

## ON THE PENALIZATION BY THE PERIMETER IN SHAPE OPTIMIZATION APPLIED TO DIRICHLET INVERSE OBSTACLE PROBLEM

FABIEN CAUBET<sup>1,\*</sup>, MARC DAMBRINE<sup>1</sup> AND JÉRÉMI DARDÉ<sup>2</sup>

**Abstract.** This paper is devoted to the understanding of regularization process in the shape optimization approach to the so-called Dirichlet inverse obstacle problem for elliptic operators. More precisely, we study two different regularizations of the very classical shape optimization approach consisting in minimizing a mismatched functional. The first one is an implicit regularization when working in the class of inclusion having a uniform  $\varepsilon$ -cone property, a natural class in shape optimization. As this regularity is not trivial to guarantee numerically, we discuss the regularization by perimeter penalization. We show that this second regularization provides a stability gain in the minimization process.

**Mathematics Subject Classification.** 35R30, 35B35, 49Q10.

Received May 15, 2025. Accepted December 1, 2025.

### 1. INTRODUCTION

**Motivations.** In this work, we are primarily interested in studying the effect of the penalization by the perimeter, in the context of shape optimization for the Dirichlet inverse obstacle problem. This regularization of Tikhonov type can be seen as an analogous one to the classical inverse Cauchy problem (also called data reconstruction problem) where the adding term is linked with the norm of the unknown and permits to obtain existence and convergence results (see [1] and reference therein). Our aim is to study in details the effect of adding a penalization term using the perimeter when we consider the inverse problem of recovering an unknown inclusion in a domain from the knowledge of boundary measurements, in particular to quantify the stability gain, if such gain exists. In other words, the question that guided us are as follows: does the Tikhonov regularization guarantee, and/or is-it necessary to guarantee, the existence of minimizers, depending on the class of admissible domains with which we are working? Does the addition of this term improve the stability of the numerical reconstruction?

More precisely, we consider a bounded smooth connected open set  $\Omega$  of  $\mathbb{R}^d$  (where  $d \in \mathbb{N}^*$  represents the dimension). In the following,  $\nu$  denotes the outward normal vector to  $\Omega$ . Let  $(g_N, g_D) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$  be a non-trivial given Cauchy pair. The inverse obstacle problem we are interested in reads as follows: find  $\omega$ ,

---

*Keywords and phrases:* Inverse obstacle problem, Tikhonov regularization, perimeter penalization, stability result, shape optimization.

<sup>1</sup> Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France.

<sup>2</sup> Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS, F-31062 Toulouse Cedex 9, France.

\* Corresponding author: [fabien.caubet@univ-pau.fr](mailto:fabien.caubet@univ-pau.fr)

a *sufficiently smooth* domain compactly contained in  $\Omega$ , and  $u \in H^1(\Omega \setminus \bar{\omega})$ , such that

$$\begin{cases} -\Delta u + u &= 0 & \text{in } \Omega \setminus \bar{\omega}, \\ u &= g_D & \text{on } \partial\Omega, \\ \partial_\nu u &= g_N & \text{on } \partial\Omega, \\ u &= 0 & \text{on } \partial\omega. \end{cases} \quad (1.1)$$

Notice that the regularity of the inclusion  $\omega$  will be specified and discussed below.

**Remark 1.1.** The aim of this article is to understand the impact of some regularization methods in solving an inverse problem. This general question is studied using the previous toy model (1.1). The question of treating other physical models is a natural one, and this will require adapting the method described in this paper to each model. Notice that, for example, we are confident that this method could be adapted to the Helmholtz equation.

The Dirichlet inverse obstacle problem is the simplest model of geometrical inverse problems, and contains the two main difficulties inherent in this class of problems: it is non-linear as we search the geometry, and it is ill-posed, as it might fail to have a solution for some pairs of Cauchy data. It is actually exponentially ill-posed, even when the geometry is known (*i.e.* the problem of reconstructing  $u$  from  $(g_N, g_D)$  knowing  $\omega$ , see [2–4]), and therefore highly sensible to small perturbations on the data. Nevertheless, identifiability holds in the class  $\omega$  Lipschitz and  $u$  continuous in a neighborhood of  $\partial\omega$  (and therefore, as a consequence of elliptic regularity, for inclusions with  $C^1$  boundaries): there exists at most one couple  $(\omega, u)$  satisfying (1.1) (see, among others, [5], Theorem 5.1 for the Laplace operator, or [6], Proposition 1 for a detailed proof in the more complex case of the Stokes system). Thus, it is reasonable to design algorithm to try to reconstruct  $\omega$  and  $u$  from the knowledge of a data  $(g_N, g_D)$  possibly affected by some noise. From now on, we will say that the data  $(g_N, g_D)$  is *compatible* if (1.1) admits a solution, and is *not compatible* otherwise.

The ill-posedness of our problem of interest is noticeable in the very poor stability estimate one can obtain for compatible data. Indeed, suppose that we have two compatible sets of data with, for simplicity, the same Neumann condition:  $(g_N, g_1)$  with corresponding solution  $(\omega_1, u_1)$ , and  $(g_N, g_2)$  with corresponding solution  $(\omega_2, u_2)$ . Set  $A_1 = \Omega \setminus \bar{\omega}_1$  and  $A_2 = \Omega \setminus \bar{\omega}_2$ . There exists a modulus of continuity  $\gamma$  with  $\gamma(t) \leq C \log(t)^\eta$  where  $C$  and  $\eta \in (0, 1]$  depend on  $\Omega$  such that (see the reference paper by Alessandrini and al [7], Thm. 2-1, p. 763)

$$\|g_1 - g_2\|_{L^2(\partial\Omega)} \leq \varepsilon \implies d_{\mathcal{H}}(A_1, A_2) \leq \gamma\left(\frac{\varepsilon}{\|g_N\|_{H^{-1/2}(\partial\Omega)}}\right), \quad (1.2)$$

where  $d_{\mathcal{H}}$  is the so-called Hausdorff distance. Such stability result is optimal (see [7]), hence the very poor stability (logarithmic stability) of the Dirichlet inverse problem.

**A shape optimization approach.** Numerous methods have been developed to tackle this problem (see, *e.g.*, [8–12], and the references therein). In our study, we focus on the shape optimization approach (used, *e.g.*, in [1, 13]), with a very standard misfit function. More precisely, let us define a set of admissible domains

$$\mathcal{A} := \{A = \Omega \setminus \bar{\omega}, \omega \text{ open}, \omega \subset K, \omega \text{ Lipschitz}\}.$$

Here,  $K$  is a known compact set satisfying the assumption  $\omega \subset K \subset \Omega$ . It is classically used to prevent the open sets to be arbitrary close to the boundary of  $\Omega$ , which would add technical difficulties. We consider the problem of minimizing the least-squares functional  $\mathcal{L} : \mathcal{A} \rightarrow \mathbb{R}^+$  given, for all  $A \in \mathcal{A}$ , by

$$\mathcal{L}(A) = \frac{1}{2} \int_{\partial\Omega} |u - g_D|^2,$$

where  $u \in H^1(A)$  is the solution of the following problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } A = \Omega \setminus \bar{\omega}, \\ \partial_\nu u = g_N & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\omega. \end{cases} \quad (1.3)$$

The classical underlying idea of the approach is that if  $A$  satisfies  $\mathcal{L}(A) = 0$ , then  $\omega = \Omega \setminus \bar{A}$  is indeed the searched obstacle as  $(\omega, u)$  satisfies (1.1). Equivalently, if the Dirichlet inverse obstacle problem admits a solution, then  $\mathcal{L}$  admits a minimizer  $A$  and  $\mathcal{L}(A) = 0$ . Now, if the data is corrupted by some noise (which is obviously the case in practical situations), the situation gets complicated: if the corrupted data is still compatible,  $\mathcal{L}$  still admits 0 as minimum, but the corresponding minimizer might be distant from the correct obstacle due to the very low stability of the problem. And if the corrupted data is not compatible, the infimum of  $\mathcal{L}$  may be still zero but it may be that no minimizer exists.

As expected, there is no miracle: the problem of minimizing  $\mathcal{L}$  is very weakly stable. The instability of the reconstruction with respect to the measurements expressed by the result of Alessandrini *et al.* (1.2) is found in the shape optimization problem. This reformulation in term of shape optimization has no regularizing effect. Afraites *et al.* have studied this in [13] by a second order shape sensitivity analysis and showed that the Riesz representative of the shape Hessian is compact: the difficulty lays at the continuous level. This fact has also been observed for the Stokes cases [14], or in the Navier–Stokes case [15] and one can expect it is true for general elliptic operators. On the example of a ring configuration [13], Proposition 3, p. 399, the eigenvalues of this shape Hessian can be explicitly calculated and are exponentially decreasing. This means that, in general, this reformulation of the original inverse problem in terms of shape optimization is expected to have a logarithmic stability similar to (1.2) for compatible data. In practice, when one considers a numerical minimization scheme for  $\mathcal{L}$ , one does not see such poor stability: when one goes in the iterations of the descent method, the boundary of the reconstructed inclusion oscillates more and more. This clearly underlines why a regularization is needed.

**The regularization methods.** Here, we study two types of regularization of this shape optimization formulation. The first one consists in enforcing more regularity on the admissible sets  $\omega$ , by forcing them to satisfy an *uniform  $\varepsilon$ -cone condition*. In other word, we replace the set  $\mathcal{A}$  by a more demanding set  $\mathcal{A}_\varepsilon$ . This alone restores stability in the minimization problem, but at the same time seems not very practical from a numerical point of view. The second one is the standard penalization by the perimeter. It consists in replacing the function  $\mathcal{L}$  by its penalized version

$$\mathcal{L}_\eta : A \in \mathcal{A} \mapsto \mathcal{L}(A) + \eta \mathcal{P}(A),$$

where  $\eta \geq 0$  is the regularization parameter and  $\mathcal{P}(A)$  is the perimeter of  $A$  (relative to  $\mathbb{R}^d$ ) given by (see, *e.g.*, [16], Def. 2.3.1)

$$\mathcal{P}(A) = \sup \left\{ \int_A \operatorname{div} \phi; \phi \in \mathcal{D}(\mathbb{R}^d), \|\phi\|_\infty \leq 1 \right\}.$$

We recall that if  $A$  is regular (that is  $\mathcal{C}^1$ ), the previous definition coincides with the usual definition of perimeter:  $\mathcal{P}(A) = \int_{\partial A} 1$  (see, *e.g.*, [16], Prop. 2.3.3). Notice again that even if it is a very common choice of regularization, it is not obvious that it restores well-posedness of the minimization problem in the class  $\mathcal{A}$ . It nevertheless has a positive impact in the minimization process, as in a certain sense, it improves the coerciveness of the functional at a minimizer. We quantify this improvement, which constitutes the main result of this study.

The first method proposed consists in strongly reducing the class of domains considered (here to the class of domains with a  $\varepsilon$  cone property). It is similar to regularization by projection. The second is based on the

addition of a geometric regularization term of the Tykhonov type. However, to make sense of this additional term, we also need to restrict the class of domains considered. This restriction is weaker, however, since a bounded domain possessing the  $\epsilon$ -cone property has finite perimeter, whereas the converse is false.

**Outline of the article** The paper is organized as follows. We first introduce the possible optimization problems related to  $\mathcal{L}$  and  $\mathcal{L}_\eta$ , discussing in particular different admissible spaces of open sets. This naturally leads to several problems for which we prove existence of minimizers, and discuss the behavior of these minimizers as the different parameters of regularization go to zero. Finally, we quantify the gain of stability obtained by penalizing the functional with the perimeter, using the second order shape derivatives of the misfit functional. To obtain this stability result, we use the general theory introduced by Dambrine *et al.* in [17, 18] and check all the required assumptions point by point.

## 2. SEVERAL CLASS OF ADMISSIBLE SETS AND SEVERAL REGULARIZATIONS

### 2.1. Four minimization problems

As mentioned above, a natural strategy in order to solve the inverse obstacle problem (1.1) is to minimize the functional

$$\mathcal{L}(A) = \frac{1}{2} \int_{\partial\Omega} |u - g_D|^2,$$

where  $u$  is the  $A$ -dependent solution to (1.3), over the previously defined set

$$\mathcal{A} = \{A = \Omega \setminus \bar{\omega}, \omega \text{ open}, \omega \subset K, \omega \text{ Lipschitz} \},$$

as we know that, by assumption, our target  $\omega$  belongs to  $\mathcal{A}$ . In other words, we are interested in the minimization problem

$$\inf_{A \in \mathcal{A}} \mathcal{L}(A). \tag{P}$$

However, as said in the introduction, problem (P) is ill-posed and highly unstable. It is therefore risky to solve this problem numerically without additional regularization.

A first method to regularize problem (P) consists in restricting the set of admissible domains. Very classically in shape optimization, for a positive real parameter  $\epsilon > 0$ , we are interested in open sets satisfying the so-called  $\epsilon$ -cone property whose definition is recalled below.

**Definition 2.1.** A bounded open set  $\Theta$  of  $\mathbb{R}^d$  is said to satisfy the  $\epsilon$ -cone property if

$$\forall x \in \partial\Theta, \exists \xi_x \text{ unit vector such that } \forall y \in \bar{\Theta} \cap \mathcal{B}(x, \epsilon), C(y, \xi_x, \epsilon) \subset \Theta,$$

where  $C(y, \xi, \epsilon) = \{z \in \mathbb{R}^d, (z - y, \xi) \geq \cos(\epsilon)|z - y| \text{ and } 0 < |z - y| < \epsilon\}$  is the cone of vertex  $y$ , of direction  $\xi$  and dimension  $\epsilon$ .

Having this in mind, we introduce the set  $\mathcal{A}_\epsilon$  of domains having an  $\epsilon$ -cone property:

$$\mathcal{A}_\epsilon = \{A := \Omega \setminus \bar{\omega}, \omega \text{ open}, \omega \subset K, \omega^c \text{ has the } \epsilon\text{-cone property}\}.$$

Then our first regularized problem of interest consists in minimizing the functional  $\mathcal{L}$  over the set  $\mathcal{A}_\varepsilon$ , for a fixed positive parameter  $\varepsilon > 0$ . We denote  $(\mathcal{P}_{\varepsilon 0})$  this second minimization problem:

$$\inf_{A \in \mathcal{A}_\varepsilon} \mathcal{L}(A). \quad (\mathcal{P}_{\varepsilon 0})$$

Let us make two comments on the set  $\mathcal{A}_\varepsilon$ . First of all, as any open set having the  $\varepsilon$ -cone property for some  $\varepsilon > 0$  is Lipschitz (see [16]), we have  $\mathcal{A}_\varepsilon \subset \mathcal{A}$  for all  $\varepsilon > 0$ . Additionally, for all  $A \in \mathcal{A}_\varepsilon$ , problem (1.3) is well-posed. Finally, for any open set  $\omega_0$  with Lipschitz boundary, there exists  $\varepsilon_0$  such that  $\omega_0$  satisfies the  $\varepsilon$ -cone property for all  $\varepsilon$  in  $(0, \varepsilon_0)$  (see [16]). Therefore, as  $\Omega$  is by assumption smooth, hence Lipschitz, there exists a positive parameter  $\varepsilon_*$  such that for all  $\varepsilon \in (0, \varepsilon_*)$ ,  $A \in \mathcal{A}_\varepsilon$  implies  $A$  has the  $\varepsilon$ -cone property. From now on, the parameter  $\varepsilon$  is supposed to be in the interval  $(0, \varepsilon_*)$ .

The second method of regularization that we consider consists in minimizing the previously defined penalized functional  $\mathcal{L}_\eta$  for some positive parameter  $\eta > 0$ . Then we have two options: minimizing  $\mathcal{L}_\eta$  over the initial set  $\mathcal{A}$ , or the restricted set  $\mathcal{A}_\varepsilon$ , which gives two new minimization problem:

$$\inf_{A \in \mathcal{A}} \mathcal{L}_\eta(A), \quad (\mathcal{P}_{0\eta})$$

and

$$\inf_{A \in \mathcal{A}_\varepsilon} \mathcal{L}_\eta(A). \quad (\mathcal{P}_{\varepsilon\eta})$$

In the following, we mainly focus on problems  $(\mathcal{P})$ ,  $(\mathcal{P}_{\varepsilon 0})$  and  $(\mathcal{P}_{\varepsilon\eta})$ . Problem  $(\mathcal{P}_{0\eta})$  corresponds to a regularization of problem  $(\mathcal{P})$  by penalization of the perimeter only. It is not clear that such penalization is sufficient to ensure the well-posedness of problem  $(\mathcal{P}_{0\eta})$ : we do not know if it is possible for a minimizing sequence to leave the class of Lipschitz domain at the limit. Indeed, in most known situations arising in shape optimization in which this perimeter penalization ensures the existence of an optimal design, the obtained minimizer is of finite perimeter but has no regularity whatsoever (see [19]). Therefore, it seems doubtful that the penalization by the perimeter guarantees the existence of a minimizer in the class of Lipschitz domains  $\mathcal{A}$ .

## 2.2. Well-posedness of problems $(\mathcal{P}_{\varepsilon 0})$ and $(\mathcal{P}_{\varepsilon\eta})$

Contrary to the reformulation of the obstacle problem as the minimization problem  $(\mathcal{P})$ , there always exists a solution to the regularized minimization problems  $(\mathcal{P}_{\varepsilon 0})$  and  $(\mathcal{P}_{\varepsilon\eta})$ . Let us emphasize the crucial assumption of the uniform cone property that insures the existence of a minimizer. Without it, the homogenization phenomenon may occur leading to non-convergent minimizing sequences.

However, a minimizer can be associated to a positive value of the reconstruction functional  $\mathcal{L}$ . Hence, it may not be a solution of the original inverse problem. This occurs, for example, if the Cauchy data are not compatible. Additionally, we cannot guarantee uniqueness of the minimizer.

**Theorem 2.2.** *The minimization problem  $(\mathcal{P}_{\varepsilon 0})$  admits a solution denoted by  $A^* \in \mathcal{A}_\varepsilon$  and the penalized problem  $(\mathcal{P}_{\varepsilon\eta})$  admits also a solution denoted by  $A_\eta^* \in \mathcal{A}$ .*

*Proof.* We focus on problem  $(\mathcal{P}_{\varepsilon 0})$ . The idea is to prove that the function  $\mathcal{L}$  is lower semi-continuous (and even continuous in our case) for one of the Hausdorff, characteristic function, or compact sets convergence (see [16], Chapter 2.2, Different topologies on domains). Let  $(A_n) \in \mathcal{A}_\varepsilon$  be a sequence of domains converging to  $A^* \in \mathcal{A}_\varepsilon$  for one of these convergences. Let us prove that  $\liminf \mathcal{L}(A_n) \geq \mathcal{L}(A^*)$ .

*Step 1: uniform bounds.* Let us consider, for all  $n \in \mathbb{N}$ , the solution  $u_n \in H^1(A_n)$  of the following problem

$$\begin{cases} -\Delta u_n + u_n = 0 & \text{in } A_n = \Omega \setminus \overline{\omega_n}, \\ \partial_\nu u_n = g_N & \text{on } \partial\Omega, \\ u_n = 0 & \text{on } \partial\omega_n, \end{cases}$$

and  $u \in H^1(A)$  solution of the same problem with  $A_n$  replaced by  $A$ , and  $\omega_n$  replaced by  $\omega$ . We extend  $u_n$  by zero in  $\omega_n$  (resp.  $u$  by zero in  $\omega$ ), and we still denote  $u_n$  (resp.  $u$ ) this extension. By definition, both  $u_n$  for all  $n$  and  $u$  are  $H^1(\Omega)$  functions, and we now show that the sequence  $(u_n)$  converges to  $u$  in  $H^1(\Omega)$ .

Let  $n \in \mathbb{N}$ . We classically have

$$\|u_n\|_{H^1(\Omega)} \leq \|g_N\|_{H^{-1/2}(\partial\Omega)}.$$

Therefore, there exist  $u^* \in H^1(\Omega)$  and a sub-sequence, which we still denote  $(u_n)$ , such that  $(u_n)$  weakly converges to  $u^*$  in  $H^1(\Omega)$ .

*Step 2: characterization of the weak limit.* We now characterize the limit  $u^*$  through a boundary value problem. Let  $\phi \in C_c^\infty(\Omega)$ , and for all  $n$  in  $\mathbb{N}$ ,  $w_n = \phi u_n$ . We have

$$\begin{cases} \Delta w_n = \Delta\phi u_n + 2\nabla\phi \cdot \nabla u_n + \phi u_n & \text{in } A_n = \Omega \setminus \overline{\omega_n}, \\ w_n = 0 & \text{on } \partial\Omega, \\ w_n = 0 & \text{on } \partial\omega_n. \end{cases}$$

Denote  $f_n = \Delta\phi u_n + 2\nabla\phi \cdot \nabla u_n + \phi u_n$  and  $f = \Delta\phi u^* + 2\nabla\phi \cdot \nabla u^* + \phi u^*$ . Since  $(u_n)$  weakly converges to  $u^*$  in  $H^1(\Omega)$ ,  $(f_n)$  converges strongly in  $H^{-1}(\Omega)$  to  $f$ . From [16], Corollary 3.2.6, Theorem 3.2.13 and Corollary 3.2.15, we deduce that  $(w_n)$  converges strongly in  $H_0^1(\Omega)$  to  $w$ , where  $w \in H_0^1(\Omega)$  solves

$$\begin{cases} \Delta w = f & \text{in } A^* = \Omega \setminus \overline{\omega^*}, \\ w = 0 & \text{on } \partial\Omega, \\ w = 0 & \text{on } \partial\omega. \end{cases}$$

We notice that  $w = \phi u^*$  (by uniqueness of the limit). Then, since

$$\Delta\phi u^* + 2\nabla\phi \cdot \nabla u^* + \phi u^* = f = \Delta w = \Delta\phi u^* + 2\nabla\phi \cdot \nabla u^* + \phi \Delta u^*,$$

for all  $\phi \in C_c^\infty(\Omega)$ , we have  $-\Delta u^* + u^* = 0$  in  $\Omega \setminus \overline{\omega^*}$ . Moreover,  $w = \phi u^* = u^* = 0$  on  $\partial\omega^*$ . It remains to determine the boundary condition on the fixed boundary  $\partial\Omega$ . To that end, we choose  $\phi$  such that  $\phi = 1$  in  $K$  and localize in the fixed domain  $\Omega \setminus K$ . Finally we have

$$\Delta((1-\phi)u_n) = \Delta(1-\phi)u_n + 2\nabla(1-\phi) \cdot \nabla u_n + (1-\phi)u_n \quad \text{in } \Omega,$$

$(1-\phi)u_n = 0$  on  $\partial K$ , and the weak convergences  $(1-\phi)u_n \rightharpoonup (1-\phi)u^*$  in  $H^1(\Omega \setminus K)$  and  $\Delta((1-\phi)u_n) \rightharpoonup \Delta((1-\phi)u^*)$  in  $L^2(\Omega \setminus K)$ . Thus, on  $\partial\Omega$ , we have

$$\partial_\nu((1-\phi)u_n) = \partial_\nu(u_n) = g_N \rightharpoonup \partial_\nu((1-\phi)u^*) = \partial_\nu(u^*)$$

weakly in  $H^{-1/2}(\partial\Omega)$ , and then  $\partial_\nu u^* = g_N$ .

To sum up,  $u^*$  satisfies

$$\begin{cases} -\Delta u^* + u^* = 0 & \text{in } A^* = \Omega \setminus \overline{\omega^*}, \\ \partial_\nu u^* = g_N & \text{on } \partial\Omega, \\ u^* = 0 & \text{on } \partial\omega^*, \end{cases}$$

which immediately implies  $u^* = u$  by uniqueness of the solution to that problem.

*Step 3: continuity of  $\mathcal{L}$ .* Let us now prove that the weak convergence  $u_n \rightharpoonup u$  is actually strong in  $H^1(\Omega)$ . We have  $u_n = (1 - \phi)u_n + \phi u_n$ . We know that  $(w_n) = (\phi u_n)$  converges strongly in  $H_0^1(\Omega)$  to  $w = \phi u$ . Moreover, setting  $h_n = (1 - \phi)u_n$ , for all  $n \in \mathbb{N}$ , and  $h = (1 - \phi)u$ , we have

$$\Delta h_n = u_n - \Delta(\phi u_n) \longrightarrow u - \Delta(\phi u) = \Delta h,$$

strongly in  $H^{-1}(\Omega)$ , and  $\partial_\nu h_n = g_N = \partial_\nu h$  on  $\partial\Omega$ . Thus, there exists a constant  $c$  such that  $h_n$  strongly converges to  $h + c$  in  $H^1(\Omega)$ . As  $h_n$  weakly converges to  $h$  in  $H^1(\Omega)$ , the constant is zero. Finally, we conclude that  $u_n = (1 - \phi)u_n + \phi u_n$  converges strongly to  $(1 - \phi)u + \phi u = u$  in  $H^1(\Omega)$ . Using this convergence, we obtain that  $\mathcal{L}$  is continuous and then  $\lim \mathcal{L}(A_n) = \mathcal{L}(A^*)$ .

*Step 4: conclusion.* Let  $(A_n)$  be a minimizing sequence. Then there exists a subsequence  $(A_{n_k})$  which converges to  $A^* \in \mathcal{A}$  in the sense of Hausdorff, of the characteristic functions and of the compact sets (see, e.g., [16], Thm. 2.4.10). Since  $\mathcal{L}$  is continuous, we have

$$\inf_{A \in \mathcal{A}} \mathcal{L}(A) = \lim \mathcal{L}(A_{n_k}) = \mathcal{L}(A^*),$$

which proves that  $A^*$  is a minimizer.

The statement for the penalized criterion follows since the perimeter  $\mathcal{P}$  is lower semicontinuous for the convergence of characteristic functions (see, e.g., [16], Prop. 2.3.7) and since  $\mathcal{L}$  is continuous.  $\square$

**Remark 2.3.** Notice that this can obviously be adapted to the case  $-\Delta u + cu = 0$ , where  $c \in L^\infty(\Omega)$  with  $c > 0$ . Moreover, let us underline that if we consider  $-\Delta u = 0$  instead of  $-\Delta u + u = 0$ , we obtain  $|u_n|_{H^1(\Omega)} \leq \|g_N\|_{H^{-1/2}(\partial\Omega)}$  instead of  $\|u_n\|_{H^1(\Omega)} \leq \|g_N\|_{H^{-1/2}(\partial\Omega)}$  in the previous proof. Then, in this case, we have to check that there exists a constant  $c \geq 0$ , independent of  $n$ , such that  $\|\nabla u_n\|_{L^2(\Omega)} \geq c \|u_n\|_{H^1(\Omega)}$  in order to conclude that there exists  $u^* \in H^1(\Omega)$  such that  $u_n \rightharpoonup u^*$ . To obtain such a uniform Poincaré inequality on the sequence  $(\Omega_n)$  of domains, we usually rely on a homogeneous Dirichlet boundary. One way of obtaining it is to assume that such condition is satisfied on a fixed part of the outer boundary. Another way is to assume that all inclusions  $\omega_n$  contain the same ball not reduced to a point. This condition is satisfied, at least for  $n$  large enough, if the limit domain  $\omega^*$  is a non-empty open set and hence contains an open ball (see [16], Prop. 2.2.15). Under one of these assumptions, Theorem 2.2 remains true for the equation without the order 0 term.

**Remark 2.4.** In this remark, we propose an alternative proof to step 2. By definition of  $u_n$ , for any test function  $v \in H^1(\Omega)$  and any integer  $n$ , it holds

$$\int_{\Omega} \nabla u_n \cdot \nabla v + u_n v = \int_{A_n} \nabla u_n \cdot \nabla v + u_n v = \int_{\partial\Omega} g_N v.$$

Passing to the limit  $n \rightarrow +\infty$  yields:

$$\int_{\Omega} \nabla u^* \cdot \nabla v + u^* v = \int_{\partial\Omega} g_N v.$$

Moreover, the triangle inequality yields

$$\begin{aligned} \left| \int_{\Omega} \mathbb{1}_{\omega_n} (u_n)^2 - \mathbb{1}_{\omega^*} (u^*)^2 \right| &\leq \left| \int_{\Omega} (\mathbb{1}_{\omega_n} - \mathbb{1}_{\omega^*}) (u^*)^2 \right| + \left| \int_{\Omega} \mathbb{1}_{\omega_n} (u_n^2 - (u^*)^2) \right| \\ &\leq \left| \int_{\Omega} (\mathbb{1}_{\omega_n} - \mathbb{1}_{\omega^*}) (u^*)^2 \right| + \left| \int_{\Omega} (u_n^2 - (u^*)^2) \right| \rightarrow 0 \end{aligned}$$

since  $u_n^2 \rightarrow (u^*)^2$  in  $L^1(\Omega)$  and  $\mathbb{1}_{\omega_n} \xrightarrow{*} \mathbb{1}_{\omega^*}$  in  $L^\infty(\Omega)$ . Then

$$0 = \int_{\omega_n} u_n^2 = \int_{\Omega} \mathbb{1}_{\omega_n} u_n^2 \rightarrow \int_{\Omega} \mathbb{1}_{\omega^*} (u^*)^2 = \int_{\omega^*} (u^*)^2.$$

Hence,  $u^* = 0$  almost everywhere in  $\omega^*$ . Hence,  $u^*$  is the extension by 0 of its restriction to  $\Omega \setminus \overline{\omega^*}$ . Since  $u^*$  is in  $H^1(\Omega)$ , its restriction to  $\Omega \setminus \overline{\omega^*}$  belongs to the space  $H_{0, \partial\omega^*}^1(\Omega \setminus \overline{\omega^*})$  of the functions in  $H^1(\Omega \setminus \overline{\omega^*})$  whose trace on  $\partial\omega^*$  is 0 in the space  $H^{1/2}(\partial\omega^*)$  (this characterization of  $H_0^1$  holds for domains in  $\mathcal{A}$ , see for example [16], Lem. 3.2.15) and satisfies

$$\int_{\Omega \setminus \overline{\omega^*}} \nabla u^* \cdot \nabla v + u^* v = \int_{\partial\Omega} g_N v.$$

We conclude as above.

### 2.3. Convergence results

In this section, we prove the principal results of convergence regarding our problems of interest. To put it simply, we prove below that family of solutions of problems  $(\mathcal{P}_{\varepsilon\eta})$  converges to solutions to corresponding problems  $(\mathcal{P}_{\varepsilon 0})$  when  $\eta$  goes to zero. And if  $(\mathcal{P})$  has a minimizer, the same situation occurs between  $(\mathcal{P}_{\varepsilon 0})$  and  $(\mathcal{P})$  as  $\varepsilon$  goes to zero, which leads, roughly speaking, to the diagram

$$(\mathcal{P}_{\varepsilon\eta}) \xrightarrow{\eta \rightarrow 0} (\mathcal{P}_{\varepsilon 0}) \xrightarrow{\varepsilon \rightarrow 0} (\mathcal{P}).$$

We also show that, for  $\varepsilon$  small enough, we can go directly from  $(\mathcal{P}_{\varepsilon\eta})$  to  $(\mathcal{P})$ , that is

$$(\mathcal{P}_{\varepsilon\eta}) \xrightarrow{\eta \rightarrow 0} (\mathcal{P})$$

Obviously, these diagrams are simplifications, as for example most of convergences are only true up to subsequences. The precise statements are given in the following propositions.

**Proposition 2.5.** *Let  $\varepsilon > 0$  be fixed. For  $\eta > 0$ , let  $A_\eta$  be a minimizer of  $(\mathcal{P}_{\varepsilon\eta})$ . There exists a sequence  $(\eta_n)$  of positive parameters converging to zero and  $A^* \in \mathcal{A}_\varepsilon$  such that  $A^*$  is a minimizer of  $(\mathcal{P}_{\varepsilon 0})$  and  $A_{\eta_n}$  converges to  $A^*$  in the Hausdorff, characteristic functions and compact set topologies.*

*Proof.* As by definition,  $(A_\eta)_{\eta>0}$  is a family of elements of  $\mathcal{A}_\varepsilon$ , there exists a sequence of positive real numbers  $(\eta_n)$  that converges to 0 and  $A^*$  in  $\mathcal{A}_\varepsilon$  such that  $A_{\eta_n}$  converges to  $A^*$  in the Hausdorff, characteristic functions and compact set topologies [16], Theorem 2.4.10. We have already seen that  $\mathcal{L}(A_{\eta_n}) \xrightarrow{n \rightarrow \infty} \mathcal{L}(A^*)$ .

Let  $A$  in  $\mathcal{A}_\varepsilon$ . We have

$$\mathcal{L}(A^*) \xleftarrow{n \rightarrow \infty} \mathcal{L}(A_{\eta_n}) \leq \mathcal{L}_{\eta_n}(A_{\eta_n}) \leq \mathcal{L}_{\eta_n}(A) = \mathcal{L}(A) + \eta_n \mathcal{P}(A) \xrightarrow{n \rightarrow \infty} \mathcal{L}(A).$$

The result follows.  $\square$

**Proposition 2.6.** *Suppose that problem  $(\mathcal{P})$  admits a minimizer  $A^*$ . There exists  $\varepsilon_0$  such that  $A^*$  is a minimizer of  $(\mathcal{P}_{\varepsilon_0})$  for all  $\varepsilon$  in  $(0, \varepsilon_0)$ .*

*Proof.* The minimizer  $A^*$  is Lipschitz by definition, hence there exists  $\varepsilon_0$  such that  $A^*$  belongs to  $\mathcal{A}_\varepsilon$  for all  $\varepsilon \in (0, \varepsilon_0)$ . Let  $\varepsilon$  be in  $(0, \varepsilon_0)$  and  $A_\varepsilon$  be a minimizer of  $(\mathcal{P}_{\varepsilon_0})$ . By definition,  $\mathcal{L}(A_\varepsilon) \leq \mathcal{L}(A^*)$ . But  $A_\varepsilon$  belongs to  $\mathcal{A}$ , hence  $\mathcal{L}(A^*) \leq \mathcal{L}(A_\varepsilon)$ . The results follows.  $\square$

**Corollary 2.7.** *Suppose that the inverse obstacle problem (1.1) admits a (necessarily unique) solution. Then*

- $\mathcal{P}$  admits a unique minimizer  $A^*$ , and  $\mathcal{L}(A^*) = 0$ ,
- for  $\varepsilon > 0$  and  $\eta > 0$ , let  $A_{\varepsilon\eta}$  be a solution of  $(\mathcal{P}_{\varepsilon\eta})$ . There exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , there exists a sequence of positive parameters  $(\eta_n)$  such that  $A_{\varepsilon\eta_n}$  converges to  $A^*$  in the Hausdorff, characteristic functions and compact set topologies.

*Proof.* The first point has already been proven in the introduction. Let us therefore focus on the second point: by Proposition 2.6, there exists  $\varepsilon_0$  such that  $A^*$  is a minimizer of  $(\mathcal{P}_{\varepsilon_0})$  for all  $\varepsilon$  in  $(0, \varepsilon_0)$ . Let  $A_\varepsilon$  be another minimizer of  $(\mathcal{P}_{\varepsilon_0})$ . We have  $\mathcal{L}(A_\varepsilon) = \mathcal{L}(A^*) = 0$ , which immediately implies  $A_\varepsilon = A^*$  by uniqueness of the solution of (1.1). Then the result is a direct consequence of Proposition 2.5.  $\square$

### 3. A INCREASED STABILITY RESULT FOR THE PENALIZED PROBLEM

As seen in the previous section, the penalization of the misfit functional by the perimeter only is *a priori* not sufficient to ensure the well-posedness of the minimization problem. At the theoretical level, we need to work in the regularized set  $\mathcal{A}_\varepsilon$  to ensure the existence of minimizers, with or without the parameter penalization.

Nevertheless, this does not mean that the perimeter regularization is useless in our context. First of all, the set  $\mathcal{A}_\varepsilon$  is not really practical at the numerical level, whereas the penalization by the perimeter is easily implementable. But more importantly, it turns out the regularization by the perimeter improves the stability of our problem. We intend to highlight that in the present section, by giving a quantitative estimate of the stability increase at a minimum.

#### 3.1. Preliminaries and reminders

In order to introduce the main ideas of this section devoted to achieving an increased stability result, let us first focus on the very classical problem of the minimization of the so-called *energy functional*. The energy functional  $\mathcal{E}$  of  $\Omega$  is defined by

$$\mathcal{E}(\Omega) = \inf \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} u; u \in H_0^1(\Omega) \right\}.$$

The energy is homogeneous of degree  $d + 2$  and a volume constraint is needed to obtain existence of a minimizer of this energy. It is well known that the minimizer is then the ball and Brasco *et al.* have even given in a quantitative version of this inequality while studying the Faber–Krahn inequality. Precisely, they have shown in [20] that there exists a dimensional constant  $\sigma(d)$  such that for any open set  $\Omega$  with finite measure

$$\mathcal{E}(\Omega) \geq \mathcal{E}(B_1) + \sigma(d)\mathcal{F}(\Omega)^2, \tag{3.1}$$

where  $\mathcal{F}$  is the *Fraenkel asymmetry* defined as

$$\mathcal{F}(\Omega) = \inf \left\{ \frac{|\Omega \Delta B|}{|B|}; B \text{ ball such that } |B| = |\Omega| \right\},$$

where the symbol  $\Delta$  stands just here for the symmetric difference between sets. This is a typical stable situation. Notice that the gap between the value at a generic domain and the value at the minimizer is expressed in terms of a crude  $L^1$  type term. This is typical of the case where no regularity assumption is made on the deformations. Let us emphasize a point already noticed by Dambrine and Lamboley in [18]: the local stability statement is directly connected to the regularity of the class of domains. For example, the previous general statement can even be more precise under additional regularity assumption. If one considers a quasispherical cap  $\Omega_h$ , that is a domain such that its boundary is the graph over the sphere  $\partial B$  of a  $C^{2,\alpha}$  function  $h$

$$\partial\Omega_h = \{x \in \mathbb{R}^d, x = (1 + h(y))y, y \in \partial B\}, \quad (3.2)$$

then there exists  $\delta(d, \alpha) > 0$  such that

$$\|h\|_{2,\alpha} \leq \delta(d, \alpha) \quad \Rightarrow \quad \mathcal{E}(\Omega_h) \geq \mathcal{E}(B_1) + \frac{1}{32d^2} \|h\|_{H^{1/2}(\partial B)}^2$$

as shown in [18]. Notice that, this type of result can be the first step to derive general stability results by using the selection principle as in [20] or geometric constraint as in [21].

Finally let us consider the regularization of  $\mathcal{E}$  by adding the penalization term  $\eta\mathcal{P}(\Omega)$ , where  $\eta > 0$ . Since the ball is the minimizer for both the energy function  $\mathcal{E}$  and the perimeter  $\mathcal{P}$ , thus the ball is also trivially a minimizer for the penalized energy  $\mathcal{E}_\eta = \mathcal{E} + \eta\mathcal{P}$ . The stability issue is then well-known thanks to the quantitative inequalities. The quantitative isoperimetric inequality has been proven by Fusco and al in [22] and states that there exists a dimensional constant  $C(d) > 0$  such that

$$\mathcal{P}(\Omega) \geq \mathcal{P}(B_1) + c(d)\mathcal{F}(\Omega)$$

holds for any Borel set  $\Omega$  with finite measure  $|B_1|$  and with a finite perimeter  $P(\Omega)$  in the distributional sense of Caccioppoli–De Giorgi. Combined with (3.1), we obtained the following quantitative stability estimate:

$$E_\eta(\Omega) \geq E_\eta(B_1) + (\sigma(d) + \eta c(d))\mathcal{F}(\Omega),$$

that holds for any open set  $\Omega$  with finite measure  $|B_1|$  and with a finite perimeter  $P(\Omega)$  in the distributional sense of Caccioppoli–De Giorgi. In the previous statement no gain is observed while it is known from a long time by numericians that adding this penalization by the perimeter permits to regularize this optimization problem.

The reason of this apparent contradiction lies in the class of domains under consideration. We aim to justify properly this fact at the continuous level by using the methods of shape sensitivity of order two. Using the results detailed in [18], we immediately get for spherical caps defined in (3.2): there exists positive numbers  $\epsilon$  and  $C(d)$  such that if  $\|h\|_{2,\alpha} < \epsilon$ , then

$$\mathcal{E}_\eta(\Omega_h) \geq \mathcal{E}_\eta(B_1) + \frac{1}{32d^2} \|h\|_{H^{1/2}(\partial B_1)}^2 + \eta C(d) \|h\|_{H^1(\partial B_1)}^2.$$

In other words, the presence of the perimeter terms increases the local stability by passing from a  $H^{1/2}$ -stability to a  $H^1$ -stability.

### 3.2. Stability result concerning the inverse obstacle problem

In order to consider the least-squares tracking objective  $\mathcal{L}$ , we need to consider more general shapes than balls. To that end, we first recall some basic facts from differential geometry and fix the notations. We follow in this paragraph the constructions first given in [23] and followed in [18]. We follow step by step the general strategy developed in [18] to prove quantitative stability results for smooth perturbations. Our aim is to apply

general Theorem 3–2 of [18] to the particular case of the objective  $\mathcal{L}_\eta$  we are interested in. Our task in the rest of this section is therefore to test the hypotheses of this theorem one by one.

In the following, we will assume that  $\omega$  is a smooth (at least  $\mathcal{C}^{2,\alpha}$ ) bounded open set and we recall that  $\nu$  denotes the outer unit normal field to  $\partial\omega$ . There is a non-negative real  $\rho(\Omega)$ , such that the application  $T_{\partial\omega}$  defined by

$$\begin{aligned} T_{\partial\omega} : \partial\omega \times \mathbb{R} &\longrightarrow \mathbb{R}^d \\ (M, V) &\longmapsto M + V\nu(M) \end{aligned}$$

is a local diffeomorphism from  $\partial\omega \times (-\rho(\omega), \rho(\omega))$  on an open tubular neighborhood of  $\partial\omega$  we will denote  $\mathcal{T}_{\partial\omega}$ . This property expresses the fact that any point  $x$  in  $\mathcal{T}_{\partial\omega}$  has a unique orthogonal projection  $p_{\partial\omega}(x)$  on  $\partial\omega$  and that the relation

$$x = p_{\partial\omega}(x) + h(x)\nu(p_{\partial\omega}(x)) \quad \text{where } |h(x)| = \|x - p_{\partial\omega}(x)\|,$$

holds. Notice that  $h$  is uniquely defined and we can define *normal graph* on  $\partial\omega$ , that is  $\omega_h$  such that

$$\partial\omega_h = \{x + h(x)\nu(x), x \in \partial\omega\}.$$

Before stating and proving the following theorem, we briefly recall the definition of our main tool, that is the shape derivatives, and fix some notations. Given a domain  $A$ , we first define for any  $V \in W^{1,\infty}(\mathbb{R}^d)$  such that  $\text{supp}(V) \subset K \subset \Omega$  and for any shape functional  $J$  the function  $\mathcal{J}_A(V) = J[(I + V)(A)]$  on a neighborhood of 0 in  $W^{1,\infty}$ . In particular, the first shape derivative is  $DJ(A) \cdot V := \mathcal{J}'_A(0)$ , a continuous linear form on  $W^{1,\infty}$  (the shape gradient), and the second order shape derivative is  $D^2J(A) \cdot [V, V] := \mathcal{J}''_A(0)$ , a continuous symmetric bilinear form on  $W^{1,\infty}$  (the shape Hessian).

In the following, we assume that the perturbation direction  $V$  belongs to  $\mathcal{C}^3(\partial\omega)$  and, taking into account the structure theorem of the shape derivatives (see [16], Thm. 5.9.2), we also assume that  $V = (V \cdot \nu)\nu$  on  $\partial\omega$  so that  $h = V \cdot \nu$  with the previous notations. In that context, our main result reads:

**Theorem 3.1.** *Let  $A_\eta^* = \Omega \setminus \overline{\omega_\eta^*}$  be a domain of class  $\mathcal{C}^{2,\alpha}$ . Then the penalized least-squares functional  $\mathcal{L}_\eta$  is twice Fréchet differentiable on a neighborhood of  $A_\eta^*$  for the  $W^{1,\infty}$  norm. Moreover, if  $A_\eta^* = \Omega \setminus \overline{\omega_\eta^*}$  is a critical shape for  $\mathcal{L}_\eta$  stable in the sense that*

$$D^2\mathcal{L}_\eta(A_\eta^*) \cdot [V, V] > 0, \quad \forall V \in W^{1,\infty}(\partial\omega_\eta^*),$$

*then there exists  $\xi > 0$  and  $c = c(\xi) > 0$  such that*

$$\mathcal{L}_\eta(A_h) \geq \mathcal{L}_\eta(A_\eta^*) + c \|h\|_{H^1(\partial\omega_\eta^*)}^2,$$

*for all  $A_h = (I + h\nu)(A_\eta^*)$  such that  $\|h\|_{\mathcal{C}^{2,\alpha}(\partial\omega_\eta^*)} \leq \xi$ .*

In other words, the presence of the penalization by the perimeter provides a local  $H^1$ -stability at a minimum of the functional, therefore making the numerical research of a minimizer slightly more stable.

*Proof.* The fact that  $\mathcal{L}_\eta$  is twice Fréchet differentiable on a neighborhood of  $A_\eta^*$  for the  $W^{1,\infty}$  norm is classical and is based on the use of the implicit function theorem (we refer for example to [16], Chapter 5 or also [13] for details).

Let us now prove the main part of the theorem which is the stability result. We recall that we consider  $V \in W^{1,\infty}(\mathbb{R}^d)$  with  $V \in \mathcal{C}^{2,\alpha}(\partial\omega_\eta^*)$ ,  $\text{supp}(V) \subset K \subset \Omega$  and with  $V = h\nu$  on  $\partial\omega_\eta^*$  and that  $A_\eta^* = \Omega \setminus \overline{\omega_\eta^*}$  is a stable critical shape for  $\mathcal{L}_\eta$ .

*Step 1: computation of the first order shape derivative.* We know that the shape derivative  $u' \in \mathbf{H}^1(A_\eta^*)$  solves (see, e.g., [16], Chapter 5 or [13])

$$\begin{cases} -\Delta u' + u' = 0 & \text{in } A_\eta^*, \\ \partial_\nu u' = 0 & \text{on } \partial\Omega, \\ u' = -\partial_\nu u h & \text{on } \partial\omega_\eta^*. \end{cases}$$

We introduce the adjoint state  $w \in \mathbf{H}^1(A)$ , solution to

$$\begin{cases} -\Delta w + w = 0 & \text{in } A_\eta^*, \\ \partial_\nu w = u - g_D & \text{on } \partial\Omega, \\ w = 0 & \text{on } \partial\omega_\eta^*. \end{cases}$$

We have

$$\begin{aligned} \mathrm{D}\mathcal{L}(A_\eta^*) \cdot V &= \int_{\partial\Omega} u'(u - g_D) = \int_{\partial\Omega} u' \partial_\nu w = \int_{A_\eta^*} \Delta w u' + \int_{A_\eta^*} \nabla w \cdot \nabla u' - \int_{\partial\omega_\eta^*} u' \partial_\nu w \\ &= \int_{A_\eta^*} \Delta u' w + \int_{A_\eta^*} \nabla u' \cdot \nabla w - \int_{\partial\omega_\eta^*} u' \partial_\nu w = \int_{\partial\Omega} w \partial_\nu u' + \int_{\partial\omega_\eta^*} w \partial_\nu u' - \int_{\partial\omega_\eta^*} u' \partial_\nu w. \end{aligned}$$

Thus, using the boundary conditions, we obtain

$$\mathrm{D}\mathcal{L}(A_\eta^*) \cdot V = \int_{\partial\omega_\eta^*} \partial_\nu u \partial_\nu w h,$$

and then, using the classical shape derivative of the perimeter

$$\mathrm{D}\mathcal{L}_\eta(A_\eta^*) \cdot V = \int_{\partial\omega_\eta^*} (\partial_\nu u \partial_\nu w + \eta \mathbf{H}) h.$$

Notice that if  $A_\eta^*$  is a critical point, then one has

$$\partial_\nu u \partial_\nu w = -\eta \mathbf{H} \quad \text{on } \partial\omega_\eta^*.$$

*Step 2: computation of the second order shape derivative.* We know that the second order shape derivative  $u'' \in \mathbf{H}^1(A_\eta^*)$  solves (see, e.g., [16], Chapter 5 or [13])

$$\begin{cases} -\Delta u'' + u'' = 0 & \text{in } A_\eta^*, \\ \partial_\nu u'' = 0 & \text{on } \partial\Omega, \\ u'' = -(\partial_\nu u' - \nabla u \cdot \nabla_\tau h) h - \partial_\nu u V \cdot (-\nabla_\tau h) & \text{on } \partial\omega_\eta^*. \end{cases}$$

Since  $V = h\nu$ , we obtain that

$$u'' = -(\partial_\nu u' - \nabla u \cdot \nabla_\tau h) h \quad \text{on } \partial\omega_\eta^*.$$

Moreover the shape derivative  $w' \in H^1(A_\eta^*)$  solves

$$\begin{cases} -\Delta w' + w' = 0 & \text{in } A_\eta^*, \\ \partial_\nu w' = u' & \text{on } \partial\Omega, \\ w' = -\partial_\nu w h & \text{on } \partial\omega_\eta^*. \end{cases}$$

We then have

$$\begin{aligned} D^2\mathcal{L}(A_\eta^*) \cdot [V, V] &= \int_{\partial\Omega} \left( u''(u - g_D) + |u'|^2 \right) = \int_{\partial\Omega} u'' \partial_\nu w + \int_{\partial\Omega} u' \partial_\nu w' \\ &= \int_{\partial\Omega} w \partial_\nu u'' + \int_{\partial\omega_\eta^*} w \partial_\nu u'' - \int_{\partial\omega_\eta^*} u'' \partial_\nu w + \int_{\partial\Omega} w' \partial_\nu u' + \int_{\partial\omega_\eta^*} w' \partial_\nu u' - \int_{\partial\omega} u' \partial_\nu w'. \end{aligned}$$

Thus, using the boundary conditions for  $u''$  the second derivative of the state, we obtain

$$D^2\mathcal{L}(A_\eta^*) \cdot [V, V] = \int_{\partial\omega_\eta^*} \partial_\nu w (\partial_\nu u' - \nabla u \cdot \nabla_\tau h) h - \int_{\partial\omega_\eta^*} \partial_\nu w \partial_\nu u' h + \int_{\partial\omega_\eta^*} \partial_\nu u \partial_\nu w' h.$$

Noticing that  $u = 0$  on the boundary  $\partial\omega_\eta^*$  of the inclusion and hence  $\nabla_\tau u = 0$  on  $\partial\omega$  and then  $\nabla u \cdot \nabla_\tau h = 0$  on  $\partial\omega_\eta^*$ , we arrive at

$$D^2\mathcal{L}(A_\eta^*) \cdot [V, V] = \int_{\partial\omega_\eta^*} \partial_\nu u \partial_\nu w' h.$$

Moreover we know that

$$D^2\mathcal{P}(\Omega) \cdot V \cdot V = \int_{\partial\omega_\eta^*} |\nabla_\tau h|^2 + \int_{\partial\omega_\eta^*} h^2 (\mathbb{H}^2 - \text{Tr}({}^t\nabla_\tau \nu \nabla_\tau \nu)),$$

where  $\mathbb{H}$  is the mean curvature (understood as the sum of the principal curvatures) of  $\partial\omega_\eta^*$ , and thus

$$D^2\mathcal{L}_\eta(A_\eta^*) \cdot [V, V] = \int_{\partial\omega_\eta^*} \partial_\nu u \partial_\nu w' h + \eta \left( \int_{\partial\omega_\eta^*} |\nabla_\tau h|^2 + \int_{\partial\omega} h^2 (\mathbb{H}^2 - \text{Tr}({}^t\nabla_\tau \nu \nabla_\tau \nu)) \right).$$

*Step 3: the shape Hessian  $D^2\mathcal{L}_\eta(A_\eta^*)$  is coercive in  $H^1$ .* We split the shape Hessian as

$$D^2\mathcal{L}_\eta(A_\eta^*) \cdot [V, V] = \ell_r(h, h) + \ell_m(h, h),$$

where we have set

$$\ell_r(h, h) = \int_{\partial\omega_\eta^*} \partial_\nu u \partial_\nu w' h + \eta \int_{\partial\omega_\eta^*} h^2 (\mathbb{H}^2 - \text{Tr}({}^t\nabla_\tau \nu \nabla_\tau \nu))$$

and

$$\ell_m(h, h) = \eta \int_{\partial\omega_\eta^*} |\nabla_\tau h|^2.$$

We check easily that  $\ell_r$  is continuous on  $L^2(\partial\omega_\eta^*)$  while  $\ell_m$  is lower semi-continuous in  $H^1(\partial\omega_\eta^*)$  and moreover satisfies the coercivity property

$$\ell_m(h, h) > c_1 |h|_{H^1(\partial\omega^*)}^2,$$

for all  $h \in C^\infty(\partial\omega_\eta^*)$ . As a consequence, we can apply Lemma 3-1 of [18] and obtain that there exists a constant  $C$  such that

$$D^2\mathcal{L}_\eta(A_\eta^*) \cdot [V, V] \geq C |h|_{H^1(\partial\omega^*)}^2.$$

*Step 4: Improved Taylor property.* We want now to prove that there exist  $\xi > 0$  and a modulus of continuity  $\mathfrak{m}$  such that for every domain  $A_h = \Omega \setminus \bar{\omega} = (I + h\nu)(A_\eta^*)$  with  $\|h\|_{C^3(\partial\omega_\eta^*)} \leq \xi$ ,

$$\left| \mathcal{L}_\eta(A_h) - \mathcal{L}_\eta(A_\eta^*) - \frac{1}{2} D^2\mathcal{L}_\eta(A_\eta^*) \cdot [h, h] \right| \leq \mathfrak{m} \left( \|h\|_{C^3(\partial\omega_\eta^*)} \right) \|h\|_{H^1(\partial\omega_\eta^*)}^2.$$

We split the functional into the two natural parts. Concerning the perimeter, we have (see [18], Prop. 4.5)

$$\mathcal{P}(A_h) = \int_{\partial\omega} 1 = \int_{\partial\omega_\eta^*} j_{\partial\omega}(h) = \mathcal{P}(A_\eta^*) + \mathcal{P}'(A_\eta^*) \cdot h + \frac{1}{2} \mathcal{P}''(A_\eta^*) \cdot [h, h] + \mathcal{O} \left( \|h\|_{W^{1,\infty}(\partial\omega_\eta^*)} \|h\|_{H^1(\partial\omega_\eta^*)}^2 \right),$$

where  $j_{\partial\omega}(h)$  is the surfacic jacobian.

Concerning the least-squares tracking, we follow the strategy of Dambrine and Lamboley [18], Section 4–2. As shown in [17, 18] the crucial step is to prove an *improved continuity property of the second order shape derivative* : namely that there exist  $\eta > 0$  and a modulus of continuity  $\omega$  such that for every domain  $\Omega_h$  with  $\|h\|_{2,\alpha} \leq \eta$ , and all  $t \in [0, 1]$  :

$$|l''(t) - l''(0)| \leq \omega(\|h\|_{2,\alpha}) \|h\|_{H^s}^2$$

where  $l : t \in [0, 1] \mapsto \mathcal{L}(A_t)$  for the path  $(A_t)_{t \in [0,1]}$  connecting  $A_\eta^*$  to  $A_h$ , and defined through its boundary

$$\partial A_t = \{x + th(x)\mathbf{n}(x), x \in \partial A_\eta^*\}.$$

We have to estimate the difference. We claim that:

$$\left| \int_{\partial A_t} \partial_{\nu_t} u_t \partial_{\nu_t} w_t' h_t - \int_{\partial A_\eta^*} \partial_\nu u \partial_\nu w' h \right| \leq \omega(\|h\|_{2,\alpha}) \|h\|_{H^{1/2}}^2.$$

The proof is technical but straightforward by following the various steps of [18]. In particular, we can use directly the estimates on geometric quantities stated in [18], Lemma 4-8. The control of the variations of the state  $u$  and of the adjoint  $w$  is a direct application of the sensitivity analysis of these functions with respect to the deformation fields. That sensitivity follows directly from the Implicit Function Theorem (see [18], Proof of Thm. 5.7.4, [24]).

Using the Taylor formula with integral remainder:

$$\mathcal{L}(A_h) - \mathcal{L}(A_\eta^*) = \mathcal{L}'(A_\eta^*) \cdot h + \frac{1}{2} \mathcal{L}''(A_\eta^*) \cdot (h, h) + \int_0^1 [l''(t) - l''(0)] (1-t) dt$$

*Step 6: conclusion.* We apply [18], Theorem 1.1 to prove that  $A_\eta^*$  is an  $H^1$ -stable local minimum of  $\mathcal{L}_\eta$  in a  $C^{2,\alpha}$ -neighborhood, that is to say there exists  $\xi > 0$  and  $c = c(\xi) > 0$  such that for all  $A = (I + h\nu)(A_\eta^*)$  such that  $\|h\|_{C^{2,\alpha}(\partial\omega_\eta^*)} \leq \xi$

$$\mathcal{L}_\eta(A) \geq \mathcal{L}_\eta(A_\eta^*) + c\|h\|_{H^1(\partial\omega_\eta^*)}^2. \quad \square$$

#### ACKNOWLEDGMENTS

This work has been partially supported by the ANR project TRECOS ANR-20-CE40-0009, by the ANR project STOIQUES ANR-24-CE40-2216), by the ANR LabEx CIMI (under grant ANR-11-LABX-0040) within the French State Programme “Investissements d’Avenir”, by the project PHC-Toubkal 51628SC and by the project MathAmSud SCIPinPDEs 22-MATH-08.

#### DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

#### REFERENCES

- [1] F. Caubet, J. Dardé and M. Godoy, On the data completion problem and the inverse obstacle problem with partial Cauchy data for Laplace’s equation. *ESAIM Control Optim. Calc. Var.* **25** (2019) Paper No. 30, 30.
- [2] J. Hadamard, Lectures on Cauchy’s problem in linear partial differential equations (1923).
- [3] F. Ben Belgacem, Why is the Cauchy problem severely ill-posed? *Inverse Probl.* **23** (2007) 823–836.
- [4] F. Ben Belgacem and H. El Fekih, On Cauchy’s problem. I. A variational Steklov–Poincaré theory. *Inverse Probl.* **21** (2005) 1915–1936.
- [5] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Vol. 93 of *Appl. Math. Sci.*, 4th expanded edn. Springer, Cham (2019).
- [6] L. Bourgeois and J. Dardé, The “exterior approach” to solve the inverse obstacle problem for the stokes system. *Inverse Probl. Imaging* **8** (2014) 23–51.
- [7] G. Alessandrini, E. Beretta, E. Rosset and S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* **29** (2000) 755–806.
- [8] L. Bourgeois and J. Dardé, A quasi-reversibility approach to solve the inverse obstacle problem. *Inverse Probl. Imaging* **4** (2010) 351–377.
- [9] J. Dardé, The ‘exterior approach’: a new framework to solve inverse obstacle problems. *Inverse Probl.* **28** (2012) 015008, 22.
- [10] R. Potthast, A survey on sampling and probe methods for inverse problems. *Inverse Probl.* **22** (2006) r1–r47.
- [11] W. Rundell, Recovering an obstacle using integral equations. *Inverse Probl. Imaging* **3** (2009) 319–332.
- [12] H. Haddar and R. Kress, Conformal mappings and inverse boundary value problems. *Inverse Probl.* **21** (2005) 935–953.
- [13] L. Afraites, M. Dambrine, K. Eppler and D. Kateb, Detecting perfectly insulated obstacles by shape optimization techniques of order two. *Discrete Contin. Dyn. Syst. Ser. B* **8** (2007) 389–416.
- [14] M. Badra, F. Caubet and M. Dambrine, Detecting an obstacle immersed in a fluid by shape optimization methods. *Math. Models Methods Appl. Sci.* **21** (2011) 2069–2101.
- [15] F. Caubet, Instability of an inverse problem for the stationary Navier–Stokes equations. *SIAM J. Control Optim.* **51** (2013) 2949–2975.
- [16] A. Henrot and M. Pierre, *Variation et optimisation de formes*, vol. 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin (2005). Une analyse géométrique. [A geometric analysis].
- [17] M. Dambrine, On variations of the shape Hessian and sufficient conditions for the stability of critical shapes. *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **96** (2002) 95–121.
- [18] M. Dambrine and J. Lamboley, Stability in shape optimization with second variation. *J. Differ. Equ.* **267** (2019) 3009–3045.
- [19] D. Bucur and G. Buttazzo, *Variational methods in shape optimization problems*, vol. 65 of *Prog. Nonlinear Differ. Equ. Appl.* Birkhäuser, Basel (2005).

- [20] L. Brasco, G. De Philippis and B. Velichkov, Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.* **164** (2015) 1777–1831.
- [21] R. Prunier, *Régularité et stabilité en optimisation de forme sous contrainte géométrique*. PhD thesis (2023). Thèse de doctorat dirigée par Lamboley, Jimmy et Bucur, Dorin Mathématiques Sorbonne université 2023.
- [22] N. Fusco, F. Maggi and A. Pratelli, The sharp quantitative isoperimetric inequality. *Ann. Math.* **168** (2008) 941–980.
- [23] M. Dambrine and M. Pierre, About stability of equilibrium shapes. *M2AN Math. Model. Numer. Anal.* **34** (2000) 811–834.
- [24] D. Henry, *Perturbation of the Boundary in Boundary-value Problems of Partial Differential Equations*. vol. 318. Cambridge University Press (2005).



**Please help to maintain this journal in open access!**

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting [subscribers@edpsciences.org](mailto:subscribers@edpsciences.org).

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.