

INTEGRAL REPRESENTATIONS OF THE SOLUTIONS TO THE HOMOGENIZED TRANSPORT EQUATIONS

MARC BRIANE^{1,*} AND JUAN CASADO-DÍAZ²

Abstract. This paper provides two general representations of the weak limit $u(t, x)$ of the solution $u_\varepsilon(t, x)$ for $(t, x) \in (0, T) \times \mathbb{R}^d$, to the linear transport equation with an oscillating regular velocity $b(x/\varepsilon)$, an initial datum $u_\varepsilon^0(x)$ and a right-hand side $f_\varepsilon(t, x)$. Our main assumption is the existence of a function $w \in C^1(\mathbb{R}^d)$ such that $b \cdot \nabla w$ is bounded from below by a positive constant. As a consequence, the dynamic flow $\Phi(t, y)$ associated with the vector field $b(y)$ induces a one-to-one mapping from $\mathbb{R} \times \Sigma$ onto \mathbb{R}^d , where Σ is the equipotential hypersurface $\{w = 0\}$. This assumption allows us to finely characterize the kernel of the differential operator $b(y) \cdot \nabla_y(\cdot)$ thanks to some quotient set $\widehat{\Sigma}/R$, where $\widehat{\Sigma}$ is a suitable compact subset of Σ . Then, using a two-scale procedure we establish two integral formulas of the limit $u(t, x)$, one over the quotient set $\widehat{\Sigma}/R$ and the other one over the d -dimensional torus \mathbb{T}^d , which involve the two-scales limits of the sequences of data $u_\varepsilon^0(x)$, $f_\varepsilon(t, x)$ and the projection of b onto the kernel of $b(y) \cdot \nabla_y(\cdot)$. Finally, we also derive two alternative expressions of the asymptotics of the flow $\lim_{\infty} \Phi(t, y)/t$ for a.e. $y \in \mathbb{T}^d$.

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1. INTRODUCTION

In this paper we study in a new perspective the classical problem of homogenization of the transport equation

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}(t, x) + b(x/\varepsilon) \cdot \nabla_x u_\varepsilon(t, x) = f_\varepsilon(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\ u_\varepsilon(0, x) = u_\varepsilon^0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where the velocity $b(y)$ is a C^1 -regular \mathbb{Z}^d -periodic divergence free vector field and the data $f_\varepsilon(t, x)$, $u_\varepsilon^0(x)$ also contain oscillations with respect to the fast variable $y = x/\varepsilon$. This problem has been widely studied since the end the eighties.

On the one hand, in their famous paper [1] Di Perna and Lions have established the deep link between general transport equations and the associated dynamics flows. Then, from the point of view of homogenization, the seminal work of Brenier [2] has highlighted the crucial role of the dynamics flow $\Phi(t, x)$ associated with the

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¹ Univ Rennes, INSA Rennes, CNRS, IRMAR – UMR 6625, France.

² Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Spain.

* Corresponding author: mbriane@insa-rennes.fr

vector field b defined by

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t, x) = b(\Phi(t, x)), & t \in \mathbb{R}, \\ \Phi(0, x) = x, & x \in \mathbb{R}^d. \end{cases} \quad (1.2)$$

Brenier has obtained the convergence of the solution to (1.1) under the ergodicity assumption of the flow Φ (see Def. 2.7 below). Then, using the two-scale procedure of Nguetseng [3], Allaire [4], Hou, Xin [5], Theorem 3.2 have proved that in dimension two, under the ergodicity assumption, the homogenized of (1.1) is still a transport equation with the average velocity \bar{b} . Moreover, Golse [6, 7] has extended these results to the locally periodic case $b_\varepsilon(x) = b(x, x/\varepsilon)$ with $\operatorname{div}_y b(x, \cdot) = 0$. A few years later, Tassa [8] has generalized the two dimensional result of [5] assuming that the vector field b satisfies the equation $\operatorname{div}(\rho b) = 0$ for some regular periodic positive function ρ . In the two-dimensional works [5, 8] the vector field b is not supposed to vanish in \mathbb{R}^2 , which leads to a simpler shear (uni-directional) flow by Kolmogorov's theorem. Actually, under the non-vanishing condition for b , Peirone [9] has proved that the limit $\lim_{\infty} \Phi(t, x)/t$ does exist for any point $x \in \mathbb{R}^2$, which allows him to recover in [10] the convergence result of [2] without the ergodicity assumption. More recently, it was obtained in [11] a necessary and sufficient condition on the flow (1.2) (see Rem. 3.1) so that the homogenized equation of (1.1) is still a transport equation with average velocity \bar{b} .

On the other hand, in 1989 answering to a De Giorgi's remark, Tartar [12] obtained in a non-periodic two-dimensional framework, a memory effect in the homogenization of the transport equation (1.1), assuming the decoupling of the variables $b_\varepsilon(x) = a_\varepsilon(x_2) \partial_{x_1}$, $f_\varepsilon = 0$ and u_ε^0 independent of ε . To this end, Tartar performed a Laplace transform in t and a Fourier transform in x_1 of equation (1.1) to derive an integral representation of the limit of the solution u_ε . The seminal Tartar's work on the memory effects through homogenization of linear transport equations has been extended by Mascarenhas [13], then by Amirat *et al.* [14–16] still assuming the decoupling of the coordinates. Alternatively, in the two-dimensional framework with a non-vanishing vector field b but without the ergodicity assumption, an integral representation [8], Theorem 4.5 of the weak limit u of the solution u_ε to (1.1) is given for the two-dimensional shear flow, while in [5], Theorem 3.2 a series expansion of u involving a countable family of ordinary differential equations is performed based on Kolmogorov's theorem. These two representations of the weak limit u in dimension two show implicitly the memory effect highlighted by Tartar [12] through both some representation of the weak limit u and a nonlocal equation satisfied by u .

In the present paper, having in mind Tartar's inquiry in [12]: “*Did we gain anything by expressing that the weak limit [of some nonlocal equation], or did we lure ourselves into playing the strange game (for a mathematician) of having a solution and looking for an equation?*”, we have definitely restricted ourselves to general integral representations of the weak limit in any dimension.

To this end, we assume that there exists a function $w \in C^1(\mathbb{R}^d)$ such that $b \cdot \nabla w$ is bounded from below by a positive constant. This is the main geometric assumption of the paper. As a consequence, we prove (see Prop. 2.1) that the flow Φ (1.2) associated with the vector field b induces the global one-to-one C^1 -mapping

$$\begin{aligned} \mathbb{R} \times \Sigma &\longrightarrow \mathbb{R}^d \\ (s, z) &\longmapsto y = \Phi(s, z) \text{ with } (s, z) = (\tau(y), \sigma(y)), \end{aligned} \quad (1.3)$$

where Σ is the regular hypersurface of \mathbb{R}^d defined as the equipotential $\{w = 0\}$, τ is a function in $C^1(\mathbb{R}^d)$ and σ is a vector-valued function in $C^1(\mathbb{R}^d; \Sigma)$. Moreover, the orbits of the flow cross transversally each equipotential $\{w = r\}$ for $r \in \mathbb{R}$.

Then, in Section 2.2 we establish (see Prop. 2.10) a complete characterization of the kernel of the operator

$$\begin{aligned} L_{\sharp}^2(\mathbb{T}^d) &\longrightarrow \mathcal{D}'(\mathbb{R}^d) \\ v &\longmapsto b(y) \cdot \nabla_y v(y), \end{aligned} \quad (1.4)$$

where $L_{\sharp}^2(\mathbb{T}^d)$ is the L^2 -space on the d -dimensional torus \mathbb{T}^d endowed with Lebesgue's measure, and $\mathcal{D}'(\mathbb{R}^d)$ is the space of distributions on \mathbb{R}^d . Then, we prove (see Thm. 2.13) that the kernel of operator (1.4) is isometrically isomorphic to the space $L_{\mu}^2(\widehat{\Sigma}/R)$, where R is a suitable equivalence relation in the compact hypersurface $\widehat{\Sigma} := \Sigma \cap \sigma([0, 1]^d)$, and μ is a finite measure on the quotient set $\widehat{\Sigma}/R$. This identification is essential, since it allows us to perform a pointwise representation on $\widehat{\Sigma}/R$ of the weak limit $u(t, x)$ of the solution $u_{\varepsilon}(t, x)$ to (1.1).

In Section 3.1, similarly to [5] we proceed by two-scale convergence to the homogenization of equation (1.1). Making a matching between the integrals over $\widehat{\Sigma}/R$ and the integrals over \mathbb{T}^d , we prove (see Thm. 3.2) that the weak limit $u(t, x)$ is given indifferently by two integral representations, one over the quotient set $\widehat{\Sigma}/R$, and the other following one over the torus \mathbb{T}^d ,

$$u(t, x) = \int_0^t \left(\int_{\mathbb{T}^d} P\hat{f}(r, x - (t-r)Pb(y), y) dy \right) dr + \int_{\mathbb{T}^d} P\hat{u}^0(x - tPb(y), y) dy, \quad (1.5)$$

where P is the orthogonal projection of the space $L_{\sharp}^2(\mathbb{T}^d)$ onto the kernel of operator (1.4), and where \hat{f} , \hat{u}^0 are respectively the two-scale limits of the sequences of data f_{ε} , u_{ε}^0 .

In particular, by virtue of the Birkhoff theorem the projection Pb is given by the two following expressions both in \mathbb{T}^d and in the quotient set $\widehat{\Sigma}/R$ (see Prop. 3.4):

- for almost-everywhere $y \in \mathbb{T}^d$,

$$Pb(y) = \lim_{t \rightarrow \infty} \frac{\Phi(t, y)}{t}; \quad (1.6)$$

- $Pb(\tilde{z}) = \eta(\tilde{z})$ for μ -almost-everywhere $\tilde{z} \in \widehat{\Sigma}/R$, where the function $\eta \in L_{\mu}^{\infty}(\widehat{\Sigma}/R)$ is given by the relation satisfied by any $v \in L_{\mu}^1(\widehat{\Sigma}/R)$,

$$\int_{\widehat{\Sigma}/R} \eta(\tilde{z}) v(\tilde{z}) d\mu(\tilde{z}) = \int_{\widehat{\Sigma}} \left(\int_{\{\sigma(y)=z\} \cap (0,1)^d} \frac{b(y)}{|J(\sigma)(y)|} d\mathcal{H}^1(y) \right) v(\pi(z)) d\mathcal{H}^{d-1}(z), \quad (1.7)$$

where $J(\sigma)$ is the Jacobian determinant of the map $\sigma : \mathbb{R}^d \rightarrow \Sigma$ defined by (1.3), π is the canonical projection from $\widehat{\Sigma}$ onto $\widehat{\Sigma}/R$, and $d\mathcal{H}^n$ is the n -dimensional Hausdorff measure.

Finally, in Section 3.4 we illustrate the previous results with the divergence free vector field $b(y) = a(Ay)\xi$, where a is a regular non-vanishing scalar function in \mathbb{R}^d , A is a matrix in $\mathbb{Z}^{d \times d}$, and ξ a non-zero vector in the kernel of A . In this case, Σ is the hyperplane $(\mathbb{R}\xi)^{\perp}$, and $\widehat{\Sigma}/R$ is a quotient group endowed with the pushforward measure of the Hausdorff measure on $\widehat{\Sigma}$ by the canonical projection on $\widehat{\Sigma}/R$. When ξ has only rational coordinates, we show that the kernel of (1.4) may be identified to the Hilbert space $L_{\sharp}^2(\mathbb{T}^{d-1})$.

Notation

- (e_1, \dots, e_d) denotes the canonical basis of \mathbb{R}^d , and $0_{\mathbb{R}^d}$ denotes the null vector of \mathbb{R}^d .
- “ \cdot ” denotes the scalar product, and $|\cdot|$ the euclidean norm in \mathbb{R}^d .
- $\mathbb{R}^{m \times n}$ for $m, n \in \mathbb{N}$, denotes the set of real matrices with m columns and n rows.
- $M^T \in \mathbb{R}^{n \times m}$ denotes the transposed matrix of the rectangular matrix $M \in \mathbb{R}^{m \times n}$.
- I_d denotes the unit matrix of $\mathbb{R}^{d \times d}$.
- A rectangular matrix M in $\mathbb{R}^{m \times n}$ whose columns are the vectors ξ^1, \dots, ξ^m in \mathbb{R}^n , is denoted by

$$M = (\xi^1 \mid \dots \mid \xi^m) \in \mathbb{R}^{m \times n}.$$

- $\xi^1 \times \cdots \times \xi^{d-1}$ denotes the cross product of $(d-1)$ vectors ξ^i in \mathbb{R}^d , and it is defined through the determinant by

$$(\xi^1 \times \cdots \times \xi^{d-1}) \cdot \xi = \det(\xi^1 \mid \cdots \mid \xi^{d-1} \mid \xi), \quad \forall \xi \in \mathbb{R}^n.$$

- \mathbb{T}^d for $d \in \mathbb{N}$, denotes the d -dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, which may be identified to the unit cube $[0, 1)^d$ in \mathbb{R}^d .
- \mathbb{S}^{d-1} denotes the unit sphere of \mathbb{R}^d .
- $|A|$ denotes the Lebesgue measure of any measurable set A in \mathbb{R}^d or in \mathbb{T}^d .
- π denotes the canonical surjection from \mathbb{R}^d onto \mathbb{T}^d .
- $C_c^k(\mathbb{R}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^d)$ with compact support in \mathbb{R}^d .
- $C_{\#}^k(\mathbb{T}^d)$ for $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$ which are \mathbb{Z}^d -periodic, *i.e.*

$$f(x+k) = f(x), \quad \forall k \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d. \quad (1.8)$$

- The gradient of a scalar function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is denoted by the vector-valued function ∇f with coordinates $\partial_{x_j} f$ for $j \in \{1, \dots, d\}$.
- The Jacobian matrix of a vector-valued function $F : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is denoted by the matrix-valued function $\nabla F \in \mathbb{R}^{n \times d}$ with entries $\partial_{x_j} F_i$ for $i \in \{1, \dots, n\}$, $j \in \{1, \dots, d\}$.
- dx denotes the Lebesgue measure on \mathbb{R}^d or on \mathbb{T}^d .
- $L_{\#}^p(\mathbb{T}^d)$ for $p \in [1, \infty]$, denotes the space of the Lebesgue measurable functions φ in $L_{\text{loc}}^p(\mathbb{R}^d)$, which are \mathbb{Z}^d -periodic dx -a.e. in \mathbb{R}^d , and $H_{\#}^1(\mathbb{T}^d)$ denotes the subspace of $L_{\#}^2(\mathbb{T}^d)$ composed of the functions φ such that $\nabla \varphi \in L_{\#}^2(\mathbb{T}^d)^d$.
- $\mathcal{D}'(\mathbb{R}^d)$ denotes the space of the distributions on \mathbb{R}^d .
- For $f \in L_{\#}^1(\mathbb{T}^d)$, we denote the mean-value of f by

$$\bar{f} := \int_{\mathbb{T}^d} f(y) dy.$$

- The abbreviations “a.e.” for almost everywhere and “s.t” for such that, will be used throughout the paper. The simple mention “a.e.” refers to the Lebesgue measure on \mathbb{R}^d .
- c denotes a positive constant which may vary from line to line.

2. THE KERNEL OF THE OPERATOR $b(y) \cdot \nabla_y(\cdot)$

2.1. The main assumption on the flow

Let \mathbb{T}^d be the torus in dimension $d \geq 2$. In the whole paper b denotes a vector field in $C_{\#}^1(\mathbb{T}^d)^d$ satisfying

$$\operatorname{div} b = 0 \quad \text{in } \mathbb{R}^d. \quad (2.1)$$

Consider the flow $\Phi \in C^1(\mathbb{R} \times \mathbb{R}^d)^d$ associated with the vector field b solution to the Cauchy problem (1.2). By the uniqueness of a solution to (1.2) the flow Φ satisfies the semi-group property

$$\Phi(s, \Phi(t, x)) = \Phi(s+t, x), \quad \forall (s, t) \in \mathbb{R}^2, \forall x \in \mathbb{R}^d. \quad (2.2)$$

Moreover, due the periodicity of b it satisfies

$$\Phi(t, x + k) = \Phi(t, x) + k, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \forall k \in \mathbb{Z}^d. \quad (2.3)$$

For any $x \in \mathbb{R}^d$, we define the orbit of the point x by

$$\Gamma_x := \{\Phi(t, x), t \in \mathbb{R}\}. \quad (2.4)$$

Recall that the set of all the orbits provide a partition of the space \mathbb{R}^d .

In the whole paper we assume the following geometric condition on the flow Φ : there exists a function $w \in C^1(\mathbb{R}^d)^d$ satisfying

$$m := \inf_{\mathbb{R}^d} (b \cdot \nabla w) > 0. \quad (2.5)$$

We have the following result.

Proposition 2.1. *Let $b \in C_{\sharp}^1(\mathbb{T}^d)^d$. Assume that condition (2.5) holds and define the C^1 -regular equipotential surface*

$$\Sigma := \{w = 0\}. \quad (2.6)$$

Then, there exist a function $\tau \in C^1(\mathbb{R})$ and a vector-valued function $\sigma \in C^1(\mathbb{R}^d; \Sigma)$ such that

$$\forall y \in \mathbb{R}^d, \exists! (s, z) \in \mathbb{R} \times \Sigma, \quad y = \Phi(s, z) \quad \text{with} \quad (s, z) = (\tau(y), \sigma(y)). \quad (2.7)$$

Remark 2.2. Assumption (2.5) means that the vector field b intersects transversally any equipotential hypersurface $\{w = r\}$ for $r \in \mathbb{R}$. As a consequence, each orbit (2.4) of the flow Φ intersects exactly once the hypersurface (2.6), which leads us to the global parametrization (2.7) of the flow.

Proof of Proposition 2.1. We start by following the procedure of the proof of [17], Theorem 2.15 to build a conductivity from a given periodic electric field. To this end, first note that the condition (2.5) combined with (1.2) implies that for any $y \in \mathbb{R}^d$,

$$\partial_t (w(\Phi(t, y))) = (b \cdot \nabla w)(\Phi(t, y)) \geq m > 0, \quad \forall t \in \mathbb{R}. \quad (2.8)$$

Hence, by the mean value theorem the function $(t \mapsto w(\Phi(t, y)))$ is a one-to-one increasing function from \mathbb{R} onto \mathbb{R} , so that there exists a unique function $\omega : \mathbb{R}^d \rightarrow \mathbb{R}$ solution to

$$w(\Phi(\omega(y), y)) = 0, \quad \forall y \in \mathbb{R}^d. \quad (2.9)$$

Since the function $((t, y) \mapsto w(\Phi(t, y)))$ is in $C^1(\mathbb{R} \times \mathbb{R}^d)$ and $\partial_t (w(\Phi(t, y))) > 0$ by (2.8), the implicit function theorem implies that the function ω belongs to $C^1(\mathbb{R}^d)$. Then, due to the semi-group property of the flow (2.2) we have

$$\Phi(-\omega(y), \Phi(\omega(y), y)) = y, \quad \forall y \in \mathbb{R}^d.$$

Finally, this combined with (2.9) and (2.6) yields the representation (2.7) with

$$\tau(y) := -\omega(y) \quad \text{and} \quad \sigma(y) := \Phi(\omega(y), y) \in \Sigma. \quad (2.10)$$

where $\tau \in C^1(\mathbb{R}^d)$ and $\sigma \in C^1(\mathbb{R}^d; \Sigma)$.

Conversely, if $y = \Phi(s, z)$ with $s \in \mathbb{R}$ and $z \in \Sigma$, by the semi-group property (2.2) we have $z = \Phi(-s, y)$ which implies that $w(\Phi(-s, y)) = 0$. Hence, by the uniqueness of the function ω satisfying (2.9), it follows that

$s = -\omega(y) = \tau(y)$ and $z = \sigma(y)$. Therefore, this shows the uniqueness in the representation (2.7) and concludes the proof of Proposition 2.1. \square

Proposition 2.3. *Let $b \in C_{\sharp}^1(\mathbb{T}^d)^d$. Then, for any $y \in \mathbb{R}^d$, the linear mapping $\nabla\sigma(y)$ is surjective from \mathbb{R}^d onto the tangent plane $T_{\sigma(y)}$ to Σ at the point $\sigma(y)$.*

Proof. Let $y \in \mathbb{R}^d$. Then, for $(s, z) := (\tau(y), \sigma(y))$ and $\zeta \in T_{\sigma(y)}$, we have $\nabla_z \Phi(s, z) \zeta = \theta(s)$, where $\theta = \theta(r)$ is the solution to the equation

$$\begin{cases} \frac{d\theta}{dr}(r) = \nabla_y b(\Phi(r, z)) \theta(r), & r \in [0, \infty), \\ \theta(0) = \zeta, \end{cases}$$

which implies that $\nabla\sigma(y) \theta(s) = \zeta$. \square

Remark 2.4. The condition (2.7) on the flow Φ implies that the associated vector field b does not vanish in \mathbb{T}^d . Indeed, if there exists some $y = \Phi(r, z) \in \Phi(\mathbb{R} \times \Sigma)$ such that $b(y) = 0_{\mathbb{R}^d}$, then the orbit Γ_y is reduced to the unit set $\{y\}$. Hence, $y = z \in \Sigma$ and $\Phi(s, z) = y$ for any $s \in \mathbb{R}$. This contradicts the uniqueness of s in (2.7).

2.2. Characterization of the kernel

Define the space

$$\mathcal{V}_{\sharp} := \{\hat{v} \in L_{\sharp}^2(\mathbb{T}^d) : b \cdot \nabla \hat{v} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d)\}, \quad \mathcal{W}_{\sharp} := \mathcal{V}_{\sharp} \cap H_{\sharp}^1(\mathbb{T}^d). \quad (2.11)$$

Our aim is to obtain a representation of the elements of \mathcal{V}_{\sharp} . To this end, we will use the following result.

Proposition 2.5. *Let $v \in L_{\text{loc}}^1(\mathbb{R}^{d+1})$. Then, the following equivalence holds*

$$\partial_s v + b \cdot \nabla_x v = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d) \quad \Leftrightarrow \quad \partial_s [v(s, \Phi(s, y))] = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d). \quad (2.12)$$

Proof of Proposition 2.5. It is enough to use that the mapping $\Psi : (s, y) \mapsto (s, \Phi(s, y))$ is a C^1 -diffeomorphism on $\mathbb{R} \times \mathbb{R}^d$. Namely, by definition of the distributional derivative, we have $\partial_s v + b \cdot \nabla_x v = 0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ if, and only, if we have for any $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_{\mathbb{R} \times \mathbb{R}^d} v (\partial_s \varphi + b \cdot \nabla_x \varphi) \, ds \, dx = 0, \quad (2.13)$$

where the change of variables $(s, x) = \Psi(s, y)$ (whose Jacobian is 1 since b has null divergence) provides

$$\int_{\mathbb{R} \times \mathbb{R}^d} v (\partial_s \varphi + b \cdot \nabla_x \varphi) \, ds \, dx = \int_{\mathbb{R} \times \mathbb{R}^d} v(s, \Phi(s, y)) \partial_s [\varphi(s, \Phi(s, y))] \, ds \, dy. \quad (2.14)$$

However, since Ψ is a C^1 -diffeomorphism, a function ϕ belongs to $C_c^1(\mathbb{R} \times \mathbb{R}^d)$ if, and only if, there exists a function $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}^d)$ such that $\phi(s, y) = \varphi(s, \Phi(s, y))$ in $\mathbb{R} \times \mathbb{R}^d$. Therefore, (2.13) and (2.14) are equivalent to $\partial_s [v(s, \Phi(s, y))] = 0$ in the distributional sense. \square

By virtue of Proposition 2.5 combined with condition (2.7) we may define the overspace \mathcal{V} of \mathcal{V}_{\sharp} by the two following ways:

$$\begin{aligned} \mathcal{V} &:= \{\hat{v} \in L_{\text{loc}}^2(\mathbb{R}^d) : b \cdot \nabla \hat{v} = 0 \text{ in } \mathbb{R}^d\} \\ &= \{\hat{v} \in L_{\text{loc}}^2(\mathbb{R}^d) : \partial_s (\hat{v}(\Phi(s, z))) = 0 \text{ in } \mathbb{R} \times \Sigma\}. \end{aligned} \quad (2.15)$$

From now on, for each function $\hat{v} \in \mathcal{V}$, we consider a representative of \hat{v} such that $\hat{v}(\Phi(\cdot, z))$ is constant in \mathbb{R} for any $z \in \Sigma \setminus N$, with $\mathcal{H}^{d-1}(N) = 0$.

Remark 2.6. The meaning of the kernel $\mathcal{V}_\#$ (2.14) is given by Ergodic Theory as follows (we refer to [18], Chapters I & II for a comprehensive introduction).

First, since b is divergence free, the Liouville theorem [18], Section II.2, Theorem 1 implies that the Lebesgue measure is invariant for the flow Φ (1.2), *i.e.* for any measurable set A of \mathbb{R}^d ,

$$|\{\Phi(s, y) : y \in A\}| = |A|, \quad \forall s \in \mathbb{R}.$$

Then, the von Neumann theorem [18], Section I.7, Theorem 4 and the Birkhoff ergodic theorem [18], Section I.2, Theorem 4 imply that the orthogonal projection P from the Hilbert space $L^2_\#(\mathbb{T}^d)$ onto the subspace $\mathcal{V}_\#$ satisfies the time average convergence

$$\frac{1}{t} \int_0^t v(\Phi(s, y)) \, ds \xrightarrow[t \rightarrow \infty]{} Pv(y) \text{ strongly in } L^2_\#(\mathbb{T}^d) \text{ and a.e. } y \in \mathbb{T}^d, \quad \forall v \in L^2_\#(\mathbb{T}^d),$$

where the function Pv is invariant for the flow in the weak sense of Proposition 2.5.

Definition 2.7. The flow Φ is said to be ergodic (with respect to Lebesgue's measure) if the space $\mathcal{V}_\#$ is reduced to the space of constant functions, or equivalently, if

$$Pv = \int_{\mathbb{T}^d} v \, dy, \quad \forall v \in \mathcal{V}_\#.$$

Now, to obtain a finer characterization of the space $\mathcal{V}_\#$, we introduce an equivalence relation between the points of the hypersurface Σ through their orbits.

Definition 2.8. We define in Σ the equivalence relation R by

$$z_1 \overset{R}{\sim} z_2 \text{ if } \exists s_1, s_2 \in \mathbb{R} \text{ s.t. } \Phi(s_1, z_1) - \Phi(s_2, z_2) = k \in \mathbb{Z}, \quad (2.16)$$

or equivalently,

$$z_1 \overset{R}{\sim} z_2 \text{ if } \Gamma_{z_1} \cap (\Gamma_{z_2} + \mathbb{Z}^d) \neq \emptyset. \quad (2.17)$$

The equivalence class of $z \in \Sigma$ is denoted by \tilde{z} .

Remark 2.9. Taking into account that by the definition (1.2) of Φ and by (2.3), equality $\Phi(s_1, z_1) - \Phi(s_2, z_2) = k$ implies

$$\Phi(s, z_1) = \Phi(s + s_2 - s_1, z_2) + k, \quad \forall s \in \mathbb{R},$$

we also have

$$z_1 \overset{R}{\sim} z_2 \Leftrightarrow \exists s \in \mathbb{R}, \exists k \in \mathbb{Z}^d \text{ s.t. } z_1 = \Phi(s, z_2) + k. \quad (2.18)$$

Then, we obtain the following characterization of the space $\mathcal{V}_\#$.

Proposition 2.10. *A function \hat{v} belongs to \mathcal{V}_{\sharp} if, and only if, $\hat{v} \in L^2_{\text{loc}}(\mathbb{R}^d)$ and there exist a representative of \hat{v} , still denoted by \hat{v} , and a subset $N \subset \Sigma$ with $\mathcal{H}^{d-1}(N) = 0$, such that $\hat{v}(y)$ is well defined for any $y \in \mathbb{R}^d \setminus \sigma^{-1}(N)$ and satisfies*

$$\hat{v}(y) = \hat{v}(z), \quad \forall y \in \mathbb{R}^d \setminus \sigma^{-1}(N), \forall z \in \widetilde{\sigma(y)} \setminus N. \quad (2.19)$$

Proof. Let \hat{v} be a function in \mathcal{V}_{\sharp} . Due to $\hat{v} \in \mathcal{V}$, we can choose a representative of the function \hat{v} which is well defined for any $y \in \mathbb{R}^d \setminus \sigma^{-1}(N)$. Take $z \in \widetilde{\sigma(y)} \setminus N$ and $s \in \mathbb{R}$, $k \in \mathbb{Z}^d$ such that $\Phi(s, z) + k = \sigma(y)$, then we have

$$\hat{v}(y) = \hat{v}(\sigma(y)) = \hat{v}(\Phi(s, z) + k) = \hat{v}(\Phi(s, z)) = \hat{v}(z).$$

Conversely, consider a function $\hat{v} \in L^2_{\text{loc}}(\mathbb{R}^d)$ satisfying (2.19) for a representative of \hat{v} and $N \subset \Sigma$ with $\mathcal{H}^{d-1}(N) = 0$. In particular, it satisfies $\hat{v}(y) = \hat{v}(\sigma(y))$ for any $y \in \mathbb{R}^d \setminus \sigma^{-1}(N)$, so that it belongs to \mathcal{V} . Now, for any $y \in \mathbb{R}^d$ and any $k \in \mathbb{Z}^d$ satisfying $y, y + k \notin \sigma^{-1}(N)$, there exist $s, r \in \mathbb{R}$ such that

$$y = \Phi(s, \sigma(y)) \quad \text{and} \quad y + k = \Phi(r, \sigma(y + k)),$$

we have

$$\sigma(y + k) \stackrel{R}{\sim} \sigma(y) \quad \text{and} \quad \sigma(y), \sigma(y + k) \notin N,$$

which by (2.19) implies that

$$\hat{v}(y) = \hat{v}(\sigma(y + k)) = \hat{v}(y + k), \quad \forall y \in \mathbb{R}^d, \forall k \in \mathbb{Z}^d \quad \text{with} \quad y, y + k \notin \sigma^{-1}(N).$$

Since by Fubini's theorem we have $|\sigma^{-1}(N)| = 0$, we conclude that \hat{v} belongs to \mathcal{V}_{\sharp} . \square

Next, the following result provides a deeper identification of the kernel \mathcal{V}_{\sharp} by passing to the Lebesgue measure in \mathbb{T}^d to a suitable measure on a compact subset of the hypersurface Σ .

Proposition 2.11. *Let $J(\sigma)$ be the product of the singular values of the matrix $\nabla\sigma(y)^T$ considered as a linear mapping from the $(d-1)$ -dimensional tangent space $T_{\sigma(y)}$ in \mathbb{R}^d . Let $\widehat{\Sigma}$ be the compact hypersurface defined by*

$$\widehat{\Sigma} := \Sigma \cap \sigma([0, 1]^d), \quad (2.20)$$

where $\sigma : \mathbb{R}^d \rightarrow \Sigma$ is the C^1 -regular function given by (2.10).

Then, we have the following change of variables formula on $\widehat{\Sigma}$,

$$\int_{\mathbb{T}^d} g(y) \hat{v}(y) \, dy = \int_{\widehat{\Sigma}} p_g(z) \hat{v}(z) \, d\mathcal{H}^{d-1}(z), \quad \forall g \in L^2_{\sharp}(\mathbb{T}^d), \forall \hat{v} \in \mathcal{V}_{\sharp}, \quad (2.21)$$

where the function p_g is defined by

$$p_g(z) := \int_{\{\sigma(y)=z\} \cap (0,1)^d} \frac{g(y)}{|J(\sigma)(y)|} \, d\mathcal{H}^1(y) \quad \text{for } z \in \widehat{\Sigma}, \quad (2.22)$$

and belongs to $L^2(\widehat{\Sigma}, d\mathcal{H}^{d-1})$.

Remark 2.12. In formula (2.22) any function g in $L^2_{\sharp}(\mathbb{T}^d)$ defined in the torus \mathbb{T}^d , is identified to a function in $L^2((0,1)^d)$ defined in the cube $(0,1)^d$.

Proof of Proposition 2.11. Since $\nabla\sigma(y)$ is surjective from \mathbb{R}^d onto $T_{\sigma(y)}$, it has rank $(d-1)$ for any $y \in \mathbb{R}^d$, so that the determinant $J(\sigma)$ does not vanish in \mathbb{R}^d . Moreover, since $\nabla\sigma$ is continuous in \mathbb{R}^d , we also have

$$\exists c > 0, \quad J(\sigma) \geq c \text{ in } (0,1)^d. \quad (2.23)$$

Therefore, by virtue of the Proposition A.1 in the Appendix below and by the identification of Remark 2.12, the change of variables formula (A.3) on the compact hypersurface $\widehat{\Sigma}$ applied with the open set $\Omega := (0,1)^d$, reads as for any $h \in L^1_{\sharp}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} h(y) \, dy = \int_{(0,1)^d} h(y) \, dy = \int_{\widehat{\Sigma}} \left(\int_{\{\sigma(y)=z\} \cap (0,1)^d} \frac{h(y)}{J(\sigma)(y)} \, d\mathcal{H}^1(y) \right) \, d\mathcal{H}^{d-1}(z). \quad (2.24)$$

Taking in equality (2.24) the function $h = g \hat{v}$ with $g \in L^2_{\sharp}(\mathbb{T}^d)$ and $\hat{v} \in \mathcal{V}_{\sharp}$, then using the equality $\hat{v}(y) = \hat{v}(\sigma(y))$, we obtain the desired formula (2.21), where by virtue of Fubini's theorem $p_g \in L^1(\widehat{\Sigma}, d\mathcal{H}^{d-1})$.

Finally, using estimate (2.23), $g \in L^2_{\sharp}(\mathbb{T}^d)$ (recall Rem. 2.12) and the compactness of $\widehat{\Sigma}$, we also get that $p_g \in L^2(\widehat{\Sigma}, d\mathcal{H}^{d-1})$. \square

Finally, by the above propositions we get the following identification of the kernel space \mathcal{V}_{\sharp} .

Theorem 2.13. *There exists a probability measure μ on the quotient set $\widehat{\Sigma}/R$ such that the linear mapping $\mathcal{J} : \mathcal{V}_{\sharp} \rightarrow L^2_{\mu}(\widehat{\Sigma}/R)$ defined by*

$$\mathcal{J}(\hat{v})(\tilde{z}) := \hat{v}(z), \quad \mu\text{-a.e. } \tilde{z} \in \widehat{\Sigma}/R, \quad (2.25)$$

is a one-to-one isometry which is characterized by the change of variables formula on $\widehat{\Sigma}/R$,

$$\int_{\mathbb{T}^d} \hat{v}(y) \, dy = \int_{\widehat{\Sigma}/R} \hat{v}(\tilde{z}) \, d\mu(\tilde{z}), \quad \forall \hat{v} \in \mathcal{V}_{\sharp}. \quad (2.26)$$

Proof. On the one hand, consider

$$\pi : \widehat{\Sigma} \rightarrow \widehat{\Sigma}/R \text{ the canonical projection of } \widehat{\Sigma} \text{ onto } \widehat{\Sigma}/R. \quad (2.27)$$

The set $\widehat{\Sigma}/R$ is endowed with the sigma-algebra

$$\mathcal{B}_R := \left\{ \tilde{A} \subset \widehat{\Sigma}/R : \pi^{-1}(\tilde{A}) \text{ is a Borel subset of } \widehat{\Sigma} \right\}.$$

Then, define the measure μ on the measured space $(\widehat{\Sigma}/R, \mathcal{B}_R)$ by

$$\mu(\tilde{A}) := \int_{\pi^{-1}(\tilde{A})} p_1(z) \, d\mathcal{H}^{d-1}(z) \quad \text{for } \tilde{A} \in \mathcal{B}_R, \quad (2.28)$$

where the non-negative function $p_1 \in L^\infty(\widehat{\Sigma}, d\mathcal{H}^{d-1})$ is defined by (2.22) with $g = 1$. Taking $g = \hat{v} = 1$ in formula (2.21) and using (2.28) with $\pi^{-1}(\widehat{\Sigma}/R) = \widehat{\Sigma}$, we get that

$$1 = \int_{\mathbb{T}^d} dy = \int_{\widehat{\Sigma}} p_1(z) d\mathcal{H}^{d-1}(z) = \mu(\widehat{\Sigma}/R),$$

which implies that μ is a probability measure on $\widehat{\Sigma}/R$. The measure μ also agrees with the pushforward measure of the measure $p_1(z) d\mathcal{H}^{d-1}$ on the hypersurface $\widehat{\Sigma}$ by the canonical projection π of $\widehat{\Sigma}$ onto the quotient set $\widehat{\Sigma}/R$, or equivalently,

$$\int_{\widehat{\Sigma}/R} v(\tilde{z}) d\mu(\tilde{z}) = \int_{\widehat{\Sigma}} v(\pi(z)) p_1(z) d\mathcal{H}^{d-1}(z), \quad \forall v \in L^2_\mu(\widehat{\Sigma}/R). \quad (2.29)$$

On the other hand, for any function $\hat{v} \in \mathcal{V}_\sharp$, equality (2.19) implies that

$$\hat{v}(z) = \hat{v}(\tilde{z}) = \hat{v}(\pi(z)) \quad \text{a.e. } z \in \widehat{\Sigma}, \quad (2.30)$$

since by definition $\tilde{z} = \pi(z)$. Therefore, equating (2.21) with (2.29) yields the desired change of variables formula (2.26).

Next, formula (2.26) is still satisfied by any product $\hat{v} \hat{w}$ with $\hat{v} \in \mathcal{V}_\sharp$ and $\hat{w} \in L^\infty_\mu(\mathbb{T}^d) \cap \mathcal{V}_\sharp$. However, since the space \mathcal{V}_\sharp is clearly stable by truncation, the space $L^\infty_\mu(\mathbb{T}^d) \cap \mathcal{V}_\sharp$ is dense into \mathcal{V}_\sharp for the L^2 -norm. Therefore, formula (2.26) also holds for any product $\hat{v} \hat{w}$ in \mathcal{V}_\sharp . This combined with (2.30) finally implies that the mapping (2.25) is a one-to-one isometry from \mathcal{V}_\sharp onto $L^2_\mu(\widehat{\Sigma}/R)$ which is characterized by formula (2.26). \square

3. THE SOLUTION TO THE HOMOGENIZED TRANSPORT EQUATION

3.1. Homogenization of the transport equation

Fix $T > 0$. We will derive a macroscopic model of the transport equation (1.1), in which the sequence f_ε is assumed to be bounded in $L^2(0, T; L^2(\mathbb{R}^d))$ and the sequence u_ε^0 is assumed to be bounded in $L^2(\mathbb{R}^d)$.

To this end, we will apply the two-scale convergence procedure of Nguetseng [3] and Allaire [4]. Recall that for any bounded sequence $v_\varepsilon(t, x)$ in $L^2(0, T; L^2(\mathbb{R}^d))$, there exists a function $\hat{v}(t, x, y)$ in $L^2(0, T; L^2(\mathbb{R}^d \times \mathbb{T}^d))$ such that, up to a subsequence of ε , one has for any $\psi(t, x, y) \in C^\infty([0, T]; C_c^\infty(\mathbb{R}^d; C_\sharp^\infty(\mathbb{T}^d)))$ with compact support in x ,

$$\int_{(0, T) \times \mathbb{R}^d} v_\varepsilon(t, x) \psi(t, x, x/\varepsilon) dt dx = \int_{(0, T) \times \mathbb{R}^d \times \mathbb{R}^d} \hat{v}(t, x, y) \psi(t, x, y) dt dx dy. \quad (3.1)$$

The sequence $v_\varepsilon(t, x)$ is said to two-scale converge to $\hat{v}(t, x, y)$, which is denoted by

$$v_\varepsilon \xrightarrow{2s} \hat{v} \quad \text{in } (0, T) \times \mathbb{R}^d.$$

Then, up to extract a subsequence, there exist a function $\hat{f} \in L^2(0, T; L^2(\mathbb{R}^d; L^2_\sharp(\mathbb{T}^d)))$ and a function $\hat{u}^0 \in L^2(\mathbb{R}^d; L^2_\sharp(\mathbb{T}^d))$ such that the two-scale convergences in the sense of (3.1) hold:

$$f_\varepsilon \xrightarrow{2s} \hat{f} \quad \text{in } (0, T) \times \mathbb{R}^d, \quad (3.2)$$

$$u_\varepsilon^0 \xrightarrow{2s} \hat{u}^0 \quad \text{in } \mathbb{R}^d. \quad (3.3)$$

Taking into account that u_ε exists and it is bounded in $L^\infty(0, T; L^2(\mathbb{R}^d))$, we can also extract a subsequence such that

$$u_\varepsilon \xrightarrow{2s} \hat{u} \quad \text{in } (0, T) \times \mathbb{R}^d, \quad (3.4)$$

for some $\hat{u} \in L^\infty(0, T; L^2(\mathbb{R}^d; L^2_\#(\mathbb{T}^d)))$, which then implies that

$$u_\varepsilon \xrightarrow{*} u(t, x) := \int_{\mathbb{T}^d} \hat{u}(t, x, y) \, dy \quad \text{in } L^\infty(0, T; L^2(\mathbb{R}^d)). \quad (3.5)$$

Our aim is to characterize the two-scale limit \hat{u} , then the function u .

Following the two-scale procedure of [5], Section 2 applied to the transport equation, for any $\hat{\varphi} \in C_c^\infty((0, T) \times \mathbb{R}^d; C^\infty_\#(\mathbb{T}^d))$, we put $\varepsilon \hat{\varphi}(t, x, x/\varepsilon)$ as test function in (1.1). Passing to the limit as ε tends to zero, we easily get that

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} b(y) \hat{u}(t, x, y) \cdot \nabla_y \hat{\varphi}(t, x, y) \, dy dx dt = 0.$$

By the definition (2.11) of the space $\mathcal{V}_\#$ the function \hat{u} satisfies

$$\hat{u}(t, x, \cdot) \in \mathcal{V}_\#, \quad \text{a.e. } (t, x) \in (0, T) \times \mathbb{R}^d. \quad (3.6)$$

Next, consider any function $\hat{\varphi} \in C^\infty(0, T; C_c^\infty(\mathbb{R}^d; \mathcal{W}_\#))$, such that $\varphi(T, x, y) = 0$ in $\mathbb{R}^d \times \mathbb{T}^d$. Taking $\hat{\varphi}(t, x, x/\varepsilon)$ as test function in (1.1) and passing to the limit as ε tends to zero, we also get that

$$\begin{aligned} & - \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \hat{u}^0 \hat{\varphi}(0, x, y) \, dy dx - \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} (\hat{u} \partial_t \hat{\varphi} + \hat{u} b \cdot \nabla_x \hat{\varphi}) \, dy dx dt \\ & = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} \hat{f} \hat{\varphi} \, dy dx dt. \end{aligned}$$

By a density argument this holds for any $\hat{\varphi} \in H^1((0, T) \times \mathbb{R}^d; \mathcal{V}_\#)$, with $\varphi(T, x, y) = 0$. Then, replacing $\hat{\varphi}(t, x, y)$ by $\psi(t, x) \hat{\varphi}(y)$ with $\hat{\varphi} \in \mathcal{V}_\#$ and $\psi \in H^1((0, T) \times \mathbb{R}^d)$, we obtain that the function \hat{u} is solution to the problem

$$\begin{cases} \hat{u} \in L^\infty(0, T; L^2(\mathbb{R}^d; \mathcal{V}_\#)) \\ \frac{d}{dt} \left(\int_{\mathbb{T}^d} \hat{u} \hat{\varphi} \, dy \right) + \operatorname{div}_x \left(\int_{\mathbb{T}^d} \hat{u} b \hat{\varphi} \, dy \right) = \int_{\mathbb{T}^d} \hat{f} \hat{\varphi} \, dy & \text{in } (0, T) \times \mathbb{R}^d, \quad \forall \hat{\varphi} \in \mathcal{V}_\#, \\ \int_{\mathbb{T}^d} (\hat{u}(0, x, y) - \hat{u}^0) \hat{\varphi} \, dy = 0 & \text{for } x \in \mathbb{R}^d, \quad \forall \hat{\varphi} \in \mathcal{V}_\#. \end{cases} \quad (3.7)$$

This problem has a unique solution (see, *e.g.*, [19], Lem. 3.3), so it characterizes the function \hat{u} . In particular, this shows that it is not necessary to extract a new subsequence for the weak convergence (3.4).

Remark 3.1. If the flow Φ (1.2) is ergodic then the problem (3.7) shows that the solution u_ε of (1.1) converges in the weak-* topology of $L^\infty(0, T; L^2(\mathbb{R}^d))$ to the solution u of the transport equation

$$\begin{cases} \frac{\partial u}{\partial t} + \bar{b} \cdot \nabla_x u = \int_{\mathbb{T}^d} \hat{f}(t, x, y) \, dy, & (t, x) \in (0, T) \times \mathbb{R}^d, \\ u(0, x) = \int_{\mathbb{T}^d} u^0(x, y) \, dy, & x \in \mathbb{R}^d, \end{cases} \quad (3.8)$$

where \bar{b} is the mean-value of b . This result was first obtained in [5], however this can be true even if Φ is not ergodic. Actually, we have proved [11], Theorem 3.2 that a sufficient and necessary condition for which the limit equation of (1.2) reads as (3.8), is given by

$$\# \left\{ \int_{\mathbb{T}^d} \rho(y) b(y) dy : \rho \in L^1_{\#}(\mathbb{T}^d), \rho(y) dy \text{ invariant probability measure for } \Phi \right\} = 1. \quad (3.9)$$

The set in (3.9) is the subset of the so-called Herman rotation set [20] (see also [21] for a complete overview) defined by

$$C_b := \left\{ \int_{\mathbb{T}^d} b(y) \nu(dy) : \nu \text{ invariant probability measure for } \Phi \right\}. \quad (3.10)$$

3.2. Integral representations of the solution

Denoting by P the orthogonal projection of $L^2_{\#}(\mathbb{T}^d)$ onto the kernel $\mathcal{V}_{\#}$ characterized by Proposition 2.10, we have the following representation result.

Theorem 3.2. *Let b a divergence free vector field in $C^1_{\#}(\mathbb{T}^d)^d$ satisfying the flow condition (2.7). Then, assuming (3.2) and (3.3), the weak limit $u(t, x)$ of the solution $u_{\varepsilon}(t, x)$ to the transport equation (1.1) is written indifferently by the two following integral formulas:*

- One over the quotient set $\widehat{\Sigma}/R$:

$$\begin{aligned} u(t, x) &= \int_0^t \left(\int_{\widehat{\Sigma}/R} P\hat{f}(r, x - (t-r)\eta(\tilde{z}), \tilde{z}) d\mu(\tilde{z}) \right) dr \\ &\quad + \int_{\widehat{\Sigma}/R} P\hat{u}^0(x - t\eta(\tilde{z}), \tilde{z}) d\mu(\tilde{z}), \end{aligned} \quad (3.11)$$

where the function $\eta \in L^{\infty}(\widehat{\Sigma}/R)$ is associated through the duality $L^1_{\mu} - L^{\infty}_{\mu}$

$$\int_{\widehat{\Sigma}} p_b(z) v(\pi(z)) d\mathcal{H}^{d-1}(z) = \int_{\widehat{\Sigma}/R} \eta(\tilde{z}) v(\tilde{z}) d\mu(\tilde{z}), \quad \forall v \in L^1_{\mu}(\widehat{\Sigma}/R). \quad (3.12)$$

Here, the vector-valued function $p_b \in L^{\infty}(\widehat{\Sigma}, d\mathcal{H}^{d-1})^d$ is defined componentwise by formula (2.22) with $g = b$.

- The other one over the torus \mathbb{T}^d :

$$\begin{aligned} u(t, x) &= \int_0^t \left(\int_{\mathbb{T}^d} P\hat{f}(r, x - (t-r)Pb(y), y) dy \right) dr \\ &\quad + \int_{\mathbb{T}^d} P\hat{u}^0(x - tPb(y), y) dy. \end{aligned} \quad (3.13)$$

Remark 3.3. In the particular case where \hat{u}^0 and f do not depend on the oscillating variable y , we have $P\hat{f} = f$ and $P\hat{u}^0 = \hat{u}^0$, since the constant functions belong to $\mathcal{V}_{\#}$. Hence, the representation (3.13) is reduced to

$$u(t, x) = \int_0^t \left(\int_{\mathbb{T}^d} f(r, x - (t-r)Pb(y)) dy \right) dr + \int_{\mathbb{T}^d} \hat{u}^0(x - tPb(y)) dy. \quad (3.14)$$

In the particular case where b satisfies the equality

$$\int_{\mathbb{T}^d} b(y) \hat{\varphi}(y) dy = \int_{\mathbb{T}^d} b(y) dy \int_{\mathbb{T}^d} \hat{\varphi}(y) dy, \quad \forall \hat{\varphi} \in \mathcal{V}_{\sharp},$$

we get that

$$Pb(y) = \bar{b} := \int_{\mathbb{T}^d} b(y) dy,$$

so that \hat{u} agrees with the solution to (3.8).

Proof of Theorem 3.2. We will apply the identification Theorem 2.13 to rewrite the limit problem (3.7) as a family of transport equations in $(0, T) \times \mathbb{R}^d$ which is indexed by $\tilde{z} \in \widehat{\Sigma}/R$ μ -a.e..

First of all, using that $\hat{u}(t, x, \cdot) \in \mathcal{V}_{\sharp}$ for a.e. $(t, x) \in (0, T) \times \mathbb{R}^d$ and applying the change of variables formula (2.26), we have for any $\varphi \in \mathcal{V}_{\sharp}$,

$$\int_{\mathbb{T}^d} \hat{u}(t, x, y) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}/R} \hat{u}(t, x, \tilde{z}) \hat{\varphi}(\tilde{z}) d\mu(\tilde{z}). \quad (3.15)$$

Let $(t, x) \in (0, T) \times \mathbb{R}^d$ and let $\varphi \in \mathcal{V}_{\sharp}$. Since any element of \mathcal{V}_{\sharp} is constant in each orbit of the flow, by formula (2.26) we can write in problem (3.7)

$$\int_{\mathbb{T}^d} \hat{f}(t, x, y) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}/R} P\hat{f}(t, x, \tilde{z}) \hat{\varphi}(\tilde{z}) d\mu(\tilde{z}), \quad (3.16)$$

$$\int_{\mathbb{T}^d} (\hat{u}(0, x, y) - \hat{u}^0(x, y)) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}/R} (\hat{u}(0, x, \tilde{z}) - P\hat{u}^0(x, \tilde{z})) \hat{\varphi}(\tilde{z}) d\mu(\tilde{z}). \quad (3.17)$$

For the second integral in (3.7), first note that by the change of variables formula (2.21) we have

$$\int_{\mathbb{T}^d} b(y) \hat{u}(t, x, y) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}} p_b(z) \hat{u}(t, x, z) \hat{\varphi}(z) d\mathcal{H}^{d-1}(z). \quad (3.18)$$

Moreover, by the boundedness of b we can proceed as above to transform the integral on $\widehat{\Sigma}$ into an integral on $\widehat{\Sigma}/R$. Indeed, recalling that the pushforward measure μ defined by (2.29) is a probability measure on $\widehat{\Sigma}/R$ (thus sigma-finite), the L^1_{μ} - L^{∞}_{μ} duality theorem provides the existence of a vector field $\eta \in L^{\infty}_{\mu}(\widehat{\Sigma}/R)^d$ such that the duality relation (3.12) is satisfied. Then, taking $v := \hat{u}(t, x, \cdot) \hat{\varphi} \in L^1_{\mu}(\widehat{\Sigma}/R)$ in (3.12), equality (3.18) also reads as

$$\int_{\mathbb{T}^d} b(y) \hat{u}(t, x, y) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}/R} \eta(\tilde{z}) \hat{u}(t, x, \tilde{z}) \hat{\varphi}(\tilde{z}) d\mu(\tilde{z}). \quad (3.19)$$

Hence, using successively (3.15), (3.16), (3.17), (3.19) and using the arbitrariness of the test function $\hat{\varphi}$ in $L^2_{\mu}(\widehat{\Sigma}/R)$ (recall the identification (2.25)), we deduce that problem (3.7) can be rewritten

$$\begin{cases} \hat{u} \in L^{\infty}(0, T; L^2(\mathbb{R}^d, L^2_{\mu}(\widehat{\Sigma}/R))), \\ \frac{\partial \hat{u}}{\partial t} + \operatorname{div}_x(\eta \hat{u}) = P\hat{f} & \text{in } (0, T) \times \mathbb{R}^d, \mu\text{-a.e. in } \widehat{\Sigma}/R, \\ \hat{u}(0, x, \tilde{z}) = P\hat{u}^0(x, \tilde{z}) & \text{for } x \in \mathbb{R}^d, \tilde{z} \in \widehat{\Sigma}/R. \end{cases} \quad (3.20)$$

Moreover, since the function η does not depend on (t, x) , we easily solve problem (3.20), which leads us to the following integral representation of \hat{u} ,

$$\hat{u}(t, x, \tilde{z}) = \int_0^t P\hat{f}(r, x - (t-r)\eta(\tilde{z}), \tilde{z}) dr + P\hat{u}^0(x - t\eta(\tilde{z}), \tilde{z}). \quad (3.21)$$

Therefore, the limit u given by the weak convergence (3.5) is given by the first representation formula (3.11).

On the other hand, the integral expressions of \hat{u} and u can be also written as integrals over the torus as follows. Performing the projection P onto \mathcal{V}_{\sharp} componentwise (recall that b is a vector field) and using the change of variables formula (2.26) between \mathbb{T}^d and $\widehat{\Sigma}/R$, we have for any function $\hat{\varphi} \in \mathcal{V}_{\sharp} \cap L^{\infty}(\mathbb{T}^d)$ and for a.e. $(t, x) \in (0, T) \times \mathbb{R}^d$,

$$\int_{\mathbb{T}^d} b(y) \hat{u}(t, x, y) \hat{\varphi}(y) dy = \int_{\widehat{\Sigma}/R} Pb(\tilde{z}) \hat{u}(t, x, \tilde{z}) \hat{\varphi}(\tilde{z}) d\mu(\tilde{z}). \quad (3.22)$$

Moreover, in view of definition (2.11) the truncation principle applies in \mathcal{V}_{\sharp} , so that $L^{\infty}(\mathbb{T}^d) \cap \mathcal{V}_{\sharp}$ is dense into \mathcal{V}_{\sharp} endowed with the $L^2(\mathbb{T}^d)$ -norm. Then, using this density result and proceeding as for the function η to derive the first representation formulas (3.21) and (3.11) from equality (3.19), we deduce the second representation formula (3.13) from equality (3.22), which concludes the proof of Theorem 3.2. \square

3.3. Link with the asymptotics of the flow

The following result establishes a link between the two representations (3.11), (3.13) of the limit $u(t, x)$ and the asymptotics of the flow Φ (1.2).

Proposition 3.4. *Let b a divergence vector field in $C_{\sharp}^1(\mathbb{T}^d)^d$, and let $\Phi(t, x)$ be the flow associated with b by (1.2). Then, we have*

$$\lim_{t \rightarrow \infty} \left(\frac{\Phi(t, y) - y}{t} \right) = \lim_{t \rightarrow \infty} \left(\frac{1}{t} \int_{\mathbb{T}^d} b(\Phi(t, y)) dy \right) = Pb(y), \quad a.e. y \in \mathbb{T}^d, \quad (3.23)$$

where Pb is the orthogonal projection of the vector field b on the kernel \mathcal{V}_{\sharp} (2.11).

Moreover, the orthogonal projection Pb can be alternatively deduced from the function p_b defined by (2.22) (with $g = b$) by the duality relation

$$\int_{\widehat{\Sigma}} p_b(z) v(\pi(z)) d\mathcal{H}^{d-1}(z) = \int_{\widehat{\Sigma}/R} Pb(\tilde{z}) v(\tilde{z}) d\mu(\tilde{z}), \quad \forall v \in L^1_{\mu}(\widehat{\Sigma}/R). \quad (3.24)$$

Proof. Since the Lebesgue measure is an invariant probability measure for the flow $\Phi(t, x)$ associated with the vector field b , there exists a vector-valued function $\zeta \in L^{\infty}(\mathbb{T}^d)^d$ such that

$$\lim_{t \rightarrow \infty} \left(\frac{\Phi(t, y) - y}{t} \right) = \zeta(y), \quad a.e. y \in \mathbb{T}^d. \quad (3.25)$$

Due the semi-group property of the flow the function ζ is clearly invariant for the flow. By Liouville's theorem (see, e.g., [18], Thm. 1, Sect. 2.2) the vector-valued ζb is divergence free in \mathbb{R}^d . Since b is regular and divergence free in \mathbb{R}^d , this means that $b \cdot \nabla \zeta = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, or equivalently, the function ζ belongs to the space \mathcal{V}_{\sharp} (2.11).

Let φ be a function in \mathcal{Y}_{\sharp} . Using successively Lebesgue and Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{T}^d} (b(y) - \zeta(y)) \varphi(y) dy &= \int_{\mathbb{T}^d} b(y) \varphi(y) dy - \int_{\mathbb{T}^d} \lim_{t \rightarrow \infty} \left(\frac{\Phi(t, y) - y}{t} \right) \varphi(y) dy \\ &= \int_{\mathbb{T}^d} b(y) \varphi(y) dy - \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{T}^d} \left(\int_0^t b(\Phi(s, x)) \varphi(y) ds \right) dy \\ &= \int_{\mathbb{T}^d} b(y) \varphi(y) dy - \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(\int_{\mathbb{T}^d} b(\Phi(s, x)) \varphi(y) dy \right) ds. \end{aligned}$$

However, since the function $\varphi \in \mathcal{Y}_{\sharp}$ is invariant for the flow Φ , we have

$$\int_{\mathbb{T}^d} b(\Phi(s, x)) \varphi(y) dy = \int_{\mathbb{T}^d} b(y) \varphi(y) dy, \quad \forall s \in \mathbb{R},$$

which implies that

$$\int_{\mathbb{T}^d} (b(y) - \zeta(y)) \varphi(y) dy = 0, \quad \forall \varphi \in \mathcal{Y}_{\sharp}.$$

Therefore, any component of the vector field $b - \zeta$ belongs to $\mathcal{Y}_{\sharp}^{\perp}$, which implies the equality $\zeta = Pb$ a.e. in \mathbb{R}^d . Moreover, relation (3.24) follows immediately from the equality (2.21) with $g = b$. \square

3.4. Illustration by the Stepanoff flow

We consider the case of the so-called Stepanoff flow [22] associated with the following divergence free vector field

$$b(x) := a(Ax) \xi \quad \text{with} \quad a \in C^1_{\sharp}(\mathbb{T}^d), \quad a \neq 0, \quad \xi \in \mathbb{S}^{d-1}, \quad A \in \mathbb{Z}^{d \times d} \text{ such that } A\xi = 0_{\mathbb{R}^d}. \quad (3.26)$$

First of all, if the vector ξ is incommensurable, *i.e.*

$$\xi \cdot k \neq 0, \quad \forall k \in \mathbb{Z}^d \setminus \{0_{\mathbb{R}^d}\}, \quad (3.27)$$

the flow Φ is known to be ergodic in the sense of Definition 2.7 (see, *e.g.*, [11], Prop. 5.1). Therefore, the space the space \mathcal{Y}_{\sharp} is reduced to the space of constant functions.

From now on, we assume that the vector ξ is commensurable. The flow $\Phi(t, x)$ associated with the vector field b is given by

$$\Phi(t, x) = t a(Ax) \xi + x, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (3.28)$$

since we have $A(\Phi(t, x)) = Ax$.

The vector field b satisfies the main assumption (2.5) with the linear function $w(x) := \xi \cdot x$, and the flow (3.28) satisfies the condition (2.7) with the hypersurface $\Sigma := \{x \in \mathbb{R}^d : \xi \cdot x = 0\}$. Then, each orbit of the flow is a line which crosses orthogonally the hyperplane Σ . Hence, it is easy to show that the functions τ, σ defined by (2.10) are given explicitly by

$$\begin{cases} \tau(y) = \frac{\xi \cdot y}{a(Ay)}, \\ \sigma(y) = y - (\xi \cdot y) \xi =: \pi(y), \end{cases} \quad \forall y \in \mathbb{R}^d, \quad (3.29)$$

where π denotes the orthogonal projection of \mathbb{R}^d on the hyperplane Σ .

Next, the equivalence relation R on Σ defined by (2.16) satisfies for any $z_1, z_2 \in \Sigma$,

$$z_1 \stackrel{R}{\sim} z_2 \Leftrightarrow \exists \in \mathbb{Z}^d, z_1 - z_2 \in k + \mathbb{R}\xi \Leftrightarrow \exists k \in \mathbb{Z}^d, z_1 - z_2 = \pi(k).$$

Hence, the orbits of the flow (3.28) are the sets

$$\Gamma_z = z + \pi(\mathbb{Z}^d) \quad \text{for } z \in \Sigma,$$

where $\pi(\mathbb{Z}^d)$ is an additive subgroup of \mathbb{R}^d . Therefore, the quotient set Σ/R is an additive subgroup of \mathbb{R}^d (recall that Σ is a vector subspace of \mathbb{R}^d), and it is given by

$$\Sigma/R = \Sigma/\pi(\mathbb{Z}^d) = \{z + \pi(\mathbb{Z}^d), z \in \Sigma\}. \quad (3.30)$$

On the other hand, the Jacobian $J(\nabla\sigma)$ is the constant 1, since the function $\sigma|_{\Sigma}$ agrees with the identity on Σ . Moreover, the function p_1 defined by (2.22) (with $g = 1$) satisfies

$$p_1(z) = d\mathcal{H}^1(\{y \in (0, 1)^d : \sigma(y) = z\}) = 1, \quad \forall z \in \widehat{\Sigma} := \sigma([0, 1]^d) \cap \Sigma, \quad (3.31)$$

since $\{y \in (0, 1)^d : \sigma(y) = z\}$ is a closed line segment of length 1. It follows that the measure μ defined by (2.28) or (2.29) is the pushforward measure given by

$$\mu(\tilde{A}) = d\mathcal{H}^{d-1}(\pi^{-1}(\tilde{A})), \quad \forall \tilde{A} \in \mathcal{B}(\widehat{\Sigma}/R). \quad (3.32)$$

Finally, by the asymptotics of the flow (3.23) with the expression (3.28) of the Stepanoff flow, we obtain the general representation formula (3.13) of the weak limit u with

$$Pb(y) = \lim_{t \rightarrow \infty} \left(\frac{\Phi(t, y) - y}{t} \right) = a(Ay)\xi, \quad \forall y \in \mathbb{T}^d. \quad (3.33)$$

Remark 3.5. By virtue of [23], Proposition 1, Chapter VI, and 1.1 the additive group $\pi(\mathbb{Z}^d)$ is a sublattice of \mathbb{R}^d if, and only if, $\xi \in \mathbb{Q}^d$. In this case $\pi(\mathbb{Z}^d)$ is a lattice of the hyperplane Σ thus of \mathbb{R}^{d-1} , and the quotient group $\Sigma/\pi(\mathbb{Z}^d)$ (recall (3.30)) is then a compact set. Moreover, by the property [23], Proposition 10, Chapter VI, and 1.3 combined with the compactness of $\widehat{\Sigma}/R$, we get the following isomorphic result

$$\Sigma/R = \Sigma/\pi(\mathbb{Z}^d) \wr \mathbb{T}^{d-1}.$$

Therefore, we deduce from Theorem 2.13 and from the definition (3.32) of the measure μ , the following isometric isomorphism

$$\xi \in \mathbb{Q}^d \cap \mathbb{S}^{d-1} \Rightarrow \mathcal{V}_{\sharp} \wr L_{\sharp}^2(\mathbb{T}^{d-1}). \quad (3.34)$$

The case where $\xi \notin \mathbb{Q}^d$ is more delicate to describe due to a more intricate structure of the group $\pi(\mathbb{Z}^d)$.

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No new data/codes were created or analyzed in this study.

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APPENDIX A. A CHANGE OF VARIABLES FORMULA ON A HYPERSURFACE

Let f be a map in $C^1(\mathbb{R}^d; \mathbb{R}^{d-1})$. According to [24], Section 3.2.1, for any $x \in \mathbb{R}^d$, one has the following polar decomposition of the matrix $\nabla f(x) \in \mathbb{R}^{(d-1) \times d}$,

$$\nabla f(x) = [\partial_{x_j} f_i(x)]_{1 \leq i \leq d-1, 1 \leq j \leq d} = S(x) O(x)^T, \quad (\text{A.1})$$

where $S(x)$ is a symmetric matrix in $\mathbb{R}^{(d-1) \times (d-1)}$ and $O(x)$ is an orthogonal matrix in $\mathbb{R}^{d \times (d-1)}$, i.e. $O(x)^T O(x) = I_{d-1}$. The Jacobian of the map f is then defined by

$$J(f)(x) := \det(\nabla f(x) \nabla f(x)^T)^{1/2} = |\det S(x)| \quad \text{for } x \in \mathbb{R}^d, \quad (\text{A.2})$$

which also agrees with the product of the singular values of $\nabla f(x)^T$. Here, we are interested in the case where f is the C^1 -regular function σ defined as a map from \mathbb{R}^d onto the hypersurface Σ , in such a way that for any $y \in \mathbb{R}^d$, the matrix $\nabla \sigma(y)^T$ is regarded as a linear mapping from the $(d-1)$ -dimensional tangent space $T_{\sigma(y)}$ in \mathbb{R}^d .

Proposition A.1. *Let σ be a C^1 -function from an open subset $\Omega \subset \mathbb{R}^d$ into a C^1 -regular hypersurface Σ of \mathbb{R}^d . Then, we have for any function $g \in L^1(\mathbb{R}^d)$,*

$$\int_{\Omega} g(x) |J(\sigma)(x)| dx = \int_{\Sigma} \left(\int_{\{\sigma(x)=z\} \cap \Omega} g(x) d\mathcal{H}^1(x) \right) d\mathcal{H}^{d-1}(z). \quad (\text{A.3})$$

Proof. It is enough to prove the result assuming that Σ is parametrized by a single mapping F which is a C^1 -diffeomorphism from an open set $U \subset \mathbb{R}^{d-1}$ onto $\Sigma \subset \mathbb{R}^d$.

Taking $G = F^{-1} \circ \sigma$, we have by virtue of the coarea formula (see [24], Thms. 2, and 3.4.3) applied to the function into brackets

$$\begin{aligned} & \int_{\Omega} [g(x) |\partial_{y_1} F(G(x)) \times \cdots \times \partial_{y_{d-1}} F(G(x))|] |J(G)(x)| dx \\ &= \int_{\Sigma} \left(\int_{\{G(x)=y\}} [g(x) |\partial_{y_1} F(G(x)) \times \cdots \times \partial_{y_{d-1}} F(G(x))|] d\mathcal{H}^1(x) \right) dy \\ &= \int_{\Sigma} \left(\int_{\{G(x)=y\}} g(x) d\mathcal{H}^1(x) \right) |\partial_{y_1} F(y) \times \cdots \times \partial_{y_{d-1}} F(y)| dy, \quad \forall g \in L^1(\Omega). \end{aligned}$$

Next, by the change of variables $z = F(y)$ onto the hypersurface Σ , we get that

$$\begin{aligned} & \int_{\Omega} [g(x) |\partial_{y_1} F(G(x)) \times \cdots \times \partial_{y_{d-1}} F(G(x))|] |J(G)(x)| dx \\ &= \int_{\Sigma} \left(\int_{\{\sigma(x)=z\}} g(x) d\mathcal{H}^1(x) \right) d\mathcal{H}^{d-1}(z). \end{aligned}$$

Therefore, to derive the desired formula (A.3) it remains to prove the identity

$$|J(\sigma)(x)| = |J(G)(x)| |\partial_{y_1} F(G(x)) \times \cdots \times \partial_{y_{d-1}} F(G(x))|, \quad \forall x \in \Omega. \quad (\text{A.4})$$

By the chain rule we have for any $x \in \Omega$,

$$\nabla \sigma(x) = \nabla F(G(x)) \nabla G(x)$$

which implies that

$$\nabla \sigma(x) \nabla \sigma(x)^T = \nabla F(G(x)) \nabla G(x) \nabla G(x)^T \nabla F(G(x))^T. \quad (\text{A.5})$$

Consider an orthonormal basis $(\xi^1, \dots, \xi^{d-1})$ in the tangent space $T(x)$ at the point $\sigma(x)$, composed of eigenvectors of the matrix $\sigma(x) \sigma(x)^T$. The corresponding eigenvalues are given by λ_i^2 for $i \in \{1, \dots, d-1\}$, where $\lambda_i > 0$ are the singular values of the matrix $\sigma(x)^T$. Also consider $\xi^d := \xi^1 \times \cdots \times \xi^{d-1}$ the unit normal to $T(x)$. The vectors (ξ^1, \dots, ξ^d) thus form an orthonormal basis of \mathbb{R}^d , while the vectors ν^i defined by

$$\nu^i = \nabla F(G(x))^T \xi^i \in \mathbb{R}^{d-1} \quad \text{for } i \in \{1, \dots, d-1\}, \quad (\text{A.6})$$

satisfy

$$\nabla G(x) \nabla G(x)^T \nu^i \cdot \nu^j = \nabla \sigma(x) \nabla \sigma(x)^T \xi^i \cdot \xi^j = \lambda_i^2 \delta_{ij}, \quad \forall i, j \in \{1, \dots, d-1\}.$$

Hence, $(\nu^1, \dots, \nu^{d-1})$ is a basis of \mathbb{R}^{d-1} which is orthogonal with respect to the scalar product defined by $\nabla G(x) \nabla G(x)^T$. Define the matrix P by its columns

$$P = (\xi^1 \mid \cdots \mid \xi^{d-1}) \in \mathbb{R}^{d \times (d-1)},$$

which by definition (A.6) satisfies

$$\nabla F(G(x))^T P = (\nu^1 \mid \cdots \mid \nu^{d-1}) \in \mathbb{R}^{(d-1) \times (d-1)}.$$

Then, multiplying (A.4) by P on the right and by P^T on the left, we get that

$$\text{diag}(\lambda_1^2, \dots, \lambda_{d-1}^2) = (\nu^1 \mid \cdots \mid \nu^{d-1})^T \nabla G(x) \nabla G(x)^T (\nu^1 \mid \cdots \mid \nu^{d-1}),$$

which implies the determinant equality

$$\lambda_1^2 \cdots \lambda_{d-1}^2 = \det(\nabla G(x) \nabla G(x)^T) \det(\nu^1 \mid \cdots \mid \nu^{d-1})^2,$$

or equivalently,

$$|J(\sigma)(x)| = |J(G)(x)| |\det(\nu^1 \mid \cdots \mid \nu^{d-1})|. \quad (\text{A.7})$$

On the other hand, by definition (A.6) we have

$$\nu_j^i = \sum_{k=1}^{d-1} \partial_{y_j} F_k(G(x)) \xi_k^i = \partial_{y_j} F_k(G(x)) \cdot \xi^i, \quad \forall i, j \in \{1, \dots, d-1\},$$

which, taking into account that $(\xi^1, \dots, \xi^{d-1})$ is an orthonormal basis of \mathbb{R}^{d-1} , implies that

$$\eta^j := \partial_{y_j} F(G(x)) = \sum_{i=1}^{d-1} (\partial_{y_j} F_k(G(x)) \cdot \xi^i) \xi^i = \sum_{i=1}^{d-1} \nu_j^i \xi^i, \quad \forall j \in \{1, \dots, d-1\}.$$

These equalities can be matrixially written as

$$(\eta^1, \dots, \eta^{d-1}) = (\xi^1 \mid \dots \mid \xi^{d-1}) (\nu^1 \mid \dots \mid \nu^{d-1})^T.$$

Therefore, we get that

$$(\xi^1 \mid \dots \mid \xi^{d-1} \mid \xi^d) \begin{pmatrix} (\nu^1 \mid \dots \mid \nu^{d-1})^T & 0_{(d-1) \times 1} \\ 0_{1 \times (d-1)} & 1 \end{pmatrix} = (\eta^1 \mid \dots \mid \eta^{d-1} \mid \xi^d),$$

which, taking the absolute value of the determinant, implies that

$$|\det(\nu^1 \mid \dots \mid \nu^{d-1})| = |\eta^1 \times \dots \times \eta^{d-1}|.$$

This combined with (A.7) proves the desired formula (A.4). □