W\big(t_0, X^{t,x,\bar u}_{t_0}\big)\big]\Big]\\
&\leqslant \mathbb{E}\Big[\overline{\lim_{h\to 0^+}}\,\frac{1}{h}\mathbb{E}_{t,t_0}\big[W\big(t_0+h, X^{t,x,\bar u}_{t_0+h}\big) - W\big(t_0, X^{t,x,\bar u}_{t_0}\big)\big]\Big]\\
&=\mathbb{E}\Big[\bar q(t_0)+\bar p(t_0).b\big(t_0, X^{t,x,\bar u}_{t_0}, \bar u_{t_0}\big)+\frac{1}{2}\mathrm{tr}\,\big(\sigma\sigma^\top(t_0, X^{t,x,\bar u}_{t_0}, \bar u_{t_0})\bar P(t_0)\big)+\mathcal{B}^{\bar u}\bar\varphi\big(t_0, X^{t,x,\bar u}_{t_0}\big)\Big].
\end{aligned}
$$

Due to the set of such points t_0 being of full measure in $[t, T-\delta]$, by applying Lemma 5.6 (needing (5.14)-(b)), for any $u(\cdot)\in\mathcal{U}_{t,T}$, we have

$$
\begin{aligned}
&\mathbb{E}\big[W\big(T-\delta, X^{t,x,\bar u}_{T-\delta}\big) - W(t,x)\big]\\
&=\int_t^{T-\delta}\mathbb{E}\Big[\bar q(s)+\bar p(s).b\big(s, X^{t,x,\bar u}_s, \bar u_s\big)+\frac{1}{2}\mathrm{tr}\,\big(\sigma\sigma^\top(s, X^{t,x,\bar u}_s, \bar u_s)\bar P(s)\big)+\mathcal{B}^{\bar u}\bar\varphi\big(s, X^{t,x,\bar u}_s\big)\Big]\mathrm{d}s.
\end{aligned}
$$

According to Lebesgue dominated convergence theorem and the continuity properties of $W(\cdot,\cdot)$ and $X^{t,x;\bar u}_\cdot$, letting $\delta\to 0$ in the above, we get

$$
\begin{aligned}
&\mathbb{E}\big[W\big(T, X^{t,x,\bar u}_T\big) - W(t,x)\big] = \mathbb{E}\big[\Phi\big(X^{t,x,\bar u}_T\big) - W(t,x)\big]\\
&\leqslant \int_t^T\mathbb{E}\Big[\bar q(s)+\bar p(s).b\big(s, X^{t,x,\bar u}_s, \bar u_s\big)+\frac{1}{2}\mathrm{tr}\,\big(\sigma\sigma^\top(s, X^{t,x,\bar u}_s, \bar u_s)\bar P(s)\big)+\mathcal{B}^{\bar u}\bar\varphi\big(s, X^{t,x,\bar u}_s\big)\Big]\mathrm{d}s\\
&\leqslant -\mathbb{E}\Big[\int_t^T f\big(s, X^{t,x,\bar u}_s, Y^{t,x,\bar u}_s, \bar p(s).\sigma(s, X^{t,x;\bar u}_s, \bar u_s), \mathcal{C}^{\bar u}W(s, X^{t,x,\bar u}_s), \bar u_s\big)\mathrm{d}s\Big]\\
&\leqslant -\mathbb{E}\Big[\int_t^T f\big(s, X^{t,x,\bar u}_s, Y^{t,x,\bar u}_s, Z^{t,x,\bar u}_s, \int_E l(e)V^{t,x,\bar u}_s(e)\nu(\mathrm{d}e), \bar u_s\big)\mathrm{d}s - A^{t,x,\bar u}_T\Big],
\end{aligned}
$$

where we have used the conditions (ii), (iii), (iv) and $A^{t,x,\bar u}_\cdot\in\mathcal{A}^2_{\mathbb{F}}(t,T;\mathbb{R})$.

Thus, for any $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$
W(t,x)\geqslant\mathbb{E}\Big[\Phi\big(X^{t,x;\bar u}_T\big)+\int_t^T f\big(s, X^{t,x,\bar u}_s, Y^{t,x,\bar u}_s, Z^{t,x,\bar u}_s, \int_E l(e)V^{t,x,\bar u}_s(e)\nu(\mathrm{d}e), \bar u_s\big)\mathrm{d}s - A^{t,x,\bar u}_T\Big] = J\big(t,x;\bar u(\cdot)\big).
$$

Combined with (5.10), we get $W(t,x) = J\big(t,x;\bar u(\cdot)\big)$, which means $\bar u(\cdot)$ is an optimal control of Problem $(\mathrm{C})_{t,x}$. ∎

Now, we shall construct the feedback optimal control of Problem $(\mathrm{C})_{t,x}$ from the viscosity solution of PIDE (3.6).

Lemma 5.8. *Assume* **(H1)** *and* **(H2)** *hold. Then the value function $W(\cdot,\cdot)$ defined by* (3.4) *is the only function in Θ, which satisfies the following: for any $(t,x)\in[0,T]\times\mathbb{R}^n$, $(q,p,P)\in D^{1,2,+}_{t+,x}W(t,x)$ and $\varphi\in\mathfrak{I}(q,p,P;W)(t,x)$*

$$
\max\big\{W(t,x)-h(t,x), -q-\inf_{u\in U}\mathcal{H}\big(t,x,W(t,x),p,P,\mathcal{B}^u\varphi(t,x),\mathcal{C}^u\varphi(t,x),u\big)\big\}\leqslant 0. \tag{5.15}
$$

Proof. From Lemma 3.12, we know the value function $W(\cdot,\cdot)\in\Theta$ is the unique viscosity solution of (3.6). Then, based on Lemma 5.5, we can obtain some test function $\varphi\in\mathfrak{I}(q,p,P;W)(t,x)$. From the Definition 3.7-(i), we have

$$\max\Big\{W(t,x)-h(t,x),-\frac{\partial\varphi}{\partial t}(t,x)-\inf_{u\in U}\mathcal{H}\big(t,x,W(t,x),\varphi_x(t,x),\varphi_{xx}(t,x),\mathcal{B}^u\varphi(t,x),\mathcal{C}^u\varphi(t,x),u\big)\Big\}\leqslant 0.$$

And thus, (5.15) holds. Further, the uniqueness comes from the uniqueness of the viscosity solution of (3.6) in Θ. ∎

Theorem 5.9. *Assume that the conditions* $\mathbf{(H_1)}$–$\mathbf{(H_3)}$ and $\mathbf{(C)}$ *hold. Let $W(\cdot,\cdot)\in C_1([0,T]\times\mathbb{R}^n)$ be the viscosity solution of the PIDE (3.6). For any $\delta>0$, assume that $W(\cdot,\cdot)$ is semi-concave in $x\in\mathbb{R}^n$ and Lipschitz continuous in $t\in[0,T-\delta]$, uniformly in x. Then, for each $(t,x)\in[0,T]\times\mathbb{R}^n$,*

$$\inf_{(q,p,P,\varphi,u)\in D^{1,2,+}_{t+,x}W(t,x)\times\mathfrak{I}(q,p,P;W)(t,x)\times U}\Big[q+\mathcal{H}\big(t,x,W(t,x),p,P,\mathcal{B}^u\varphi(t,x),\mathcal{C}^u\varphi(t,x),u\big)\Big]\geqslant W(t,x)-h(t,x).$$

Furthermore, if $\mathbb{u}(\cdot,\cdot)\in\mathscr{U}_{t,T}$ and for all $(t,x)\in[0,T]\times\mathbb{R}^n$, $\mathbb{q}$, $\mathbb{p}$, $\mathbb{P}$ are measurable functions satisfying $(\mathbb{q}(t,x),\mathbb{p}(t,x),\mathbb{P}(t,x))\in D^{1,2,+}_{t+,x}W(t,x)$, and

$$\begin{cases}\text{(i) } \mathbb{p}(s,X_s^{t,x,\mathbb{u}}).\sigma\big(s,X_s^{t,x,\mathbb{u}},\mathbb{u}(s,X_s^{t,x,\mathbb{u}})\big)=\mathbb{Z}_s^{t,x,\mathbb{u}},\ \text{ a.e. } s\in[t,T],\ \mathbb{P}\text{-a.s.};\\ \text{(ii) } W\big(s,X_s^{t,x,\mathbb{u}}+\gamma(s,X_s^{t,x,\mathbb{u}},\mathbb{u}(s,X_s^{t,x,\mathbb{u}}),e)\big)-W(s,X_s^{t,x,\mathbb{u}})=V_s^{t,x,\mathbb{u}}(e),\ e\in E;\\ \text{(iii) } \mathbb{E}\Big[\displaystyle\int_t^T\Big(\mathbb{q}(s,X_s^{t,x,\mathbb{u}})+\mathcal{H}\big(s,X_s^{t,x,\mathbb{u}},Y_s^{t,x,\mathbb{u}},\Xi(s,X_s^{t,x,\mathbb{u}}),\mathcal{B}^{\mathbb{u}}\varphi(s,X_s^{t,x,\mathbb{u}}),\mathcal{C}^{\mathbb{u}}\varphi(s,X_s^{t,x,\bar{u}}),\mathbb{u}(s,X_s^{t,x,\mathbb{u}})\big)\Big)\mathrm{d}s\Big]\\ \hfill\leqslant 0,\quad \textit{with some } \varphi(\cdot,\cdot)\in\mathfrak{I}(\mathbb{q},\mathbb{p},\mathbb{P};W)\big(s,X_s^{t,x,\mathbb{u}}\big),\end{cases}\tag{5.16}$$

where $\Xi(\cdot,X_\cdot^{t,x;\mathbb{u}})=\big(\mathbb{p}(\cdot,X^{t,x;\mathbb{u}}(\cdot)),\mathbb{P}(\cdot,X^{t,x;\mathbb{u}}(\cdot))\big)$ and $X_\cdot^{t,x;\mathbb{u}}$, $\big(Y_\cdot^{t,x;\mathbb{u}},Z_\cdot^{t,x;\mathbb{u}},V_\cdot^{t,x;\mathbb{u}}(\cdot),A_\cdot^{t,x;\mathbb{u}}\big)$ satisfy (5.1) and (5.8) with $\mathbb{u}(\cdot,X_\cdot^{t,x;\mathbb{u}})\in\mathcal{U}_{t,T}$, respectively. Then, $\mathbb{u}(\cdot,\cdot)$ is an optimal feedback control law of Problem $(C)_{t,x}$.

Proof. The first result can be directly obtained from the uniqueness of the viscosity solution of PIDE (3.6) and Lemma 5.8.

Subsequently, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, set

$$\bar{u}(s):=\mathbb{u}(s,X_s^{t,x;\mathbb{u}}),\quad \bar{q}(s):=\mathbb{q}(s,X_s^{t,x;\mathbb{u}}),\quad \bar{p}(s):=\mathbb{p}(s,X_s^{t,x;\mathbb{u}}),\quad \bar{P}(s):=\mathbb{P}(s,X_s^{t,x;\mathbb{u}}),\quad s\in[t,T],$$

then, they satisfy (i), (ii) and (iii) in Theorem 5.7. So $\bar{u}(\cdot)$ is an optimal control, *i.e.*, $\mathbb{u}(\cdot,\cdot)$ is an optimal feedback control law. ∎

Remark 5.10. *In Theorems 5.7 and 5.9, we now consistently use the space $C_1([0,T]\times\mathbb{R}^n)$, which is natural for the verification framework. This choice is justified since the value function $W(\cdot,\cdot)$ has been shown to possess linear growth, and hence lies in $C_1([0,T]\times\mathbb{R}^n)\subseteq\Theta$. Uniqueness in Θ thus implies uniqueness in $C_1([0,T]\times\mathbb{R}^n)$ ensuring a consistent transition between the two settings.*

Now, we have a look at the procedures of finding the optimal feedback control law. From Theorem 5.9, we can get the candidate of optimal feedback control law by minimizing

$$q+\mathcal{H}\big(t,x,W(t,x),p,P,\mathcal{B}^u\varphi(t,x),\mathcal{C}^u\varphi(t,x),u\big)$$

over $D^{1,2,+}_{t+,x}W(t,x)\times\mathfrak{I}(q,p,P;W)(t,x)\times U$. Further, to ensure the candidate to be the true optimal feedback control law, we need to make sure the candidate $\mathbb{u}(\cdot,\cdot)$ belongs to $\mathscr{U}_{t,T}$ and (5.16) is vaild under $(\mathbb{q},\mathbb{p},\mathbb{P})\in D^{1,2,+}_{t+,x}W(t,x)$ and $\varphi\in\mathfrak{I}(\mathbb{q},\mathbb{p},\mathbb{P};W)(t,x)$.

Finally, we provide a concrete example to illustrate the applicability of the verification theorem within the viscosity solution framework for determining whether a given admissible control is optimal. For simplicity, we consider the one-dimensional case.

Example 5.11. *Let the control domain be $U = [-4, -2]$, and denote by $\mathcal{U}_{t,T}$ be the set of all admissible controls on $[t, T]$, consisting of U-valued, $\mathbb{F}$-predictable stochastic processes. For any initial pair $(t, x) \in [0, T] \times \mathbb{R}$, consider the controlled system,*

$$\begin{cases} dX_s = X_s u_s \mathrm{d}s + X_s \mathrm{d}B_s + \displaystyle\int_E (1 \wedge |e|) X_{s-} \tilde{N}(\mathrm{d}s, \mathrm{d}e), \quad s \in [t, T], \\ X_t = x, \end{cases} \tag{5.17}$$

and the following RBSDE with jumps,

$$\begin{cases} Y_s = -|X_T| - \displaystyle\int_t^T \big(Y_r + Z_r\big)\mathrm{d}r - (A_T - A_s) + \int_t^T Z_r \mathrm{d}B_r + \int_t^T V_r(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \quad s \in [t, T], \\ Y_s \leqslant -\dfrac{1}{2}|X_s|, \quad \text{a.e. } s \in [t, T], \\ \displaystyle\int_t^T \big(-\frac{1}{2}|X_r| - Y_r\big)\mathrm{d}A_r = 0. \end{cases} \tag{5.18}$$

It is clear that both systems (5.17) and (5.18) are well-posed. Then for any $(t, x) \in [0, T] \times \mathbb{R}$ and $u(\cdot) \in \mathcal{U}_{t,T}$, we define the cost functional as $J(t, x; u(\cdot)) := Y_t$. Our control problem (denoted by Problem $(\mathbb{C})_{t,x}$) is to minimize $J(t, x; u(\cdot))$ over $u(\cdot) \in \mathcal{U}_{t,T}$, and the corresponding value function is denoted by $W(\cdot, \cdot)$.

Next, we introduce the following PIDE of HJB type,

$$\begin{cases} \max\Big\{W(t, x) + \dfrac{1}{2}|x|, -\dfrac{\partial}{\partial t}W(t, x) - \inf\limits_{u \in [-4,-2]}\Big(\dfrac{1}{2}x^2 W_{xx}(t, x) + xuW_x(t, x) + W(t, x) + xW_x(t, x) \\ \qquad + \displaystyle\int_E \big[W\big(t, x + (1 \wedge |e|)x\big) - W(t, x) - (1 \wedge |e|)xW_x(t, x)\big]\nu(\mathrm{d}e)\Big)\Big\} = 0, \ (t, x) \in [0, T] \times \mathbb{R}, \\ W(T, x) = -|x|, \quad x \in \mathbb{R}. \end{cases} \tag{5.19}$$

We first claim that

$$W(t, x) = -|x|, \qquad (t, x) \in [0, T] \times \mathbb{R}, \tag{5.20}$$

is a viscosity solution of PIDE (5.19) in the sense of Definition 3.7.

Indeed, for any $t \in [0, T]$, $x \neq 0$, $W(t, x)$ is differentiable, with

$$W_t(t, x) = 0, \quad W_x(t, x) = \begin{cases} -1, & (t, x) \in [0, T] \times \mathbb{R}^+, \\ 1, & (t, x) \in [0, T] \times \mathbb{R}^-, \end{cases} \quad W_{xx}(t, x) = 0.$$

Substituting these derivatives into PIDE (5.19) shows that the equation is satisfied identically. Therefore, for $x \neq 0$, $W(\cdot, \cdot)$ in (5.20) is a classical solution of (5.19), and hence also a viscosity solution.

When $t \in [0, T]$, $x = 0$, we now verify that $W(\cdot, \cdot)$ is a viscosity solution of (5.19). Note first that the terminal condition $W(T, x) = -|x|$ holds for all $x \in \mathbb{R}$.

To show the subsolution property, fix any $t \in [0, T]$, let $\varphi(\cdot, \cdot) \in C^3_{l,b}([0, T] \times \mathbb{R}; \mathbb{R})$ be such that $W - \varphi$ attains its local maximum at $(t, 0)$. Without loss of generality, we may assume that this local maximum is 0 (otherwise, *we can translate the function* φ*). For any such test functions* $\varphi(\cdot,\cdot)$*, we have*

$$\frac{\partial}{\partial t}\varphi(t,0)=\lim_{\Delta t\to 0}\frac{\varphi(t+\Delta t,0)-\varphi(t,0)}{\Delta t}\geqslant \lim_{\Delta t\to 0}\frac{W(t+\Delta t,0)-W(t,0)}{\Delta t}=0,\quad t\in[0,T].$$

Then,

$$\begin{aligned}&\max\Big\{W(t,x)+\frac{1}{2}|x|,-\frac{\partial}{\partial t}\varphi(t,x)-\inf_{u\in[-4,-2]}\Big(\frac{1}{2}x^2\varphi_{xx}(t,x)+xu\varphi_x(t,x)+W(t,x)+x\varphi_x(t,x)\\&\qquad\qquad+\int_E\big[W\big(t,x{+}(1\wedge|e|)x\big)-W(t,x)-(1\wedge|e|)x\varphi_x(t,x)\big]\nu(\mathrm{de})\Big)\Big\}\Big|_{x=0}\\&=\max\Big\{0,-\frac{\partial}{\partial t}\varphi(t,0)\Big\}=0.\end{aligned}$$

This shows that $W(\cdot,\cdot)$ *is a viscosity subsolution of* (5.19) *at* $(t,0)$ *in the sense of Definition* 3.7*-(i).*

To establish the supersolution property, we consider test functions for which $W-\varphi$ *attains a local minimum at* $(t,0)$*. Following analogous reasoning, we obtain* $\frac{\partial}{\partial t}\varphi(t,0)\leqslant 0$*. Evaluating the HJB operator, we get*

$$\begin{aligned}&\max\Big\{W(t,x)+\frac{1}{2}|x|,-\frac{\partial}{\partial t}\varphi(t,x)-\inf_{u\in[-4,-2]}\Big(\frac{1}{2}x^2\varphi_{xx}(t,x)+xu\varphi_x(t,x)+W(t,x)+x\varphi_x(t,x)\\&\qquad+\int_E\big[W\big(t,x{+}(1\wedge|e|)x\big)-W(t,x)-(1\wedge|e|)x\varphi_x(t,x)\big]\nu(\mathrm{de})\Big)\Big\}\Big|_{x=0}\\&=\max\Big\{0,-\frac{\partial}{\partial t}\varphi(t,0)\Big\}\geqslant 0.\end{aligned}$$

This confirms that $W(\cdot,\cdot)$ *is a viscosity supersolution of* (5.19) *at* $(t,0)$ *according to Definition* 3.7*-(ii). Therefore, from all the above discussions,* $W(\cdot,\cdot)$ *in* (5.20) *is verified to be a viscosity solution of* (5.19)*. Combining both cases, we conclude that* $W(\cdot,\cdot)$ *in* (5.20) *is a viscosity solution of* (5.19) *on* $[0,T]\times\mathbb{R}$*.*

Moreover, it is straightforward to check that $W(\cdot,\cdot)\in C_1([0,T]\times\mathbb{R})$*, is semi-concave in* $x\in\mathbb{R}$ *and Lipschitz continuous in* $t\in[0,T]$*.*

Having established that W *is a viscosity solution, we next verify that the admissible control* $\bar u(\cdot)\equiv -2$ *is optimal for Problem* $(\mathbb{C})_{0,0}$ *by checking the conditions in Theorem* 5.7*. Note that the corresponding trajectory* $\bar X_\cdot = X_\cdot^{t,0;\bar u}\equiv 0$*. In this case,*

$$D^{1,2,+}_{t+,x}W\big(s,\bar X_s\big)=[0,+\infty)\times[-1,1]\times[0,+\infty),\quad s\in[0,T].$$

Taking $\big(\bar q(s),\bar p(s),\bar P(s)\big)=(0,1,1)\in D^{1,2,+}_{t+,x}W\big(s,\bar X(s)\big)$*,* $s\in[0,T]$*, we can readily verify that*

$$\begin{aligned}&\bar p(s).\bar X_s=0=\bar Z_s,\ \text{ a.e. } s\in[t,T],\ \mathbb{P}\text{-a.s.};\\&W\big(s,\bar X_{s-}+(1\wedge|e|)\bar X_{s-}\big)-W\big(s,\bar X_{s-}\big)=0=\bar V_s(e),\ e\in E,\ \text{ a.e., }\ s\in[t,T],\ \mathbb{P}\text{-a.s.}\end{aligned}$$

Moreover, for any $\varphi(\cdot,\cdot)\in\mathfrak{I}(\bar q,\bar p,\bar P;W)\big(s,X_s^{t,x,\bar u}\big)$*, we have* $\mathcal{B}^{\bar u}\varphi(s,\bar X_s)=\mathcal{C}^{\bar u}\varphi(s,\bar X_s)=0$*, and consequently,*

$$\mathbb{E}\Big[\int_t^T\Big(\bar q(s){+}\mathcal{H}\big(s,\bar X_s,\bar Y_s,\bar p(s),\bar P(s),\mathcal{B}^{\bar u}\varphi(s,\bar X_s),\mathcal{C}^{\bar u}\varphi(s,\bar X_s),\bar u_s\big)\Big)\mathrm{d}s\Big]\leqslant 0.$$

This verifies conditions (i)-(iv) of Theorem 5.7. We also note that under the control $\bar{u}(\cdot)$, *the solution* $(\bar{Y}, \bar{Z}, \bar{V}, \bar{A})$ *of* (5.18) *is* $(0,0,0,0)$. *Therefore, by Theorem 5.7, we conclude that* $(\bar{X}, \bar{u}(\cdot))$ *is indeed an optimal pair for Problem* $(\mathbb{C})_{0,0}$.

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