ESAIM: COCV 32 (2026) 60
https://doi.org/10.1051/cocv/2026042

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

UNIQUE CONTINUATION AND LOCALIZED DAMPING EFFECTS IN NON-HOMOGENEOUS TIMOSHENKO SYSTEMS

Leonardo Gustavo Ronchin Alves, Michele de Oliveira Alves and Rodrigo Nunes Monteiro*

Abstract. We study the stability of contraction semigroups generated by the Timoshenko system with non-homogeneous coefficients and localized frictional damping. Formulated as an abstract Cauchy problem, the system is shown to be well-posed *via* semigroup theory. Our main contribution is a Unique Continuation Property (UCP) for the resolvent equation associated with the system, which allows us to establish the strong stability of the corresponding semigroup. Combining this with resolvent estimates, we establish precise semigroup decay rates, which are exponential or polynomial depending on the interaction between the non-homogeneous coefficients and the support of the localized damping coefficient.

Mathematics Subject Classification. 35B40, 35Q74.

Received September 16, 2025. Accepted May 21, 2026.

1. Introduction

1.1. Non-homogeneous Timoshenko system

Semigroup theory provides a natural framework to study the energy decay of dissipative evolution equations. For many hyperbolic systems, the long-time behavior of solutions is determined by spectral properties of the generator, together with resolvent estimates on the imaginary axis. The Timoshenko beam system is a classical model in structural mechanics that fits this framework. In this work, we study this system and describe its asymptotic behavior from the semigroup perspective.

The model under consideration is governed by the partial differential equations

$$\begin{aligned} \rho_1(x)\varphi_{tt} - [k(x)(\varphi_x + \psi)]_x = 0 \ \text{ in } \ (0,\infty)\times(0,L), \\ \rho_2(x)\psi_{tt} - [b(x)\psi_x]_x + k(x)(\varphi_x+\psi) + \gamma(x)\psi_t = 0 \ \text{ in } \ (0,\infty)\times(0,L). \end{aligned} \tag{1.1}$$

To the governing system (1.1) we add initial conditions

$$\varphi(0,\cdot) = \varphi_0,\ \varphi_t(0,\cdot) = \varphi_1,\ \psi(0,\cdot) = \psi_0,\ \psi_t(0,\cdot) = \psi_1, \tag{1.2}$$

Keywords and phrases: Unique continuation property, localized damping, non-constant coefficients.

Department of Mathematics, Universidade Estadual de Londrina, 86.057-970 Londrina, PR, Brazil.

* Corrersponding author: monteirorn@uel.br

and Dirichlet boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = 0. \tag{1.3}$$

The Timoshenko system describes transverse vibrations of a flexible beam of length $L > 0$. The unknown functions $\varphi = \varphi(t,x)$ and $\psi = \psi(t,x)$ represent, respectively, the transversal displacement and rotation angle of a filament of the beam. The positive functions $\rho_1 = \rho_1(x)$, $\rho_2 = \rho_2(x)$, $k = k(x)$ and $b = b(x)$ represent the physical properties of the material and are given by $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k = k^{'}GA$ and $b = EI$. Here, ρ is the mass density per unit of the reference area, A is the cross-sectional area, I is the second moment of area of the cross section, $k^{'}$ is the shear factor coefficient, G is the shear modulus and E is the modulus of elasticity. Furthermore, $\gamma = \gamma(x)$ is a non-negative function responsible for the localization of the frictional damping ψ_t.

1.2. Literature

We now provide a brief review of the literature closely related to problem (1.1)–(1.3), with emphasis on the number and type of dissipation mechanisms considered, as well as on the presence of non-homogeneous coefficients.

One dissipative effect. We begin with the well-known stability number for the Timoshenko system, defined by

$$\chi = \frac{k}{\rho_1} - \frac{b}{\rho_2}.$$

Typically, for partially dissipative problems, the energy decays exponentially when the wave propagation speeds are equal, that is, when $\chi = 0$, and polynomially when $\chi \neq 0$.

When the damping acts on the rotation equation $(1.1)_2$, several results have been obtained. In [1], the author studied the case of linear frictional damping and proved that the energy decays exponentially when $\chi = 0$. When the propagation speeds differ, that is when $\chi \neq 0$, it was shown that the energy does not decay exponentially, and no explicit decay rate was obtained. In [2], the analysis was extended to the non-homogeneous coefficients case with linear frictional damping, yielding exponential stability under the assumption

$$\chi(x) = \frac{k(x)}{\rho_1(x)} - \frac{b(x)}{\rho_2(x)} \text{ for all } x \in (0,L). \tag{1.4}$$

In [3] the same problem was considered and the exponential decay was proved when $\chi(x) = 0$ for all $x \in (0,L)$, while showing that no exponential decay holds when $\chi(x) \neq 0$ for all $x \in (0,L)$. In [4], the Timoshenko system with linear localized damping was investigated showing that the exponential stability holds in the equal-speed case $\chi = 0$ while in the unequal speed case $\chi \neq 0$ the polynomial rate of decay of order $\left(\ln(t)\, t^{-1}\right)^{\frac{1}{l}} \ln(t)$, with $l > 2$, as $t \to +\infty$, was obtained. The thermoelastic Timoshenko system with non-homogeneous coefficients was studied in [5], where exponential stability was shown when $\chi(x) = 0$ for all x on some subinterval $I \subset (0,L)$, and otherwise a polynomial rate of order $t^{-\frac{1}{2}}$, as $t \to +\infty$, was obtained.

Other works considered nonlinear damping mechanisms. In [6], it was shown that when $\chi = 0$, the decay rate of the energy depends on the growth of the damping function near the origin, while for $\chi \neq 0$ with linear damping, it is shown that the decay rate is of polynomial type, specifically t^{-N}, with $N \in \mathbb{N}$, as $t \to +\infty$. In [7], nonlinear localized damping was analyzed, establishing convergence to zero without an explicit rate.

Different from these works, [8] studied the problem with frictional dissipation acting on the transversal displacement equation $(1.1)_1$ proving exponential decay when $\chi = 0$ and polynomial decay of order $t^{-\frac{1}{2}}$, as $t \to +\infty$, when $\chi \neq 0$.

Two dissipative effects. When both equations in system (1.1) are subject to distinct damping mechanisms, exponential stability is often obtained. Several works addressed this setting with linear frictional damping [9], nonlinear localized damping [10, 11] for constant coefficients and [12] for non-homogeneous coefficients, as well as boundary damping [13]. In all these cases, uniform energy decay was established.

1.3. Contributions

Unique Continuation Property (UCP). When dealing with localized dissipation, the UCP is an essential tool. Previous works, such as [10, 12], employed different approaches depending on the coefficients. In [10], with constant coefficients, the UCP follows from a distributional version of Holmgren Uniqueness Theorem [14], whereas in [12], with non-homogeneous coefficients, it follows from results in [15]. In both cases, the UCP was employed to derive observability inequalities *via* contradiction arguments. In contrast, in the present work we prove a new UCP (see Thm. 1.2) for the resolvent equation, established directly from observability estimates rather than classical results, which is then used to demonstrate the strong stability of the associated semigroup.

Interaction between dissipation and non-homogeneous coefficients. To the best of our knowledge, no previous study has established stability properties for the Timoshenko system with a single localized dissipative effect in combination with non-homogeneous coefficients. This gap in the literature motivates our second main contribution, which is to show that stability depends on how the equal wave speed condition (1.4) holds locally and how it interacts with the support of the damping function γ. This local interaction plays a fundamental role in determining the qualitative behavior of the solutions (see Thms. 1.3 and 1.4).

1.4. Statement of main results

Here, we study the problem (1.1)–(1.3) under the following assumptions.

Assumption (A.1). The positive functions satisfy ρ_1, ρ_2 and b belong to $W^{1,\infty}(0,L)$ and k belongs to $W^{2,\infty}(0,L)$. Moreover, all these functions are bounded below by a positive constant.

Assumption (A.2). The non-negative function γ belongs to $L^\infty(0,L)$ and there exists a sub-interval $(\ell_0,\ell_1) \subset (0,L)$ satisfying

$$0 < \gamma_0 \leqslant \gamma(x), \ \forall\, x \in (\ell_0,\ell_1).$$

Before describing our main result, we shall establish that the problem (1.1)–(1.3) defines a differential operator which is the generator of a strongly continuous contraction semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$ on $\mathscr{H}$ (see Thm. 2.2). Indeed, for the unknown vector function $\mathbf{U} = (\varphi, \Phi, \psi, \Psi)$ with $\Phi = \varphi_t$, and $\Psi = \psi_t$, the problem (1.1)–(1.3) is equivalent to the following abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U} = \mathscr{A}\mathbf{U} \text{ and } \mathbf{U}(0) = (\varphi_0, \varphi_1, \psi_0, \psi_1) = \mathbf{U}_0, \tag{1.5}$$

where $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ is the differential operator given

$$\mathscr{A}\mathbf{U} = \begin{pmatrix} \Phi \\ \rho_1^{-1}[k(\varphi_x + \psi)]_x \\ \Psi \\ \rho_2^{-1}[b\psi_x]_x \,+\, k\rho_2^{-1}(\varphi_x + \psi) \,-\, \rho_2^{-1}\gamma\,\Psi \end{pmatrix} \tag{1.6}$$

defined on the Hilbert space $\mathscr{H}$ given by

$$\mathscr{H} = H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L),$$

with

$$D(\mathscr{A}) = \Big\{ \mathbf{U} \in \mathscr{H} \mid \varphi, \psi \in H^2(0,L) \cap H_0^1(0,L) \text{ and } \Phi, \Psi \in H_0^1(0,L) \Big\}.$$

The space $\mathscr{H}$ is equipped with the norm (for simplicity of notation $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(0,L)}$)

$$\|\mathbf{U}\|_{\mathscr{H}}^2 = \|k^{\frac{1}{2}}(\varphi_x + \psi)\|_{L^2}^2 + \|b^{\frac{1}{2}}\psi_x\|_{L^2}^2 + \|\rho_1^{\frac{1}{2}}\Phi\|_{L^2}^2 + \|\rho_2^{\frac{1}{2}}\Psi\|_{L^2}^2 \tag{1.7}$$

and

$$\|\mathbf{U}\|_{D(\mathscr{A})} = \|\mathscr{A}\mathbf{U}\|_{\mathscr{H}}.$$

Remark 1.1. Assumption $(A.1)$ ensures that the norm (1.7) is equivalent to the standard norm on $\mathscr{H}$.

Main results. With the above notations, we are now ready to present our main contribution. To establish the strong stability of the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$, a unique continuation property (UCP) for the associated resolvent equation is required. This is essential, as the condition $i\mathbb{R} \subset \varrho(\mathscr{A})$, where $\varrho(\mathscr{A})$ denotes the resolvent set of $\mathscr{A}$, does not follow from the compactness of the resolvent operator alone, in contrast to problems where the damping acts on the whole domain (see Rem. 4.3). Our first main result is stated in the following theorem. We note that this result extends beyond the present setting and applies to other types of localized dissipation. Moreover, the proof can be adapted to the Bresse system.

Theorem 1.2. [Unique Continuation Property] *Under the assumptions $(A.1)$-$(A.2)$, let $\mathbf{U} \in D(\mathscr{A})$ be a solution to the resolvent equation $(i\lambda I - \mathscr{A})\mathbf{U} = 0$, for some $\lambda \in \mathbb{R}$. If there exists an interval $(p,q) \subset (\ell_0, \ell_1)$ where $\mathbf{U}(x) = 0$ for all $x \in (p,q)$, then $\mathbf{U}$ must be identically zero.*

To state our next main results precisely, we recall the function χ, originally introduced in the introduction, and defined by

$$\chi(x) = \frac{k(x)}{\rho_1(x)} - \frac{b(x)}{\rho_2(x)}. \tag{1.8}$$

The following main result establishes sufficient conditions for the exponential stability of the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$. The condition requires that the function χ in (1.8) vanishes on an arbitrary open subset where the dissipation is active, namely, on a subset of (ℓ_0, ℓ_1).

Theorem 1.3. [Exponential Decay Rates] *Under assumptions $(A.1)$-$(A.2)$ and let χ be defined as in (1.8). Suppose there exists a non-empty open interval $(p,q) \subset (\ell_0, \ell_1)$, such that, $\chi = 0$ on (p,q). Then the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$ decays exponentially.*

The third main result establishes polynomial decay rates for the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$. We point out that, although the techniques employed are not new, they allow us to improve the decay rates previously obtained in [4].

Theorem 1.4. [Polynomial Decay Rates] *Under assumptions $(A.1)$-$(A.2)$ and let χ be defined as in (1.8). Suppose that $\chi \neq 0$ for all $x \in (\ell_0, \ell_1)$. Then there exists a constant $C > 0$, such that, the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$ satisfies the polynomial decay rate*

$$\|S_{\mathscr{A}}(t)\mathbf{U}_0\|_{\mathscr{H}} \leqslant C t^{-\frac{1}{2}} \|\mathbf{U}_0\|_{D(\mathscr{A})}, \ \forall\, t > 0.$$

Figures 1 and 2 illustrate examples of functions that satisfy the conditions stated in Theorems 1.3 and 1.4, respectively.

Orientation. The development of the present manuscript is divided into three sections. Section 2 is devoted to the well-posedness of the problem (1.1)–(1.3) (equivalent to (1.5)). Section 3 contains the proof of the unique continuation result stated in Theorem 1.2, and Section 4 is dedicated to the proof of the main Theorems 1.3 and 1.4. We also emphasize that all constants will be denoted by the same symbol C, with any dependence on specific parameters indicated by an appropriate subscript.

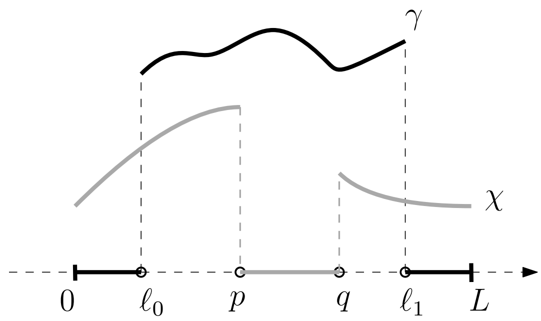

FIGURE 1. Theorem 1.3.

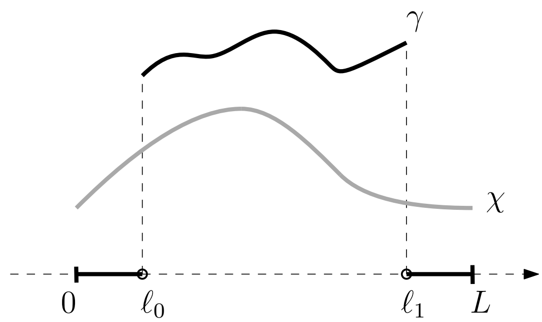

FIGURE 2. Theorem 1.4.

2. Well-posedness

The main goal of this section is to establish the well-posedness of problem (1.1)–(1.3), which is equivalent to obtaining the well-posedness for the Cauchy problem (1.5). To achieve this, we shall use the Lumer–Phillips theorem. The next lemma collects all the assumptions required for the operator $\mathscr{A}$, defined in (1.6), to generate a strongly continuous contraction semigroup on $\mathscr{H}$.

Lemma 2.1. *Assume that $\rho_1, \rho_2, k, b \in W^{1,\infty}(0, L)$ and $\gamma \in L^\infty(0, L)$. Then*
1. *$\mathscr{A}$ is a dissipative operator on $\mathscr{H}$.*
2. *$0 \in \varrho(\mathscr{A})$.*

Proof. Straightforward calculations imply that

$$\mathrm{Re}\big(\mathscr{A}\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}} = -\int_0^L \gamma|\Psi|^2 \mathrm{d}x, \ \forall\, \mathbf{U} \in D(\mathscr{A}). \tag{2.1}$$

This shows that $\mathscr{A}$ is a dissipative operator on $\mathscr{H}$.

To prove Item 2, we will demonstrate the $\mathscr{A}^{-1}$ exists and is a bounded operator. Let $\mathbf{F} = (f_1, f_2, f_3, f_4) \in \mathscr{H}$. Taking

$$\Phi = -f_1, \ \Psi = -f_3,$$

it remains to find $\varphi, \psi \in H^2(0, L) \cap H_0^1(0, L)$ such that

$$[k(\varphi_x + \psi)]_x = -\rho_1 f_2, \tag{2.2}$$
$$[b\psi_x]_x - k(\varphi_x + \psi) = -\rho_2 f_4 + \gamma f_3. \tag{2.3}$$

In the Hilbert space $\mathscr{H}_0 = H_0^1(0,L) \times H_0^1(0,L)$ endowed with the usual norm, we define the sesquilinear form $a : \mathscr{H}_0 \times \mathscr{H}_0 \to \mathbb{C}$ by

$$a\big((\varphi,\psi),(\varphi^*,\psi^*)\big) = \int_0^L k(\varphi_x+\psi)\overline{(\varphi_x^*+\psi^*)}\,\mathrm{d}x + \int_0^L b\psi_x\overline{\psi_x^*}\,\mathrm{d}x.$$

The form a is continuous, that is there exists a constant $C>0$, such that

$$|a\big((\varphi,\psi),(\varphi^*,\psi^*)\big)| \leqslant C\|(\varphi,\psi)\|_{\mathscr{H}_0}\|(\varphi^*,\psi^*)\|_{\mathscr{H}_0}.$$

Also, note that a is coercive. Indeed, the norm endowed in $\mathscr{H}$ is equivalent to usual norm, which allow us to conclude the existence of a constant $C>0$, such that

$$\begin{aligned}\|(\varphi,\psi)\|_{\mathscr{H}_0}^2 &= \|\varphi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2\\ &\leqslant C\Big(\|k^{\frac12}(\varphi_x+\psi)\|_{L^2}^2 + \|b^{\frac12}\psi_x\|_{L^2}^2\Big)\\ &= C\,a\big((\varphi,\psi),(\varphi,\psi)\big).\end{aligned}$$

Next, we define the continuous anti-linear functional defined on $\mathscr{H}_0$

$$\alpha(\varphi^*,\psi^*) = \int_0^L \rho_1 f_2\overline{\varphi^*}\,\mathrm{d}x + \int_0^L (\rho_2 f_4 - \gamma f_3)\overline{\psi^*}\,\mathrm{d}x.$$

By Lax–Milgran Theorem, for every $(\varphi^*,\psi^*)\in\mathscr{H}_0$, that exists a unique solution $(\varphi,\psi)\in\mathscr{H}_0$, such that

$$a((\varphi,\psi),(\varphi^*,\psi^*)) = \alpha(\varphi^*,\psi^*).$$

Moreover, $(\varphi,\psi)\in[H^2(0,L)\cap H_0^1(0,L)]^2$ and satisfies the equations in (2.2)–(2.3). To see this, we take $\varphi^*\in C_0^\infty(0,L)$ and $\psi^*=0$. Then it follows that

$$\varphi_{xx} = -k^{-1}\rho_1 f_2 + -k^{-1}k'(\varphi_x+\psi) + \psi_x \in L^2(0,L).$$

The above and the fact that $\rho_1, k \in W^{1,\infty}(0,L)$ imply that $\varphi\in H^2(0,L)\cap H_0^1(0,L)$. The same argument applied to ψ, allows us to conclude that $\psi\in H^2(0,L)\cap H_0^1(0,L)$ and consequently, that (2.2)–(2.3) are satisfied. This also allows us to conclude that $\mathscr{A}^{-1}$ exists.
It remains to show that the operator $\mathscr{A}^{-1}$ is continuous. Multiplying in $L^2(0,L)$ the equations (2.2) and (2.3) by φ and ψ, respectively, and then adding up the results, we obtain

$$\|\mathbf{U}\|_{\mathscr{H}}^2 = \|\rho_1^{\frac12} f_1\|_{L^2}^2 + \|\rho_2^{\frac12} f_3\|_{L^2}^2 + \int_0^L \rho_1 f_2\overline{\varphi}\,\mathrm{d}x + \int_0^L [\rho_2 f_4 - \gamma f_3]\overline{\psi}\,\mathrm{d}x.$$

From the assumptions $\rho_1,\rho_2\in W^{1,\infty}(0,L)$ and $\gamma\in L^\infty(0,L)$ there exists a constant $C>0$, such that

$$\|\mathbf{U}\|_{\mathscr{H}}^2 \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{F}\|_{\mathscr{H}}^2.$$

Therefore, $\mathscr{A}^{-1}$ is a continuous operator.

This concludes the proof. □

The following result states that the problem (1.1)–(1.3) is well-posed. The proof is a consequence of the combination of Lemma 2.1 and the Lumer–Philips Theorem.

Theorem 2.2. *Assume that $\rho_1, \rho_2, k, b \in W^{1,\infty}(0,L)$ and $\gamma \in L^\infty(0,L)$. The operator $\mathscr{A}$ defined in (1.6) generates a strongly continuous contraction semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$ on $\mathscr{H}$.*

3. Unique continuation property

In this section, we will prove the unique continuation result stated in Theorem 1.2. As mentioned earlier, this result is crucial for establishing that $i\mathbb{R} \subset \varrho(\mathscr{A})$, which is a fundamental condition for Theorems 1.3 and 1.4. Before, we present observability inequalities, which will be important not only for the proof of Theorem 1.2 but also for extending the local estimates obtained in the preceding sections.

3.1. Observability inequalities

The goal of this section is to establish observability inequalities for the proposed model. To this end, we utilize the observability inequality for the homogeneous Timoshenko system, as stated in Proposition 3.12 from reference [5] (see also Lem. 3.2 from reference [16]).

In this scenario let $\mathbf{U} \in D(\mathscr{A})$ and $\mathbf{F} = (f_1, f_2, f_3, f_4) \in \mathscr{H}$ satisfying the resolvent equation

$$(i\lambda I - \mathscr{A})\mathbf{U} = \mathbf{F}. \tag{3.1}$$

Taking

$$g_1 = f_1, \quad g_2 = \rho_1 f_2, \quad g_3 = f_3, \quad g_4 = \rho_2 f_4 - \gamma\Psi.$$

Then $\mathbf{G} = (g_1, g_2, g_3, g_4) \in \mathscr{H}$, since $\rho_1, \rho_2 \in W^{1,\infty}(0,L)$ and $\gamma \in L^\infty(0,L)$. Hence, we conclude that $\mathbf{U} \in D(\mathscr{A})$ also satisfies the equation

$$(i\lambda I - \mathscr{A}_1)\mathbf{U} = \mathbf{G},$$

where $\mathscr{A}_1$ is the abstract operator associated with the homogeneous Timoshenko system ($\gamma = 0$) with domain $D(\mathscr{A}_1) = D(\mathscr{A})$. Therefore, Proposition 3.12 proved in [5] implies that there exist constants $C_0, C_1 > 0$, such that, for any $0 \leqslant a_1 < a_2 \leqslant L$ the following estimates hold true

$$E_{\mathbf{U}}(a_j) \leqslant C_0 \int_{a_1}^{a_2} E_{\mathbf{U}}(x)\mathrm{d}x + C_0\|\mathbf{G}\|^2_{\mathscr{H}} \ (j = 1, 2), \tag{3.2}$$

$$\int_{a_1}^{a_2} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant C_1 E_{\mathbf{U}}(a_j) + C_1\|\mathbf{G}\|^2_{\mathscr{H}} \ (j = 1, 2), \tag{3.3}$$

where, $E_{\mathbf{U}}(\cdot)$ is defined by

$$E_{\mathbf{U}}(\cdot) = \left|[k^{\frac{1}{2}}(\varphi_x + \psi)](\cdot)\right|^2 + \left|(b^{\frac{1}{2}}\psi_x)(\cdot)\right|^2 + \left|(\rho_1^{\frac{1}{2}}\Phi)(\cdot)\right|^2 + \left|(\rho_2^{\frac{1}{2}}\Psi)(\cdot)\right|^2. \tag{3.4}$$

Note that, from the definition of $E_{\mathbf{U}}(\cdot)$ and taking $0 \leqslant a_1 < a_2 \leqslant L$, we promptly find

$$\int_{a_1}^{a_2} E_{\mathbf{U}}(x)\,\mathrm{d}x = \|k^{\frac{1}{2}}(\varphi_x + \psi)\|^2_{L^2(a_1,a_2)} + \|b^{\frac{1}{2}}\psi_x\|^2_{L^2(a_1,a_2)} + \|\rho_1^{\frac{1}{2}}\Phi\|^2_{L^2(a_1,a_2)} + \|\rho_2^{\frac{1}{2}}\Psi\|^2_{L^2(a_1,a_2)}. \tag{3.5}$$

The above discussion proves the following statement.

Proposition 3.1. *Suppose that the assumptions* (*A*.1) *hold and* $\gamma \in L^\infty(0,L)$*. If* $\mathbf{U} \in D(\mathscr{A})$ *satisfies the resolvent equation* (3.1)*, then there exists a constant* $C > 0$*, such that, for any* $0 \leqslant a_1 < a_2 \leqslant L$

$$E_{\mathbf{U}}(a_j) \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{F}\|^2_{\mathscr{H}} + C\int_{a_1}^{a_2} E_{\mathbf{U}}(x)\,\mathrm{d}x \ (j=1,2), \tag{3.6}$$

$$\int_{a_1}^{a_2} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{F}\|^2_{\mathscr{H}} + CE_{\mathbf{U}}(a_j) \ (j=1,2). \tag{3.7}$$

Proof. From estimates (3.2) and (3.3), we only need to estimate the norm $\|\mathbf{G}\|_{\mathscr{H}}$. More precisely, by the definition of the norm in $\mathscr{H}$, it is enough to estimate

$$\|\mathbf{G}\|^2_{\mathscr{H}} = \|k^{\frac{1}{2}}(f_{1,x}+f_3)\|^2_{L^2} + \|b^{\frac{1}{2}}f_{3,x}\|^2_{L^2} + \|\rho_1^{\frac{3}{2}}f_2\|^2_{L^2} + \|\rho_2^{\frac{1}{2}}(\rho_2 f_4 - \gamma\Psi)\|^2_{L^2}. \tag{3.8}$$

In view of (3.8), we only need to detail the estimate of the last term. First, using (2.1), we obtain

$$\int_0^L \gamma|\Psi|^2\mathrm{d}x = -\operatorname{Re}\big(\mathscr{A}\mathbf{U},\mathbf{U}\big)_{\mathscr{H}} = \operatorname{Re}\big(\mathbf{F},\mathbf{U}\big)_{\mathscr{H}} \leqslant \|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}.$$

This implies

$$\|\rho_2^{\frac{1}{2}}(\rho_2 f_4 - \gamma\Psi)\|^2_{L^2} \leqslant C\int_0^L \rho_2|f_4|^2\mathrm{d}x + C\int_0^L \gamma|\Psi|^2\mathrm{d}x \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{F}\|^2_{\mathscr{H}}. \tag{3.9}$$

Plugging (3.9) into (3.8), we conclude that

$$\|\mathbf{G}\|^2_{\mathscr{H}} \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{F}\|^2_{\mathscr{H}}.$$

Therefore, inserting the estimate for $\|\mathbf{G}\|_{\mathscr{H}}$ into (3.2) and (3.3), we infer that (3.6) and (3.7) hold true. □

3.2. Proof of Theorem 1.2

With the estimates in Proposition 3.1 at hand, we are able to prove the first main result.

Proof of Theorem 1.2. Suppose that there exists an interval $(p,q) \subset (\ell_0,\ell_1)$, such that, $\mathbf{U}(x) = 0$ for all $x \in (p,q)$. It then follows that $E_{\mathbf{U}}(\cdot) = 0$ in (p,q). Next, we consider the decomposition

$$\|\mathbf{U}\|^2_{\mathscr{H}} = \int_0^L E_{\mathbf{U}}(x)\,\mathrm{d}x = \int_0^p E_{\mathbf{U}}(x)\,\mathrm{d}x + \int_p^q E_{\mathbf{U}}(x)\,\mathrm{d}x + \int_q^L E_{\mathbf{U}}(x)\,\mathrm{d}x. \tag{3.10}$$

Applying estimates (3.6) and (3.7) from Proposition 3.1, first with $a_1 = 0$, $a_2 = p$, and $\mathbf{F} = 0$, and subsequently with $a_1 = p$, $a_2 = q$, and $\mathbf{F} = 0$, we obtain

$$\int_0^p E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant CE_{\mathbf{U}}(p) \leqslant C\int_p^q E_{\mathbf{U}}(x)\,\mathrm{d}x = 0.$$

Similarly, using (3.6) and (3.7), first with $a_1 = q$, $a_2 = L$, and $\mathbf{F} = 0$, and then with $a_1 = p$, $a_2 = q$, and $\mathbf{F} = 0$, we obtain

$$\int_q^L E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant CE_{\mathbf{U}}(q) \leqslant C\int_p^q E_{\mathbf{U}}(x)\,\mathrm{d}x = 0.$$

Combining these estimates with (3.10) yields

$$\|\mathbf{U}\|_{\mathscr{H}}^2 \leqslant C \int_p^q E_{\mathbf{U}}(x)\,\mathrm{d}x = 0.$$

Hence $\|\mathbf{U}\|_{\mathscr{H}} = 0$, and therefore $\mathbf{U} = 0$. This completes the proof. □

4. Proof of Theorems 1.3 and 1.4

The proof of the Theorems 1.3 and 1.4 will proceed by demonstrating that the generator $\mathscr{A}$ in (1.6) of the C_0-semigroup of contractions $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$ established by Theorem 2.2 has no spectrum on the imaginary axis, that is $i\mathbb{R} = \{i\lambda \mid \lambda \in \mathbb{R}\} \subset \varrho(\mathscr{A})$, and there exists a constant $C > 0$, such that

$$\|(i\lambda I - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \leqslant C|\lambda|^N. \tag{4.1}$$

According to Pruss Theorem [17], for $N = 0$, we obtain exponential decay. Furthermore, by the Borichev–Tomilov Theorem [18], for $N > 1$, we achieve the polynomial decay rate $t^{\frac{1}{N}}$, as $t \to \infty$.

Firstly, note that the resolvent equation of operator $\mathscr{A}$ is given by taking $\lambda \in \mathbb{R}$, $\mathbf{F} = (f_1, f_2, f_3, f_4) \in \mathscr{H}$ and assuming that there is $\mathbf{U} = (\varphi, \Phi, \psi, \Psi) \in D(\mathscr{A})$, such that

$$(i\lambda I - \mathscr{A})\mathbf{U} = \mathbf{F} \tag{4.2}$$

or equivalently

$$i\lambda\varphi - \Phi = f_1, \tag{4.3}$$
$$i\lambda\rho_1\Phi - [k(\varphi_x + \psi)]_x = \rho_1 f_2, \tag{4.4}$$
$$i\lambda\psi - \Psi = f_3, \tag{4.5}$$
$$i\lambda\rho_2\Psi - [b\psi_x]_x + k(\varphi_x + \psi) + \gamma\Psi = \rho_2 f_4. \tag{4.6}$$

Remark 4.1. The proof of the estimate (4.1) is equivalent to showing that there exists a constant $C > 0$, such that, the solution $\mathbf{U} \in D(\mathscr{A})$ of the resolvent equation (4.2) satisfies

$$\|\mathbf{U}\|_{\mathscr{H}} \leqslant C|\lambda|^N \|\mathbf{F}\|_{\mathscr{H}}, \ \forall \lambda \in \mathbb{R} \text{ and } |\lambda| \geqslant 1. \tag{4.7}$$

Orientation of the proof.

1. As discussed earlier, the main consequence of Theorem 1.2 is to show that $i\mathbb{R} \subset \varrho(\mathscr{A})$. This will be established in the next section.

2. To establish the resolvent condition (4.7), we begin by employing multiplier techniques in combination with carefully chosen cut-off functions. This allows us to estimate separately each component of the norm of $\mathbf{U} \in D(\mathscr{A})$, leading to a collection of local estimates, as developed in Section 4.2.1. The argument is then completed in Section 4.2.2, where the observability inequalities established in Proposition 3.1 yield the corresponding global estimates, and therefore provide the desired resolvent estimate (4.7).

4.1. Strong stability

Lemma 4.2. *If Assumptions (A.1) and (A.2) are satisfied, then $i\mathbb{R} \subset \varrho(\mathscr{A})$.*

Proof. Since $D(\mathscr{A})$ is compactly embedded in $\mathscr{H}$, the spectrum $\sigma(\mathscr{A})$ consists only of eigenvalues. Therefore, if any $i\lambda \in \sigma(\mathscr{A}) \cap i\mathbb{R}$ exists, there is an eigenvector $\mathbf{U} \in D(\mathscr{A})$, such that

$$(i\lambda I - \mathscr{A})\mathbf{U} = 0.$$

Multiplying the above equality by $\mathbf{U}$, we find

$$i\lambda \|\mathbf{U}\|_{\mathscr{H}}^2 - \big(\mathscr{A}\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}} = 0.$$

The dissipative property (2.1) implies that

$$0 = \mathrm{Re}\big(\mathscr{A}\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}} = -\int_0^L \gamma |\Psi|^2 \,\mathrm{d}x.$$

From Assumption $(A.2)$, we obtain

$$\Psi = 0 \text{ in } (\ell_0, \ell_1). \tag{4.8}$$

We feed this information back into resolvent equation (4.5) to show that

$$\psi = 0 \text{ in } (\ell_0, \ell_1). \tag{4.9}$$

Using the identities (4.8) and (4.9) in (4.6), we obtain

$$k(\varphi_x + \psi) = 0 \text{ in } (\ell_0, \ell_1).$$

The preceding identity, together with the resolvent equation (4.4), allows us to conclude that

$$\Phi = 0 \text{ in } (\ell_0, \ell_1) \tag{4.10}$$

Next, from identity (4.10) and resolvent equation (4.3), we obtain

$$\varphi = 0 \text{ in } (\ell_0, \ell_1). \tag{4.11}$$

Therefore, from (4.8), (4.9), (4.10) and (4.11), we find that

$$\mathbf{U}(x) = 0, \ \forall x \in (\ell_0, \ell_1).$$

Finally, we recall Theorem 1.2 to conclude that $\mathbf{U} = 0$, which leads to a contradiction. $\square$

Remark 4.3. We emphasize once again the crucial role of the UCP. In the presence of localized dissipation, the condition $i\mathbb{R} \subset \varrho(\mathscr{A})$ cannot be deduced from the compactness of the resolvent of $\mathscr{A}$ by itself. As shown in the preceding proof, $\|\gamma^{\frac{1}{2}}\Psi\|_{L^2(0,L)} = 0$ does not imply that $\Psi = 0$ in $(0, L)$, making UCP indispensable for establishing the resolvent set condition.

4.2. Proof of resolvent operator condition (4.1)

4.2.1. Local estimates

Cut-off functions. Let $L_0 \in (\ell_0, \ell_1)$ and $\delta > 0$ be chosen such that

$$(L_0 - \delta, L_0 + \delta) \subset (\ell_0, \ell_1).$$

Later, in the proofs of Theorems 1.3 and 1.4, we will choose and fix L_0 and δ so that

$$(L_0-\delta, L_0+\delta) \subset (p,q) \subset (\ell_0, \ell_1).$$

Furthermore, we consider cut-off functions $s_1, s_2 \in C^2(0,L)$ satisfying

$$\operatorname{supp}(s_1) \subset (L_0-\delta, L_0+\delta),\ \ s_1 = 1 \text{ in } \left[L_0-\frac{\delta}{2}, L_0+\frac{\delta}{2}\right] \ \text{ and } \ 0 \leqslant s_1 \leqslant 1$$

and

$$\operatorname{supp}(s_2) \subset \left(L_0-\frac{\delta}{2}, L_0+\frac{\delta}{2}\right),\ \ s_2 = 1 \text{ in } \left[L_0-\frac{\delta}{3}, L_0+\frac{\delta}{3}\right] \ \text{ and } \ 0 \leqslant s_2 \leqslant 1.$$

Local estimate for the component Ψ.

Lemma 4.4. *Let $\rho_1, \rho_2, k, b \in W^{1,\infty}(0,L)$ and $\gamma \in L^\infty(0,L)$. If $\mathbf{U} \in D(\mathscr{A})$ solves the resolvent equation (4.2), then*

$$\int_0^L \gamma |\Psi|^2 \,\mathrm{d}x \leqslant \|\mathbf{U}\|_{\mathscr{H}} \|\mathbf{F}\|_{\mathscr{H}}.$$

Proof. Note that

$$\big((i\lambda I - \mathscr{A})\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}} = i\lambda \|\mathbf{U}\|^2_{\mathscr{H}} - \big(\mathscr{A}\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}}.$$

Taking the real part and using Lemma 2.1 (Item 1.), we obtain

$$\int_0^L \gamma |\Psi|^2 \mathrm{d}x = \operatorname{Re}\big((i\lambda I - \mathscr{A})\mathbf{U}, \mathbf{U}\big)_{\mathscr{H}} = \operatorname{Re}\big(\mathbf{F}, \mathbf{U}\big)_{\mathscr{H}} \leqslant \|\mathbf{U}\|_{\mathscr{H}} \|\mathbf{F}\|_{\mathscr{H}}.$$

□

Lemma 4.5. *Let $\rho_1, \rho_2, k, b \in W^{1,\infty}(0,L)$ and assume that Assumption (A.2) is satisfied. If $\mathbf{U} \in D(\mathscr{A})$ solves the resolvent equation (4.2), then there exists a constant $C > 0$, such that*

$$\int_0^L s_1 \rho_2 |\Psi|^2 \,\mathrm{d}x \leqslant C \|\mathbf{U}\|_{\mathscr{H}} \|\mathbf{F}\|_{\mathscr{H}}.$$

Proof. Note that

$$\int_0^L s_1 \rho_2 |\Psi|^2 \mathrm{d}x = \int_{L_0-\delta}^{L_0+\delta} s_1 \rho_2 |\Psi|^2 \mathrm{d}x = \int_{L_0-\delta}^{L_0+\delta} s_1 \rho_2 \frac{\gamma}{\gamma} |\Psi|^2 \mathrm{d}x \leqslant \frac{C}{\gamma_0} \int_0^L \gamma |\Psi|^2 \mathrm{d}x.$$

The conclusion follows from Lemma 4.4. □

Local estimate for the component ψ_x.

Lemma 4.6. *Let Assumptions (A.1)-(A.2) hold. If $|\lambda| \geqslant 1$ and $\mathbf{U} \in D(\mathscr{A})$ solves the resolvent equation (4.2), then there exists a constant $C > 0$, such that*

$$\int_0^L s_1 b |\psi_x|^2 \,\mathrm{d}x \leqslant C\Big(\|\mathbf{U}\|_{\mathscr{H}} \|\mathbf{F}\|_{\mathscr{H}} + \|\mathbf{U}\|^{\frac{3}{2}}_{\mathscr{H}} \|\mathbf{F}\|^{\frac{1}{2}}_{\mathscr{H}}\Big).$$

Proof. Multiplying in $L^2(0,L)$ the resolvent equation (4.6) by $s_1\psi$, we obtain

$$i\lambda\int_0^L s_1\rho_2\Psi\overline{\psi}\mathrm{d}x-\int_0^L s_1[b\psi_x]_x\overline{\psi}\mathrm{d}x+\int_0^L s_1k(\varphi_x+\psi)\overline{\psi}\mathrm{d}x+\int_0^L s_1\gamma\Psi\overline{\psi}\mathrm{d}x=\int_0^L s_1\rho_2 f_4\overline{\psi}\mathrm{d}x.$$

After integrating by parts and using resolvent equation (4.5), we find

$$\begin{aligned}\int_0^L s_1 b|\psi_x|^2\mathrm{d}x=&-\frac{i}{\lambda}\int_0^L s_1' b\psi_x\overline{\Psi}\mathrm{d}x-\frac{i}{\lambda}\int_0^L s_1' b\psi_x\overline{f_3}\mathrm{d}x+\int_0^L s_1\rho_2 f_4\overline{\psi}\mathrm{d}x\\&+\int_0^L s_1\rho_2|\Psi|^2\,\mathrm{d}x+\int_0^L s_1\rho_2\Psi\overline{f_3}\,\mathrm{d}x-\int_0^L s_1\gamma\Psi\overline{\psi}\,\mathrm{d}x\\&-\frac{i}{\lambda}\int_0^L s_1k(\varphi_x+\psi)\overline{\Psi}\,\mathrm{d}x-\frac{i}{\lambda}\int_0^L s_1k(\varphi_x+\psi)\overline{f_3}\,\mathrm{d}x.\end{aligned}\tag{4.12}$$

To conclude, we estimate the right-hand side of (4.12). First, by Hölder and Poincaré inequalities, together with Lemma 4.4 and the assumption $|\lambda|\geqslant 1$, we obtain

$$\left|-\frac{i}{\lambda}\int_0^L s_1' b\psi_x\overline{\Psi}\mathrm{d}x\right|=\left|-\frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta}s_1' b\psi_x\frac{\gamma}{\gamma}\overline{\Psi}\mathrm{d}x\right|\leqslant C\|b^{\frac12}\psi_x\|_{L^2}\|\gamma^{\frac12}\Psi\|_{L^2}\leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac32}\|\mathbf{F}\|_{\mathscr{H}}^{\frac12},$$

$$\left|\int_0^L s_1\gamma\Psi\overline{\psi}\mathrm{d}x\right|\leqslant C\|b^{\frac12}\psi_x\|_{L^2}\|\gamma^{\frac12}\Psi\|_{L^2}\leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac32}\|\mathbf{F}\|_{\mathscr{H}}^{\frac12}$$

and

$$\left|-\frac{i}{\lambda}\int_0^L s_1k(\varphi_x+\psi)\overline{\Psi}\,\mathrm{d}x\right|=\left|\frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta}s_1k(\varphi_x+\psi)\frac{\gamma}{\gamma}\overline{\Psi}\,\mathrm{d}x\right|\leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac32}\|\mathbf{F}\|_{\mathscr{H}}^{\frac12}.$$

The remaining products involving components of $\mathbf{U}$ and $\mathbf{F}$ are treated similarly. Repeating the previous argument based on Hölder and Poincaré inequalities, we find a constant $C>0$, such that, the remaining contributions on the right-hand side of (4.12) are bounded by $C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}$.

The conclusion follows from the preceding estimates and Lemma 4.5. □

Local estimate for the component $(\varphi_x+\psi)$.

Lemma 4.7. *Let Assumptions (A.1)-(A.2) hold. If $|\lambda|\geqslant 1$ and $\mathbf{U}\in D(\mathscr{A})$ solves the resolvent equation (4.2), then there exists a constant $C>0$, such that*

$$\int_0^L s_1k|\varphi_x+\psi|^2\,\mathrm{d}x\leqslant C\Big(\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}+\|\mathbf{U}\|_{\mathscr{H}}^{\frac32}\|\mathbf{F}\|_{\mathscr{H}}^{\frac12}\Big)+C|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\,\mathrm{d}x.$$

Proof. Multiplying in $L^2(0,L)$ the resolvent equation (4.6) by $s_1(\varphi_x+\psi)$ and using the fact that $\mathrm{supp}(s_1)\subset[L_0-\delta,L_0+\delta]$, we find

$$\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x=\int_0^L s_1\rho_2 f_4\overline{(\varphi_x+\psi)}\,\mathrm{d}x-i\lambda\int_0^L s_1\rho_2\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x$$

$$
-\int_{L_0-\delta}^{L_0+\delta} s_1' b\psi_x\overline{(\varphi_x+\psi)}\,\mathrm{d}x-\int_0^L s_1 b\psi_x\overline{(\varphi_x+\psi)_x}\,\mathrm{d}x-\int_0^L s_1\gamma\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x.
$$

Using resolvent equation (4.4), we obtain

$$
[\varphi_x+\psi]_x=\frac{i\lambda\rho_1}{k}\Phi-\frac{\rho_1}{k}f_2-\frac{k'}{k}(\varphi_x+\psi).
$$

Now, we use resolvent equations (4.3) and (4.5)

$$
\begin{aligned}
\int_0^L s_1 k|\varphi_x+\psi|^2\mathrm{d}x=&\int_0^L s_1\rho_2 f_4\overline{(\varphi_x+\psi)}\,\mathrm{d}x-\int_0^L s_1\gamma\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x\\
&+\int_0^L s_1\rho_2\Psi\overline{(f_{1,x}+f_3)}\,\mathrm{d}x+\int_0^L s_1\rho_2|\Psi|^2\,\mathrm{d}x\\
&+\int_0^L s_1 b\rho_1 k^{-1}\psi_x\,\overline{f_2}\,\mathrm{d}x+\sum_{j=1}^3 I_j,
\end{aligned}
\tag{4.13}
$$

where

$$
I_1=\int_{L_0-\delta}^{L_0+\delta} s_1\rho_2\Psi\overline{\Phi_x}\,\mathrm{d}x,
$$

$$
I_2=i\lambda\int_{L_0-\delta}^{L_0+\delta} s_1 b\rho_1 k^{-1}\psi_x\overline{\Phi}\,\mathrm{d}x
$$

and

$$
I_3=-\int_{L_0-\delta}^{L_0+\delta}(s_1'-s_1k'k^{-1})b\psi_x\overline{(\varphi_x+\psi)}\,\mathrm{d}x.
$$

In the following, we shall estimate I_1, I_2 and I_3. Using resolvent equations (4.3) and (4.5), we obtain

$$
\begin{aligned}
I_1+I_2=&\int_0^L s_1\left[\rho_2-b\rho_1k^{-1}\right]\Psi\overline{\Phi_x}\,\mathrm{d}x-\int_{L_0-\delta}^{L_0+\delta}(s_1b\rho_1k^{-1})'\Psi\overline{\Phi}\mathrm{d}x+\int_0^L s_1b\rho_1k^{-1}f_{3,x}\overline{\Phi}\,\mathrm{d}x\\
=&-i\lambda\int_0^L s_1\rho_2\rho_1k^{-1}\chi\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x\ -\int_0^L s_1\rho_2\rho_1k^{-1}\chi\Psi\overline{(f_{1,x}-f_3)}\,\mathrm{d}x\\
&-\int_0^L s_1\rho_2\rho_1k^{-1}\chi|\Psi|^2\,\mathrm{d}x-\int_{L_0-\delta}^{L_0+\delta}(s_1b\rho_1k^{-1})'\Psi\overline{\Phi}\,\mathrm{d}x+\int_0^L s_1b\rho_1k^{-1}f_{3,x}\overline{\Phi}\,\mathrm{d}x.
\end{aligned}
\tag{4.14}
$$

Since $\mathrm{supp}(s_1)\subset[L_0-\delta,L_0+\delta]$ and $\rho_2\rho_1k^{-1}\in L^\infty(0,L)$, we may apply Hölder and Ponicaré inequalities to obtain

$$
\begin{aligned}
\left|i\lambda\int_0^L s_1\rho_2\rho_1k^{-1}\chi\Psi\overline{(\varphi_x+\psi)}\mathrm{d}x\right|&\leqslant C|\lambda|\left(\int_0^L s_1|\chi\Psi|^2\right)^{\frac12}\left(\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x\right)^{\frac12}\\
&\leqslant\frac12\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x+C|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\mathrm{d}x.
\end{aligned}
$$

Next, using the fact that $(s_1 b\rho_1 k^{-1})' \in L^\infty(0,L)$ and invoking Lemma 4.4, we infer that

$$\left|\int_{L_0-\delta}^{L_0+\delta} (s_1 b\rho_1 k^{-1})'\Psi\overline{\Phi}\,\mathrm{d}x\right| \leqslant C\|\rho_1^{\frac{1}{2}}\Phi\|_{L^2}\|\gamma^{\frac{1}{2}}\Psi\|_{L^2} \leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac{3}{2}}\|\mathbf{F}\|_{\mathscr{H}}^{\frac{1}{2}}.$$

The remaining products on the right-hand side of (4.14) are controlled in the same way and satisfy the bound $C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}$, for some constant $C>0$. Collecting these estimates, we arrive at

$$\begin{aligned} |I_1+I_2| \leqslant \frac{1}{2}\int_0^L s_1 k|\varphi_x+\psi|^2\,\mathrm{d}x + C|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\,\mathrm{d}x \\ + C\Big(\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + \|\mathbf{U}\|_{\mathscr{H}}^{\frac{3}{2}}\|\mathbf{F}\|_{\mathscr{H}}^{\frac{1}{2}}\Big). \end{aligned} \tag{4.15}$$

Now, denoting $h=(s_1'-s_1k'k^{-1})b \in W^{1,\infty}(0,L)$. From resolvent equations (4.4) and (4.5), we get

$$\begin{aligned} I_3 &= \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} hf_{3,x}\overline{(\varphi_x+\psi)}\,\mathrm{d}x + \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} h\Psi_x\overline{(\varphi_x+\psi)}\,\mathrm{d}x \\ &= \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} hf_{3,x}\overline{(\varphi_x+\psi)}\,\mathrm{d}x - \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} h'\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x \\ &\quad - \int_{L_0-\delta}^{L_0+\delta} h\rho_1 k^{-1}\Psi\overline{\Phi}\mathrm{d}x + \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} hk'k^{-1}\Psi\overline{(\varphi_x+\psi)}\mathrm{d}x \\ &\quad + \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} h\rho_1 k^{-1}\Psi\overline{f_2}\,\mathrm{d}x. \end{aligned} \tag{4.16}$$

The final step is to estimate the right-hand side of (4.16). First, using Hölder inequality and Lemma 4.4, we obtain

$$\begin{aligned} \left|\frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} h'\Psi\overline{(\varphi_x+\psi)}\,\mathrm{d}x\right| &= \left|\frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} h'\frac{\gamma}{\gamma}\Psi\,\frac{k}{k}\overline{(\varphi_x+\psi)}\,\mathrm{d}x\right| \\ &\leqslant C\|k^{\frac{1}{2}}(\varphi_x+\psi)\|_{L^2}\|\gamma^{\frac{1}{2}}\Psi\|_{L^2} \\ &\leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac{3}{2}}\|\mathbf{F}\|_{\mathscr{H}}^{\frac{1}{2}}. \end{aligned}$$

Similarly, we find

$$\left|-\int_{L_0-\delta}^{L_0+\delta} h\rho_1 k^{-1}\Psi\overline{\Phi}\mathrm{d}x + \frac{i}{\lambda}\int_{L_0-\delta}^{L_0+\delta} hk'k^{-1}\Psi\overline{(\varphi_x+\psi)}\mathrm{d}x\right| \leqslant C\|\mathbf{U}\|_{\mathscr{H}}^{\frac{3}{2}}\|\mathbf{F}\|_{\mathscr{H}}^{\frac{1}{2}}.$$

It remains to deal with the other terms appearing on the right-hand side of (4.16). These are estimated by a direct application of Hölder inequality, together with the boundedness of the coefficients. Consequently, there exists a constant $C>0$, such that, the contribution of all such terms is bounded by $C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}$. Combining this with the previous estimates yields

$$|I_3| \leqslant C\Big(\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + \|\mathbf{U}\|_{\mathscr{H}}^{\frac{3}{2}}\|\mathbf{F}\|_{\mathscr{H}}^{\frac{1}{2}}\Big). \tag{4.17}$$

Therefore, returning to (4.13), and using estimates (4.15) and (4.17), we conclude Lemma 4.7. □

Local estimate for the component Φ.

Lemma 4.8. *Let Assumptions (A.1)-(A.2) hold. If $\mathbf{U} \in D(\mathscr{A})$ solves the resolvent equation (4.2), then exists a constant $C > 0$, such that*

$$\int_0^L s_2\rho_1|\Phi|^2\,\mathrm{d}x \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}} + C\|\mathbf{U}\|_{\mathscr{H}}\left(\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Proof. Multiplying resolvent equation (4.4) by $s_2\varphi$ and using (4.3), we obtain

$$i\lambda\int_0^L s_2\rho_1\Phi\overline{\varphi}\,\mathrm{d}x - \int_0^L s_2[k(\varphi_x+\psi)]_x\overline{\varphi}\,\mathrm{d}x = \int_0^L s_2\rho_1 f_2\overline{\varphi}\,\mathrm{d}x.$$

Then, from resolvent equation (4.3) follows that

$$\begin{aligned}\int_0^L s_2\rho_1|\Phi|^2\mathrm{d}x = &\int_0^L s_2k|\varphi_x+\psi|^2\,\mathrm{d}x - \int_0^L s_2k(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x\\ &-\int_{L_0-\frac{\delta}{2}}^{L_0+\frac{\delta}{2}} s_2'k(\varphi_x+\psi)\overline{\varphi}\,\mathrm{d}x - \int_0^L s_2\rho_1\Phi\overline{f_1}\,\mathrm{d}x\\ &-\int_0^L s_2\rho_1 f_2\overline{\varphi}\,\mathrm{d}x.\end{aligned} \tag{4.18}$$

Using that $\mathrm{supp}(s_2) \subset [L_0-\frac{\delta}{2}, L_0+\frac{\delta}{2}]$ and $s_1 = 1$ in $[L_0-\frac{\delta}{2}, L_0+\frac{\delta}{2}]$, we obtain

$$\int_0^L s_2k|\varphi_x+\psi|^2\,\mathrm{d}x = \int_{L_0-\frac{\delta}{2}}^{L_0+\frac{\delta}{2}} s_1s_2k|\varphi_x+\psi|^2\,\mathrm{d}x \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\left(\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x\right)^{\frac{1}{2}},$$

$$\left|\int_0^L s_2k(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x\right| \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\left(\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

and

$$\left|\int_{L_0-\frac{\delta}{2}}^{L_0+\frac{\delta}{2}} s_2'k(\varphi_x+\psi)\overline{\varphi}\,\mathrm{d}x\right| \leqslant C\|\mathbf{U}\|_{\mathscr{H}}\left(\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

We now turn to the remaining terms on the right-hand side of (4.18). A straightforward application of Hölder inequality shows that there exists a constant $C > 0$, such that, the total contribution of these terms is bounded by $C\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}$. Combining this bound with the estimates obtained above yields the conclusion of Lemma 4.8. □

Combination of the estimates from Lemmas 4.5–4.8.

In the following result, we will make estimates for $a_1 = L_0 - \frac{\delta}{3}$ and $a_2 = L_0 + \frac{\delta}{3}$. In this case, we have

$$s_1 = s_2 = 1 \ \text{ in } \ \left[L_0 - \frac{\delta}{3}, L_0 + \frac{\delta}{3}\right].$$

Theorem 4.9. *Let Assumptions (A.1)-(A.2) hold. If $\mathbf{U}\in D(\mathscr{A})$ solves the resolvent equation (4.2), then for any arbitrary sub-interval (p,q) of (ℓ_0,ℓ_1) and any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$, such that*

$$\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant \begin{cases} \varepsilon\|\mathbf{U}\|^2_{\mathscr{H}}+C_\varepsilon\|\mathbf{F}\|^2_{\mathscr{H}}, & \text{if } \chi(x)=0 \text{ for all } x\in(p,q),\\ \varepsilon\|\mathbf{U}\|^2_{\mathscr{H}}+C_\varepsilon|\lambda|^4\|\mathbf{F}\|^2_{\mathscr{H}}, & \text{if } \chi(x)\neq 0 \text{ for all } x\in(p,q). \end{cases} \tag{4.19}$$

Proof. We begin by taking $L_0\in(p,q)$ and $\delta>0$, such that

$$(L_0-\delta,L_0+\delta)\subset(p,q)\subset(\ell_0,\ell_1).$$

Our first objective is to recover the localized energy term on the left-hand side of (4.19). To this end, by combining Lemmas 4.4 and 4.6, we infer that

$$\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant \int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x+\int_0^L s_2\rho_1|\Phi|^2\mathrm{d}x + C\Big(\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}\ +\ \|\mathbf{U}\|^{\frac{3}{2}}_{\mathscr{H}}\|\mathbf{F}\|^{\frac{1}{2}}_{\mathscr{H}}\Big).$$

Next, appealing to Lemma 4.8 and applying Young inequality, we obtain the estimate

$$\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{4}\|\mathbf{U}\|^2_{\mathscr{H}}+C\Big(1+\frac{1}{\varepsilon}\Big)\int_0^L s_1k|\varphi_x+\psi|^2\mathrm{d}x + C\Big(\ \|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}\ +\ \|\mathbf{U}\|^{\frac{3}{2}}_{\mathscr{H}}\|\mathbf{F}\|^{\frac{1}{2}}_{\mathscr{H}}\ \Big).$$

Finally, we use Lemma 4.7 to control the remaining term, whence

$$\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant C\Big(1+\frac{1}{\varepsilon}\Big)|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\,\mathrm{d}x\ +\ \frac{\varepsilon}{4}\|\mathbf{U}\|^2_{\mathscr{H}} + C\Big(1+\frac{1}{\varepsilon}\Big)\Big(\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}+\|\mathbf{U}\|^{\frac{3}{2}}_{\mathscr{H}}\|\mathbf{F}\|^{\frac{1}{2}}_{\mathscr{H}}\Big).$$

A further application of Young inequality yields

$$\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x \leqslant \frac{\varepsilon}{2}\|\mathbf{U}\|^2_{\mathscr{H}}+C\Big(1+\frac{1}{\varepsilon}\Big)|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\,\mathrm{d}x+C_{\varepsilon,1}\|\mathbf{F}\|^2_{\mathscr{H}} \tag{4.20}$$

where

$$C_{\varepsilon,1}=C\left(\frac{1}{\varepsilon}\left(1+\frac{1}{\varepsilon}\right)^2+\frac{1}{\varepsilon^3}\left(1+\frac{1}{\varepsilon}\right)^4\right).$$

We are now in a position to conclude the argument.

If $\chi=0$ in (p,q) then the integral term in (4.20) vanishes, and (4.20) immediately implies $(4.19)_1$ with $C_\varepsilon=C_{\varepsilon,1}$.

Assume next that $\chi\neq 0$ in (p,q). In order to derive $(4.19)_2$, it suffices to estimate the localized term appearing on the right-hand side of (4.20). Since $\rho_2\in W^{1,\infty}(0,L)$ is bounded below and $\chi\in W^{1,\infty}(0,L)$, we find from

Lemma 4.4

$$\begin{aligned}C\Big(1+\frac{1}{\varepsilon}\Big)|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}|\chi\Psi|^2\mathrm{d}x &\leqslant C\Big(1+\frac{1}{\varepsilon}\Big)|\lambda|^2\int_{L_0-\delta}^{L_0+\delta}\rho_2|\Psi|^2\mathrm{d}x\\ &\leqslant C\Big(1+\frac{1}{\varepsilon}\Big)|\lambda|^2\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}\\ &\leqslant \frac{\varepsilon}{2}\|\mathbf{U}\|^2_{\mathscr{H}}+\frac{C}{\varepsilon}\Big(1+\frac{1}{\varepsilon}\Big)^2|\lambda|^4\|\mathbf{F}\|^2_{\mathscr{H}}.\end{aligned}$$

Substituting this estimate into (4.20) we obtain $(4.19)_2$ with

$$C_\varepsilon = C_{\varepsilon,1}+\frac{C}{\varepsilon}\Big(1+\frac{1}{\varepsilon}\Big)^2.$$

This completes the proof. □

4.2.2. Global estimates

Theorem 4.10. *Let Assumptions (A.1)-(A.2) hold. If $\mathbf{U}\in D(\mathscr{A})$ solves the resolvent equation (4.2), then for any arbitrary sub-interval (p,q) of (ℓ_0,ℓ_1) there exists a constant $C>0$, such that*

$$\|\mathbf{U}\|_{\mathscr{H}}\leqslant\begin{cases}C\|\mathbf{F}\|_{\mathscr{H}}, & \text{if } \chi(x)=0 \text{ for all } x\in(p,q),\\ C|\lambda|^2\|\mathbf{F}\|_{\mathscr{H}}, & \text{if } \chi(x)\neq 0 \text{ for all } x\in(p,q).\end{cases}$$

Proof. Note that, from the definitions (3.5) and (3.4), we find

$$\|\mathbf{U}\|^2_{\mathscr{H}}=\int_0^L E_{\mathbf{U}}(x)\,\mathrm{d}x=\int_0^{L_0-\frac{\delta}{3}}E_{\mathbf{U}}(x)\,\mathrm{d}x+\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}}E_{\mathbf{U}}(x)\,\mathrm{d}x+\int_{L_0+\frac{\delta}{3}}^{L}E_{\mathbf{U}}(x)\,\mathrm{d}x. \tag{4.21}$$

We next estimate the right-hand side of (4.21). Using (3.6) and (3.7) from Proposition 3.1 and then applying Young inequality, we obtain

$$\begin{aligned}\int_0^{L_0-\frac{\delta}{3}}E_{\mathbf{U}}(x)\,\mathrm{d}x &\leqslant C\,\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}+C\|\mathbf{F}\|^2_{\mathscr{H}}+CE_{\mathbf{U}}\left(L_0-\frac{\delta}{3}\right)\\ &\leqslant \frac{1}{4}\|\mathbf{U}\|^2_{\mathscr{H}}+C\|\mathbf{F}\|^2_{\mathscr{H}}+C\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}}E_{\mathbf{U}}(x)\,\mathrm{d}x\end{aligned} \tag{4.22}$$

and, analogously

$$\begin{aligned}\int_{L_0+\frac{\delta}{3}}^{L}E_{\mathbf{U}}(x)\,\mathrm{d}x &\leqslant C\,\|\mathbf{U}\|_{\mathscr{H}}\|\mathbf{F}\|_{\mathscr{H}}+C\|\mathbf{F}\|^2_{\mathscr{H}}+CE_{\mathbf{U}}\left(L_0+\frac{\delta}{3}\right)\\ &\leqslant \frac{1}{4}\|\mathbf{U}\|^2_{\mathscr{H}}+C\|\mathbf{F}\|^2_{\mathscr{H}}+C\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}}E_{\mathbf{U}}(x)\,\mathrm{d}x.\end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23) in (4.21), we obtain

$$\|\mathbf{U}\|_{\mathscr{H}}^2 \leqslant C\|\mathbf{F}\|_{\mathscr{H}}^2 + C\int_{L_0-\frac{\delta}{3}}^{L_0+\frac{\delta}{3}} E_{\mathbf{U}}(x)\,\mathrm{d}x. \tag{4.24}$$

The conclusion follows from estimate (4.24) and Theorem 4.9 after choosing $\varepsilon > 0$ sufficiently small.

This completes the proof. □

4.3. Completion of the proof of Theorems 1.3 and 1.4

Proof of Theorems 1.3 *and* 1.4*.* We are now in a position to conclude the proofs of the main results. Lemma 4.2 yields the strong stability of the semigroup $\{S_{\mathscr{A}}(t)\}_{t\geqslant 0}$. Moreover, Theorem 4.10 provides the resolvent estimate (4.7) in the precise form required by the abstract frequency domain criterion. If $(p,q) \subset (\ell_0, \ell_1)$ is an interval on which the hypothesis of Theorem 1.3 holds then (4.7) is satisfied with $N = 0$. On the other hand, if $\chi(x) \neq 0$ for all $x \in (\ell_0, \ell_1)$, then (4.7) holds with $N = 2$. Consequently, we deduce exponential decay in the first case and the polynomial decay rate asserted in Theorem 1.4 in the second case.

This completes the proof of Theorems 1.3 and 1.4. □

Data availability statement

All data generated or analyzed during this study are included in this published article.

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