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TURNPIKE PROPERTY OF A LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM IN LARGE HORIZONS WITH REGIME SWITCHING II. NONHOMOGENEOUS CASES

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Abstract. This paper is concerned with an optimal control problem for a nonhomogeneous linear stochastic differential equation having regime switching with a quadratic functional in the large time horizon. This is a continuation of the paper [H. Mei, R. Wang and J. Yong, Turnpike property of stochastic linear-quadratic optimal control problems in large horizons with regime switching. I. Homogeneous cases. preprint], in which the strong turnpike property was established for homogeneous linear systems with purely quadratic cost functionals. We extend the results to the current situation. It turns out that some of the results are new even for the cases without regime switchings.

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1. Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t);\, t \geqslant 0\}$ and a Markov chain $\alpha(\cdot)$ with a finite state space $\mathcal{M} = \{1, 2, 3, \cdots, m_0\}$ are defined, for which they are assumed to be independent. The generator of $\alpha(\cdot)$ is denoted by $(\lambda_{\imath\jmath})_{m_0 \times m_0}$ (see below for details). We now denote by $\mathbb{F}^W = \{\mathscr{F}_t^W\}_{t\geqslant 0}$ (resp. $\mathbb{F}^\alpha = \{\mathscr{F}_t^\alpha\}_{t\geqslant 0}$, $\mathbb{F} = \{\mathscr{F}_t\}_{t\geqslant 0}$) the usual augmentation of the natural filtration generated by $W(\cdot)$ (resp. by $\alpha(\cdot)$, and by $(W(\cdot), \alpha(\cdot))$). Consider the following *state equation* which is a controlled linear stochastic differential equation (SDE, for short), with regime switchings:

$$\begin{cases} \mathrm{d}X(t) = \big[A(\alpha(t))X(t) + B(\alpha(t))u(t) + b(t)\big]\mathrm{d}t \\ \qquad\qquad + \big[C(\alpha(t))X(t) + D(\alpha(t))u(t) + \sigma(t)\big]dW(t), \quad t \in [0, T], \\ X(0) = x, \quad \alpha(0) = \imath, \end{cases} \tag{1.1}$$

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For the coefficients of the state equation (1.1), we adopt the following basic assumptions:

(H1) Let $A, C : \mathcal{M} \to \mathbb{R}^{n\times n}$ and $B, D : \mathcal{M} \to \mathbb{R}^{n\times m}$ be measurable.

(H2) Let $b(\cdot), \sigma(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$.

Here, for any Euclidean space $\mathbb{H}$ (such as $\mathbb{R}^n, \mathbb{R}^{n\times m}$, *etc.*),

$$L^2_{\mathbb{F}}(0,T;\mathbb{H}) = \Big\{\varphi : [0,T]\times\Omega \to \mathbb{H} \bigm| \varphi(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \\ \mathbb{E}\int_0^T |\varphi(t)|^2_{\mathbb{H}} dt < \infty\Big\},$$

and

$$L^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{H}) = \bigcap_{T>0} L^2_{\mathbb{F}}(0,T;\mathbb{H}).$$

Also, since $\mathcal{M}$ is finite, $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are automatically bounded.

In (1.1), any $(x, \imath) \in \mathbb{R}^n \times \mathcal{M} \equiv \mathscr{D}$ is called an *initial pair*, $u(\cdot)$, called a *control*, is selected from the space

$$\mathscr{U}[0,T] = L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).$$

It is well-known that for each $(x, \imath) \in \mathscr{D}$ and $u(\cdot) \in \mathscr{U}[0,T]$, under (H1) and (H2), (1.1) admits a unique solution $X(\cdot) \equiv X(\cdot\,; x, \imath; u(\cdot))$, called the *state process*. Clearly, $X(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$.

To measure the performance of a control $u(\cdot) \in \mathscr{U}[0,T]$, we introduce the following cost functional

$$J_T(x,\imath;u(\cdot)) = \mathbb{E}\Big(\int_0^T g(t, X(t), \alpha(t), u(t))\mathrm{d}t\Big), \tag{1.2}$$

where

$$g(t,x,\imath,u) = \frac{1}{2}\Big\langle \begin{pmatrix} Q(\imath) & S(\imath)^\top \\ S(\imath) & R(\imath)\end{pmatrix}\begin{pmatrix} x \\ u\end{pmatrix}, \begin{pmatrix} x \\ u\end{pmatrix}\Big\rangle + \Big\langle \begin{pmatrix} q(t) \\ r(t)\end{pmatrix}, \begin{pmatrix} x \\ u\end{pmatrix}\Big\rangle,$$

with $Q(\cdot), S(\cdot), R(\cdot)$ being suitable matrix valued maps and some stochastic processes $q(\cdot), r(\cdot)$. Here, the superscript $\top$ denotes the transpose of matrices; $\langle\cdot\,,\cdot\rangle$ denotes the inner product of two vectors (possibly in different spaces). In what follows, we denote $\mathbb{S}^n$, $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ to be the sets of all $(n\times n)$ symmetric, positive semi-definite, and positive definite matrices, respectively. For the weights in the cost functional (1.2), we adopted the following assumptions.

(H3) Suppose that $Q(\imath) \in \mathbb{S}^n_{++}$, $R(\imath) \in \mathbb{S}^m_{++}$, $S(\imath) \in \mathbb{R}^{n\times m}$ such that

$$Q(\imath) - S(\imath)^\top R(\imath)^{-1} S(\imath) \in \mathbb{S}^n_{++}.$$

(H4) Let $q(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ and $r(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$.

Now it is natural to consider the following optimal control problem, under (H1)–(H4).

Problem (LQ)$_T$. For a given initial pair $(x,\imath) \in \mathscr{D}$, find a control $\bar{u}^{x,\imath}_T(\cdot) \in \mathscr{U}[0,T]$ such that

$$J_T(x,\imath;\bar{u}^{x,\imath}_T(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J_T(x,\imath;u(\cdot)) \equiv V_T(x,\imath).$$

The above problem is referred to as a (nonhomogeneous) *linear-quadratic* (LQ, for short) *optimal control problem* over a finite horizon with regime switchings (see [1, 2] for examples). We call $\bar u_T^{x,\imath}(\cdot)$ an *open-loop optimal control process*, $\bar X_T^{x,\imath}(\cdot)$ the corresponding *open-loop optimal state process*, and $(\bar X_T^{x,\imath}(\cdot),\bar u_T^{x,\imath}(\cdot))$ the *open-loop optimal pair* of Problem $(\mathrm{LQ})_T$, respectively. In addition, we call $V_T(x,\imath)$ the *value function* of Problem $(\mathrm{LQ})_T$.

Under some general mild assumptions (which will be given in later sections), it can be proved that Problem $(\mathrm{LQ})_T$ admits a unique open-loop optimal control $\bar u_T^{x,\imath}(\cdot)\in\mathscr U[0,T]$ with the optimal state process $\bar X_T^{x,\imath}(\cdot)\equiv X(\cdot\,;x,\imath;\bar u_T^{x,\imath}(\cdot))\in L^2_{\mathbb F}(0,T;\mathbb R^n)$. For the cases without regime switchings, people found that, under the so-called stabilizability condition (see below), for some stochastic processes $(\bar X_\infty(\cdot),\bar u_\infty(\cdot))$, and some constants $\beta,K>0$, all are independent of $0<T<\infty$, (in what follows, $K>0$ will be a generic constant which can be different from line to line) such that

$$\mathbb E\big(|\bar X_T^{x,\imath}(t)-\bar X_\infty(t)|^2+|\bar u_T^{x,\imath}(t)-\bar u_\infty(t)|^2\big)\leqslant K(\mathrm e^{-\beta t}+\mathrm e^{-\beta(T-t)}),\quad \forall t\in[0,T]. \tag{1.3}$$

Such an asymptotic behavior of the optimal pair $(\bar X_T^{x,\imath}(\cdot),\bar u_T^{x,\imath}(\cdot))$ as $T\to\infty$ is called the *strong turnpike property* (STP, for short) of Problem $(\mathrm{LQ})_T$. The main feature of (1.3) is that the (open-loop) optimal pair $(\bar X_T^{x,\imath}(\cdot),\bar u_T^{x,\imath}(\cdot))$ will be very close to a T-independent pair $(\bar X_\infty(\cdot),\bar u_\infty(\cdot))$ for all t in the middle range of $[0,T]$ (*i.e.*, $t\in[\varepsilon T,(1-\varepsilon)T]$ for some $\varepsilon\in(0,\frac12)$.)

Research on turnpike phenomenon was begun by Ramsey [3] in 1928, followed by von Neumann [4] in 1945, and Dorfman–Samuelson–Solow [5] in 1958, who coined the name. Since then, the turnpike property has been found to hold for a large class of (deterministic, finite or infinite dimensional) optimal control problems. Numerous relevant results can be found in [6–16] and the references cited therein. Since the beginning of 1970, several authors studied the problem from the portfolio aspect showing that for certain maximization problems of the utility for investments, the turnpike properties were established, mainly under proper assumptions on the utility functions (see [17–25]). Recently, a systematic investigation for continuous-time stochastic optimal LQ control problems was begun by the work of Sun–Wang–Yong in early of 2020 [26], followed by the works [27–33]. In particular, turnpike property for stochastic LQ control problems with regime switching has been studied by the authors in [34] when the linear SDE is homogeneous and the cost functional is purely quadratic. Naturally, one may ask if the results of [34] are true for nonhomogeneous problems, with the cost functional also having linear terms. The purpose of the current paper is to give a positive answer to this question, with additional techniques.

More precisely, combing those in [34], with some additional assumptions (see below), we will refine (1.3) as follows: there exist a function $h(\cdot)\geqslant 0$ and constants $\beta,K>0$, all are independent of $0<T<\infty$, such that the following refined STP holds:

$$\begin{aligned}&\mathbb E\Big[|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x_\infty,\imath}(t)|^2+\int_0^t|\bar u_T^{x,\imath}(s)-\bar u_\infty^{x_\infty,\imath}(s)|^2ds\Big]\\&\quad\leqslant K\Big[\mathrm e^{-\beta t}|x-x_\infty|^2+\mathrm e^{-\beta(T-t)}\Big(\mathrm e^{-\beta t}|x|^2+h(t)\Big)\Big],\qquad\forall t\in[0,T],\end{aligned}\tag{1.4}$$

with $(x,\imath),(x_\infty,\imath)\in\mathscr D$ being two possibly different initial pairs. In particular, if we strengthen (H2) and (H4) to the following:

(H2)′ Let $b(\cdot),\sigma(\cdot),q(\cdot)\in L^\infty_{\mathbb F^\alpha}(0,T;\mathbb R^n)$ and $r(\cdot)\in L^\infty_{\mathbb F^\alpha}(0,T;\mathbb R^m)$ for any $T>0$.

Then the above (1.4) can be strengthened to the following:

$$\begin{aligned}&\mathbb E\Big[|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x_\infty,\imath}(t)|^2+|\bar u_T^{x,\imath}(t)-\bar u_\infty^{x_\infty,\imath}(t)|^2\Big]\\&\quad\leqslant K\Big[\mathrm e^{-\beta t}|x-x_\infty|^2+\mathrm e^{-\beta(T-t)}\Big(\mathrm e^{-\beta t}|x|^2+h(t)\Big)\Big],\qquad\forall t\in[0,T],\end{aligned}$$

Now, we indicate three types of asymptotic behaviors of the open-loop optimal pair to the relevant Problem $(\text{LQ})_T$:

• **Homogeneous Case**: Let $b(\cdot), \sigma(\cdot), q(\cdot), r(\cdot)$ be all 0, and (H1), (H3) hold. In this case, $h(t) \equiv 0$. This case, fully treated in [34], is singled out since it catches most of essential features of the problem. A crucial step is to obtain the convergence of the solutions to differential Riccati equations (DREs, for short) to that of the algebraic Riccati equation (ARE, for short).

• **Integrable Case**: $b(\cdot), \sigma(\cdot), q(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$ and $r(\cdot) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^m)$. In this case, $h(\cdot)$ is a non-negative integrable function on $[0, \infty)$. Due to the appearance of the nonhomogeneous (square integrable) terms $b(\cdot), \sigma(\cdot)$ in the state equation (1.1) and linear (square integrable) weights $q(\cdot), r(\cdot)$ in the cost functional (1.2), some backward stochastic differential equations (BSDEs, for short) will be involved. From this case, we can see how far one can go (by this approach). This (even for the cases without switching) is new in the literature, since they were assumed to be constants [26, 32] or periodic [33].

• **Local-Integrable Case**: For any $0 < T < \infty$, $b(\cdot), \sigma(\cdot), q(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ and $r(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ with some additional assumptions. In this case, we can take $h(t) \equiv 1$. This is new again, even for the optimal control problems without regime switchings. Without doubt, the LQ ergodic control problem will be involved. The previous results in [26, 32, 33] are some special cases of that in the current paper.

The rest of the paper is arranged as follows. In Section 2, we recall results for Problem $(\text{LQ})_T$, with $0 < T < \infty$. Section 3 is to devote to some asymptotic behavior of the open-loop optimal pair $(\bar{X}^{x,\imath}_T(\cdot), \bar{u}^{x,\imath}_T(\cdot))$ as $T \to \infty$, under the stabilizability condition, including the identification of the limit pair $(\bar{X}^{x,\imath}_\infty(\cdot), \bar{u}^{x,\imath}_\infty(\cdot))$. Then our main results on STP are proved in Section 4. In particular, we verify the optimality of $(\bar{X}^{x,\imath}_\infty(\cdot), \bar{u}^{x,\imath}_\infty(\cdot))$ in two different cases: integrable and local-integrable cases in Section 5. Then some concluding remarks are made in Section 6. Finally, some proofs are relegated in Section 7.

2. Optimal control of problem $(\text{LQ})_T$

In this section, we will recall the optimal control and its closed-loop representation for Problem $(\text{LQ})_T$ with $0 < T < \infty$. Our results are some special cases of those found in [2].

Recall that $\alpha(\cdot)$ is a Markov chain whose state space $\mathcal{M}$ is finite. Thus, we may let its generator be $(\lambda_{\imath\jmath})_{m_0 \times m_0} \in \mathbb{R}^{m_0 \times m_0}$, which is a real matrix so that the following hold:

$$\lambda_{\imath\jmath} > 0, \quad \imath \neq \jmath; \qquad \sum_{\jmath=1}^{m_0} \lambda_{\imath\jmath} = 0, \quad \imath \in \mathcal{M}. \tag{2.1}$$

We now proceed with a *martingale measure of Markov chain* $\alpha(\cdot)$. For $\imath \neq \jmath$, we define

$$\begin{aligned} &\widetilde{M}_{\imath\jmath}(t) := \sum_{0 \le s \leqslant t} \mathbf{1}_{[\alpha(s^-)=\imath]} \mathbf{1}_{[\alpha(s)=\jmath]} \equiv \text{accumulative jump number from } \imath \text{ to } \jmath \text{ in } (0, t], \\ &\langle \widetilde{M}_{\imath\jmath} \rangle(t) := \int_0^t \lambda_{\imath\jmath} \mathbf{1}_{[\alpha(s^-)=\imath]} ds, \quad M_{\imath\jmath}(t) := \widetilde{M}_{\imath\jmath}(t) - \langle \widetilde{M}_{\imath\jmath} \rangle(t), \qquad t \geqslant 0. \end{aligned}$$

The above $M_{\imath\jmath}(\cdot)$ is a square-integrable martingale (with respect to $\mathbb{F}^\alpha$). For convenience, we let

$$M_{\imath\imath}(t) = \widetilde{M}_{\imath\imath}(t) = \langle \widetilde{M}_{\imath\imath} \rangle(t) = 0, \qquad t \geqslant 0.$$

Then $\{M_{ıȷ}(\cdot) \bigm| ı,ȷ\in\mathcal{M}\}$ is the *martingale measure* of Markov chain $\alpha(\cdot)$. If $\mathbb{H}$ is a Euclidean space and $F:\mathcal{M}\to\mathbb{H}$ is measurable, then

$$d[F(\alpha(t))]=\Lambda[F](\alpha(t))\mathrm{d}t+\sum_{ı,ȷ\in\mathcal{M}}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(t^-)=ı\}}dM_{ıȷ}(t), \tag{2.2}$$

where (see (2.1))

$$\Lambda[F](ı)=\sum_{ȷ\neq ı}\lambda_{ıȷ}F(ȷ)\equiv\sum_{ȷ\in\mathcal{M}}\lambda_{ıȷ}[F(ȷ)-F(ı)].$$

This is a special case of [35], Section 2.7, or [36], Section 2.2. In fact,

$$\begin{aligned}
&F(\alpha(t))-F(\alpha(0))\\
&=\sum_{0\leqslant s\leqslant t}[F(\alpha(s))-F(\alpha(s^-))]=\sum_{0\leqslant s\leqslant t}\sum_{ı,ȷ\in\mathcal{M}}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(s)=ȷ\},\alpha(s^-)=ı\}}\\
&=\int_0^t\sum_{ı,ȷ\in\mathcal{M}}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(s^-)=ı\}}d\widetilde{M}_{ıȷ}(s)=\int_0^t\sum_{ı,ȷ\in\mathcal{M}}\lambda_{ıȷ}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(s^-)=ı\}}ds\\
&+\int_0^t\sum_{ı,ȷ\in\mathcal{M}}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(s^-)=ı\}}d\Big(\widetilde{M}_{ıȷ}(s)-\lambda_{ıȷ}s\Big)\\
&=\int_0^t\Lambda[F](\alpha(s))ds+\int_0^s\sum_{ı,ȷ\in\mathcal{M}}[F(ȷ)-F(ı)]\mathbf{1}_{\{\alpha(s^-)=ı\}}dM_{ıȷ}(s).
\end{aligned}$$

Thus, we have (2.2). Here we note that $\int_0^t\Lambda[F](\alpha(s))ds=\int_0^t\Lambda[F](\alpha(s-))ds$ because there are at most countable many jumps on each path of $\alpha(\cdot)$.

Now, let $\mathbb{F}_-$ be the smallest filtration containing $\{\mathcal{F}_t^W\}_{t\geqslant 0}$ and $\{\mathcal{F}_{t-}^\alpha\}_{t\geqslant 0}$ augmented with all $\mathbb{P}$-null sets. To define the stochastic integral with respect to such a martingale measure, we need to introduce the following Hilbert spaces

$$\begin{aligned}
M_{\mathbb{F}_-}^2(t,T;\mathbb{H})=\Big\{&\varphi(\cdot\,,\cdot)=(\varphi(\cdot\,,1),\cdots,\varphi(\cdot\,,m_0))\bigm|\varphi(\cdot\,,\cdot)\text{ is }\mathbb{H}\text{-valued and }\mathbb{F}_-\text{-measurable}\\
&\text{with }\mathbb{E}\int_t^T\sum_{ı\neq ȷ}|\varphi(s,ȷ)|^2\lambda_{ıȷ}\mathbf{1}_{[\alpha(s)=ı]}ds<\infty,\quad\forall ı,ȷ\in\mathcal{M}\Big\}.
\end{aligned}$$

Now, for any $\varphi(\cdot)\in M_{\mathbb{F}_-}^2(t,T;\mathbb{H})$, we define its stochastic integral against dM by the following:

$$\int_t^T\varphi(s)dM(s):=\sum_{ȷ\neq ı}\int_{[t,T]}\varphi(s,ȷ)\mathbf{1}_{[\alpha(s^-)=ı]}dM_{ıȷ}(s),$$

whose quadratic variation is

$$\mathbb{E}\Big(\int_t^T\varphi(s)dM(s)\Big)^2=\mathbb{E}\int_t^T\sum_{ı\neq ȷ}|\varphi(s,ȷ)|^2\lambda_{ıȷ}\mathbf{1}_{[\alpha(s)=ı]}ds.$$

Now we state the following results concerning Problem $(\mathrm{LQ})_T$, whose proof can be found in [2, 37] (see also [38]).

Proposition 2.1. *Let* (H1)–(H4) *hold.*

(i) *For each $\imath\in\mathcal{M}$, the following DRE admits a unique uniformly regular solution $P_T(\cdot,\imath)\in C(0,T;\mathbb{S}^n_{++})$ ($\imath\in\mathcal{M}$):*

$$\begin{cases}\dot P_T+\Lambda[P_T]+P_TA+A^\top P_T+C^\top P_TC+Q\\ -(P_TB+C^\top P_TD+S^\top)(R+D^\top P_TD)^{-1}(B^\top P_T+D^\top P_TC+S)=0, \quad t\in[0,T],\\ P_T(T)=0,\end{cases}\tag{2.3}$$

i.e., it is a solution of (2.3) *and for some T-independent constant $\delta>0$, it holds*

$$\widetilde R_T(t,\imath)\equiv R(\imath)+D^\top(\imath)P_T(t,\imath)D(\imath)\geqslant\delta I,\quad \forall(t,\imath)\in[0,T]\times\mathcal{M}.$$

(ii) *There exists a unique adapted solution $(\eta_T(\cdot),\zeta_T(\cdot),\zeta_T^M(\cdot))\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times M^2_{\mathbb{F}_-}(0,T;\mathbb{R}^n)$ solving the following BSDE on $[0,T]$:*

$$\begin{cases}d\eta_T(t)=-\Big(A^{\Theta_T}(t,\alpha(t))^\top\eta_T(t)+C^{\Theta_T}(t,\alpha(t))^\top\zeta_T(t)+\varphi_T(t,\alpha(t))\Big)\mathrm{d}t\\ \qquad\qquad +\zeta_T(t)dW(t)+\zeta_T^M(t)dM(t),\\ \eta_T(T)=0,\end{cases}\tag{2.4}$$

where

$$\begin{aligned}&A^{\Theta_T}(t,\imath)=A(\imath)+B(\imath)\Theta_T(t,\imath),\qquad C^{\Theta_T}(t,\imath)=C(\imath)+D(\imath)\Theta_T(t,\imath),\\ &\Theta_T(t,\imath)=-\widetilde R_T(t,\imath)^{-1}[B(\imath)^\top P_T(t,\imath)+D(\imath)^\top P_T(t,\imath)C(\imath)+S(\imath)],\\ &\varphi_T(t,\imath)=P_T(t,\imath)b(t)+C^{\Theta_T}(t,\imath)^\top P_T(t,\imath)\sigma(t)+\Theta_T(t,\imath)^\top r(t)+q(t).\end{aligned}\tag{2.5}$$

(iii) *For each $(x,\imath)\in\mathscr{D}$, the unique open-loop optimal control $\bar u_T^{x,\imath}(\cdot)\in\mathscr{U}[0,T]$ admits a closed-loop representation*

$$\bar u_T^{x,\imath}(t)=\Theta_T(t,\alpha(t))\bar X^{x,\imath}(t)+v_T(t,\alpha(t)),\qquad t\in[0,T],$$

where $\bar X^{x,\imath}(\cdot)$ is the corresponding optimal state process and

$$v_T(t,\imath)=-\widetilde R_T(t,\imath)^{-1}[D(\imath)^\top P_T(t,\imath)\sigma(t)+B(\imath)^\top\eta_T(t)+D(\imath)^\top\zeta_T(t)+r(t)]$$

for $(t,\imath)\in[0,T]\times\mathcal{M}$. In this case, the optimal closed-loop system reads

$$\begin{cases}d\bar X_T^{x,\imath}(t)=[A^{\Theta_T}(t,\alpha(t))\bar X_T^{x,\imath}(t)+B(\alpha(t))v_T(t,\alpha(t))+b(t)]\mathrm{d}t\\ \qquad\qquad +[C^{\Theta_T}(t,\alpha(t))\bar X_T^{x,\imath}(t)+D(\alpha(t))v_T(t,\alpha(t))+\sigma(t)]dW(t),\\ \bar X_T^{x,\imath}(0)=x,\qquad \alpha(0)=\imath.\end{cases}\tag{2.6}$$

Having the open-loop optimal pair $(\bar X_T^{x,\imath}(\cdot),\bar u_T^{x,\imath}(\cdot))$ of Problem $(\mathrm{LQ})_T$, our next goal is to obtain the asymptotic behavior of this optimal pair as $T\to\infty$. This will be carefully investigated in the following section.

3. Asymptotic behavior of optimal controls

In this section, we will investigate the behavior of the open-loop optimal pair $(\bar{X}_T^{x,\imath}(\cdot), \bar{u}_T^{x,\imath}(\cdot))$ as $T \to \infty$. A so-called stabilizability condition is required. Then we will recall the asymptotic behavior of $\Theta_T(\cdot)$. This part has been fully studied in [34]. For readers' convenience, we recall the main results here. Based on this, we will further derive the asymptotic behavior for $v_T(\cdot)$.

Let

$$\begin{aligned}\boldsymbol{\Theta} &= \Big\{\Theta : \mathcal{M} \to \mathbb{R}^{m\times n} \mid \Theta(\cdot) \text{ is measurable}\Big\},\\ \boldsymbol{\Sigma} &= \Big\{\Sigma : \mathcal{M} \to \mathbb{S}^n_{++} \mid \Sigma(\cdot) \text{ is measurable}\Big\},\end{aligned}$$

and let us consider the following linear SDE with a regime switching governed by a Markov chain:

$$\begin{cases} \mathrm{d}X(t) = A(\alpha(t))X(t)\mathrm{d}t + C(\alpha(t))X(t)dW(t), & t \in [0,\infty),\\ X(0) = x, \quad \alpha(0) = \imath. \end{cases}$$

The above system is denoted by $[A, C]$. Under (H1), such a system is well-posed. If $X(\cdot) \equiv X(\cdot\,; x, \imath)$ is the solution of the above corresponding to $(x, \imath) \in \mathscr{D}$. We now introduce the following definition.

Definition 3.1. (i) System $[A, C]$ is said to be *stable* if for any $(x, \imath) \in \mathscr{D}$, $X(\cdot\,; x, \imath) \in L^2_{\mathbb{F}}(0, \infty; \mathbb{R}^n)$,

(ii) System $[A, C]$ is said to be *dissipative* if one could find a $\Sigma(\cdot) \in \boldsymbol{\Sigma}$ and a $\delta > 0$ so that

$$\Big(\Lambda[\Sigma] + \Sigma A + A^\top \Sigma + C^\top \Sigma C\Big)(\imath) \leqslant -\delta\Sigma(\imath), \qquad \imath \in \mathcal{M}. \tag{3.1}$$

The following definition is adopted from [1].

Definition 3.2. (i) System $[A, C; B, D]$ is said to be *stabilizable* if one can find a map $\Theta(\cdot) \in \boldsymbol{\Theta}$, so that for $[A^\Theta, C^\Theta]$ is stable, where (see (2.5))

$$A^\Theta(\imath) = A(\imath) + B(\imath)\Theta(\imath), \quad C^\Theta(\imath) = C(\imath) + D(\imath)\Theta(\imath).$$

In this case, the map $\Theta(\cdot)$ is called a *stabilizer* of $[A, C; B, D]$. The set of all possible stabilizers of system $[A, C; B, D]$ is denoted by $\mathbf{S}[A, C; B, D]$.

(ii) The map $\Theta(\cdot) \in \boldsymbol{\Theta}$ is called a *dissipating strategy* of system $[A, C; B, D]$ if there exists a $\delta > 0$ and a $\Sigma(\cdot) \in \boldsymbol{\Sigma}$ such that (3.1) holds with $[A, C]$ replaced by $[A^\Theta, C^\Theta]$, *i.e.*,

$$\Big(\Lambda[\Sigma] + \Sigma A^\Theta + (A^\Theta)^\top \Sigma + (C^\Theta)^\top \Sigma C^\Theta\Big)(\imath) \leqslant -\delta\Sigma(\imath), \qquad \imath \in \mathcal{M}.$$

Since the state space $\mathcal{M}$ of the Markov chain $\alpha(\cdot)$ is finite, the following is true (see [1], Prop. 3.7).

Proposition 3.3. *System $[A, C; B, D]$ is stabilizable if and only if it admits a dissipating strategy.*

It is known that stabilizability of $[A, C; B, D]$ is necessary for studying LQ problems in an infinite time horizon even for the problems without regime switchings [1, 38]. Thus, we accept the following assumption.

(H5) System $[A, C; B, D]$ is stabilizable, *i.e.*, $\mathbf{S}[A, C; B, D] \neq \varnothing$.

To find the limit behavior of $P_T(t,\imath)$ as well as $\Theta_T(t,\imath)$, we introduce the following ARE:

$$\begin{aligned}&\Lambda[P_\infty]+P_\infty A+A^\top P_\infty+C^\top P_\infty C+Q\\&\quad-(B^\top P_\infty+D^\top P_\infty C+S)^\top(R+D^\top P_\infty D)^{-1}(B^\top P_\infty+D^\top P_\infty C+S)=0.\end{aligned}\tag{3.2}$$

The following is the key result obtained in [34]. The main feature is the convergence.

Proposition 3.4. *Let* (H1), (H3) *and* (H5) *hold. Then*

(i) *ARE* (3.2) *admits a unique regular solution* $P_\infty(\cdot):\mathcal{M}\to\mathbb{S}^n_{++}$, *i.e., it is a solution of* (3.2) *such that*

$$\widetilde{R}_\infty(\imath)\equiv R(\imath)+D(\imath)^\top P_\infty(\imath)D(\imath)\geqslant\delta I,\quad \imath\in\mathcal{M},$$

for some $\delta>0$, *and*

$$\Theta_\infty(\cdot)=-\widetilde{R}_\infty(\cdot)^{-1}[B(\cdot)^\top P_\infty(\cdot)+D(\cdot)^\top P_\infty(\cdot)C(\cdot)+S(\cdot)]\in\mathbf{S}[A,C;B,D],$$

i.e., there exists a $\delta>0$ *and* $\Sigma_\infty(\cdot)\in\Sigma$ *such that (by Prop. 3.3)*

$$\Big(\Lambda[\Sigma_\infty]+\Sigma_\infty A^{\Theta_\infty}+(A^{\Theta_\infty})^\top\Sigma_\infty+(C^{\Theta_\infty})^\top\Sigma_\infty C^{\Theta_\infty}\Big)(\imath)\leqslant-\delta\Sigma_\infty(\imath),\quad\forall\imath\in\mathcal{M}.\tag{3.3}$$

(ii) *For any given* $t\in[0,\infty)$, *the following convergence holds*

$$P_T(t,\imath)=P_{T-t}(0,\imath)\nearrow P_\infty(\imath),\qquad \text{as } T\nearrow\infty,\quad\forall\imath\in\mathcal{M}.$$

Moreover, there exists a $\delta>0$ *so that (for some absolute constants* $K,\delta>0$*)*

$$0\leqslant P_\infty(\imath)-P_T(t,\imath)\leqslant K\mathrm{e}^{-\delta(T-t)}I,\qquad t\in[0,T],\tag{3.4}$$

and consequently,

$$|\Theta_\infty(\imath)-\Theta_T(t,\imath)|\leqslant K\mathrm{e}^{-\delta(T-t)},\qquad t\in[0,T],\tag{3.5}$$

(iii) *There exists a constant* $0<T_0<T$ *with* $T-T_0\geqslant0$ *large enough such that*

$$\begin{aligned}&\Lambda[\Sigma_\infty(\cdot)](\imath)+\Sigma_\infty(\imath)A^{\Theta_T}(t,\imath)+A^{\Theta_T}(t,\imath)^\top\Sigma_\infty(\imath)\\&\qquad+C^{\Theta_T}(t,\imath)^\top\Sigma_\infty(\imath)C^{\Theta_T}(t,\imath)\leqslant-\frac{\delta}{2}\Sigma_\infty(\imath),\qquad t\in[0,T-T_0],\end{aligned}\tag{3.6}$$

where A^{Θ_T} *and* C^{Θ_T} *are given by* (2.5).

Proof. (i) and (ii) are derived in [34]. (iii) is concluded from (3.5) and inequality (3.3) since

$$\begin{aligned}&|A^{\Theta_T}(t,\imath)-A^{\Theta_\infty}(\imath)|+|C^{\Theta_T}(t,\imath)-C^{\Theta_\infty}(\imath)|\\&\leqslant\big(|B(\imath)|+|D(\imath)|\big)|\Theta_T(t,\imath)-\Theta_\infty(\imath)|\leqslant K\mathrm{e}^{-\delta(T-t)}.\end{aligned}$$

The choice of $T_0>0$ is such that $K\mathrm{e}^{-\delta(T-t)}\leqslant K\mathrm{e}^{-\delta T_0}$ is small enough for all $t\in[0,T-T_0]$. ∎

We note that when t is close to T, say, $0 < T - t \leqslant T_0$, the right-hand sides of (3.4) and (3.5) might not be small. In other words, only if t is far away from T, say, $T - t \geqslant T_0$, the right-hand sides of (3.4) and (3.5) will be small, and (3.6) will be true.

Now, we introduce the following assumption.

(H6) Let $b(\cdot), \sigma(\cdot), q(\cdot) \in L^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$, and $r(\cdot) \in L^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$.

Under the dissipativity assumption, we are able to further derive the following proposition concerning with the existence and uniqueness of the adapted solutions to BSDEs (2.4), together with several useful estimates. The proof of the proposition is quite lengthy and will be given in Section 7. Write

$$\xi(t) \equiv \mathbb{E}[|b(t)|^2 + |\sigma(t)|^2 + |q(t)|^2 + |r(t)|^2]. \tag{3.7}$$

Proposition 3.5. *Let* (H1)–(H6) *hold. Let $\delta > 0$ given by* (3.3) *and T_0 be that in* (iii) *of Proposition 3.4. Then it follows that*

$$\begin{aligned}&\mathbb{E}|\eta_T(t)|^2 + \mathbb{E}\int_t^T e^{-\frac{\delta}{4}(s-t)}\sum_{\jmath\neq\imath}\lambda_{\imath\jmath}|\zeta^M_T(s,\jmath)|^2\mathbf{1}_{[\alpha(s)=\imath]}ds\\&\quad+\mathbb{E}\int_t^T e^{-\frac{\delta}{4}(s-t)}|\zeta_T(s)|^2ds \leqslant K\int_t^T e^{-\frac{\delta}{4}(s-t)}\xi(s)ds.\end{aligned} \tag{3.8}$$

For any $T' > T > T_0$, it also holds that

$$\begin{aligned}&\mathbb{E}|\eta_T(t)-\eta_{T'}(t)|^2 + \mathbb{E}\int_t^T e^{-\frac{\delta}{4}(s-t)}\sum_{\jmath\neq\imath}\lambda_{\imath\jmath}|\zeta^M_T(s,\jmath)-\zeta^M_{T'}(s,\jmath)|^2\mathbf{1}_{[\alpha(s)=\imath]}ds\\&\quad+\mathbb{E}\int_t^T e^{-\frac{\delta}{4}(s-t)}|\zeta_T(s)-\zeta_{T'}(s)|^2ds \leqslant K e^{-\frac{\delta}{8}(T-t)}\int_t^{T'} e^{-\frac{\delta}{4}(s-t)}\xi(s)ds.\end{aligned} \tag{3.9}$$

The above proposition, (3.9) particularly, suggests the following assumption.

(H6)$'$ For $\delta > 0$ given by (iii) of Proposition 3.5, the following holds:

$$\sup_{\tau\in[0,\infty)}\int_0^\infty e^{-\frac{\delta}{4}|\tau-s|}\xi(s)ds < \infty.$$

Even though (H6)$'$ is a little stronger than (H6), we see that (H6)$'$ holds if $\xi(\cdot)$ is measurable and bounded. Thus, (H6)$'$ covers most interesting cases.

Under (H6)$'$, taking $T \to \infty$ in (2.4), formally, we have the following BSDE on $[0,\infty)$:

$$\begin{aligned}d\eta_\infty(t) =& -\Big(A^{\Theta_\infty}(t,\alpha(t))^\top\eta_\infty(t) + C^{\Theta_\infty}(t,\alpha(t))^\top\zeta_\infty(t) + \varphi_\infty(t,\alpha(t))\Big)dt\\&+\zeta_\infty(t)dW(t) + \zeta^M_\infty(t)dM(t),\end{aligned} \tag{3.10}$$

with

$$\varphi_\infty(t,\imath) = P_\infty(t,\imath)b(t) + C^{\Theta_\infty}(t,\imath)^\top P_\infty(t,\imath)\sigma(t) + \Theta_\infty(t,\imath)^\top r(t) + q(t).$$

A similar argument to the proof of Proposition 3.5 can show that (3.10) admits a unique solution BSDE

$$(\eta_\infty(\cdot),\zeta_\infty(\cdot),\zeta^M_\infty(\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times M^2_{\mathbb{F}_-}(0,T;\mathbb{R}^n),$$

for any $T > 0$. Moreover, (3.8) holds for $T = \infty$ and (3.9) holds for any $T > T_0$ and $T' = \infty$.

Remark 3.6. (1) It is not necessary that

$$(\eta_\infty(\cdot),\zeta_\infty(\cdot),\zeta_\infty^M(\cdot))\in L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\times L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\times M^2_{\mathbb{F}_-}(0,\infty;\mathbb{R}^n).$$

(2) Even though it follows that $\eta_T(T)=0$ in (2.4), it should be noted that the infinite-horizon BSDE (3.10) does not necessarily admit an terminal condition such as

$$\lim_{t\to\infty}\mathbb{E}|\eta_\infty(t)|^2=0. \tag{3.11}$$

A simple example is that $b(\cdot),\sigma(\cdot),q(\cdot),r(\cdot)$ are constants. While (3.11) holds if a stronger assumption $\int_0^\infty \xi(s)ds<\infty$ is satisfied (compared to (H6)$'$).

With the help of $\eta_\infty(\cdot)$, now we can define the following closed-loop control

$$\bar u_\infty^{x,\imath}(t)=\Theta_\infty(\alpha(t))X(t)+v_\infty(t,\alpha(t)). \tag{3.12}$$

where

$$\begin{cases}\Theta_\infty(\imath)=-\widetilde R(t,\imath)^{-1}(B^\top(\imath)P_\infty(\imath)+D^\top(\imath)P_\infty(\imath)C(\imath)+S(\imath)),\\ v_\infty(t,\imath)=-\widetilde R(t,\imath)^{-1}(D^\top(\imath)P_\infty(\imath)\sigma(t)+B^\top(\imath)\eta_\infty(t)+D^\top(\imath)\zeta_\infty(t)+r(t)).\end{cases}$$

Then the corresponding state process by $\bar X_\infty^{x,\imath}(\cdot)$ satisfies

$$\begin{cases}d\bar X_\infty^{x,\imath}(t)=[A^{\Theta_\infty}(\alpha(t))\bar X_\infty^{x,\imath}(t)+B(\alpha(t))v_\infty(t,\alpha(t))+b(t)]\mathrm{d}t\\ \qquad\qquad +[C^{\Theta_\infty}(\alpha(t))\bar X_\infty^{x,\imath}(t)+D(\alpha(t))v_\infty(t,\alpha(t))+\sigma(t)]dW(t),\\ \bar X_\infty^{x,\imath}(0)=x,\qquad \alpha(0)=\imath.\end{cases} \tag{3.13}$$

Our key result lies in deriving the estimate between $\bar X_\infty^{x,\imath}(\cdot)$ and $\bar X_T^{x,\imath}(\cdot)$ (see (2.6)). This will be carefully studied in the next section. Before the end of this section, some estimates are presented in the following proposition. The proof is posted in Section 7.

Proposition 3.7. *Let* (H1)–(H4) *and* (H6)$'$ *hold. Then for any* $t\in[0,T]$*, we have*

$$\mathbb{E}\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}[|\zeta_T(s)|^2+|\zeta_\infty(s)|^2]\mathrm{d}t\leqslant K\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds, \tag{3.14}$$

$$\mathbb{E}\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}|\zeta_T(s)-\zeta_\infty(s)|^2ds\leqslant K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds,$$

$$\mathbb{E}\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}|v_T(s)-v_\infty(s)|^2ds\leqslant K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)\mathrm{d}r, \tag{3.15}$$

$$\mathbb{E}\int_0^T |\zeta_T(s)|^2+|\zeta_\infty(s)|^2dt\leqslant K\int_0^T\xi(s)ds+K(T+1)\sup_{s\geqslant0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r,$$

$$\mathbb{E}\int_0^T |\zeta_T(s)-\zeta_\infty(s)|^2ds\leqslant K\sup_{s\geqslant0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r, \tag{3.16}$$

$$\mathbb{E}\big[|\bar X_T^{x,\imath}(t)|^2+|\bar X_\infty^{x,\imath}(t)|^2\big]\leqslant K\Big(\mathrm{e}^{-\frac{\delta}{2}t}|x|^2+\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big). \tag{3.17}$$

4. Strong turnpike property

In this section, we are going to state and prove the main result of this paper.

Theorem 4.1. *Let* (H1)–(H5) *and* (H6)′ *hold. Let* $(\bar X_T^{x,\imath}(\cdot),\bar u_T^{x,\imath}(\cdot))$ *be the open-loop optimal pair of Problem* $(\mathrm{LQ})_T$ *corresponding to* $(x,\imath)\in\mathscr{D}$ *(see (2.6)), and let* $(\bar X_\infty^{x_\infty,\imath}(\cdot),\bar u_\infty^{x_\infty,\imath}(\cdot))$ *be the state-control pair corresponding initial couple* $(x_\infty,\imath)\in\mathscr{D}$ *so that* $\bar u_\infty^{x_\infty,\imath}(\cdot)$ *given by (3.12) (see (3.13)). Then it follows that*

$$\begin{aligned}&\mathbb{E}(|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x_\infty,\imath}(t)|^2+\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}|\bar u_T^{x,\imath}(s)-\bar u_\infty^{x_\infty,\imath}(s)|^2ds\\&\leqslant K\mathrm{e}^{-\frac{\delta}{4}t}|x_\infty-x|^2+K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\Big(\mathrm{e}^{-\frac{\delta}{4}t}|x|^2+\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big).\end{aligned}\tag{4.1}$$

for all $t\in[0,T]$.

Here let us remark that the function $h(t)$ in (1.4) admits the following form

$$h(t)=\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds.$$

In particular, when $\xi(\cdot)=0$, $h(\cdot)=0$, (4.1) reduces to the homogeneous case studied in [34]. For different $\xi(\cdot)$, $h(\cdot)$ admits different integrability so that the optimal couple verifies different types of optimality. More details will be presented in the next section.

Before presenting the proof, some observations should be made here. Taking $t=0$ and $t=T$, the right-hand side of (4.1) respectively reads

$$K\Big[|x_\infty-x|^2+\mathrm{e}^{-\frac{\delta}{2}T}\Big(|x|^2+\int_0^\infty \mathrm{e}^{-\frac{\delta}{2}r}\xi(r)\mathrm{d}r\Big)\Big],$$

and

$$K\Big[\mathrm{e}^{-\frac{\delta}{2}T}|x_\infty-x|^2+\Big(\mathrm{e}^{-\frac{\delta}{2}T}|x|^2+\int_0^\infty \mathrm{e}^{-\frac{\delta}{2}|T-r|}\xi(r)\mathrm{d}r\Big)\Big],$$

which might not be small. However, for any $\varepsilon\in(0,\frac12)$, if $t\in[\varepsilon T,(1-\varepsilon)T]$ (a middle range of $[0,T]$), then the right-hand side of (4.1) can be estimated as follows:

$$\begin{aligned}&K\Big[\mathrm{e}^{-\frac{\delta}{2}t}|x_\infty-x|^2+\mathrm{e}^{-\frac{\delta}{2}(T-t)}\Big(\mathrm{e}^{-\frac{\delta}{2}t}|x|^2+\int_t^\infty \mathrm{e}^{-\frac{\delta}{2}r}\xi(r)\mathrm{d}r\Big)\Big]\\&\leqslant K\Big[\mathrm{e}^{-\frac{\delta}{2}\varepsilon T}|x_\infty-x|^2+\mathrm{e}^{-\frac{\delta}{2}\varepsilon T}\Big(\mathrm{e}^{-\frac{\delta}{2}\varepsilon T}|x|^2+\int_t^\infty \mathrm{e}^{-\frac{\delta}{2}r}\xi(r)\mathrm{d}r\Big)\Big]\to 0,\end{aligned}$$

as $T\to\infty$. This exactly describes what we call the turnpike property of our LQ problem. Now let us turn to the proof.

Proof of Theorem 4.1. In what follows, we will suppress the superscript $(x,\imath)$ and $(x_\infty,\imath)$, together with $(t,\alpha(t))$. Set

$$\begin{cases}\widehat X(t)=\bar X_\infty(t)-\bar X_T(t),\quad \widehat x=x_\infty-x,\quad \widehat u(t)=\bar u_\infty(t)-\bar u_T(t),\\ \widehat\Theta(t)=\Theta_\infty(\alpha(t))-\Theta_T(t,\alpha(t)),\qquad \widehat v(t)=v_\infty(t,\alpha(t))-v_T(t,\alpha(t)),\end{cases}$$

with $(\bar X_T(\cdot), \bar u_T(\cdot))$ being defined in (2.6) and $(\bar X_\infty(\cdot), \bar u_\infty(\cdot))$ being defined in (3.13). Since

$$\widehat u(t) = \Theta_\infty \widehat X(t) + \widehat\Theta(t)\bar X_T(t) + \widehat v(t),$$

we need only to estimate $\widehat X(\cdot), \widehat v(\cdot)$ in certain sense and $\bar X_T(\cdot)$ uniformly bounded. The rest of this paper aims to realize this. Note that

$$\begin{aligned}
d\widehat X(t) &= d[\bar X_\infty(t) - \bar X_T(t)]\\
&= \Big[\Big(A^{\Theta_\infty}\bar X_\infty(t) + Bv_\infty(t) + b(t)\Big) - \Big(A^{\Theta_T}\bar X_T(t) + Bv_T(t) + b(t)\Big)\Big]\mathrm{d}t\\
&\quad + \Big[\Big(C^{\Theta_\infty}\bar X_\infty(t) + Dv_\infty(t) + \sigma(t)\Big) - \Big(C^{\Theta_T}\bar X_T(t) + Dv_T(t) + \sigma(t)\Big)\Big]dW(t)\\
&= \Big(A^{\Theta_\infty}\widehat X(t) + B\widehat\Theta(t)\bar X_T(t) + B\widehat v(t)\Big)\mathrm{d}t + \Big(C^{\Theta_\infty}\widehat X(t) + D\widehat\Theta(t)\bar X_T(t) + D\widehat v(t)\Big)dW(t).
\end{aligned}$$

Now, let $\Sigma_\infty(\cdot) \in \mathbb{\Theta}$ satisfy (3.3). Applying Itô's formula to $t \mapsto \langle \Sigma_\infty(\alpha(t))\widehat X(t), \widehat X(t)\rangle$, we have

$$\begin{aligned}
&\frac{d}{ds}\mathbb{E}\langle\Sigma_\infty(\alpha(t))\widehat X(t), \widehat X(t)\rangle\\
=&\mathbb{E}\langle\Big(\Lambda[\Sigma_\infty] + \Sigma_\infty A^{\Theta_\infty} + (A^{\Theta_\infty})^\top\Sigma_\infty + (C^{\Theta_\infty})^\top\Sigma_\infty C^{\Theta_\infty}\Big)\widehat X(t), \widehat X(t)\rangle\\
&\quad +2\mathbb{E}\langle\Sigma_\infty[B\widehat\Theta(t)\bar X_T(t) + B\widehat v(t)], \widehat X(t)\rangle + 2\mathbb{E}\langle\Sigma_\infty C^{\Theta_\infty}\widehat X(t), D\widehat v(t) + D\widehat\Theta(t)\bar X_T(t)\rangle\\
&\quad +\mathbb{E}\langle\Sigma_\infty[D\widehat v(t) + D\widehat\Theta(t)\bar X_T(t)], D\widehat v(t) + D\widehat\Theta(t)\bar X_T(t)\rangle\\
\leqslant& -\frac{\delta}{2}\mathbb{E}\langle\Sigma_\infty(\alpha(t))\widehat X(t), \widehat X(t)\rangle + K\mathbb{E}|\widehat\Theta(t)\bar X_T(t)|^2 + K\mathbb{E}|\widehat v(t)|^2.
\end{aligned}$$

By Gronwall's inequality, using (3.17) and (3.15), we have

$$\begin{aligned}
&\mathbb{E}|\widehat X(t)|^2 \leqslant K\mathbb{E}\langle\Sigma_\infty(\alpha(t))\widehat X(t), \widehat X(t)\rangle\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}\langle\Sigma(\imath)\widehat x, \widehat x\rangle + K\int_0^t \mathrm{e}^{-\frac{\delta}{2}(t-s)}\mathbb{E}\Big(|\widehat\Theta(s)\bar X_T(s)|^2 + |\widehat v(s)|^2\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|\widehat x|^2 + K\int_0^t \mathrm{e}^{-\frac{\delta}{2}(t-s)}\mathbb{E}\Big(\mathrm{e}^{-\frac{\delta}{2}(T-s)}|\bar X_T(s)|^2 + |\widehat v(s)|^2\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|\widehat x|^2 + K\int_0^t \mathrm{e}^{-\frac{\delta}{2}(t-s)}\mathbb{E}|\widehat v(s)|^2 ds\\
&\qquad +K\int_0^t \mathrm{e}^{-\frac{\delta}{2}(t-s)}\mathrm{e}^{-\frac{\delta}{2}(T-s)}\Big(\mathrm{e}^{-\frac{\delta}{2}s}|x|^2 + \int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|\widehat x|^2 + K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\Big(\mathrm{e}^{-\frac{\delta}{4}t}|x|^2 + \int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big).
\end{aligned} \tag{4.2}$$

Note that

$$\begin{aligned}
\mathbb{E}|\widehat u(s)| &= \mathbb{E}|\Theta_T(s)\bar X_T(s) - \Theta_\infty(s)\bar X_\infty(s) + \bar v_T(s) - \bar v_\infty(s)|\\
&\leqslant \mathbb{E}\Big(|\Theta_\infty(s)|\,|\widehat X(s)| + |\widehat\Theta(s)|\,|\bar X_T(s)| + |\widehat v(s)|\Big).
\end{aligned} \tag{4.3}$$

Using the estimates obtained in (4.2), (3.17) and (3.15), it follows that

$$
\begin{aligned}
&\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}|\widehat{u}(s)|^2 ds = \int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}|\bar{u}_T^{x,\imath}(s) - \bar{u}_\infty^{x_\infty,\imath}(s)|^2 ds \\
&\leqslant K\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}\Big(|\Theta_\infty(s)|^2|\widehat{X}(s)|^2 + |\widehat{\Theta}(s)|^2|\bar{X}_T(s)|^2 + |\widehat{v}(s)|^2\Big)ds \\
&\leqslant K\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}\Big(|\Theta_\infty(s)|^2|\widehat{X}(s)|^2 + |\widehat{\Theta}(s)|^2|\bar{X}_T(s)|^2 + |\widehat{v}(s)|^2\Big)ds \\
&\leqslant K\mathrm{e}^{-\frac{\delta}{4}t}|\widehat{x}|^2 + K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\Big(\mathrm{e}^{-\frac{\delta}{4}t}|x|^2 + \int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big).
\end{aligned}
\tag{4.4}
$$

Our main result holds from (4.2) and (4.4). □

Until now, we have proven the STP for the optimal pair in Problem $(\mathrm{LQ})_T$. We can see the limit pair $(\bar{X}_\infty^{x,\imath}(\cdot), \bar{u}_\infty^{x,\imath}(\cdot))$ is identified by taking $T \to \infty$ for $(\Theta_T(\cdot), v_T(\cdot))$. Naturally, the next is to verify the optimality of the $(\bar{X}_\infty^{x,\imath}(\cdot), \bar{u}_\infty^{x,\imath}(\cdot))$ in some appropriate sense.

5. Optimality of $(\bar{X}_\infty^{x,\imath}(\cdot), \bar{u}_\infty^{x,\imath}(\cdot))$

In this section, we will construct the appropriate optimal control problems for which $(\bar{X}_\infty^{x,\imath}(\cdot), \bar{u}_\infty^{x,\imath}(\cdot))$ is the optimal couple.

5.1. Local-integrable case

In this subsection, we work with local-integrable cases under (H1)–(H5) and (H6)′. First we claim that

$$
\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T \xi(t)\mathrm{d}t < \infty. \tag{5.1}
$$

Write

$$
h_1(t) = \int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\xi(s)ds.
$$

It is obvious that $h_1 \leq h$ and (H6)' implies that $h_1(\cdot)$ is bounded. Direct calculation yields that

$$
\frac{1}{T}\int_0^T \xi(t)\mathrm{d}t = \frac{1}{T}h_1(T) + \frac{\delta}{4T}\int_0^T h_1(t)\mathrm{d}t.
$$

Then (5.1) holds because $h_1(\cdot)$ is bounded.

In this case, we can show that $\bar{u}_\infty^{x,\imath}(\cdot)$ through the limit process is the optimal control for the following ergodic control problem. Define

$$
\mathscr{U}_{loc}[0,\infty) = \bigcap_{T\geq 0}\mathscr{U}[0,T].
$$

Problem (LQ)$_E$. For a given initial state $(x,\imath) \in \mathscr{D}$, find a control $u_E(\cdot) \in \mathscr{U}_{loc}[0,\infty)$ such that

$$
J_{\mathrm{E}}(x,\imath;u_E(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}_{loc}[0,\infty)} J_E(x,\imath;u(\cdot)) =: V_{\mathrm{E}}(x,\imath),
$$

where the ergodic cost is defined by

$$J_E(x,\imath;u(t)) := \varliminf_{T\to\infty}\frac{1}{T}J_{\mathrm{E}}(x,\imath;u_E(\cdot)).$$

Proposition 5.1. *Suppose* (H1)–(H5), (H6)$'$. *For any* $(x,\imath)\in\mathscr{D}$, $\bar u_\infty^{x,\imath}(\cdot)$ *is the optimal control and* $\bar X_\infty^{x,\imath}(\cdot)$ *is the corresponding optimal trajectory for Problem (LQ)*$_E$. *Moreover,* $J_E(x,\imath;\bar u_\infty^{x,\imath}(\cdot))$ *is finite. Note that the optimal couple for Problem (LQ)*$_E$ *may not be unique.*

Proof. We suppress the top index $(x,\imath)$ in the proof. Note that

$$\begin{aligned}&|\bar u_T(t)|+|\bar u_\infty(t)|\\&\leqslant K\Big(|X_T(t)|+|\eta_T(t)|+|\zeta_T(t)|+|X_\infty(t)|+|\eta_\infty(t)|+|\zeta_\infty(t)|+|r(t)|+|\sigma(t)|\Big),\end{aligned}$$

and

$$\begin{aligned}&|\bar u_T(t)-\bar u_\infty(t)|\leqslant|\Theta_\infty(\bar X_T-X_\infty)|+|(\Theta_\infty-\Theta_T)X_T(t))|+|v_T(t)-v_\infty(t)|\\&\leqslant K|\bar X_T-X_\infty|+K|\eta_T(t)-\eta_\infty(t)|+K|\zeta_T(t)-\zeta_\infty(t)|\\&\quad+K\mathrm{e}^{-\frac{\delta}{2}(T-t)}\Big(|X_T(t)|+|\eta_T(t)|+|\zeta_T(t)|+|r(t)|+|\sigma(t)|\Big)\end{aligned}$$

Applying all the estimates in Proposition 3.7, (3.9) and (4.1), it follows that

$$\varlimsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T|\bar u_T(t)|^2+|\bar u_\infty(t)|^2dt<\infty\text{ and }\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T|\bar u_T(t)-\bar u_\infty(t)|^2dt=0.\tag{5.2}$$

Next, we see

$$\begin{aligned}&\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),X(t),u(t))\mathrm{d}t\geqslant\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),\bar X_T(t),\bar u_T(t))\mathrm{d}t\\&\geqslant\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),\bar X_\infty(t),\bar u_\infty(t))\mathrm{d}t\\&\quad-\frac{K}{T}\int_0^T\Big(\mathbb{E}[|\bar X_T(t)-\bar X_\infty(t)|^2+|\bar u_T(t)-\bar u_\infty(t)|^2]\\&\qquad\cdot\mathbb{E}[1+|\bar X_T(t)|^2+|\bar X_\infty(t)|^2+|\bar u_T(t)|^2+|\bar u_\infty^{x,\imath}(t)|^2]\Big)^{\frac12}\mathrm{d}t\\&\quad-\frac{K}{T}\int_0^T\Big(\mathbb{E}[|\bar X_T(t)-\bar X_\infty(t)|^2+|\bar u_T(t)-\bar u_\infty(t)|^2]\Big)^{\frac12}\\&\qquad\cdot\Big(\mathbb{E}[1+|\bar X_T(t)|^2+|\bar X_\infty(t)|^2+|\bar u_\infty(t)|^2+|\bar u_T(t)|^2]\Big)^{\frac12}\mathrm{d}t.\end{aligned}$$

Taking $T\to\infty$, it follows that for any $u(\cdot)\in\mathscr{U}_{loc}[0,\infty)$,

$$\begin{aligned}&\varliminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),X(t),u(t))\mathrm{d}t\geqslant\varliminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),\bar X_T^{x,\imath}(t),\bar u_T^{x,\imath}(t))\mathrm{d}t\\&=\varliminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T g(t,\alpha(t),\bar X_\infty^{x,\imath}(t),\bar u_\infty^{x,\imath}(t))\mathrm{d}t.\end{aligned}$$

Moreover, the uniform boundedness of $\mathbb{E}|\bar{X}_\infty|^2$ and (5.2) together imply that

$$\begin{aligned}
&\frac{1}{T}\int_0^T \mathbb{E}|\bar{X}_\infty(t)|^2 dt < K, \quad \frac{1}{T}\int_0^T \mathbb{E}|\bar{u}_\infty(t)|^2 dt < K,\\
&\frac{1}{T}\int_0^T \mathbb{E}|\langle q(t), \bar{X}_\infty(t)\rangle| \mathrm{d}t \leqslant \frac{1}{T}\int_0^T \xi(t) + \mathbb{E}|\bar{X}_\infty(t)|^2 dt < K,\\
&\frac{1}{T}\int_0^T \mathbb{E}|\langle r(t), \bar{u}_\infty(t)\rangle| \mathrm{d}t \leqslant \frac{1}{T}\int_0^T \xi(t) + \mathbb{E}|\bar{u}_\infty(t)|^2 dt < K.
\end{aligned}$$

Therefore $J_E(x,\imath;\bar{u}_\infty(t))$ is finite. Moreover, $\bar{u}_\infty(t)$ is the optimal control process in $\mathscr{U}_{loc}[0,\infty)$ and $\bar{X}_\infty(t)$ is the corresponding trajectory for Problem $(\mathrm{LQ})_E$. □

Finally, when $b(\cdot), \sigma(\cdot), q(\cdot), r(\cdot)$ are bounded $\mathbb{F}^\alpha$-measurable (instead of $\mathbb{F}$-measurable), one can easily see that $\zeta_T(t) = 0$ for all $T > 0$ and $0 \leqslant t \leqslant T$. Then all the estimates on ζ_T are not necessary. For such a particular case, without essential difficulties, (4.1) can be refined to

$$\begin{aligned}
&\mathbb{E}(|\bar{X}_T^{x,\imath}(t) - \bar{X}_\infty^{x_\infty,\imath}(t)|^2 + |\bar{u}_T^{x,\imath}(t) - \bar{u}_\infty^{x_\infty,\imath}(t)|^2\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|x_\infty - x|^2 + K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\Big(\mathrm{e}^{-\frac{\delta}{4}t}|x|^2 + \int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big).
\end{aligned}$$

The above matches the results obtained in [26] and [32] where $b(\cdot), \sigma(\cdot), q(\cdot), r(\cdot)$ are assumed to be deterministic constants. From this, we can see that those previous results in [26] and [32] are some special cases of those in the current paper, even without switching states.

5.2. Integrable case

In this subsection, we particularly work with the refined integrable cases by assuming

(H7). $b(\cdot), \sigma(\cdot), q(\cdot) \in L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n), \quad r(\cdot) \in L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$.

It is obvious that (H7) is stronger than (H6)$'$. Recall that the definition of $\xi(\cdot)$ in (3.7), $\xi(\cdot)$ is integrable on $[0,\infty)$ with

$$\int_0^\infty \mathrm{e}^{-\beta|s-t|}\xi(s)ds \leqslant \int_0^\infty \xi(s)ds < \infty,$$

for any $\beta > 0$. All the previous results are true.

Recall that

$$\bar{u}_\infty^{x,\imath}(t) = \Theta_\infty(\alpha(t))X(t) + \bar{v}_\infty(t,\alpha(t)).$$

where

$$\begin{cases}
\Theta_\infty = -(R + D^\top P_\infty D)^{-1}(B^\top P_\infty + D^\top P_\infty C + S) \in \mathbf{S}[A,C;B,D],\\
\bar{v}_\infty(t,\imath) = -(R + D^\top P_\infty D)^{-1}(D^\top P_\infty \sigma + B^\top \eta_\infty + D^\top \zeta_\infty + r).
\end{cases}$$

We will see that (3.12) is the optimal control for a LQ problem on an infinite horizon.

To define the problem, we need the following set of admissible controls

$$\mathscr{U}_{ad}^{x,\imath}[0,\infty] = \Big\{u(\cdot) \in L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^m) \Big| X(\cdot;x,\imath,u(\cdot)) \in L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\Big\}$$

where $X(\cdot;x,i,u(\cdot))$ is the solution of (1.1) with initial $(x,\imath)$ and control $u(\cdot)$. For each $u(\cdot)\in\mathscr{U}_{ad}^{x,\imath}[0,\infty]$, we define the following cost functional

$$J_\infty(x,\imath;u(\cdot))=\mathbb{E}\Big(\int_0^\infty g(t,X(t),\alpha(t),u(t))\mathrm{d}t\Big).$$

It can be easily seen that the cost functional is well-defined. We have the following LQ optimization problem on $[0,\infty]$.

Problem (LQ)$_\infty$. For a given initial $(x,\imath)\in\mathscr{D}$, find a control $\bar u_\infty^{x,\imath}(\cdot)\in\mathscr{U}_{ad}^{x,\imath}[0,\infty]$ such that

$$J_\infty(x,\imath;\bar u_\infty^{x,\imath}(\cdot))=\inf_{u(\cdot)\in\mathscr{U}_{ad}^{x,\imath}[0,\infty]}J_\infty(x,\imath;u(\cdot))\equiv V_\infty(x,\imath).$$

Now let us verify the optimality of (3.12) for Problem (LQ)$_\infty$ in the following proposition.

Proposition 5.2. *Under* (H1)–(H5)*, and* (H7)*,* $(\bar X_\infty^{x,\imath}(\cdot),\bar u_\infty^{x,\imath}(\cdot))$ *is the unique optimal pair for Problem (LQ)*$_\infty$*.*

The above proposition is a special case studied in [1] and hence the proof is omitted. Now let us compare Proposition 5.1 with Proposition 5.2. When (H7) holds, the optimal pairs for Problem (LQ)$_E$ are not unique and $(\bar X_\infty^{x,\imath}(\cdot),\bar u_\infty^{x,\imath}(\cdot))$ is one of them. While $(\bar X_\infty^{x,\imath}(\cdot),\bar u_\infty^{x,\imath}(\cdot))$ is the unique optimal pair for Problem (LQ)$_\infty$. This is why we single out this particular case in this section. We also conclude the following corollary immediately.

Corollary 5.3. *Under (H1)–(H5) and (H7), it follows that*

$$\overline{\lim_{T\to\infty}}\Big(\mathbb{E}\int_0^T|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x,\imath}(t)|^2+|\bar u_T^{x,\imath}(t)-\bar u_\infty^{x,\imath}(t)|^2\Big)\mathrm{d}t=0.$$

We can see that the above result is stronger that in Proposition 5.1 (see (5.2) in particular).

Proof. Write $h(t)=\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-t|}\xi(s)ds$. It is straightforward to see that (H7) yields that $\int_0^\infty h(t)\mathrm{d}t<\infty$. Note that

$$\begin{aligned}\int_0^T h(t)\mathrm{e}^{-\frac{\delta}{8}(T-t)}\mathrm{d}t&=\int_0^{T/2}h(t)\mathrm{e}^{-\frac{\delta}{8}(T-t)}\mathrm{d}t+\int_{T/2}^T h(t)\mathrm{e}^{-\frac{\delta}{8}(T-t)}\mathrm{d}t\\ &\leqslant \mathrm{e}^{-\frac{\delta}{16}T}\int_0^\infty h(t)\mathrm{d}t+\int_{T/2}^\infty h(t)\mathrm{d}t\to 0,\ \text{as } T\to\infty.\end{aligned}$$

Then we have

$$\overline{\lim_{T\to\infty}}\,\mathbb{E}\int_0^T|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x,\imath}(t)|^2dt=0.$$

By (4.3), it follows that

$$\begin{aligned}&\overline{\lim_{T\to\infty}}\,\mathbb{E}\int_0^T|\bar u_T^{x,\imath}(t)-\bar u_\infty^{x,\imath}(t)|^2dt\\ &\leqslant K\,\overline{\lim_{T\to\infty}}\,\mathbb{E}\int_0^T|\bar X_T^{x,\imath}(t)-\bar X_\infty^{x,\imath}(t)|^2+\mathrm{e}^{\delta(T-t)}\mathbb{E}|\bar X_T^{x,\imath}(t)|^2+|v_T(t)-v_\infty(t)|^2dt\\ &=K\,\overline{\lim_{T\to\infty}}\,\mathbb{E}\int_0^T|v_T(t)-v_\infty(t)|^2dt\end{aligned}$$

$$\leqslant K \varlimsup_{T\to\infty} \mathbb{E}\int_0^T |\eta_T(t)-\eta_\infty(t)|^2+|\zeta_T(t)-\zeta_\infty(t)|^2+\mathrm{e}^{-\delta(T-t)}\Big(|\zeta_T(t)|^2+|\eta_T(t)|^2+\xi(t)\Big)\mathrm{d}t$$
$$=0.$$

In the above, we have used the boundedness of $\mathbb{E}|\bar{X}_T^{x,\imath}(t)|^2$ and $\mathbb{E}|\bar{\eta}_T(t)|^2$ (see (3.17) and (3.8)), (3.9), (3.14), (3.16) and (H6)$'$. ☐

Before we finish this section, it is worth remarking that even without the switching Markov chain, our results are not studied in [26] or [32] where b, σ, q, r are assumed to be deterministic and constants. Our assumption (H7) allows those nonhomogeneous terms to be stochastic. With some appropriate integrability conditions, we derive a new form of STP compared to that in [26] and [32].

6. Concluding remarks

In this paper, we obtained the turnpike property for LQ optimal control in an infinite horizon with a regime-switching state when the system is nonhomogeneous. We see that the comparing limit pair admits different optimality for different optimal control problems, depending on the integrability of the optimal solution over the infinite horizon. Those relate to three different cases: homogeneous case, integrable case, and local-integrable case. Even for the problem without switching, our results provide more accurate bounds under weaker assumptions compared to the previous results in the literature.

7. Proofs

In this section, we present the proofs of some results.

Proof of Proposition 3.5. Due to the linearity of the BSDE, the existence and uniqueness of the adapted solution triple $(\eta_T(\cdot), \zeta_T(\cdot), \zeta_T^M(\cdot)) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times M^2_{\mathbb{F}_-}(0,T;\mathbb{R}^n)$ is standard. Thus, we only need to establish the estimates. We split the proof into several steps.

Step 1. Dissipativity of the modified system. Note that the (homogenous) closed-loop system $[A^{\Theta_T}, C^{\Theta_T}]$ may not be dissipative for some states of the Markov chain (see (2.6)). The key step in our proof is to seek a modified and equivalent system that is dissipative for all the states of the Markov chain. Then the estimates can be derived in a classical way. To this end, let $\Sigma_\infty(\imath)\in\Sigma$ be that in Proposition 3.4 (i), satisfying (3.3). Set

$$E(\imath,\jmath)=\Sigma_\infty(\jmath)^{\frac{1}{2}}-\Sigma_\infty(\imath)^{\frac{1}{2}}\in\mathbb{S}^n.$$

It is clear that $E(\imath,\jmath)$ is well-defined and symmetric. By (2.2), we have

$$d[\Sigma_\infty(\alpha(t))^{\frac{1}{2}}]=\Lambda[\Sigma_\infty^{\frac{1}{2}}](\alpha(t))\mathrm{d}t+\sum_{\imath,\jmath\in\mathcal{M}}E(\imath,\jmath)\mathbf{1}_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t),$$

Let $X(\cdot)$ be the solution of the homogeneous system $[A^{\Theta_T}, C^{\Theta_T}]$. We define

$$\widetilde{X}_T(t)=\Sigma_\infty(\alpha(t))^{\frac{1}{2}}X_T(t),\qquad t\in[0,T].$$

Then, by Itô's formula, we have

$$\begin{aligned}
d\widetilde{X}_T(t) &= d[\Sigma_\infty(\alpha(t))^{\frac{1}{2}}X_T(t)]\\
&= \Big(\Lambda[\Sigma_\infty^{\frac{1}{2}}](\alpha(t))X_T(t) + \Sigma_\infty(\alpha(t))^{\frac{1}{2}}A^{\Theta_T}(t,\alpha(t))X_T(t)\Big)\mathrm{d}t\\
&\quad + \Sigma_\infty(\alpha(t))^{\frac{1}{2}}C^{\Theta_T}(t,\alpha(t))X_T(t)dW(t) + \sum_{\imath,\jmath\in\mathcal{M}} E(\imath,\jmath)X_T(t^-)\mathbf{1}_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t)\\
&\equiv \Big(\Lambda[\Sigma_\infty^{\frac{1}{2}}](\alpha(t))\Sigma_\infty(\alpha(t))^{-\frac{1}{2}} + \Sigma_\infty(\alpha(t))^{\frac{1}{2}}A^{\Theta_T}(t,\alpha(t))\Sigma_\infty(\alpha(t))^{-\frac{1}{2}}\Big)\widetilde{X}_T(t)\mathrm{d}t\\
&\quad + \Sigma_\infty(\alpha(t))^{\frac{1}{2}}C^{\Theta_T}(t,\alpha(t))\Sigma_\infty(\alpha(t))^{-\frac{1}{2}}\widetilde{X}_T(t)dW(t)\\
&\quad + \sum_{\imath,\jmath\in\mathcal{M}} E(\imath,\jmath)\Sigma_\infty(\alpha(t))^{-\frac{1}{2}}\widetilde{X}_T(t^-)\mathbf{1}_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t)\\
&= \widetilde{A}^{\Theta_T}\widetilde{X}(t)\mathrm{d}t + \widetilde{C}^{\Theta_T}\widetilde{X}(t)dW(t) + \sum_{\imath,\jmath\in\mathcal{M}}\widetilde{E}(\imath,\jmath)\widetilde{X}_T(t^-)\mathbf{1}_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t),
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{A}^{\Theta_T}(t,\imath) = \Lambda[\Sigma_\infty^{\frac{1}{2}}](\imath)\Sigma_\infty(\imath)^{-\frac{1}{2}} + \Sigma_\infty(\imath)^{\frac{1}{2}}A^{\Theta_T}(t,\imath)\Sigma_\infty(\imath)^{-\frac{1}{2}},\\
&\widetilde{C}^{\Theta_T}(t,\imath) = \Sigma_\infty(\imath)^{\frac{1}{2}}C^{\Theta_T}(t,\imath)\Sigma_\infty(\imath)^{-\frac{1}{2}}, \quad \widetilde{E}(\imath,\jmath) = E(\imath,\jmath)\Sigma_\infty(\imath)^{-\frac{1}{2}}.
\end{aligned}$$

Thus, we obtain the following new SDE

$$\begin{cases}
d\widetilde{X}_T(t) = \widetilde{A}^{\Theta_T}\widetilde{X}(t)\mathrm{d}t + \widetilde{C}^{\Theta_T}\widetilde{X}(t)dW(t) + \displaystyle\sum_{\imath,\jmath\in\mathcal{M}}\widetilde{E}(\imath,\jmath)\widetilde{X}_T(t^-)\mathbf{1}_{(\alpha(t^-)=\imath)}dM_{\imath\jmath}(t),\\
\hfill t\in[0,T],\\
\widetilde{X}_T(0) = \Sigma(\imath)^{\frac{1}{2}}x.
\end{cases}$$

Note that $|\widetilde{X}_T(t)|^2 = \langle\Sigma_\infty(\alpha(t))X_T(t), X_T(t)\rangle$, and by Itô's formula, we have

$$\begin{aligned}
\frac{d}{\mathrm{d}t}\mathbb{E}|\widetilde{X}(t)|^2 &= \mathbb{E}\Big\langle\Big(\widetilde{A}^{\Theta_T} + (\widetilde{A}^{\Theta_T})^\top + (\widetilde{C}^{\Theta_T})^\top\widetilde{C}^{\Theta_T}\\
&\qquad + \sum_{\jmath\neq\alpha(t)}\lambda_{\alpha(t)\jmath}\widetilde{E}(\alpha(t),\jmath)^\top\widetilde{E}(\alpha(t),\jmath)\Big)\widetilde{X}_T(t), \widetilde{X}_T(t)\Big\rangle\\
&= \frac{d}{\mathrm{d}t}\langle\Sigma_\infty(\alpha(t))X_T(t), X_T(t)\rangle\\
&= \mathbb{E}\Big\langle\Big(\Lambda[\Sigma_\infty](\alpha(t)) + \Sigma_\infty(\alpha(t))A^{\Theta_T} + (A^{\Theta_T})^\top\Sigma_\infty(\alpha(t))\\
&\qquad + (C^{\Theta_T})^\top\Sigma_\infty(\alpha(t))C^{\Theta_T}\Big)X_T(t), X_T(t)\Big\rangle\\
&\leqslant -\frac{\delta}{2}\mathbb{E}\langle\Sigma_\infty(\alpha(t))X_T(t), X_T(t)\rangle = -\frac{\delta}{2}|\widetilde{X}_T(t)|^2, \qquad t\in[0,T-T_0].
\end{aligned}$$

Thus, we have for $t\in[0,T-T_0]$,

$$\widetilde{A}^{\Theta_T} + (\widetilde{A}^{\Theta_T})^\top + (\widetilde{C}^{\Theta_T})^\top\widetilde{C}^{\Theta_T} + \sum_{\jmath\neq\imath}\lambda_{\imath\jmath}\widetilde{E}(\imath,\jmath)^\top\widetilde{E}(\imath,\jmath) \leqslant -\frac{\delta}{2}I. \tag{7.1}$$

This means $\widetilde{X}_T(\cdot)$ itself is dissipative on $[0,T-t_0]$, which will be very important below.

Step 2. BSDEs for the modified adapted solution. Again, let $\Sigma_\infty(\cdot)\in\Sigma$ be that in Proposition 3.4 (i), satisfying (3.3). Let $(\eta_T(\cdot),\zeta_T(\cdot),\zeta_T^M(\cdot))$ be the adapted solution to BDSE (2.4). Now, we write

$$\begin{aligned}&\widetilde\eta_T(t)=\Sigma_\infty(\alpha(t))^{-\frac12}\eta_T(t),\quad \widetilde\zeta_T(t)=\Sigma_\infty(\alpha(t))^{-\frac12}\zeta_T(t),\\&\widetilde\zeta_T^M(t,\jmath)=\Sigma_\infty(\jmath)^{-\frac12}\zeta_T^M(t,\jmath)-\Sigma_\infty(\jmath)^{-\frac12}E(\alpha(t^-),\jmath)\widetilde\eta_T(t^-),\qquad t\in[0,T],\\&\widetilde\varphi_T(t)=\Sigma_\infty(\alpha(t))^{-\frac12}\varphi_T(t,\alpha(t)),\end{aligned}\tag{7.2}$$

Then we claim that $(\widetilde\eta_T(\cdot),\widetilde\zeta_T(\cdot),\widetilde\zeta_T^M(\cdot))$ is the adapted solution of the following BSDE (compared with (2.4)):

$$\begin{cases}d\widetilde\eta_T(t)=-\Big(\widetilde A^{\Theta_T}(t,\alpha(t))^\top\widetilde\eta_T(t)+\widetilde C^{\Theta_T}(t,\alpha(t))^\top\widetilde\zeta_T(t)+\widetilde\varphi_T(t)\\\qquad\qquad+\displaystyle\sum_{\imath,\jmath\in\mathcal M}\widetilde E(\imath,\jmath)^\top\widetilde\zeta_T^M(t,\jmath)\lambda_{\imath\jmath}\mathbf 1_{\{\alpha(t)=\imath\}}\Big)\mathrm dt+\widetilde\zeta_T(t)dW(t)+\widetilde\zeta_T^M(t)dM(t),\\\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,T],\\\widetilde\eta_T(T)=\vartheta.\end{cases}\tag{7.3}$$

Here ϑ is a $\mathcal F_T$ measurable random variable with finite second moment. In fact, noting $\eta_T(t)=\Sigma_\infty(\alpha(t))^{\frac12}\widetilde\eta_T(t)$, using Itô's formula, we have

$$\begin{aligned}d\eta_T(t)&=\Big(d[\Sigma_\infty(\alpha(t))^{\frac12}]\Big)\widetilde\eta_T(t^-)+\Sigma_\infty(\alpha(t^-))^{\frac12}d\widetilde\eta_T(t)\\&\quad+\sum_{\imath,\jmath\in\mathcal M}E(\imath,\jmath)\widetilde\zeta_T^M(t,\jmath)\lambda_{\imath\jmath}\mathbf 1_{\{\alpha(t)=\imath\}}\mathrm dt+\sum_{\imath,\jmath\in\mathcal M}E(\imath,\jmath)\widetilde\zeta_T^M(t,\jmath)\mathbf 1_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t)\\&=\Big(\Lambda[\Sigma_\infty(\cdot)^{\frac12}](\alpha(t))\widetilde\eta_T(t)\mathrm dt+\sum_{\imath,\jmath\in\mathcal M}E(\imath,\jmath)\mathbf 1_{\{\alpha(t^-)=\imath\}}\widetilde\eta(t^-)dM_{\imath\jmath}(t)\Big)\\&\quad+\Sigma_\infty(\alpha(t))^{\frac12}\Big[-\Big((\widetilde A^{\Theta_T})^\top\widetilde\eta_T(t)+(\widetilde C_T^{\Theta_T})^\top\widetilde\zeta_T(t)+\widetilde\varphi_T(t)\\&\quad+\sum_{\imath,\jmath\in\mathcal M}\widetilde E(\imath,\jmath)^\top\widetilde\zeta_T^M(t,\jmath)\lambda_{\imath\jmath}\mathbf 1_{\{\alpha(t)=\imath\}}\Big)\mathrm dt+\widetilde\zeta_T(t)dW(t)+\sum_{\imath,\jmath\in\mathcal M}\widetilde\zeta_T^M(t,\jmath)\mathbf 1_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t)\Big]\\&\quad+\sum_{\imath,\jmath\in\mathcal M}E(\imath,\jmath)\widetilde\zeta_T^M(t,\jmath)\lambda_{\imath\jmath}\mathbf 1_{\{\alpha(t)=\imath\}}\mathrm dt+\sum_{\imath,\jmath\in\mathcal M}E(\imath,\jmath)\widetilde\zeta_T^M(t,\jmath)\mathbf 1_{\{\alpha(t^-)=\imath\}}dM_{\imath\jmath}(t)\\&=-\Big[-\Lambda[\Sigma_\infty^{\frac12}](\alpha(t))\widetilde\eta_T(t)+\Sigma(\alpha(t))^{\frac12}\Big((\widetilde A^{\Theta_T})^\top\widetilde\eta_T(t)+(\widetilde C^{\Theta_T})^\top\widetilde\zeta_T(t)\\&\quad+\widetilde\varphi_T(t)\Big)\mathrm dt\Big]+\Sigma_\infty(\alpha(t))^{\frac12}\widetilde\zeta_T(t)dW(t)\\&\quad+\sum_{\imath,\jmath\in\mathcal M}\Big(\Sigma_\infty(\jmath)^{\frac12}\widetilde\zeta_T^M(t,\jmath)+E(\imath,\jmath)\widetilde\eta_T(t^-)\Big)\mathbf 1_{(\alpha(t^-)=\imath)}dM_{\imath\jmath}(t)\\&=-[(A^{\Theta_T})^\top\eta_T(t)+(C^{\Theta_T})^\top\zeta_T(t)+\varphi_T(t,\alpha(t))]\mathrm dt+\zeta_T(t)dW(t)+\zeta_T^M(t)dM(t),\end{aligned}$$

where (note (7.2))

$$\begin{cases}\zeta_T(t)=\Sigma_\infty(\alpha(t))^{\frac12}\widetilde\zeta_T(t),\\\zeta_T^M(t,\jmath)=\Sigma_\infty(\jmath)^{\frac12}\widetilde\zeta_T^M(t,\jmath)+E(\alpha(t^-),\jmath)\widetilde\eta_T(t^-).\end{cases}$$

In the above,

$$\begin{aligned}&\Sigma_\infty(\alpha(t))^{-\frac{1}{2}}\sum_{\imath,\jmath\in\mathcal{M}}\widetilde{E}(\imath,\jmath)^\top\mathbf{1}_{\{\alpha(t)=\imath\}}=\sum_{\imath,\jmath\in\mathcal{M}}\Sigma_\infty(\imath)^{-\frac{1}{2}}\Big(E(\imath,\jmath)\Sigma(\imath)^{\frac{1}{2}}\Big)^\top\mathbf{1}_{\{\alpha(t)=\imath\}},\\&=\sum_{\imath,\jmath\in\mathcal{M}}E(\imath,\jmath)^\top\mathbf{1}_{\{\alpha(t)=\imath\}}=\sum_{\imath,\jmath\in\mathcal{M}}E(\imath,\jmath).\end{aligned}$$

By the uniqueness of a linear BSDE, the above calculation yields that $(\widetilde{\eta}_T(\cdot),\widetilde{\zeta}_T(\cdot),\widetilde{\zeta}_T^M(\cdot))$ is the adapted solution of the BSDE (7.3) by taking $\vartheta=0$.

Step 3. Dissipation inequality for $\widetilde{\eta}_T(\cdot)$. By Itô's formula, for $t\in[0,T-T_0]$, we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}\mathbb{E}|\widetilde{\eta}_T(t)|^2&=\Big(-2\langle\widetilde{\eta}_T(t),\widetilde{A}^{\Theta_T}(t,\alpha(t))^\top\widetilde{\eta}_T(t)+\widetilde{C}^{\Theta_T}(t,\alpha(t))^\top\widetilde{\zeta}_T(t)+\widetilde{\varphi}_T(t)\rangle\\&\quad+\sum_{\imath\neq\jmath}E(\imath,\jmath)\widetilde{\zeta}_T^M(t,\jmath)\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}}\rangle+|\widetilde{\zeta}_T(t)|^2dt+\sum_{\imath\neq\jmath}\lambda_{\imath\jmath}|\widetilde{\zeta}_T^M(t,\jmath)|^2\mathbf{1}_{[\alpha(t)=\imath]}\Big)\mathrm{d}t\\&=-\mathbb{E}\Big(\langle[\widetilde{A}^{\Theta_T}+(\widetilde{A}^{\Theta_T})^\top]\widetilde{\eta}_T(t),\widetilde{\eta}_T(t)\rangle-|\widetilde{C}^{\Theta_T}\widetilde{\eta}_T(t)|^2+|\widetilde{\zeta}_T(t)-\widetilde{C}^{\Theta_T}(\alpha(t))\widetilde{\eta}_T(t)|^2\\&\quad+\sum_{\imath\neq\jmath}\Big(|\widetilde{\zeta}_T^M(t,\jmath)|^2-2\langle E(\imath,\jmath)\widetilde{\eta}_T(t),\widetilde{\zeta}_T^M(t,\jmath)\rangle\Big)\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t^-)=\imath\}}-2\mathbb{E}\langle\widetilde{\varphi}_T(t),\widetilde{\eta}_T(t)\rangle\Big)\\&=-\mathbb{E}\Big[\langle\Big(\widetilde{A}^{\Theta_T}+(\widetilde{A}^{\Theta_T})^\top+(\widetilde{C}^{\Theta_T})^\top\widetilde{C}^{\Theta_T}+\sum_{\imath\neq\jmath}E(\imath,\jmath)E(\imath,\jmath)\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}}\Big)\widetilde{\eta}_T(t),\widetilde{\eta}_T(t)\rangle\\&\quad+|\widetilde{\zeta}_T(t)-\widetilde{C}^{\Theta_T}\widetilde{\eta}_T(t)|^2+\sum_{\imath,\jmath\in\mathcal{M}}|\zeta_T^M(t,\jmath)-E(\imath,\jmath)\widetilde{\eta}_T(t)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}}-2\mathbb{E}\langle\widetilde{\varphi}_T(t),\widetilde{\eta}_T(t)\rangle\Big)\\&\geqslant\mathbb{E}\Big(\frac{\delta}{4}|\widetilde{\eta}_T(t)|^2+|\widetilde{\zeta}_T(t)-\widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t)|^2+\sum_{\imath\neq\jmath}|\widetilde{\zeta}_T^M(t,\jmath)-E(\imath,\jmath)\widetilde{\eta}_T(t)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t^-)=\imath\}}\\&\quad-K\mathbb{E}|\widetilde{\varphi}_T(t)|^2\Big).\end{aligned}$$

Where we have used the dissipativity of $\widetilde{X}(\cdot)$, *i.e.*, (7.1), in the last step. Thus, for all $t\in[0,T-T_0]$, the following dissipativity inequality holds:

$$\begin{aligned}\frac{d}{\mathrm{d}t}\mathbb{E}|\widetilde{\eta}_T(t)|^2\geqslant\mathbb{E}\Big(&\frac{\delta}{4}|\widetilde{\eta}_T(t)|^2+|\widetilde{\zeta}_T(t)-\widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t)|^2\\&+\sum_{\imath\neq\jmath}|\widetilde{\zeta}_T^M(t,\jmath)-E(\imath,\jmath)\widetilde{\eta}_T(s)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}}-K\xi(t)\Big).\end{aligned}\tag{7.4}$$

On $[T-T_0,T]$, using the boundedness of A^{Θ_T} and C^{Θ_T}, it is standard to derive that

$$\begin{aligned}\frac{d}{\mathrm{d}t}\mathbb{E}|\widetilde{\eta}_T(t)|^2\geqslant\mathbb{E}\Big(&-K|\widetilde{\eta}_T(t)|^2+|\widetilde{\zeta}_T(t)-\widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t)|^2\\&+\sum_{\imath\neq\jmath}|\widetilde{\zeta}_T^M(t,\jmath)-E(\imath,\jmath)\widetilde{\eta}_T(s)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}}-K\xi(t)\Big).\end{aligned}\tag{7.5}$$

Step 4. Boundedness of $(\eta_T(\cdot),\zeta_T(\cdot),\zeta_T^M(\cdot))$. By (7.5), Gronwall's inequality implies that for $t\in[T-t_0,T]$,

$$\begin{aligned}&\mathbb{E}|\widetilde{\eta}_T(t)|^2+\int_t^T \mathrm{e}^{K(s-t)}\mathbb{E}|\widetilde{\zeta}_T(s)+C^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2ds\\&\leqslant K\int_t^T \mathrm{e}^{K(s-t)}\xi(s)ds+\mathrm{e}^{K(T-t)}\mathbb{E}|\vartheta|^2.\end{aligned}$$

Because $0\leqslant s-t\leqslant t_0$ for $t\in[T-t_0,T]$, the above is equivalent to

$$\begin{aligned}&\mathbb{E}|\widetilde{\eta}_T(t)|^2+\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}|\widetilde{\zeta}_T(s)+C^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2ds\\&\leqslant K\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.\end{aligned}\tag{7.6}$$

In particular, we have

$$\begin{aligned}&\mathbb{E}|\widetilde{\eta}_T(T-T_0)|^2+\int_{T-T_0}^T \mathrm{e}^{-\frac{\delta}{4}(s-T+T_0)}\mathbb{E}|\widetilde{\zeta}_T(s)+C^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2ds\\&\leqslant K\int_{T-T_0}^T \mathrm{e}^{-\frac{\delta}{4}(s-(T-T_0))}\xi(s)ds+K\mathbb{E}|\vartheta|^2.\end{aligned}\tag{7.7}$$

By (7.4) and (7.7), Gronwall's inequality implies that for $t\in[0,T-T_0]$,

$$\begin{aligned}\mathbb{E}|\widetilde{\eta}_T(t)|^2\ &+\textstyle\int_t^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}|\widetilde{\zeta}_T(s)-\widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2ds\\&+\int_t^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}\sum_{\imath,\jmath\in\mathcal{M}}|\widetilde{\zeta}_T^M(s,\jmath)-E(\imath,\jmath)\widetilde{\eta}_T(s)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(s)=\imath\}}ds\\&\leqslant \mathrm{e}^{-\frac{\delta}{4}(T-T_0-t)}\mathbb{E}|\widetilde{\eta}_T(T-T_0)|^2+K\int_t^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds\\&\leqslant K\mathrm{e}^{-\frac{\delta}{4}(T-T_0-t)}\int_{T-T_0}^T\mathrm{e}^{-\frac{\delta}{4}(s-(T-T_0))}\zeta(s)ds+K\int_t^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2\\&\leqslant K\int_t^T\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.\end{aligned}\tag{7.8}$$

Combining (7.6) and (7.8), by $\eta_T(t)=\Sigma(t,\alpha(t))\widetilde{\eta}_T(t)$, we obtain that $\mathbb{E}|\eta_T(\cdot)|^2$ is uniformly bounded on $[0,T]$, *i.e.*,

$$\mathbb{E}|\eta_T(t)|^2\leqslant K\int_t^T\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.\tag{7.9}$$

Next, from (7.6) and (7.8), for $t\in[0,T]$, we have (see (7.2) again)

$$\begin{aligned}&\int_t^T\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}|\zeta_T(s)|^2ds\leqslant K\int_t^T\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}|\widetilde{\zeta}_T(s)|^2ds\\&\leqslant K\mathrm{e}^{-\frac{\delta}{4}(T-T_0-t)}\int_{T-T_0}^T\mathrm{e}^{-\frac{\delta}{4}(s-(T-T_0))}\mathbb{E}|\widetilde{\zeta}_T(s)|^2ds+K\int_t^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}|\widetilde{\zeta}_T(s)|^2ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant K\mathrm{e}^{-\frac{\delta}{4}(T-T_0-t)}\int_{T-T_0}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-(T-T_0))}\mathbb{E}\Big(|\widetilde{\zeta}_{T}(s)-\widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_{T}(s)|^2+|\widetilde{\eta}_{T}(s)|^2\Big)ds\\
&\quad+K\int_{t}^{T-T_0}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}\Big(|\widetilde{\zeta}_{T}(s)-\widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_{T}(s)|^2+|\widetilde{\eta}_{T}(s)|^2\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{4}(T-T_0-t)}\Big(\int_{T-T_0}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-(T-T_0))}\zeta(s)ds+\int_{T-T_0}^{T}\int_{s}^{T}\mathrm{e}^{-\frac{\delta}{4}(r-s)}\xi(r)drds\Big)\\
&\quad+K\Big(\int_{t}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+\int_{t}^{T-T_0}\int_{s}^{T}\mathrm{e}^{-\frac{\delta}{4}(r-s)}\xi(r)drds\Big)+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2\\
&\leqslant K\int_{t}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\int_{t}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}\big[\sum_{\imath\neq\jmath}\lambda_{\imath\jmath}|\zeta_{T}^{M}(s,\jmath)|^2\mathbf{1}_{[\alpha(s)=\imath]}\big]ds\\
&=\int_{t}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathbb{E}\Big[\sum_{\imath\neq\jmath}|\widetilde{\zeta}_{T}^{M}(s,\jmath)+E(\imath,\jmath)\widetilde{\eta}_{T}(s)|^2\lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(s)=\imath\}}\Big]ds\\
&\leqslant K\int_{t}^{T}\mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.
\end{aligned}
\tag{7.10}
$$

Combining (7.9)–(7.10), we get (3.8) by taking $\vartheta=0$.

Step 5. Stability estimates. For $T'>T\geqslant T_0$, we have (recall (2.4))

$$
\begin{aligned}
d(\eta_T-\eta_{T'})\;&=-\Big((A^{\Theta_T}\eta_T+C^{\Theta_T}\zeta_T+\varphi_T-(A^{\Theta_{T'}}\eta_{T'}+C^{\Theta_{T'}}\zeta_{T'}+\varphi_{T'})\Big)\mathrm{d}t\\
&\qquad\qquad+(\zeta_T-\zeta_{T'})dW+(\zeta_T^M-\zeta_{T'}^M)dM\\
&=-\Big((A^{\Theta_{T'}})^\top(\eta_T-\eta_{T'})+C^{\Theta_{T'}}(\zeta_T-\zeta_{T'})\\
&\qquad+(A^{\Theta_T}-A^{\Theta_{T'}})\eta_T+(C^{\Theta_T}-C^{\Theta_{T'}})\zeta_T+\varphi_T-\varphi_{T'}\Big)\mathrm{d}t\\
&\qquad+(\zeta_T-\zeta_{T'})dW+(\zeta_T^M-\zeta_{T'}^M)dM\\
&\equiv-\Big((A^{\Theta_{T'}})^\top(\eta_T-\eta_{T'})+C^{\Theta_{T'}}(\zeta_T-\zeta_{T'})+\Delta_{T,T'}(t)\Big)\mathrm{d}t\\
&\qquad+(\zeta_T-\zeta_{T'})dW+(\zeta_T^M-\zeta_{T'}^M)dM,
\end{aligned}
\tag{7.11}
$$

where

$$
\begin{aligned}
&\Delta_{T,T'}(s)=(A^{\Theta_T}-A^{\Theta_{T'}})^\top\eta_T+(C^{\Theta_T}-C^{\Theta_{T'}})^\top\zeta_T+\varphi_T-\varphi_{T'},\\
&\varphi_T=P_Tb+(C^{\Theta_T})^\top P_T\sigma+q+\Theta_T^\top r,\quad \varphi_{T'}=P_{T'}b+(C^{\Theta_{T'}})^\top P_{T'}\sigma+q+\Theta_{T'}^\top r.
\end{aligned}
$$

Since $T' > T$, we have $P_T(t,\imath) \leqslant P_{T'}(t,\imath) \leqslant P_\infty(t,\imath)$, it follows that

$$0 \leqslant P_{T'}(t,\imath) - P_T(t,\imath) \leqslant P_\infty(t,\imath) - P_T(t,\imath) \leqslant K\mathrm{e}^{-\delta(T-t)} I.$$

For $t \in [0,T]$, we have

$$\begin{aligned}
\mathbb{E}|\Delta_{T,T'}(t)|^2 &\leqslant \mathbb{E}\Big(|(A^{\Theta_T} - A^{\Theta_{T'}})^\top \eta_T + (C^{\Theta_T} - C^{\Theta_{T'}})^\top \zeta_T| + |\varphi_T - \varphi_{T'}|\Big)^2 \\
&\leqslant \mathbb{E}\Big(|B(\Theta_T - \Theta_{T'})\eta_T| + |D(\Theta_T - \Theta_{T'})\zeta_T| \\
&\qquad + |(P_T - P_{T'})b| + |(C^{\Theta_T} P_T - C^{\Theta_{T'}} P_{T'})\sigma| + |(\Theta_T - \Theta_{T'})^\top r|\Big)^2 \\
&\leqslant K\mathrm{e}^{-2\delta(T-t)}\mathbb{E}\Big(|b| + |\sigma| + |r| + |\eta_T| + |\zeta_T|\Big)^2 \\
&\leqslant K\mathrm{e}^{-2\delta(T-t)}\Big(\xi(t) + \mathbb{E}|\eta_T(t)|^2 + \mathbb{E}|\zeta_T(t)|^2\Big).
\end{aligned} \tag{7.12}$$

Note that (7.11) is parallel with BSDE (7.3) with different nonhomogeneous terms and terminal condition only. Similar to Steps 2–3, it follows that

$$\begin{aligned}
&\mathbb{E}|\eta_T(t) - \eta_{T'}(t)|^2 + \mathbb{E}\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)}|\zeta_T(s) - \zeta_{T'}(s)|^2 ds \\
&\qquad + \mathbb{E}\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)} \sum_{\jmath\neq\imath} \lambda_{\imath\jmath}|\zeta_T^M(s,\jmath) - \zeta_{T'}^M(s,\jmath)|^2 \mathbf{1}_{[\alpha(s)=\imath]} ds \\
&\leqslant K\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathrm{e}^{-2\delta(T-s)}\mathbb{E}|\Delta_{T,T'}|^2 ds + K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\eta_T(T) - \eta_{T'}(T)|^2 \\
&\leqslant K\int_t^T \mathrm{e}^{-\frac{\delta}{4}(s-t)}\mathrm{e}^{-2\delta(T-s)}\Big(\xi(s) + \mathbb{E}|\eta_T(s)|^2 + \mathbb{E}|\zeta_T(s)|^2\Big)ds + K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\eta_T(T) - \eta_{T'}(T)|^2 \\
&\leqslant \mathrm{e}^{-\frac{\delta}{8}(T-t)}\int_t^{T'} \mathrm{e}^{-\frac{\delta}{4}(s-t)}\xi(s)ds.
\end{aligned} \tag{7.13}$$

In the last step, we use $\eta_T(T) = 0$ and (7.9) (taking $T = T'$ and $t = T$). □

Proof of Proposition 3.7. (1) We consider the BSDE (7.3). By (7.4) and (7.5), we have for $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}|\widetilde{\eta}_T(t)|^2 + \frac{\delta}{4}\mathbb{E}|\widetilde{\eta}_T(t)|^2 \geqslant \mathbb{E}|\widetilde{\zeta}_T(t) - \widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t)|^2 \\
&\quad + \mathbb{E}\sum_{\imath\neq\jmath} |\widetilde{\zeta}_T^M(t,\jmath) - E(\imath,\jmath)\widetilde{\eta}_T(t)|^2 \lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(t)=\imath\}} - K\xi(t) - K\mathbb{E}|\widetilde{\eta}_T(t)|^2 \mathbf{1}_{t\in[T-T_0,T]}.
\end{aligned}$$

Gronwall's inequality implies that

$$\begin{aligned}
\mathbb{E}|\widetilde{\eta}_T(t)|^2 - \mathrm{e}^{-\frac{\delta}{4}t}|\widetilde{\eta}_T(0)|^2 &\geqslant \int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\Big(\mathbb{E}|\widetilde{\zeta}_T(s) - \widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2 \\
&\quad + \sum_{\imath\neq\jmath} |\widetilde{\zeta}_T^M(s,\jmath) - E(\imath,\jmath)\widetilde{\eta}_T(s)|^2 \lambda_{\imath\jmath}\mathbf{1}_{\{\alpha(s)=\imath\}} - K\xi(s) + K\mathbb{E}|\widetilde{\eta}_T(s)|^2 \mathbf{1}_{s\in[T-T_0,T]}\Big)ds.
\end{aligned}$$

Hence, for $t \in [0,T]$, (see (7.2) again)

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}|\zeta_T(s)|^2 ds \leqslant K\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}|\widetilde{\zeta}_T(s)|^2 ds\\
&\leqslant K\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}\Big(|\widetilde{\zeta}_T(s)-\widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2+|\widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2\Big)\mathrm{d}r\\
&\leqslant K\mathbb{E}\Big[|\widetilde{\eta}_T(t)|^2+\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\Big(|\widetilde{\eta}_T(s)|^2+\xi(s)\Big)ds+\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}|\widetilde{\eta}_T(s)|^2\mathbf{1}_{s\in[T-T_0,T]}ds\Big]\\
&\leqslant K\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds+K\mathrm{e}^{-\frac{\delta}{4}(T-t)}\mathbb{E}|\vartheta|^2.
\end{aligned} \tag{7.14}$$

Taking $\vartheta=0$, we get (3.14).

(2). Consider (7.11) and (7.12), and take $T'=\infty$. By virtue of (7.14), using (7.13), we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}|\zeta_T(s)-\zeta_\infty(s)|^2 ds\\
&\leqslant K\mathbb{E}\Big[|\widetilde{\eta}_T(t)-\widetilde{\eta}_\infty(t)|^2+\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\Big(|\widetilde{\eta}_T(s)-\widetilde{\eta}_\infty(s)|^2+|\Delta_{T,\infty}(s)|^2\Big)ds+\int_0^t \mathbb{E}|\widetilde{\eta}_T(s)-\widetilde{\eta}_\infty(s)|^2\mathbf{1}_{s\in[T-T_0,T]}ds\Big]\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds.
\end{aligned}$$

(3) Note that

$$\begin{aligned}
&\mathbb{E}|v_\infty(s,\alpha(s))-v_T(s,\alpha(s))|^2\\
&=\mathbb{E}\big|\widetilde{R}_\infty(\alpha(s))^{-1}[D^\top P_\infty(\alpha(s))\sigma(s)+B^\top\eta_\infty(s)+D^\top\zeta_\infty(s)+r(s)]\\
&\qquad -\widetilde{R}_T(\alpha(s))^{-1}[D^\top P_T(s,\alpha(s))\sigma(s)+B^\top\eta_T(s)+D^\top\zeta_T(s)+r(s)]\big|^2\\
&\leqslant K\mathbb{E}\Big[\big|\widetilde{R}_\infty(\alpha(s))^{-1}\big|\Big(|P_\infty(\alpha(s))-P_T(s,\alpha(s))|\,|\sigma(s)|+|\eta_\infty(s)-\eta_T(s)|+|\zeta_\infty(s)-\zeta_T(s)|\Big)\\
&\quad +\big|\widetilde{R}_\infty(\alpha(s))^{-1}-\widetilde{R}_T(s,\alpha(s))^{-1}\big|\Big(|\eta_\infty(s)|+|\zeta_\infty(s)|+|\sigma(s)|+|r(s)|\Big)\Big]^2\\
&\leqslant K\mathbb{E}\Big[|P_\infty(\alpha(s))-P_T(s,\alpha(s))|^2|\sigma(s)|^2+|\eta_\infty(s)-\eta_T(s)|^2+|\zeta_\infty(s)-\zeta_T(s)|^2\Big)\\
&\quad +\big|P_\infty(\alpha(s))-P_T(s,\alpha(s))\big|^2\Big(|\eta_\infty(s)|^2+|\zeta_\infty(s)|^2+|\sigma(s)|^2+|r(s)|^2\Big)\Big]\\
&\leqslant K\mathbb{E}\Big[\Big(|\eta_\infty(s)-\eta_T(s)|^2+|\zeta_\infty(s)-\zeta_T(s)|^2\\
&\quad +\mathrm{e}^{-2\delta(T-s)}\Big(|\eta_\infty(s)|^2+|\zeta_\infty(s)|^2+|\sigma(s)|^2+|r(s)|^2\Big)\Big].
\end{aligned}$$

Hence, for $t\in[0,T]$, we have

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\mathbb{E}|v_\infty(s,\alpha(s))-v_T(s,\alpha(s))|^2 ds\\
&\leqslant K\mathbb{E}\int_0^t \mathrm{e}^{-\frac{\delta}{4}(t-s)}\Big(|\eta_\infty(s)-\eta_T(s)|^2+|\zeta_\infty(s)-\zeta_T(s)|^2\\
&\qquad\qquad +\mathrm{e}^{-2\delta(T-s)}(|\eta_\infty(s)|^2+|\zeta_\infty(s)|^2+|\sigma(s)|^2+|r(s)|^2)\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{8}(T-t)}\Big(\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds\Big).
\end{aligned}$$

(4) By (7.4) and (7.5), we have

$$\frac{d}{dt}\mathbb{E}|\widetilde{\eta}_T(t)|^2 \geqslant \mathbb{E}|\widetilde{\zeta}_T(t) - \widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t)|^2 - K\xi(t) - K\mathbb{E}|\widetilde{\eta}_T(t)|^2\mathbf{1}_{t\in[T-T_0,T]}.$$

Integrating both sides on $[0,T]$ implies that

$$\begin{aligned}&\mathbb{E}|\widetilde{\eta}_T(T)|^2 - |\widetilde{\eta}_T(0)|^2\\ &\geqslant \int_0^T \mathbb{E}|\widetilde{\zeta}_T(s) - \widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2 - K\xi(s) - K\mathbb{E}|\widetilde{\eta}_T(s)|^2\mathbf{1}_{s\in[T-T_0,T]}\Big)ds.\end{aligned}$$

Hence, taking $\vartheta = 0$,

$$\begin{aligned}\int_0^T \mathbb{E}|\zeta_T(s)|^2 ds &\leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_T(s)|^2 ds\\ &\leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_T(s) - \widetilde{C}^{\Theta_T}(s,\alpha(s))\widetilde{\eta}_T(s)|^2 ds + K\int_0^T \mathbb{E}|\widetilde{\eta}_T(s)|^2 ds\\ &\leqslant K\int_0^T \xi(s)ds + K\int_0^T \mathbb{E}|\widetilde{\eta}_T(s)|^2 ds + \mathbb{E}|\widetilde{\eta}_T(T)|^2\\ &\leqslant K\int_0^T \xi(s)ds + K(T+1)\sup_{s\geqslant 0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r.\end{aligned}$$

Similarly, one has

$$\frac{d}{dt}\mathbb{E}|\widetilde{\eta}_\infty(t)|^2 \geqslant \mathbb{E}|\widetilde{\zeta}_\infty(t) - \widetilde{C}^{\Theta_\infty}(t,\alpha(t))\widetilde{\eta}_\infty(t)|^2 - K\xi(t).$$

Therefore, we have

$$\begin{aligned}\int_0^T \mathbb{E}|\zeta_\infty(s)|^2 ds &\leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_\infty(s)|^2 ds\\ &\leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_\infty(s) - \widetilde{C}^{\Theta_\infty}(s,\alpha(s))\widetilde{\eta}_T(s)|^2 ds + K\int_0^T \mathbb{E}|\widetilde{\eta}_\infty(s)|^2 ds\\ &\leqslant K\int_0^T \xi(s)ds + K\int_0^T \mathbb{E}|\widetilde{\eta}_\infty(s)|^2 ds + \mathbb{E}|\widetilde{\eta}_\infty(T)|^2\\ &\leqslant K\int_0^T \xi(s)ds + K(T+1)\sup_{s\geqslant 0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r.\end{aligned}$$

(5). By (7.11) and (7.12) (letting $T' = \infty$), we have

$$\frac{d}{dt}\mathbb{E}|\widetilde{\eta}_T(t) - \widetilde{\eta}_\infty(t)|^2 \geqslant \mathbb{E}|\widetilde{\zeta}_T(t) - \widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_T(t) - \widetilde{\zeta}_\infty(t) + \widetilde{C}^{\Theta_\infty}(t,\alpha(t))\widetilde{\eta}_\infty(t)|^2 - K\mathbb{E}|\Delta_{T,\infty}(t)|^2.$$

Thus, it follows that

$$\int_0^T \mathbb{E}|\zeta_T(s) - \zeta_\infty(s)|^2 ds \leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_T(s) - \widetilde{\zeta}_\infty(s)|^2 ds$$

$$\begin{aligned}
&\leqslant K\int_0^T \mathbb{E}|\widetilde{\zeta}_{T}(t)-\widetilde{C}^{\Theta_T}(t,\alpha(t))\widetilde{\eta}_{T}(t)-\widetilde{\zeta}_{T}(t)+\widetilde{C}^{\Theta_\infty}(t,\alpha(t))\widetilde{\eta}_{\infty}(t)|^2ds\\
&\quad+K\int_0^T \mathbb{E}|\widetilde{\eta}_{T}(s)-\widetilde{\eta}_{\infty}(s)|^2ds+\mathbb{E}|\widetilde{\eta}_{T}(T)-\widetilde{\eta}_{\infty}(T)|^2\\
&\leqslant K\int_0^T \mathbb{E}|\Delta_{T,\infty}(t)|^2ds+K\int_0^T \mathrm{e}^{-\frac{\delta}{2}(T-s)}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|r-s|}\xi(r)drds+\mathbb{E}|\widetilde{\eta}_{\infty}(T)|^2\\
&\leqslant K\int_0^T \mathrm{e}^{-2\delta(T-t)}\Big(\xi(t)+\mathbb{E}|\eta_{T}(t)|^2+\mathbb{E}|\zeta_{T}(t)|^2\Big)ds+K\sup_{s\geqslant 0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r\\
&\leqslant K\sup_{s\geqslant 0}\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|s-r|}\xi(r)\mathrm{d}r
\end{aligned}$$

In the last step, we used (3.14) (with $t=T$) for $\zeta_{T}(\cdot)$.

(6) We suppress the index $(x,\imath)$ in this part of proof. Note that

$$\begin{cases} d\bar{X}_{T}(t)=[A^{\Theta_T}(t,\alpha(t))\bar{X}_{T}(t)+B(\alpha(t))v_{T}(t,\alpha(t))+b(t)]\mathrm{d}t\\ \qquad\qquad +[C^{\Theta_T}(t,\alpha(t))\bar{X}_{T}(t)+D(\alpha(t))v_{T}(t,\alpha(t))+\sigma(t)]dW(t),\\ \bar{X}_{T}(0)=x,\end{cases}$$

with $A^{\Theta_T}(\cdot\,,\alpha(\cdot))$ and $C^{\Theta_T}(\cdot\,,\alpha(\cdot))$ being given by (2.5). Now, let $\Sigma_\infty(\cdot)\in\Sigma$ satisfy (3.3). Applying Itô's formula to the map $t\mapsto\langle\Sigma_\infty(\alpha(t))\bar{X}_{T}(t),\bar{X}_{T}(t)\rangle$, we have, (see (3.6)) suppressing $\alpha(t)$,

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\mathbb{E}\langle\Sigma_\infty(\alpha(t))\bar{X}_{T}(ts),\bar{X}_{T}(t)\rangle\\
&=\mathbb{E}\Big\langle\Big(\Lambda[\Sigma_\infty]+\Sigma_\infty A^{\Theta_T}+(A^{\Theta_T})^\top\Sigma_\infty+(C^{\Theta_T})^\top\Sigma_\infty C^{\Theta_T}\Big)\bar{X}_{T}(t),\bar{X}_{T}(t)\Big\rangle\\
&\quad+2\mathbb{E}\langle\Sigma_\infty(Bv_{T}(t)+b(t)),\bar{X}(t)\rangle+2\mathbb{E}\langle\Sigma_\infty C^{\Theta_T}\bar{X}_{T}(t),Dv_{T}(t,\alpha(t))+\sigma(t)\rangle\\
&\quad+\mathbb{E}\langle\Sigma_\infty(Dv_{T}(t,\alpha(t))+\sigma(t),Dv_{T}(t,\alpha(t))+\sigma(t)\rangle\\
&\leqslant-\frac{\delta}{2}\mathbb{E}\langle\Sigma_\infty(\alpha(t))\bar{X}_{T}(t),\bar{X}_{T}(t)\rangle+K\mathbb{E}\Big(|b(t)|^2+|\sigma(t)|^2+|v_{T}(t,\alpha(t))|^2\Big).
\end{aligned}$$

Note that

$$\mathbb{E}|v_{T}(t,\alpha(t))|^2\leq K(\xi(t)+\mathbb{E}|\eta_{T}(t)|^2+\mathbb{E}|\zeta_{T}(t)|^2,$$

By Gronwall's inequality, we have

$$\begin{aligned}
&\mathbb{E}\langle\Sigma_\infty(\alpha(t))\bar{X}_{T}(t),\bar{X}_{T}(t)\rangle\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|x|^2+K\int_0^t \mathrm{e}^{-\frac{\delta}{2}(t-s)}\mathbb{E}\Big(|b(s)|^2+|\sigma(s)|^2+|v_{T}(s,\alpha(s))|^2\Big)ds\\
&\leqslant K\mathrm{e}^{-\frac{\delta}{2}t}|x|^2+K\int_0^\infty \mathrm{e}^{-\frac{\delta}{4}|t-s|}\xi(s)ds,\qquad 0\leqslant t<T.
\end{aligned}$$

Therefore, (3.17) holds for $\bar{X}_{T}(\cdot)$. The proof for $\bar{X}_\infty(\cdot)$ is identical. □

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