

## ON EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR FRACTIONAL LOGARITHMIC SCHRÖDINGER EQUATION WITH STEEP POTENTIAL WELL

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**Abstract.** In this paper, we study the following fractional Schrödinger equation

$$(-\Delta)^s u + \lambda V(x)u = u \log u^2, \quad x \in \mathbb{R}^N,$$

where  $0 < s < 1$ ,  $\lambda > 0$  and  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a measurable potential satisfying some assumptions. Since the logarithmic term is singular at the origin, the energy functional is not of class  $C^1$ . To overcome this difficulty, the nonsmooth critical point theory developed by Szulkin is used. Employing variational methods, the existence of a nonnegative least energy solution for the problem is established for sufficient large  $\lambda$ 's. In addition, we prove that such solutions converge to a least energy solution of the limit problem defined on a bounded domain as  $\lambda \rightarrow \infty$ .

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### 1. INTRODUCTION

The aim of this paper is to study the existence and concentration of solutions for the following fractional logarithmic Schrödinger equation with steep potential well

$$(-\Delta)^s u + \lambda V(x)u = u \log u^2, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where  $0 < s < 1$ ,  $\lambda > 0$ ,  $N \geq 2s$ , and  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is a continuous nonnegative function satisfying some assumptions to be specified below.

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*Keywords and phrases:* Fractional Schrödinger equation, logarithmic nonlinearity, steep potential well, least energy solution, concentration phenomenon.

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Here  $(-\Delta)^s$  is the usual fractional Laplacian, defined as

$$\begin{aligned} (-\Delta)^s u(x) &= C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \\ &= C(N, s) \lim_{r \rightarrow 0} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N, \end{aligned} \quad (1.2)$$

where  $C(N, s)$  is a positive dimensional constant depending on  $N$  and  $s$ , while P.V. stands for the Cauchy principle value. The nonlocal operator  $(-\Delta)^s$  has attracted great interests in pure mathematical research and concrete real-world applications: indeed, it has been used to describe phase transition phenomena [1], minimal surfaces [2–4], population dynamics [5], Markov processes [6] and fractional quantum mechanics [7]. Moreover,  $(-\Delta)^s$  can be regarded as the infinitesimal generator of rotationally invariant  $2s$ -stable Lévy process [8].

Problem (1.1) arises from the study of the following time-dependent Schrödinger equation

$$i \frac{\partial \eta}{\partial t} = (-\Delta)^s \eta + (W(x) + \omega)\eta - \eta \log |\eta|^2, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N. \quad (1.3)$$

As we look for the standing wave solutions of the form  $\eta(t, x) = e^{-i\omega t} u(x)$ , where  $\omega$  is a constant, we can transform (1.3) into (1.1). We refer to [9–14] and references therein for more physical background.

There are many researches on the existence and concentration of solutions for the following Schrödinger equations

$$-\Delta u + \lambda a(x)u = g(x, u), \quad x \in \mathbb{R}^N, \quad (1.4)$$

where  $\lambda > 0$  and the potential  $a(x)$  satisfies the following hypotheses:

- (H<sub>1</sub>)  $a \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$  and  $a(x) \geq 0$  for all  $x \in \mathbb{R}^N$ ;
- (H<sub>2</sub>)  $\Omega := \text{int } a^{-1}(0) \neq \emptyset$  has a smooth boundary and  $\bar{\Omega} = a^{-1}(0)$ ;
- (H<sub>3</sub>) There exists  $M_0 > 0$  such that

$$\mu(\{x \in \mathbb{R}^N : a(x) \leq M_0\}) < \infty,$$

where  $\mu$  denotes the Lebesgue measure in  $\mathbb{R}^N$ .

Hypotheses (H<sub>1</sub>) – (H<sub>3</sub>) are first proposed by Bartsch and Wang [15] with the nonlinearity  $g(x, u) = u^p - u$ , where  $N \geq 3$  and  $1 < p < (N + 2)/(N - 2)$ . They prove that (1.4) has a least energy solution  $u_{\lambda, p}$  for  $\lambda$  sufficiently large. They also show that for any sequence  $(\lambda_n)$ , when  $\lambda_n \rightarrow +\infty$ , the sequence of solutions  $(u_{\lambda_n})$  strongly converges to  $u_\Omega$  in  $H^1(\mathbb{R}^N)$ , where  $u_\Omega$  is the positive least energy solution of the limit problem defined on  $\Omega$ . Furthermore, if  $\Omega$  is bounded, they show that when  $p$  sufficiently approaches the Sobolev critical exponent  $2^* = 2N/(N - 2)$ , (1.4) has at least  $\text{cat}(\Omega)$  solutions for  $\lambda$  large enough.

Inspired by the results above, Clapp and Ding [16] consider problem (1.4) with the critical nonlinearity  $g(x, u) = \kappa u + u^{2^*-1}$ , where  $N \geq 4$ ,  $\kappa > 0$  and  $a(x)$  satisfies (H<sub>1</sub>) – (H<sub>3</sub>) with  $\Omega$  bounded. The authors establish the existence and concentration of least energy solutions. In addition, the multiplicity of solutions for (1.4) is also given. For more results about problems of this type, we refer the reader to [17–21] and the references therein.

On the other hand, there are few results for stationary Schrödinger equations with logarithmic nonlinearity of the form

$$-\Delta u + V(x)u = Q(x)u \log u^2, \quad x \in \mathbb{R}^N. \quad (1.5)$$

When  $V$  and  $Q$  are spatially 1-period functions of the variables  $x_1, \dots, x_N$ ,  $Q \in C^1(\mathbb{R}^N)$ ,  $\min_{\mathbb{R}^N} Q > 0$  and  $\min_{\mathbb{R}^N} (V + Q) > 0$ . Squassina and Szulkin [22] establish the existence of the least energy solution for (1.5) and prove that (1.5) has infinitely many geometrically distinct solutions. Tanaka and Zhang [23] show the existence of infinitely many multi-bump solutions for (1.5) which are geometrically distinct. When  $Q \equiv 1$  and  $V$  has an asymptotic behavior at infinity, existence and multiplicity of solutions for (1.5) are given by Ji and Szulkin [24]. As for more results about the logarithmic Schrödinger equation, we refer the reader to [25–29] and the references therein.

Recently, Alves, de Morais Filho and Figueiredo [30] study the following logarithmic Schrödinger equation with steep potential well

$$-\Delta u + \lambda V(x)u = u \log u^2, \quad x \in \mathbb{R}^N, \quad (1.6)$$

where  $\lambda > 0$  and  $V(x)$  has a potential well. The authors use variational methods to establish the existence and concentration of least energy solutions for (1.6). Based on their work, it is natural to ask whether the results above still hold for the fractional logarithmic Schrödinger equation (1.1). This paper is to fill this gap.

The assumptions on  $V$  are as follows:

- (V<sub>1</sub>)  $V$  is measurable and  $V(x) \geq 0$  for all  $x \in \mathbb{R}^N$ ;
- (V<sub>2</sub>)  $\Omega := \text{int } V^{-1}(0) \neq \emptyset$  is bounded with smooth boundary and  $\bar{\Omega} = V^{-1}(0)$ ;
- (V<sub>3</sub>) There exists  $M_0 > 0$  such that

$$\mu(\{x \in \mathbb{R}^N : V(x) \leq M_0\}) < \infty,$$

where  $\mu$  denotes the Lebesgue measure in  $\mathbb{R}^N$ .

The existence result is stated in the following.

**Theorem 1.1.** *Assume that (V<sub>1</sub>) – (V<sub>3</sub>) hold. Then there exists  $\lambda^* > 0$  such that (1.1) has a nonnegative least energy solution  $u_\lambda$  for  $\lambda \geq \lambda^*$ .*

We remark that the sign of the solution is found by exploiting the energy doubling, introduced by Weth [31] in the local setting, see Remark 3.13 below and by Deng and Shuai [32] in the nonlocal one.

In addition, we also give the concentration phenomenon of least energy solutions for (1.1).

**Theorem 1.2.** *Assume that (V<sub>1</sub>) – (V<sub>3</sub>) hold. Then there exists  $u_\Omega \in H^s(\mathbb{R}^N)$  such that if  $\lambda_n \rightarrow +\infty$ , then  $u_{\lambda_n} \rightarrow u_\Omega$  in  $H^s(\mathbb{R}^N)$ , where  $(u_{\lambda_n})$  are nonnegative least energy solutions of (1.1) and  $u_\Omega$  is a nonnegative least energy solution of the limit problem*

$$\begin{cases} (-\Delta)^s u = u \log u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.7)$$

**Remark 1.3.** It is not difficult to find a potential  $V$  satisfying (V<sub>1</sub>) – (V<sub>3</sub>) in  $\mathbb{R}^N$ . For instance, there is the potential  $V$  defined as

$$V(x) = \begin{cases} 0, & |x| < 1, \\ |x|^r - 1, & |x| \geq 1, \end{cases}$$

for some  $r \in (1, +\infty)$ .

Before proving the results above, it is necessary to point out the difficulties in this paper. For this, let us start by describing the functional setting: for  $s \in (0, 1)$ , we introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

with the norm

$$\|u\|_{H^s} := ([u]_s^2 + |u|_2^2)^{\frac{1}{2}}, \quad [u]_s := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

The Euler-Lagrange functional  $I : H^s(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$  related to (1.1) is defined as

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda V(x) + 1) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx.$$

According to the logarithmic Sobolev inequality [33], for each  $u \in H^s(\mathbb{R}^N)$ , one has

$$\int_{\mathbb{R}^N} u^2 \log \left( \frac{u^2}{|u|_2^2} \right) dx + \left( N + \frac{N}{s} \log \alpha + \log \frac{s\Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2s})} \right) |u|_2^2 \leq \frac{\alpha^2}{\pi^s} |(-\Delta)^{\frac{s}{2}} u|_2^2, \quad (1.8)$$

where  $\alpha > 0$  is arbitrary. It is obvious that  $\int_{\mathbb{R}^N} u^2 \log u^2 dx < +\infty$ . However, the functional  $I$  is not well defined in  $H^s(\mathbb{R}^N)$  because of the singularity of logarithmic term at the origin. For example, if we choose  $u \in H^s(\mathbb{R}^N)$  satisfying

$$u(x) = \begin{cases} 0, & |x| \leq 2, \\ (|x|^{N/2} \log |x|)^{-1}, & |x| \geq 3, \end{cases}$$

it is straightforward to check that  $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$  is not well defined, so  $I$  would not be of class  $\mathcal{C}^1$ . Hence, the classical critical point theory cannot be applied directly.

**Remark 1.4.** When considering on a bounded domain  $\Omega \subset \mathbb{R}^N$ , one can easily prove that the functional  $I$  is  $\mathcal{C}^1$  differentiable. In fact, for any  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that  $|u \log u^2| \leq C_\epsilon(1 + |u|^{1+\epsilon})$  for any  $u$ , so that  $I$  is  $\mathcal{C}^1$  in  $H^s(\Omega)$ .

To overcome the difficulty of dealing with a nonsmooth functional, there are some methods which can be found in the literature. Cazenave [12] chooses a reflexive Banach space  $W$  equipped with a Luxemburg type norm to achieve that  $I : W \rightarrow \mathbb{R}^N$  is well defined and of class  $\mathcal{C}^1$ . Wang and Zhang [34] prove the relation between the power-law scalar field equation and the logarithmic-law scalar field equation, showing that one can study the logarithmic-law equation with the help of the power-law equation. Squassina and Szulkin [22] use the nonsmooth critical point theory developed by Szulkin [35], whose idea is to decompose the functional  $I$  into the sum of a  $\mathcal{C}^1$  functional and a convex lower semicontinuous functional. In this paper, we will use the nonsmooth critical point theory [35] to overcome the lack of smoothness of the functional. We also recall that such an approach, based on nonsmooth critical point theory, has been successfully used to study a related equation under a different viewpoint in [17], and also to study equations in bounded domains in [36].

On the other hand, the presence of the fractional Laplacian  $(-\Delta)^s$  may bring some difficulties, due to its nonlocal nature, which makes the identity in equation (1.1) dependent on the integral in  $\mathbb{R}^N$ . Moreover, it is not straightforward to choose the work space for the limit problem (1.7), since several different notions of fractional Laplacians in bounded domains are available. Here, coherently with definition (1.2), the natural setting is the

one popularized by Song and Vondáček [37] and Servadei and Valdinoci [38], which will be properly introduced later on.

The proofs of Theorems 1.1 and 1.2 mainly use variational methods. First, we prove the existence of a nonsmooth  $(PS)$  sequence  $(u_n)$  at the mountain pass level. Then we introduce the Nehari manifold and obtain that the mountain pass level equals the infimum of the functional on the manifold. Moreover, for each  $u$  which belongs to the work space, there exists a unique projection on the Nehari manifold. Based on this result, we show that for  $\lambda$  large enough,  $(u_n)$  admits a strongly convergent subsequence. Hence, (1.1) has a nonnegative least energy solution. As for concentration result, we choose a new work space to study the limit problem (1.7). By introducing the related notion of  $(PS)_{d,\infty}$  sequence, we prove that as the sequence  $\lambda_n \rightarrow +\infty$ , the least energy solutions  $(u_{\lambda_n})$  of (1.1) converge to the least energy solution of (1.7).

**Notations.** From now on, unless specifically mentioned, we will adhere to the following notations:

- $B_r(y)$  is an open ball with center  $y \in \mathbb{R}^N$  and radius  $r > 0$ ;
- $H^{-s}(\mathbb{R}^N)$  is the dual space of  $H^s(\mathbb{R}^N)$ ;
- $|\cdot|_p$  stands for the standard norm in  $L^p(\mathbb{R}^N)$  with  $1 \leq p \leq \infty$ ;
- $o_n(1)$  denotes a sequence with  $o_n(1) \rightarrow 0$  as  $n \rightarrow \infty$ ;
- $C, C_1, C_2, C_3, \dots$  denote any positive constants;
- “ $\rightharpoonup$ ” and “ $\rightarrow$ ” denote the weak and strong convergence in the function space, respectively;
- $2_s^* = 2N/(N - 2s)$  is the critical embedding exponent for fractional Sobolev space.

## 2. PRELIMINARIES AND VARIATIONAL SETTING

In this paper, we consider the following Hilbert space

$$E_\lambda := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}$$

equipped with the norm

$$\|u\|_\lambda := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (\lambda V(x) + 1)u^2 dx \right)^{\frac{1}{2}}$$

and the scalar product

$$\langle u, v \rangle_\lambda := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} (\lambda V(x) + 1)uv dx.$$

In view of the assumption  $(V_1)$ , it is obvious that the embedding  $E_\lambda \hookrightarrow H^s(\mathbb{R}^N)$  is continuous, so that the embedding  $E_\lambda \hookrightarrow L^q(\mathbb{R}^N)$  is continuous for any  $q \in [2, 2_s^*]$ , see [39], Theorem 6.7.

**Definition 2.1** (Weak solution). We say that  $u \in E_\lambda$  is a weak solution of equation (1.1), if  $u$  satisfies  $u^2 \log u^2 \in L^1(\mathbb{R}^N)$  and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} V(x)u\varphi dx = \int_{\mathbb{R}^N} u\varphi \log u^2 dx,$$

for any  $\varphi \in C_0^\infty(\mathbb{R}^N)$ .

The Euler-Lagrange functional  $J_\lambda : E_\lambda \rightarrow \mathbb{R} \cup \{+\infty\}$  associated with (1.1) is defined by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda V(x) + 1) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx \\ &= \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx. \end{aligned}$$

Thanks to the logarithmic Sobolev inequality (1.8), it is easy to verify that  $J_\lambda(u) > -\infty$  for all  $u \in E_\lambda$ . However, there exists some  $u \in E_\lambda$  such that  $\int_{\mathbb{R}^N} u^2 \log u^2 dx = -\infty$ , so the functional  $J_\lambda$  fails to be of  $\mathcal{C}^1$  class. For this reason, concerning the functional approach, here we follow the ideas already used in Squassina and Szulkin [22] and in Ji and Szulkin [24].

For  $\delta \in (0, e^{-3/2})$  we define

$$F_1(t) := \begin{cases} 0, & t = 0, \\ -\frac{1}{2} t^2 \log t^2, & 0 < |t| < \delta, \\ -\frac{1}{2} t^2 (\log \delta^2 + 3) + 2\delta|t| - \frac{1}{2} \delta^2, & |t| > \delta \end{cases}$$

and

$$F_2(t) := \begin{cases} 0, & |t| < \delta, \\ \frac{1}{2} t^2 \log(t^2/\delta^2) + 2\delta|t| - \frac{3}{2} t^2 - \frac{1}{2} \delta^2, & |t| > \delta. \end{cases}$$

It is obvious that  $F_1, F_2 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$  and

$$F_2(t) - F_1(t) = \frac{1}{2} t^2 \log t^2, \quad t \in \mathbb{R},$$

with the obvious trivial extension at  $t = 0$ .

Thus, for each  $\lambda > 0$ , the functional  $J_\lambda$  can be rewritten as

$$J_\lambda(u) = \Phi_\lambda(u) + \Psi(u), \tag{2.1}$$

where

$$\Phi_\lambda(u) := \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbb{R}^N} F_2(u) dx$$

and

$$\Psi(u) := \int_{\mathbb{R}^N} F_1(u) dx.$$

Due to the fact that there exists  $C_p > 0$  such that  $|F_2'(t)| \leq C_p |t|^{p-1}$  for any  $t \in \mathbb{R}$  with  $p \in (2, 2_s^*)$ , by standard arguments, it is straightforward to check that  $\Phi_\lambda \in \mathcal{C}^1(E_\lambda, \mathbb{R})$ . Observe that  $\Psi$  is convex and non-negative. Besides, by Fatou's lemma,  $\Psi$  is lower semicontinuous. As a result, we have decomposed the functional  $J_\lambda$  as the sum of a  $\mathcal{C}^1$  functional  $\Phi_\lambda$  and a convex lower semicontinuous functional  $\Psi$ .

**Remark 2.2.** Since

$$F_1'(t)t = \begin{cases} -t^2 \log t^2 - t^2 & \text{if } |t| < \delta, \\ -t^2(\log \delta^2 + 3) + 2\delta|t| & \text{if } |t| \geq \delta, \end{cases}$$

the restriction  $\delta \in (0, e^{-3/2})$  ensures that  $F_1'(t)t \geq 0$  for all  $t \in \mathbb{R}$ . Easier calculations show that  $F_2'(t)t \geq 0$  for all  $t \in \mathbb{R}$ , as well.

We recall the following definitions from [22], which are derived from those in [35].

**Definition 2.3.** Let  $X$  be a real Banach space,  $X'$  be its dual space and  $\langle \cdot, \cdot \rangle$  be the duality pair between  $X'$  and  $X$ . Let  $J = \Phi + \Psi$ , where  $\Phi \in C^1(X, \mathbb{R})$ ,  $\Psi : X \rightarrow (-\infty, +\infty]$  is convex, lower semicontinuous and  $\Psi \not\equiv +\infty$ .

- (i) The set  $D(J) := \{u \in X : J(u) < +\infty\}$  is called the effective domain of  $J$ .
- (ii) The set

$$\partial J(u) := \{\omega \in X' : \langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq \langle \omega, v - u \rangle, \forall v \in X\}.$$

is called the subdifferential of  $J$  at  $u$ .

- (iii) A point  $u \in X$  is said to be a critical point of  $J$  if  $u \in D(J)$  and  $0 \in \partial J(u)$ , i.e.

$$\langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X.$$

- (iv)  $(u_n)$  is a Palais–Smale sequence at level  $d$  for  $J$  ( $(PS)_d$  sequence for short) if  $(J(u_n)) \rightarrow d$  and there exists a number sequence  $\varepsilon_n \rightarrow 0^+$  such that

$$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X$$

- (v)  $J$  satisfies the Palais–Smale condition at level  $d$  ( $(PS)_d$  condition for short) if each Palais–Smale sequence at level  $d$  has a convergent sequence.

For our study, we set  $X = E_\lambda$  and  $J_\lambda = \Phi_\lambda + \Psi$  defined as (2.1).

With the same proof of [22], Lemma 2.4, we can obtain the following lemma.

**Lemma 2.4.** *If  $u \in D(J_\lambda)$ , then there exists a unique  $\omega \in E_\lambda'$  such that  $\partial J_\lambda(u) = \{\omega\}$ , i.e.*

$$\langle \Phi_\lambda'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq \langle \omega, v - u \rangle, \quad \forall v \in E_\lambda.$$

Moreover,

$$\langle \Phi_\lambda'(u), z \rangle + \int_{\mathbb{R}^N} F_1'(u)z dx = \langle \omega, z \rangle \text{ for all } z \in E_\lambda \text{ such that } F_1'(u)z \in L^1(\mathbb{R}^N).$$

**Definition 2.5.** We denote by  $J_\lambda'(u)$  the unique element in  $\partial J_\lambda(u)$ .

Thanks to Lemma 2.4, the following immediate results can be given, see [24], Lemma 2.4.

**Lemma 2.6.**

- (i) *If  $u \in D(J_\lambda)$ , then  $u$  is a critical point of  $J_\lambda$  if and only if  $u$  is a weak solution of (1.1).*
- (ii) *If  $(u_n)$  is a Palais–Smale sequence, then  $\|J_\lambda'(u_n)\| < \infty$  and  $J_\lambda'(u_n) \rightarrow 0$ .*

### 3. THE EXISTENCE OF A LEAST ENERGY SOLUTION

In this section we prove the existence of a least energy solution to problem (1.1) by applying the Mountain Pass Theorem. However, since  $J_\lambda$  is nonsmooth, the classical Mountain Pass theorem [40], Theorem 1.15 cannot be applied to give the existence of Palais–Smale sequences. For this reason, here will use the following non-smooth version of Mountain Pass theorem without Palais–Smale condition [41], Theorem 3.1.

In this section, we will show the existence and compactness of Palais–Smale sequences. Regarding the non-smooth functional  $J_\lambda$ , the classical Mountain Pass theorem [40], Theorem 1.15 cannot be applied to give the existence of Palais–Smale sequence. Here we will use the non-smooth version of Mountain Pass theorem without Palais–Smale condition [41], Theorem 3.1.

**Theorem 3.1.** *Let  $X$  be a real Banach space and  $J : X \rightarrow (-\infty, +\infty]$  be a functional such that  $J(u) = \Phi(u) + \Psi(u)$ , where  $\Phi \in C^1(X, \mathbb{R})$  and  $\Psi : X \rightarrow (-\infty, +\infty]$  is convex, lower semicontinuous and  $\Psi \not\equiv +\infty$ . Moreover,  $J$  satisfies the following mountain pass geometry assumptions:*

- (i)  $J(0) = 0$  and  $J|_{\partial B_\rho} \geq \alpha$  for some constants  $\rho, \alpha > 0$ .
- (ii)  $J(e) \leq 0$  for some  $e$  with  $\|e\| > \rho$ .

Then there exists a Palais–Smale sequence  $(u_n)$  at level  $c$  for  $J$ , i.e.  $J(u_n) \rightarrow c$  and

$$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

where  $\varepsilon_n \rightarrow 0^+$ ,

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J(\gamma(t)) \geq \alpha$$

and

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

**Lemma 3.2.** *The functional  $J_\lambda = \Phi_\lambda + \Psi$  defined in (2.1) satisfies the mountain pass geometry*

- (i)  $J_\lambda(0) = 0$  and  $J_\lambda|_{\partial B_\rho} \geq \alpha$  for some constants  $\rho, \alpha > 0$ .
- (ii)  $J_\lambda(e) \leq 0$  for some  $e \in E_\lambda$  with  $\|e\| > \rho$ .

*Proof.* Obviously,  $J_\lambda(0) = 0$ . Since  $\Psi$  is non-negative, then for every  $u \in E_\lambda$  we have

$$J_\lambda(u) \geq \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbb{R}^N} F_2(u) dx.$$

Recall that there exists  $C_p > 0$  such that  $|F_2'(t)| \leq C_p |t|^{p-1}$  for  $t \in \mathbb{R}$  with  $p \in (2, 2_s^*)$ . So we know that

$$|F_2(t)| \leq \frac{C_p}{p} |t|^p, \quad \forall t \in \mathbb{R}.$$

From the embedding  $E_\lambda \hookrightarrow L^p(\mathbb{R}^N)$ , one has

$$J_\lambda(u) \geq \frac{1}{2} \|u\|_\lambda^2 - C(N, p, s) \|u\|_\lambda^p,$$

where  $C(N, p, s)$  is a positive constant dependent on  $N, p$  and  $s$ . Since  $p > 2$ , there exists  $\rho > 0$  such that

$$J_\lambda(u) \geq \frac{1}{2}\rho^2 - C(N, p, s)\rho^p = \alpha > 0$$

for all  $u \in E_\lambda$  with  $\|u\|_\lambda = \rho$ , so  $J_\lambda$  satisfies (i).

For every  $u \in D(J_\lambda) \setminus \{0\}$  and  $t > 0$ ,

$$\begin{aligned} J_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \frac{t^2}{2} \int_{\mathbb{R}^N} u^2 \log(tu)^2 \\ &= t^2 \left( \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 - \log t \int_{\mathbb{R}^N} u^2 \right) \\ &= t^2 \left( J_\lambda(u) - \log t \int_{\mathbb{R}^N} u^2 \right) \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Hence,  $J_\lambda$  satisfies (ii) and the proof is complete.  $\square$

According to Theorem 3.1, for each  $\lambda > 0$ , there exists a sequence  $(u_n) \subset E_\lambda$  such that

$$J_\lambda(u_n) \rightarrow c_\lambda$$

and

$$\langle \Phi'_\lambda(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|_\lambda, \quad \forall v \in E_\lambda \quad (3.1)$$

with  $\varepsilon_n \rightarrow 0^+$ . In view of Lemma 2.6(ii), we get that  $\|J'_\lambda(u_n)\| < \infty$  and  $J'_\lambda(u_n) \rightarrow 0$ . The mountain pass value  $c_\lambda$  is defined by

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma_\lambda := \{\gamma \in \mathcal{C}([0, 1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e\}.$$

From Lemma 3.2 we get that

$$c_\lambda \geq \alpha > 0, \quad \text{uniformly for } \lambda > 0. \quad (3.2)$$

For  $u \in D(J_\lambda) \setminus \{0\}$ , one has

$$\begin{aligned} \langle J'_\lambda(u), u \rangle &= \|u\|_\lambda^2 - \int_{\mathbb{R}^N} F'_2(u)u dx + \int_{\mathbb{R}^N} F'_1(u)u dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} \lambda V(x)u^2 dx - \int_{\mathbb{R}^N} u^2 \log u^2 dx. \end{aligned}$$

Next, we introduce the Nehari manifold

$$\mathcal{N}_\lambda := \{u \in D(J_\lambda) \setminus \{0\} : \langle J'_\lambda(u), u \rangle = 0\}.$$

Using an argument analogous to that in [22], we can prove that

$$c_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u). \quad (3.3)$$

We have the following property.

**Lemma 3.3.** *For every  $\lambda > 0$  and  $u \in D(J_\lambda) \setminus \{0\}$ , there exists a unique positive number  $t_u$  such that  $t_u u \in \mathcal{N}_\lambda$ . Furthermore,  $J_\lambda(t_u u) > J_\lambda(tu)$  for all  $t > 0$  and  $t \neq t_u$ ; in particular,*

$$J_\lambda(u) = \max_{t > 0} J_\lambda(tu)$$

when  $u \in \mathcal{N}_\lambda$ , i.e. when  $t_u = 1$ . Moreover,  $J_\lambda(tu)$  is strictly increasing on  $(0, t_u)$  and strictly decreasing on  $(t_u, +\infty)$ .

*Proof.* For every  $\lambda > 0$  and  $u \in D(J_\lambda) \setminus \{0\}$ , we consider the fiber mapping  $t \mapsto J_\lambda(tu)$  defined by

$$\begin{aligned} J_\lambda(tu) &= \frac{t^2}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{t^2}{2} \int_{\mathbb{R}^N} (\lambda V(x) + 1) u^2 dx - \frac{t^2}{2} \int_{\mathbb{R}^N} u^2 \log(tu)^2 dx \\ &= t^2 J_\lambda(u) - t^2 \log t \int_{\mathbb{R}^N} u^2 dx \end{aligned}$$

for  $t > 0$ . From the proof of Lemma 3.2, we know that  $J_\lambda(tu) > 0$  if  $t > 0$  small enough and  $J_\lambda(tu) < 0$  if  $t > 0$  large enough. Computing the derivative of  $J_\lambda(tu)$ , we have

$$\frac{\langle J'_\lambda(tu), tu \rangle}{t} = \frac{d}{dt} J_\lambda(tu) = 2t J_\lambda(u) - 2t \log t \int_{\mathbb{R}^N} u^2 dx - t \int_{\mathbb{R}^N} u^2 dx. \quad (3.4)$$

Therefore,

$$tu \in \mathcal{N}_\lambda \iff \frac{d}{dt} J_\lambda(tu) = 0;$$

this equation is satisfied by the unique root  $t_u > 0$  of the following algebraic equation

$$J_\lambda(u) = \frac{2 \log t_u + 1}{2} \int_{\mathbb{R}^N} u^2 dx.$$

Moreover, taking advantage of the sign of  $J_\lambda(tu)$  for small and large  $t$ 's, we get that  $J_\lambda(tu)$  is strictly increasing on  $(0, t_u)$  and strictly decreasing on  $(t_u, +\infty)$ . Hence,  $t_u$  is a maximum point of  $J_\lambda(tu)$ ,  $t > 0$ .  $\square$

Next, we show that the functional  $J_\lambda$  satisfies the  $(PS)_{c_\lambda}$  condition, namely that the  $(PS)_{c_\lambda}$  sequence found by Theorem 3.1 has a convergent subsequence.

**Lemma 3.4.** *The sequence  $(u_n)$  is bounded in  $E_\lambda$ .*

*Proof.* Since  $(u_n)$  is a  $(PS)_{c_\lambda}$  sequence of  $J_\lambda$ , we get

$$c_\lambda + o_n(1)(1 + \|u_n\|_\lambda) \geq J_\lambda(u_n) - \frac{1}{2} \langle J'_\lambda(u_n), u_n \rangle = \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx,$$

so that

$$\|u_n\|_2^2 \leq C_1 + o_n(1)\|u_n\|_\lambda. \quad (3.5)$$

By the logarithmic Sobolev inequality (1.8), if  $\alpha > 0$  is sufficiently small, one has that for any  $u \in H^s(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \leq \frac{1}{2}[u]_s^2 + C_2(\log \|u\|_2^2 + 1)\|u\|_2^2. \quad (3.6)$$

Thus, by (3.6) and (3.5), we have

$$\begin{aligned} C_3 \geq J_\lambda(u_n) &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda V(x) + 1) u_n^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \log u_n^2 dx \\ &\geq \frac{1}{4} \|u_n\|_\lambda^2 - C_4(\log \|u_n\|_2^2 + 1)\|u_n\|_2^2 \geq \frac{1}{4} \|u_n\|_\lambda^2 - C_5(1 + \|u_n\|_\lambda^r), \end{aligned}$$

where  $r \in (1, 2)$ . Indeed, for any  $\epsilon > 0$  there exists  $C_\epsilon$  such that  $|u_n|_2 \log |u_n|_2 \leq C_\epsilon + |u_n|_2^{1+\epsilon}$ , and choosing  $\epsilon$  small enough, we get the claim.

Hence,  $(u_n)$  is bounded in  $E_\lambda$ .  $\square$

From Lemma 3.4 we can suppose that there exists  $u_\lambda \in E_\lambda$  such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda \quad \text{in } E_\lambda, \\ u_n &\rightarrow u_\lambda \quad \text{in } L_{loc}^q(\mathbb{R}^N) \text{ for } q \in [1, 2_s^*), \\ u_n &\rightarrow u_\lambda \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

As usual, we have the following

**Lemma 3.5.** *For any  $\lambda > 0$  and for every  $\varphi \in C_0^\infty(\mathbb{R}^N)$  we have  $\langle J'_\lambda(u_\lambda), \varphi \rangle = 0$ .*

*Proof.* Take  $\varphi \in C_0^\infty(\mathbb{R}^N)$ . Of course  $\langle J'_\lambda(u_n), \varphi \rangle = o_n(1)$ . Passing to the limit, by the weak convergence, we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} \lambda V(x) u \varphi dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n \varphi \log u_n^2 dx,$$

Since  $\varphi$  has compact support and  $|u_n \log u_n^2| \leq C_1 + C_2|u_n|^2$ , by the strong convergence in the support of  $\varphi$ , we finally get

$$\langle J'_\lambda(u_\lambda), \varphi \rangle = 0 \text{ for every } \varphi \in C_c^\infty(\mathbb{R}^N).$$

$\square$

**Lemma 3.6.** *For each  $\lambda > 0$ ,*

$$\liminf_{n \rightarrow \infty} \|u_n\|_\lambda^2 \geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n^2 dx \geq \alpha.$$

where  $\alpha$  is given in Lemma 3.2.

*Proof.* In view of Lemma 3.4,  $(u_n)$  is a bounded  $(PS)_{c_\lambda}$  sequence of  $J_\lambda$ , so

$$J_\lambda(u_n) + o_n(1) = J_\lambda(u_n) - \frac{1}{2} \langle J'_\lambda(u_n), u_n \rangle = \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx.$$

Since  $J_\lambda(u_n) \rightarrow c_\lambda \geq \alpha$ , by definition of  $\|\cdot\|_\lambda$ , we have

$$\frac{1}{2} \|u_n\|_\lambda^2 \geq \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 dx = c_\lambda + o_n(1) \geq \alpha + o_n(1),$$

which concludes the proof.  $\square$

**Remark 3.7.** The previous estimate shows that, according to Lions' definition, vanishing cannot occur.

Inspired by [15], Lemma 2.5, we have the following lemma.

**Lemma 3.8.** *For any  $\varepsilon > 0$ , there exists  $\lambda^* = \lambda^*(\varepsilon) > 0$  and  $R > 0$  such that*

$$\limsup_{n \rightarrow \infty} \int_{B_R^c(0)} |u_n|^q dx \leq \varepsilon$$

for every  $\lambda \geq \lambda^*$  and  $q \in [2, 2_s^*)$ , where  $B_R^c(0) := \{x \in \mathbb{R}^N : |x| > R\}$ .

*Proof.* For a given  $R > 0$ , set

$$A(R) := \{x \in \mathbb{R}^N : |x| > R \text{ and } V(x) \geq M_0\}$$

and

$$B(R) := \{x \in \mathbb{R}^N : |x| > R \text{ and } V(x) < M_0\},$$

where  $M_0$  is given in  $(V_3)$ . On one hand, by Lemma 3.4,

$$\begin{aligned} \int_{A(R)} |u_n|^2 dx &\leq \frac{1}{\lambda M_0 + 1} \int_{\mathbb{R}^N} (\lambda V(x) + 1) |u_n|^2 \\ &\leq \frac{1}{\lambda M_0 + 1} \|u_n\|_\lambda^2 \leq \frac{C_0}{\lambda M_0 + 1} \leq \frac{\varepsilon}{2}, \end{aligned}$$

for every  $\lambda \geq \lambda^*$ , where

$$\lambda^* = \frac{1}{M_0} \left( \frac{2C_0}{\varepsilon} - 1 \right).$$

On the other hand, using the Hölder inequality and the embedding  $E_\lambda \hookrightarrow L^{2_s^*}$ , we have

$$\begin{aligned} \int_{B(R)} |u_n|^2 dx &\leq \left( \int_{B(R)} |u_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \left( \int_{B(R)} 1 dx \right)^{\frac{2_s^*}{N}} \\ &\leq |u_n|_{2_s^*}^2 [\mu(B(R))]^{\frac{2_s^*}{N}} \\ &\leq C \|u_n\|_\lambda^2 [\mu(B(R))]^{\frac{2_s^*}{N}} \\ &\leq C [\mu(B(R))]^{\frac{2_s^*}{N}}. \end{aligned}$$

From (V<sub>3</sub>), we know that there exists  $R > 0$  large enough such that  $\mu(B(R))$  will be arbitrarily small. So we have

$$\int_{B(R)} |u_n|^2 dx \leq \frac{\varepsilon}{2}.$$

Thus,

$$\int_{B_R^c(0)} |u_n|^2 dx \leq \varepsilon.$$

So far we have showed the desired inequality for the case  $q = 2$ . As for the case  $q \in (2, 2_s^*)$ , it is enough to recall the fractional version of the Gagliardo-Nirenberg inequality

$$|u|_q \leq C |(-\Delta)^{\frac{s}{2}} u|_2^\beta |u|_2^{1-\beta}, \quad (3.7)$$

where  $\beta$  satisfies

$$\frac{1}{q} = \frac{\beta}{2_s^*} + \frac{1-\beta}{2},$$

and recall that  $(u_n)$  is bounded in  $E_\lambda$ . The proof is complete.  $\square$

**Lemma 3.9.**  $u_\lambda \not\equiv 0$  for every  $\lambda \geq \lambda^*$ , where  $\lambda^* = \lambda_\varepsilon^*$  is defined in Lemma 3.8.

*Proof.* From the fact that  $\langle J'_\lambda(u_n), u_n \rangle \rightarrow 0$  and  $F'_1(u_n)u_n \geq 0$  (thanks to the choice of  $\delta$ ), we get that

$$\|u_n\|_\lambda^2 \leq \int_{\mathbb{R}^N} F'_2(u_n)u_n dx + o_n(1).$$

Recall that there exists  $C_p > 0$  such that  $|F'_2(t)| \leq C_p |t|^{p-1}$  for  $t \in \mathbb{R}$  with  $p \in (2, 2_s^*)$ . Thus,

$$\|u_n\|_\lambda^2 \leq C_p \int_{\mathbb{R}^N} |u_n|^p dx + o_n(1) = C_p \int_{B_R(0)} |u_n|^p dx + C_p \int_{B_R^c(0)} |u_n|^p dx + o_n(1). \quad (3.8)$$

Taking  $\varepsilon \leq \frac{\alpha}{2C_p}$  in Lemma 3.8, we have

$$\limsup_{n \rightarrow \infty} \int_{B_R^c(0)} |u_n|^p dx \leq \frac{\alpha}{2C_p}.$$

Now, we argue by contradiction. Suppose that  $u_\lambda \equiv 0$ . Since  $u_n \rightarrow u_\lambda$  in  $L^p_{loc}(\mathbb{R}^N)$ , then

$$\lim_{n \rightarrow \infty} \int_{B_R(0)} |u_n|^p dx = 0.$$

From (3.8), we have

$$\limsup_{n \rightarrow \infty} \|u_n\|_\lambda^2 \leq \frac{\alpha}{2},$$

which leads to a contradiction with Lemma 3.6.  $\square$

Based on Lemmas 3.9 and 3.3, we know that  $u_\lambda \in D(J_\lambda) \setminus \{0\}$  and there exists a unique  $t_\lambda > 0$  such that  $t_\lambda u_\lambda \in \mathcal{N}_\lambda$ . The following lemma shows that  $t_\lambda$  is uniformly bounded from above.

**Lemma 3.10.**  $t_\lambda \in (0, 1]$  for any  $\lambda > 0$ .

*Proof.* According to (3.4) in Lemma 3.3,  $\langle J'_\lambda(tu_\lambda), u_\lambda \rangle$  is positive on  $(0, t_\lambda)$  and negative on  $(t_\lambda, +\infty)$ . So it is sufficient to prove that  $\langle J'_\lambda(u_\lambda), u_\lambda \rangle \leq 0$ . Take a truncation function  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$  such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_1(0) \quad \text{and} \quad \phi \equiv 0 \text{ in } \mathbb{R}^N \setminus B_2(0).$$

If  $R > 0$ , set  $\phi_R = \phi(\cdot/R)$ . Then we know that  $|\nabla \phi_R| \leq \frac{C}{R}$  for some  $C > 0$ . It follows from  $\langle J'_\lambda(u_n), u_n \phi_R \rangle \rightarrow 0$  that

$$\begin{aligned} & \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (u_n \phi_R) dx + \int_{\mathbb{R}^N} \lambda V(x) u_n^2 \phi_R dx + \int_{\mathbb{R}^N} F'_1(u_n) u_n \phi_R dx \\ &= \int_{\mathbb{R}^N} F'_2(u_n) u_n \phi_R dx + o_n(1). \end{aligned} \tag{3.9}$$

Recall that  $u_n \rightarrow u_\lambda$  in  $L^q_{loc}(\mathbb{R}^N)$  for  $q \in [1, 2_s^*)$  and  $|F'_2(t)| \leq C_p |t|^{p-1}$  for all  $t \in \mathbb{R}$  with  $p \in (2, 2_s^*)$ ; then we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_2(u_n) u_n \phi_R dx = \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda \phi_R dx.$$

Similarly, since  $|F'_1(t)| \leq (1 + C_p) |t|^{p-1}$  with  $p \in (2, 2_s^*)$ , one has

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_1(u_n) u_n \phi_R dx = \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda \phi_R dx.$$

We observe that

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} (u_n \phi_R) dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(u_n(x) \phi_R(x) - u_n(y) \phi_R(y))}{|x - y|^{N+2s}} dx dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \phi_R(x) dx dy \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+2s}} u_n(y) dx dy \\ &:= I + II, \end{aligned}$$

and

$$I \geq \int_{B_R(0)} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dy dx = \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx.$$

Thus, (3.9) implies that

$$\begin{aligned} & \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \int_{B_R(0)} \lambda V(x) u_n^2 dx + II + \int_{\mathbb{R}^N} F'_1(u_n) u_n \phi_R dx \\ & \leq \int_{\mathbb{R}^N} F'_2(u_n) u_n \phi_R dx + o_n(1). \end{aligned}$$

Thanks to the weak lower continuity of the norm, one has

$$\begin{aligned} & \int_{B_R(0)} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 dx + \int_{B_R(0)} \lambda V(x) u_\lambda^2 dx + \limsup_{n \rightarrow \infty} II + \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda \phi_R dx \\ & \leq \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda \phi_R dx. \end{aligned}$$

By Remark 2.2, by using Fatou's lemma and recalling the  $\phi_R \geq 0$  in  $\mathbb{R}^N$ , we obtain that

$$\liminf_{R \rightarrow \infty} \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda \phi_R dx \geq \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda dx$$

and

$$\int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda \phi_R dx \leq \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx.$$

Then

$$\begin{aligned} & \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_\lambda^2 dx + \limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} II + \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda dx \\ & \leq \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx. \end{aligned} \tag{3.10}$$

Now, we claim that

$$\limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} II = \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} II = 0.$$

Using the Hölder inequality and recalling that  $u_n$  is bounded, we obtain that

$$\begin{aligned} |II| & \leq \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(y) dx dy \right)^{\frac{1}{2}} \\ & \leq C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy \right)^{\frac{1}{2}}, \end{aligned}$$

so it is sufficient to prove that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy = 0.$$

Next, we decompose the domain

$$\begin{aligned} \mathbb{R}^N \times \mathbb{R}^N & = [(\mathbb{R}^N \setminus B_{2R}(0)) \times (\mathbb{R}^N \setminus B_{2R}(0))] \cup (B_{2R}(0) \times \mathbb{R}^N) \cup [(\mathbb{R}^N \setminus B_{2R}(0)) \times B_{2R}(0)] \\ & := K_R^1 \cup K_R^2 \cup K_R^3. \end{aligned}$$

Obviously,

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy &= \int \int_{K_R^1} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy \\ &+ \int \int_{K_R^2} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy \\ &+ \int \int_{K_R^3} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy. \end{aligned}$$

Since  $\phi_R \equiv 0$  on  $\mathbb{R}^N \setminus B_{2R}(0)$ , then

$$\int \int_{K_R^1} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy = 0. \quad (3.11)$$

Moreover,  $\phi_R \in [0, 1]$  and  $|\nabla \phi_R| \leq \frac{C}{R}$  in  $\mathbb{R}^N$ , so that by the Lagrange Theorem

$$\begin{aligned} \int \int_{K_R^2} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy &= \int_{B_{2R}(0)} \int_{\{|y-x|>R\}} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dy dx \\ &+ \int_{B_{2R}(0)} \int_{\{|y-x|\leq R\}} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dy dx \\ &\leq 4 \int_{B_{2R}(0)} \int_{\{|y-x|>R\}} \frac{u_n^2(x)}{|x - y|^{N+2s}} dy dx \\ &+ CR^{-2} \int_{B_{2R}(0)} \int_{\{|y-x|\leq R\}} \frac{u_n^2(x)}{|x - y|^{N+2s-2}} dy dx \\ &\leq C_1 R^{-2s} \int_{B_{2R}(0)} u_n^2(x) dx + C_2 R^{-2s} \int_{B_{2R}(0)} u_n^2(x) dx \\ &= C_3 R^{-2s} \int_{B_{2R}(0)} u_n^2(x) dx. \end{aligned}$$

Due to the fact that  $u_n \rightarrow u_\lambda$  in  $L_{loc}^q(\mathbb{R}^N)$  for  $q \in [1, 2_s^*)$  and  $u_\lambda \in H^s(\mathbb{R}^N)$ , one has

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} C_3 R^{-2s} \int_{B_{2R}(0)} u_n^2(x) dx = \lim_{R \rightarrow \infty} C_3 R^{-2s} \int_{B_{2R}(0)} u_\lambda^2(x) dx = 0,$$

which implies that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int \int_{K_R^2} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy = 0. \quad (3.12)$$

Proceeding as in the proof of [42], Lemma 2.6, we can also show that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int \int_{K_R^3} \frac{|\phi_R(x) - \phi_R(y)|^2}{|x - y|^{N+2s}} u_n^2(x) dx dy = 0. \quad (3.13)$$

Combining with (3.11), (3.12) and (3.13), we have shown the assertion. Finally, from (3.10), we get

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_\lambda|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) u_\lambda^2 dx + \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda dx \leq \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx,$$

which concludes the proof.  $\square$

**Lemma 3.11.** *If  $\lambda \geq \lambda^*$ , then*

$$u_n \rightarrow u_\lambda \quad \text{in } E_\lambda.$$

*Proof.* Since  $(u_n)$  is  $(PS)_{c_\lambda}$  sequence of  $J_\lambda$ , we have

$$c_\lambda + o_n(1) = J_\lambda(u_n) - \frac{1}{2} \langle J'_\lambda(u_n), u_n \rangle = \frac{1}{2} \int_{\mathbb{R}^N} |u_n|^2 dx. \quad (3.14)$$

Using Fatou's lemma, one has

$$c_\lambda \geq \frac{1}{2} \int_{\mathbb{R}^N} |u_\lambda|^2 dx. \quad (3.15)$$

Notice that  $t_\lambda u_\lambda \in \mathcal{N}_\lambda$ , so that from (3.3) and Lemma 3.10 we obtain

$$c_\lambda \leq J_\lambda(t_\lambda u_\lambda) = J_\lambda(t_\lambda u_\lambda) - \frac{1}{2} \langle J'_\lambda(t_\lambda u_\lambda), t_\lambda u_\lambda \rangle = \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} |u_\lambda|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^N} |u_\lambda|^2 dx \leq c_\lambda.$$

Hence,  $t_\lambda = 1$ , namely  $u_\lambda \in \mathcal{N}_\lambda$  and  $J_\lambda(u_\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} |u_\lambda|^2 dx = c_\lambda$ . The last equality, (3.14) and (3.15), imply that

$$u_n \rightarrow u_\lambda \quad \text{in } L^2(\mathbb{R}^N).$$

It follows from the fractional version of the Gagliardo-Nirenberg inequality (3.7) that

$$u_n \rightarrow u_\lambda \quad \text{in } L^q(\mathbb{R}^N) \text{ with } q \in [2, 2_s^*].$$

Due to the fact that  $|F'_2(t)| \leq C_p |t|^{p-1}$  for all  $t \in \mathbb{R}$  with  $p \in (2, 2_s^*)$ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_2(u_n) u_n dx = \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx. \quad (3.16)$$

In view of  $\langle J'_\lambda(u_\lambda), u_\lambda \rangle = 0$ , we also obtain

$$\|u_\lambda\|_\lambda^2 + \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda dx = \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx. \quad (3.17)$$

From the fact that  $\langle J'_\lambda(u_n), u_n \rangle \rightarrow 0$ , by (3.17) and (3.16), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} F'_1(u_n) u_n dx \right) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_2(u_n) u_n dx \\ &= \int_{\mathbb{R}^N} F'_2(u_\lambda) u_\lambda dx = \|u_\lambda\|_\lambda^2 + \int_{\mathbb{R}^N} F'_1(u_\lambda) u_\lambda dx. \end{aligned}$$

By the weak lower semicontinuity of the norm and by Remark 2.2 (which permits to apply Fatou's lemma), we obtain

$$\liminf_{n \rightarrow \infty} \|u_n\|_\lambda^2 \geq \|u_\lambda\|_\lambda^2 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_1(u_n)u_n dx \geq \int_{\mathbb{R}^N} F'_1(u_\lambda)u_\lambda dx,$$

so that we obtain

$$\|u_n\|_\lambda^2 \rightarrow \|u_\lambda\|_\lambda^2$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_1(u_n)u_n dx = \int_{\mathbb{R}^N} F'_1(u_\lambda)u_\lambda dx.$$

The proof is complete.  $\square$

**Lemma 3.12.** *Let  $u \in D(J_\lambda)$  be a sign-changing solution of (1.1), then  $J_\lambda(u) > 2c_\lambda$ .*

*Proof.* Set  $u^+ = \max\{u, 0\} \geq 0$  and  $u^- = \min\{u, 0\} \leq 0$ , so that  $u = u^+ + u^-$ . Note that  $u \in \mathcal{N}_\lambda$  and it is easy to check that

$$\begin{aligned} \langle J'_\lambda(u), u^+ \rangle &= \langle J'_\lambda(u^+), u^+ \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy = 0, \\ \langle J'_\lambda(u), u^- \rangle &= \langle J'_\lambda(u^-), u^- \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy = 0. \end{aligned}$$

Clearly,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy \leq \int_{u(x)>0} \int_{u(y)<0} \frac{u^+(x)u^-(y)}{|x-y|^{N+2s}} dx dy < 0,$$

which implies that  $\langle J'_\lambda(u^+), u^+ \rangle < 0$  and  $\langle J'_\lambda(u^-), u^- \rangle < 0$ , so that  $u^+, u^- \notin \mathcal{N}_\lambda$ . From Lemma 3.3, there exist  $t^\pm = t_{u^\pm} > 0$  such that  $t^\pm u^\pm \in \mathcal{N}_\lambda$ . From (3.4) we have  $\frac{d}{dt} J_\lambda(tu^\pm)|_{t=1} = \langle J'_\lambda(u^\pm), u^\pm \rangle < 0$ , and so from Lemma 3.3 we obtain that  $t^\pm < 1$ . Hence, from (3.3), we have

$$\begin{aligned} 2c_\lambda &\leq J_\lambda(t^+u^+) + J_\lambda(t^-u^-) \\ &= J_\lambda(t^+u^+) - \frac{1}{2} \langle J'_\lambda(t^+u^+), t^+u^+ \rangle + J_\lambda(t^-u^-) - \frac{1}{2} \langle J'_\lambda(t^-u^-), t^-u^- \rangle \\ &= \frac{(t^+)^2}{2} \int_{\mathbb{R}^N} (u^+)^2 dx + \frac{(t^-)^2}{2} \int_{\mathbb{R}^N} (u^-)^2 dx \\ &< \frac{1}{2} \int_{\mathbb{R}^N} (u^+)^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (u^-)^2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (u^+ + u^-)^2 dx = \frac{1}{2} \int_{\mathbb{R}^N} u^2 dx = J_\lambda(u) - \frac{1}{2} \langle J'_\lambda(u), u \rangle = J_\lambda(u), \end{aligned}$$

which concludes the proof.  $\square$

**Remark 3.13.** It is easy to verify that if  $u \in D(J_\lambda)$  changes sign, we have

$$J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-) - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}} dx dy > J_\lambda(u^+) + J_\lambda(u^-).$$

However, if  $u$  is also a solution of (1.1), we cannot obtain that  $J_\lambda(u) > 2c_\lambda$  directly by this way, since  $u^+, u^- \notin \mathcal{N}_\lambda$ . This situation is crucially different from the case  $s = 1$  (classical Laplacian), where decompositions of the form

$$J(u) = J(u^+) + J(u^-) \text{ and } \langle J'(u), u \rangle = \langle J'(u^+), u^+ \rangle + \langle J'(u^-), u^- \rangle$$

are in general available, see [31] and [32].

Finally, we are ready to conclude with the

*Proof of Theorem 1.1*. By Lemma 3.11, for  $\lambda \geq \lambda^*$ , where  $\lambda^*$  is mentioned in Lemma 3.8, we have

$$u_n \rightarrow u_\lambda \text{ in } E_\lambda.$$

Since  $\Psi$  is lower semicontinuous, passing to the limit in (3.1), by (3.16), this convergence implies that

$$\langle u_\lambda, v - u_\lambda \rangle_\lambda - \int_{\mathbb{R}^N} F'_2(u_\lambda)v dx + \int_{\mathbb{R}^N} F'_2(u_\lambda)u_\lambda dx + \int_{\mathbb{R}^N} F_1(v) dx - \int_{\mathbb{R}^N} F_1(u_\lambda) dx \geq 0,$$

for every  $v \in E_\lambda$ , namely

$$\langle \Phi'_\lambda(u_\lambda), v - u_\lambda \rangle + \Psi(v) - \Psi(u_\lambda) \geq 0.$$

It follows from Definition 2.3(iii) that  $u_\lambda \in E_\lambda$  is a critical point of  $J_\lambda$  with  $J_\lambda(u_\lambda) = c_\lambda$ . From Lemma 2.6(i),  $u_\lambda$  is a weak solution of (1.1) and in view of Lemma 3.12, we obtain that  $u_\lambda \geq 0$  or  $u_\lambda \leq 0$  in  $\mathbb{R}^N$ . Without loss of generality, we can assume that  $u_\lambda \geq 0$  in  $\mathbb{R}^N$ , and so the theorem is proved.  $\square$

#### 4. CONCENTRATION PHENOMENON

In this section, we will show the concentration phenomenon of the least energy solution  $u_\lambda$  of (1.1) as  $\lambda$  diverges. First, we need to recall some notions related to fractional Sobolev spaces in bounded domains.

Recalling that  $\Omega = V^{-1}(0) \subset \mathbb{R}^N$  is a bounded smooth domain, we set  $\mathcal{Q} := (\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathcal{O}$ , where

$$\mathcal{O} := (\Omega^c \times \Omega^c) \subset (\mathbb{R}^N \times \mathbb{R}^N) \text{ and } \Omega^c = \mathbb{R}^N \setminus \Omega.$$

Inspired by [38], let us introduce the fractional space

$$Y := \left\{ u \in L^2(\Omega) : \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

equipped with the norm

$$\|u\|_Y = \left( \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\Omega} u^2 dx \right)^{\frac{1}{2}}.$$

Finally, we set

$$Y_0 := \{u \in Y : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

**Lemma 4.1.**  $\Psi$  is of class  $\mathcal{C}^1$  in  $Y_0$ .

*Proof.* Since  $|F_1'(t)| \leq (1 + C_p)|t|^{p-1}$  with  $p \in (2, 2_s^*)$ , we can use an argument similar to the one in [40], Lemma 2.16 and prove that  $\Psi \in \mathcal{C}^1(Y_0, \mathbb{R})$ .  $\square$

Now let us consider the limit problem (1.7). The Euler-Lagrange functional associated with (1.7) is  $J_\Omega : Y_0 \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} J_\Omega(u) &:= \frac{1}{2} \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{2} \int_{\Omega} u^2 dx - \frac{1}{2} \int_{\Omega} u^2 \log u^2 dx \\ &= \frac{1}{2} \|u\|_Y^2 - \int_{\Omega} F_2(u) dx + \int_{\Omega} F_1(u) dx. \end{aligned}$$

From Lemma 4.1, it is simple to check that  $J_\Omega \in \mathcal{C}^1(Y_0, \mathbb{R})$ . Similar to the argument in Lemma 3.2, we can verify that  $J_\Omega$  satisfies the mountain pass geometry. Define

$$c_\Omega := \inf_{\gamma \in \Gamma_\Omega} \sup_{t \in [0,1]} J_\Omega(\gamma(t)),$$

where

$$\Gamma_\Omega := \{\gamma \in \mathcal{C}([0, 1], Y_0) : \gamma(0) = 0, J_\Omega(\gamma(1)) < 0\}.$$

If  $u \in Y_0$ , then  $u \in Y$  and also  $u \in H^s(\mathbb{R}^N)$ . It follows from (V<sub>2</sub>) that

$$\int_{\mathbb{R}^N} V(x)u^2 dx = \int_{\Omega} V(x)u^2 dx + \int_{\mathbb{R}^N \setminus \Omega} V(x)u^2 dx = 0.$$

Hence,  $u \in E_\lambda$ , which yields that  $Y_0 \subset E_\lambda$ . By the definitions of  $c_\lambda$  in (3.3) and  $c_\Omega$ , we have

$$c_\lambda \leq c_\Omega. \tag{4.1}$$

Related to (1.7), we can introduce the Nehari manifold

$$\mathcal{N}_\Omega := \{u \in Y_0 \setminus \{0\} : \langle J'_\Omega(u), u \rangle = 0\}.$$

Mimicking the argument in [41], Lemma 3.3 for a related local logarithmic equation, we can prove that

$$c_\Omega = \inf_{u \in \mathcal{N}_\Omega} J_\Omega(u).$$

Similarly to Lemmas 3.12 and 3.3, the following lemmas can be shown.

**Lemma 4.2.** *Suppose that  $u \in Y_0$  is a solution of (1.1) with  $J_\Omega(u) \in [c_\Omega, 2c_\Omega)$ , then  $u$  does not change sign.*

**Lemma 4.3.** *For  $u \in Y_0 \setminus \{0\}$ , there exists a unique positive number  $t_u$  such that  $t_u u \in \mathcal{N}_\Omega$ . Furthermore,  $J_\Omega(t_u u) > J_\Omega(tu)$  for all  $t > 0$  and  $t \neq t_u$ . In particular,*

$$J_\Omega(u) = \max_{t > 0} J_\Omega(tu)$$

for  $t_u = 1$ , i.e.  $u \in \mathcal{N}_\Omega$ .

**Definition 4.4.** Let  $d \in \mathbb{R}$ . A sequence  $(u_{\lambda_n}) \subset H^s(\mathbb{R}^N)$  is called a  $(PS)_{d,\infty}$  sequence for the functional family  $(J_\lambda)$ , if there exists a sequence  $\lambda_n \rightarrow +\infty$  such that  $u_{\lambda_n} \in E_{\lambda_n}$ ,  $J_{\lambda_n}(u_{\lambda_n}) \rightarrow d$  and

$$\langle \Phi'_{\lambda_n}(u_{\lambda_n}), v - u_{\lambda_n} \rangle + \Psi(v) - \Psi(u_{\lambda_n}) \geq -\varepsilon_n \|v - u_{\lambda_n}\|_{\lambda_n}, \quad \forall v \in E_{\lambda_n} \quad (4.2)$$

with  $\varepsilon_n \rightarrow 0^+$ .

**Lemma 4.5.** Let  $(u_{\lambda_n}) \subset H^s(\mathbb{R}^N)$  be a  $(PS)_{d,\infty}$  sequence for the functional family  $(J_\lambda)$ , then there exists  $u_\Omega$  such that

$$u_{\lambda_n} \rightarrow u_\Omega \quad \text{in } L^q_{loc}(\mathbb{R}^N), \text{ for } q \in [2, 2_s^*].$$

Moreover,  $u_\Omega = 0$  a.e. in  $\mathbb{R}^N \setminus \Omega$  and  $u_\Omega \in Y_0$ .

*Proof.* From the proof of Lemma 3.4 we see that

$$d \geq C \|u_{\lambda_n}\|_{\lambda_n}^2 \quad (4.3)$$

for some  $C > 0$  independent of  $\lambda_n$ , so that  $(\|u_{\lambda_n}\|_{\lambda_n})$  is bounded in  $\mathbb{R}$ , which implies that  $(u_{\lambda_n})$  is bounded in  $H^s(\mathbb{R}^N)$ . So there exists  $u_\Omega \in H^s(\mathbb{R}^N)$  such that

$$u_{\lambda_n} \rightharpoonup u_\Omega \quad \text{in } H^s(\mathbb{R}^N)$$

and

$$u_{\lambda_n} \rightarrow u_\Omega \quad \text{in } L^q_{loc}(\mathbb{R}^N), \text{ for } q \in [2, 2_s^*].$$

For each  $m \in \mathbb{N}$ , define

$$C_m := \left\{ x \in \mathbb{R}^N : V(x) \geq \frac{1}{m} \right\}.$$

Then

$$\int_{C_m} |u_{\lambda_n}|^2 dx \leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x) |u_{\lambda_n}|^2 dx \leq \frac{m}{\lambda_n} \|u_{\lambda_n}\|_{\lambda_n}^2 = o_n(1).$$

Using Fatou's Lemma, we immediately find that

$$\int_{C_m} |u_\Omega|^2 dx = 0$$

for every  $m \in \mathbb{N}$ . In view of  $(V_2)$ , we have  $\mathbb{R}^N \setminus \Omega = \cup_{m=1}^{+\infty} C_m$ , so

$$\int_{\mathbb{R}^N \setminus \Omega} |u_\Omega|^2 dx = 0,$$

so that  $u_\Omega = 0$  a.e. in  $\mathbb{R}^N \setminus \Omega$  and thus  $u_\Omega \in Y_0$ . □

**Lemma 4.6.** *Let  $(u_{\lambda_n}) \subset H^s(\mathbb{R}^N)$  be a  $(PS)_{d,\infty}$  sequence for the functional family  $(J_\lambda)$ , then either  $d = 0$  or  $d \geq c_\Omega$ . Furthermore, if  $d = c_\Omega$ , then*

$$u_{\lambda_n} \rightarrow u_\Omega \quad \text{in } H^s(\mathbb{R}^N),$$

where  $u_\Omega$ , given in Lemma 4.5, is a least energy solution of (1.7).

*Proof.* In view of (4.3), we have  $d \geq 0$ . If  $d = 0$ , we immediately find that

$$\|u_{\lambda_n}\|_{\lambda_n}^2 \rightarrow 0.$$

If  $d > 0$ , similar to the argument in Lemma 3.9, then  $u_\Omega \neq 0$ . By Lemma 4.3, there exists a unique positive number  $t_{u_\Omega}$  such that  $t_{u_\Omega}u_\Omega \in \mathcal{N}_\Omega$ . Repeating the argument in the proof of Lemma 3.10, we obtain  $t_{u_\Omega} \in (0, 1]$ . Since

$$d + o_n(1) = J_{\lambda_n}(u_{\lambda_n}) = J_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} \langle J'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle = \frac{1}{2} \int_{\mathbb{R}^N} u_{\lambda_n}^2 dx, \quad (4.4)$$

then we use Fatou's lemma to obtain

$$d \geq \frac{1}{2} \int_{\Omega} |u_\Omega|^2 dx \geq \frac{t_{u_\Omega}^2}{2} \int_{\Omega} |u_\Omega|^2 dx = J_\Omega(t_{u_\Omega}u_\Omega) - \frac{1}{2} \langle J'_\Omega(t_{u_\Omega}u_\Omega), t_{u_\Omega}u_\Omega \rangle \geq c_\Omega. \quad (4.5)$$

Now suppose that  $d = c_\Omega$ . According to the inequality above, we know that  $t_{u_\Omega} = 1$  and  $u_\Omega \in \mathcal{N}_\Omega$ . Combining (4.4) and (4.5), we have

$$u_{\lambda_n} \rightarrow u_\Omega \quad \text{in } L^2(\mathbb{R}^N). \quad (4.6)$$

Moreover,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_2(u_{\lambda_n})u_{\lambda_n} dx = \int_{\mathbb{R}^N} F'_2(u_\Omega)u_\Omega dx. \quad (4.7)$$

Then we have

$$\langle J'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle = \|u_{\lambda_n}\|_{\lambda_n}^2 - \int_{\mathbb{R}^N} F'_2(u_{\lambda_n})u_{\lambda_n} dx + \int_{\mathbb{R}^N} F'_1(u_{\lambda_n})u_{\lambda_n} dx + o_n(1). \quad (4.8)$$

and

$$\begin{aligned} \langle J'_{\lambda_n}(u_\Omega), u_\Omega \rangle &= \|u_\Omega\|_{\lambda_n}^2 - \int_{\mathbb{R}^N} F'_2(u_\Omega)u_\Omega dx + \int_{\mathbb{R}^N} F'_1(u_\Omega)u_\Omega dx + o_n(1) \\ &= \|u_\Omega\|_Y^2 - \int_{\Omega} F'_2(u_\Omega)u_\Omega dx + \int_{\Omega} F'_1(u_\Omega)u_\Omega dx + o_n(1). \end{aligned} \quad (4.9)$$

Hence, by (4.7), one has

$$\lim_{n \rightarrow \infty} \left( \|u_{\lambda_n}\|_{\lambda_n}^2 + \int_{\mathbb{R}^N} F'_1(u_{\lambda_n})u_{\lambda_n} dx \right) = \|u_\Omega\|_Y^2 + \int_{\Omega} F'_1(u_\Omega)u_\Omega dx.$$

From (4.6), we have

$$\|u_{\lambda_n}\|_{\lambda_n}^2 \rightarrow \|u_\Omega\|_Y^2$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F'_1(u_{\lambda_n}) u_{\lambda_n} dx = \int_{\Omega} F'_1(u_\Omega) u_\Omega dx.$$

Thus,  $u_{\lambda_n} \rightarrow u_\Omega$  in  $E_{\lambda_n}$  and  $u_{\lambda_n} \rightarrow u_\Omega$  in  $H^s(\mathbb{R}^N)$ . By (4.2), (4.7) and the lower semicontinuity of  $\Psi$ , we have

$$\begin{aligned} & \int_{\mathcal{Q}} \frac{(u_\Omega(x) - u_\Omega(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy - \int_{\mathcal{Q}} \frac{|u_\Omega(x) - u_\Omega(y)|^2}{|x - y|^{N+2s}} dx dy \\ & + \int_{\Omega} u_\Omega(v - u_\Omega) dx - \int_{\mathbb{R}^N} F'_2(u_\Omega)(v - u_\Omega) dx + \int_{\mathbb{R}^N} F_1(v) dx - \int_{\mathbb{R}^N} F_1(u_\Omega) dx \geq 0, \quad \forall v \in Y_0. \end{aligned}$$

Therefore, together with  $J_\Omega(u_\Omega) = c_\Omega$ , we conclude that  $u_\Omega$  is a least energy solution of (1.7).  $\square$

**Remark 4.7.** We notice that the convergence proved above fits in the framework of a version of the notion of asymptotically critical points introduced in [43] and [44].

*Proof of Theorem 1.2.* Consider a sequence  $(\lambda_n) \rightarrow +\infty$  as  $n \rightarrow \infty$ . For convenience, for each  $n \in \mathbb{N}$  we set  $u_n := u_{\lambda_n}$ , which is a solution of (1.1) with  $J_{\lambda_n}(u_n) = c_{\lambda_n}$  by Theorem 1.1. From (4.1),  $c_{\lambda_n} \leq c_\Omega$  for every  $n \in \mathbb{N}$  and  $(c_{\lambda_n})$  is evidently a non decreasing sequence. So

$$\lim_{n \rightarrow \infty} J_{\lambda_n}(u_n) = \lim_{n \rightarrow \infty} c_{\lambda_n} \rightarrow d \leq c_\Omega \quad (4.10)$$

for some  $d$ . Recall that  $(u_n)$  satisfies

$$\langle u_n, v - u_n \rangle_{\lambda_n} - \int_{\mathbb{R}^N} F'_2(u_n) v dx + \int_{\mathbb{R}^N} F'_2(u_n) u_n dx + \int_{\mathbb{R}^N} F_1(v) dx - \int_{\mathbb{R}^N} F_1(u_n) dx \geq 0, \quad (4.11)$$

for  $v \in E_{\lambda_n}$ . Together with (4.10) and (4.11), we know that  $(u_n)$  is a  $(PS)_{d, \infty}$  sequence of the functional family  $(J_{\lambda_n})$  according to definition 4.2. By (3.2), we have  $c_{\lambda_n} \geq \alpha > 0$ , so  $d \geq \alpha$ . By Lemma 4.6 we get that  $d = c_\Omega$ . Thus,  $(u_n)$  is a  $(PS)_{c_\Omega, \infty}$  sequence of the functional family  $(J_{\lambda_n})$ . Using Lemmas 4.5 and 4.6, there exists  $u_\Omega \in H^s(\mathbb{R}^N)$  such that, up to a subsequence,  $u_n \rightarrow u_\Omega$  in  $H^s(\mathbb{R}^N)$  and a.e. in  $\mathbb{R}^N$ . Besides,  $u_\Omega$  is a least energy solution of (1.7) and  $u_\Omega = 0$  a.e. in  $\mathbb{R}^N \setminus \Omega$ . Finally, since  $u_n \geq 0$  in  $\Omega$ , by the pointwise convergence we immediately get that  $u \geq 0$  in  $\Omega$ , as claimed.  $\square$

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#### DATA AVAILABILITY STATEMENT

No data were used for the research described in this article.

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