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ON THE STRUCTURE OF OPTIMAL SOLUTIONS OF CONSERVATION LAWS AT A JUNCTION WITH ONE INCOMING AND ONE OUTGOING ARC

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Abstract. We consider a min–max problem for strictly concave conservation laws on a 1–1 network, with inflow controls acting at the junction. We investigate the minimization problem for a functional measuring the total variation of the flow of the solutions at the node, among those solutions that maximize the time integral of the flux. To formulate this problem we establish a regularity result showing that the total variation of the boundary-flux of the solution of an initial-boundary value problem is controlled by the total variation of the initial datum and of the flux of the boundary datum. In the case the initial datum is monotone, we show that the flux of the entropy weak solution at the node provides an optimal inflow control for this min–max problem. We also exhibit two prototype examples showing that, in the case where the initial datum is not monotone, the flux of the entropy weak solution is no more optimal.

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1. Introduction

In this note, we consider an optimal control problem for scalar conservation laws evolving on a 1–1 network consisting of a junction (node) with one incoming and one outgoing edge. We model the incoming edge with the half-line $I_1 \doteq (-\infty, 0)$, and the outgoing one with the half-line $I_2 \doteq (0, +\infty)$, so that the junction is sitting at $x = 0$. On each edge I_i the evolution of the state variable $u_i(t,x)$ is governed by a scalar conservation law supplemented with some initial condition $\overline{u}_i \in \mathbf{L}^\infty(I_i;\mathbb{R})$, $i = 1, 2$. This yields to study the Cauchy problems on the two semilines I_i, $i = 1, 2$:

$$\begin{cases}\partial_t u_1 + \partial_x f(u_1) = 0, & x < 0,\, t > 0,\\ u_1(0,x) = \overline{u}_1(x), & x < 0,\end{cases} \tag{1.1}$$

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$$\begin{cases} \partial_t u_2 + \partial_x f(u_2) = 0, & x > 0,\, t > 0, \\ u_2(0,x) = \overline{u}_2(x), & x > 0, \end{cases} \tag{1.2}$$

where we assume that the flux $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable, (uniformly) strictly concave map. The equations (1.1), (1.2) are usually coupled through a Kirkhoff type condition at the boundary $x = 0$:

$$f(u_1(t,0)) = f(u_2(t,0)) \qquad \text{for a.e. } \; t > 0. \tag{1.3}$$

Note that condition (1.3) implies that the total mass is conserved. Thus, it is not equivalent to reformulate our problem as a single balance law on the real line, adding, in the same spirit as in [1], a source term of the type $\eta(t)\delta(x)$, where δ is the Dirac measure, centered at $x = 0$.

Here, and throughout the following, we adopt the notation $u_1(t,0) \doteq \lim_{x\to 0^-} u_1(t,x)$ and $u_2(t,0) \doteq \lim_{x\to 0^+} u_2(t,x)$, for the one-sided limit of $u_i(t,\cdot)$ at $x = 0$. The equation (1.3) expresses the conservation of the total mass through the node. Note that the system (1.1), (1.2), and (1.3), admits in general infinitely many entropy admissible solutions. One may recover the uniqueness of solutions to the nodal Cauchy problem (1.1), (1.2), (1.3), requiring that the flux trace at the junction

$$\gamma(t) \doteq f(u_1(t,0)) = f(u_2(t,0)), \qquad t > 0, \tag{1.4}$$

satisfies some further condition. Such conditions are expressed in terms of time-dependent germs [2, 3], or introducing the notion of flux-limited solutions for Hamilton-Jacobi equations [4] (exploiting the equivalence between the two classes of equations [5]), or are obtained *via* vanishing viscosity approximations [6]. Here, we follow the different perspective introduced in [7] to analyze control problems for conservation laws models at a junction with n incoming and m outgoing edges. Namely, we regard the flux trace γ in (1.4) as an *inflow control* acting at the junction. More precisely, for a fixed $T > 0$, we consider $\gamma \in \mathbf{L}^\infty((0,T);\mathbb{R})$ as an *admissible inflow control* if there exists a pair of boundary data $k_1, k_2 \in \mathbf{L}^\infty((0,T);\mathbb{R})$, such that, letting $u_1 : [0,T) \times (-\infty, 0) \to \mathbb{R}$, $u_2 : [0,T) \times (0,+\infty) \to \mathbb{R}$, be the entropy admissible weak solutions of the mixed initial boundary value problems

$$\begin{cases} \partial_t u_1 + \partial_x f(u_1) = 0, & x < 0,\, t > 0, \\ u_1(0,x) = \overline{u}_1(x), & x < 0, \\ u_1(t,0) = k_1(t), & t > 0, \end{cases} \tag{1.5}$$

$$\begin{cases} \partial_t u_2 + \partial_x f(u_2) = 0, & x > 0,\, t > 0, \\ u_2(0,x) = \overline{u}_2(x), & x > 0, \\ u_2(t,0) = k_2(t), & t > 0, \end{cases} \tag{1.6}$$

the corresponding fluxes satisfy the constraint (1.4). The Dirichlet boundary conditions in (1.5), (1.6) are interpreted in the relaxed sense of Bardos, Le Roux, Nédélec [8] which, for strictly concave fluxes, are equivalent to the conditions (iii) stated in Definition 2.1 (in Sect. 2) of entropy admissible solution of the mixed initial-boundary value problem (*cfr.* [9]). Given a fixed initial datum

$$\overline{u}(x) = \begin{cases} \overline{u}_1(x), & \text{if} \quad x < 0, \\ \overline{u}_2(x), & \text{if} \quad x > 0, \end{cases} \tag{1.7}$$

for any admissible inflow control $\gamma \in \mathbf{L}^\infty((0,T);\mathbb{R})$, we will denote by

$$u(t,x;\gamma) \doteq \begin{cases} u_1(t,x), & \text{if} \quad x<0, \\ u_2(t,x), & \text{if} \quad x>0, \end{cases} \tag{1.8}$$

the unique solution of the nodal Cauchy problem (1.1), (1.2), (1.3), (1.4).

Optimization issues for conservation laws evolving on networks has attracted an increasing attention in the last two decades (see the monographs [10–12] and the references therein). The interest on these problems is motivated by a wide range of applications in different areas such as data flow and telecommunication [13], gas or water pipelines [14, 15], production processes [16–19], biological resources [20], and blood circulation [21]. In particular, optimal control problems for conservation laws with concave flux have found a natural application in traffic flow models, firstly introduced in the seminal papers by Lighthill, Whitham and Richards (LWR model) [22, 23] see also [43–46]. Several optimization problems related to different traffic performance indexes have been considered in the literature to improve the mean travel time in a given road segment [24–28], to diminish the queue length due to a red light or stop sign [29], to minimise stop-and-go waves [30–32], or to get a fuel consumption reduction [33, 34].

Within the above setup, in the general case of a junction with n incoming and m outgoing edges we have established the existence of optimal inflow controls for various functionals depending on the value of the solution at the node [7], and along the incoming or outgoing edges [35].

The object of this note is to derive a characterization of optimal inflow controls for a class of optimal control problems on a 1–1 network. Namely, we first consider the problem of maximizing the total flux at the junction

$$\sup_{\gamma\in\mathcal{U}} \int_0^T f(u_1(t,0;\gamma))\mathrm{d}t = \sup_{\gamma\in\mathcal{U}} \int_0^T \gamma(t)\ \mathrm{d}t, \tag{1.9}$$

where $\mathcal{U}$ denotes the class of admissible inflow controls. Our first result (Thm. 2.5) shows that, letting $\widetilde{u}:[0,T)\times\mathbb{R}\to\mathbb{R}$ denote the entropy weak solution of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x\in\mathbb{R},\, t>0, \\ u(0,x) = \overline{u}(x), & x\in\mathbb{R}, \end{cases} \tag{1.10}$$

the corresponding flux $\widetilde{\gamma} \doteq f\circ\widetilde{u}(\cdot,0)$ provides a maximizer of (1.9). Since the optimization problem (1.9) admits in general more than one maximizer, we next address a minimization problem for a functional $\mathcal{F}$ measuring the total variation of the solution at the node (see Def. (2.13))

$$\inf_{\gamma\in\mathcal{U}_{max}} \mathcal{F}\big(u(\cdot,\cdot;\gamma)\big), \tag{1.11}$$

where $\mathcal{U}_{max}$ denotes the set of the maximizers for (1.9). The goal of this min–max problem is to keep as small as possible the oscillations of the flux of the solution at the node among those solutions that maximize the time integral of the flux. To formulate this minimization problem we need first to ensure that the flux of the solutions of the initial boundary value problems (1.5), (1.6), evaluated at the boundary $x=0$, is a function of bounded variation ($\mathbf{BV}$), provided that the initial data $\overline{u}_i$, and the flux of the boundary data $f\circ k_i$, are $\mathbf{BV}$ functions as well. This regularity result is established in Theorem A.1 of the Appendix. Note that this paper does not address the well-posedness of a Cauchy problem on a junction. Rather, we aim to optimize traffic performance by using inflows and outflows as boundary control functions. To achieve this, we rely on existing well-posedness results for conservation laws with boundary conditions. From a practical point of view, traffic control can be implemented in a variety of ways, which includes traffic lights timing, yielding and stop signs, integrated vehicular and roadside sensors so to modulate the amount of incoming fluxes entering the junction.

The main results of this note (Thms. 5.5, 5.8) show that, in the case where the initial datum $\overline{u}$ is monotone, the flux $\widetilde{\gamma}$ of the entropy weak solution of (1.10) provides an optimal inflow control for the min–max problem (1.11). On the other hand, in Section 4 we discuss two prototype examples showing that, in the case where the initial datum $\overline{u}$ is not monotone, the flux $\widetilde{\gamma}$ is no more optimal. This leads us to formulate the following:

Conjecture 1.1. *If the flux $\widetilde{\gamma}$ of the entropy weak solution of* (1.10) *is not monotone on $(0, T)$, then an optimal control for* (1.11) *must be a piecewise constant flux of a non-entropic solution of* (1.10)*. Moreover, such a solution admits a shock discontinuity emerging tangentially from the junction $x = 0$.*

From the perspective of traffic model applications, our result is quite natural. Specifically, when the traffic on both incoming and outgoing roads exhibits homogeneous features (meaning it is either in acceleration or in deceleration regime), no control is needed at the road intersection in order to enhance the overall performance. However, when the regime on the roads is heterogeneous, so presenting both deceleration and acceleration areas, then implementing a different condition at the road intersection can significantly improve car circulation.

The analysis of the optimization problems considered in this note relies on the method of generalized characteristics developed by Dafermos [36, 37] for conservation laws with convex or concave fluxes, and applied to the present setting of a 1–1 network. Clearly, other techniques, as the wave-front tracking (cfr. [42, 47]), can be useful for such analysis. However, it is not directly applicable, since it requires piecewise constant approximations. Here we have restricted the study of these optimization problems to the case where the dynamics along the incoming and outgoing edges is described by the same conservation law. As a natural next step, we plan to investigate the structure of solutions for this type of optimal control problems when the flux of the incoming and outgoing edges is different (in order to describe traffic flow dynamics on roads of varying amplitudes or surface conditions). In this different scenario we will exploit the analysis pursued in [38, 39] on the structure of solutions to conservation laws with discontinuous flux. We expect to derive conditions (on some classes of initial data) which ensure the optimality of inflow controls associated to the so-called *interface connections* (A, B). Moreover, other possible extensions are: the case of general junctions with m incoming edges and n outgoing ones and the case of balance laws with source terms at the intersection.

Recent results on the structure of optimal solutions for another class of control problems on a 1−1 network were obtained in [5] exploiting the equivalence between conservation laws and Hamilton Jacobi equations; see also [41].

The note is organized as follows. In Section 2 we recall some preliminary results on initial-boundary value problems, present the general control setting, and state the main results of the note. In Section 3 we prove the optimality for (1.9) of the flux of the entropy weak solution of (1.10). In Section 4 we propose two examples of non-monotone initial data $\overline{u}$ for which the entropy weak solution of (1.10) is not an optimal solution of (1.11). In Section 5 we establish the main results of the note on the optimality for (1.11) of the flux of the entropy weak solution of (1.10) when the initial datum is monotone. Finally, in the Appendix A we provide a proof of the **BV**-regularity of the trace of the flux of the solution of an initial-boundary value problem.

2. Preliminaries and main results

In traffic flow models, usually the traffic density takes values in a compact domain $\Omega = [0, u^{\max}]$, where $u^{\max}$ denotes the maximum possible density (bump-to-bump) inside the road. The fundamental diagram $f : \Omega \to \mathbb{R}$ has the expression $f(u) = uv(u)$ where v is the average speed of vehicles. In most models the velocity v satisfies $v'(u) \leq \overline{v} < 0$ for all u, and $u \to uv'(u)$ is a non-increasing map. Because of the finite propagation speed, it is not restrictive to assume that the domain of f is the whole real line. Thus, throughout the note we make the general assumption that the flux function is a twice continuously differentiable map $f : \mathbb{R} \to \mathbb{R}$, which is (uniformly) strictly concave, and that attains its global maximum at the point θ, *i.e.*

$$f(\theta) = \max_{u \in \mathbb{R}} f(u). \tag{2.1}$$

We recall that a function $g : (a, b) \to \mathbb{R}$, $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, of bounded variation (**BV**) admits left limits at any point $t \in (a, b]$, and right limits at any point $t \in [a, b)$. We will denote them with $g(t^-)$ and $g(t^+)$, respectively. We define $\mathrm{TV}_{(a,b)}\{g\}$ as the *essential variation* of g on the interval (a,b) which coincides with the *pointwise variation* on (a,b) of the right continuous or left continuous representative of g (see [48], Sect. 3.2). More explicitly for the right continuous representative g we write:

$$\mathrm{TV}_{(a,b)}\{g\} \doteq \sup_{a<t_0<t_1<\cdots<t_N<b} \left\{ \sum_{\ell=1}^{N} |g(t_\ell) - g(t_{\ell-1})| \right\}. \tag{2.2}$$

We will sometimes also use the notation, for $(c,d) \subset (a,b)$,

$$\mathrm{TV}_{[c,d]}\{g\} \doteq \mathrm{TV}_{(c,d)}(g) + |g(c^+) - g(c^-)| + |g(d^+) - g(d^-)|. \tag{2.3}$$

2.1. Admissible junction controls

Following [7], we introduce in this section the set of admissible junction inflow controls associated to pairs of solutions of the initial boundary value problems (1.5), (1.6). We recall (see [7], Rem. 2.3) that an entropy weak solution $u_1(t,\cdot)$ of (1.5) ($u_2(t,\cdot)$ of (1.6)) satisfies a-priori BV-bounds (due to Oleĭnik-type inequalities) and admits the one-sided limit $u_1(t,0^-)$ ($u_2(t,0^+)$) at any fixed time $t>0$. As observed in the introduction, Throughout the paper we will adopt the notation $u_1(t,0) \doteq u_1(t,0^-)$, $u_2(t,0) \doteq u_2(t,0^+)$. Moreover, we will use the notations $I_1 \doteq (-\infty,0)$, $I_2 \doteq (0,+\infty)$, $J_1 \doteq [\theta,+\infty)$, $J_2 \doteq (-\infty,\theta]$. The Dirichlet boundary conditions at $x=0$ must be interpreted in the relaxed sense of Bardos, Le Roux, Nédélec [8]. Therefore, we shall adopt the following definition of entropy admissible weak solution of the mixed initial-boundary value problem (1.5), (1.6) (*cfr.* [9, 49]).

Definition 2.1. Given $\overline{u}_i \in \mathbf{L}^\infty(I_i;\mathbb{R})$, $T>0$, and $k_i \in \mathbf{L}^\infty\big((0,T);J_i\big)$, we say that a function $u_i \in \mathbf{C}\big([0,T);\mathbf{L}^1_{\mathrm{loc}}(I_i;\mathbb{R})\big)$ is an entropy admissible weak solution of (1.5) (resp. of (1.6)) if :

(i) For every $t>0$, $u_i(t) \in \mathbf{BV}_{\mathrm{loc}}(I_i;\mathbb{R})$ and admits the one-sided limit at $x=0$.

(ii) u_i is a weak entropic solution of (1.1) (resp. of (1.2)) with initial data $\overline{u}_i$, *i.e.* for all $\varphi \in \mathbf{C}^1_{\mathbf{c}}\big([0,T)\times I_i;\mathbb{R}\big)$ there holds

$$\int_0^T \int_{I_i} \Big[u_i\,\partial_t\varphi + f(u_i)\,\partial_x\varphi\Big](t,x)\,\mathrm{d}x\mathrm{d}t + \int_{I_i} \overline{u}_i(x)\varphi(0,x)\mathrm{d}x = 0,$$

and the Lax entropy condition is satisfied

$$u_i(t,x^-) \le u_i(t,x^+), \qquad t>0,\ x\in I_i.$$

(iii) The boundary condition in (1.5) is verified if, for a.e. $t>0$, there holds:

$$\begin{aligned} &\text{either} && u_1(t,0) = k_1(t), \\ &\text{or} && f'(u_1(t,0))>0 \quad\text{and}\quad f(u_1(t,0)) \le f\big(k_1(t)\big); \end{aligned} \tag{2.4}$$

the boundary condition in (1.6) is verified if, for a.e. $t>0$, there holds:

$$\begin{aligned} &\text{either} && u_2(t,0) = k_2(t), \\ &\text{or} && f'(u_2(t,0))<0 \quad\text{and}\quad f(u_2(t,0)) \le f\big(k_2(t)\big). \end{aligned}$$

Remark 2.2. In Definition 2.1 we are considering only boundary data that have characteristics entering the domain, that is $k_1 \in \mathbf{L}^\infty((0,T);[\theta,+\infty))$ and $k_2 \in \mathbf{L}^\infty((0,T);(-\infty,\theta])$. As observed in [9], this is not a restrictive assumption. Indeed, if one considers more general boundary data $k_i \in \mathbf{L}^\infty((0,T);\mathbb{R})$, one can recover the same entropy admissible weak solutions $u_i(t,x)$, $i=1,2$, of (1.5), (1.6), replacing the boundary data k_i with the normalized boundary data $\widetilde{k}_i$ defined by

$$\widetilde{k}_i(t) \doteq \begin{cases} f_+^{-1}\big(f(u_1(t,0))\big), & \text{if} \quad i=1,\\ f_-^{-1}\big(f(u_2(t,0))\big), & \text{if} \quad i=2, \end{cases} \tag{2.5}$$

where we denote as f_-, f_+ the restrictions of f to the intervals $(-\infty,\theta]$ and $[\theta,+\infty]$, respectively.

Now we are in the position to formulate the definition of admissible inflow control.

Definition 2.3. Given $T>0$, $\overline{u}\in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, as in (1.7), with $\overline{u}_i\in \mathbf{L}^\infty(I_i;\mathbb{R})$, $i=1,2$, we say that $\gamma\in \mathbf{L}^\infty((0,T);\mathbb{R})$ is an admissible junction inflow control if there exist boundary data $k_1\in \mathbf{L}^\infty\left((0,T);[\theta,+\infty)\right)$, $k_2\in \mathbf{L}^\infty\left((0,T);(-\infty,\theta]\right)$ such that, for a.e. $t\in[0,T]$, there holds

$$\gamma(t)=f(u_1(t,0))=f(u_2(t,0)), \tag{2.6}$$

where u_1 is the entropy weak solution to (1.5) with initial datum $\overline{u}_1$ and boundary datum k_1, while u_2 is the entropy weak solution to (1.6) with initial datum $\overline{u}_2$ and boundary datum k_2.

Let us denote

$$u(t,x;\gamma)=\begin{cases} u_1(t,x), & x<0\\ u_2(t,x), & x>0 \end{cases} \tag{2.7}$$

so that $\gamma(t)=f(u(t,0;\gamma))$ for a.e. t.

Remark 2.4. Observe that, if $\widetilde{u}:[0,T)\times\mathbb{R}\to\mathbb{R}$ denotes the entropy admissible weak solution to (1.10), with initial datum $\overline{u}\in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ as in (1.7), $\overline{u}_i\in \mathbf{L}^\infty(I_i;\mathbb{R})$ with $i=1,2$, then the flux $f\circ\widetilde{u}(\cdot\,;0)$ is an admissible junction inflow control. In fact, one can immediately check that letting $\widetilde{u}_1$, $\widetilde{u}_2$ denote the restriction of $\widetilde{u}$ to $\{x<0\}$, $\{x>0\}$, respectively, then we have:

- $\widetilde{u}_1$ is the entropy weak solution to (1.5) with initial datum $\overline{u}_1$ and boundary datum
$$k_1(t)\doteq f_+^{-1}\big(f(\widetilde{u}_1(t,0))\big),\quad t\in(0,T);$$
- $\widetilde{u}_2$ is the entropy weak solution to (1.6) with initial datum $\overline{u}_2$ and boundary datum
$$k_2(t)\doteq f_-^{-1}\big(f(\widetilde{u}_2(t,0))\big),\quad t\in(0,T);$$
- for a.e. $t\in(0,T)$,
$$f(\widetilde{u}_1(t,0))=f(\widetilde{u}_2(t,0)). \tag{2.8}$$

Then, it follows that the map $\widetilde{\gamma}(t)\doteq f(\widetilde{u}_1(t,0))$, $t\in(0,T)$, satisfies the conditions of Definition 2.3. Therefore, we have

$$\widetilde{u}(t,x)=u(t,x;\widetilde{\gamma}\,)\qquad \forall\ (t,x)\in[0,T)\times\mathbb{R},$$

with $u(t,x;\widetilde{\gamma}\,)$ defined as in (2.7) replacing u_i with $\widetilde{u}_i$.

2.2. The control problem

For every fixed $T > 0$, given $\overline{u} \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, we introduce the set of admissible controls according to Definition 2.3:

$$\mathcal{U} \doteq \mathcal{U}(\overline{u}) \doteq \Big\{\gamma \in \mathbf{L}^\infty\big((0,T);\,\mathbb{R}\big) \;\big|\; \gamma \text{ is an admissible junction inflow control}\Big\}, \tag{2.9}$$

and we consider the maximization problem

$$\sup_{\gamma\in\mathcal{U}} \int_0^T f(u_1(t,0;\gamma))\,\mathrm{d}t = \sup_{\gamma\in\mathcal{U}} \int_0^T \gamma(t)\,\mathrm{d}t. \tag{2.10}$$

Our first result is the following:

Theorem 2.5. *Given $T > 0$ and $\overline{u} \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, let $\widetilde{u}$ be the entropy admissible weak solution to the Cauchy problem (1.10). Then the composite function $\widetilde{\gamma} \doteq f \circ \widetilde{u}(\cdot\, 0)$ solves the maximization problem (2.10), i.e. there holds*

$$\int_0^T f(\widetilde{u}(t,0))\,\mathrm{d}t = \max_{\gamma\in\mathcal{U}} \int_0^T \gamma(t)\,\mathrm{d}t.$$

The proof of this result is given in Section 3. One can easily verify that the optimization problem (2.10) admits other maximizers different from $\widetilde{\gamma}$, which are junction inflow controls that present rarefaction waves or shocks generated at the node $x = 0$ at positive times (see Sect. 4). Since a usual objective in traffic management is to keep the traffic flow as fluent as possible, it is then natural to investigate the existence of maximizers of (2.10) that minimize a cost functional measuring the total variation of the corresponding solution at the node $x = 0$ (*e.g.* see [7, 30]).

Therefore for every fixed $T > 0$, given $\overline{u} \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, we restrict the set of maximizers of (2.10) to the set of inflow controls:

$$\mathcal{U}_{max} \doteq \mathcal{U}_{max}(\overline{u}) \doteq \left\{\bar{\gamma} \in \mathcal{U} \cap \mathbf{BV}\big((0,T);\,\mathbb{R}\big) \;\middle|\; \int_0^T f(u_1(t,0;\bar{\gamma}))\mathrm{d}t = \max_{\gamma\in\mathcal{U}} \int_0^T f(u_1(t,0;\gamma))\mathrm{d}t\right\} \tag{2.11}$$

which are admissible for the functional we are going to minimize. For this reason we will assume $\overline{u} \in \mathbf{BV}(\mathbb{R};\mathbb{R})$, so to guarantee that this set is not empty, see the following Remark.

Remark 2.6. If $\overline{u} \in \mathbf{BV}(\mathbb{R};\mathbb{R})$, and $\widetilde{u}$ is the entropy admissible weak solution to (1.10), then the associated boundary inflow control $\widetilde{\gamma} \doteq f \circ \widetilde{u}(\cdot, 0)$ is a function of bounded variation on $(0,T)$ as a consequence of a recent result obtained in [40], Theorem 1.1. More precisely the result is the following: for every $T > 0$ and for every $x \in \mathbb{R}$, the map $t \to f(\widetilde{u}(t,x))$ is in $\mathbf{BV}((0,T);\mathbb{R})$, with

$$\mathrm{TV}_{(0,T)}\, f(\widetilde{u}(\cdot,x)) \leq C(\mathrm{TV}(\overline{u}), \|f' \circ \overline{u}\|_{\mathbf{L}^\infty}), \tag{2.12}$$

where $C(\mathrm{TV}(\overline{u}), \|f' \circ \overline{u}\|_{\mathbf{L}^\infty})$ denotes an explicit constant only depending on $\mathrm{TV}(\overline{u})$ and on the Lipschitz constant of $\|f' \circ \overline{u}\|_{\mathbf{L}^\infty}$.

Thus, by Remark 2.4, we have $\widetilde{\gamma} \in \mathcal{U}_{max}$ and in particular the set $\mathcal{U}_{max}$ is not empty.

In Appendix A we complement the result [40], Theorem 1.1 recalled in Remark 2.6 with a regularity result for solutions of initial-boundary value problems. More precisely in Theorem A.1 we show that if $\overline{u}_1 \in \mathbf{BV}(\mathbb{R};\mathbb{R})$, $k_1 \in \mathbf{L}^\infty\left((0,T);[\theta,+\infty)\right)$ with $f(k_1) \in \mathbf{BV}((0,T);\mathbb{R})$ then the flow map $t \mapsto f\big(u_1(\cdot,0)\big)$ of the entropy admissible weak solution of (1.5) is in $\mathbf{BV}((0,T);\mathbb{R})$.

This implies that in order to obtain admissible boundary inflow control $\gamma \in \mathcal{U}$ that has bounded total variation it is sufficient to choose boundary data k_1, k_2 in (1.5), (1.6) with $f(k_i) \in \mathbf{BV}((0,T);\mathbb{R})$.

Remark 2.7. Notice that to construct admissible inflow controls it is not necessary that the boundary data k_i are in $\mathbf{BV}((0,T);\mathbb{R})$, but only that $f(k_i)$ are in $\mathbf{BV}((0,T);\mathbb{R})$. The requirement $f(k_i) \in \mathbf{BV}((0,T);\mathbb{R})$ is less restrictive: indeed in general the entropy solution $\widetilde{u}$ satisfies $f(\widetilde{u}(\cdot,0)) \in \mathbf{BV}((0,T);\mathbb{R})$, but it does not satisfy $\widetilde{u}(\cdot,0) \in \mathbf{BV}((0,T);\mathbb{R})$ as shown by an explicit counterexample in [40].

Observe that, if $\gamma \in \mathbf{BV}\big((0,T);\,\mathbb{R}\big)$ is an admissible junction inflow control, according to Definition 2.3 we get that

$$\mathrm{TV}_{(0,T)}(f\big(u_1(\cdot,0;\gamma)\big)) = \mathrm{TV}_{(0,T)}(f(u_2(\cdot,0;\gamma))),$$

where the total variation is defined in (2.2). We then introduce the following functional: for any $\gamma \in \mathcal{U}_{max}$, let

$$\begin{aligned}\mathcal{F}_{[0,T]}(\gamma) \doteq\ & \mathrm{TV}_{(0,T)}(f(u_1(\cdot,0;\gamma)))\\ &+ |f(\overline{u}(0^-)) - \gamma(0^+)| + |f(u_1(T,0;\gamma)) - \gamma(T^-)|\\ &+ |f(\overline{u}(0^+)) - \gamma(0^+)| + |f(u_2(T,0;\gamma)) - \gamma(T^-)|.\end{aligned} \tag{2.13}$$

This definition of the functional $\mathcal{F}$ aims to take into account also variations of the flux at the node $x=0$ which occur at the beginning and at the end of the interval of time $[0,T]$.

We study the following minimization problem

$$\inf_{\gamma\in\mathcal{U}_{max}} \mathcal{F}_{[0,T]}(\gamma). \tag{2.14}$$

Due to Theorem 2.5 and Remark 2.6 we get that

$$\inf_{\gamma\in\mathcal{U}_{max}} \mathcal{F}_{[0,T]}(\gamma) = \inf_{\gamma\in\mathcal{U}^M_{max}} \mathcal{F}_{[0,T]}(\gamma),$$

where

$$\mathcal{U}^M_{max} = \Big\{\gamma \in \mathcal{U}_{max} \;\big|\; \mathcal{F}_{[0,T]}(\gamma) \le M \doteq \mathcal{F}_{[0,T]} f(\widetilde{u}(\cdot,0))\Big\}.$$

Note that this set is compact in $\mathbf{L}^1_{loc}$, as proved in [7], so we conclude the following existence result for (2.14).

Theorem 2.8. *Given $T>0$, and $\overline{u} \in \mathbf{BV}(\mathbb{R};\mathbb{R})$, there exists a solution to* (2.14)*.*

In [7] existence of solutions to (2.10) and (2.14) have been proved in much more generality, for networks with n incoming roads and m outgoing roads.

2.3. Main result

The main result of the note shows that the entropy admissible weak solution solves the minimization problem (2.14) when the initial datum $\overline{u}$ is monotone. This provides an optimal control characterization of the entropic solution for monotone initial data. The precise statement of our result is the following.

Theorem 2.9. *Let $\widetilde{u}$ be the unique entropy admissible weak solution to* (1.10)*. If the initial datum $\overline{u}$ is monotone, then*

$$\min_{\gamma\in\mathcal{U}^M_{max}} \mathcal{F}_{[0,T]}(\gamma) = \mathcal{F}_{[0,T]}(\widetilde{\gamma}),$$

where $\widetilde{\gamma} = f(\widetilde{u}(\cdot,0))$.

This Theorem follows directly from Theorem 5.5 for monotone non-decreasing initial data and from Theorem 5.8 for monotone non-increasing initial data.

On the other hand, without the monotonicity assumption on the initial data, we cannot expect that the entropy solution minimizes the total variation functional $\mathcal{F}_{[0,T]}$. In particular in Section 4, we provide two prototypical examples of non-monotone initial data for which the minimum in (2.14) is achieved by a not entropic solution.

3. The entropy admissible weak solution maximizes the flow

In this section we provide the proof of Theorem 2.5.

We start with a lemma, which will be useful in the following.

Lemma 3.1.

1. *Let u_1, v_1 be entropy admissible weak solutions to (1.5), with the same initial datum $\overline{u} \in \mathbf{L}^\infty((-\infty, 0); \mathbb{R})$ and such that $u_1(T, x) = v_1(T, x)$ for almost every $x \leq 0$. Then $\int_0^T f(u_1(t, 0))\,\mathrm{d}t = \int_0^T f(v_1(t, 0))\,\mathrm{d}t$.*
2. *Let u_2, v_2 be entropy admissible weak solutions to (1.6), with the same initial datum $\overline{u} \in \mathbf{L}^\infty((0, +\infty); \mathbb{R})$ and such that $u_2(T, x) = v_2(T, x)$ for almost every $x \geq 0$. Then $\int_0^T f(u_2(t, 0))\,\mathrm{d}t = \int_0^T f(v_2(t, 0))\,\mathrm{d}t$.*

Proof. We prove only statement 1, since the proof of 2 is completely analogous.

Take $L > \max\left\{\|f'(u_1)\|_{\mathbf{L}^\infty([0,T]\times(-\infty,0))}, \|f'(u_2)\|_{\mathbf{L}^\infty([0,T]\times(-\infty,0))}\right\}$ and observe that for all $x < -LT$, the values of entropy admissible weak solutions u_1, v_1 to (1.5) are determined only by the initial data $\overline{u}$: this means that $f(u_1(t, x)) = f(u_2(t, x))$ for a.e. $t \in [0, T]$ and $x < -LT$.

Then, by the Divergence Theorem we find that

$$\begin{aligned}\int_0^T f(u_1(t,0))\mathrm{d}t &= \int_0^T f(u_1(t,-LT^+))\mathrm{d}t + \int_{-LT}^0 (\overline{u}(x) - u_1(T,x))\mathrm{d}x \\ &= \int_0^T f(v_1(t,-LT^+))\mathrm{d}t + \int_{-LT}^0 (\overline{u}(x) - v_1(T,x))\mathrm{d}x = \int_0^T f(v_1(t,0))\mathrm{d}t.\end{aligned}$$

□

Now we are ready to prove Theorem 2.5. We repeat the statement for more clarity.

Theorem 3.2. *Given $\overline{u} \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R})$, let $\widetilde{u}$ be the unique entropy admissible solution to (1.10). Then, for every $T > 0$, we have that $f(\widetilde{u}(\cdot, 0))$ solves the maximization problem (2.10). More precisely, there holds*

$$\int_0^T f(\widetilde{u}(t,0))\,\mathrm{d}t = \max_{\gamma\in\mathcal{U}} \int_0^T \gamma(t)\,\mathrm{d}t.$$

Proof. Consider an arbitrary $\gamma \in \mathcal{U}$, and define the corresponding solution $u(t, x; \gamma)$ as in (2.7) in Definition 2.3. From now on, we denote $u_1(t, x) = u(t, x; \gamma)$ for $x < 0$ and $u_2(t, x) = u(t, x; \gamma)$ for $x > 0$. Note that, by Definition 2.3,

$$f(u_2(t,0)) = f(u_1(t,0)) \qquad \text{for a.e. } t \in (0,T).$$

Let $T > 0$ be fixed. Consider the minimal backward characteristic $\widetilde{\xi}$ for $\widetilde{u}$ passing at $(T, 0)$. Without loss of generality we may assume that $\widetilde{\xi}(0) \leq 0$, which implies that all the values of $\widetilde{u}(T, x)$, for $x < 0$, are determined by $\overline{u}_{|(-\infty,0)}$. We claim that

$$\int_0^T f\left(\widetilde{u}(s,0)\right)\,\mathrm{d}s \geq \int_0^T f\left(u_1(s,0)\right)\,\mathrm{d}s. \tag{3.1}$$

In the case $\widetilde{\xi}(0) > 0$, one can prove with the same arguments that

$$\int_0^T f\left(\widetilde{u}(s,0)\right)\,\mathrm{d}s \geq \int_0^T f\left(u_2(s,0)\right)\,\mathrm{d}s.$$

Assume first that there exists a backward characteristic ξ for u_1 passing at $(T,0^-)$ with $\xi(0) < 0$. In this case we have that the value of u on the line (T,x), for $x \leq 0$, is completely determined by the initial datum $\overline{u}$: therefore $\widetilde{u}(T,x) = u_1(T,x)$ for a.e. $x \leq 0$. By Lemma 3.1 applied to $\widetilde{u}, u_1$ we get

$$\int_0^T f(u_1(t,0))\mathrm{d}t = \int_0^T f(\widetilde{u}(t,0))\mathrm{d}t,$$

which implies the claim (3.1).

Assume now that all the backward characteristics ξ for u_1 passing at $(T,0^-)$ satisfy $\xi(0) \geq 0$. Define now the point

$$\bar{x} = \inf\left\{x \leq 0\colon\ f'(u_1(T,z^+))T \leq z \ \text{ for a.e. } z \in (x,0)\right\}.$$

We deduce that necessarily $u_1(T,x) = \widetilde{u}(T,x)$ for a.e. $x < \bar{x}$. Moreover, $u_1(T,x) > \theta$ for a.e. $x \in (\bar{x},0)$ since all characteristics for u_1 passing at (T,x) for $x \in (\bar{x},0)$ must cross $x = 0$ and have strictly negative slope. This implies that $u_1(T,x) \geq \widetilde{u}(T,x)$ for a.e. $x \in (\bar{x},0)$ since either $\widetilde{u}(T,x) \leq \theta$ or the negative slope of the backward characteristics for $\widetilde{u}$ at (T,x) is higher. In conclusion choosing L as above,

$$\int_{-LT}^0 \widetilde{u}(T,x)\mathrm{d}x \leq \int_{-LT}^0 u_1(T,x)\mathrm{d}x.$$

So, repeating the argument based on the Divergence Theorem we have that

$$\begin{aligned}\int_0^T f(u_1(t,0))\mathrm{d}t &= \int_0^T f(u_1(t,-LT))\mathrm{d}t + \int_{-LT}^0 \overline{u}(x)\mathrm{d}x - \int_{-LT}^0 u_1(T,x)\mathrm{d}x \\ &\leq \int_0^T f(\widetilde{u}(t,-LT))\mathrm{d}t + \int_{-LT}^0 \overline{u}(x)\mathrm{d}x - \int_{-LT}^0 \widetilde{u}(T,x)\mathrm{d}x = \int_0^T f(\widetilde{u}(t,0))\mathrm{d}t\end{aligned}$$

proving the claim (3.1). □

4. Non-entropic solutions to the min–max problem (2.14)

In this section we propose two examples showing that, in general, the entropy solution to (1.10) does not satisfy the min–max problem (2.14). They are characterized by an initial datum $\overline{u}$ piecewise constant but not monotone. In Example 4.1 we consider an initial condition $\overline{u}$ piecewise constant with two points of discontinuity located at the left of the interface $x = 0$. Instead, in Example 4.2 one point of discontinuity is located at the left of the interface, while the second one at the right. The general idea consists in producing waves from the interface $x = 0$, which interact with shocks and rarefaction fronts coming from the initial data, so to achieve the maximum amount of possible cancellations.

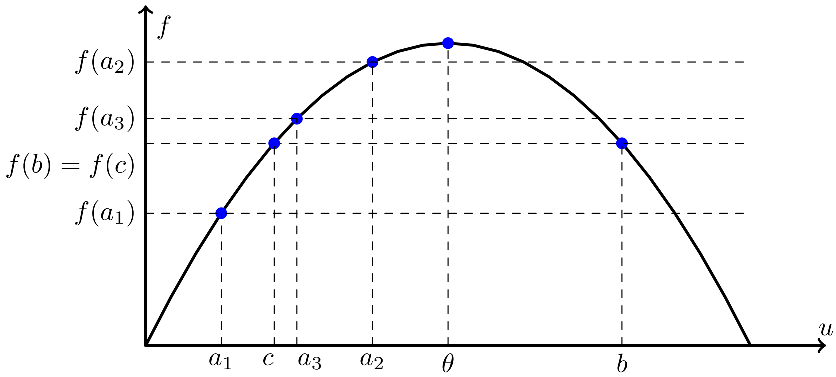

FIGURE 1. The configurations of Example 4.1.

Example 4.1. Consider the Cauchy problem (1.10) with initial condition

$$\overline{u}(x) = \begin{cases} a_1, & \text{if } x < \overline{x}_1, \\ a_2, & \text{if } \overline{x}_1 < x < \overline{x}_2, \\ a_3, & \text{if } x > \overline{x}_2, \end{cases}$$

where $\overline{x}_1 < \overline{x}_2 < 0$ and

$$a_1 < a_3 < a_2 < \theta; \tag{4.1}$$

see Figure 1. Suppose also that

$$-\frac{\overline{x}_2}{f'(a_2)} < -\frac{\overline{x}_1}{\lambda} < T, \tag{4.2}$$

where $\lambda = \frac{f(a_2)-f(a_1)}{a_2-a_1}$. The discontinuity at the point $\overline{x}_1$ originates a shock wave with positive speed given by λ. Instead, at the point $\overline{x}_2$ a rarefaction wave with positive speed originates. Assumption (4.2) implies that the shock wave exiting $\overline{x}_1$ reaches the interface $x = 0$ before interacting with the rarefaction wave. Hence, such interaction happens for $x > 0$ generating waves with positive speed due to (4.1). Therefore, the entropy solution $\widetilde{u}$ in the region $x < 0$ is given by

$$\widetilde{u}(t,x) = \begin{cases} a_1, & \text{if } x < \overline{x}_1 + \lambda t, \\ a_2, & \text{if } \overline{x}_1 + \lambda t < x < \overline{x}_2 + f'(a_2)t, \\ (f')^{-1}\left(\frac{x-\overline{x}_2}{t}\right), & \text{if } f'(a_2)t < x - \overline{x}_2 < f'(a_3)t, \\ a_3, & \text{if } x > \overline{x}_2 + f'(a_3)t; \end{cases} \tag{4.3}$$

see Figure 2. Denote with $\widetilde{\gamma}(t) = f\left(\widetilde{u}(t,0^-)\right)$ the boundary control associated with the entropy solution $\widetilde{u}$. Note that $\widetilde{\gamma}(t)$ is not monotone. Recalling the definition of the functional $\mathcal{F}_{[0,T]}$, see (2.13), we get

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = 2f(a_2) - f(a_3) - f(a_1). \tag{4.4}$$

Choose boundary data b and c such

$$c < \theta < b, \qquad f(a_1) < f(b) = f(c) \leq f(a_3) < f(a_2), \qquad f(a_3) - f(b) < f(a_2) - f(a_3), \tag{4.5}$$

see Figure 1, and consider the constant boundary data $k_1 = b, k_2 = c$.

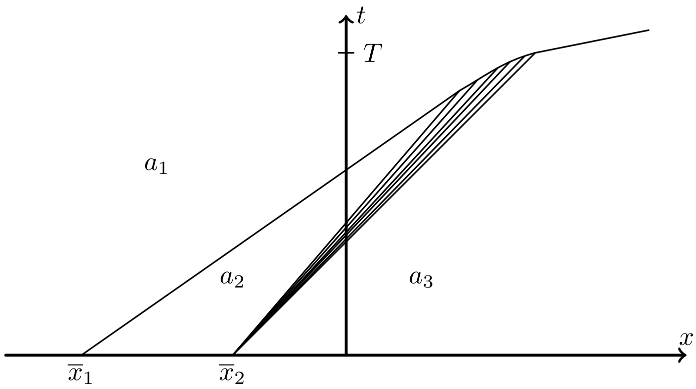

FIGURE 2. The entropy solution $\widetilde{u}$, defined in (4.3), of Example 4.1.

The choices (4.1) and (4.5) imply that the wave generated by the jump (a_1, b) is a shock with positive speed, while the waves generated by the jumps (a_2, b) and (a_3, b) are both shocks with negative speed. Note that it is possible to select the points $\overline{x}_1$ and $\overline{x}_2$ such that the wave generated at $\overline{x}_1$ arrives at the interface $x = 0$, after interacting with the wave (a_2, b), at time $\tau < T$. Therefore, the maps u_1 and u_2 solving respectively the initial-boundary value problems (1.5) and (1.6) with controls k_i are

$$u_1(t,x) = \begin{cases} a_1, & \text{if } (t,x) \in A_1, \\ a_2, & \text{if } (t,x) \in A_2, \\ (f')^{-1}\left(\frac{x-\overline{x}_2}{t}\right), & \text{if } (t,x) \in A_4, \\ a_3, & \text{if } (t,x) \in A_3, \\ b, & \text{otherwise}, \end{cases} \tag{4.6}$$

and

$$u_2(t,x) = \begin{cases} c, & \text{if } \max\left\{0, \frac{f(c)-f(a_1)}{c-a_1}(t-\tau)\right\} < x < \frac{f(c)-f(a_3)}{c-a_3}\, t, \\ a_3, & \text{if } x > \frac{f(c)-f(a_3)}{c-a_3}\, t, \\ a_1, & \text{if } x < \frac{f(c)-f(a_1)}{c-a_1}(t-\tau), \end{cases} \tag{4.7}$$

where the sets A_1, A_2, A_3, A_4 are suitable subsets of the region $x < 0$, see Figure 3.

Note that $\gamma(t) = f(u_1(t,0)) = f(u_2(t,0)) \in \mathcal{U}$. Moreover, since $u_1(T,x) = \widetilde{u}(T,x)$ for a.e. $x \le 0$, by Lemma 3.1 and Theorem 3.2, γ is a solution to the maximization problem (2.10). Moreover, by (4.5),

$$\mathcal{F}_{[0,T]}(\gamma) = |f(b) - f(a_1)| + |f(a_3) - f(b)| + |f(a_3) - f(c)| = 2f(a_3) - f(a_1) - f(b). \tag{4.8}$$

Hence, (4.4), (4.5), and (4.8) imply that

$$\begin{aligned} \mathcal{F}_{[0,T]}(\gamma) - \mathcal{F}_{[0,T]}(\widetilde{\gamma}) =& 2f(a_3) - f(a_1) - f(b) - 2f(a_2) + f(a_3) + f(a_1) \\ =& 2f(a_3) - 2f(a_2) + f(a_3) - f(b) \\ <& 2f(a_3) - 2f(a_2) + f(a_2) - f(a_3) = f(a_3) - f(a_2) < 0, \end{aligned}$$

showing that the entropy solution $\widetilde{u}$ is not optimal for the functional $\mathcal{F}_{[0,T]}$.

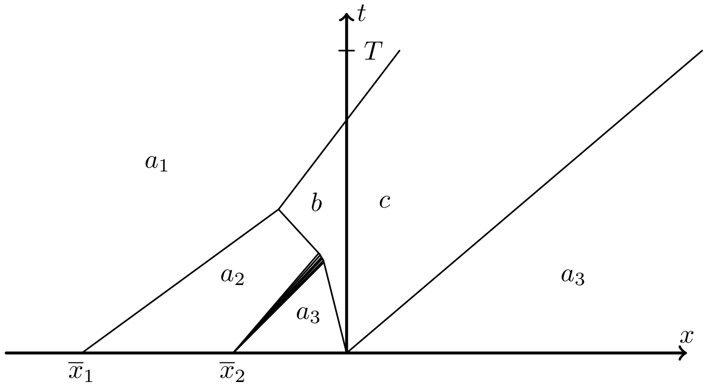

FIGURE 3. The solution u_1 and u_2, defined in (4.6) and in (4.7), of Example 4.1.

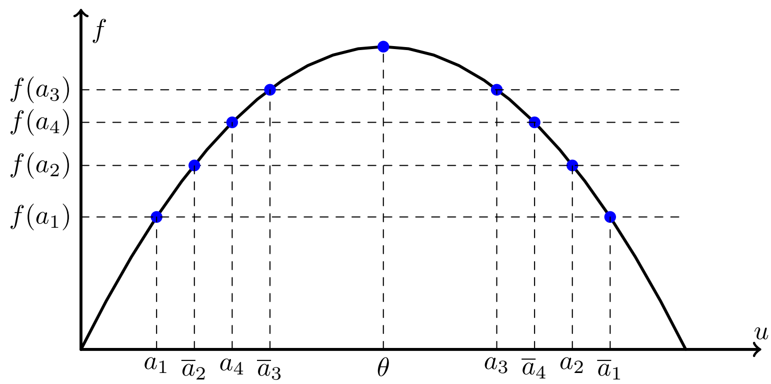

FIGURE 4. The configuration of Example 4.2.

We conclude observing that in this case we may compute explicitly the solution of the minimization problem (2.14). Indeed, if we take $c = a_3$ in (4.5), we obtain

$$\mathcal{F}_{[0,T]}(\gamma) = f(a_3) - f(a_1) = \min_{\gamma \in \mathcal{U}^M_{max}} \mathcal{F}_{[0,T]}(\gamma).$$

The shock discontinuity generated by (a_3, b), with $a_3 = c$, emerges tangentially from the junction $x = 0$.

Example 4.2. Consider the Cauchy problem (1.10) with initial condition

$$\overline{u}(x) = \begin{cases} a_1, & \text{if } x < \overline{x}_1, \\ a_2, & \text{if } \overline{x}_1 < x < \overline{x}_2, \\ a_3, & \text{if } x > \overline{x}_2, \end{cases} \tag{4.9}$$

where $\overline{x}_1 < 0 < \overline{x}_2$ and the densities

$$a_1 < \theta < a_3 < a_2, \quad \text{with} \quad f(a_1) < f(a_2) < f(a_3); \tag{4.10}$$

see Figure 4.

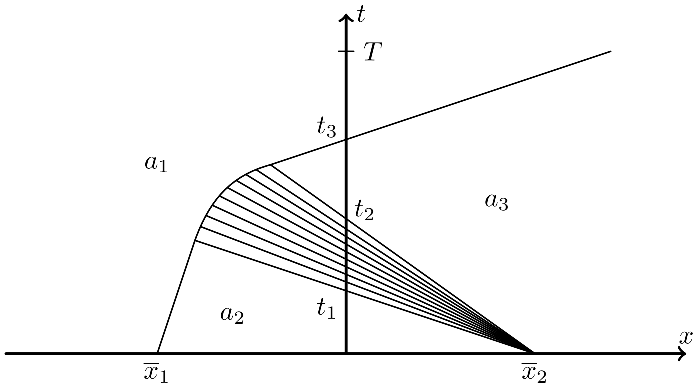

FIGURE 5. The entropy solution $\widetilde{u}$ of Example 4.2. At the location $\overline{x}_1$ a shock wave, connecting the states a_1 and a_2, originates. At the position $\overline{x}_2$ a rarefaction wave, connecting a_2 and a_3, is generated. This wave interacts with the interface $x = 0$ in the time interval $[t_1, t_2]$. In the semiplane $x < 0$ the shock wave interacts with the rarefaction one and the resulting shock travels with positive speed and reaches the interface $x = 0$ at time $t_3 < T$.

The discontinuity at the point $\overline{x}_1$ originates a shock wave with positive speed given by $\lambda = \frac{f(a_1)-f(a_2)}{a_1-a_2}$, whereas at the point $\overline{x}_2$ a rarefaction wave with negative speed originates. Choosing appropriately the points $\overline{x}_1$ and $\overline{x}_2$, by (4.10), we may assume that the rarefaction wave interacts with the shock wave in the semiplane $x < 0$, being completely canceled in the semiplane $x < 0$, and the resulting shock wave interacts with the interface $x = 0$ at time $t_3 > t_2$; see Figure 5. We define the first and the latter time of interaction of the rarefaction wave exiting $\overline{x}_2$ with $x = 0$ as t_1, t_2:

$$t_1 \doteq -\frac{\overline{x}_2}{f'(a_2)} < -\frac{\overline{x}_2}{f'(a_3)} \doteq t_2. \tag{4.11}$$

We may compute explicitly the value t_3 as follows:

$$t_3 \doteq \frac{a_1\overline{x}_1 - a_3\overline{x}_2 + a_2(\overline{x}_2 - \overline{x}_1)}{f(a_3) - f(a_1)}. \tag{4.12}$$

The explicit formula (4.12) can be obtained by applying the Divergence Theorem to the vector field $(\widetilde{u}, f(\widetilde{u}))$ in the rectangular domain $(0, t_3) \times (\overline{x}_1, \overline{x}_2)$.

Recall that $t_3 > t_2$. Without loss of generality we can assume that $t_3 < T$; see Figure 5.

Denote with $\widetilde{\gamma}(t) = f\left(\widetilde{u}(t, 0)\right)$ the boundary control associated with the entropy solution $\widetilde{u}$. Note that $\widetilde{\gamma}(t)$ is nondecreasing in $(0, t_2)$ and nonincreasing in (t_2, T) and moreover $\max \widetilde{\gamma} = f(a_3)$. Recalling the definition of the functional $\mathcal{F}_{[0,T]}$, see (2.13), we get

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = 2f(a_3) - f(a_1) - f(a_2). \tag{4.13}$$

Note that in every interval $(0, t)$ with $t_3 < t \leq T$ the flow $f(\widetilde{u}(t, 0))$ is not monotone.

We provide now a construction for a boundary inflow control $\gamma(t)$ which is a solution to the maximization problem (2.10) and $\mathcal{F}_{[0,T]}(\gamma) < \mathcal{F}_{[0,T]}(\widetilde{\gamma})$.

The construction of the boundary data is quite involved and interesting on its own in our opinion, so we provide it in the following Proposition.

Proposition 4.3. *Let $\overline{u}$ as defined in (4.9). Then there exist boundary data k_1, k_2 such that the solutions u_1, u_2 to the boundary value problems (1.5), (1.6) are associated to an admissible boundary inflow control $\gamma(t) = f(u_1(t,0)) = f(u_2(t,0))$ which is a solution to the maximization problem (2.10) and satisfies $\mathcal{F}_{[0,T]}(\gamma) < \mathcal{F}_{[0,T]}(\widetilde{\gamma})$.*

In particular, the entropy admissible solution $\widetilde{u}$ with initial data $\overline{u}$ is not a solution to min–max problem (2.14).

Proof. Define $\overline{a}_2, \overline{a}_3$ such that $\overline{a}_2 < \overline{a}_3 < \theta$ and $f(a_2) = f(\overline{a}_2)$, $f(a_3) = f(\overline{a}_3)$; see Figure 4.

We aim to construct the boundary data k_i, depending on a new time $t_4 \in (t_1, t_2)$ and on densities a_4 and $\overline{a}_4$ with

$$\begin{aligned} a_1 < \overline{a}_2 < a_4 < \overline{a}_3 < \theta < a_3 < \overline{a}_4 < a_2 < \overline{a}_1, \\ f(a_1) = f(\overline{a}_1) < f(a_2) = f(\overline{a}_2) < f(a_4) = f(\overline{a}_4) < f(a_3) = f(\overline{a}_3), \end{aligned} \tag{4.14}$$

see Figure 4, such that the flux at $x = 0$ of the corresponding solutions u_1, u_2 is given by

$$f(u_1(t,0)) = f(u_2(t,0)) = \begin{cases} f(a_2), & t \in (0, t_4), \\ f(a_4), & t \in (t_4, t_6), \\ f(a_1), & t \in (t_6, T), \end{cases}$$

where t_6 is defined as

$$t_6 \doteq \frac{-(a_2 - a_1)\overline{x}_1 + t_4(f(a_4) - f(a_2))}{f(a_4) - f(a_1)}. \tag{4.15}$$

More precisely the solution u_1 would enjoy the following properties

1.1 Up to the time t_1, its flux trace is constantly $f(u_1(t,0)) = f(\widetilde{u}(t,0))$. In the interval (t_1, t_4), it holds $f(u_1(t,0)) = f(a_2)$.
1.2 At time t_4 a rarefaction wave, connecting the state a_2 with $\overline{a}_4$ (recall (4.14)), enters $x < 0$.
1.3 In the interval (t_4, t_6) the flow $f(u_1(\cdot, 0))$ is constantly equal to $f(\overline{a}_4)$, where t_6 is the time at which the shock wave obtained by the interaction between the shock originated at $\overline{x}_1$ and the rarefaction originated at t_4 crosses the axis $x = 0$, see Figure 7. The time t_6 can be explicitly computed with the formula (4.15), by applying the Divergence Theorem to the vector field $(u_1, f(u_1))$ in the triangular domain with vertexes $(t_6, 0)$, $(0, -f'(a_1)t_6)$ and $(0, 0)$.
We claim that actually $t_3 < t_6 < T$, where t_3 is defined in (4.12) and t_6 is defined in (4.15).
1.4 In the interval (t_6, T) the flow $f(u_1(t,0))$ is constantly equal to $f(\widetilde{u}(t,0)) = f(a_1)$.

On the other side, the features of the solution u_2 would be the following:

2.1 Up to time t_1 its flux trace is constantly $f(u_2(t,0)) = f(\widetilde{u}(t,0)) = f(a_2)$.
2.2 At time t_1 a shock ξ, connecting the states $\overline{a}_2, a_2$ and entering $x > 0$, is originated.
We claim that this shock remains for all times in $x > 0$, never crossing the axis $x = 0$.
2.3 In the interval (t_1, t_4), it is constantly $f(u_2(t,0)) = f(\overline{a}_2) = f(a_2)$.
2.4 In the interval (t_4, t_6) the flow $f(u_2(t,0))$ is constantly equal to $f(a_4)$, where t_6 is defined in (4.15).
2.5 In the interval (t_6, T) the flow $f(u_2(t,0))$ is constantly equal to $f(\widetilde{u}(t,0)) = f(a_1)$.

Such solutions u_1, u_2 are represented in Figure 7.

Observe that for these solutions we have that $\gamma(t) = f(u_1(t,0)) = f(u_2(t,0))$, and

$$\max_{t \in [0,T]} \gamma(t) = f(a_4) < f(a_3) = \max_{t \in [0,T]} \widetilde{\gamma}(t).$$

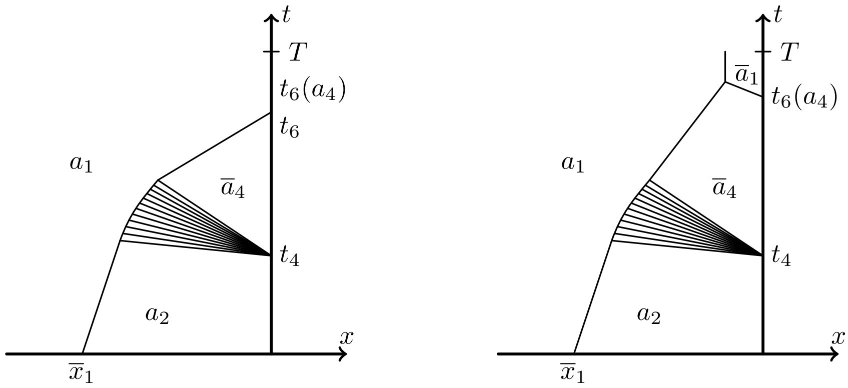

FIGURE 6. Two possible configurations for the solution u_1 of Example 4.2, constructed with boundary control $f(k_1(t))$, where the boundary datum k_1 is given in (4.20). At left the case $t_6(a_4) \geq t_6$. At right the case $t_6(a_4) < t_6$. Here $t_6(a_4)$ is defined in (4.19), while t_6 is defined in (4.15).

Moreover, $u_1(t,0), u_2(t,0)$ coincide with $\widetilde{u}(t,0)$ on $(0,t_1)\cup(t_6,T)$ and $u_1(T,x)=\widetilde{u}(T,x)$ for a.e. $x<0$. Therefore, due to Lemma 3.1, $\gamma(t)$ is a solution of the maximization problem (2.10). Moreover, $\mathcal{F}_{[0,T]}(\gamma) < \mathcal{F}_{[0,T]}(\widetilde{\gamma})$. More explicitly

$$\mathcal{F}_{[0,T]}(\gamma) = |f(a_2)-f(a_4)| + |f(a_4)-f(a_1)| = 2f(a_4) - f(a_1) - f(a_2),$$

and hence recalling (4.13)

$$\mathcal{F}_{[0,T]}(\gamma) - \mathcal{F}_{[0,T]}(\widetilde{\gamma}) = 2f(a_4) - 2f(a_3) < 0.$$

Therefore, to conclude the proof it is sufficient to show that we can actually choose t_4, a_4, and $\overline{a}_4$ so that properties 1.1–1.4, 2.1–2.5 above are verified and in particular

$$t_3 < t_6 < T \tag{4.16}$$

the shock ξ originated at t_1 and entering $x>0$ remains for all times in $x>0$. (4.17)

Recalling that $t \mapsto f(\widetilde{u}(t,0))$ is continuously increasing for $t\in(t_1,t_2)$ between $f(a_2)$ and $f(a_3)$ and that $f(\widetilde{u}(t,0)) \equiv f(a_3)$ for $t\in(t_2,t_3)$, we have that there exists a unique $t_4\in(t_1,t_2)$ such that

$$\int_{t_1}^{t_3} f(\widetilde{u}(t,0))\mathrm{d}t = (t_4-t_1)f(a_2) + (t_3-t_4)f(a_3). \tag{4.18}$$

Note that for $a\in(\overline{a}_2,\overline{a}_3)$ it holds that

$$\int_{t_1}^{t_3} f(\widetilde{u}(t,0))\mathrm{d}t = (t_4-t_1)f(a_2) + (t_3-t_4)f(a_3) > (t_4-t_1)f(a_2) + (t_3-t_4)f(a).$$

For every $a\in(\overline{a}_2,\overline{a}_3)$ let us define $t_6(a) > t_3$ such that

$$\int_{t_1}^{t_6(a)} f(\widetilde{u}(t,0))\mathrm{d}t = (t_4-t_1)f(a_2) + (t_6(a)-t_4)f(a) \tag{4.19}$$

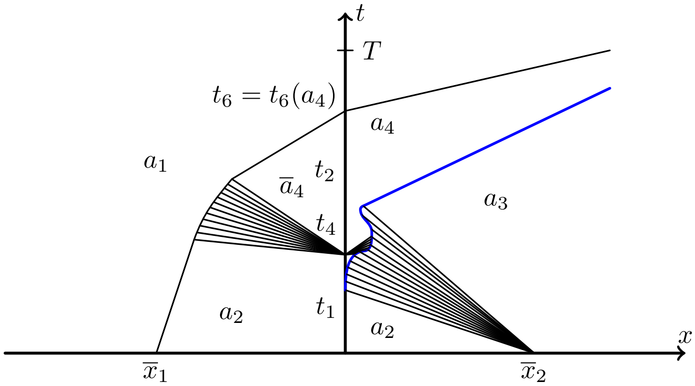

FIGURE 7. The solution u of Example 4.2, constructed with boundary control $f(k_1(t)) = f(k_2(t))$, where the boundary data k_1 and k_2 are given in (4.20).

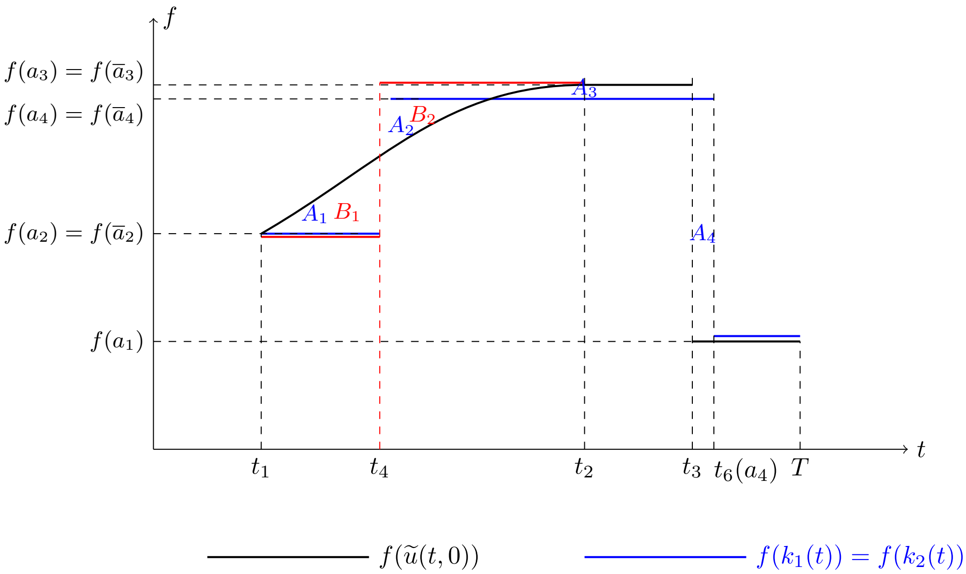

FIGURE 8. The construction related to the conditions (4.18) and (4.19). The time t_4 is such that the area B_1 is equal to the area B_2. The value $f(a_4)$ and $t_6(a_4)$ are such that the area of $A_1 + A_3$ is equal to the area of $A_2 + A_4$.

see Figure 8. We observe that

$$\lim_{a \to \overline{a}_3} t_6(a) = t_3,$$

and so we choose $a = a_4$ so to have $t_6(a_4) \in (t_3, T)$. Then define

$$k_1(t) = \begin{cases} a_2, & t \in (0, t_4), \\ \overline{a}_4, & t \in (t_4, t_6(a_4)), \\ \overline{a}_1, & t \in (t_6(a_4), T), \end{cases} \qquad k_2(t) = \begin{cases} a_2, & t \in (0, t_1), \\ \overline{a}_2, & t \in (t_1, t_4), \\ a_4, & t \in (t_4, t_6(a_4)), \\ a_1, & t \in (t_6(a_4), T). \end{cases} \tag{4.20}$$

Note that the solution u_1 has two possible configurations, according to the fact that $t_6(a_4) \geq t_6$ or $t_6(a_4) < t_6$, where $t_6(a_4)$ is defined in (4.19) while t_6 in (4.15); see Figure 6. By construction, we have

$$\int_0^t f(\widetilde{u}(t,0))\mathrm{d}t = \int_0^t f(k_2(t))\mathrm{d}t = \int_0^t f(k_1(t))\mathrm{d}t \qquad \forall\, t \in (0, t_1]. \tag{4.21}$$

and

$$\int_0^t f(\widetilde{u}(t,0))\mathrm{d}t > \int_0^t f(k_2(t))\mathrm{d}t = \int_0^t f(k_1(t))\mathrm{d}t \qquad \forall\, t \in (t_1, t_6(a_4)). \tag{4.22}$$

We prove that $t_6(a_4) = t_6$ and (4.16). Since $t_6(a_4) > t_3$ and using (4.19) and the Divergence Theorem applied to the vector field $(\widetilde{u}, f(\widetilde{u}))$ on $(0, t_6(a_4)) \times (\overline{x}_1, 0)$, we obtain

$$t_4 f(a_2) + (t_6(a_4) - t_4) f(a_4) = \int_0^{t_6(a_4)} f(\widetilde{u}(t,0))\mathrm{d}t = t_6(a_4) f(a_1) + \overline{x}_1(a_1 - a_2)$$

we directly obtain that $t_6(a_4) = t_6$, where t_6 is defined in (4.15), and then (4.16). Hence, the structure of u_1 is that given in Figure 6, left.

We prove (4.17). Assume by contradiction that the shock generated at t_1 and entering $x > 0$ hits again $x = 0$ at a time $t^* > t_1$.

If $t^* \leq t_3$, then $u_2(t^*, x) = \widetilde{u}(t^*, x)$ for a.e. $x > 0$. By Lemma 3.1, we get that

$$\int_0^{t^*} f(\widetilde{u}(t,0))\mathrm{d}t = \int_0^{t^*} f(u_2(t,0))\mathrm{d}t.$$

This condition, together with (4.19), (4.21) and the fact that $f(u_2(t,0)) = f(k_2(t))$ for $t \in (0, t^*)$ implies a contradiction.

If $t^* > t_3$, then necessarily $u_2(t^*, 0) = a_3$. Moreover, either $\lim_{t\to(t^*)^-} f(u_2(t,0)) = f(a_4) < f(a_3)$ or $\lim_{t\to(t^*)^-} f(u_2(t,0)) = f(a_1) < f(a_3)$. This condition implies that the slope of the shock arriving in t^* is positive, giving a contradiction.

We conclude with the following observation: the function $a \to t_6(a)$ is decreasing (see Fig. 8) and so in particular there exists a value $\overline{a}_4 \in (\overline{a}_2, \overline{a}_3)$ such that $t_6(\overline{a}_4) = T$. Note that $f(\overline{a}_4)$ is the minimum value among all possible values $f(a_4)$ associated with the previous construction. The associated inflow admissible control $\overline{\gamma}$ satisfies the same properties as before, in particular it is a maximizer of (2.10), and we have

$$\mathcal{F}_{[0,T]}(\overline{\gamma}) = |f(a_2) - f(\overline{a}_4)| + |f(\overline{a}_4) - f(a_1)| = 2f(\overline{a}_4) - f(a_1) - f(a_2).$$

Note that $\mathcal{F}_{[0,T]}(\overline{\gamma})$ is the minimum value achieved by $\mathcal{F}_{[0,T]}(\gamma)$ among all inflow admissible controls γ associated to non-entropic solutions constructed with boundary data of the type in (4.20). $\square$

5. The entropy solution solves the min–max problem (2.14) in the monotone case

In this Section we provide the main result of the paper, showing that if the initial datum $\overline{u}$ is monotone, then the entropy weak solution to (1.10) is a solution to the max-min problem (2.14). In particular, no procedures as the ones used in previous Section 4 can be implemented.

We start with two preliminary lemmata based on the Divergence Theorem.

Lemma 5.1. *1. Assume $\widetilde{u}$ satisfies*

$$\widetilde{u}(T,x) \leq \theta \qquad \text{for a.e. } x \in (-\infty, 0).$$

Let u be a solution to the maximization problem (2.10), associated to an admissible boundary inflow control γ. Then $u(T,0^-;\gamma) = \widetilde{u}(T,0^-)$.

2. Assume $\widetilde{u}$ satisfies

$$\widetilde{u}(T,x) \geq \theta, \qquad \text{for a.e. } x \in (0, +\infty).$$

Let u be a solution to the maximization problem (2.10), associated to an admissible boundary inflow control γ. Then $u(T,0^+;\gamma) = \widetilde{u}(T,0^+)$.

Proof. We prove only case 1. since case 2. is completely analogous.

Let $u(t,x;\gamma)$ be a solution to the maximization problem (2.10) associated with γ. Since γ is fixed, from now on we drop the dependence on γ, and we denote the solution as $u(t,x)$.

Let L be such that $L > \max\{\|f'(u)\|_{\mathbf{L}^\infty([0,T]\times(-\infty,0))}, \|f'(\widetilde{u})\|_{\mathbf{L}^\infty([0,T]\times(-\infty,0))}\}$. Then arguing exactly as in the proof of Lemma 3.1 we have that $f(u(t,-LT^+)) = f(\widetilde{u}(t,-LT^+))$ for a.e. $t \in [0,T]$.

Now we claim that $u(T,x) = \widetilde{u}(T,x)$ for $x \in (-LT, 0)$. Assume by contradiction that there exists a continuity point $\overline{x} \in (-LT,0)$ for $u, \widetilde{u}$ such that $u(T,\overline{x}) \neq \widetilde{u}(T,\overline{x})$. Since the initial datum for u and $\widetilde{u}$ is the same, there exists $\overline{t} \in (0,T)$ such that the points $u(T,\overline{x})$ and $u(\overline{t},0)$ share the same generalized characteristic line. Since the generalized characteristic lines cannot cross, for every $t \in [\overline{t},T]$, the characteristic passing through $(t,0)$ has negative speed. Therefore for every $t \in [\overline{t},T]$, we have $u(t,0) > \theta$, and then also $u(T,x) > \theta \geq \widetilde{u}(T,x)$ for every $x \in [\overline{x},0]$.

On the basis of the previous argument, we define

$$\hat{x} \doteq \inf\{x \in [-LT,0] \;:\; u(T,x) \neq \widetilde{u}(T,x)\} = \inf\{x \in [-LT,0] \;:\; u(T,x) > \widetilde{u}(T,x)\}.$$

Therefore, by the previous argument we get that $u(T,x) > \theta \geq \widetilde{u}(T,x)$ for every $x \in (\hat{x},0]$. Using the Divergence Theorem, we get

$$\begin{aligned}
\int_0^T f(u(t,0))\mathrm{d}t &= \int_0^T f(u(t,-LT))\mathrm{d}t + \int_{-LT}^0 u(0,x)\mathrm{d}x - \int_{-LT}^0 u(T,x)\mathrm{d}x \\
&= \int_0^T f(\widetilde{u}(t,-LT))\mathrm{d}t + \int_{-LT}^0 \overline{u}_1(x)\mathrm{d}x - \int_{-LT}^0 u(T,x)\mathrm{d}x \\
&< \int_0^T f(\widetilde{u}(t,-LT))\mathrm{d}t + \int_{-LT}^0 \overline{u}_1(x)\mathrm{d}x - \int_{-LT}^0 \widetilde{u}(T,x)\mathrm{d}x \\
&= \int_0^T f(\widetilde{u}(t,0))\mathrm{d}t,
\end{aligned}$$

that contradicts the fact that $\widetilde{u}$ is a solution to (2.10). Therefore, we get that $u(T,x) = \widetilde{u}(T,x)$ for a.e. $x \in (-\infty,0)$. ☐

Lemma 5.2. *Assume $\widetilde{u}(T,\cdot)$ is monotone non-increasing. Let u be a solution to the maximization problem (2.10), associated to an admissible boundary inflow control γ. Then $\widetilde{u}(T,\cdot)$ is continuous. Moreover,*

1. *if $\widetilde{u}(T,0) \geq \theta$ we have $u(T,0^+;\gamma) = \widetilde{u}(T,0)$;*
2. *if $\widetilde{u}(T,0) \leq \theta$ we have $u(T,0^-;\gamma) = \widetilde{u}(T,0)$.*

Proof. Since $\widetilde{u}$ is the entropy admissible solution and $\widetilde{u}(T,\cdot)$ is non-increasing, then Oleinik type estimates imply that $\widetilde{u}(T,\cdot)$ is continuous. Moreover, also the initial datum $\overline{u}$ is monotone non-increasing. Indeed, due to the fact that $\widetilde{u}(T,\cdot)$ is non-increasing, for every $x \in \mathbb{R}$, there hold $\overline{u}(x^+) = \widetilde{u}(T,\xi^+(x)) \leq \widetilde{u}(T,\xi^-(x)) = \overline{u}(x^-)$, where $\xi^+(x), \xi^-(x)$ are respectively the intersection with $t = T$ of the maximal and minimal characteristic issuing from x.

We prove only case 1. since case 2. is completely analogous. Therefore, we assume that $\widetilde{u}(T,0) \geq \theta$. Define $\bar{x} \in [0,+\infty]$ as $\bar{x} = \inf\{x \geq 0 : \widetilde{u}(T,x) = \theta\}$ and denote with $u(t,x;\gamma)$ a solution to the maximization problem (2.10) associated with γ.

We start with the case $\widetilde{u}(T,0) = \theta$, so that $\bar{x} = 0$. In this case $f(\widetilde{u}(t,0)) = f(\theta)$ for every $t \in (0,T)$. So u necessarily satisfies $\int_0^T f(u(t,0^+;\gamma))\mathrm{d}t = f(\theta)T$, since it is a solution to (2.10). This immediately implies that $u(T,0^+;\gamma) = \widetilde{u}(T,0) = \theta$.
Indeed, if $u(T,0^+;\gamma) < \theta$, then there exists $\delta > 0$ such that $u(T,x;\gamma) < \theta$ for $x \in (0,\delta)$, which implies that $\tau < T$, where $\tau = \inf\{t \in [0,T] : u(s,0^+) < \theta \text{ for a.e. } s \in [t,T]\}$. This contradicts the fact that $f(u(t,0^+)) = f(\theta)$ for a.e. $t \in [0,T]$.
On the other hand it is also not possible that $u(T,0^+;\gamma) > \theta$, since this would contradict the fact that $u, \widetilde{u}$ share the same initial datum and $\widetilde{u}(T,0) = \theta$.

Assume now $\widetilde{u}(T,0) > \theta$, so that $\bar{x} > 0$, and suppose by contradiction that $u(T,0^+;\gamma) \neq \widetilde{u}(T,0)$. In this case there exists $\delta > 0$ such that $u(T,x;\gamma) \leq \theta$ for every $x \in (0,\delta)$. Indeed, assume by contradiction that $u(T,0^+;\gamma) > \theta$, $u(T,0^+;\gamma) \neq \widetilde{u}(T,0^+) > \theta$. Let $\alpha \doteq \min\{f'(u(T,0^+;\gamma)), f'(\widetilde{u}(T,0^+))\}$, then the line $x = (t-T)\alpha$ is a genuine characteristic for both solutions u and $\widetilde{u}$. Since $u, \widetilde{u}$ share the same initial data, then they have to coincide on $x > (t-T)\alpha$, and this in particular would imply $u(T,0^+) = u(T,0^+;\gamma)$ in contradiction with the assumption $u(T,0^+;\gamma) > \theta$.

Define $\tau \doteq \inf\{t \in [0,T] : u(t,0^+;\gamma) \leq \theta\}$. Fix

$$L > \max\left\{\|f'(u)\|_{\mathbf{L}^\infty((0,T)\times(0,+\infty))}, \|f'(\widetilde{u})\|_{\mathbf{L}^\infty((0,T)\times(0,+\infty))}\right\}.$$

Let ξ be the forward characteristic issuing from $(\tau,0)$ and define $\hat{x} = \xi(T)$, so that $\hat{x} \in (0,LT]$. Then necessarily $u(T,x;\gamma) = \widetilde{u}(T,x)$ for $x \in (\hat{x},LT)$ and $u(T,x;\gamma) \leq \theta$ for $x \in (0,\hat{x})$. We claim that

$$\int_0^{\hat{x}} u(T,x;\gamma)\mathrm{d}x < \int_0^{\hat{x}} \widetilde{u}(T,x)\mathrm{d}x. \tag{5.1}$$

In the case $\hat{x} \leq \bar{x}$, we have $u(T,x) \leq \theta < \widetilde{u}(T,x)$ for a.e. $x \in (0,\hat{x})$ and so (5.1) holds.
In the case $\hat{x} > \bar{x}$ we have that $u(T,x;\gamma) \leq \theta$ and $\widetilde{u}(T,x) \leq \theta$ for a.e. $x \in (\bar{x},\hat{x})$. Moreover, if $x \in (\bar{x},\hat{x})$, then $f'(u(T,x^+;\gamma)) \geq \frac{x}{T}$ and $f'(\widetilde{u}(T,x)) \leq \frac{x}{T}$; hence $u(T,x^+;\gamma) \leq \widetilde{u}(T,x)$ and so (5.1) holds.

Reasoning as in Lemma 5.1, using of the Divergence Theorem and (5.1), we deduce that

$$\int_0^T f(\widetilde{u}(t,0^+))\mathrm{d}t > \int_0^T f(u(t,0;\gamma))\mathrm{d}t,$$

which contradicts the fact that $u(\cdot,\cdot;\gamma)$ is a solution to the maximization problem (2.10). □

5.1. Monotone non-decreasing initial datum

In this section we consider the case in which the initial datum may only generate shock waves or compression waves.

We start computing in the next two lemmata the functional $\mathcal{F}_{[0,T]}$ defined in (2.13) for the entropy solution.

Lemma 5.3. *Assume that $\overline{u}$ is monotone non-decreasing and let $\widetilde{u}$ be the unique entropy solution to (1.10). Then the following holds:*

1. *for every $\hat{x} \in \mathbb{R}$, the map $t \mapsto f(\widetilde{u}(t,\hat{x}))$ is non-increasing,*
2. $f(\widetilde{u}(T^-,0)) \geq f(\widetilde{u}(T,0^+)), f(\widetilde{u}(T,0^-))$.

In particular, denoting $\widetilde{\gamma} = f(\widetilde{u}(\cdot,0))$, we have that

$$\begin{aligned}\mathcal{F}_{[0,T]}(\widetilde{\gamma}) =& \mathrm{TV}_{(0,T)}(f(\widetilde{u}(\cdot,0))) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)| \\ &+ 2\widetilde{\gamma}(T^-) - f(\widetilde{u}(T,0^+)) - f(\widetilde{u}(T,0^-)) \\ =& \widetilde{\gamma}(0^+) - \widetilde{\gamma}(T^-) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)| \\ &+ 2\widetilde{\gamma}(T^-) - f(\widetilde{u}(T,0^+)) - f(\widetilde{u}(T,0^-)).\end{aligned}$$

Proof. We start proving that $t \to f(\widetilde{u}(t,\hat{x}))$ is non-increasing. Without loss of generality we assume that $\hat{x} = 0$.

By contradiction assume that there exist $0 < t_1 < t_2$ such that $f(\widetilde{u}(t_1,0)) < f(\widetilde{u}(t_2,0))$. We may assume without loss of generality that they are continuity points. Note that the Lax condition implies that the map $t \mapsto f(\widetilde{u}(t,0))$ can have only downward jumps. Hence, we may assume that there exists a continuity point $\bar{t} \in (t_1,t_2)$ such that

$$f(\widetilde{u}(t_1,0)) < f(\widetilde{u}(\bar{t},0)) < f(\widetilde{u}(t_2,0)). \tag{5.2}$$

Denote with ξ_1, ξ_2 the backwards generalized characteristics exiting from $(t_1,0)$ and $(t_2,0)$. We have some possibilities.

(i) $\xi_1(0) < \xi_2(0)$. In this case we necessarily deduce that $\xi_2(0) > 0$, otherwise ξ_1 and ξ_2 cross each other. Thus, $\widetilde{u}(t_2,0) > \theta$. The monotonicity of $\overline{u}$ implies that $\widetilde{u}(t_1,0) = \overline{u}(\xi_1(0)) \leq \overline{u}(\xi_2(0)) = \widetilde{u}(t_2,0)$. Since $f(\widetilde{u}(t_1,0)) < f(\widetilde{u}(t_2,0))$ we deduce that $\widetilde{u}(t_1,0) < \theta$ and $\xi_1(0) < 0$.

Denote with $\bar{\xi}$ the backward generalized characteristic exiting from $(\bar{t},0)$.
If $\bar{\xi}(0) \leq 0$, then $\bar{\xi}(0) < \xi_1(0)$ (otherwise $\bar{\xi}$ crosses ξ_1) and so $\widetilde{u}(\bar{t},0) \leq \widetilde{u}(t_1,0) < \theta$, violating (5.2).
If $\bar{\xi}(0) > 0$, then $\bar{\xi}(0) \leq \xi_2(0)$ (otherwise $\bar{\xi}$ crosses ξ_2) and so $\widetilde{u}(t_1,0) < \theta < \widetilde{u}(\bar{t},0) \leq \widetilde{u}(t_2,0)$, violating (5.2).

Therefore, this case is not possible.

(ii) $\xi_1(0) = \xi_2(0)$. So in $\xi_1(0)$ the initial data $\overline{u}$ would have a downward jump, which is not possible by the assumption on monotonicity.

(iii) $\xi_1(0) > \xi_2(0)$. In this case we necessarily deduce that $\xi_2(0) < 0$, otherwise ξ_1 and ξ_2 cross each other. Thus, $\widetilde{u}(t_2,0) < \theta$. The monotonicity of $\overline{u}$ implies that $\widetilde{u}(t_1,0) = \overline{u}(\xi_1(0)) \geq \overline{u}(\xi_2(0)) = \widetilde{u}(t_2,0)$. Since $f(\widetilde{u}(t_1,0)) < f(\widetilde{u}(t_2,0))$ we deduce that $\widetilde{u}(t_1,0) > \theta$ and $\xi_1(0) > 0$.

Consider now $\bar{t} \in (t_1,t_2)$ such that (5.2) holds and $(\bar{t},0)$ point of continuity for $\widetilde{u}$. Denote with $\bar{\xi}$ the backward generalized characteristic exiting from $(\bar{t},0)$.
If $\bar{\xi}(0) \geq 0$, then $\bar{\xi}(0) > \xi_1(0)$ (otherwise $\bar{\xi}$ crosses ξ_1) and so $\widetilde{u}(\bar{t},0) \geq \widetilde{u}(t_1,0)$, violating (5.2).
If $\bar{\xi}(0) < 0$, then $\bar{\xi}(0) \geq \xi_2(0)$ (otherwise $\bar{\xi}$ crosses ξ_2) and so $\widetilde{u}(t_2,0) \leq \widetilde{u}(\bar{t},0) < \theta < \widetilde{u}(t_1,0)$, violating (5.2).

Therefore, this case is not possible.

So, the proof of 1. is concluded.

Now we show that $f(\widetilde{u}(T^-,0)) \geq f(\widetilde{u}(T,0^+)), f(\widetilde{u}(T,0^-))$. We show that $f(\widetilde{u}(T^-,0)) \geq f(\widetilde{u}(T,0^-))$ (the case $f(\widetilde{u}(T^-,0)) \geq f(\widetilde{u}(T,0^+))$ is completely analogous).

Let us consider an increasing sequence $x_n \to 0^-$ such that (T,x_n) are continuity points for $\widetilde{u}$ and moreover $f(\widetilde{u}(T,x_n)) \to f(\widetilde{u}(T,0^-))$. We denote with ξ_n the backward genuine characteristic passing through (T,x_n).

We distinguish three cases.

(i) There exists $\bar{t} < T$ such that all points $(t,0)$ for $t \in (\bar{t},T)$ are discontinuity points for $\widetilde{u}$. In this case there exists a shock located in $x = 0$ in the interval $(\bar{t},T)$, so $\widetilde{u}(t,0^-) \leq \theta$ and $\widetilde{u}(t,0^+) \geq \theta$ for all $t \in (\bar{t},T)$. We consider a sequence $t_n \to T^-$ and the backward minimal characteristics $\widetilde{\xi}_n$ from $(t_n, 0)$. Then necessarily $\xi_n(0) \le \widetilde{\xi}_n(0) \le \bar{x}$, where $\bar{x}$ is defined in (5.3). Then, we conclude by the monotonicity of $\overline{u}$, we get that $\widetilde{u}(T, x_n) \le \widetilde{u}(t_n, 0^-) \le \theta$, from which we deduce $f(\widetilde{u}(T, 0^-)) \le f(\widetilde{u}(T^-, 0))$.

(ii) There exists $\bar{t} < T$ such that there exists a dense subset of $(\bar{t}, T)x\{0\}$ of continuity points for $\widetilde{u}$.

Assume that $\widetilde{u}(t, 0) \le \theta$ for $t \to T^-$ and $(t, 0)$ continuity point. Let us denote $\widetilde{\xi}$ is the backward genuine characteristic passing through $(t, 0)$.

In this case, $\widetilde{u}(T, x_n) \le \theta$, and we proceed exactly as in case 1.

(iii) There exists $\bar{t} < T$ such that there exists a dense subset of $(\bar{t}, T) \times \{0\}$ of continuity points for $\widetilde{u}$.

Assume that $\widetilde{u}(t, 0) > \theta$ for $t \to T^-$ and $(t, 0)$ continuity point. Let us denote $\widetilde{\xi}$ is the backward genuine characteristic passing through $(t, 0)$, for $t \in (\bar{t}, T)$, such that $(t, 0)$ continuity point.

If $\widetilde{u}(T, x_n) > \theta$, then we proceed as in case 1, by symmetry.

It remains to consider the case $\widetilde{u}(T, x_n) \le \theta$. Assume that there exists $\tau \in (t, T)$ such that $\widetilde{\xi}(\tau) = 0$. Then this would imply $f'(\widetilde{(}\tau, 0^-)) < \widetilde{\xi}'(\tau)$, in contradiction with Lax conditions. Moreover, due to the fact that $\widetilde{u}(T, 0^-) \le \theta$, we conclude that necessarily $\widetilde{\xi}(T) = 0$ with $\widetilde{\xi}'(T) \ge 0$. This means that $\widetilde{u}$ admits a shock discontinuity at $(T, 0)$, connecting the left state $\widetilde{u}(T, 0^-)$ with the right state $\widetilde{u}(T^-, 0)$, that has slope $\widetilde{\xi}'(T) \ge 0$, which implies that $f(\widetilde{u}(T^-, 0)) \ge f(\widetilde{u}(T, 0^-))$.

□

Note that since $\overline{u}$ is non-decreasing, there exists $\bar{x} \in [-\infty, +\infty]$ such that

$$\bar{x} \doteq \sup\{x \in \mathbb{R} : \overline{u}(x) \le \theta\}. \tag{5.3}$$

Lemma 5.4. *Assume that $\overline{u}$ is monotone non-decreasing and let $\bar{x} \in [-\infty, +\infty]$ as in (5.3).*

Let $\widetilde{u}$ be the unique entropy solution to (1.10) and $\widetilde{\gamma} = f(\widetilde{u}(\cdot, 0))$. We denote $\bar{\xi}$ the unique forward generalized characteristic issuing from $\bar{x}$ at time 0.

We get

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = \begin{cases} \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^-)), & \text{if } \bar{x} \in [0, +\infty] \text{ and } \bar{\xi}(T) > 0, \\ \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^+)), & \text{if } \bar{x} \in [0, +\infty) \text{ and } \bar{\xi}(T) < 0, \\ \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^+)), & \text{if } \bar{x} \in [-\infty, 0] \text{ and } \bar{\xi}(T) < 0, \\ \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^-)), & \text{if } \bar{x} \in (-\infty, 0] \text{ and } \bar{\xi}(T) > 0, \end{cases} \tag{5.4}$$

and finally if $\bar{\xi}(T) = 0$,

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) \le f(\overline{u}(0^+)) - f(\widetilde{u}(T, 0^-)) + f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^+)). \tag{5.5}$$

Proof. Throughout the proof we rely on the properties of generalized characteristics, see [36]. Note that $f(\widetilde{u}(t, 0^-)) = f(\widetilde{u}(t, 0^+))$ for a.e. t and $f(\widetilde{u}(\cdot, 0^-)), f(\widetilde{u}(\cdot, 0^+)) \in \mathbf{BV}(0, T)$.

By the definition (2.13), and Lemma 5.3 we have that

$$\begin{aligned} \mathcal{F}_{[0,T]}(\widetilde{\gamma}) &= \widetilde{\gamma}(0^+) - \widetilde{\gamma}(T^-) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)| \\ &\quad + 2\widetilde{\gamma}(T^-) - f(\widetilde{u}(T, 0^+)) - f(\widetilde{u}(T, 0^-)) \\ &= \widetilde{\gamma}(0^+) + \widetilde{\gamma}(T^-) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)| \\ &\quad - f(\widetilde{u}(T, 0^+)) - f(\widetilde{u}(T, 0^-)). \end{aligned}$$

We consider different cases depending on the value of $\bar{x}$.

1. **Either** $\bar{x} = +\infty$**, or** $\bar{x} \geq 0$ **and** $\bar{\xi}(T) > 0$**.**
   Observe that $\widetilde{u}(T, x) \leq \theta$, for a.e. $x \leq 0$.
   We start considering $\bar{x} > 0$. Since $\overline{u}$ is monotone non-decreasing and $\overline{u}(x) \leq \theta$ for a.e. $x \leq \bar{x}$, we have that $f(\overline{u}(0^-)) = \widetilde{\gamma}(0^+)$ and $\widetilde{\gamma}(0^+) = f(\overline{u}(0^-)) \leq f(\overline{u}(0^+))$. Note also that $f(\widetilde{u}(T, 0^-)) \leq \widetilde{\gamma}(T^-) = f(\widetilde{u}(T, 0^+))$. Hence, we deduce that

$$\begin{aligned}\mathcal{F}_{[0,T]}(\widetilde{\gamma}) &= \widetilde{\gamma}(0^+) - \widetilde{\gamma}(T^-) + \widetilde{\gamma}(T^-) - f(\widetilde{u}(T, 0^-)) + f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)\\ &= f(\overline{u}(0^+)) - f(\widetilde{u}(T, 0^-)),\end{aligned}$$

   proving (5.4).
   For $\bar{x} = 0$, we observe that $\widetilde{\gamma}(0^+) = \min(f(\overline{u}(0^-), f(\overline{u}(0^+))$. Then we may proceed as before, getting

$$\begin{aligned}\mathcal{F}_{[0,T]}(\widetilde{\gamma}) &= \widetilde{\gamma}(0^+) - \widetilde{\gamma}(T^-) + \widetilde{\gamma}(T^-) - f(\widetilde{u}(T, 0^-)) + \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)\\ &= \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T, 0^-)).\end{aligned}$$

2. $\bar{x} \geq 0$ **and** $\bar{\xi}(T) < 0$**.**
   Note that in this case $\widetilde{u}(T, x) \geq \theta$ for a.e. $x \geq 0$.
   We get as in case 1 we have that $\widetilde{\gamma}(0^+) = \min(f(\overline{u}(0^-)), f(\overline{u}(0^+))$ whereas $f(\widetilde{u}(T, 0^+)) \leq \widetilde{\gamma}(T^-) = f(\widetilde{u}(T, 0^-))$. In conclusion

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = \max(f(\overline{u}(0^-)), f(\overline{u}(0^+)) - f(\widetilde{u}(T, 0^+)).$$

3. **Either** $\bar{x} = -\infty$ **or** $\bar{x} \leq 0$ **and** $\bar{\xi}(T) < 0$**.**
   In this case we observe that $u(T, x) \geq \theta$ for a.e. $x \geq 0$. We proceed as in item 1 by symmetry.
4. $\bar{x} \leq 0$ **and** $\bar{\xi}(T) > 0$**.**
   Note that in this case $\widetilde{u}(T, x) \leq \theta$ for a.e. $x \leq 0$. We proceed as in item 2 by symmetry.
5. $\bar{\xi}(T) = 0$**.**
   Note that in this case $\widetilde{u}(T, x) \geq \theta$ for a.e. $x \geq 0$ and $\widetilde{u}(T, x) \leq \theta$ for a.e. $x \leq 0$.
   Arguing as above, we have that $\widetilde{\gamma}(0^+) = \min(f(\overline{u}(0^-)), f(\overline{u}(0^+)))$.
   So we get

$$\begin{aligned}\mathcal{F}_{[0,T]}(\widetilde{\gamma}) &= \widetilde{\gamma}(0^+) - \widetilde{\gamma}(T^-) + 2\widetilde{\gamma}(T^-) - f(\widetilde{u}(T, 0^-)) - f(\widetilde{u}(T, 0^+))\\ &\quad + \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+))\\ &= \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) + \widetilde{\gamma}(T^-)) - f(\widetilde{u}(T, 0^-)) - f(\widetilde{u}(T, 0^+))\\ &\leq \max(f(\overline{u}(0^+), f(\overline{u}(0^-)) + \widetilde{\gamma}(0^+) - f(\widetilde{u}(T, 0^-)) - f(\widetilde{u}(T, 0^+))\\ &= f(\overline{u}(0^+) + f(\overline{u}(0^-) - f(\widetilde{u}(T, 0^-)) - f(\widetilde{u}(T, 0^+)).\end{aligned}$$

□

Based on the previous lemma we show that in the case the initial datum generates only shock waves, the entropy solution solves the minimization problem (2.14).

Theorem 5.5. *Assume that $\overline{u}$ is monotone non-decreasing. Let $\widetilde{u}$ be the unique entropy solution to* (1.10)*. Then $\widetilde{u}$ is a solution to* (2.10) *and also of* (2.14) *for all $M \geq M_0 \doteq \mathcal{F}_{[0,T]}\{f(\widetilde{u}(\cdot, 0))\}$. In particular*

$$\min_{\gamma \in \mathcal{U}^M_{max}} \mathcal{F}_{[0,T]}(\gamma) = \mathcal{F}_{[0,T]}(\widetilde{\gamma}).$$

Proof. For every $\gamma \in \mathcal{U}^M_{max}$, we consider the associated function $u(t, x; \gamma)$, see Definition 2.3.

We fix $\gamma \in \mathcal{U}^M_{max}$, and we drop the dependence on γ on the function u. Recalling (2.13), we get that

$$\begin{cases} \mathcal{F}_{[0,T]}(\gamma) \geq |f(\overline{u}(0^+)) - f(u(T,0^-))| + |f(\overline{u}(0^-)) - f(u(T,0^+))|, \\ \mathcal{F}_{[0,T]}(\gamma) \geq |f(\overline{u}(0^+)) - f(u(T,0^+))| + |f(\overline{u}(0^-)) - f(u(T,0^-))|, \\ \mathcal{F}_{[0,T]}(\gamma) \geq |\max(f(\overline{u}(0^+)), f(\overline{u}(0^-))) - f(u(T,0^+))|, \\ \mathcal{F}_{[0,T]}(\gamma) \geq |\max(f(\overline{u}(0^+)), f(\overline{u}(0^-))) - f(u(T,0^-))|. \end{cases} \tag{5.6}$$

We consider $\bar{x}$ as in (5.3), and we denote $\bar{\xi}$ the unique forward generalized characteristic issuing from $\bar{x}$ at time 0. We divide the proof in several cases, using the same notation of Lemma 5.4.

1. **Either $\bar{x} = +\infty$, or $\bar{x} \geq 0$ and $\bar{\xi}(T) > 0$.**
   Since $\widetilde{u}(T,x) \leq \theta$ for a.e. $x \leq 0$, by Lemma 5.1, we get that $f(u(T,0^-)) = f(\widetilde{u}(T,0^-))$. So recalling inequality (5.6), and by (5.4) we conclude that
   $$\begin{aligned} \mathcal{F}_{[0,T]}(\gamma) &\geq |\max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(u(T,0^-))| \\ &= |\max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T,0^-))| = \mathcal{F}_{[0,T]}(\widetilde{\gamma}). \end{aligned}$$
2. **$\bar{x} \geq 0$ and $\bar{\xi}(T) < 0$.**
   Since $\widetilde{u}(T,x) \geq \theta$ for a.e. $x \geq 0$, by Lemma 5.1, we get that $f(u(T,0^+)) = f(\widetilde{u}(T,0^+))$. So recalling inequality (5.6), and by (5.4) we conclude that
   $$\begin{aligned} \mathcal{F}_{[0,T]}(\gamma) &\geq |\max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(u(T,0^+))| \\ &= |\max(f(\overline{u}(0^+), f(\overline{u}(0^-)) - f(\widetilde{u}(T,0^+))| = \mathcal{F}_{[0,T]}(\widetilde{\gamma}). \end{aligned}$$
3. **Either $\bar{x} = -\infty$ or $\bar{x} \leq 0$ and $\bar{\xi}(T) < 0$.**
   It is the symmetric case to case 1.
4. **$\bar{x} \leq 0$ and $\bar{\xi}(T) > 0$.**
   It is the symmetric case to case 2.
5. **$\xi(T) = 0$.**
   In this case $\widetilde{u}(x,T) \leq \theta$ a.e. for $x < 0$ and $\widetilde{u}(x,T) \geq \theta$ a.e. for $x > 0$. Therefore, by Lemma 5.1 we get $f(u(T,0^-)) = f(\widetilde{u}(T,0^-))$ and $f(u(T,0^+)) = f(\widetilde{u}(T,0^+))$. Recalling (5.5) and (5.6) we get
   $$\begin{aligned} \mathcal{F}_{[0,T]}(\widetilde{\gamma}) &\leq f(\overline{u}(0^+)) - f(\widetilde{u}(T,0^-)) + f(\overline{u}(0^-)) - f(\widetilde{u}(T,0^+)) \\ &\leq |f(\overline{u}(0^+)) - f(u(T,0^-))| + |f(\overline{u}(0^-)) - f(u(T,0^+))| \leq \mathcal{F}_{[0,T]}(\gamma). \end{aligned}$$

□

5.2. Monotone non-increasing initial datum

In this section we consider the case in which the initial datum may only generate rarefaction waves.

As in the previous section, we compute $\mathcal{F}_{[0,T]}$ for the entropy solution $\widetilde{u}$ in this setting. We start computing in the next two lemmata the functional $\mathcal{F}_{[0,T]}$ for the entropy solution.

Lemma 5.6. *Assume $\overline{u}$ is monotone non-increasing and let us consider $\widetilde{u}$ the unique entropy solution to* (1.10).

Then, for every $t > 0$, the function $\widetilde{u}(t,\cdot)$ is continuous and for every $\hat{x} \in \mathbb{R}$, the map $t \mapsto f(\widetilde{u}(t,\hat{x}))$ is non-decreasing.

In particular, denoting $\widetilde{\gamma} = f(\widetilde{u}(\cdot,0))$, we have that

$$\begin{aligned} \mathcal{F}_{[0,T]}(\widetilde{\gamma}) &= \mathrm{TV}_{(0,T)}(f(\widetilde{u}(\cdot,0))) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)| \\ &= \widetilde{\gamma}(T) - \widetilde{\gamma}(0) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)|. \end{aligned}$$

Proof. As before, we rely on the properties of generalized characteristics, see [36].

We show that the function $\widetilde{u}(t, \cdot)$ is continuous.

We fix $t = T$, but the argument is completely analogous for a generic t. Assume by contradiction that there exists $\widetilde{x} \in \mathbb{R}$ such that $\widetilde{u}(T, \widetilde{x}^-) \neq \widetilde{u}(T, \widetilde{x}^+)$. Denote with ξ^- and ξ^+ respectively the minimal and the maximal backward characteristics from $(T, \widetilde{x})$. Since $\xi^-(0) < \xi^+(0)$ and $\overline{u}$ non-increasing, we get $\overline{u}(\xi^-(0)^+) \geq \overline{u}(\xi^+(0)^-)$. This is not compatible with the fact that $\dot{\xi}^-(t) = f'(\widetilde{u}(T, \bar{x}^-)) \leq \dot{\xi}^+(t) = f'(\widetilde{u}(T, \bar{x}^+))$ if $\overline{u}(\xi^-(0)^+) \leq \theta$ and with the fact that $\dot{\xi}^-(t) = f'(\widetilde{u}(T, \bar{x}^-)) \geq \dot{\xi}^+(t) = f'(\widetilde{u}(T, \bar{x}^+))$ if $\overline{u}(\xi^+(0)^-) \geq \theta$ (see [37], (11.1.13) and Thm. 11.1.3).

We prove now that the map $t \to f(\widetilde{u}(t, \hat{x}))$ is non-decreasing. Without loss of generality we assume that $\hat{x} = 0$. By contradiction assume that there exist $0 < t_1 < t_2 \leq T$ such that $f(\widetilde{u}(t_1, 0)) > f(\widetilde{u}(t_2, 0))$.

Denote with ξ_1, ξ_2 the genuine backward characteristics from $(t_1, 0)$ and $(t_2, 0)$.

1. If $\xi_1(0) = \xi_2(0)$, then a rarefaction wave is generated at $\xi_1(0)$. In particular $\widetilde{u}(t_1, 0) = (f')^{-1}(-\xi_1(0)/t_1)$ and $\widetilde{u}(t_2, 0) = (f')^{-1}(-\xi_1(0)/t_2)$ and the traces $\widetilde{u}(t_1, 0), \widetilde{u}(t_2, 0)$ belong to $[\overline{u}(\xi_1(0)^+), \overline{u}(\xi_1(0)^-)]$.

   If $\xi_1(0) \leq 0$, we have that $\overline{u}(\xi_1(0)^-) \leq \theta$ and then $\widetilde{u}(t_1, 0), \widetilde{u}(t_2, 0) \leq \theta$. Using the fact that f is strictly concave, we get that $\widetilde{u}(t_1, 0^-) = (f')^{-1}(-\xi_1(0)/t_1) < (f')^{-1}(-\xi_1(0)/t_2) = \widetilde{u}(t_2, 0^-)$, and so we conclude that $f(\widetilde{u}(t_1, 0)) < f(\widetilde{u}(t_2, 0))$, in contradiction with our assumption.

   If $\xi_1(0) > 0$, we have that $\widetilde{u}(t_1, 0), \widetilde{u}(t_2, 0) \geq \theta$, and we argue exactly as before, by symmetry and obtain $f(\widetilde{u}(t_1, 0)) < f(\widetilde{u}(t_2, 0))$, in contradiction with our assumption.

   So, this case is not possible.
2. If $\xi_1(0) > \xi_2(0)$, then, necessarily $\xi_2(0) \leq 0$, and $\widetilde{u}(t_2, 0) \leq \theta$. Moreover by monotonicity of $\overline{u}$, we have that $\widetilde{u}(t_1, 0) \leq \overline{u}(\xi_1(0)^-) \leq \overline{u}(\xi_2(0)^+) \leq \widetilde{u}(t_2, 0) \leq \theta$. Therefore, we conclude that $f(\widetilde{u}(t_1, 0^-)) \leq f(\widetilde{u}(t_2, 0^-))$, in contradiction with our assumption.
3. If $\xi_1(0) < \xi_2(0)$, then, necessarily $\xi_1(0) \geq 0$, and $\widetilde{u}(t_1, 0), \widetilde{u}(t_2, 0) \geq \theta$. Recalling the monotonicity of $\overline{u}$, we get $\widetilde{u}(t_1, 0) \geq \overline{u}(\xi_1(0)^+) \geq \overline{u}(\xi_2(0)^-) \geq \widetilde{u}(t_2, 0) \geq \theta$. So we conclude that $f(\widetilde{u}(t_1, 0^-)) \leq f(\widetilde{u}(t_2, 0^-))$, in contradiction with our assumption.

So, none of the previous cases is possible, and the proof is concluded. □

Note that since $\overline{u}$ is non-increasing, there exists $\bar{x} \in [-\infty, +\infty]$ such that

$$\bar{x} \doteq \sup \{x \in \mathbb{R} : \overline{u}(x) \geq \theta\}. \tag{5.7}$$

Lemma 5.7. *Assume that $\overline{u}$ is monotone non-increasing. Let $\widetilde{u}$ be the unique entropy solution to (1.10) and let $\widetilde{\gamma} = f(\widetilde{u}(\cdot, 0^-))$. Then*

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = \begin{cases} f(\widetilde{u}(T, 0)) - \min\{f(\overline{u}(0^+)), f(\overline{u}(0^-))\}, & \text{if } \bar{x} \neq 0, \\ 2f(\theta) - f(\overline{u}(0^+)) - f(\overline{u}(0^-)), & \text{if } \bar{x} = 0, \end{cases} \tag{5.8}$$

where $\bar{x}$ has been defined in (5.7).

Proof. By Lemma 5.3 we have that

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = f(\widetilde{u}(T, 0)) - \widetilde{\gamma}(0^+) + |f(\overline{u}(0^-)) - \widetilde{\gamma}(0^+)| + |f(\overline{u}(0^+)) - \widetilde{\gamma}(0^+)|.$$

We consider different cases depending on the value of $\bar{x}$ as defined in (5.7).

1. $\bar{x} = 0$**.**

   In this case a rarefaction wave is generated at 0, and $\widetilde{u}(t, 0) = \theta$ for $t > 0$, so $f(\widetilde{u}(T, 0)) = \widetilde{\gamma}(0) = \theta$. Then

$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = \theta - \theta + \theta - f(\overline{u}(0^-)) + \theta - f(\overline{u}(0^+)),$$

   which give (5.8).

2. $\bar{x} < 0$.
   In this case $\overline{u}(0^+) \le \overline{u}(0^-) \le \theta$. Therefore, $\widetilde{\gamma}(0^+) = f(\overline{u}(0^-)) \ge f(\overline{u}(0^+))$.
   Therefore,
$$\mathcal{F}_{[0,T]}(\widetilde{\gamma}) = f(\widetilde{u}(T,0)) - \widetilde{\gamma}(0^+) + \widetilde{\gamma}(0^+) - f(\overline{u}(0^+)) = f(\widetilde{u}(T,0)) - f(\overline{u}(0^+)).$$
3. $\bar{x} > 0$.
   In this case $\theta \le \overline{u}(0^+) \le \overline{u}(0^-)$. Therefore, $\widetilde{\gamma}(0^+) = f(\overline{u}(0^+)) \ge f(\overline{u}(0^-))$ and we conclude as above.

□

Based on the previous lemmata we show that also in the case the initial datum generates only rarefaction waves, the entropy solution solves the minimization problem (2.14).

Theorem 5.8. *Assume that $\overline{u}$ is monotone non-increasing. Let $\widetilde{u}$ be the unique entropy solution to* (1.10)*. Then $\widetilde{u}$ is a solution to* (2.10) *and also of* (2.14) *for all $M \ge M_0 \doteq \mathcal{F}_{[0,T]}\{f(\widetilde{u}(\cdot,0))\}$. In particular*
$$\min_{\gamma\in\mathcal{U}^M_{max}} \mathcal{F}_{[0,T]}(\gamma) = \mathcal{F}_{[0,T]}(\widetilde{\gamma}).$$

Proof. For every $\gamma \in \mathcal{U}^M_{max}$, we consider the associated function $u(t,x;\gamma)$, see Definition 2.3.

We fix $\gamma \in \mathcal{U}^M_{max}$, and we drop the dependence on γ on the function u. Recalling the definition of $\mathcal{F}_T$ in (2.13), we have that
$$\begin{cases} \mathcal{F}_{[0,T]}(\gamma) \ge |f(u(T,0^+)) - \min(f(\overline{u}(0^+)), f(\overline{u}(0^-)))|, \\ \mathcal{F}_{[0,T]}(\gamma) \ge |f(u(T,0^-)) - \min f(\overline{u}(0^+)), f(\overline{u}(0^-)))|. \end{cases} \tag{5.9}$$

We proceed with the different cases, depending on $\bar{x}$, defined in (5.7), as in Lemma 5.7.

1. $\bar{x} = 0$.
   We observe that if $\bar{x} = 0$, then $\widetilde{u}(0,t) = \theta$ for $t \in (0,T)$. This implies that $\int_0^T f(\widetilde{u}(t,0))\mathrm{d}t = f(\theta)T = (\max f)T$, therefore every other solution u to the maximization problem (2.10) necessarily coincides with $\widetilde{u}$.
2. $\bar{x} \ne 0$.
   Note that due to the fact that $\overline{u}$ is monotone non-increasing, then only rarefaction waves can be generated, and $\widetilde{u}(T,\cdot)$ remains monotone non-increasing. We may apply then Lemma 5.2, which implies that either $f(\widetilde{u}(T,0)) = f(u(T,0^+))$ or $f(\widetilde{u}(T,0)) = f(u(T,0^-))$. We conclude then using (5.9) and (5.8) in Lemma 5.7.

□

6. Conclusions

In this paper, we considered an optimization problem for a conservation law on a 1–1 junction using the inflows as controls. In particular, we looked for solutions that maximize the flow at the junction and, at the same time, minimize the total variation of the flux at $x = 0$.

If the initial datum is monotone, then we showed, using the generalized characteristics technique, that the flux of the entropy weak solution at the node provides an optimal inflow control for this problem. We also exhibited two prototype examples showing that, in the case where the initial datum is not monotone, the flux of the entropy weak solution is no more optimal. Finally, a conjecture about optimal solutions in the case of non-monotone data is presented.

The analysis in this paper can be seen as the first step of a more general class of min–max control problem. Indeed, various generalizations can be studied: the case of different fluxes, of general junctions, or of balance laws with source terms.

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Appendix A.

Throughout the section, we denote by $u(t,0) = u(t,0^+)$ the trace of the solution of the initial-boundary value problem at $x = 0$.

Theorem A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be of class $\mathbf{C^1}$, strictly concave, and satisfying (2.1). Given $\overline{u} \in \mathbf{BV}((0,+\infty);\mathbb{R})$, and $k \in \mathbf{L}^\infty\left((0,+\infty);(-\infty,\theta]\right)$, assume that $f \circ k \in \mathbf{BV}((0,+\infty);\mathbb{R})$. Let $u \in \mathbf{L}^\infty\left((0,+\infty)^2;\mathbb{R}\right)$ be the entropy admissible weak solution of the initial-boundary value problem*

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x > 0,\, t > 0, \\ u(0,x) = \overline{u}(x), & x > 0, \\ u(t,0) = k(x), & t > 0. \end{cases} \tag{A.1}$$

Then, we have

$$\mathrm{TV}_{(0,+\infty)}\, f(u(\cdot,0)) \leq 2\,\mathrm{TV}_{(0,+\infty)}(f \circ k) + \|f'(\overline{u})\|_\infty\, \mathrm{TV}_{(0,+\infty)}(\overline{u}) + \mathrm{Osc}(f), \tag{A.2}$$

with

$$\operatorname{Osc}(f) \doteq \sup\Big\{\big|f\circ k(t) - f\circ \overline{u}(x)\big|,\quad t>0,\ x>0\Big\}. \tag{A.3}$$

Proof. We divide the proof in several steps.

Step 1. Properties of the solution u at the boundary.

Since $f\circ k \in \mathbf{BV}$ and $k(t)\le\theta$ for a.e. $t>0$, it follows that k admits at most countably many discontinuities and one-sided limits $k(t^{\pm})$ at any time $t>0$, because $f\circ k$ enjoys these properties and $(f_-)^{-1}$ is continuous (f_- defined as in Sect. 2.1). We may assume that k is right continuous *i.e.* that $k(t)=k(t^+)$ for all $t>0$.

Let $t>0$ such that $u(t,0)\le\theta$. Then the following holds:

(i) Due to the boundary condition in (A.1) (*e.g.* see [9], Sect. 2.1), it holds $u(t,0)=k(t)$ if $u(\cdot,0)$ is continuous at t, otherwise $u(\cdot,0)$ admits one-sided limits, and we have $u(t,0)=u(t^-,0)=k(t^-)$, $u(t^+,0)=k(t)$.

(ii) If $u(t,0)<\theta$, tracing backward generalized characteristics starting at points (t,x), with $x>0$ arbitrary close to 0, we deduce that

$$\exists\ \tau'<t \qquad s.t. \qquad u(s,0)<\theta \quad \forall\ s\in(\tau',t). \tag{A.4}$$

(iii) If $u(t,0)=\theta$, then, by item (i), it holds $u(t^-,0)=\theta=k(t)$. Moreover, we deduce that

$$\exists\ \tau'<t \qquad s.t. \qquad u(s,0)\le\theta \quad \forall\ s\in(\tau',t). \tag{A.5}$$

If it were not true, there would exist a sequence $\{t_n\}_n$, with $t_n\uparrow t$, such that $u(t_n,0)\downarrow\theta$ and so $f'(u(t_n,0))\uparrow 0$. Considering the maximal backward characteristics starting at the points $(t_n,0)$, this would imply that they must cross each other in the domain $\{x>0\}$ for n sufficiently large, which is not possible.

Step 2. Definition of the generalized characteristics issuing from the boundary.

We will say that a map $\xi\colon[\tau_1,\tau_2]\to\mathbb{R}$, $\tau_2\in\mathbb{R}^+\cup+\infty$, is a generalized characteristic starting at the point $(\tau_1,0)$ if:

– $\xi(\tau_1)=0$;
– either $\tau_2=+\infty$ or $\xi(\tau_2)=0$;
– $\xi(t)>0$ for all $t\in(\tau_1,\tau_2)$ and the restriction of ξ to (τ_1,τ_2) is a generalized characteristic in the classical sense (*e.g.* see [36, 37]).

For all $t>0$ such that

$$u(t,0)<\theta \quad\text{ or }\quad k(t)<k(t^-)=u(t,0)=\theta,\quad\text{ or }\quad k(t)<\theta<u(t,0)\le\pi_+(k(t)), \tag{A.6}$$

where $\pi_+(u)\doteq(f_+)^{-1}(f(u))$ (f_+ defined as in Sect. 2.1) we denote $s\to\xi_t(s)$ the *maximal forward generalized characteristic* starting at $(t,0)$, *i.e.* the forward generalized characteristic starting at $(t,0)$ with slope

$$\lambda_t \doteq \begin{cases} f'(u(t,0)), & \text{if}\quad u(t,0)<\theta,\\[2ex] \dfrac{f(u(t,0))-f(k(t))}{u(t,0)-k(t)}, & \begin{array}{l}\text{if}\quad k(t)<\theta<u(t,0)\le\pi_+(k(t)),\\ \text{or}\quad k(t)<k(t^-)=u(t,0)\le\theta.\end{array}\end{cases} \tag{A.7}$$

We denote $\operatorname{Dom}(\xi_t)=[t,\hat t\,]$, $\hat t\in\mathbb{R}^+\cup+\infty$, the domain of existence of ξ_t. Observe that $\lambda_t\ge0$ by definition (A.7). Moreover, if $\hat t\in\mathbb{R}^+$, then we have $\xi_t(\hat t\,)=0$.

Step 3. Definition of τ^* and computation of total variation of $f(u(t,0))$ in $(\tau^*,+\infty)$.

Let

$$\tau^*\doteq\inf\Big\{t>0\ \Big|\ \text{(A.6) holds and}\quad \operatorname{Dom}(\xi_t)=[t,+\infty)\Big\}. \tag{A.8}$$

Note that if (A.6) does not hold for any $t > 0$, then there is no forward generalized characteristics starting at $(t, 0)$, and we set in this case $\tau^* = +\infty$. Moreover, it may happen that for all $t > 0$ for which (A.6) holds, it holds that $\mathrm{Dom}(\xi_t) = [t, \hat{t}]$ with $\hat{t} \in \mathbb{R}^+$. Also in this case by definition $\tau^* = +\infty$.

Let $\tau^* < +\infty$. Observe that

$$u(t, 0) \leq \theta \qquad \forall\, t > \tau^*. \tag{A.9}$$

In fact, if $u(t, 0) > \theta$ at some $t > \tau^*$, we could trace the maximal backward characteristics ϑ_t starting at $(t, 0)$, which would meet at some point $x > 0$ a forward generalized characteristic $\xi_{t'}$ starting at some $(t', 0)$, $t' \in (\tau^*, t)$. But a backward characteristic cannot cross a forward generalized characteristic. We deduce from (A.9), recalling property (i) at **Step 1**, that

$$\mathrm{TV}_{(\tau^*,+\infty)}\, f(u(\cdot, 0)) = \mathrm{TV}_{(\tau^*,+\infty)}(f \circ k). \tag{A.10}$$

We observe that if $\tau^* < +\infty$, actually it is the minimum in (A.8) since the set of generalized characteristic is closed with respect to uniform convergence.

Step 4. The variation of $f(u(t, 0))$ at $t = \tau^* < +\infty$.

We show first of all that

$$u(\tau^*, 0) > \theta. \tag{A.11}$$

If it were not true, either $u(\tau^*, 0) < \theta$ or $u(\tau^*, 0) = \theta$, and we are going to show that both these possibilities cannot verify.

1. Assume $u(\tau^*, 0) < \theta$. Then, by (A.4), there exists $\tau' < \tau^*$ with $u(t, 0) < \theta$ for $t \in (\tau', \tau^*)$. Then, consider the maximal forward generalized characteristic ξ_t starting at some point $(t, 0)$, $t \in (\tau', \tau^*)$. Observe that, if $\mathrm{Dom}(\xi_t) = [t, \hat{t}]$, $\hat{t} \leq \tau^*$, then we would have $\xi_t(\hat{t}\,) = 0,\ \ \xi_t'(\hat{t}\,) \leq 0$. Hence, the Lax condition implies that $f'(u(\hat{t}, 0)) \leq \xi_t'(\hat{t}\,) \leq 0$ which in turns gives $u(\hat{t}, 0) \geq \theta$, in contradiction with (A.4) since $\hat{t} \in (\tau', \tau^*)$. If $\mathrm{Dom}(\xi_t) = [t, \hat{t}]$, $\hat{t} > \tau^*$, then $\hat{t} < +\infty$ by minimality of τ^*, recalling definition (A.8). Therefore, there exists $\widetilde{\tau} > \tau^*$ such that $\xi_t(\widetilde{\tau}\,) = \xi_{\tau^*}(\widetilde{\tau}\,)$ and then $\xi_t(s)$ coincides with $\xi_{\tau^*}(s)$ for all $s \geq \widetilde{\tau}$. So again it will give $\hat{t} = +\infty$ in contradiction with definition (A.8).
2. Assume $u(\tau^*, 0) = \theta$. By property (A.5), we may define

$$\tau_1 \doteq \inf \Big\{ \tau' \in (0, \tau^*\,) \ \big|\ u(t, 0) \leq \theta \quad \forall\, t \in (\tau', \tau^*) \Big\}. \tag{A.12}$$

By definition (A.12) it follows that

$$u(\tau_1, 0) > \theta. \tag{A.13}$$

In fact, if $u(\tau_1, 0) \leq \theta$, by the same arguments above we would deduce that there exists $\tau' < \tau_1$ such that $u(t, 0) \leq \theta$ for all $t \in (\tau', \tau_1)$, contradicting the definition (A.12). Moreover, observe that, relying again on properties (i)-(ii) at **Step 1**, and because of definition (A.12), we have $u(\tau_1^+, 0) = k(\tau_1)$, and

$$k(\tau_1) < \theta, \qquad\qquad \pi_+(k(\tau_1)) \geq u(\tau_1, 0). \tag{A.14}$$

In fact, if either $k(\tau_1) = \theta$, or $\pi_+(k(\tau_1)) < u(\tau_1, 0)$, then the initial boundary Riemann problem on $\{t \geq \tau_1,\, x \geq 0\}$, with initial datum $u(\tau_1, 0)$, and boundary datum $k(\tau_1)$, would be solved by the constant function $u(\tau_1, 0) > \theta$, which is incompatible with the fact that $u(\tau_1^+, 0) \leq \theta$ by definition (A.12). Therefore, due to (A.12), (A.14), condition (A.6) is verified at time $t = \tau_1$, and we may consider the maximal forward generalized characteristic ξ_{τ_1} starting at $(\tau_1, 0)$, $\tau_1 < \tau^*$. By the same arguments of the previous case we deduce that $\mathrm{Dom}(\xi_{\tau_1}) = [\tau_1, +\infty)$, which is contrast with the minimality of τ^* in Definition (A.8).

Observe then that due to (A.11) and to the Definition of τ^*, it holds that $f(u((\tau^*)^+, 0)) = f(k((\tau^*)^+))$, and that $f(u(\tau^*, 0)) \in \big[f(\overline{u}((x^*)^-), f(\overline{u}((x^*)^+)\big]$ where x^* is the position at $t = 0$ of the maximal backward characteristic starting

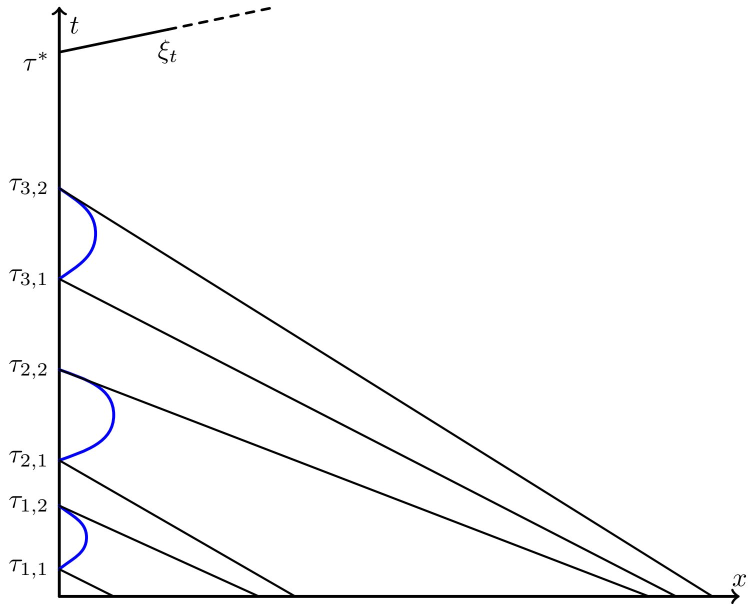

FIGURE A.1. A possible configuration, in the (t,x) plane, related to the intervals $(\tau_{n,1},\tau_{n,2}) \subseteq (0,\tau^*)$, described in (A.16)–(A.17), with $n = 1,2,3$. The maximal forward generalized characteristics ξ_t (see Step 2), for $t > \tau^*$, is defined on $[t,+\infty)$. Note that the generalized backward characteristics starting form points $(t,0)$, with time $t < \tau^*$ not in $(\tau_{n,1},\tau_{n,2})$, enter the domain and reach, at time 0, a point $x > 0$.

at $(\tau^*,0)$. We then have that

$$\left|f(u((\tau^*)^+,0)) - f(u(\tau^*,0))\right| \leq \mathrm{Osc}(f), \tag{A.15}$$

with $\mathrm{Osc}(f)$ defined as in (A.3).

Step 5. Structure of the trace of the solution $u(\cdot,0)$ in $(0,\tau^*)$.

We claim that there exist at most countably many intervals $(\tau_{n,1},\tau_{n,2}) \subset [0,\tau^*]$, $n \in \mathbb{N}$, with the property

$$u(\tau_{n,i},0) > \theta, \quad i = 1,2, \qquad u(t,0) \leq \theta, \qquad \forall\, t \in (\tau_{n,1},\tau_{n,2}), \tag{A.16}$$

such that

$$\Big\{t \in (0,\tau^*) \;\Big|\; u(t,0) \leq \theta\Big\} = \bigcup_n (\tau_{n,1},\tau_{n,2}); \tag{A.17}$$

see Figure A.1.

We start fixing any time $\tau \in (0,\tau^*)$ where $u(\tau,0) \leq \theta$. We claim that there exist $\tau_1 < \tau_2 \leq \tau^*$ such that $\tau \in (\tau_1,\tau_2)$ and

$$u(\tau_i,0) > \theta, \quad i = 1,2, \qquad u(t,0) \leq \theta, \qquad \forall\, t \in (\tau_1,\tau_2). \tag{A.18}$$

We observe that since $u(\tau,0) \leq \theta$, by properties (A.4), (A.5) there exists $\tau' < \tau$ such that $u(t,0) \leq \theta$ for $t \in (\tau',\tau)$. Let us define

$$\tau_1 \doteq \inf\Big\{\tau' \in (0,\tau) \;\Big|\; u(t,0) \leq \theta \quad \forall\, t \in (\tau',\tau)\Big\}.$$

Then, $\pi_+(k(\tau_1)) \geq u(\tau_1,0) > \theta \geq k(\tau_1)$.

Next, consider the maximal forward generalized characteristic ξ_{τ_1} starting at $(\tau_1, 0)$. Since $\tau_1 < \tau < \tau^*$, it follows from the minimality of τ^* in (A.8) that $\mathrm{Dom}(\xi_{\tau_1}) = [\tau_1, \tau_2]$, for some $\tau_2 \leq \tau^2$, and $\xi_{\tau_1}(\tau_1) = \xi_{\tau_1}(\tau_2) = 0$. Arguing as in (A.9) we deduce that

$$u(t,0) \leq \theta \qquad \forall\, t \in (\tau_1, \tau_2).$$

Observe that by Lax condition we have $f'(u(\tau_2, 0)) \leq \xi'_{\tau_1}(\tau_2) \leq 0$, which implies $u(\tau_2, 0) \geq \theta$. We will show that necessarily $u(\tau_2, 0) > \theta$. Indeed, $u(\cdot, 0)$ admits a discontinuity at $t = \tau_2$ connecting the left state $u(\tau_2^-, 0)$ with the right state $u(\tau_2, 0)$, and having slope $\xi'_{\tau_1}(\tau_2)$. If $u(\tau_2, 0) = \theta$, then it follows that $\xi'_{\tau_1}(\tau_2) = 0$, but there is no discontinuity with zero slope and right state θ. Therefore, it must be $u(\tau_2, 0) > \theta$ and then necessarily $\tau_2 \leq \tau^*$. This concludes the proof of the claim (A.18).

Finally observe that the interval (τ_1, τ_2) is associated to the discontinuity curve $(t, \xi_{\tau_1}(t))$ starting at $(\tau_1, 0)$ and ending at $(\tau_2, 0)$. Since any solution admits at most countably many curves of discontinuity, repeating the construction of the interval (τ_1, τ_2) starting from any time $\tau \in (0, \tau^*)$, we deduce the existence of (at most) countably many intervals $(\tau_{n,1}, \tau_{n,2}) \subset [0, \tau^*]$, $n \in \mathbb{N}$, enjoying the properties (A.16), (A.17).

Step 6. The computation of the total variation of $f(u(t,0))$ on a single interval $[\tau_{n,1}, \tau_{n,2}]$.
Let $\tau_1 < \tau_2 \leq \tau^*$ be such that (A.18) holds.

We show that

$$\begin{aligned}\big|f(u(\tau_2,0)) - f(u(\tau_2^-,0))\big| + \big|f(u(\tau_1^+,0)) - f(u(\tau_1,0))\big| \\ \leq \big|f(u(\tau_2,0)) - f(u(\tau_1,0))\big| + \mathrm{TV}_{(\tau_1,\tau_2)}(f\circ k).\end{aligned} \tag{A.19}$$

Observe that by property (i) at **Step 1**, and because of (A.18), we have $f(u(\tau_2^-, 0)) = f(k(\tau_2^-))$, $f(u(\tau_1^+, 0)) = f(k(\tau_1))$.

Notice that, by the analysis at **Step 5**, since there is a shock discontinuity at $(0, \tau_1)$ with nonnegative slope connecting the left state $u(\tau_1^+, 0)$ with the right state $u(\tau_1, 0)$, it follows that

$$f(u(\tau_1^+,0)) \leq f(u(\tau_1,0)). \tag{A.20}$$

Similarly, we have

$$f(u(\tau_2^-,0)) \geq f(u(\tau_2,0)), \tag{A.21}$$

since at $(0, \tau_2)$ there is a shock discontinuity with nonnegative slope connecting the left state $u(\tau_2^-, 0)$ with the right state $u(\tau_2, 0)$. Therefore, we have

$$\begin{aligned}&\big|f(u(\tau_2,0)) - f(u(\tau_2^-,0))\big| + \big|f(u(\tau_1^+,0)) - f(u(\tau_1,0))\big| \\ &\qquad = f(u(\tau_2^-,0)) - f(u(\tau_1^+,0)) + f(u(\tau_1,0)) - f(u(\tau_2,0)) \\ &\qquad \leq \big|f(u(\tau_2,0)) - f(u(\tau_1,0))\big| + \big|f(u(\tau_2^-,0)) - f(u(\tau_1^+,0))\big|.\end{aligned}$$

This implies that (A.19) holds.

Now, we compute the total variation. Consider a $(N+1)$-tupla of times $t_0 \leq \tau_1 < t_1 < \cdots < t_{N-1} < \tau_2 \leq t_N$, with

$$u(t_0,0) > \theta, \qquad\qquad u(t_N,0) > \theta. \tag{A.22}$$

So, recalling (A.19), we compute

$$\begin{aligned}&\sum_{k=0}^{N-1} \big|f(u(t_{k+1},0)) - f(u(t_k),0)\big| \\ &\leq \big|f(u(t_N,0)) - f(u(\tau_2,0))\big| + \big|f(u(\tau_1,0)) - f(u(t_0,0)) \\ &\quad + \big|f(u(\tau_2,0)) - f(u(\tau_2^-,0))\big| + \mathrm{TV}_{(\tau_1,\tau_2)}(f\circ k) + \big|f(u(\tau_1^+,0)) - f(u(\tau_1,0))\big|\end{aligned}$$

$$\leq \big|f(u(t_N,0)) - f(u(\tau_2,0))\big| + \big|f(u(\tau_1,0)) - f(u(t_0,0))\big| \\ +\big|f(u(\tau_2,0)) - f(u(\tau_1,0))\big| + 2\,\mathrm{TV}_{(\tau_1,\tau_2)}(f\circ k). \tag{A.23}$$

Step 7. Computation of the total variation of $f(u(t,0))$ on the entire interval $(0,\tau^*)$. Let $\big\{\tau_{n,1},\tau_{n,2}\big\}_n$, be as in (A.16), (A.17).

We fix a $(N+1)$-tupla of times $0\leq t_0<t_1<t_2<\cdots<t_N\leq\tau^*$.

Since t_j are finite, there exists a finite number $\bar n\geq 0$ of intervals $(\tau_{n,1},\tau_{n,2})$ which contains at least one of the t_j. Note that $\bar n$ could also be 0 (in particular this will always happen when (A.6) for any $t>0$).

From now on we assume $\bar n>0$, the case $\bar n=0$ can be treated in an analogous (and simpler) way. Up to relabeling the indexes we may assume that this holds for $n=1,\dots,\bar n$, with $\tau_{n,1}<\tau_{n+1,2}$ for all $n=1,\dots,\bar n-1$.

Therefore, for any $n\in\{1,\dots,\bar n\}$, we have that the intersection $(\tau_{n,1},\tau_{n,2})\cap\{t_0,\dots,t_N\}$ is not empty and contains $m_n\geq 1$ elements, that up to relabeling we denote as the ordered sequence

$$\{t_{n,1},t_{n,2},\dots,t_{n,m_n}\}=(\tau_{n,1},\tau_{n,2})\cap\{t_0,\dots,t_N\}.$$

In particular, we have that

$$u(t_{n,j},0)\leq\theta,\qquad j=1,\dots,m_n,\ n=1,\dots,\bar n.$$

Moreover, for every $n=1,\dots,\bar n-1$ there exists a finite number $p_n\geq 0$ of t_j which are contained in $[\tau_{n,2},\tau_{n+1,1}]$. Let us denote them as the ordered sequence

$$\{t_{n,n+1,1},\dots,t_{n,n+1,p_n}\}=[\tau_{n,2},\tau_{n+1,1}]\cap\{t_0,\dots,t_N\},\qquad n=1,\dots,\bar n-1.$$

Finally, there are at most $p_0\geq 0$ times t_j such that $t_j\leq\tau_{1,1}$. We label them as $\{t_{0,1,1},\dots,t_{0,1,p_0}\}$. Analogously there are at most $p_{\bar n}\geq 0$ times t_j such that $t_j\geq\tau_{\bar n,2}$, and we label them as $\{t_{\bar n,\bar n+1,1},\dots,t_{\bar n,\bar n+1,p_{\bar n}}\}$ (maintaining the order).

In particular, we have that

$$u(t_{n,n+1,j},0)>\theta,\qquad j=1,\dots,p_n,\ n=0,\dots,\bar n+1.$$

In conclusion, we have relabeled the $(N+1)-$tupla of times as follows:

$$\{t_0,t_1,\dots,t_k,\dots t_N\}=\bigcup_{n=0}^{\bar n+1}\bigcup_{j=1}^{p_n}\{t_{n,n+1,j}\}\cup\bigcup_{n=1}^{\bar n}\bigcup_{j=1}^{m_n}\{t_{n,j}\}.$$

Then, using this relabelling and applying estimate (A.23) at the previous **Step 6**, we find

$$\begin{aligned}&\sum_{k=0}^{N-1}|f(u(t_{k+1},0))-f(u(t_k,0))| \\ &\leq\sum_{n=0}^{\bar n+1}\sum_{j=1}^{p_n-1}|f(u(t_{n,n+1,j+1},0))-f(u(t_{n,n+1,,j},0))| \\ &\quad+\sum_{n=0}^{\overline{n}-1}|f(u(\tau_{n+1,1},0))-f(u(t_{n,n+1,p_n},0))|+\sum_{n=1}^{\overline{n}+1}|f(u(\tau_{n,2},0))-f(u(t_{n,n+1,1},0))| \\ &\quad+\sum_{n=1}^{\overline{n}}|f(u(\tau_{n,1},0))-f(u(\tau_{n,2},0))|+2\sum_{n=1}^{\overline{n}}\mathrm{TV}_{(\tau_{n,1},\tau_{n,2})}(f\circ k).\end{aligned}\tag{A.24}$$

Let $\vartheta_{n,n+1,j}(t)$ be the maximal backward characteristic starting at $(t_{n,n+1,j},0)$, for $j=1,\dots,p_n$ and $n=0,\dots,\bar n+1$, and denote $x_{n,n+1,j}=\vartheta_{n,n+1,j}(0)$. Then the initial datum $\overline{u}$ will be either continuous at $x_{n,n+1,j}$, or it will admit at

$x_{n,n+1,j}$ a downward jump $\overline{u}(x^-_{n,n+1,j}) > \overline{u}(x^+_{n,n+1,j})$ generating a rarefaction. In particular, for every $j = 1, \dots, p_n - 1$, we have

$$|f(u(t_{n,n+1,j}, 0)) - f(u(t_{n,n+1,j+1}, 0))| \leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{[x_{n,n+1,j}, x_{n,n+1,j+1}]}(\overline{u}), \tag{A.25}$$

where $\operatorname{TV}_{[a,b]}(\overline{u})$ is defined in (2.3).

Similarly, we may consider the maximal backward characteristic $\zeta_{n,i}$ starting at $(\tau_{n,i}, 0)$, $i = 1, 2$, set $z_{n,i} = \zeta_{n,i}(0)$, $i = 1, 2$, and conclude that

$$\begin{aligned}
|f(u(\tau_{n+1,1}, 0)) - f(u(t_{n,n+1,p_n}, 0))| &\leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{[z_{n+1,1}, x_{n,n+1,p_n}]}(\overline{u}),\\
|f(u(\tau_{n,2}, 0)) - f(u(t_{n,n+1,1}, 0))| &\leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{[z_{n,2}, x_{n,n+1,1}]}(\overline{u}),\\
|f(u(\tau_{n,1}, 0)) - f(u(\tau_{n,2}, 0))| &\leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{[z_{n,2}, z_{n,1}]}(\overline{u}).
\end{aligned} \tag{A.26}$$

If $x_{n,n+1,j} \neq x_{n,n+1,j+1} \neq z_{n,i}$ for all i, j, relying on (A.25), (A.26), we derive from (A.24) that

$$\sum_{k=0}^{N-1} \big|f(u(t_{k+1}, 0)) - f(u(t_k, 0))\big| \leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{(0,+\infty)}(\overline{u}) + 2 \operatorname{TV}_{(0,\tau^*)}(f \circ k). \tag{A.27}$$

Note that it could happen that $x_{n,n+1,j-1} < x_{n,n+1,j} = x_{n,n+1,j+k} < x_{n,n+1,j+k+1}$ for some $j, k > 0$. In this case, instead of relying on the estimates in (A.25), we estimate all together

$$\sum_{i=j-1}^{j+k} \big|f(u(t_{n,n+1,i}, 0)) - f(u(t_{n,n+1,i+1}, 0))\big| \leq \big\|f'(\overline{u})\big\|_\infty \operatorname{TV}_{[x_{n,n+1,j-1},\, x_{n,n+1,j+k+1}]}(\overline{u}). \tag{A.28}$$

Similarly, we treat the cases in which $z_{n,2} < x_{n,n+1,1} = x_{n,n+1,1+k} < x_{n,n+1,2+k}$, for some k, getting

$$\begin{aligned}
&\big|f(u(\tau_{n,2}, 0)) - f(u(t_{n,n+1,1}, 0))\big| + \sum_{i=1}^{2+k} \big|f(u(t_{n,n+1,i}, 0)) - f(u(t_{n,n+1,i+1}, 0))\big|\\
&\qquad \leq \big\|f'(\overline{u})\big\|_\infty \operatorname{TV}_{[z_{n,2},\, x_{n,n+1,2+k}]}(\overline{u})
\end{aligned}$$

or $x_{n,n+1,j-1} < x_{n,n+1,j} = x_{n,n+1,j+k} < x_{n,n+1,j+k+1} < z_{n+1,1}$, for some j, or $z_{n,2} = z_{n,1}$ or $z_{n,2} = x_{n,n+1,1}$ or $z_{n+1,1} = x_{n,n+1,p_n}$ for some n. Thus, in any case, combining (A.25), (A.26)with (A.28), we derive the bound (A.24).

Step 8. Conclusion.

Given any $(N+1)$-tupla of times $t_0 < t_1 < t_2 < \cdots < t_N$, combining (A.10) with (A.27) we deduce

$$\begin{aligned}
\sum_{k=0}^{N-1} \big|f(u(t_{k+1}, 0)) - f(u(t_k, 0))\big| &\leq \|f'(\overline{u})\|_\infty \operatorname{TV}_{(0,+\infty)}(\overline{u}) + 2 \operatorname{TV}_{(0,+\infty)}(f \circ k)\\
&\quad + \big|f(u((\tau^*)^+, 0)) - f(u(\tau^*, 0))\big|.
\end{aligned}$$

From this, recalling estimates (A.10) and (A.15), we get the proof of the theorem. □

Remark A.2. We observe that the assumptions in Theorem A.1 are sharp.

Observe that if $f \circ k \notin \mathbf{BV}((0, +\infty); \mathbb{R})$ then the result is not true.

Let $k \in \mathbf{L}^\infty((0, +\infty); (-\infty, \theta])$, with $f \circ k \notin \mathbf{BV}((0, +\infty); \mathbb{R})$, and $\overline{u} \in \mathbf{BV}((0, +\infty); (-\infty, \theta])$. Then the entropy admissible weak solution u of the initial-boundary value problem (A.1) will take values in $(-\infty, \theta]$ as well, and we have $u(\cdot, 0) = k$; see [9]. Therefore, in this case we have $f \circ u(\cdot, 0) = f \circ k \notin \mathbf{BV}((0, +\infty); \mathbb{R})$.

Next, we show with a counterexample that if $\overline{u} \notin \mathbf{BV}((0, +\infty); \mathbb{R})$ then the result is not true. Consider the flux $f(u) = -\frac{u^2}{2}$, the boundary datum $k(t) \equiv 0$ and the initial datum

$$\overline{u}(x) \doteq \begin{cases} \frac{1}{2}, & x \in \left(\frac{4}{3}\frac{1}{2^n}, \frac{1}{2^{n-1}}\right), n \geq 1,\\ 1, & \text{otherwise.} \end{cases} \tag{A.29}$$

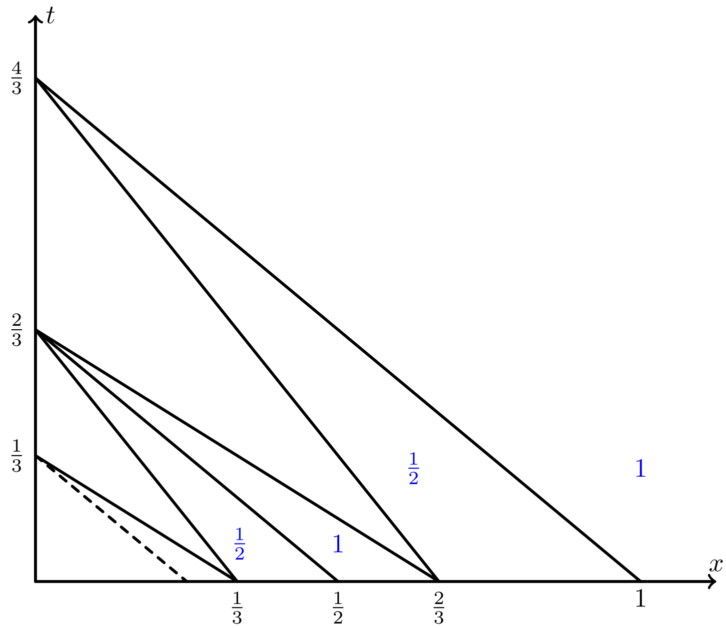

FIGURE A.2. The solution of Remark A.2. At the positions $x = \frac{1}{2^{n-1}}$, with $n \geq 1$, shock waves with speed $-\frac{3}{4}$ are originated. Instead at the positions $x = \frac{4}{3}\frac{1}{2^n}$, with $n \geq 1$, rarefaction waves are originated. All these waves do not intersect together inside the region $x > 0$.

Clearly we have $\overline{u} \notin \mathbf{BV}(0, +\infty)$. By direct computation one can check that the trace of the solution satisfies:

$$u(t,0) = \begin{cases} 1, & t > \frac{4}{3}, \\ \frac{2}{32^{n-1}t}, & t \in \left(\frac{4}{3}\frac{1}{2^n}, \frac{4}{3}\frac{1}{2^{n-1}}\right), n \geq 1; \end{cases}$$

see Figure A.2. In particular observe that

$$u\left(\left(\frac{4}{3}\frac{1}{2^n}\right)^+, 0\right) = 1 \qquad u\left(\left(\frac{4}{3}\frac{1}{2^{n-1}}\right)^-, 0\right) = \frac{1}{2}$$

which implies immediately that

$$\mathrm{TV}_{(0,4/3)}\, u(t,0) \geq \sum_{n=0}^{+\infty}\left(1 - \frac{1}{2}\right) = +\infty$$

and similarly

$$\mathrm{TV}_{(0,4/3)}\, f(u(t,0)) \geq \sum_{n=0}^{+\infty}\left(\frac{1}{2} - \frac{1}{8}\right) = +\infty.$$

Therefore, $f(u(\cdot,0)) \notin \mathbf{BV}((0,4/3);\mathbb{R})$.