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APPROXIMATION OF ELLIPTIC EQUATIONS WITH INTERIOR SINGLE-POINT DEGENERACY AND ITS APPLICATION TO WEAK UNIQUE CONTINUATION PROPERTY

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Abstract. This paper investigates the weak unique continuation property (WUCP) for a class of high-dimensional elliptic equations with interior point degeneracy. First, we establish well-posedness results in weighted function spaces. Then, using an innovative approximation method, we derive the three sphere inequality at the degenerate point. Finally, we apply the three sphere inequality to prove WUCP for two different cases.

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1. Introduction

The unique continuation properties for uniformly elliptic equations have been extensively studied in the literature [1–16]. There are two types of unique continuation properties: the strong unique continuation property (SUCP) and the weak unique continuation property (WUCP). Below, we briefly recall these two properties.

Let $P(x, \partial)$ be a uniformly elliptic operator. The strong unique continuation property (SUCP) states that if $P(x, \partial)u = 0$ in a domain $\Omega \subset \mathbb{R}^N$, and there exists a point $x_0 \in \Omega$ such that u vanishes to infinite order, meaning

$$\int_{B_r(x_0)} u^2 \mathrm{d}x = O(r^k) \text{ as } r \to 0, \text{ for every } k \in \mathbb{N},$$

where $B_r(x_0)$ denotes a ball in Ω centered at x_0 with radius r, then $u \equiv 0$ in Ω.

The weak unique continuation property (WUCP) states that if $P(x, \partial)u = 0$ in Ω, and $u = 0$ in an open subset $\omega \subset \Omega$, then $u \equiv 0$ in Ω. It is easy to see that the WUCP requires less stringent conditions compared to the SUCP.

Keywords and phrases: Degenerate elliptic equations, weak unique continuation property.

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Notably, unique continuation does not hold universally for all uniformly elliptic equations (*e.g.* [16]). Furthermore, the analysis of unique continuation properties becomes significantly more challenging for degenerate elliptic equations compared to uniformly elliptic ones. Currently, there are two methods — the three sphere inequality [1, 8, 9, 14, 15, 17, 18] and Carleman estimates [2–4, 12, 13, 16], which are effective in dealing with certain special cases [3, 11, 19–21].

The three sphere inequality states that for a harmonic (or subharmonic) function $u(x)$ defined in a region containing three concentric balls B_{r_1}, B_{r_2} and B_{r_3} with $r_1 < r_2 < r_3$, the maximum value $H(r)$ of $u(x)$ on the intermediate sphere B_{r_2} can be bounded by a weighted geometric mean of the maximum values on the inner and outer spheres:

$$H(r_2) \leq (H(r_1))^{\mu}(H(r_3))^{1-\mu},$$

where $\mu \in (0, 1)$ is determined by the radii. The three sphere inequality is developed on the basis of the doubling inequality, which was originally introduced by Garofalo and Lin in [9]. The author of [15] provides a detailed introduction to the doubling inequality, the three sphere inequality, and their applications in unique continuation. In [19], the author primarily investigates the unique continuation properties of a specific class of second-order elliptic operators that degenerate on manifolds of arbitrary codimension, using the doubling inequality. The focus is on the model operator

$$P_\alpha = \Delta_z + |z|^{2\alpha}\Delta_t, \ \alpha > 0,$$

in $\mathbb{R}^n \times \mathbb{R}^m$, which is elliptic outside a degeneracy manifold ($\{0\} \times \mathbb{R}^m$) but degenerates on it. The author establishes SUCP using Carleman estimates and introduces a quantitative version of SUCP that bypasses Carleman estimates, instead relying on the doubling inequality. Similarly, in [22], the doubling inequality is also applied to study the unique continuation properties of solutions to degenerate Schrödinger equations influenced by singular potentials and weighted settings. In [23], SUCP is established for a class of degenerate elliptic operators with Hardy-type potentials using Carleman estimates. This work extends the results of [19] but does not yield a quantitative conclusion. By contrast, in [24] a quantitative three-sphere-type inequality is established for solutions to Baouendi–Grushin type equations with a zeroth-order potential. Notably, the three sphere inequality appears to be more effective for studying quantitative weak unique continuation properties.

In this paper, we shall consider the weak unique continuation properties for the elliptic equation with degenerate interior point by approximation. It is well known that the solution spaces of degenerate elliptic equations belong to weighted Sobolev spaces [5–7, 25, 26]. A natural approach is to approximate a solution of a degenerate elliptic equation by a sequence of solutions to uniformly elliptic equations [5, 6]. This method is feasible in weighted spaces and applies to high-dimensional cases, but it heavily relies on the Calderón-Zygmund decomposition, which can compromise certain desirable properties of the weight function. For instance, the approximating weight functions may lack differentiability, which is crucial when using the three sphere inequality to prove WUCP, requiring the approximating weight functions to be at least Lipschitz continuous. Another approximation approach, similar to that in [27–29], involves constructing a non-degenerate coefficient $|x + \epsilon|^\alpha$ over the entire domain Ω to approximate the degenerate coefficient $|x|^\alpha$. However, this method is suitable for one-dimensional degenerate equations but not for the high-dimensional problems we aim to study. For the problem we consider in this paper, local estimates are required to approximate the solution (see Lem. 3.5).

While the idea of approximation has been utilized in many works, our method is fundamentally different from those in the existing literature. First, one of our main contributions is the introduction of an alternative approximation method for a specific class of weight functions with a single degenerate interior point. Our approximation is achieved by constructing a carefully designed non-degenerate weight function to approximate the degenerate weight function within a small local region B_ϵ rather than the entire domain Ω. This ensures that the weight function remains differentiable in high-dimensional settings. For a detailed discussion, refer to Section 3. Second, in the proof of WUCP, we consider two cases: $0 \in \omega$ and $0 \notin \omega$. For case $0 \in \omega$, the result is obtained using the three sphere inequality at both degenerate and non-degenerate points. For the more challenging case $0 \notin \omega$, we apply Schauder estimates to address the difficulties arising from the degenerate point being excluded from ω. Finally, we derive a WUCP result.

It is worth noting that in most works (see [27–29]), the SUCP is typically achieved using the doubling inequality. However, this paper employs the more robust three sphere inequality. Although we do not present results on SUCP here, we have demonstrated it in another working paper using an annular estimate method.

We organize the paper as follows: In Section 2, we present several well-posedness results. In Section 3, we provide a detailed explanation of the construction of the approximation and introduce the preliminary lemmas required for proving the three sphere inequality at the degenerate point. In Section 4, we establish the three sphere inequality at the degenerate point and prove WUCP for two cases: $0 \in \omega$ and $0 \notin \omega$.

2. Preliminary results

Let us consider the following equation

$$\begin{cases} -\mathrm{div}(w\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a domain containing the origin $(0 \in \Omega)$, and its boundary $\partial\Omega$ is of class C^2. The weight function is given by $w = |x|^\alpha$, with a fixed $\alpha \in (0, 2)$, and f is a given function such that $f \in L^2(\Omega; w^{-1})$. The weighted Sobolev space $L^2(\Omega; w)$ is defined as follows:

$$L^2(\Omega; w) = \left\{ u(x) \mid u \text{ is measurable, and } \int_\Omega u^2 w \mathrm{d}x < \infty \right\}.$$

The inner product on $L^2(\Omega; w)$ is

$$(u, v)_{L^2(\Omega;w)} = \int_\Omega uvw \mathrm{d}x,$$

and the norm on $L^2(\Omega; w)$ is

$$\|u\|_{L^2(\Omega;w)} = \left(\int_\Omega u^2 w \mathrm{d}x \right)^{\frac{1}{2}}.$$

It is well known that $(L^2(\Omega; w), (\cdot, \cdot)_{L^2(\Omega;w)})$ (see [30]) is a Hilbert space and $(L^2(\Omega; w), \|\cdot\|_{L^2(\Omega;w)})$ is a Banach space.

Set

$$H^1_w(\Omega) = \left\{ u \in L^2(\Omega) \colon \frac{\partial u}{\partial x_i} \in L^2(\Omega; w), i = 1, \cdots, N \right\},$$

where $\frac{\partial u}{\partial x_i}, i = 1, \cdots, N$ are the distributional partial derivatives, the inner product on $H^1_w(\Omega)$ is

$$\begin{aligned} (u, v)_{H^1_w(\Omega)} &= \int_\Omega uvw \mathrm{d}x + \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} w \mathrm{d}x \\ &= (u, v)_{L^2(\Omega;w)} + (\nabla u, \nabla v)_{L^2(\Omega;w)} \end{aligned}$$

and the norm is

$$\|u\|_{H^1_w(\Omega)} = \left(\int_\Omega u^2 w \mathrm{d}x + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 w \mathrm{d}x \right)^{\frac{1}{2}}.$$

Define

$$H^1_{w,0}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^1_w(\Omega)}},$$

where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of test functions. Denote by $H_w^{-1}(\Omega)$ the dual space of $H^1_{w,0}(\Omega)$. This space is a subspace of $\mathcal{D}'(\Omega)$, the space of distributions on Ω. It is well known that $(H^1_w(\Omega), (\cdot,\cdot)_{H^1_w(\Omega)})$ forms a Hilbert space, while $(H^1_w(\Omega), \|\cdot\|_{H^1_w(\Omega)})$ is a Banach space.

Next, we aim to establish some well-posedness results for equation (2.1). First, we introduce some notations that will be used:

$$\Omega^\epsilon = \{x \in \Omega \mid |x| > \epsilon\}, \ B_\epsilon = \{x \in \Omega \mid |x| < \epsilon\}.$$

Similar to the proof of Lemma 3.1 in [31] or Proposition 2.1 (1) in [26], we can easily derive the following weighted Hardy inequality.

Lemma 2.1. *For any $N \geq 2$ and $\alpha \in (0,2)$, if $u \in H^1_{w,0}(\Omega)$, then we have*

$$(N-2+\alpha) \left(\int_\Omega |x|^{\alpha-2} u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \leq 2 \left(\int_\Omega |x|^\alpha \nabla u \cdot \nabla u \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.2}$$

Moreover, let $m := \sup_{x\in\Omega} |x| + 1$. We can also derive

$$\left(\int_\Omega u^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \leq \frac{2m^{1-\frac{\alpha}{2}}}{N-2+\alpha} \left(\int_\Omega |x|^\alpha \nabla u \cdot \nabla u \,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.3}$$

Proof. Let $u \in H^1_{w,0}(\Omega)$. Then $u = 0$ on $\partial\Omega$ in the sense of traces. Moreover, for each $\epsilon > 0$, the restriction of u to the open set Ω^ϵ belongs to $W^{1,2}(\Omega^\epsilon)$, and the trace of u on $\partial\Omega^\epsilon$ is well-defined and bounded in $L^2(\partial\Omega^\epsilon)$. Applying the divergence theorem on Ω^ϵ, we obtain

$$\begin{aligned} 2\int_{\Omega^\epsilon} |x|^{\alpha-2}(x\cdot\nabla u)u \,\mathrm{d}x &= \int_{\Omega^\epsilon} |x|^{\alpha-2} x \cdot \nabla(u^2) \,\mathrm{d}x \\ &= \int_{\partial\Omega} |x|^{\alpha-2} u^2 (x\cdot\nu) \,\mathrm{d}s + \int_{\partial B_\epsilon} |x|^{\alpha-2} u^2 (x\cdot\nu) \,\mathrm{d}s \\ &\quad - \int_{\Omega^\epsilon} \mathrm{div}\left(|x|^{\alpha-2}x\right) u^2 \,\mathrm{d}x. \end{aligned}$$

Since u $= 0$ on $\partial\Omega$, the boundary integral over $\partial\Omega$ vanishes. Noting that $x\cdot\nu = -|x|$ on ∂B_ϵ and $\mathrm{div}\left(|x|^{\alpha-2}x\right) = (N-2+\alpha)|x|^{\alpha-2}$, we deduce

$$2\int_{\Omega^\epsilon} |x|^{\alpha-2}(x\cdot\nabla u)u \,\mathrm{d}x = -\int_{\partial B_\epsilon} |x|^{\alpha-1} u^2 \,\mathrm{d}s - (N-2+\alpha)\int_{\Omega^\epsilon} |x|^{\alpha-2} u^2 \,\mathrm{d}x.$$

Rearranging terms gives

$$(N-2+\alpha)\int_{\Omega^\epsilon}|x|^{\alpha-2}u^2\,\mathrm{d}x \le -2\int_{\Omega^\epsilon}|x|^{\alpha-2}(x\cdot\nabla u)u\,\mathrm{d}x.$$

Applying the Cauchy–Schwarz inequality, we estimate

$$\begin{aligned}-2\int_{\Omega^\epsilon}|x|^{\alpha-2}(x\cdot\nabla u)u\,\mathrm{d}x &\le 2\int_{\Omega^\epsilon}|x|^{\frac{\alpha}{2}-1}|u|\,|x|^{\frac{\alpha}{2}}|\nabla u|\,\mathrm{d}x\\ &\le 2\left(\int_{\Omega^\epsilon}|x|^{\alpha-2}u^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\Omega^\epsilon}|x|^{\alpha}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Thus, we obtain

$$(N-2+\alpha)\left(\int_{\Omega^\epsilon}|x|^{\alpha-2}u^2\,\mathrm{d}x\right)^{1/2} \le 2\left(\int_{\Omega^\epsilon}|x|^{\alpha}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}. \tag{2.4}$$

Passing to the limit as $\epsilon\to 0^+$, we use the almost everywhere convergence of $|x|^{\alpha-2}u^2\chi_{\Omega^\epsilon}\to|x|^{\alpha-2}u^2$, and Fatou's lemma yields

$$\int_\Omega|x|^{\alpha-2}u^2\,\mathrm{d}x \le \liminf_{\epsilon\to 0}\int_{\Omega^\epsilon}|x|^{\alpha-2}u^2\,\mathrm{d}x.$$

Since $|x|^\alpha|\nabla u|^2\in L^1(\Omega)$, and taking limits in (2.4) yields

$$(N-2+\alpha)\left(\int_{\Omega}|x|^{\alpha-2}u^2\,\mathrm{d}x\right)^{1/2} \le 2\left(\int_{\Omega}|x|^{\alpha}|\nabla u|^2\,\mathrm{d}x\right)^{1/2},$$

thus establishing inequality (2.2).

Finally, to prove inequality (2.3), observe that

$$|u(x)|^2\le m^{2-\alpha}|x|^{\alpha-2}|u(x)|^2,\quad \forall\, x\in\Omega,$$

where $m:=\sup_{x\in\Omega}|x|+1$. Integrating over Ω and applying (2.2) gives

$$\left(\int_\Omega u^2\,\mathrm{d}x\right)^{1/2}\le\frac{2m^{1-\alpha/2}}{N-2+\alpha}\left(\int_\Omega|x|^\alpha|\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

The proof is thus complete. □

From Lemma 2.1, it is also evident that space $H^1_{w,0}(\Omega)$ is embedded into space $L^2(\Omega)$. Next, we will prove that this embedding is compact.

Lemma 2.2. *The embedding $H^1_{w,0}(\Omega)\subset\subset L^2(\Omega)$ is compact.*

Proof. To establish the compactness of the embedding it suffices to show that if $\{u_n\}$ is a sequence converging weakly to zero in $H^1_{w,0}(\Omega)$ as $n\to\infty$, then $\|u_n\|_{L^2(\Omega)}\to 0$ as $n\to\infty$.

Since $H^1_{w,0}(\Omega)$ is continuously embedded in $L^2(\Omega)$ by Lemma 2.1, $L^2(\Omega)^*\subset H^1_{w,0}(\Omega)^*$ and hence $\{u_n\}$ converges weakly to zero in $L^2(\Omega)$.

Consider $\epsilon > 0$. If $\{u_n\}$ does not converge weakly to zero in $W^{1,2}(\Omega^\epsilon)$, there exist $f \in W^{1,2}(\Omega^\epsilon)^*$, a subsequence $\{u_{n_k}\}$ and $\delta > 0$ such that $|f(u_{n_k})| \geq \delta$ for all n_k. Passing to a further subsequence if necessary, we can suppose that $\{u_{n_k}\}$ converges weakly to an element v in $W^{1,2}(\Omega^\epsilon)$. Thus $\{u_{n_k}\}$ converges weakly to v in $L^2(\Omega^\epsilon)$ and so $v = 0$ a.e. on Ω^ϵ since $\{u_n\}$ converges weakly to zero in $L^2(\Omega)$ and hence also on $L^2(\Omega^\epsilon)$. But then $f(u_{n_k}) \to f(v) = f(0) = 0$ as $n_k \to \infty$, contradicting with $|f(u_{n_k})| \geq \delta$ for all n_k. Hence $\{u_n\}$ converges weakly to zero in $W^{1,2}(\Omega^\epsilon)$ and therefore $\|u_n\|_{L^2(\Omega^\epsilon)} \to 0$ as $n \to \infty$. From the above it follows that

$$\limsup_{n\to\infty} \|u_n\|^2_{L^2(\Omega)} = \limsup_{n\to\infty} \int_{B_\epsilon} |u_n|^2 \, dx. \tag{2.5}$$

But from [32] and (2.3) in Lemma 2.1, we have

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{H^1_{w,0}(\Omega)}, \; 1 \leq q \leq \frac{2N}{N-2+\alpha}, \tag{2.6}$$

then (taking $q > 2$)

$$\int_{B_\epsilon} |u_n|^2 \, dx \leq \left(\int_{B_\epsilon} |1|^{\frac{q}{q-2}} \mathrm{d}x\right)^{\frac{q-2}{q}} \left(\int_{B_\epsilon} \left(|u_n|^2\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{2}{q}} \leq |B_\epsilon|^{\frac{q-2}{q}} \|u_n\|^2_{L^q(\Omega)}.$$

The weak convergence of $\{u_n\}$ in $H^1_{w,0}(\Omega)$ and (2.6) imply that this sequence is bounded in $L^q(\Omega)$, then (note that $q > 2$)

$$\int_{B_\epsilon} |u_n|^2 \, dx \leq C|B_\epsilon|^{\frac{q-2}{q}} \|u_n\|^2_{H^1_{w,0}(B_\epsilon)} \leq C|B_\epsilon|^{\frac{q-2}{q}}.$$

Letting $\epsilon \to 0^+$ in (2.5) shows that $\|u_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$, completing the proof. □

Next, we use the Lax-Milgram theorem to show that equation (2.1) has a unique weak solution $u \in H^1_{w,0}(\Omega)$ in the sense of

$$\int_\Omega (\nabla u \cdot \nabla v) w \mathrm{d}x = \int_\Omega f v \mathrm{d}x$$

for all $v \in H^1_{w,0}(\Omega)$.

Lemma 2.3. *For each $f \in L^2(\Omega; w^{-1})$, there exists a unique solution for the equation (2.1).*

Proof. Denote

$$B_w[u, v] = \int_\Omega (\nabla u \cdot \nabla v) w \mathrm{d}x \text{ for all } u, v \in H^1_{w,0}(\Omega).$$

It is easily verified that $B_w[\cdot, \cdot] : H^1_{w,0}(\Omega) \times H^1_{w,0}(\Omega) \to \mathbb{R}$ is a bilinear form.

On one hand, we have

$$|B_w[u, v]| \leq \|u\|_{H^1_{w,0}(\Omega)} \|v\|_{H^1_{w,0}(\Omega)},$$

and $B_w[u,u] = \|u\|^2_{H^1_{w,0}(\Omega)}$. On the other hand, we have

$$\left|\int_\Omega fv\mathrm{d}x\right| \le \|f\|_{L^2(\Omega;w^{-1})}\|v\|_{L^2(\Omega;w)} \le C\|f\|_{L^2(\Omega;w^{-1})}\|v\|_{H^1_{w,0}(\Omega)}$$

by Cauchy inequality and (2.3) in Lemma 2.1, *i.e.*, $f : H^1_{w,0}(\Omega) \to \mathbb{R}$ is a bounded linear functional on $H^1_{w,0}(\Omega)$.
Finally, by the Lax-Milgram theorem, we obtain that there exists a unique $u \in H^1_{w,0}(\Omega)$ satisfying (2.1). ☐

3. Approximations

Our approach is to approximate the solution of a degenerate equation by a sequence of solutions to non-degenerate equations that satisfy the uniform ellipticity condition.

Let

$$w_\epsilon = \begin{cases} |x|^\alpha, & |x| \ge \epsilon, \\ (\frac{3}{4}|x|^2 + \frac{1}{4}\epsilon^2)^{\frac{\alpha}{2}}, & |x| \le \epsilon. \end{cases} \tag{3.1}$$

Then it is clear that $w_\epsilon \in C^{0,1}(\overline{\Omega})$ since $\alpha \in (0,2)$, w_ϵ is a radial convex function on $\mathbb{R}^N$ and nondecreasing on $[0,\infty)$, and $(\frac{\epsilon}{2})^\alpha \le w_\epsilon \le \epsilon^\alpha$ in B_ϵ, and

$$\nabla w_\epsilon = \begin{cases} \alpha|x|^{\alpha-2}x, & |x| > \epsilon, \\ \alpha\left(\frac{3}{4}|x|^2 + \frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-1}\frac{3}{4}x, & |x| < \epsilon. \end{cases}$$

Remark 3.1. It is worth noting that our approximation method is different from the one used in other literature (see [27–29]), such as setting a non-degenerate coefficient $|x+\epsilon|^\alpha$ to approximate the degenerate coefficient $|x|^\alpha$, which takes the form $|x+\epsilon|^\alpha$ over the entire domain Ω. However, in this paper, our setup of w_ϵ ensures that the approximate coefficients do not depend on ϵ outside B_ϵ while approximating the original degenerate coefficient within B_ϵ. This allows us to achieve better estimates of the solution and obtain improved regularity results, even in the high-dimensional case.

Remark 3.2. In this paper, we assume $\alpha \in (0,2)$ in order to work in a suitable functional framework for well-posedness and for our approximation argument. In fact, it is pointed out in [26] that, when $\alpha > 2$, the associated operator may no longer have a discrete spectrum.

In the proof of Lemma 2.1 (the Hardy-type inequality) we only need $\alpha > 0$. However, the condition $\alpha < 2$ is essential in Lemma 2.2 to establish the compact embedding $H^1_{w,0}(\Omega) \subset\subset L^2(\Omega)$. This compactness is then used in the proof of Lemma 3.5 to extract a strongly convergent subsequence of the approximating solutions. Therefore, the restriction $\alpha \in (0,2)$ is natural for our approach to the approximation scheme and to the subsequent unique continuation analysis.

For each $k \in \mathbb{N}$, we denote $w_{\frac{1}{k}}$ by w_k and consider the following approximate equation

$$\begin{cases} -\mathrm{div}(w_k\nabla u_k) = f_k, & \text{in } \Omega, \\ u_k = 0, & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

with $f_k \in L^2(\Omega; w_k^{-1})$. We say that $u_k \in H^1_{w_k,0}(\Omega)$ is a *weak solution* of (3.2), if

$$\int_\Omega (\nabla u_k \cdot \nabla v)w_k\mathrm{d}x = \int_\Omega f_k v\mathrm{d}x$$

for all $v \in H^1_{w_k,0}(\Omega)$.

We note that $H^1_{w_k,0}(\Omega) = H^1_0(\Omega)$ since $(\frac{1}{k})^\alpha \le w_k \le m^\alpha$ ($m := \sup_{x\in\Omega}|x| + 1$ defined in Lem. 2.1) for each $k \in \mathbb{N}$, where $H^1_0(\Omega)$ is the classical Sobolev space.

As Lemma 2.1, we provide a proof of the Hardy inequality for the non-degenerate equation.

Lemma 3.3. *Let $u \in H^1_{w_k,0}(\Omega)$. Then*

$$(N+\alpha-2)\|w_k^{\frac{1}{2}-\frac{1}{\alpha}}u\|_{L^2(\Omega)} \le 2\|u\|_{H^1_{w_k,0}(\Omega)}. \tag{3.3}$$

Moreover, we have

$$\|u\|_{L^2(\Omega;w_k)} \le \frac{2m}{N+\alpha-2}\||\nabla u|\|_{L^2(\Omega;w_k)}. \tag{3.4}$$

Proof. Denote $\epsilon = \frac{1}{k}$. We shall prove

$$(N+\alpha-2)\|u\|_{L^2(\Omega;w_\epsilon)} \le 2\|u\|_{H^1_{w_\epsilon,0}(\Omega)}$$

for each $u \in H^1_{w_\epsilon,0}(\Omega)$.

Since $w_\epsilon \in C^{0,1}(\overline{\Omega})$ (*i.e.*, $w_\epsilon \in W^{1,\infty}(\Omega)$), we have

$$\begin{aligned}
&2\int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}u(x\cdot\nabla u)\mathrm{d}x = \int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}x\cdot\nabla u^2\mathrm{d}x\\
&= \int_\Omega \operatorname{div}\left(w_\epsilon^{1-\frac{2}{\alpha}}u^2x\right)\mathrm{d}x - \int_\Omega u^2\operatorname{div}\left(w_\epsilon^{1-\frac{2}{\alpha}}x\right)\mathrm{d}x\\
&= -(N+\alpha-2)\int_\Omega u^2w_\epsilon^{1-\frac{2}{\alpha}}\mathrm{d}x - \frac{2-\alpha}{4}\epsilon^2\int_{B_\epsilon}\left(\frac{3}{4}|x|^2+\frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-2}u^2\mathrm{d}x.
\end{aligned}$$

Then

$$\begin{aligned}
&(N+\alpha-2)\int_\Omega u^2w_\epsilon^{1-\frac{2}{\alpha}}\mathrm{d}x\\
&\le -2\int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}u(x\cdot\nabla u)\mathrm{d}x = -2\int_\Omega\left(w_\epsilon^{\frac{1}{2}-\frac{1}{\alpha}}u\right)\left(w_\epsilon^{\frac{1}{2}-\frac{1}{\alpha}}x\cdot\nabla u\right)\mathrm{d}x\\
&\le 2\left(\int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}u^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}|x|^2|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le 2\left(\int_\Omega w_\epsilon^{1-\frac{2}{\alpha}}u^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u|^2w_\epsilon\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}$$

by $\alpha\in(0,2)$ and $|x|^2 \le \frac{3}{4}|x|^2+\frac{1}{4}\epsilon^2$ on B_ϵ. This shows that

$$(N+\alpha-2)\|w_\epsilon^{\frac{1}{2}-\frac{1}{\alpha}}u\|_{L^2(\Omega)} \le 2\||\nabla u|\|_{L^2(\Omega;w_\epsilon)}.$$

Finally, by (3.3) and $\frac{1}{4}\epsilon^2 \le \frac{3}{4}|x|^2+\frac{1}{4}\epsilon^2 \le m^2$ in Ω, we get (3.4). □

To prove that the solution of the non-degenerate equation converges weakly in the solution space to the solution of the degenerate equation, we first show that the approximate solutions are bounded.

Lemma 3.4. *Let u_k be a solution of (3.2) with $f_k \in L^2(\Omega; w_k^{-1})$. For each $k \in \mathbb{N}$, then*

$$\|u_k\|_{H^1_{w_k,0}(\Omega)} \le C\|f_k\|_{L^2(\Omega;w_k^{-1})},$$

where the constant $C > 0$ depends only on α, N and Ω.

Proof. Let $u_k \in H^1_{w_k,0}(\Omega)$ be the test function. Then

$$\int_\Omega |\nabla u_k|^2 w_k \mathrm{d}x \le \left(\int_\Omega |f_k w_k^{-1}|^2 w_k \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u_k^2 w_k \mathrm{d}x\right)^{\frac{1}{2}}.$$

By (3.4) in Lemma 3.3, we get

$$\|u_k\|_{H^1_{w_k,0}(\Omega)} \le C\|f_k\|_{L^2(\Omega;w_k^{-1})},$$

where the constant $C > 0$ depends only on α, N and Ω. □

Now, we are ready to show the existence of a solution to the degenerate equation (2.1) by using the approximation *via* the solutions of the non-degenerate equations.

Lemma 3.5. *Let $u_k \in H^1_{w_k,0}(\Omega)$ be the solution of (3.2) with $f_k = f$, where $f \in L^2(\Omega; w^{-1})$ and $k \in \mathbb{N}$. Then, there is a $u_0 \in H^1_{w,0}(\Omega)$ such that*

$$u_k \rightharpoonup u_0 \text{ weakly in } H^1_{w,0}(\Omega), \tag{3.5}$$

and

$$u_k \to u_0 \text{ strongly in } L^2(\Omega). \tag{3.6}$$

Moreover, u_0 is the unique solution of (2.1) with $f \in L^2(\Omega; w^{-1})$.

Proof. Since $f \in L^2(\Omega; w^{-1})$, we have

$$\int_\Omega (f w_k^{-1})^2 w_k \mathrm{d}x = \int_\Omega (f w^{-1})^2 (w w_k^{-1}) w \mathrm{d}x \le \int_\Omega (f w^{-1})^2 w \mathrm{d}x$$

according to $w \le w_k$ for all $k \in \mathbb{N}$. *i.e.*, $f \in L^2(\Omega; w_k^{-1})$ for all $k \in \mathbb{N}$. From Lemma 3.4, for each $k \in \mathbb{N}$, since $w \le w_k$ for all $k \in \mathbb{N}$, we have

$$\|u_k\|_{H^1_{w,0}(\Omega)} \le \|u_k\|_{H^1_{w_k,0}(\Omega)} \le C\|f\|_{L^2(\Omega;w_k^{-1})} \le C\|f\|_{L^2(\Omega;w^{-1})},$$

where the constant $C > 0$ depends only on α, N and Ω. Then there exists a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, still denote by itself, and $\widetilde{u}_0 \in H^1_{w,0}(\Omega)$, such that

$$u_k \rightharpoonup \widetilde{u}_0 \text{ weakly in } H^1_{w,0}(\Omega),$$

and

$$u_k \to \widetilde{u}_0 \text{ strongly in } L^2(\Omega).$$

by $H^1_{w,0}(\Omega) \subset\subset L^2(\Omega)$ is compact (see Lem. 2.2).

Now, we prove $\widetilde{u}_0$ satisfies the equation (2.1) with $f \in L^2(\Omega; w^{-1})$.
Let $\psi \in \mathcal{D}(\Omega)$. Let $\epsilon > 0$. By (3.5), there exists $k_0 \in \mathbb{N}$, such that

$$\left| \int_\Omega (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x - \int_\Omega (\nabla \widetilde{u}_0 \cdot \nabla \psi) w \mathrm{d}x \right| < \frac{1}{2}\epsilon \text{ when } k \geq k_0. \tag{3.7}$$

Since u_k is a solution of (3.2) for each $k \in \mathbb{N}$, we have

$$\int_\Omega (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x = \int_\Omega f \psi \mathrm{d}x. \tag{3.8}$$

Note that $w_k = w$ on $\Omega \setminus B_{\frac{1}{k}}$ for $k \geq k_0$, we have

$$\int_\Omega (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x = \int_{\Omega \setminus B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x + \int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x. \tag{3.9}$$

By Lemma 3.4 we have

$$\begin{aligned} \left| \int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x \right| &\leq \left(\int_{B_{\frac{1}{k}}} |\nabla u_k|^2 w_k \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_{\frac{1}{k}}} |\nabla \psi|^2 w_k \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \left(\sup_{x \in \Omega} |\nabla \psi(x)| \right) \left(\int_\Omega |\nabla u_k|^2 w_k \mathrm{d}x \right)^{\frac{1}{2}} w_k(B_{\frac{1}{k}})^{\frac{1}{2}} \\ &\leq C_{\psi,N,\alpha} \|f\|_{L^2(\Omega; w^{-1})} \frac{1}{k^{\frac{\alpha+N}{2}}}, \end{aligned} \tag{3.10}$$

where $C_{\psi,N,\alpha} > 0$ is a constant that depends only on ψ, Ω, N and α, and $w_k(B_{\frac{1}{k}})^{\frac{1}{2}} = \left(\int_{B_{\frac{1}{k}}} w_k \mathrm{d}x \right)^{\frac{1}{2}}$. Hence we can assume

$$\left| \int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x \right| < \frac{1}{4}\epsilon \text{ when } k \geq k_0\,. \tag{3.11}$$

On the other hand, by $w \leq w_k$, and by the same argument as for (3.10), we have

$$\begin{aligned} \left| \int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x \right| &\leq \int_{B_{\frac{1}{k}}} |\nabla u_k| |\nabla \psi| w_k \mathrm{d}x \\ &\leq \left(\int_{B_{\frac{1}{k}}} |\nabla u_k|^2 w_k \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_{\frac{1}{k}}} |\nabla \psi|^2 w_k \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq C_{\psi,N,\alpha} \|f\|_{L^2(\Omega; w^{-1})} \frac{1}{k^{\frac{\alpha+N}{2}}}, \end{aligned}$$

hence we also can assume

$$\left|\int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x\right| < \frac{1}{4}\epsilon \text{ when } k \geq k_0. \tag{3.12}$$

From (3.7), (3.8), (3.9), (3.11) and (3.12), we obtain

$$\begin{aligned}
&\left|\int_\Omega (\nabla \widetilde{u}_0 \cdot \nabla \psi) w \mathrm{d}x - \int_\Omega f\psi \mathrm{d}x\right| \\
\leq &\left|\int_\Omega (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x - \int_\Omega (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x\right| \\
&+ \left|\int_\Omega (\nabla \widetilde{u}_0 \cdot \nabla \psi) w \mathrm{d}x - \int_\Omega (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x\right| \\
&+ \left|\int_\Omega (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x - \int_\Omega f\psi \mathrm{d}x\right| \\
\leq &\left|\int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w_k \mathrm{d}x\right| + \left|\int_{B_{\frac{1}{k}}} (\nabla u_k \cdot \nabla \psi) w \mathrm{d}x\right| + \frac{1}{2}\epsilon < \epsilon.
\end{aligned}$$

This implies that

$$\int_\Omega (\nabla \widetilde{u}_0 \cdot \nabla \psi) w \mathrm{d}x = \int_\Omega f\psi \mathrm{d}x.$$

This proves that $\widetilde{u}_0$ is a solution of (2.1).

Finally, by the uniqueness of the solution of the equation (2.1), we get $\widetilde{u}_0 = u_0$. This completes the proof of the lemma. □

Next, we transform the inhomogeneous problem (2.1) into a boundary value problem to facilitate the subsequent proof of the three sphere inequality, and the proof of WUCP.

Corollary 3.6. *Let $u_0 \in H^1_w(\Omega)$ be a solution of the equation*

$$\begin{cases} -\mathrm{div}(w\nabla u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $g \in H^{\frac{3}{2}}(\partial\Omega)$ is a given function. Let $u_k \in H^1_{w_k}(\Omega)$ be a solution of the following equation

$$\begin{cases} -\mathrm{div}(w_k\nabla u_k) = 0, & \text{in } \Omega, \\ u_k = g, & \text{on } \partial\Omega. \end{cases}$$

Then we have

$$u_k \to u_0 \text{ strongly in } L^2(\Omega).$$

Proof. We first prove that (3.13) admits a solution. Let $v \in H^2(\Omega)$ be a solution of the following equation

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega, \\ v = g, & \text{on } \partial\Omega\,. \end{cases}$$

Let $R_0 > 0$ with $B_{3R_0} \subseteq \Omega$ and take the cut-off function $\zeta \in C_0^\infty(\Omega)$ such that

$$0 \le \zeta \le 1, \quad \zeta = 0 \text{ on } B_{R_0}, \quad \zeta = 1 \text{ on } \Omega \setminus B_{2R_0}, \quad |\nabla\zeta| \le \frac{C}{R_0},$$

where $C > 0$ is a generic constant. Set $v_0 = \zeta v$. Since $\operatorname{supp} v_0 \subset \Omega \setminus B_{R_0}$, the weight $w(x) = |x|^\alpha$ and its reciprocal w^{-1} are bounded on $\operatorname{supp} v_0$. In particular, $\operatorname{div}(w\nabla v_0) \in L^2(\Omega) \subset L^2(\Omega; w^{-1})$. Therefore, by Lemma 2.3 there exists a unique $\widetilde{u}_0 \in H^1_{w,0}(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(w\nabla\widetilde{u}_0) = \operatorname{div}(w\nabla v_0), & \text{in } \Omega, \\ \widetilde{u}_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

We then define $u_0 := \widetilde{u}_0 + v_0$. By construction, $u_0 \in H^1_w(\Omega)$, $u_0 = g$ on $\partial\Omega$, and

$$-\operatorname{div}(w\nabla u_0) = -\operatorname{div}(w\nabla\widetilde{u}_0) - \operatorname{div}(w\nabla v_0) = 0 \quad \text{in } \Omega,$$

so u_0 is a weak solution of (3.13).

By Lemma 3.5, there exists a sequence $u_{k,0} \in H^1_{w_k,0}(\Omega)$ that satisfies

$$\begin{cases} -\operatorname{div}(w_k\nabla u_{k,0}) = \operatorname{div}(w\nabla v_0), & \text{in } \Omega, \\ u_{k,0} = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{aligned} &u_{k,0} \rightharpoonup \widetilde{u}_0 \text{ weakly in } H^1_{w,0}(\Omega), \\ &u_{k,0} \to \widetilde{u}_0 \text{ strongly in } L^2(\Omega). \end{aligned}$$

Denote $u_k = u_{k,0} + v_0$. Notice that $v_0 = 0$ on B_{R_0} and $w = w_k$ for $k > \frac{1}{R_0}$, then u_k is the solution of the following system

$$\begin{cases} -\operatorname{div}(w_k\nabla u_k) = 0, & \text{in } \Omega, \\ u_k = g, & \text{on } \partial\Omega, \end{cases}$$

and

$$u_k \to u_0 \text{ strongly in } L^2(\Omega).$$

This completes the proof of Corollary 3.6. □

Before proving the three sphere inequality, we present some preliminary results. The proof of the degenerate three sphere inequality is more complex than that of the standard one, and the following corollary will play a crucial role in establishing the degenerate version.

Corollary 3.7. *Let u_0, u_k $(k \in \mathbb{N})$ be defined as in Lemma 3.5 or Corollary 3.6. Then for any $\eta > 0$ with $B_\eta \subseteq \Omega$, we have*

$$\int_{B_\eta} w_k u_k^2 \mathrm{d}x \to \int_{B_\eta} w u_0^2 \mathrm{d}x \text{ as } k \to \infty, \tag{3.14}$$

and

$$R_k := \frac{\alpha}{2k^2} \int_{B_{\frac{1}{k}}} \left(\frac{3}{4}|y|^2 + \frac{1}{4k^2}\right)^{\frac{\alpha}{2}-1} u_k^2 \mathrm{d}y \overset{k\to\infty}{\longrightarrow} 0. \tag{3.15}$$

Proof. Since $\eta > 0$, let $k \in \mathbb{N}$ be sufficiently large such that $\frac{1}{k} < \frac{1}{2}\eta$. Since $w = w_k$ in $B_\eta \setminus B_{\frac{1}{k}}$, by Lemma 3.5 and the uniform continuity property, we see that

$$\begin{aligned}
&\left|\int_{B_\eta} w_k u_k^2 \mathrm{d}x - \int_{B_\eta} w u_0^2 \mathrm{d}x\right| \\
&\le \left|\int_{B_\eta} w_k u_k^2 \mathrm{d}x - \int_{B_\eta} w u_k^2 \mathrm{d}x\right| + \left|\int_{B_\eta} w u_k^2 \mathrm{d}x - \int_{B_\eta} w u_0^2 \mathrm{d}x\right| \\
&\le \max_{|x|\le\frac{1}{k}} \left[\left(\frac{3}{4}|x|^2 + \frac{1}{4k^2}\right)^{\frac{\alpha}{2}} - |x|^\alpha\right] \|u_k\|^2_{L^2(B_\eta)} + \left|\int_{B_\eta} w u_k^2 \mathrm{d}x - \int_{B_\eta} w u_0^2 \mathrm{d}x\right| \\
&\le C \max_{|x|\le\frac{1}{k}} \left[\left(\frac{3}{4}|x|^2 + \frac{1}{4k^2}\right)^{\frac{\alpha}{2}} - |x|^\alpha\right] + \left|\int_{B_\eta} w u_k^2 \mathrm{d}x - \int_{B_\eta} w u_0^2 \mathrm{d}x\right| \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This proves the (3.14). Now, we are going to prove (3.15).
Note that $\frac{1}{4k^2} \le \frac{3}{4}|y|^2 + \frac{1}{4k^2} \le \frac{1}{k^2}$ in $B_{\frac{1}{k}}$, we have

$$\frac{\alpha}{2}\int_{B_{\frac{1}{k}}} w_k u_k^2 \mathrm{d}x \le R_k \le 2\alpha \int_{B_{\frac{1}{k}}} w_k u_k^2 \mathrm{d}x,$$

so, we only need to show that

$$\int_{B_{\frac{1}{k}}} w_k u_k^2 \mathrm{d}x \overset{k\to\infty}{\longrightarrow} 0.$$

Indeed, by (3.6), we have

$$\begin{aligned}
\int_{B_{\frac{1}{k}}} w_k u_k^2 \mathrm{d}x &\le \left|\int_{B_{\frac{1}{k}}} w_k u_k^2 \mathrm{d}x - \int_{B_{\frac{1}{k}}} w u_0^2 \mathrm{d}x\right| + \int_{B_{\frac{1}{k}}} w u_0^2 \mathrm{d}x \\
&\le \left|\int_{B_{\frac{1}{k}}} (w_k u_k^2 - w_k u_0^2) \mathrm{d}x\right| + \left|\int_{B_{\frac{1}{k}}} (w_k - w) u_0^2 \mathrm{d}x\right| + \int_{B_{\frac{1}{k}}} w u_0^2 \mathrm{d}x \\
&\le \frac{1}{k^\alpha}\left|\int_{B_{\frac{1}{k}}} u_k^2 \mathrm{d}x - \int_{B_{\frac{1}{k}}} u_0^2 \mathrm{d}x\right| + 2m^\alpha \int_{B_{\frac{1}{k}}} u_0^2 \mathrm{d}x + \int_{B_{\frac{1}{k}}} w u_0^2 \mathrm{d}x \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

□

4. Three sphere inequality

We now proceed to prove the degenerate three-sphere inequality, again by means of an approximation argument. We emphasize that the approximation results in Section 3 were formulated for the inhomogeneous boundary value problem (2.1), in order to establish well-posedness for the operator $-\mathrm{div}(w\nabla\cdot)$. From this section on, however, our interest lies in quantitative estimates and weak unique continuation properties for the operator itself. Accordingly, we only consider weak solutions of the homogeneous equation

$$-\mathrm{div}(w\nabla v) = 0 \quad \text{in } \Omega. \tag{4.1}$$

First, we introduce some notation. For $r > 0$ we set $B_r := B(0, r)$ and fix $0 < \varepsilon < r$. Following the standard notation in the study of quantitative unique continuation (see, for instance, [9, 24]), for such a solution v of (4.1) we define the associated height and energy functionals by

$$H(r) = \int_{B_r} w|v(y)|^2 \mathrm{d}y, \quad D(r) = \int_{B_r} w|\nabla v(y)|^2 (r^2 - |y|^2)\mathrm{d}y, \tag{4.2}$$

and

$$H_\epsilon(r) = \int_{B_r} w_\epsilon |v(y)|^2 \mathrm{d}y, \quad D_\epsilon(r) = \int_{B_r} w_\epsilon |\nabla v(y)|^2 (r^2 - |y|^2)\mathrm{d}y,$$

where w_ϵ defined in (3.1). Let

$$\Phi_\epsilon(r) = \begin{cases} \frac{D_\epsilon(r)}{H_\epsilon(r)}, & \text{if } H_\epsilon \neq 0, \\ 0, & \text{if } H_\epsilon = 0. \end{cases}$$

We first prove the following three sphere inequality for the uniformly elliptic operator with ϵ, which contains additional terms $\frac{1}{2}\int_{r_1}^{r_2} \frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\mathrm{d}r$ and $\frac{1}{2}\int_{r_2}^{r_3} \frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\mathrm{d}r$, compared to the standard form of the three sphere inequality.

Lemma 4.1. *Let $R_0 > 0$ with $B_{2R_0} \subseteq \Omega$, and let $\epsilon \in (0, \frac{1}{2}R_0)$ be small enough. Let v be a solution of $\mathrm{div}(w_\epsilon \nabla v) = 0$ in B_{R_0}. Then for any $0 < 2\epsilon < r_1 < r_2 < r_3 < R_0$, we have*

$$\begin{aligned} &\frac{1}{r_1^{-\alpha} - r_2^{-\alpha}} \left(\log \frac{H_\epsilon(r_2)}{H_\epsilon(r_1)} + \frac{1}{2}\int_{r_1}^{r_2} \frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\mathrm{d}r \right) \\ \leq\ &\frac{1}{r_2^{-\alpha} - r_3^{-\alpha}} \left(\log \frac{H_\epsilon(r_3)}{H_\epsilon(r_2)} + \frac{1}{2}\int_{r_2}^{r_3} \frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\mathrm{d}r \right), \end{aligned} \tag{4.3}$$

where

$$R_\epsilon = \frac{\alpha}{2}\epsilon^2 \int_{B_\epsilon} \left(\frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2 \right)^{\frac{\alpha}{2}-1} v^2 \mathrm{d}y.$$

Proof. It is obvious that $v \in H^2(B_{R_0})$ since $(\frac{\epsilon}{2})^\alpha \leq w_\epsilon$ on Ω, *i.e.*, $-\mathrm{div}(w_\epsilon \nabla v) = 0$ in B_{R_0} is indeed a uniformly elliptic equation. We divide our proof into the following steps.
Step 1. We compute $H'_\epsilon(r)$ and $D_\epsilon(r)$. It is clear that

$$H'_\epsilon(r) = \int_{\partial B_r} w_\epsilon |v(y)|^2 \mathrm{d}\sigma(y). \tag{4.4}$$

By

$$\int_{B_r} \operatorname{div}\big(w_\epsilon(\nabla v)v(r^2-|y|^2)\big)\mathrm{d}y = \int_{\partial B_r} w_\epsilon(\nabla v\cdot\nu)v(r^2-|y|^2)\mathrm{d}\sigma(y) = 0,$$

and

$$\begin{aligned}&\int_{B_r} \operatorname{div}\big(w_\epsilon(\nabla v)v(r^2-|y|^2)\big)\mathrm{d}y\\ =&\int_{B_r}\operatorname{div}(w_\epsilon\nabla v)v(r^2-|y|^2)\mathrm{d}y+\int_{B_r} w_\epsilon|\nabla v|^2(r^2-|y|^2)\mathrm{d}y\\ &-2\int_{B_r} w_\epsilon v\nabla v\cdot y\mathrm{d}y\end{aligned}$$

we get

$$D_\epsilon(r) = 2\int_{B_r} w_\epsilon v\nabla v\cdot y\mathrm{d}y$$

by $\operatorname{div}(w_\epsilon\nabla v)=0$.
Step 2. We compute $\frac{H'_\epsilon(r)}{H_\epsilon(r)}$. Set

$$G(y) = r^2-|y|^2,$$

then

$$G|_{\partial B_r}=0,\quad \nabla G=-2y,\quad \frac{\partial G}{\partial\nu}\bigg|_{\partial B_r}=-2r.$$

We compute

$$\begin{aligned}&\int_{B_r}\operatorname{div}(w_\epsilon\nabla v^2)G\mathrm{d}y\\ =&\int_{B_r}\operatorname{div}(w_\epsilon G\nabla v^2)\mathrm{d}y-\int_{B_r} w_\epsilon\nabla v^2\cdot\nabla G\mathrm{d}y\\ =&\int_{\partial B_r} w_\epsilon G\nabla v^2\cdot\nu\mathrm{d}\sigma(y)-\int_{B_r} w_\epsilon\nabla v^2\cdot\nabla G\mathrm{d}y\\ =&-\int_{B_r} w_\epsilon\nabla G\cdot\nabla v^2\mathrm{d}y=-\int_{B_r}\operatorname{div}(w_\epsilon v^2\nabla G)\mathrm{d}y+\int_{B_r} v^2\operatorname{div}(w_\epsilon\nabla G)\mathrm{d}y\\ =&\,2r\int_{\partial B_r} w_\epsilon v^2\mathrm{d}\sigma(y)-2(N+\alpha)\int_{B_r} w_\epsilon v^2\mathrm{d}y\\ &+\frac{\alpha}{2}\epsilon^2\int_{B_\epsilon}\left(\frac{3}{4}|y|^2+\frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-1}v^2\mathrm{d}y.\end{aligned}$$

Since

$$\int_{B_r} \mathrm{div}(w_\epsilon \nabla v^2) G \mathrm{d}y = 2\int_{B_r} \mathrm{div}(w_\epsilon v \nabla v) G \mathrm{d}x$$
$$= 2\int_{B_r} w_\epsilon |\nabla v|^2 G \mathrm{d}x + 2\int_{B_r} v \mathrm{div}(w_\epsilon \nabla v) G \mathrm{d}x = 2D_\epsilon(r)$$

by (4.4), we have

$$H_\epsilon'(r) = \frac{N+\alpha}{r} H_\epsilon(r) + \frac{1}{r} D_\epsilon(r) - \frac{1}{2r} R_\epsilon,$$

where

$$R_\epsilon = \frac{\alpha}{2}\epsilon^2 \int_{B_\epsilon} \left(\frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-1} v^2 \mathrm{d}y.$$

This implies

$$\frac{H_\epsilon'(r)}{H_\epsilon(r)} = \frac{N+\alpha}{r} + \frac{1}{r}\frac{D_\epsilon(r)}{H_\epsilon(r)} - \frac{1}{2r}\frac{R_\epsilon}{H_\epsilon(r)}. \tag{4.5}$$

Step 3. We compute $D_\epsilon'(r)$. Now,

$$D_\epsilon'(r) = 2r\int_{B_r} w_\epsilon |\nabla v(y)|^2 \mathrm{d}y.$$

On one hand, we have

$$\int_{B_r} \mathrm{div}\big[w_\epsilon |\nabla v(y)|^2 (r^2 - |y|^2) y\big] \mathrm{d}y$$
$$= \int_{\partial B_r} w_\epsilon |\nabla v(y)|^2 \left(r^2 - |y|^2\right) y \cdot \nu \mathrm{d}\sigma(y) = 0.$$

On the other hand, we have

$$\int_{B_r} \mathrm{div}\big[w_\epsilon |\nabla v(y)|^2 (r^2 - |y|^2) y\big] \mathrm{d}y$$
$$= (N+\alpha)\int_{B_r} w_\epsilon |\nabla v(y)|^2 (r^2 - |y|^2) \mathrm{d}y$$
$$- \frac{\alpha}{4}\epsilon^2 \int_{B_\epsilon} \left(\frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-1} |\nabla v(y)|^2 (r^2 - |y|^2) \mathrm{d}y$$
$$- 2\int_{B_r} w_\epsilon |y|^2 |\nabla v(y)|^2 \mathrm{d}y + \int_{B_r} w_\epsilon (y \cdot \nabla |\nabla v(y)|^2)(r^2 - |y|^2) \mathrm{d}y.$$

Now, by the equation $\nabla v \cdot \nabla(y \cdot \nabla v) = |\nabla v|^2 + \frac{1}{2} y \cdot \nabla |\nabla v|^2$ and $\operatorname{div}(w_\epsilon \nabla v) = 0$ and $\operatorname{div}(w_\epsilon \nabla v) = 0$, we get

$$\begin{aligned}
&\int_{B_r} w_\epsilon y \cdot \nabla |\nabla v(y)|^2 (r^2 - |y|^2) \mathrm{d}y \\
= &-2 \int_{B_r} w_\epsilon |\nabla v|^2 (r^2 - |y|^2) \mathrm{d}y + 2 \int_{B_r} w_\epsilon \nabla v \cdot \nabla (y \cdot \nabla v)(r^2 - |y|^2) \mathrm{d}y \\
= &-2 D_\epsilon(r) + 2 \int_{B_r} \operatorname{div} \left[w_\epsilon (r^2 - |y|^2)(y \cdot \nabla v) \nabla v \right] \mathrm{d}x \\
&- 2 \int_{B_r} (y \cdot \nabla v) \operatorname{div} \left[w_\epsilon (r^2 - |y|^2) \nabla v \right] \mathrm{d}y \\
= &-2 D_\epsilon(r) - 2 \int_{B_r} (y \cdot \nabla v) \operatorname{div}(w_\epsilon \nabla v)(r^2 - |y|^2) \mathrm{d}y + 4 \int_{B_r} w_\epsilon (y \cdot \nabla v)^2 \mathrm{d}y \\
= &-2 D_\epsilon(r) + 4 \int_{B_r} w_\epsilon (y \cdot \nabla v)^2 \mathrm{d}y.
\end{aligned}$$

Then

$$(N + \alpha - 2) D_\epsilon(r) - 2 \int_{B_r} w_\epsilon |y|^2 |\nabla v(y)|^2 \mathrm{d}y + 4 \int_{B_r} w_\epsilon (y \cdot \nabla v)^2 \mathrm{d}y - \widetilde{R}_\epsilon = 0,$$

where

$$\widetilde{R}_\epsilon = \frac{\alpha}{4} \epsilon^2 \int_{B_\epsilon} \left(\frac{3}{4} |y|^2 + \frac{1}{4} \epsilon^2 \right)^{\frac{\alpha}{2} - 1} |\nabla v(y)|^2 (r^2 - |y|^2) \mathrm{d}y.$$

Since

$$D_\epsilon(r) = r^2 \int_{B_r} w_\epsilon |\nabla v|^2 \mathrm{d}y - \int_{B_r} w_\epsilon |y|^2 |\nabla v|^2 \mathrm{d}y,$$

we obtain that

$$-\int_{B_r} w_\epsilon |y|^2 |\nabla v|^2 \mathrm{d}y = D_\epsilon(r) - r^2 \int_{B_r} w_\epsilon |\nabla v|^2 \mathrm{d}y,$$

and

$$(N + \alpha) D_\epsilon(r) = 2 r^2 \int_{B_r} w_\epsilon |\nabla v|^2 \mathrm{d}y - 4 \int_{B_r} w_\epsilon (y \cdot \nabla v)^2 \mathrm{d}y + \widetilde{R}_\epsilon.$$

Hence,

$$D'_\epsilon(r) = \frac{N + \alpha}{r} D_\epsilon(r) + \frac{4}{r} \int_{B_r} w_\epsilon (y \cdot \nabla v)^2 \mathrm{d}y - \frac{1}{r} \widetilde{R}_\epsilon.$$

Step 4. We prove $r^\alpha \Phi_\epsilon(r)$ is a nondecreasing function for $r>0$. Note that $R_\epsilon \geq 0, \widetilde{R}_\epsilon \geq 0$, we obtain

$$\begin{aligned} & H_\epsilon^2(r)\Phi_\epsilon'(r) = D_\epsilon'(r)H_\epsilon(r) - D_\epsilon(r)H_\epsilon'(r) \\ = & \frac{4}{r}\left[\left(\int_{B_r} w_\epsilon (y\cdot\nabla v)^2 \mathrm{d}x\right)\left(\int_{B_r} w_\epsilon v^2 \mathrm{d}x\right) - \left(\int_{B_r} w_\epsilon v y\cdot\nabla v \mathrm{d}x\right)^2\right] \\ & -\frac{1}{r}\widetilde{R}_\epsilon H_\epsilon(r) + \frac{1}{2r}D_\epsilon(r)R_\epsilon \\ \geq & -\frac{1}{r}\widetilde{R}_\epsilon H_\epsilon(r). \end{aligned}$$

Since $\frac{\epsilon^2}{4} \leq \frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2 \leq \epsilon^2$, we have

$$\begin{aligned} \widetilde{R}_\epsilon &= \frac{\alpha}{4}\epsilon^2 \int_{B_\epsilon}\left(\frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}-1} |\nabla v(y)|^2 (r^2 - |y|^2)\mathrm{d}y \\ &\leq \alpha \int_{B_\epsilon}\left(\frac{3}{4}|y|^2 + \frac{1}{4}\epsilon^2\right)^{\frac{\alpha}{2}} |\nabla v(y)|^2 (r^2 - |y|^2)\mathrm{d}y \leq \alpha D_\epsilon(r), \end{aligned}$$

which implies that

$$\Phi_\epsilon'(r) \geq -\frac{\alpha}{r}\Phi_\epsilon(r).$$

Hence

$$r^\alpha \Phi_\epsilon(r) \text{ is a nondecreasing function for } r>0. \tag{4.6}$$

Step 5. Conclusion of the proof. Again, by (4.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H_\epsilon(r) = \frac{H_\epsilon'(r)}{H_\epsilon(r)} = \frac{1}{r}\left((N+\alpha) + \Phi_\epsilon(r) - \frac{1}{2}\frac{R_\epsilon}{H_\epsilon(r)}\right).$$

Then, for $0<r_1<r_2$, we have

$$\begin{aligned} \log\frac{H_\epsilon(r_2)}{H_\epsilon(r_1)} &= \int_{r_1}^{r_2} \frac{1}{r}\left((N+\alpha) + \Phi_\epsilon(r) - \frac{1}{2}\frac{R_\epsilon}{H_\epsilon(r)}\right)\mathrm{d}r \\ &= \int_{r_1}^{r_2} \frac{1}{r^{1+\alpha}}\left(r^\alpha(N+\alpha) + r^\alpha\Phi_\epsilon(r) - \frac{1}{2}\frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\right)\mathrm{d}r, \end{aligned}$$

and hence, by (4.5), we obtain

$$\log\frac{H_\epsilon(r_2)}{H_\epsilon(r_1)} + \frac{1}{2}\int_{r_1}^{r_2}\frac{R_\epsilon}{rH_\epsilon(r)}\mathrm{d}r \leq \alpha^{-1}(r_1^{-\alpha} - r_2^{-\alpha})\big(r_2^\alpha(N+\alpha) + r_2^\alpha\Phi_\epsilon(r_2)\big). \tag{4.7}$$

We note that the integral $\int_{r_1}^{r_2}\frac{R_\epsilon}{rH_\epsilon(r)}\mathrm{d}x$ is meaningful since $R_\epsilon \leq 2\alpha H_\epsilon(r)$ for all $r>0$ and all $\epsilon>0$. Now, for $r_2<r_3<R_0$, we have

$$\log\frac{H_\epsilon(r_3)}{H_\epsilon(r_2)} = \int_{r_2}^{r_3} \frac{1}{r^{1+\alpha}}\left(r^\alpha(N+\alpha) + r^\alpha\Phi_\epsilon(r) - \frac{1}{2}\frac{r^\alpha R_\epsilon}{H_\epsilon(r)}\right)\mathrm{d}r,$$

and hence, by (4.6), we obtain

$$\log\frac{H_\epsilon(r_3)}{H_\epsilon(r_2)}+\frac{1}{2}\int_{r_2}^{r_3}\frac{R_\epsilon}{rH_\epsilon(r)}\mathrm{d}r\geq\alpha^{-1}(r_2^{-\alpha}-r_3^{-\alpha})\big(r_2^\alpha(N+\alpha)+r_2^\alpha\Phi_\epsilon(r_2)\big). \tag{4.8}$$

Combining (4.7) and (4.8), for all $\epsilon>0$, we get

$$\begin{aligned}&\frac{1}{r_1^{-\alpha}-r_2^{-\alpha}}\left(\log\frac{H_\epsilon(r_2)}{H_\epsilon(r_1)}+\frac{1}{2}\int_{r_1}^{r_2}\frac{R_\epsilon}{rH_\epsilon(r)}\mathrm{d}r\right)\\ \leq&\frac{1}{r_2^{-\alpha}-r_3^{-\alpha}}\left(\log\frac{H_\epsilon(r_3)}{H_\epsilon(r_2)}+\frac{1}{2}\int_{r_2}^{r_3}\frac{R_\epsilon}{rH_\epsilon(r)}\mathrm{d}r\right).\end{aligned}$$

This completes the proof of the lemma. □

By a limiting argument, we obtain the following degenerate three sphere inequality.

Theorem 4.2. *Let $R_0>0$ with $B_{2R_0}\subseteq\Omega$, and let u_0 be a weak solution of*

$$-\mathrm{div}(w\nabla u_0)=0\quad\text{in }B_{R_0}.$$

For $r>0$ set

$$H(r):=\int_{B_r}w\,|u_0(y)|^2\,\mathrm{d}y.$$

Then, for any $0<r_1<r_2<r_3<R_0$, we have

$$\frac{1}{r_1^{-\alpha}-r_2^{-\alpha}}\log\frac{H(r_2)}{H(r_1)}\;\leq\;\frac{1}{r_2^{-\alpha}-r_3^{-\alpha}}\log\frac{H(r_3)}{H(r_2)}. \tag{4.9}$$

Moreover, there exists $\mu\in(0,1)$ such that

$$H(r_2)\leq[H(r_1)]^\mu[H(r_3)]^{1-\mu}.$$

Proof. Without loss of generality, we assume that $u_0\neq0$ in B_{R_0}. By the standard regularity enhancement method for elliptic equations in [33], we have $u_0\in H^2(B_{R_0}\backslash B_{\frac{R_0}{2}})$ and $u_0\in H^{\frac{3}{2}}(\partial B_{R_0})$, and by Corollary 3.6, there exists $u_k\in H^1_{w_k}(B_{R_0})$ $(k\in\mathbb{N})$ satisfying

$$\begin{cases}-\mathrm{div}(w_k\nabla u_k)=0, & \text{in } B_{R_0},\\ u_k=u_0, & \text{on }\partial B_{R_0},\end{cases}$$

and

$$u_k\to u_0\text{ strongly in }L^2(B_{R_0}). \tag{4.10}$$

By Lemma 4.1, we get (4.3). Replacing ϵ by $\epsilon = \frac{1}{k}$, we see

$$\begin{aligned}&\frac{1}{r_1^{-\alpha}-r_2^{-\alpha}}\left(\log\frac{H_k(r_2)}{H_k(r_1)}+\frac{1}{2}\int_{r_1}^{r_2}\frac{R_k}{rH_k(r)}\mathrm{d}r\right)\\ \leq\ &\frac{1}{r_2^{-\alpha}-r_3^{-\alpha}}\left(\log\frac{H_k(r_3)}{H_k(r_2)}+\frac{1}{2}\int_{r_2}^{r_3}\frac{R_k}{rH_k(r)}\mathrm{d}r\right),\end{aligned}$$

where

$$H_k(r)=\int_{B_r} w_k u_k^2\mathrm{d}x,\quad R_k=\frac{\alpha}{2k^2}\int_{B_{\frac{1}{k}}}\left(\frac{3}{4}|y|^2+\frac{1}{4k^2}\right)^{\frac{\alpha}{2}-1}u_k^2\mathrm{d}y.$$

From (4.10) and Corollary 3.7, letting $k\to\infty$, we get

$$\frac{1}{r_1^{-\alpha}-r_2^{-\alpha}}\log\frac{H(r_2)}{H(r_1)}\leq\frac{1}{r_2^{-\alpha}-r_3^{-\alpha}}\log\frac{H(r_3)}{H(r_2)}.$$

This proves (4.9).

Finally, taking

$$\mu=\frac{r_1^{-\alpha}-r_2^{-\alpha}}{r_1^{-\alpha}-r_3^{-\alpha}}=\left(\frac{r_3}{r_2}\right)^{\alpha}\frac{\left(\frac{r_2}{r_1}\right)^{\alpha}-1}{\left(\frac{r_3}{r_1}\right)^{\alpha}-1}, 1-\mu=\frac{r_2^{-\alpha}-r_3^{-\alpha}}{r_1^{-\alpha}-r_3^{-\alpha}}=\left(\frac{r_1}{r_2}\right)^{\alpha}\frac{1-\left(\frac{r_2}{r_3}\right)^{\alpha}}{1-\left(\frac{r_1}{r_3}\right)^{\alpha}},$$

yields

$$H(r_2)\leq(H(r_1))^{\mu}(H(r_3))^{1-\mu}.$$

This completes the proof of the theorem. □

Below, we present the most standard and commonly used form of the degenerate three sphere inequality.

Corollary 4.3. *Assume the conditions in Theorem 4.2 hold. Then*

$$\int_{B_r}u_0^2w\mathrm{d}x\leq\left(\int_{B_{\frac{r}{2}}}u_0^2w\mathrm{d}x\right)^{\mu}\left(\int_{B_{2r}}u_0^2w\mathrm{d}y\right)^{1-\mu}$$

for $0<r<\frac{R_0}{2}$ *with* $\mu=\frac{4^\alpha-2^\alpha}{4^\alpha-1}$.

Proof. Taking $r_1=\frac{r}{2}, r_2=r, r_3=2r$ in Theorem 4.2 produces the desired conclusion. □

We have already obtained the three sphere inequality at the degenerate point 0. To derive an estimate over the entire domain, we will now present the three sphere inequality at the non-degenerate point. Before doing so, we note the following fundamental fact: although we cannot improve the regularity in the region near the degenerate point 0, on the non-degenerate region $\Omega\setminus B_{2r_0}$ the regularity of the solution can be improved by standard arguments, according to [33], Section 6.3.1. Therefore, we can assume that the solution v of (4.1) belongs to $H^2(\Omega\setminus B_{2r_0})$.

Lemma 4.4. *Let* Γ *be a non-empty open subset of* $\partial\Omega$*. Let* r_0, r_1, r_2, r_3 *be four real numbers such that* $0<r_1<r_0<r_2<r_3<\frac{R_0}{8}$*. Suppose that* $y_0\in\Omega, |y_0|>r_0$ *satisfies the following three conditions:*

i) $B(y_0, r) \cap \Omega$ *is star-shaped with respect to* y_0 *for all* $r \in (0, \frac{R_0}{4})$,
ii) $B(y_0, r) \subseteq \Omega$ *for all* $r \in (0, r_0)$,
iii) $B(y_0, r) \cap \partial\Omega \subseteq \Gamma$ *for all* $r \in (r_0, \frac{R_0}{2})$.
If u *is a solution of* (4.1) *with* $u \in H^2(\Omega \setminus B_{2r_0})$ *and* $u = 0$ *on* Γ, *then there exists* $\mu \in (0,1)$ *such that*

$$\int_{B(y_0,r_2)\cap\Omega} u^2 \mathrm{d}x \le C \left(\int_{B(y_0,r_1)} u^2 \mathrm{d}x \right)^{\mu} \left(\int_{B(y_0,r_3)\cap\Omega} u^2 \mathrm{d}x \right)^{1-\mu},$$

and $C > 0$ *is a constant that only depends on* r_0, R_0 *and* N.

Proof. Set $\widehat{\Omega} = \Omega \setminus \overline{B}(0, 2r_0)$. We note that $\mathcal{A}u = 0$ on $\widehat{\Omega}$ is a uniformly elliptic equation since

$$2^\alpha r_0^\alpha |\xi|^2 \le \sum_{i,j=1}^{N} |x|^\alpha \xi_i \xi_j \le \left(\sup_{x \in \widehat{\Omega}} |x|^\alpha \right) |\xi|^2, \ \forall \xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$$

and Ω is a bounded domain. Similar to [1, 14, 15, 17, 18], we obtain Lemma 4.4. □

Corollary 4.5. *Assume the conditions in Lemma 4.4 hold. Then*

$$\int_{B(y_0,r_2)\cap\Omega} u^2 w \, \mathrm{d}x \le C \left(\int_{B(y_0,r_1)} u^2 w \, \mathrm{d}x \right)^{\mu} \left(\int_{B(y_0,r_3)\cap\Omega} u^2 w \, \mathrm{d}x \right)^{1-\mu}. \tag{4.11}$$

Proof. Since $|y_0| > r_0$ and $r_3 < \frac{R_0}{8}$, in our application all balls $B(y_0, r_i)$, $i = 1, 2, 3$, are contained in the nondegenerate region $\Omega \setminus B(0, 2r_0)$. On this region the weight $w(x) = |x|^\alpha$ is uniformly bounded from above and below, hence there exist positive constants c_1, c_2, depending only on r_0, R_0, N, α, such that

$$c_1 \int_B u^2 \, \mathrm{d}x \ \le \ \int_B u^2 w \, \mathrm{d}x \ \le \ c_2 \int_B u^2 \, \mathrm{d}x$$

for $B = B(y_0, r_i)$, $i = 1, 2, 3$. Applying Lemma 4.4 to the unweighted integrals $\int_{B(y_0,r_i)} u^2 \, \mathrm{d}x$ and then using the above equivalence, we obtain (4.11) after absorbing c_1, c_2 into the constant C. □

In the proof of WUCP, we consider two cases: when the degenerate point lies inside ω or outside ω ($0 \notin \omega$). To deal with the latter one, we present a result analogous to the Schauder estimate. Specifically, we estimate the integral over the small ball containing the origin by the integral over an annular region surrounding the ball. In such a way, we control the integral in the degenerate region by the integral in the non-degenerate region. Similar Schauder estimates in the context of degenerate equations can be found in [34–36].

Theorem 4.6. *Let* $R_0 > 0$ *with* $B_{2R_0} \subseteq \Omega$. *Let* u *be a solution to* $\operatorname{div}(w\nabla u) = 0$ *in* B_{R_0}. *Then there exists* $C > 0$ *that is independent of* r $(r < R_0)$ *and* u, *such that*

$$\int_{B_{\frac{r}{2}}} u^2 w \mathrm{d}x \le \frac{C}{r^2} \int_{B_r \setminus B_{\frac{3}{4}r}} u^2 w \mathrm{d}x.$$

Proof. Let $\zeta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying

$$\zeta = 1 \text{ on } B_{\frac{3}{4}r}, \text{ and } \zeta = 0 \text{ on } \mathbb{R}^N - B_r, \text{ and } |\nabla\zeta| \le \frac{C}{r} \text{ on } B_r \setminus B_{\frac{3}{4}r},$$

where $C > 0$ is a generic constant.

Using $\zeta^2 u$ as the test function, we have

$$0 = \int_{B_{R_0}} \nabla u \cdot \nabla(\zeta^2 u) w \mathrm{d}x.$$

This implies that

$$\begin{aligned} \int_{B_{R_0}} \zeta^2 |\nabla u|^2 w \mathrm{d}x &= -2 \int_{B_{R_0}} \zeta u (\nabla \zeta \cdot \nabla u) w \mathrm{d}x \\ &\le \frac{1}{2} \int_{B_{R_0}} \zeta^2 |\nabla u|^2 w \mathrm{d}x + 4 \int_{B_{R_0}} u^2 |\nabla \zeta|^2 w \mathrm{d}x \end{aligned}$$

by Cauchy inequality, *i.e.*,

$$\int_{B_{R_0}} \zeta^2 |\nabla u|^2 w \mathrm{d}x \le 8 \int_{B_{R_0}} u^2 |\nabla \zeta|^2 w \mathrm{d}x.$$

From which, we obtain that

$$\begin{aligned} \int_{B_{R_0}} |\nabla(\zeta u)|^2 w \mathrm{d}x &\le 2 \int_{B_{R_0}} |\nabla \zeta|^2 u^2 w \mathrm{d}x + 2 \int_{B_{R_0}} \zeta^2 |\nabla u|^2 w \mathrm{d}x \\ &\le 18 \int_{B_{R_0}} u^2 |\nabla \zeta|^2 w \mathrm{d}x. \end{aligned}$$

Note that $\zeta u \in H^1_{w,0}(\Omega)$, by (2.3) in Lemma 2.1, we obtain

$$\int_{B_{R_0}} (\zeta u)^2 w \mathrm{d}x \le C \int_{B_{R_0}} u^2 |\nabla \zeta|^2 w \mathrm{d}x.$$

This implies that

$$\int_{B_{\frac{r}{2}}} u^2 w \mathrm{d}x \le \frac{C}{r^2} \int_{B_r \setminus B_{\frac{3}{4}r}} u^2 w \mathrm{d}x$$

according to the definition of ζ. $\square$

Next, we provide the proof of WUCP, which is characterized by the following two equivalent theorems.

Theorem 4.7. *Let Γ be a non-empty open subset of $\partial\Omega$ and let ω be a non-empty open subset of Ω. Then, for each $D \subseteq \Omega$ satisfying $\partial D \cap \partial\Omega \subset\subset \Gamma$ and $\overline{D} \setminus (\Gamma \cap \partial D) \subseteq \Omega$, there exists $\mu \in (0,1)$, such that for any solution $v \in H^1_w(\Omega)$ of (4.1) with $v = 0$ on Γ, we have*

$$\int_D v^2 w \mathrm{d}y \le C \left(\frac{1}{\epsilon}\right)^{\frac{1-\mu}{\mu}} \int_\omega v^2 w \mathrm{d}y + \epsilon \int_\Omega v^2 w \mathrm{d}y \tag{4.12}$$

for any $\epsilon > 0$, where $C > 0$ is a constant independent of v.

Proof. We divide the proof into the following steps.
Step 1. There are two cases that we should consider: one is $0 \in \omega$, the other is $0 \notin \omega$. In what follows, we denote $k \in \mathbb{N}$ an arbitrary integer.

Case 1. Assume $0 \in \omega$. We choose $r_0 > 0$ such that r_0 satisfies the conditions of Corollary 4.5 and $B(0, r_0) \subseteq \omega$. Since Ω is connected, then there exists a compact set $K \subseteq D$, such that $B(q, r_0) \subseteq D$ for all $q \in K$, and $D \subseteq \bigcup_{q \in K} B(q, 2r_0)$ and $B(q, 2r_0) \cap \partial\Omega \subseteq \Gamma$ for all $q \in K$. Hence, there exists a sequence of balls $\{B(q_j, r_0)\}_{j=0,1,\cdots,k}$, such that the following conditions hold

$$B(q_{j+1}, r_0) \subseteq B(q_j, 2r_0) \text{ for all } j = 0, 1, \cdots, k-1, \quad \text{and } q_0 = 0, q_k = q. \tag{4.13}$$

Note that $w \geq r_0^\alpha$ on $\Omega \setminus B_{r_0}$, then there exists $C > 0$ that is independent of v, such that

$$\begin{aligned}
&\int_{B(q_k, r_0)} v^2 w \mathrm{d}y \ \left(\text{or}, \int_{B(q_k, 2r_0) \cap D} v^2 w \mathrm{d}y\right) \\
&\leq C \left(\int_{B(q_k, r_0)} v^2 w \mathrm{d}y\right)^{\mu_1} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1} \\
&\leq C \left(\int_{B(q_{k-1}, 2r_0)} v^2 w \mathrm{d}y\right)^{\mu_1} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1},
\end{aligned}$$

where μ_1 is the exponent in Corollary 4.5 and we used Corollary 4.5 in the first inequality and (4.13) in the second. Then, we apply Corollary 4.5 again to $\int_{B(q_{k-1}, 2r_0)} v^2 w \mathrm{d}y$, which implies

$$\begin{aligned}
\int_{B(q_k, r_0)} v^2 w \mathrm{d}y &\leq C \left(\int_{B(q_{k-1}, 2r_0)} v^2 w \mathrm{d}y\right)^{\mu_1} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1} \\
&\leq C \left(\int_{B(q_{k-1}, r_0)} v^2 \mathrm{d}y\right)^{\mu_1^2} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1^2} \\
&\leq C \left(\int_{B(q_{k-2}, 2r_0)} v^2 \mathrm{d}y\right)^{\mu_1^2} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1^2},
\end{aligned}$$

Repeating the use of Corollary 4.5 and (4.13) in this way, we can deduce that

$$\begin{aligned}
\int_{B(q_k, r_0)} v^2 w \mathrm{d}y &\leq C \left(\int_{B(q_{k-2}, 2r_0)} v^2 \mathrm{d}y\right)^{\mu_1^2} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1^2} \\
&\leq \cdots \\
&\leq C \left(\int_{B(q_1, r_0)} v^2 w \mathrm{d}y\right)^{\mu_1^k} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1^k} \\
&\leq C \left(\int_{B(q_0, 2r_0)} v^2 w \mathrm{d}y\right)^{\mu_1^k} \left(\int_\Omega v^2 w \mathrm{d}y\right)^{1-\mu_1^k}.
\end{aligned}$$

Since $q_0 = 0$, by applying Corollary 4.3 to $\int_{B(q_0,2r_0)} v^2 w\mathrm{d}y$, we conclude that

$$\int_{B(q_k,r_0)} v^2 w\mathrm{d}y \le C\left(\int_{B(q_0,2r_0)} v^2 w\mathrm{d}y\right)^{\mu_1^k}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu_1^k}.$$
$$\le C\left(\int_{B(0,r_0)} v^2 w\mathrm{d}y\right)^{\mu_1^k\mu_2}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu_1^k\mu_2},$$

where μ_2 is the exponent in Corollary 4.3.

Case 2. Assume $0 \notin \overline{\omega}$. We choose $r_0 > 0$ and $q_0 \in \omega$ such that r_0 satisfies the conditions of Corollary 4.5 and $B(q_0, 2r_0) \subseteq \omega$ and $B(q_0, r_0) \cap B(0, r_0) = \emptyset$. Note that in contrast to the choice made in Case 1, we now take $q_0 \neq 0$. Choosing $q_1, \cdots, q_k$ and let $q_k = 0$ such that $|q_j - q_{j-1}| < r_0$ for $j = 1, \cdots, k$, and $B(q_j, 2r_0) \subseteq \Omega$ for $j = 0, 1, \cdots, k$. Then, for $q_k = 0$, there exists $C > 0$ that is independent of v, such that

$$\int_{B(q_k,2r_0)} v^2 w\mathrm{d}y \le \frac{C}{r_0^2}\int_{A_{3r_0,\frac{5}{2}r_0}} v^2 w\mathrm{d}y \tag{4.14}$$

by Theorem 4.6 and $r_0 < \frac{R_0}{4}$, where $A_{3r_0,\frac{5}{2}r_0} = \{x \in \mathbb{R}^N : \frac{5}{2}r_0 < |x| < 3r_0\}$ is an annulus. Note that

$$f(q) = \int_{B(q,r_0)\cap A_{3r_0,\frac{5}{2}r_0}} v^2 w\mathrm{d}y,\ \forall q \in \partial B\left(0, \frac{5}{2}r_0\right)$$

is a continuous function, then there exists $q_{k-1} \in \partial B(0, \frac{5}{2}r_0)$ such that

$$f(q_{k-1}) = \max_{q\in\partial B(0,\frac{5}{2}r_0)} f(q).$$

Hence, there exists a constant $C \in \mathbb{Z}^+$ (the constant C depends only on N and r_0), such that $A_{3r_0,\frac{5}{2}r_0}$ can be covered by C numbers $B(q, r_0)$ with $q \in \partial B(0, \frac{5}{2}r_0)$. Moreover,

$$\int_{A_{3r_0,\frac{5}{2}r_0}} v^2 w\mathrm{d}y \le C\int_{B(q_{k-1},r_0)} v^2 w\mathrm{d}y. \tag{4.15}$$

Since Ω is connected, then there exists a compact set $K \subseteq D \setminus B(0, 2r_0)$, such that $B(q, r_0) \subseteq D \setminus B(0, r_0)$ for all $q \in K$, and $D \setminus B(0, 2r_0) \subseteq \bigcup_{q\in K} B(q, 2r_0)$ and $B(q, 2r_0) \cap \partial\Omega \subseteq \Gamma$ for all $q \in K$. Hence, there exists a sequence of balls $\{B(q_j, r_0)\}_{j=0,1,\cdots,k-1}$, such that the following conditions hold

$$B(q_{j+1}, r_0) \subseteq B(q_j, 2r_0) \text{ for all } j = 0, 1, \cdots, k - 2.$$

Now, we use Corollary 4.5 by $k-2$ times to obtain

$$\begin{aligned}\int_{B(q_{k-1},r_0)} v^2 w\mathrm{d}y &\le \int_{B(q_{k-2},2r_0)} v^2 w\mathrm{d}y \\ &\le C\left(\int_{B(q_{k-3},r_0)} v^2 w\mathrm{d}y\right)^{\mu_1}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu_1} \\ &\le \cdots \le C\left(\int_{B(q_0,r_0)} v^2 w\mathrm{d}y\right)^{\mu_1^{k-2}}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu_1^{k-2}},\end{aligned} \tag{4.16}$$

where μ_1 is the exponent in Corollary 4.5. Finally, together (4.14), (4.15) and (4.16) we have

$$\int_{B(0,2r_0)} v^2 w\mathrm{d}y \le C\left(\int_{B(q_0,r_0)} v^2 w\mathrm{d}y\right)^{\mu}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu}, \tag{4.17}$$

where the constant $C>0$ is independent of v but depends on r_0, and $\mu=\mu_1^{k-2}$.

For each point $q_m \in K \subseteq D\setminus B(0,2r_0)$, using Corollary 4.5 by m times, we obtain

$$\begin{aligned}&\int_{B(q_m,r_0)} v^2 w\mathrm{d}y \left(\text{or}, \int_{B(q_m,2r_0)\cap D} v^2 w\mathrm{d}y\right)\\ &\le C\left(\int_{B(q_0,r_0)} v^2 w\mathrm{d}y\right)^{\mu_1^m}\left(\int_\Omega v^2 \mathrm{d}y\right)^{1-\mu_1^m},\end{aligned}$$

where μ_1 is defined in Corollary 4.5.

Step 2. By Case 1 and Case 2 in Step 1, using the fact that K is compact and the finite covering theorem, we obtain that

$$\int_D v^2 w\mathrm{d}y \le C\left(\int_\omega v^2 w\mathrm{d}y\right)^{\mu}\left(\int_\Omega v^2 w\mathrm{d}y\right)^{1-\mu}, \tag{4.18}$$

where $C>0$ is a constant that is independent of v.

Step 3. The proof of (4.12) is standard. We denote

$$A=\int_D v^2 w\mathrm{d}y \neq 0, \quad B=\int_\omega v^2 w\mathrm{d}y, \quad E=\int_\Omega v^2 w\mathrm{d}y.$$

Then, by Step 2, there exists $C>0$ and $\mu\in(0,1)$ such that $A\le CB^\mu E^{1-\mu}$, *i.e.*,

$$A \le C^{\frac{1}{\mu}} B\left(\frac{E}{A}\right)^{\frac{1-\mu}{\mu}}.$$

Now, if $\frac{E}{A}\le\frac{1}{\epsilon}$, then $A\le \epsilon E$. This implies (4.12). This completes the proof of Theorem 4.7. □

Lastly, we present an equivalent result to Theorem 4.7.

Theorem 4.8. *Let Γ be a non-empty open subset of $\partial\Omega$ and let ω be a non-empty open subset of Ω. Then, for each $D \subseteq \Omega$ satisfying $\partial D \cap \partial\Omega \subset\subset \Gamma$ and $\overline{D} \setminus (\Gamma \cap \partial D) \subseteq \Omega$, there exists $\mu \in (0,1)$, such that for any solution $u \in H^1_w(\Omega)$ of (4.1) with $u = 0$ on Γ, we have*

$$\int_D u^2 w\mathrm{d}y \le C\left(\int_\omega u^2 w\mathrm{d}y\right)^\mu \left(\int_\Omega u^2 w\mathrm{d}y\right)^{1-\mu}, \tag{4.19}$$

where $C > 0$ is a constant independent of u.

Proof. Assume (4.19) is true, we just need to follow the *Step 3* in Theorem 4.7 to derive (4.12).

Conversely, assume (4.12) is true, we denote

$$A = \int_D v^2 w\mathrm{d}y \neq 0, \quad B = \int_\omega v^2 w\mathrm{d}y, \quad E = \int_\Omega v^2 w\mathrm{d}y,$$

choose $\epsilon = \frac{1}{2}\frac{A}{E}$, then $A \le 2CB^\mu E^{1-\mu}$, and we obtain (4.19). ◻

Finally, we provide WUCP for the degenerate elliptic operator.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2-domain with $0 \in \Omega$, and let $w(x) = |x|^\alpha$ with $\alpha \in (0,2)$. Assume that $u \in H^1_w(\Omega)$ is a weak solution of*

$$-\mathrm{div}(w\nabla u) = 0 \quad \text{in } \Omega,$$

and that there exists a nonempty open set $\omega \subset \Omega$ such that $u = 0$ a.e. in ω. Then $u \equiv 0$ in Ω.

Proof. Since Ω is open, for any $x \in \Omega$ there exists $r_x > 0$ such that

$$\overline{B(x, r_x)} \subset \Omega.$$

Set $D = B(x, r_x)$. Then $D \subset \Omega$ and $\partial D \cap \partial\Omega = \emptyset$, so the assumptions in Theorems 4.7 and 4.8 are satisfied (in particular, $\partial D \cap \partial\Omega \subset\subset \Gamma$ holds trivially). Applying Theorem 4.7 (equivalently, Thm. 4.8) with this choice of D and the open set ω, we obtain

$$\int_{B(x,r_x)} u^2 w \,\mathrm{d}x \;\le\; C\Big(\int_\omega u^2 w \,\mathrm{d}x\Big)^\mu \Big(\int_\Omega u^2 w \,\mathrm{d}x\Big)^{1-\mu}.$$

If $u = 0$ a.e. in ω, then the right-hand side is zero, hence $\int_{B(x,r_x)} u^2 w \,\mathrm{d}x = 0$, and therefore $u = 0$ a.e. in $B(x, r_x)$. Because $x \in \Omega$ was arbitrary, it follows that $u \equiv 0$ in Ω.

◻

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Conflicts of interest

The authors declare that they have no competing interests and that no data availability statement is applicable.

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