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RISK-AVERSE OPTIMAL CONTROL OF SEMILINEAR ELLIPTIC PDES *, **

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Abstract. In this paper, we consider the optimal control of semilinear elliptic PDEs with random inputs. These problems are often nonconvex, infinite-dimensional stochastic optimization problems for which we employ risk measures to quantify the implicit uncertainty in the objective function. In contrast to previous works in uncertainty quantification and stochastic optimization, we provide a rigorous mathematical analysis demonstrating higher solution regularity (in stochastic state space), continuity and differentiability of the control-to-state map, and existence, regularity and continuity properties of the control-to-adjoint map. Our proofs make use of existing techniques from PDE-constrained optimization as well as concepts from the theory of measurable multifunctions. We illustrate our theoretical results with two numerical examples motivated by the optimal doping of semiconductor devices.

Résumé. Dans cet article, nous considérons le contrôle optimal des EDP elliptiques semi-linéares avec entrées aléatoires. Ces problèmes sont des programmes stochastiques non convexes, de dimension infinie, pour lesquels nous utilisons des mesures de risque pour gérer l'incertitude implicite dans la fonction objectif. Contrairement aux travaux antérieurs traitant de la quantification de l'incertitude et de l'optimisation stochastique, nous proposons une analyse mathématique rigoureuse démontrant une régularité de la solution plus élevée (dans l'espace des variables d'état stochastiques), l'analyse de la sensibilité de l'état par rapport au contrôle, et l'existence, la régularité et les propriétés de continuité de l'application état-adjoint. Nos preuves utilisent des techniques provenant de l'optimisation avec contraintes d'EDP ainsi que des concepts de la théorie des multifonctions mesurables. Nous illustrons nos résultats théoriques par deux exemples numériques motivés par le dopage optimal de dispositifs semi-conducteurs.

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Introduction

Whether a product of noisy data, unverifiable assumptions or unknown operating conditions, uncertainty is omnipresent in engineering and science applications. As a result, one often models the underlying systems using, e.g., partial differential equations (PDEs) with random coefficients, right-hand sides, and boundary conditions (cf., the complex applications in [10, 23, 49, 57, 58]). When considering optimal design and control problems constrained by such models, it is critical that any resulting design or control is resilient to uncertainty. In this paper, we achieve this by formulating these problems as infinite-dimensional risk-averse stochastic optimization problems. In particular, we employ regular measures of risk [52] as a means of treating the implicit uncertainty in the objective function. We consider the optimization problem,

$$\min_{z\in Z_{\mathrm{ad}}} \mathcal{R}[\mathcal{J}(S(z))] + \wp(z), \tag{1}$$

where $z \in Z$ are deterministic controls, Z_{ad} is the admissible set of controls, $\wp$ is the control cost, $\mathcal{R}$ is a functional that maps a set of random variables into the extended real numbers, and $\mathcal{J}$ is an uncertain objective function or cost that depends on the random-field PDE solution $S(z)$. As a motivating example, we consider the optimal control of the semilinear elliptic PDE with random inputs

$$-\nabla\cdot(\kappa(\omega,x)\nabla u(\omega,x)) + c(\omega,x)u(\omega,x) + N(u(\omega,x),\omega,x) = [B(\omega)z](x) + b(\omega,x), \qquad x\in D,\ \text{a.a. } \omega\in\Omega \tag{2a}$$

$$\kappa(\omega,x)\frac{\partial u}{\partial n}(\omega,x) = 0, \qquad x\in\partial D,\ \text{a.a. } \omega\in\Omega \tag{2b}$$

where $D \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is the physical domain with boundary ∂D and ω is an element of some probability space defined on the set of outcomes Ω. In the subsequent analysis, it will be clear that our results hold for a significantly more general class of second-order elliptic differential equations.

Stochastic PDE-constrained optimization, optimal control under uncertainty, and related topics in uncertainty quantification, are rapidly growing fields within applied mathematics, see e.g., [19, 53, 55, 59]. Until recently, much of the work has focused on numerical approximation and solution algorithms for the risk-neutral case ($\mathcal{R} = \mathbb{E}$). For example, the authors in [13,18] develop reduced-order model approaches whereas the authors in [11, 12] develop spatial multigrid algorithms with sparse-grid collocation. In a similar vein, [38] proposes a multilevel optimization algorithm based on sparse-grid collocation. Furthermore, in [39, 40], the authors develop a globally convergent trust-region optimization algorithm based on adaptive sparse-grid collocation. More recently, multiple authors have investigated the use of low-rank tensor approximation as a means of reducing computational complexity [9, 31]. For example, the authors in [9] investigate the use of low-rank tensor approximation to minimize the mean-plus-variance measure of risk,

$$\mathcal{R}[X] = \mathbb{E}[X] + c\mathbb{E}[(X - \mathbb{E}[X])^2], \quad c \geq 0. \tag{3}$$

The focus on risk-averse PDE-constrained optimization has been quite recent, cf. [41–43]. In [41], the authors consider the average value-at-risk, $\mathcal{R} = \mathrm{AVaR}_\beta$ for $0 < \beta < 1$, which, roughly speaking, is the average of the $(1 - \beta) \times 100\%$ largest scenarios. AVaR_β is nonsmooth and to enable the use of derivative-based optimization algorithms, the authors propose multiple smoothing techniques and study the effects of this smoothing. In [43], the authors provide a general theory of existence and optimality for risk-averse PDE-constrained optimization. Note that the concrete example there involves a linear PDE, for which the issues of measurability, continuity and differentiability with respect to the controls, and higher regularity are easily verified. Finally, in [42], the authors develop a systematic approach for regularizing nonsmooth risk measures using the infimal convolution. Their analysis includes error estimates, variational convergence properties, consistency of minima and stationary points as well as rates of convergence.

The main purpose of this paper is to provide a rigorous analysis of the control-to-state map in the context of general semilinear elliptic PDEs with uncertain inputs. This includes an investigation of the properties of the reduced gradient and the associated adjoint state mapping. In many works, these properties are either assumed or easily verified due to linearity. However, the actual verification of these assumptions in the nonlinear setting is nontrivial. To an extent, some of the proof techniques are similar to standard approaches in the deterministic setting, e.g., as in [36, 46, 60] and the references therein. However, since the PDE solution depends implicitly on the random inputs, it is necessary to prove measurability and integrability properties before addressing continuity and differentiability issues. To prove measurability, we provide a novel proof that makes use of the versatile theory of measurable multifunctions.

The paper is structured as follows. First, we introduce the necessary notation and definitions. This includes a short discussion on risk measures and their smooth approximation, monotone operators, and measurable multifunctions. We then analyze the solution mapping for a general class of nonlinear operator equations whose structure is motivated by (2). Following this, we introduce a general class of optimal control problems in Section 3. Here, we present the main theoretical results, which include existence of minimizers and first-order optimality theory. Finally, we verify our results using two examples related to the optimal doping of a semiconductor device.

1. Notation, Definitions and Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where Ω is the set of outcomes, $\mathcal{F} \subseteq 2^\Omega$ is a σ-algebra of events and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure. The abbreviations "a.e." and "a.a." stand for "almost everywhere" and "almost all" with respect to $\mathbb{P}$, respectively. Moreover, we refer to "$\mathcal{F}$-measurable" simply as "measurable" and denote by $\mathbb{E}[y]$ the expected value of a random variable y defined on $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e., y is measurable). In addition, we assume that the control space Z is a real reflexive Banach space and denote the set of admissible controls by $Z_{\text{ad}} \subseteq Z$. We assume that Z_{ad} is nonempty, closed and convex. Finally, we assume that the physical domain $D \subset \mathbb{R}^d$ is an open and bounded set with Lipschitz boundary ∂D. We denote the deterministic solution space for our PDE by $U := H^1(D)$. Here, $H^1(D)$ is the usual Sobolev space of $L^2(D)$-functions whose weak derivatives are also in $L^2(D)$. Recall that U is a separable Hilbert space, cf. [1].

We characterize the random field solutions of our PDE as elements of a Bochner space, cf. [35]. The Bochner space $L^p(\Omega, \mathcal{F}, \mathbb{P}; W)$ consists of strongly measurable functions mapping Ω into a Banach space W with p finite moments for $p = [1, \infty)$. When $p = \infty$, $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; W)$ consists of essentially bounded W-valued strongly measurable functions. The Bochner space $L^p(\Omega, \mathcal{F}, \mathbb{P}; W)$ is a Banach space with norm

$$\|v\|_{L^p(\Omega,\mathcal{F},\mathbb{P};W)} = \mathbb{E}\left[\|v\|_W^p\right]^{1/p} \text{ for } p \in [1, \infty) \quad \text{and} \quad \|v\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P};W)} = \operatorname*{ess\,sup}_{\omega\in\Omega} \|v(\omega)\|_W.$$

When $W = \mathbb{R}$, we set $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) = L^p(\Omega, \mathcal{F}, \mathbb{P})$. The uncertain objective function $\mathcal{J}(S(\cdot))$ in (1) will ultimately map Z into the space $\mathcal{X} := L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p \in [1, \infty)$. For most results, we state the integrability p explicitly. Otherwise, we simply write $\mathcal{X}$.

For any real Banach spaces X and Y, we denote the space of bounded linear operators between X and Y by $\mathcal{L}(X, Y)$. We denote the topological dual space of X by $X^* := \mathcal{L}(X, \mathbb{R})$ and the associated duality pairing by $\langle \cdot, \cdot \rangle_{X^*,X}$. For any $A \in \mathcal{L}(X, Y)$ we denote by $A^* \in \mathcal{L}(Y^*, X^*)$ the adjoint (conjugate) of A. Throughout, we use "$\to$" to denote convergence with respect to the norm topology, "$\rightharpoonup$" to denote convergence with respect to the weak topology and "$\rightharpoonup^*$" to denote convergence with respect to the weak* topology.

1.1. Background

For the analysis below, we require several definitions and results from the fields of convex analysis (cf. [5, Ch. 9]) and maximal monotone operators (cf. [8, Ch. 2]). A set $\Gamma \subset X \times X^*$ is said to be *monotone* provided

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle_{X^*,X} \geq 0, \forall\, (x_i, x_i^*) \in \Gamma,\ i = 1, 2.$$

The set Γ is said to be maximal monotone provided it is not properly contained in any other monotone subset of $X \times X^*$. A multifunction (set-valued map) $A : X \rightrightarrows X^*$ is then (maximal) monotone provided its graph

$$\operatorname{Gr} A := \{(x, x^*) \in X \times X^* \mid x^* \in A(x)\}$$

is a (maximal) monotone set. In order to prove measurability of our PDE solutions, we further require several notions from the theory of measurable multifunctions, which we state here for convenience. Let (T, Σ, μ) be a complete σ-finite measure space and suppose X is a separable Banach space. We say that a function $f : T \to X$ is *Borel measurable* if for any Borel subset $B \subseteq X$, we have

$$f^{-1}(B) := \{t \in T \mid f(t) \in B\} \in \Sigma.$$

Since X endowed with its norm topology is a complete separable metric space, we have that Borel measurability is equivalent to *strong measurability* [17, Sect. III.0]. Hence, we simply refer to a Borel measurable function as measurable. Similarly, we say that a multifunction $F : T \rightrightarrows X$ with nonempty closed images is *measurable* if for any Borel subset $B \subseteq X$ we have

$$F^{-1}(B) := \{t \in T \mid F(t) \cap B \neq \emptyset\} \in \Sigma.$$

Note that by Theorem 8.1.4 in [6], we can replace the Borel subset B in our definition by any open subset or any closed subset of X. Moreover, we have that the graph of F, $\operatorname{Gr} F$, belongs to the product σ-algebra generated by Σ and the Borel subsets of X. Finally, we will use Filippov's Theorem to prove measurability of our PDE solution. For the reader's convenience, we state Filippov's Theorem here.

Theorem 1.1. (Filippov's Theorem, [6, Thm. 8.2.10]) *Let (T, Σ, μ) be a complete σ-finite measure space and let X, Y be complete separable metric spaces. Additionally, suppose $g : T \times X \to Y$ is a Carathéodory function, i.e., $g(t, \cdot)$ is continuous for a.a. $t \in T$ and $g(\cdot, x)$ is measurable for all $x \in X$. Finally, let $\Gamma : T \rightrightarrows X$ be a measurable multifunction with nonempty closed images. Then for any measurable function $h : T \to Y$ satisfying $h(t) \in g(t, \Gamma(t))$ for a.a. $t \in T$, there exists a measurable function $\gamma : T \to X$ satisfying $\gamma(t) \in \Gamma(t)$ for all $t \in T$ and $h(t) = g(t, \gamma(t))$ for a.a. $t \in T$.*

1.2. Measures of Risk

As mentioned in the introduction, we focus on the so-called *regular measures of risk* [52] to incorporate risk preference in (1). The functional $\mathcal{R} : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}} := (-\infty, +\infty]$ with $p \in [1, \infty)$ is a regular measure of risk if it satisfies the following conditions:

- (R1) $\mathcal{R}$ is proper, closed and convex;
- (R2) $\mathcal{R}[C] = C$ for all constant random variables $C \in \mathbb{R}$;
- (R3) $\mathcal{R}[X] > \mathbb{E}[X]$ for all nonconstant $X \in \mathcal{X}$.

In addition to the conditions for regular measures of risk, further characterizations exist. For example, the functional $\mathcal{R}$ is *coherent* [4] provided it satisfies the following properties:

- (C1) *Subadditivity*: If $X, X' \in \mathcal{X}$, then $\mathcal{R}[X + X'] \leq \mathcal{R}[X] + \mathcal{R}[X']$;
- (C2) *Monotonicity*: If $X, X' \in \mathcal{X}$ and $X \geq X'$ a.e., then $\mathcal{R}[X] \geq \mathcal{R}[X']$;
- (C3) *Translation equivariance*: If $C \in \mathbb{R}$ and $X \in \mathcal{X}$, then $\mathcal{R}[X + C] = \mathcal{R}[X] + C$;
- (C4) *Positive homogeneity*: If $t > 0$ and $X \in \mathcal{X}$, then $\mathcal{R}[tX] = t\mathcal{R}[X]$.

Coherent measures of risk form perhaps the most well-known and popular class of risk measures. However, regular measures of risk provide a convenient minimal set of assumptions to ensure many essential properties hold. For example, if $\mathcal{R}$ satisfies (R1), then the Fenchel-Moreau Theorem implies $\mathcal{R} = \mathcal{R}^{**}$ where

$$\mathcal{R}^{**}(X) := \sup_{\theta \in (L^p(\Omega, \mathcal{F}, \mathbb{P}))^*} \{\mathbb{E}[\theta X] - \mathcal{R}^*(\theta)\} \qquad \text{and} \qquad \mathcal{R}^*(\theta) := \sup_{Y \in L^p(\Omega, \mathcal{F}, \mathbb{P})} \{\mathbb{E}[\theta Y] - \mathcal{R}(\theta)\}.$$

In addition to (R1), if $\mathcal{R}$ satisfies (C2) and (C3) then $\operatorname{dom}\mathcal{R}^* \subseteq \{\theta \in (L^p(\Omega,\mathcal{F},\mathbb{P}))^* \,|\, \mathbb{E}[\theta]=1,\ \theta \geq 0 \text{ a.e.}\}$, cf. the discussions in [56, Chap. 6]. Furthermore, property (R3) suggests that the *risk-neutral* functional, $\mathcal{R}=\mathbb{E}$, may not adequately model risk preference and hence is not *risk averse* (even though it is a coherent measure of risk). Finally, as observed in [42], coherent measures of risk are continuously Fréchet differentiable if and only if there exists $\vartheta \in (L^p(\Omega,\mathcal{F},\mathbb{P}))^*$ with $\vartheta \geq 0$ a.e. and $\mathbb{E}[\vartheta]=1$ such that $\mathcal{R}[X]=\mathbb{E}[\vartheta X]$ for all $X \in L^p(\Omega,\mathcal{F},\mathbb{P})$.

Popular measures of risk include the mean-plus-variance (3) and the mean-plus-standard deviation. These functionals are regular measures of risk but do not satisfy (C2) and (C4), and (C2), respectively. On the other hand, the average value-at-risk,

$$\mathcal{R}[X] = \mathrm{AVaR}_\beta[X] := \frac{1}{1-\beta}\int_\beta^1 F_X^{-1}(\alpha)\,\mathrm{d}\alpha = \inf_{t\in\mathbb{R}}\left\{t + \frac{1}{1-\beta}\mathbb{E}[\max\{0, X-t\}]\right\}, \quad \beta \in (0,1),$$

where F_X denotes the distribution function of X and F_X^{-1} its quantile function, is both a regular and coherent measure of risk. We note here that it is possible to treat the case when $p=\infty$. In this setting, it is common to consider $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ endowed with the weak* topology and $L^1(\Omega,\mathcal{F},\mathbb{P})$ endowed with the norm topology as paired topological vector spaces. Furthermore, we modify (R1) and assume that $\mathcal{R}$ is weak* closed. To avoid these technical complications, we assume throughout that $p<\infty$ and refer the reader to [56, Sec. 6.3 p. 283].

1.3. **Approximating Measures of Risk**

In order to avoid applying nonsmooth optimization algorithms like bundle methods [54] to our nonsmooth, nonconvex infinite-dimensional optimization problem (1), we proposed in [41] and [42] two smoothing approaches: density smoothing and epi-regularization, respectively. We briefly describe the latter, as we will use it in Section 4. Let $\Phi : \mathcal{X} \to \overline{\mathbb{R}}$ be a proper, closed, and convex functional and $\mathcal{R}$ a regular measure of risk. Then for $\varepsilon > 0$, we define the epi-regularized measure of risk as

$$\mathcal{R}_\varepsilon^\Phi[X] = \inf_{Y\in\mathcal{X}}\left\{\mathcal{R}[X-Y] + \varepsilon\Phi\left[\varepsilon^{-1}Y\right]\right\} = \inf_{Y\in\mathcal{X}}\left\{\mathcal{R}[Y] + \varepsilon\Phi\left[\varepsilon^{-1}(X-Y)\right]\right\}.$$

Epi-regularized measures of risk have many advantageous analytical and numerical properties that hold under relatively mild assumptions. For example, by letting $\mathcal{X}=L^2(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{R}=\mathrm{AVaR}_\beta$, and $\Phi[X]:=\mathbb{E}[X]+\frac{1}{2}\mathbb{E}[X^2]$, the resulting epi-regularized measure of risk is continuously Fréchet differentiable and is given by

$$\mathcal{R}_\varepsilon^\Phi[X] = \inf_{t\in\mathbb{R}}\{t + \mathbb{E}[v_{\beta,\epsilon}(X-t)]\} \quad \text{where} \quad v_{\beta,\varepsilon}(x) = \begin{cases} -\frac{\varepsilon}{2}, & \text{if } x \leq -\varepsilon \\ \frac{1}{2\varepsilon}x^2 + x, & \text{if } x \in \left(-\varepsilon, \frac{\varepsilon\beta}{1-\beta}\right) \\ \frac{1}{1-\beta}\left(x - \frac{\varepsilon\beta^2}{2(1-\beta)}\right), & \text{if } x \geq \frac{\varepsilon\beta}{1-\beta}. \end{cases}$$

In general, one can show that $\left\{\mathcal{R}_\varepsilon^\Phi\right\}_{\varepsilon>0}$ converges in the sense of Mosco to $\mathcal{R}$ and (under appropriate assumption of $\mathcal{J}$ and $\wp$) that the approximation of (1) defined by

$$\min_{z\in Z_{\mathrm{ad}}} \mathcal{R}_\varepsilon^\Phi[\mathcal{J}(S(z))] + \wp(z), \tag{4}$$

is consistent. This means (for $\epsilon \downarrow 0$) that weak accumulation points of minimizers $z_\varepsilon^\star$ to (4) are optimal for (1) and weak accumulation points of stationary points to (4) are stationary for (1). We refer the interested reader to [42] for a comprehensive study of epi-regularized measures of risk.

2. Analysis of the Control-to-State Mapping

In this section, we prove the required properties of the control-to-state map $S(z)$ needed for the analysis of the optimization problem as well as for the derivation of function-space-based optimization algorithms.

2.1. **Assumptions**

Returning to (1), we consider $S(z)$ to be the solution to the general parametric equation: For each $z \in Z$, find $u(\omega) = [S(z)](\omega) \in U$ such that

$$e(u, z; \omega) := \mathbf{A}(\omega)u + \mathbf{N}(u, \omega) - \mathbf{B}(\omega)z - \mathbf{b}(\omega) \ni 0 \quad \text{for a.a. } \omega \in \Omega. \tag{5}$$

We first state pointwise assumptions on the operators in (5) and then strengthen these assumptions to account for measurablility. The stated assumptions are often easily verified in practice as we will demonstrate in our numerical examples. We impose the following assumptions on the operators.

Assumption 2.1 (Pointwise Characterization of the Problem Data in (5))**.**

(1) Let $\mathbf{A} : \Omega \to \mathcal{L}(U, U^*)$ satisfy $\mathbf{A}(\omega)$ is monotone for a.a. $\omega \in \Omega$ and there exists $\gamma > 0$ and a random variable $C : \Omega \to [0, \infty)$ with $C > 0$ a.e. such that

$$\langle \mathbf{A}(\cdot)u, u\rangle_{U^*,U} \ge C\|u\|_U^{1+\gamma} \quad \text{a.e. } \forall\, u \in U. \tag{6}$$

(2) Let $\mathbf{b} : \Omega \to U^*$.
(3) Let $\mathbf{N} : U \times \Omega \rightrightarrows U^*$ satisfy $\mathbf{N}(\cdot, \omega)$ is maximal monotone with $\mathbf{N}(0, \omega) = \{0\}$ for a.a. $\omega \in \Omega$.
(4) Let $\mathbf{B} : \Omega \to \mathcal{L}(Z, U^*)$ satisfy $\mathbf{B}(\omega)$ is completely continuous for a.a. $\omega \in \Omega$, i.e.,

$$z_n \rightharpoonup z \quad \text{in } Z \qquad \Longrightarrow \qquad \mathbf{B}(\omega)z_n \to \mathbf{B}(\omega)z \quad \text{in } U^* \text{ for a.a. } \omega \in \Omega.$$

Note that if Assumption 2.1 holds, then $\mathbf{A}(\cdot)$ is a coercive and continuous monotone operator from U into U^* a.e. In addition, since Z is reflexive, we have that $\mathbf{B}(\cdot)$ is a compact operator a.e. [22, Prop. VI.3.3(b)]. Next, we state the required measurability and integrability assumptions for the operators defining $e(\cdot, \cdot; \omega)$.

Assumption 2.2 (Measurability and Integrability of the Operators in (2))**.** *Let Assumption 2.1 hold and suppose there exists s, $t \in [1, \infty]$ with*

$$1 + \frac{1}{\gamma} \le s < \infty \quad \text{and} \quad t \ge \frac{s}{\gamma(s-1) - 1}$$

such that $\mathbf{A}(\cdot)u \in L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ for all $u \in U$, $\mathbf{N}(\cdot, \omega)$ is single-valued and continuous for a.a. $\omega \in \Omega$ and $\mathbf{N}(u, \cdot) \in L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ for all $u \in U$, $\mathbf{B} \in L^s(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}(Z, U^*))$, $\mathbf{b} \in L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ and $C^{-1} \in L^t(\Omega, \mathcal{F}, \mathbb{P})$.

Note that Assumption 2.2 requires $\mathbf{A}$ be a strongly measurable operator-valued function whereas $\mathbf{B}$ is a uniformly measurable operator-valued function [35, Def. 3.5.5]. In addition, the operator $e(\cdot, \cdot; \omega)$ in equation (5) becomes single valued a.e. on $U \times Z$. In the subsequent section, we prove measurability and integrability of solutions to $e(u, z; \omega) = 0$ for fixed $z \in Z$. In fact, we prove that there exists $q \in [1, \infty]$ (depending on s, t and γ) such that $S(z) \in \mathcal{U} := L^q(\Omega, \mathcal{F}, \mathbb{P}; U)$ for all $z \in Z$.

Finally, to obtain first-order optimality conditions for (1), we require the following differentiability assumptions on $\mathbf{N}$.

Assumption 2.3 (Differentiability of $\mathbf{N}(\cdot, \omega)$)**.** *In addition to Assumption 2.2, we assume that $\mathbf{N}(\cdot, \omega)$ is single-valued and continuously Fréchet differentiable from U into U^* for a.a. $\omega \in \Omega$ with partial derivative $\mathbf{N}'(u, \omega)$ which defines a bounded, nonnegative linear operator from U into U^* a.e. for all $u \in U$. Moreover, we assume that $\mathbf{A}$ and $u \mapsto \mathbf{N}(u, \cdot)$ are continuous maps from $\mathcal{U}$ into $L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ and $u \mapsto \mathbf{N}'(u, \cdot)$ is a continuous map from $\mathcal{U}$ into $L^{qs/(q-s)}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}(U, U^*))$.*

2.2. Application of Assumptions to the Semilinear Elliptic PDE (2)

In this section, we translate the assumptions in Section 2.1 to the semilinear elliptic PDE (2). To write (2) in the form (5), we define the linear elliptic operator $\mathbf{A}(\omega)$ by

$$\langle \mathbf{A}(\omega)u, v\rangle_{U^*,U} = \int_D \{\kappa(\omega,x)\nabla u(x)\cdot\nabla v(x) + c(\omega,x)u(x)v(x)\}\,\mathrm{d}x.$$

Similarly, we define the nonlinear operator $\mathbf{N}(\cdot,\omega)$ by

$$\langle \mathbf{N}(u,\omega), v\rangle_{U^*,U} = \int_D N(u(x),\omega,x)v(x)\,\mathrm{d}x.$$

Finally, we denote the control operator and forcing term by

$$\langle \mathbf{B}(\omega)z, v\rangle_{U^*,U} = \int_D [B(\omega)z](x)v(x)\,\mathrm{d}x \quad \text{and} \quad \langle \mathbf{b}(\omega), v\rangle_{U^*,U} = \int_D b(\omega,x)v(x)\,\mathrm{d}x,$$

respectively. Assumption 2.1 is then satisfied if the following conditions hold on the problem data in (2). Let $\kappa(\omega,\cdot)$, $c(\omega,\cdot) \in L^\infty(D)$ for a.a. $\omega\in\Omega$ satisfy: $\exists\,\kappa_0 > 0$ and $c_0 > 0$ such that $\kappa_0 \le \kappa(\omega,x)$ and $c_0 \le c(\omega,x)$ for a.a. $\omega\in\Omega$ and $x\in D$. In this case, $\gamma = 1$ and $C = \min\{\kappa_0, c_0\}$. In addition, if $N(\cdot,\omega,x) : \mathbb{R}\to\mathbb{R}$ is continuous and monotonically increasing with $N(0,\omega,x) = 0$ for a.a. $\omega\in\Omega$ and a.a. $x\in D$, then we can easily show the monotonicity assumption holds. For example, this is the case if

$$N(u,\omega,x) = c(\omega,x)(\sinh(u) - u). \tag{7}$$

In general, continuity can be ensured provided the usual growth conditions are satisfied, cf. Theorems 1 and 4 in [33]. Finally, if $b(\omega,\cdot) \in L^r(D)$ with $r > d/2$ for a.a. $\omega\in\Omega$, then Assumption 2.1.2 holds due to the Sobolev embedding theorem and the assumed regularity of ∂D. Moreover, Assumption 2.2 holds if, e.g., $b \in L^\infty(\Omega,\mathcal{F},\mathbb{P};L^2(D))$ and κ, $c \in L^\infty(\Omega,\mathcal{F},\mathbb{P};L^\infty(D))$. Note, however, that the assumption on $\mathbf{N}$ does not hold for all $u\in U$, when $\mathbf{N}$ is generated by the hyperbolic sine (7). For this we require additional regularity of our solutions, e.g., $U = H^2(D)$ or any space that continuously embeds into $L^\infty(D)$. To conclude, the assumptions on the bounded linear operator $\mathbf{A}$ in Assumption 2.3 are rather harmless. However, as is well-known, the differentiability of nonlinear superposition operators is a delicate matter, see e.g., [3]. For superposition operators generated by monotone polynomials of arbitrary degree, we can easily verify the assumptions by exploiting the regularity properties of H^1-functions implied by the Sobolev embedding theorem, provided $D\subset\mathbb{R}^2$ and ∂D is sufficiently regular. However for the case where $N(u,\omega,x)$ is given by (7), we would again need to appeal to higher solution regularity.

2.3. Properties of the Control-to-State Map

Our first result in this subsection demonstrates that under Assumption 2.1, the generalized equation (5) has a unique solution for a.a. $\omega\in\Omega$. Then, under the stronger assumptions in Assumption 2.2, we show that the solution is integrable.

Proposition 2.1. *Let Assumption 2.1 hold. Then,* $\mathbf{A}(\omega) + \mathbf{N}(\cdot,\omega)$ *is surjective from* U *into* U^* *for a.a.* $\omega\in\Omega$. *In particular, there exists a solution to* (5) *in* U *for any* $z\in Z$. *Moreover, this solution is unique. For a given* $z\in Z$, *we denote the unique solution as* $[S(z)](\omega)\in U$ *for a.a.* $\omega\in\Omega$.

Proof. The surjectivity of $\mathbf{A}(\omega)+\mathbf{N}(\cdot,\omega)$ is a direct consequence of Corollary 32.25 in [64]. Therefore, (5) has a solution in U a.e. Now suppose u, $u' : \Omega\to U$ are both solutions of (5). The a.e. coercivity of $\mathbf{A}$ and the a.e. monotonicity of $\mathbf{N}$ ensure that

$$0 = \langle \mathbf{A}(\cdot)(u-u') + (v-v'), u-u'\rangle_{U^*,U} \ge C\|u-u'\|_U^{1+\gamma} \quad \text{a.e.}$$

where v, $v' : \Omega \to U^*$ are such that $v(\omega) \in \mathbf{N}(u,\omega)$, $v'(\omega) \in \mathbf{N}(u',\omega)$, $0 = \mathbf{A}(\omega)u(\omega) + v(\omega) - \mathbf{B}(\omega)z - \mathbf{b}(\omega)$ and $0 = \mathbf{A}(\omega)u'(\omega) + v'(\omega) - \mathbf{B}(\omega)z - \mathbf{b}(\omega)$ for a.a. $\omega \in \Omega$. This ensures that $u = u'$ a.e. as desired. □

By requiring stronger assumptions on the components of (5), we can prove measurability and integrability of the unique solution from Proposition 2.1.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold, and define*

$$q := \frac{s\gamma}{1 + s/t}. \tag{8}$$

Then, $S(z) \in L^q(\Omega, \mathcal{F}, \mathbb{P}; U)$ for all $z \in Z$. We denote $\mathcal{U} := L^q(\Omega, \mathcal{F}, \mathbb{P}; U)$.

Proof. Proposition 2.1 ensures that $S(z) : \Omega \to U$ exists and is unique for all $z \in Z$. We now prove measurability of $S(z)$. Note that $\omega \mapsto (\mathbf{A}(\omega)u + \mathbf{N}(u,\omega))$ and $\omega \mapsto (\mathbf{B}(\omega)z + \mathbf{b}(\omega))$ are measurable for all $u \in U$ and $z \in Z$, and $u \mapsto (\mathbf{A}(\omega)u + \mathbf{N}(u,\omega))$ is continuous from U into U^* for a.a. $\omega \in \Omega$. Therefore, applying Theorem 1.1 with (T, Σ, μ) replaced by $(\Omega, \mathcal{F}, \mathbb{P})$, $X = U$, $Y = U^*$, $g(\omega, u) = (\mathbf{A}(\omega)u + \mathbf{N}(u,\omega))$, $\Gamma : \Omega \rightrightarrows U$ defined by $\Gamma(\omega) = U$ for all $\omega \in \Omega$ and $h(\omega) = \mathbf{B}(\omega)z + \mathbf{b}(\omega)$ (which satisfies $h(\omega) \in g(\omega, \Gamma(\omega))$ by Proposition 2.1 for a.a. $\omega \in \Omega$) ensures that there exists a measurable function $\gamma : \Omega \to U$ such that $\gamma(\omega) \in \Gamma(\omega)$ and $e(\gamma(\omega), z; \omega) = 0$ for a.a. $\omega \in \Omega$. Since $S(z)$ is unique, we have that $S(z) = \gamma$ a.e. and $S(z)$ is measurable in U.

To conclude, we prove that $S(z)$ is Bochner integrable. Since $S(z)$ is measurable, the a.e. coercivity of $\mathbf{A}$ and the a.e. monotonicity of $\mathbf{N}$ ensure

$$C\|S(z)\|_U^{1+\gamma} \le \langle \mathbf{B}(\cdot)z + \mathbf{b}, S(z)\rangle_{U^*,U} \le \|\mathbf{B}(\cdot)z + \mathbf{b}\|_{U^*}\|S(z)\|_U \quad \text{a.e.} \tag{9}$$

Therefore, we have that

$$\|S(z)\|_U^q \le C^{-\frac{q}{\gamma}}\|\mathbf{B}(\cdot)z + \mathbf{b}\|_{U^*}^{\frac{q}{\gamma}} \quad \text{a.e.} \tag{10}$$

Taking the expectation of (10), applying Hölder's inequality with a, $b \ge 1$ satisfying

$$\frac{1}{a} + \frac{1}{b} = 1 \quad \text{and} \quad \frac{s\gamma}{s\gamma - q} \le a \le \frac{t\gamma}{q}$$

(such an a exists by the assumptions on s, t, and q), and applying Theorem 3.7.4 in [35] ensures that $S(z) \in L^q(\Omega, \mathcal{F}, \mathbb{P}; U)$ as desired. □

Theorem 2.1 guarantees that the control-to state map $S : Z \to \mathcal{U}$ is well defined. In fact, it follows from (9) that $S(z) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U)$ provided $t = \infty$ and $\|\mathbf{B}(\cdot)z + \mathbf{b}(\cdot)\|_{U^*}$ is essentially bounded.

Remark 2.1. *As discussed in Section 2.2, one often must replace U with a more regular space V to obtain the continuity and differentiability of $\mathbf{N}(\cdot,\omega)$. However, since $\mathbf{A}(\omega)$ is not V-coercive in general, it would only be possible in seemingly restricted settings to prove the integrability of $S(z)$ as a map from Ω into V.*

In order to derive optimality conditions, we need to investigate the continuity and differentiability properties of S. First, we prove a technical lemma concerning the continuity of $\mathbf{B}$. It is also important to understand the adjoint of $\mathbf{B}$ since it ultimately appears in the reduced gradient of the objective functional. Under Assumption 2.2, the pointwise adjoint satisfies $\mathbf{B}(\cdot)^* \in L^s(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}(U, Z^*))$ by Theorem 3.8.1 in [35]. Since the solution satisfies $S(z) \in \mathcal{U}$, it is natural to wonder what the properties of $\mathbf{B}(\cdot)^*$ as an operator on $\mathcal{U}$ are. In particular, we are interested in the integrability and continuity properties of $\lambda \mapsto \mathbf{B}(\cdot)^*\lambda$ for any $\lambda \in \mathcal{U}$. We formalized these properties in the subsequent result.

Lemma 2.1. *Let Assumptions 2.1 and 2.2 hold. Then the linear operator $\mathbf{B}$ as a map from Z into $L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ is bounded and completely continuous. Moreover, the linear operator $\mathbf{B}(\cdot)^*$ as a map from $\mathcal{U}$ into $L^p(\Omega, \mathcal{F}, \mathbb{P}; Z^*)$ is bounded.*

Proof. We first show that $\mathbf{B}(\cdot)$ is bounded from Z into $L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$. For any $z \in Z$, we have that

$$\|\mathbf{B}(\cdot)z\|_{U^*}^s \leq \|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}^s \|z\|_Z^s \quad \text{a.e.}$$

Since $\mathbf{B} \in L^s(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{L}(Z, U^*))$, we may take the expectation of both sides and thus obtain:

$$\mathbb{E}[\|\mathbf{B}(\cdot)z\|_{U^*}^s] \leq \mathbb{E}[\|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}^s] \|z\|_Z^s,$$

which implies the boundedness from Z into $L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$. Now to show complete continuity, let $z_k \rightharpoonup z$ in Z. By the a.e. complete continuity of $\mathbf{B}(\cdot)$, we have that $\|\mathbf{B}(\cdot)(z_k - z)\|_{U^*} \to 0$ a.e. Again by the integrability of $\|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}$ and by the boundedness of $\|z_k - z\|_Z$ (since z_k weakly converges), there exists $M \geq 0$ such that $\|\mathbf{B}(\cdot)(z_k - z)\|_{U^*}^s \leq M \|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}^s$ for all k, the right-hand side of which is $\mathbb{P}$-integrable. Therefore, the Lebesgue Dominated Convergence Theorem ensures that $\mathbf{B}(\cdot)(z_k - z) \to 0$ in $L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ as desired. Now consider the operator $\mathbf{B}(\cdot)^*$. Applying Hölder's inequality with $a = s/p$ and $b = s/(s-p)$ yields

$$\mathbb{E}[\|\mathbf{B}(\cdot)^*\lambda\|_{Z^*}^p] \leq \mathbb{E}[\|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}^{ap}]^{1/a} \mathbb{E}[\|\lambda\|_U^{bp}]^{1/b} = \mathbb{E}[\|\mathbf{B}(\cdot)\|_{\mathcal{L}(Z,U^*)}^s]^{p/s} \mathbb{E}[\|\lambda\|_U^q]^{(s-p)/s} \quad \forall\, \lambda \in \mathcal{U}$$

which ensures that $\mathbf{B}(\cdot)^* \in \mathcal{L}(\mathcal{U}, L^p(\Omega, \mathcal{F}, \mathbb{P}; Z^*))$. □

Using Lemma 2.1, we can show that S is completely continuous.

Proposition 2.2. *Let Assumptions 2.1 and 2.2 hold. If $z_k \rightharpoonup z$ in Z, then $S(z_k) \to S(z)$ in U a.e. and $S(z_k) \to S(z)$ in $\mathcal{U}$, i.e., S is completely continuous.*

Proof. Let $z_k \rightharpoonup z$ in Z be an arbitrary weakly converging sequence. Let $u = S(z)$ and $u_k = S(z_k)$, then by the a.e. monotonicity of $\mathbf{N}$ and the a.e. coercivity of $\mathbf{A}$, we have

$$\|\mathbf{B}(\cdot)(z_k - z)\|_{U^*} \geq \frac{\langle (\mathbf{A}(\cdot)u_k + \mathbf{N}(u_k, \cdot)) - (\mathbf{A}(\cdot)u - \mathbf{N}(u, \cdot)), u_k - u \rangle_{U^*,U}}{\|u_k - u\|_U} \geq C\|u_k - u\|_U^\gamma \;\text{ a.e.} \tag{11}$$

By the a.e. complete continuity of $\mathbf{B}$, the left-hand side of (11) converges to zero a.e. proving the first claim. Now, by Lemma 2.1, the left-hand side of (11) converges to zero in $L^s(\Omega, \mathcal{F}, \mathbb{P})$, and thus the integrability of C^{-1} and Hölder's inequality using a and b as in the proof of Theorem 2.1 ensure $\|u_k - u\|_U \to 0$ in $L^q(\Omega, \mathcal{F}, \mathbb{P})$ as desired. □

In order to derive first-order optimality conditions, we need S to be continuously Fréchet differentiable.

Proposition 2.3. *Let Assumptions 2.1, 2.2 and 2.3 hold. Then $z \mapsto S(z)$ is continuous Fréchet differentiable from Z into $\mathcal{U}$.*

Proof. By Assumption 2.3, $\mathbf{N}'(u, \cdot)$ is a bounded nonnegative linear operator from U to U^* for all $u \in U$ a.e. and hence maximally monotone a.e. Applying Corollary 32.25 in [64] ensures that $\mathbf{A}(\omega) + \mathbf{N}'([S(z)](\omega), \omega)$ is surjective. Therefore, we deduce the solvability (indeed unique solvability) of the equation

$$(\mathbf{A} + \mathbf{N}'(S(z), \cdot))v = w \;\text{ a.e.} \tag{12}$$

for any $w \in \mathcal{U}^*$. In fact, we may proceed exactly as in the proof of Theorem 2.1, using now $(\mathbf{A} + \mathbf{N}'(S(z), \cdot))$ as the operator in the application of Fillipov's theorem, in order to ensure that that the solution to (12) is in $\mathcal{U}$. It then follows from the Implicit Function Theorem (see e.g., [63, Theorem 4.B]) that S is continuously Fréchet differentiable, where for fixed $h \in Z$, $d = S'(z)h \in \mathcal{U}$ solves the sensitivity equation

$$(\mathbf{A} + \mathbf{N}'(S(z), \cdot))d = \mathbf{B}(\cdot)h \;\text{ a.e.} \tag{13}$$

as desired. □

2.4. **Regularity with Respect to Random Inputs**

To conclude this section, we discuss the regularity of $S(z)$ with respect to the random inputs with an eye toward numerical approximation. For this discussion, we assume that the uncertainty in $\mathbf{A}$, $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{b}$ can be parametrized by a random vector $\xi : \Omega \to \Xi$ where $\Xi := \xi(\Omega) \subseteq \mathbb{R}^M$. That is, $\mathbf{A}(\omega) := \widehat{\mathbf{A}}(\xi(\omega))$, $\mathbf{N}(\cdot,\omega) := \widehat{\mathbf{N}}(\cdot,\xi(\omega))$, $\mathbf{B}(\omega) := \widehat{\mathbf{B}}(\xi(\omega))$, $\mathbf{b}(\omega) := \widehat{\mathbf{b}}(\xi(\omega))$ for appropriately chosen operators $\widehat{\mathbf{A}}$, $\widehat{\mathbf{N}}$, $\widehat{\mathbf{B}}$, and $\widehat{\mathbf{b}}$, cf. the finite-dimensional noise assumption [7]. With respect to (2), this permits the change of variables

$$\widehat{\mathbf{A}}(\xi)u + \widehat{\mathbf{N}}(u,\xi) = \widehat{\mathbf{B}}(\xi)z + \widehat{\mathbf{b}}(\xi) \quad \forall\, \xi \in \Xi. \tag{14}$$

Note that a unique solution to (14) exists by Theorem 2.1. We denote this solution by $\widehat{S}(z)$. Now, define $E(u,\xi) := \widehat{\mathbf{A}}(\xi)u + \widehat{\mathbf{N}}(u,\xi) - (\widehat{\mathbf{B}}(\xi)z + \widehat{\mathbf{b}}(\xi))$ for a fixed $z \in Z$ and let $(u_0,\xi_0) \in U \times \Xi$ satisfy $E(u_0,\xi_0) = 0$. Moreover, suppose E is continuously Fréchet differentiable on an open set of $U \times \Xi$ containing (u_0,ξ_0) (we denote the partial derivatives by E_u and E_ξ). Then, if $E_u(u_0,\xi_0)$ has a bounded inverse, the Implicit Function Theorem [63, Thm. 4.B] ensures the existence of open neighborhoods $V(u_0) \subseteq U$ and $V(\xi_0) \subseteq \Xi$ and a unique continuous function $w : V(\xi_0) \to V(u_0)$ such that $w(\xi_0) = u_0$ and $E(w(\xi),\xi) = 0$ for all $\xi \in V(\xi_0)$, i.e., $w(\xi) = [\widehat{S}(z)](\xi)$. Additionally, w is continuously Fréchet differentiable with derivative

$$w'(\xi) = -E_u(w(\xi),\xi)^{-1}E_\xi(w(\xi),\xi) \quad \forall\, \xi \in V(\xi_0).$$

If E is m-times continuously Fréchet differentiable, then w is also m-times continuously Fréchet differentiable. Under Assumptions 2.1, 2.2 and 2.3, we have that $E_u(u_0,\xi_0) = [\widehat{\mathbf{A}}(\xi_0) + \widehat{\mathbf{N}}'(u_0,\xi_0)]$ is invertible due to monotonicity. By assuming continuous differentiability of E with respect to ξ, we can then prove that $\widehat{S}(z) : \Xi \to U$ is (at least locally) continuously differentiable. Note that if Ξ is the infinite product of one-dimensional intervals, then additional care is required to ensure differentiability. See, e.g., [34] for a detailed analysis in the context of elliptic semilinear PDEs.

These differentiability results are important when approximating (2) using, e.g., polynomial chaos [20,32,62] or stochastic collocation [38–40,48]. Similarly, the higher-order differentiability with respect to ξ can permit the use of numerical solution and optimization methods based on Taylor expansions as in [2,25–27,44]. Finally, in both stochastic optimization and uncertainty quantification, one must often approximate integrals of the form

$$\mathbb{E}[\nu(\mathcal{J}(\widehat{S}(z)))]$$

for some function $\nu : \mathbb{R} \to \mathbb{R}$. In order to exploit the desirable convergence properties of, e.g., sparse grids and deterministic quadrature, higher regularity of the integrand is also needed. Unfortunately, for many interesting risk measures, such as the average value-at-risk and mean-plus-semideviation, the associated functions ν are nonsmooth and thus require special treatment (see, e.g., [42]).

3. Analysis of the Optimal Control Problem

In this section, we analyze the optimal control problem (1). We first derive the associated adjoint equation and discuss the properties of the control-to-adjoint map $\Lambda(z)$. We then prove the existence of minimizers and derive optimality conditions. Many of the subsequent results only require a subset of the assumptions in Section 2.1 to hold. As such, we will explicitly state when each assumption is required.

3.1. **Differentiability, Existence of Minimizers, and First-Order Optimality Conditions**

The composite mapping $\mathcal{J} \circ S$ is often highly nonlinear. To guarantee continuity and differentiability, we require the usual assumptions of nonlinear superposition operators.

Assumption 3.1 (Measurability and Differentiability of the Objective Functions in (1))**.**

(1) Let $J : U \times \Omega \to \mathbb{R}$ be a Carathéodory function, i.e., $J(u, \cdot)$ is measurable for all $u \in U$ and $J(\cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$, let the superposition operator

$$[\mathcal{J}(u)](\omega) := J(u(\omega), \omega)$$

be continuous from $\mathcal{U}$ into $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with

$$p := \frac{sq}{s+q},$$

and let $\wp : Z \to \mathbb{R}$ be convex and continuous.

(2) Let J be continuously Fréchet differentiable with respect to $u \in U$ for a.a. $\omega \in \Omega$. Moreover, let the superposition operator $\mathcal{J}$ be continuously Fréchet differentiable from $\mathcal{U}$ into $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with derivative

$$\mathcal{J}'(u) = J_u(u(\cdot), \cdot) \in L^{pq/(q-p)}(\Omega, \mathcal{F}, \mathbb{P}; U^*) \quad \text{for } u \in \mathcal{U}.$$

Here, J_u denotes the partial derivative of J with respect to u.

Since U is separable, to ensure J and hence $\mathcal{J}$, satisfy Assumption 3.1 it is necessary and sufficient that the assumptions in Theorems 1 (cf. Krasnosel'skii's Theorem) and 7 in [33] are satisfied. See also [41]. We are now able to prove the existence of minimizers to (1). For readability, we set

$$F(z) := \mathcal{J}(S(z)), \quad g(z) := \mathcal{R}[F(z)], \quad f(z) := g(z) + \wp(z)$$

Proposition 3.1. *Let Assumptions 2.1, 2.2 and 3.1.1 hold, and suppose that $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ is closed and convex. If either $Z_{\rm ad}$ is bounded or $f : Z \to \mathbb{R}$ is coercive, then (1) has an optimal solution.*

Proof. Since $\mathcal{R}$ is finite, closed and convex, it is continuous, cf. [24, Cor. 2.5]. Therefore, Proposition 2.2 ensures that g is weakly continuous. In addition, $\wp$ is convex and continuous, and hence weakly lower semicontinuous. The result then follows from the direct method of calculus of variations, cf. [5, Thm. 3.2.1]. □

In order to derive first-order optimality conditions, we show that F is continuously Fréchet differentiable whenever Assumption 2.3 holds.

Proposition 3.2. *Let Assumptions 2.1, 2.2, 2.3, and 3.1 hold. Then F is continuously Fréchet differentiable from Z into $\mathcal{U}^*$ with derivative*

$$F'(z) = \mathbf{B}(\cdot)^*\Lambda(z) \tag{15}$$

where $\Lambda(z) = \lambda \in \mathcal{U}$ solves

$$(\mathbf{A} + \mathbf{N}'(S(z), \cdot))^*\lambda = J_u(S(z), \cdot) \ \text{ a.e.} \tag{16}$$

Proof. In light of Proposition 2.3, $S(z)$ is continuously Fréchet differentiable and its derivative satisfies the sensitivity equation (13). Under Assumption 3.1, the chain rule ensures that F is continuously Fréchet differentiable with derivative

$$F'(z) = S'(z)^*\mathcal{J}'(S(z)) = S'(z)^*J_u(S(z), \cdot).$$

Utilizing equation (13), we see that (15) holds where $\Lambda(z)$ solves (16). Once again applying the arguments used in the proof of Theorem 2.1, this time with the adjoint operator $(\mathbf{A} + \mathbf{N}'(S(z), \cdot))^*$, we see that $\Lambda(z) \in \mathcal{U}$ exists and is unique for each $z \in Z$. □

In addition to the differentiability of F, we have the following continuity property. This property is useful for the analysis of algorithms as well as in limiting results for variational regularization schemes.

Proposition 3.3. *Let Assumptions 2.1, 2.2, 2.3 and 3.1 hold. For any $z_k \rightharpoonup z$ in Z, the solution to the adjoint equation satisfies $\Lambda(z_k) \to \Lambda(z)$ in U a.e. and $\Lambda(z_k) \to \Lambda(z)$ in $\mathcal{U}$, i.e., Λ is completely continuous. In particular, F' is completely continuous.*

Proof. Let $z_k \rightharpoonup z$ in Z be arbitrary and denote $u_k = S(z_k)$, $u = S(z)$, $\lambda_k = \Lambda(z_k)$ and $\lambda = \Lambda(z)$. By the a.e. coercivity of $\mathbf{A}$ and the a.e. nonnegativity of $\mathbf{N}'$, we have that

$$\begin{aligned} C\|\lambda_k - \lambda\|_U^{1+\gamma} &\le \langle (J_u(u,\cdot) - J_u(u_k,\cdot)) + (\mathbf{N}'(u,\cdot)^*\lambda - \mathbf{N}'(u_k,\cdot)^*\lambda_k), \lambda_k - \lambda\rangle_{U^*,U} \ \text{ a.e.} \\ &= \langle (J_u(u,\cdot) - J_u(u_k,\cdot)) + (\mathbf{N}'(u_k,\cdot)^*(\lambda - \lambda_k) + (\mathbf{N}'(u,\cdot)^* - \mathbf{N}'(u_k,\cdot)^*)\lambda), \lambda_k - \lambda\rangle_{U^*,U} \ \text{ a.e.} \\ &\le \langle (J_u(u,\cdot) - J_u(u_k,\cdot)) + (\mathbf{N}'(u,\cdot)^* - \mathbf{N}'(u_k,\cdot)^*)\lambda, \lambda_k - \lambda\rangle_{U^*,U} \ \text{ a.e.} \\ &\le \|(J_u(u,\cdot) - J_u(u_k,\cdot)) + (\mathbf{N}'(u,\cdot) - \mathbf{N}'(u_k,\cdot))^*\lambda\|_{U^*}\|\lambda_k - \lambda\|_U \ \text{ a.e.} \end{aligned}$$

This bound combined with the a.e. continuity of J_u and $\mathbf{N}'$ and Proposition 2.2 ensures that $\lambda_k \to \lambda$ in U a.e. Moreover, since $pq/(q-p) = s$, we can apply the Cauchy-Schwarz inequality to the right hand side of

$$\mathbb{E}\left[\|\lambda_k - \lambda\|_U^q\right] \le \mathbb{E}\left[C^{-\frac{q}{\gamma}}\|(J_u(u,\cdot) - J_u(u_k,\cdot)) + (\mathbf{N}'(u,\cdot) - \mathbf{N}'(u_k,\cdot))^*\lambda\|_{U^*}^{\frac{q}{\gamma}}\right]$$

with a and b as in the proof of Theorem 2.1. Therefore, Proposition 2.2 and the continuity of J_u and $\mathbf{N}'$ ensure that $\lambda_k \to \lambda$ in $\mathcal{U}$. In light of (15), the complete continuity of F' then follows from Lemma 2.1. □

Using Proposition 3.3, we can derive first-order optimality conditions.

Proposition 3.4. *Let Assumptions 2.1, 2.2, 2.3 and 3.1 hold, and suppose $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ is closed and convex. Finally suppose that either $Z_{\rm ad}$ is bounded or $f : Z \to \mathbb{R}$ is coercive. Then for any optimal solution $z^\star$ to* (1)*, there exists $\vartheta^\star \in \partial\mathcal{R}(F(z^\star))$ such that*

$$\mathbb{E}[\langle F'(z^\star), z - z^\star\rangle_{Z^*,Z}\,\vartheta^\star] + \wp'(z^\star; z - z^\star) \ge 0, \ \forall z \in Z_{\rm ad}. \tag{17}$$

Proof. Proposition 3.1 ensures the existence of $z^\star$. Moreover, Proposition 5.2 in [24] ensures that $\mathcal{R}$ is subdifferentiable. Now, since $\wp$ is convex and continuous, it is directionally differentiable. The rest is standard, cf. the discussion following Proposition 3.13 in [43]. □

Based on (17), we obtain a more explicit first-order optimality system using the previous results, i.e., if $z^\star \in Z_{\rm ad}$ is an optimal solution to (1), then there exists a triple $(u^\star, \lambda^\star, \vartheta^\star)$ such that

$$\mathbf{A}(\cdot)u^\star + \mathbf{N}(u^\star,\cdot) = \mathbf{B}(\cdot)z^\star + \mathbf{b}(\cdot) \ \text{ a.e.,} \tag{18a}$$

$$(\mathbf{A} + \mathbf{N}'(u^\star,\cdot))^*\lambda^\star = J_u(u^\star,\cdot) \ \text{ a.e.,} \tag{18b}$$

$$\mathbb{E}[\langle \mathbf{B}(\cdot)^*\lambda^\star, z - z^\star\rangle_{Z^*,Z}\,\vartheta^\star] + \wp'(z^\star; z - z^\star) \ge 0 \ \forall\, z \in Z_{\rm ad}, \tag{18c}$$

$$\mathcal{R}[Y] - \mathcal{R}[\mathcal{J}(u^\star)] - \mathbb{E}[\vartheta^\star(Y - \mathcal{J}(u^\star)] \ge 0 \ \forall\, Y \in \mathcal{X}. \tag{18d}$$

In contrast to deterministic PDE-constrained optimization there is an additional nonlinear coupling in (18c) due to the risk measure and the subgradient inequality (18d). In light of this nontrivial coupling, the direct solution of this system using, e.g., a semismooth Newton method remains a challenging open problem. Of course, for differentiable risk measures, we have that $\vartheta^\star = \mathcal{R}'[\mathcal{J}(u^\star)]$ and therefore we can remove (18d).

4. Examples

In this section, we present two optimal control examples. The first is a stochastic extension of a "canonical" semilinear elliptic control problem. The forward problem may be seen as a rough approximation of the PDE in the second example. We fully analyze its properties and verify the necessary assumptions. The second example is considerably more challenging from a theoretical perspective, yet closer to a real application. In both cases, we numerically solve the optimization problems and compare the results.

Both examples share the general problem formulation given in (1) and (2), differing only in the nonlinear term $\mathbf{N}$. We describe this formulation here. Let the open, bounded set $D \subset \mathbb{R}^2$ denote the physical domain. We consider the optimal control problem (1) with

$$J(u,\omega) := \frac{1}{2}\int_{D_o} \max\{0, 1-u\}^2 \, \mathrm{d}x \qquad \text{and} \qquad \wp(z) := \frac{\alpha}{2}\int_D z^2 \, \mathrm{d}x.$$

where $D_o \subseteq D$ and $\alpha > 0$. Continuing, we assume that κ, $c \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfy the assumptions in Section 2.2. That is, $\exists\, \kappa_0 > 0$ and $c_0 > 0$ such that $\kappa_0 \le \kappa(\omega)$ and $c_0 \le c(\omega)$ for a.a. $\omega \in \Omega$. Moreover, we let $b : \Omega \to U^*$ such that $\mathbf{b} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ and define the action of the operator B as the solution $d(\omega, \cdot) = B(\omega)z \in H^1(D)$ to weak form of the linear elliptic PDE

$$-r(\omega)\Delta d(\omega, x) + d(\omega, x) = z(x), \qquad x \in D \tag{19a}$$

$$r(\omega)\frac{\partial d}{\partial n}(\omega, x) = 0, \qquad x \in \partial D \tag{19b}$$

for a.a. $\omega \in \Omega$ where $r \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfies: $\exists\, r_0 > 0$ such that $r(\omega) \ge r_0$ for a.a. $\omega \in \Omega$. Under the stated assumptions on κ and c, $\mathbf{A}$ satisfies

$$\min\{\kappa_0, c_0\}\|u\|_U^2 \le \langle \mathbf{A}(\cdot)u, u\rangle_{U^*,U} \;\; \text{a.e.}$$

and

$$|\langle \mathbf{A}(\cdot)u, v\rangle_{U^*,U}| \le \|\max\{\kappa, c\}\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}\langle u, v\rangle_U \;\; \text{a.e.}$$

for all u, $v \in U$. Thus, $\mathbf{A}$ is an a.e. coercive, monotone, bounded linear operator from U into U^* with $\gamma = 1$ and $C = \min\{\kappa_0, c_0\} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $t = \infty$. Moreover, $\mathbf{A}(\cdot)u \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ for all $u \in U$. Turning to $\mathbf{B}$, we note that $B(\omega)$ is a compact operator from $L^2(D)$ into $L^2(D)$ for a.a. $\omega \in \Omega$ [45, Sect. 22.3] and hence the existence of $r_0 > 0$, the Lebesgue Dominated Convergence Theorem [30] and the compact embedding of $L^2(D)$ into U^* ensure that $\mathbf{B}$ is a compact operator, i.e., $\mathbf{B}$ is completely continuous. By similar coercivity arguments as those for $\mathbf{A}$, we have that

$$r_0\|Bz\|_U^2 \le \int_D r\nabla(Bz)\cdot\nabla(Bz) + (Bz)^2 \, \mathrm{d}x = \int_D z(Bz)\, \mathrm{d}x \le \|z\|_{L^2(D)}\|Bz\|_{L^2(D)}$$

which implies $\mathbf{B}z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ for all $z \in Z$. It follows then from (9) that $S(z) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U)$.

In the forthcoming concrete examples, we consider nonlinearities of the form

$$\mathbf{N}(u,\omega) = c(\omega)\partial\Theta(u) \tag{20}$$

where $\Theta : L^2(D) \to \overline{\mathbb{R}}$ is proper, convex and closed. Recall here that we require that $\mathbf{N}(\cdot, \omega)$ is single-valued and continuous for a.a. $\omega \in \Omega$ and $\mathbf{N}(u, \cdot) \in L^s(\Omega, \mathcal{F}, \mathbb{P}; U^*)$ for all $u \in U$. This, combined with the fact that $t = \infty$ ensures that we may take s arbitrarily large. Note that we could prove continuity of S as a map from Z into $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U)$, but it is not necessarily the case that S is completely continuous into $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; U)$. In our examples, Θ has the specific form

$$\Theta(u) = \int_D g(u)\, \mathrm{d}x. \tag{21}$$

The subsequent result is somewhat standard (see, e.g., [56, Prop. 6.18]) and provides assumptions on $g : \mathbb{R} \to \mathbb{R}$ that ensure Θ is proper, closed and convex, and hence $\mathbf{N}(\cdot, \omega)$ is maximally monotone for a.a. $\omega \in \Omega$.

Proposition 4.1. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is convex. Then the functional Θ defined in (21) is proper, closed and convex. Additionally, $\partial\Theta(u)$ is maximally monotone and*

$$\partial\Theta(u) = \{w \in L^2(D) \,|\, w(x) \in \partial g(u(x)) \text{ for a.e. } x \in D\}.$$

Proof. This is a standard result on subdifferentiability of integral functionals. See e.g. [51]. □

If $\partial\Theta$ is non-empty, single-valued and continuous on U, then Theorem 2.1 ensures the existence of unique solutions $S(z)$ in $\mathcal{U} = L^q(\Omega, \mathcal{F}, \mathbb{P}; U)$ with q arbitrarily large. Additionally, Proposition 2.2 ensures that the control-to-state map S is completely continuous. To conclude this section, we investigate the properties of the objective functional. First note that the Cauchy-Schwarz inequality and Young's inequality ensure

$$0 \le J(u,\omega) := \frac{1}{2}\int_D \max\{0, 1-u(x)\}^2 \,\mathrm{d}x \le \frac{1}{2}\int_D (1-u(x))^2 \,\mathrm{d}x \le |D| + \|u\|^2_{L^2(D)}.$$

Moreover, given $u \in U = H^1(D)$ we have $0 \le J(u,\omega) \le |D| + \|u\|_U^2$. Then since $\mathcal{U}$ is continuously embedded into $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ it follows from Krasnosel'skii's theorem, see e.g., [33, Th. 4], that $\mathcal{J} : \mathcal{U} \to \mathcal{X} := L^1(\Omega, \mathcal{F}, \mathbb{P})$ is continuous. Now, to see that we have the necessary differentiability properties, we begin by noting that $\psi(x) = \frac{1}{2}\max\{0,x\}^2$ is convex and continuously differentiable with derivative $\psi'(x) = \max\{0, x\}$. Therefore, Theorems 4 and 7 in [33] ensure that $\Psi : L^2(D) \to L^1(D)$ with $[\Psi(u)](x) = \psi(u(x))$ is continuously Fréchet differentiable from $L^2(D)$ into $L^1(D)$. Moreover, since U is continuously embedded into $L^2(D)$ we have $\Psi : U \to L^1(D)$ is continuously Fréchet differentiable with derivative $\Psi'(u) = \max\{0,u\} \in U$. Hence, $\mathcal{J}$ is also continuously Fréchet differentiable from U into $\mathbb{R}$ and the derivative satisfies: there exists $K > 0$ (due to the compact embedding of $L^2(D)$ into U^*) such that

$$\|\mathcal{J}'(u)\|_{U^*} \le K\|\max\{0, 1-u\}\|_{L^2(D)} \le K\|1-u\|_{L^2(D)} \le K\|1-u\|_U \le K(|D| + \|u\|_U) \quad \forall\, u \in U.$$

Again employing Theorems 4 and 7 in [33], we have that $\mathcal{J} : \mathcal{U} \to \mathcal{X}$ is continuously Fréchet differentiable.

4.1. **Cubic Nonlinearity**

In this subsection, we consider the cubic nonlinearity,

$$N(y,\omega) = c(\omega)\left(\frac{1}{3!}y^3 - y\right) \quad\implies\quad \langle \mathbf{N}(u,\omega), v\rangle_{U^*,U} = c(\omega)\int_D \left(\frac{1}{3!}u^3 - u\right) v \,\mathrm{d}x,$$

which is maximally monotone by Proposition 4.1 with $g(y) = \frac{1}{4!}y^4 - \frac{1}{2}y^2$. Now, since ∂D is Lipschitz, it follows from the Sobolev embedding theorem that $u \in U$ satisfies $u \in L^q(D)$ for any $q \in [1,\infty)$, e.g., $q = 6$. Hence, $u^3 \in L^2(D)$ and consequently $u^3 \in U^*$. These facts ensure that Θ is finite on U and hence subdifferentiable with

$$\partial\Theta(u) = \left\{\tfrac{1}{3!}u^3 - u \in U^*\right\}$$

for all $u \in U$. This demonstrates that $\mathbf{N}(\cdot,\omega)$ is a single-valued operator that is maximal monotone for a.a. $\omega \in \Omega$. In addition, we can also prove continuous Fréchet differentiability of $\mathbf{N}$. To this end, we show for $u, h \in U$ that

$$\|\mathbf{N}(u+h,\cdot) - \mathbf{N}(u,\dot{c}) - \mathbf{G}(u,\cdot)h\|_{U^*} = o(\|h\|_U^2), \quad \text{where} \quad \mathbf{G}(u,\omega) := c(\omega)\left(\frac{1}{2}u^2 - 1\right).$$

Let $v \in U$ with $\|v\|_U = 1$. Then it follows from repeated applications of the Cauchy-Schwarz and Hölder inequalities that

$$|\langle \mathbf{N}(u+h,\cdot) - \mathbf{N}(u,\cdot) - \mathbf{G}(u,\cdot)h, v\rangle_{U^*,U}| \le K(\|h\|^3_{L^6(D)}\|v\|_{L^2(D)} + \|h\|^2_{L^6(D)}\|u\|_{L^3(D)}\|v\|_{L^3(D)}),$$

where $K > 0$ is a constant (depending on $\|c\|_{L^\infty(\Omega,\mathcal{F},\mathbb{P})}$), cf. the arguments in [36]. Then using the Sobolev embedding theorem we have

$$|\langle \mathbf{N}(u+h,\cdot) - \mathbf{N}(u,\cdot) - \mathbf{G}(u,\cdot)h, v\rangle_{U^*,U}| \leq K'\|h\|_U^2(\|h\|_U + \|u\|_U)\|v\|_U,$$

where $K' > 0$ is a constant. Consequently, the continuous Fréchet differentiability follows from Theorem 7 in [33]. Therefore, Theorem 2.1 guarantees the existence of a unique solution that by Proposition 2.2 is completely continuous and by Propositions 3.2 and 3.3 is continuously Fréchet differentiable with completely continuous derivative.

4.2. Exponential Nonlinearity

In this section, we consider an example application whose structure is in some sense similar to the problem in Section 4.1. However, without additional analysis of the regularity of the solutions to the PDE, cf. the discussion in Section 2.2, we cannot directly verify the necessary assumptions to ensure measurability and integrability of the pointwise solution.

The optimal design of semiconductor devices is an important application in which one tries to improve current flow over device contacts by modifying the doping profile; see Figure 1 for the basic geometry of the underlying device. More specifically, manufacturers introduce impurities (dopant) into the silicon in order to influence the flow of electrons through the device. In the context of deterministic PDE-constrained optimization, many contributions can be found in the literature; see, e.g., [14–16, 28, 29, 50]. In this section, we present numerical results for a simplified semiconductor design problem in which we replace the typical drift-diffusion physics with the Poisson-Boltzmann equation, see [37] for a comprehensive theoretical and numerical study.

The nonlinearity in this section is

$$N(y,\omega) = c(\omega)(\sinh(y) - y) \quad \Longrightarrow \quad \langle \mathbf{N}(u,\omega), v\rangle_{U^*,U} = c(\omega)\int_D (\sinh(u) - u)v\,\mathrm{d}x$$

and gives rise to the Poisson-Boltzmann equation. We note that the Taylor expansion for the hyperbolic sine is given by

$$\frac{1}{2}(e^y - e^{-y}) = \sinh(y) = y + \frac{y^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!}.$$

Therefore, the semilinear PDE in Section 4.1 includes a third-order approximation of the Poisson-Boltzmann nonlinearity. Now, $\mathbf{N}(\cdot,\omega)$ is maximally monotone for a.a. $\omega \in \Omega$ by Proposition 4.1 with $g(y) = (\cosh(y) - \frac{1}{2}y^2 - 1)$, and for any $u \in \operatorname{dom}\Theta$, we have

$$\partial\Theta(u) = \{(\sinh(u) - u) \in U^*\}.$$

On the other hand, if $u \notin \operatorname{dom}\Theta$, then $\partial\Theta(u) = \emptyset$. Nevertheless, Theorem 2.1 ensures the existence of a pointwise solution and, by restricting ourselves to a more regular (yet still separable) solution space, Filippov's theorem would guarantee measurability. Unfortunately, the necessary integrability, continuity, and differentiability properties needed for our analysis are not easily verified.

4.3. Numerical Results

For our numerical results, we fix $D = (0, 0.6) \times (0, 0.2)$, $D_o = (0.5, 0.6) \times (0.167, 0.2)$ and $\alpha = 10^{-2}$. Additionally, we specify the uncertain coefficients as

$$\kappa(\omega) = 2.5 \times 10^{\xi_1(\omega)}, \quad c(\omega) = 1.45 \times 10^{\xi_2(\omega)} \quad \text{and} \quad r(\omega) = 10^{\xi_3(\omega)},$$

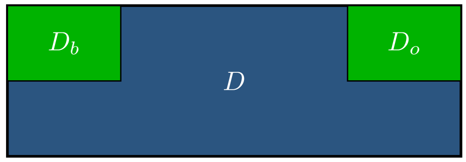

FIGURE 1. Physical domain D depicting observation and source regions D_o and D_b, respectively.

where ξ_1 is uniformly distributed on $[-2,-1]$, ξ_2 is uniformly distributed on $[-1,0]$ and ξ_3 is uniformly distributed on $[-4,-1]$. The fixed source term in $\mathbf{b}$ is generated by

$$b(\omega, x) = 12 \cdot \mathbb{1}_{D_b}(x),$$

where $D_b = (0, 0.1) \times (0.167, 0.2)$ and $\mathbb{1}_{D_b}$ denotes the characteristic function of the set D_b (i.e., b is deterministic). Figure 1 depicts the physical domain D with D_o and D_b highlighted. We discretize both the semilinear linear and linear PDEs using Q1 finite elements on a uniform 60×20 mesh of quadrilaterals. We further approximate (1) with respect to the random inputs $\xi = (\xi_1, \xi_2, \xi_3)$ using 1000 Monte Carlo samples. We note that a rigorous convergence analysis of our sampled-based finite-element approximation is beyond the scope of this paper and is left as future work. However, we mention that the authors in [47] provide a similar analysis in the context of stochastic approximation (stochastic gradient descent) for the risk neutral problem ($\mathcal{R} = \mathbb{E}$).

For the subsequent numerical experiments, we choose the risk measure to be $\mathcal{R} \equiv \mathrm{AVaR}_\beta$. We solve the resulting risk-averse stochastic optimization problem using epi-regularization with $\Phi[X] = \frac{1}{2}\mathbb{E}[X^2] + \mathbb{E}[X]$ combined with a matrix-free trust-region Newton method [21]. Since the epi-regularized risk measure is not guaranteed to be twice continuously Fréchet differentiable, we utilize generalized Hessians based on the Newton derivative [61] to formulate the trust-region subproblem.

In Figure 2, we investigate the effect of epi-regularization on the optimal controls for the cubic nonlinearity (left) and the exponential nonlinearity (right). We solved (1) for $\beta \in \{0.1, 0.2, \ldots, 0.9\}$ and $\varepsilon \in$

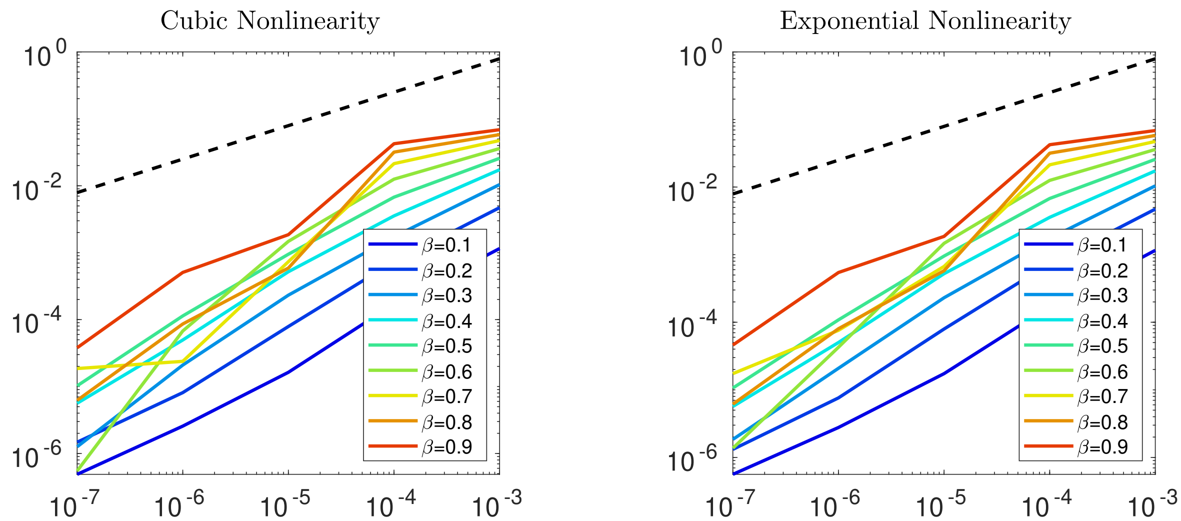

FIGURE 2. Epi-regularization error in the optimal controls for $\beta \in \{0.1, 0.2, \ldots, 0.9\}$ and $\varepsilon \in \{10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}\}$. The dashed line depicts the theoretical convergence rate of $\frac{1}{2}$. As seen in both images, the epi-regularization errors decays slightly faster than the theoretical rate.

$\{10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}\}$. As depicted in Figure 2, the epi-regularization errors decay slightly faster than the theoretical rate of $\frac{1}{2}$ (dashed line) from Theorem 6 in [42] even though these examples may not satisfy the assumptions of Theorem 6 in [42]. This suggests that either our examples are locally strongly convex or the assumptions of Theorem 6 in [42] can be relaxed. To conclude, we investigate the effect of the AVaR confidence level β on our ability to mitigate and reduce uncertainty. In Figures 3 and 4, we plot the cumulative distribution functions for $(F(z^\star) + \wp(z^\star))$ (left) and $F(z^\star)$ (right) where $z^\star$ denotes the minimizer in (1) for $\beta \in \{0.1, 0.2, \ldots, 0.9\}$. These plots clearly demonstrate that as β approaches one (i.e., our risk preference becomes more conservative), the variability in the uncertain objective function decreases. In addition, as seen in the right images of Figures 3 and 4, increasing β results in random variables $F(z^\star)$ that stochastically dominate those associated with smaller β, effectively mitigating the uncertainty associated with the random objective function.

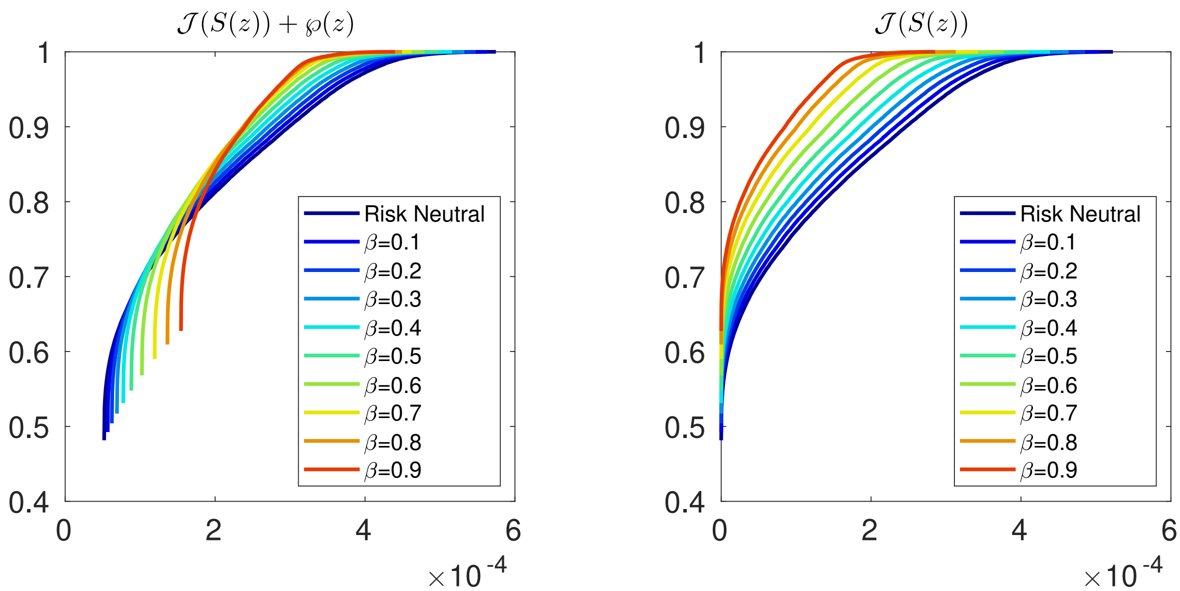

Figure 3. Left: Cumulative distribution function for the random objective plus control penalty evaluated at the optimal controls for the cubic nonlinearity. Right: Cumulative distribution function for the random objective evaluated at the optimal controls for the cubic nonlinearity.

5. Conclusion and Outlook

We have demonstrated that the analysis of the solutions to nonlinear parametric PDEs, either with or without uncertainty, should not be taken lightly. Simply assuming that one has, e.g., measurability or differentiability with respect to the parameters may be incorrect or, at least, nontrivial to verify. Although the need to work with more regular spaces to obtain continuity and differentiability of the control-to-state mapping for semilinear PDEs is known, the question of measurability or integrability with respect to exogenously determined (random) inputs is rarely studied. This situation is even more sensitive for PDEs with nonmonotone nonlinearities. In this case, it is unclear how to guarantee measurability, let alone integrability—even when the nonlinearity is a nonmonotone polynomial. Ignoring these infinite-dimensional properties in PDE-constrained optimization and optimal control often leads to a lack of efficiency (e.g., mesh dependence) of numerical solution algorithms.

In light of this, we have demonstrated for a certain class of semilinear elliptic equations that the control-to-state map is integrable, completely continuous and continuously Fréchet differentiable with completely continuous derivatives. With these properties, one can show that the optimal solutions and stationary points computed

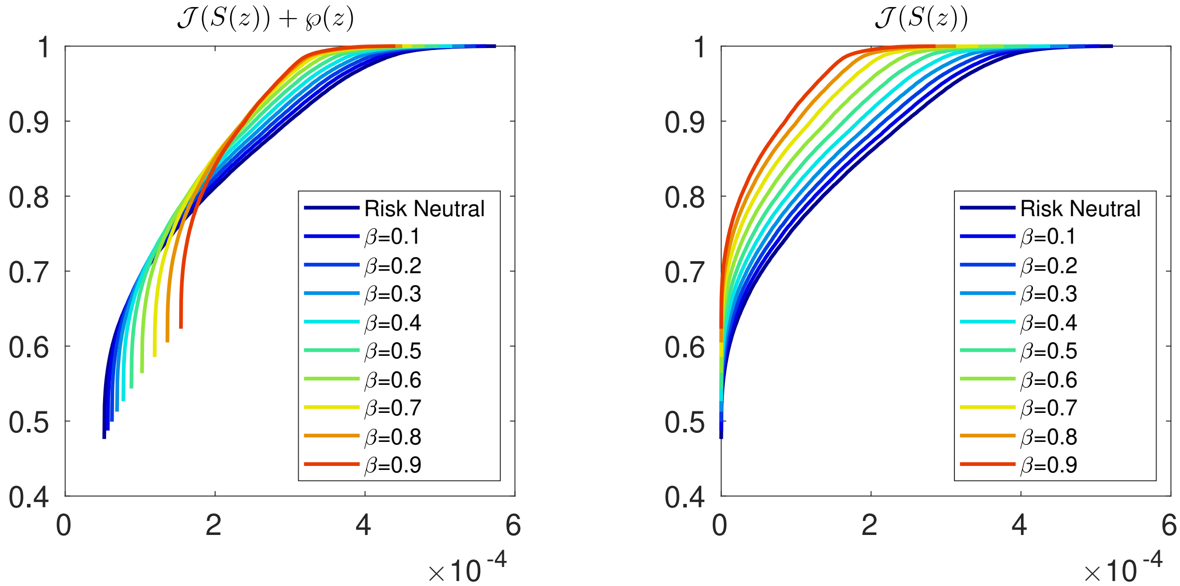

Figure 4. Left: Cumulative distribution function for the random objective plus control penalty evaluated at the optimal controls for the exponential nonlinearity. Right: Cumulative distribution function for the random objective evaluated at the optimal controls for the exponential nonlinearity.

using epi-regularized risk measures converge to optimal solutions and stationary points, respectively, of (1). We have demonstrated these results with two examples. The first conforms to our stated assumptions whereas the second does not. In both cases, we numerically verify that the epi-regularized solutions achieve the theoretical convergence rate even though we are unable to prove the requisite assumptions.

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