PRINCIPAL EIGENVALUES OF FULLY NONLINEAR INTEGRO-DIFFERENTIAL ELLIPTIC EQUATIONS WITH A DRIFT TERM

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ABSTRACT. We study existence of principal eigenvalues of a fully nonlinear integro-differential elliptic equations with a drift term via the Krein-Rutman theorem and regularity estimates up to boundary of viscosity solutions. We also show simplicity of eigenfunctions in the viscosity sense by using a nonlocal version of the ABP estimate and a "sweeping lemma".

1. INTRODUCTION

The main scope of this paper is to study regularity of viscosity solutions and spectral properties of non-divergence integro-differential equations. To be more precise, we consider non-local elliptic equations with a drift term having the following form

$$Iu(x) = \inf_{a\in A} \sup_{b\in B} \left\{ L_{K_{a,b}} u(x) + c_{a,b}(x) \cdot \nabla u(x) \right\} = 0, \tag{1.1}$$

where $\{L_{K_{a,b}}\}_{a\in A, b\in B}$ is a family of integro-differential operators defined by

$$L_{K_{a,b}} = \int_{\mathbb{R}^n} \delta(u, x; y) K_{a,b}(y)\, dy,$$
$$\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x).$$

The function $c_{a,b}$ is assumed to be uniformly bounded in Ω and the family of kernels $\{K_{a,b}\}_{a\in A,b\in B}$ is symmetric and comparable with the respective kernel of the fractional laplacian $-(-\Delta)^s$, for $s \in (0, 1)$.

Equations of type (1.1) arise from stochastic control problems, namely, in competitive stochastic games with two or more players, which are allowed to choose from different strategies at every step in order to maximize the expected value of some functions at the first exit point of a domain, see for instance [43] and [34] in the context of jump processes. Integro-differential equations like (1.1) have been studied intensively in the last years, see [15, 16, 39, 41] and references therein. We also refer to the work of Chang-Lara [19], where the author considers the case containing a drift term and uniform not-symmetric kernels.

In this article, we assume the operator I under the same hypothesis as in [19], see also [16] for equation without drift. We will be interested, as a starting point, in studying the equation $-Iu = f$ in a given domain Ω, u being a function vanishing outside the domain, and f is assumed to be a continuous function. This problem, and a generalization to possibly non-symmetric kernels, was treated in [19]. For

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other previous existence and interior regularity results we refer to [4, 15, 16, 30, 41], see also references therein.

In this manuscript we discuss C^α regularity up to the boundary by using the ideas in [5], which allow us to analyze the behavior of the maximal Pucci nonlocal operator for a power of the distance function near the boundary. Then, by using these results, we aim at establishing the existence of the principal eigenvalues corresponding to operator $-I$ with Dirichlet boundary conditions via the classical Krein-Rutman theorem for compact operator, see [1, 29, 32]. Beside the existence of the eigen-pair we also aim to give some characterization relations with the maximum principle and simplicity.

In this article, we will be focused on the principal *half-eigenvalues* of non-local fully nonlinear elliptic operator $-I$. Eigenvalue problems have been extensively studied for nonlinear operators, and the literature on this subject is quite vast. We provide for a brief review by way of introduction. In [35], Pucci first noticed the phenomena of nonlinear operators possessing two principal half-eigenvalue (which are often referred also as semi-eigenvalue, or demi-eigenvalue). A similar behavior for Sturm-Liouville equations was also remarked by Berestycki [6]. An important contribution dealing with this kind of problems was made by Lions [31], who used stochastic methods to study principal half-eigenvalues of certain Bellman operators. The ideas of Berestycki, Nirenberg and Varadhan [11] are also of essential relevance, since they pointed out deep connections between the maximum principle and principal eigenvalues of linear operators. Furthermore, the question about existence of principal eigenvalues of Pucci extremal operator was dealt by Felmer and Quaas in [23]. Regarding principal eigenvalues for fully nonlinear uniformly elliptic operators in non-divergence form, the problem was addressed by Quaas and Sirakov [36, 37]. Ishii and Yoshimura [28] and Armstrong [2] showed analogous results for not necessarily convex operators, such as Bellman-Isaacs operator. Finally, Birindelli and Demengel [7, 8] showed similar results for certain nonlinear degenerated elliptic operators.

For further information and some examples with different situation on principal eigenvalues and eigenfunction of nonlinear elliptic operators can be found in [2] and [36]. We notice that these examples also can be reproduce for the nonlocal setting after our main results.

We make the convention that any time we say that a non-regular function satisfies an (in)equality, we shall mean it is satisfied in the viscosity sense. See for instance [16, 19] for definitions and properties of these issues.

With this in mind, following the definitions in [11] (see also [2, 36] in the case of fully elliptic operators), we define the following finite quantities (see Lemma 5.3 for more more details):

$$\begin{aligned}\lambda_1^+(I,\Omega) = \sup\{\lambda : \exists v \in C(\bar\Omega)\cap L^1(\omega_s), v>0 \text{ in } \Omega \text{ and } v\geq 0 \text{ in } \mathbb{R}^n\setminus\Omega \\ \text{such that } Iv+\lambda v\leq 0 \text{ in } \Omega\},\\ \lambda_1^-(I,\Omega) = \sup\{\lambda : \exists v \in C(\bar\Omega)\cap L^1(\omega_s), v<0 \text{ in } \Omega \text{ and } v\leq 0 \text{ in } \mathbb{R}^n\setminus\Omega \\ \text{such that } Iv+\lambda v\geq 0 \text{ in } \Omega\},\end{aligned}$$

where the weight function ω_s is given in Section 2 in such way that the operator for viscosity solution is well-defined. Then $\lambda_1^+(I,\Omega)$ and $\lambda_1^-(I,\Omega)$ will be the principal half-eigenvalues of $-I$ in Ω.

We state now our main results. The first one reads as follows.

Theorem 1.1. *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$ and assume $s \in (\frac{1}{2}, 1)$. Then there exists functions $\phi^+, \phi^- \in C^\gamma(\bar{\Omega})$ for some $0 < \beta < 1$, such that $\phi^+ > 0$ and $\phi^- < 0$ in Ω, and which satisfy*

$$\begin{cases} -I\phi^+ = \lambda_1^+(I,\Omega)\phi^+ & \text{in } \Omega, \\ -I\phi^- = \lambda_1^-(I,\Omega)\phi^- & \text{in } \Omega, \\ \phi^+ = \phi^- = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

From now on, we say that the eigenvalue $\lambda_1^+(I,\Omega)$ (resp. $\lambda_1^-(I,\Omega)$) has a corresponding eigenfunction $\phi^+ > 0$ (resp. $\phi^- < 0$).

In our next result, by using an Aleksandrov-Bakelman-Pucci (ABP) estimate and owing some techniques from [11] (see also [2] and [36]), we prove a generalized simplicity result of eigenfunctions.

Theorem 1.2. *Let Ω be a C^2 bounded domain of $\mathbb{R}^n$ and assume $s \in (\frac{1}{2}, 1)$. Assume there exists a viscosity solution $u \in C(\bar{\Omega}) \cap L^1(\omega_s)$ of*

$$\begin{cases} -Iu \leq \lambda_1^+(I,\Omega)u & \text{in } \Omega, \\ u(x_0) > 0, \quad u \leq 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

for some $x_0 \in \Omega$. Then $u = t\phi^+$ for some $t \in \mathbb{R}$. If a function $v \in C(\bar{\Omega}) \cap L^1(\omega_s)$ satisfies the reverse inequalities in (1.2), with $\lambda_1^+(I,\Omega)$ replaced $\lambda_1^-(I,\Omega)$, then $v = t\phi^-$ for some $t \in \mathbb{R}$.

The main tool, as mentioned before, to obtain the two principal half-eigenvalues is a version of the classical Krein-Rutman theorem [1, 29, 32] that apply for a class of compact operators. Notice that the uniqueness (simplicity) results of [32] fails, as can be seen in [1]. In [1] there is also a review of Krein-Rutman type theorems.

The compactness issue of the operator in our case is be based on regularity estimates up to boundary. Regularity up to boundary of solutions involving a special class of integro-differential operators was tackled by Ros-Oton and Serra [39].

In our paper, we obtain regularity up to the boundary for viscosity solutions to integro-differential operators with a gradient term. We remark that in our article we only consider the case $s \in (1/2, 1)$ since we need that the first order term is of lower order with respect to the nonlocal operator. Observe that, when the coefficient $c_{a,b} \equiv 0$ in Ω, we can assure Theorems 1.1 and 1.2 to be true for the whole range $s \in (0, 1)$. Moreover in both cases zero order term can be included without any big inconvenient. Similar results (including a gradient term) are known just in the case of the fractional Laplacian, see [9], where a Green function (which is available in that particular case) is crucial to get the existence of the eigen-pair.

It is worthy to mention that the study of principal eigenvalues is a subject of relevance since it is the starting point to treat Rabinowitz bifurcation-type results, solutions at resonance, Ladezman-Lazer type results and Ambrosetti-Prodi phenomena, see for example [3, 24, 25, 38, 42] and references therein. In a recent preprint [20] some of these results were extended to non-local operators. Here we apply principal eigenvalues to establish decay estimates for parabolic nonlocal equations.

This article is organized as follows. In Section 2, we recall some definitions and some useful and known results. Regularity up to the boundary for the Dirichlet problem involving the operator (1.1) is obtained in Section 3. Section 4 is devoted to prove the ABP estimate. We prove our main theorems, Theorems 1.1 and 1.2, in Section 5. Finally, in Section 6 we provide for an application of principal eigenvalues to parabolic nonlocal equations.

2. Preliminaries

To be precise about the formulas we presented in the introduction, we need to ask for an integrability condition for the kernels around the origin. Throughout the paper we denote $\mathcal{L}$ the class of all the linear operators given in (1.1), and, given $L \in \mathcal{L}$ we assume that the operator $Lu(x)$ is defined for $u \in C^{1,1}(x) \cap L^1(\omega_s)$, where

$$\omega_s(dy) = \min\{1, |y|^{-(n+2s)}\}dy.$$

We notice that the family of extremal Pucci operator for a function u is computed at a point x by

$$\mathcal{M}^+_{\mathcal{L}} u(x) = \sup_{L\in\mathcal{L}} Lu(x), \qquad \mathcal{M}^-_{\mathcal{L}} u(x) = \inf_{L\in\mathcal{L}} Lu(x).$$

Observe that $\mathcal{L}$ and $\mathcal{M}^\pm_{\mathcal{L}}$ depend on s and on some additional parameters depending on the boundedness of the kernel, but we do not make it explicit to do not overcharge the notation.

We also say that an operator I defined over a domain $\Omega \subset \mathbb{R}^n$ is elliptic with respect to the family of linear operators $\mathcal{L}$ if for every $x \in \Omega$ and any pair of functions u and v where $Iu(x)$ and $Iv(x)$ can be evaluated, then also $Lu(x)$ and $Lv(x)$ are well defined and

$$\mathcal{M}^-_{\mathcal{L}}(u - v)(x) \le Iu(x) - Iv(x) \le \mathcal{M}^+_{\mathcal{L}}(u - v)(x).$$

In this context, Lu is continuous in $B_r(x_0)$ if $u \in C^2(B_r(x_0)) \cap L^1(\omega_s)$. Stability properties of I depend on Iu being continuous when u is sufficient regular, in this case, $C^2(B_r(x_0)) \cap L^1(\omega_s)$ is a reasonable requirement. As in [16], we define continuous elliptic operators as follows.

Definition 2.1. We say that I is a *continuous operator, elliptic with respect to* $\mathcal{L} = \mathcal{L}(\mathcal{K})$ in Ω if,

(1) I is an elliptic operator with respect to $\mathcal{L}$ in Ω,
(2) $Iu(x)$ is well defined for any $u \in C^{1,1}(x) \cap L^1(\omega_s)$ and $x \in \Omega$,
(3) Iu is continuous in $B_r(x_0)$ for any $u \in C^2(B_r(x_0)) \cap L^1(\omega_s)$ and $B_r(x_0) \subset \Omega$.

In the hypothesis we have introduced we see that the non-local term of the family $\mathcal{L}$ is actually obtained bounding our kernels by multiples of the kernel of the fractional Laplacian. Along the paper, unless it is stated otherwise, it is assumed that $s \in (\frac{1}{2}, 1)$.

2.1. **Hypothesis.** We assume the following hypothesis on the family $\mathcal{L}$ depending on a family of kernels $\mathcal{K}$ and some additional parameters in the following way:

(H1) Every $I \in \mathcal{L}$ is of the form $I = L_K + c_{a,b} \cdot \nabla$ for $K \in \mathcal{K}$.
(H2) There are constants $\lambda \le \Lambda$, such that for every $K \in \mathcal{K}$,

$$\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}$$

for $\Lambda \ge \lambda > 0$.

(H3) The operator I is positively 1-homogeneous in u, that is, $I(tu) = tI(u)$, for $t \geq 0$.

(H4) There is a constant $c^+ > 0$ such that

$$|c_{a,b}| \leq c^+ \text{ uniformly in } \Omega.$$

In this settings we can also write

$$\mathcal{M}^{\pm}_{\mathcal{L}} u(x) = \mathcal{M}^{\pm}_{\mathcal{K}} u(x) \pm c^+ |Du(x)|,$$

with

$$\mathcal{M}^{+}_{\mathcal{K}} u(x) = \sup_{K \in \mathcal{K}} L_K u(x) = \int_{\mathbb{R}^n} \frac{S^+(\delta(u,x,y))}{|y|^{n+2s}}\, dy, \tag{2.1}$$

$$\mathcal{M}^{-}_{\mathcal{K}} u(x) = \sup_{K \in \mathcal{K}} L_K u(x) = \int_{\mathbb{R}^n} \frac{S^-(\delta(u,x,y))}{|y|^{n+2s}}\, dy, \tag{2.2}$$

where given $t \in \mathbb{R}$ we denote

$$S^+(t) = \Lambda t_+ - \lambda t_-, \qquad S^-(t) = \lambda t_+ - \Lambda t_-.$$

An example of operator satisfying all the previous hypothesis is

$$\mathcal{L} = \{L_c = -(-\Delta)^s + c \cdot \nabla \,:\, |c| \leq c^+\}$$

where the fractional Laplacian is defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|y|^{n+2s}}\, dy.$$

We notice that the operator $\mathcal{L}$ we have defined belongs to the more general class treated in [19], where no symmetry assumption on the kernels is made.

Let us fix some notations we will use along the paper. From now on, we define for $\delta > 0$ the set

$$\Omega_\delta := \{y \in \Omega \,:\, d(y) < \delta\}.$$

Also, along this paper we denote $d(x)$ the distance of x to $\partial\Omega$, that is,

$$d(x) := dist(x, \partial\Omega), \qquad x \in \Omega.$$

It is well known that d is Lipschitz continuous in Ω with Lipschitz constant equal to 1. From now on, we will assume that $\partial\Omega$ is C^2, then, in light of [26, Lemma 14.16], d can be considered to be a C^2 function in a neighborhood of $\partial\Omega$.

We modify it outside this neighborhood to make it a C^2 function (still with Lipschitz constant 1), and we extend it to be zero outside Ω.

Then, we define our barrier function as follows

$$\xi(x) = \begin{cases} d(x)^\beta & \text{if } x \in \Omega_\delta, \\ \ell & \text{if } x \in \Omega \setminus \Omega_\delta, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.3}$$

for $\beta > 0$ and a function ℓ such that ξ is positive and C^2 in Ω.

2.2. **Preliminary results.** In this section we present some results concerning the family $\mathcal{L}$. We denote the set of upper (resp. lower) semicontinuous functions in Ω by $USC(\Omega)$ (resp. $LSC(\Omega)$). Then, we recall the notion of viscosity solution in this setting, considered in [16]. See also [19].

Definition 2.2. Given a non local operator I and a function $f:\Omega\to\mathbb{R}$, we say that $u\in LSC(\Omega)\cap L^1(\omega_s)$ is a *super-solution (sub-solution)* to

$$Iu \geq (\leq) f \quad \text{in the viscosity sense in } \Omega,$$

if for every point $x_0\in\Omega$ and any neighborhood V of x_0 with $\bar{V}\subset\Omega$ and for any $\varphi\in C^2(\bar{V})$ such that $u(x_0)=\varphi(x_0)$ and

$$u(x)<\varphi(x) \quad (\text{resp. } u(x)>\varphi(x)) \quad \text{for all } x\in V\setminus\{x_0\}$$

the function v defined by

$$v(x)=u(x) \quad \text{if } x\in\mathbb{R}^n\setminus V \quad \text{and} \quad v(x)=\varphi(x) \quad \text{if } x\in V$$

satisfies

$$Iv(x_0)\geq f(x_0) \quad (\text{resp. } -Iv(x_0)\leq f(x_0)).$$

Additionally, $u\in C(\Omega)\cap L^1(\omega_s)$ is a *viscosity solution* to $Iu=f$ in Ω if it is simultaneously a sub-solution and a super-solution.

Remark 2.3. Regarding the previous definition,

(1) as in the usual case, we may consider inequality instead strict inequality

$$u(x)\geq\varphi(x) \quad \text{for all } x\in V\setminus x_0,$$

and "in some neighborhood V of x_0" instead "in all neighborhood";

(2) other definitions and their equivalence can be founded in [4].

A useful tool in our computations is the following comparison principle between sub and super-solution proved in [19, Corollary 2.9].

Lemma 2.4 (Comparison principle)**.** *Let $u\in LSC(\Omega)\cap L^1(\omega_s)$ and $v\in USC(\Omega)\cap L^1(\omega_s)$ be a super-solution and a sub-solution, respectively, of the same equation $Iw=f$ in Ω . Then $u\geq v$ in $\mathbb{R}^n\setminus\Omega$ implies $u\geq v$ in $\mathbb{R}^n$.*

Also, a result related to the difference of solutions is proved in in [19] .

Theorem 2.5. *Let I be a uniformly elliptic operator with respect to $\mathcal{L}$, and f, g continuous functions. Given $u\in LSC(\Omega)\cap L^1(\omega_s)$ and $v\in USC(\Omega)\cap L^1(\omega_s)$ such that $Iu\leq f$ and $Iv\geq g$ hold in Ω in the viscosity sense, then $\mathcal{M}^-_{\mathcal{L}}(u-v)\leq f-g$ also holds in Ω in the viscosity sense.*

By using the Perron's method together with the comparison principle it follows the existence and uniqueness of solution for the operator I in the viscosity sense, see [19] (or [4] in a similar setting). Notice that barrier can be construct by using a power of the distance function see Lemma 3.3 below.

Theorem 2.6. *Given a domain $\Omega\subset\mathbb{R}^n$ with the exterior ball condition, a continuous operator I with respect to $\mathcal{L}$, and f and g bounded and continuous functions (in fact g only need to be assumed continuous on $\partial\Omega$), then the Dirichlet problem*

$$\begin{cases} Iu=f & \text{in } \Omega, \\ u=g & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{2.4}$$

has a unique bounded viscosity solution u.

A stability result for sub-solutions is stated in [19]. Naturally, a corresponding result holds for super-solutions by changing u to $-u$. As a corollary, the stability under uniform limits follows.

Proposition 2.7. *Let $\{f_k\}$ be a sequence of continuous functions and $u_k \in LSC(\Omega)\cap L^1(\omega_s)$ be a sequence of functions in $\mathbb{R}^n$ such that*

(a) $Iu_k \le f_k$ *in* Ω,
(b) $u_k \to u$ *locally uniformly in* Ω,
(c) $u_k \to u$ *in* $L^1(\omega_s)$,
(d) $f_k \to f$ *locally uniformly in* Ω,
(e) $|u_k(x)| \le C$ *for every* $x \in \Omega$.

Then $Iu \le f$ *in* Ω.

Another useful result is the following version of the strong maximum principle.

Theorem 2.8 (Strong Maximum Principle)**.** *Let $u \in LSC(\Omega)\cap L^1(\omega_s)$ be a viscosity super-solution of $-\mathcal{M}^-_{\mathcal{L}} u \ge 0$ in Ω, $u \ge 0$ in $\mathbb{R}^n$. Then either $u > 0$ in Ω or $u \equiv 0$ in Ω.*

Proof. The proof follows similarly as in [5, Lemma 7], and we omit it here. □

3. Regularity

In this section we prove regularity up to the boundary for the equation

$$\begin{cases} -Iu = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.1}$$

As usual, global regularity will follow by studying the regularity both in $\Omega\setminus\Omega_\delta$ and Ω_δ, where, for a fixed $\delta > 0$ small enough we denote Ω_δ a $\delta-$neighborhood of Ω.

In order to reach the regularity estimates, we first prove in Lemma 3.3 lower and upper bounds of the extremal Pucci operators defined in (2.1) for powers of the distance to the boundary. For that end, we recall the following useful lemma stated in [39, Proposition 2.7]. Given $\beta \in (0, 2s)$, we denote $\varphi^\beta : \mathbb{R} \to \mathbb{R}$ the function

$$\varphi^\beta(x) := (x_+)^\beta. \tag{3.2}$$

Lemma 3.1. *Given $s \in (0,1)$, for $\beta \in (0, 2s)$ the function (3.2) satisfies*

$$\mathcal{M}^+_{\mathcal{K}}(\varphi^\beta) = c^+(\beta)x^{\beta-2s} \quad \text{and} \quad \mathcal{M}^-_{\mathcal{K}}(\varphi^\beta) = c^-(\beta)x^{\beta-2s} \quad \text{in } \{x > 0\}.$$

The constants c^+ and c^- depend on s, β and n, and are continuous as functions of the variables s and β in $\{0 < s \le 1,\, 0 < \beta < 2s\}$. Moreover, there are $\beta_1 \le \beta_2$ in $(0, 2s)$ such that

$$c^+(\beta_1) = 0 \quad \text{and} \quad c^-(\beta_2) = 0.$$

Furthermore,

$$c^+(\beta) < 0 \ \text{ if } \ \beta \in (0, \beta_1), \qquad c^+(\beta) > 0 \ \text{ if } \ \beta \in (\beta_1, 2s)$$

$$c^-(\beta) < 0 \ \text{ if } \ \beta \in (0, \beta_2), \qquad c^-(\beta) > 0 \ \text{ if } \ \beta \in (\beta_2, 2s).$$

In particular, for the fractional Laplacian $-(-\Delta)^s$ it holds that $\beta_1 \le s \le \beta_2$.

Remark 3.2. An alternative approach for proving Lemma 3.3 is the study of the strict convexity of the real-valued function

$$c^+(\tau) = \int_{\mathbb{R}} \frac{S^+((1+t)_+^\tau + (1-t)_+^\tau - 2)}{|t|^{1+2s}} dt,$$

which is well-defined for $\tau \in (0, 2s)$ and $c^+(0) < 0$. See for example [5].

We state the behavior of the extremal operators regarding the barrier function ξ defined in (2.3) that corresponds to a smooth extension of the distance function so that the nonlocal operator is well defined. The proof of the lemma below is based on a contradiction argument that reduces the problem to a one dimensional one, and together with Lemma 3.1 gives the result. A direct approach is possible but the computations are more complicated, see [21] for a direct approach in the case of the fractional Laplacian for the case of negative power of the distance.

Lemma 3.3. *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$ and $s \in (0, 1)$. Then there exist $C, \delta > 0$ such that*

(a) $\mathcal{M}_{\mathcal{K}}^+(\xi(x)) \geq C d^{\beta-2s}(x)$ *if* $\beta \in (\beta_1, 2s)$,
(b) $\mathcal{M}_{\mathcal{K}}^+(\xi(x)) \leq -C d^{\beta-2s}(x)$ *if* $\beta \in (0, \beta_1)$,
(c) $\mathcal{M}_{\mathcal{K}}^-(\xi(x)) \geq C d^{\beta-2s}(x)$ *if* $\beta \in (\beta_2, 2s)$,
(d) $\mathcal{M}_{\mathcal{K}}^-(\xi(x)) \leq -C d^{\beta-2s}(x)$ *if* $\beta \in (0, \beta_2)$

for $x \in \Omega_\delta$, where $0 < \beta_1 < \beta_2 < 2s$ are given in Lemma 3.1.

Proof. Let us prove (a). By contradiction, let us assume that the conclusion of the lemma is not true. Then there exist $\beta \in (\beta_1, 2s)$ and a sequence of points $x_n \in \Omega$ converging, up to some subsequence, to some point $x \in \partial\Omega$ (i.e., $d(x_n) \to 0$) satisfying

$$\lim_{n\to+\infty} d(x_n)^{2s-\beta} \mathcal{M}_{\mathcal{K}}^+(d^\beta(x_n)) \leq 0. \tag{3.3}$$

Without loss of generality we will assume that in x, the interior normal is given by e_N, the last vector of the canonical basis of $\mathbb{R}^n$.

Equation (3.3) says that

$$d(x_n)^{2s-\beta} \mathcal{M}_{\mathcal{K}}^+(d^\beta(x_n)) = \int_{\mathbb{R}^n} \frac{S^+(\delta(d^\beta, x_n, y))}{d_n^{\beta-2s}|y|^{N+2s}} dy \leq o(1). \tag{3.4}$$

Denoting for simplicity $d_n := d(x_n)$, and performing the change of variables $y = d_n z$, we can rewrite the integral in (3.4) as

$$\int_{\mathbb{R}^n} \frac{S^+\left(\left(\frac{d(x_n+d_nz)}{d_n}\right)^\beta + \left(\frac{d(x_n-d_nz)}{d_n}\right)^\beta - 2\right)}{|z|^{N+2s}} dz. \tag{3.5}$$

In order to perform our computations we split the previous integral as

$$\int_{|z|\geq L} \frac{S^+(g(z))}{|z|^{n+2s}} dz + \int_{|z|\leq \eta} \frac{S^+(g(z))}{|z|^{n+2s}} dz + \int_{\eta\leq|z|\leq L} \frac{S^+(g(z))}{|z|^{n+2s}} dz := I_1 + I_2 + I_3$$

where L and η are fixed positive values and

$$g(z) := \left(\frac{d(x_n + d_n z)}{d_n}\right)^\beta + \left(\frac{d(x_n - d_n z)}{d_n}\right)^\beta - 2.$$

Before passing to the limit in the integral (3.5), let us estimate lower bounds of I_1 and I_2 (we also need upper bounds in order to prove items (b) and (d)). For that end, observe that

$$\begin{aligned}|I_1| &\leq \int_{\{|z|\geq L\}} \frac{|S^+(g(z))|}{|z|^{n+2s}}\, dz \leq \Lambda \int_{\{|z|\geq L\}} \frac{g^+(z)}{|z|^{n+2s}} dz + \lambda \int_{\{|z|\geq L\}} \frac{g^-(z)}{|z|^{n+2s}} dz \\ &\leq \Lambda \int_{\{|z|\geq L\}} \frac{|g(z)|}{|z|^{n+2s}} dz := \Lambda I_1'\end{aligned}$$

and analogously, $|I_2| \leq \Lambda \int_{\{|z|\leq \eta\}} \frac{|g(z)|}{|z|^{n+2s}} dz := \Lambda I_2'$.

Consequently, it is sufficient to find bounds for I_1' and I_2'.

Let us deal with I_1'. Observe that when $x_n + d_n z \in \Omega$, we have by the Lipschitz property of d that $d(x_n + d_n z) \leq d_n(1 + |z|)$. Of course, the same is true when $x_n + d_n z \notin \Omega$ and it similarly follows that $d(x_n - d_n z) \leq d_n(1 + |z|)$. Thus, taking $L > 0$ we obtain for large n

$$I_1' \leq 2 \int_{|z|\geq L} \frac{1 + (1 + |z|)^\beta}{|z|^{N+2s}} dz. \tag{3.6}$$

Observe that the previous expression tends to zero as $L \to +\infty$.

On the other hand, since d is smooth in a neighborhood of the boundary, when $|z| \leq L$ and $x_n + d_n z \in \Omega$, we obtain by Taylor's theorem

$$d(x_n + d_n z) = d_n + d_n \nabla d(x_n) z + \Theta_n(d_n, z) d_n^2 |z|^2, \tag{3.7}$$

where Θ_n is uniformly bounded, i.e, $-C \leq \Theta_n \leq C$ for some positive constant C. Hence

$$|d(x_n + d_n z) - (d_n + d_n \nabla d(x_n) z)| \leq C d_n^2 |z|^2. \tag{3.8}$$

Now choose $\eta \in (0, 1)$ small enough. Since $d(x_n) \to 0$ and $|\nabla d| = 1$ in a neighborhood of the boundary, we can assume that

$$\nabla d(x_n) \to e \text{ as } n \to +\infty \text{ for some unit vector } e. \tag{3.9}$$

Without loss of generality, we may take $e = e_N$, the last vector of the canonical basis of $\mathbb{R}^n$. If we restrict z further to satisfy $|z| \leq \eta$, we obtain $1 + \nabla d(x_n) z \sim 1 + z_N \geq 1 - \eta > 0$ for large n, since $|z_N| \leq |z| \leq \eta$. Therefore, inequality (3.8) is also true when $x_n + d_n z \notin \Omega$ for large n (depending only on η). Moreover, by using again Taylor's theorem

$$|(1 + \nabla d(x_n) z \pm C d_n |z|^2)^\beta - (1 + \beta \nabla d(x_n) z)| \leq C|z|^2,$$

for large enough n. Thus from (3.8),

$$\left| \left(\frac{d(x_n + d_n z)}{d_n} \right)^\beta - (1 + \beta \nabla d(x_n) z) \right| \leq C|z|^2,$$

for large enough n. A similar inequality is obtained for the term involving $d(x_n - d_n z)$, i.e.,

$$\left| \left(\frac{d(x_n - d_n z)}{d_n} \right)^\beta - (1 - \beta \nabla d(x_n) z) \right| \leq C|z|^2,$$

for large enough n. Therefore we deduce that I_2' can be bounded as

$$I_2' \leq C \int_{|z|\leq \eta} |z|^{2(1-s)-N} dz. \tag{3.10}$$

Observe that the previous expression tends to zero as $\eta \to 0$.

We finally observe that it follows from the above discussion (more precisely from (3.7) and (3.9) with $e = e_N$) that for $\eta \le |z| \le L$

$$\frac{d(x_n \pm d_n z)}{d_n} \to (1 \pm z_N)_+ \qquad \text{as } n \to +\infty \tag{3.11}$$

and by dominated convergence we arrive at

$$\lim_{n\to\infty} \int_{\eta\le|z|\le L} \frac{S^+(g(z))}{|z|^{N+2s}} dz = \int_{\eta\le|z|\le L} \frac{S^+((1+z_N)_+^\beta + (1-z_N)_+^\beta - 2)}{|z|^{N+2s}} dz. \tag{3.12}$$

Therefore by using (3.6) and (3.10) to bound by below (3.4), as $n \to +\infty$, from (3.12) we get

$$-C\Lambda \int_{|z|\ge L} \frac{2 + (1+|z|)^\beta}{|z|^{N+2s}} dz + \int_{\eta\le|z|\le L} \frac{S^+((1+z_N)_+^\beta + (1-z_N)_+^\beta - 2)}{|z|^{N+2s}} dz$$
$$-C\Lambda \int_{|z|\le\eta} \frac{1}{|z|^{N-2(1-s)}} dz \le 0.$$

Letting now $\eta \to 0$ and then $L \to +\infty$ we have

$$\int_{\mathbb{R}^n} \frac{S^+((1+z_N)_+^\beta + (1-z_N)_+^\beta - 2))}{|z|^{N+2s}} dz \le 0.$$

It is well-known, with the use of Fubini's theorem and a change of variables, that this integral can be rewritten as a one-dimensional integral

$$c^+(\beta) = \int_{\mathbb{R}} \frac{S^+((1+t)_+^\beta + (1-t)_+^\beta - 2)}{|t|^{1+2s}} dt \le 0, \tag{3.13}$$

for $\beta \in (\beta_1, 2s)$, which contradicts Lemma 3.1.

The proofs of (b), (c) and (d) are analogous. □

The following lemma is a key in order to obtain the boundary regularity for (3.1).

Lemma 3.4. *Let u be a solution of (3.1) with $s \in (\frac{1}{2}, 1)$, then there exists $\delta > 0$ and $\beta \in (0, \beta_1)$ such that*

$$|u(x)| \le C d(x)^\beta \quad \forall x \in \Omega_\delta,$$

where C is a positive constant.

Proof. First, we claim that there exist $\delta > 0$, $\beta \in (0, \beta_1)$ and a positive constant C such that

$$I\xi(x) \le -C d(x)^{\beta - 2s} \qquad \text{in } \Omega_\delta \tag{3.14}$$

provided that $s > 1/2$.

We apply Lemma 3.3. For $\delta > 0$ small enough, given $x \in \Omega_\delta$ it holds that

$$\mathcal{M}_{\mathcal{K}}^+ \xi \le -C d(x)^{\beta-2s} \quad \text{in } \Omega_\delta$$

for some $C > 0$ and $\beta \in (0, \beta_1)$. Now, since $D\xi(x) = C\beta d(x)^{\beta-1}$, we have

$$\begin{aligned} I\xi(x) &\le \mathcal{M}_{\mathcal{L}}^+ \xi(x) = \mathcal{M}_{\mathcal{K}}^+ \xi(x) + c^+ |D\xi(x)| \\ &\le -C d(x)^{\beta-2s} + C d(x)^{\beta-1} \\ &\le -C d(x)^{\beta-2s} \end{aligned}$$

whenever $\beta - 2s < \beta - 1$, that is, $s > \frac{1}{2}$, from where claim (3.14) follows.

Moreover, δ can be taken small enough such that

$$-I\xi(x) \geq f(x) \quad \text{in } \Omega_\delta.$$

Since, for some positive constant L, $\xi(x) = \ell(x) \geq L$ in $\Omega \setminus \Omega_\delta$, we can take C such that $C\xi(x) \geq CL \geq \|u\|_{L^\infty(\Omega)}$ for $x \in \Omega \setminus \Omega_\delta$. By using that u and ξ vanish in Ω^c we conclude that

$$C\xi(x) \geq u(x) \quad \text{in } \Omega_\delta^c.$$

From the comparison principle given in Lemma 2.4 we obtain that

$$Cd(x)^\beta = C\xi(x) \geq u(x) \quad \text{in } \Omega_\delta$$

and the result follows.

Repeating the same argument with $-u$ we find the result. □

In [19], by applying a diminish of oscillation argument (see for instance [15, 16]) the following interior Hölder's regularity for (3.1) is proved.

Lemma 3.5 (Interior regularity, [19])**.** *Let $f \in L^\infty$ and u be a viscosity solution of*

$$\begin{cases} -Iu = f & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{3.15}$$

Then $u \in C^\alpha(B_{1/2})$ for some universal $\alpha \in (0,1)$, and satisfies,

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}) \tag{3.16}$$

for some universal C.

Finally, combining the interior and boundary regularity given in Lemmas 3.5 and 3.4, by an standard ball covering argument it follows the next result. A version of this type of arguments can be found in [27] where the fractional p-Laplacian is studied.

Theorem 3.6 (Global regularity)**.** *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $f \in L^\infty(\Omega)$ and let u be a viscosity solution of* (3.1) *with $s \in (\frac{1}{2}, 1)$. Then $u \in C^\gamma(\bar{\Omega})$ for $\gamma = \min\{\alpha, \beta\}$, where α and β are given in Lemmas 3.5 and 3.4, respectively.*

4. Refined ABP estimate for nonlocal equation

The Aleksandrov-Bakelman-Pucci (ABP) estimate is a key ingredient in our arguments. Such an estimate is the bridge connecting estimates in measure with pointwise estimates. Here we follow [12], which gives even an improvement of the ABP estimate by using a weak Harnack inequality.

Following [12], for a domain $\Omega \in \mathbb{R}^n$ (not necessarily bounded) and $0 < \sigma < 1$ we define the quantity $R(\Omega)$ to be the smallest positive constant R such that

$$\text{meas}(B_R(x) \setminus \Omega) \geq \sigma \text{meas}(B_R(x)) \quad \text{for all } x \in \Omega.$$

If such a radius R does not exist, we define $R(\Omega) = +\infty$. Notice that is not difficult to see that

$$R(\Omega) \leq C(n, \sigma)|\Omega|^{1/n}.$$

We start with a version of the weak Harnack inequality that follows directly from [40, Theorem 6.1] and some scaling. In particular it means that our improvement of the ABP estimate below holds for a more general class of extremal equation as the treated in [40].

Theorem 4.1 (Weak Harnack inequality)**.** *Let $u \in C(\bar{B}_{2R})$ is a viscosity solution of*

$$\mathcal{M}^+_{\mathcal{L}}(u) \geq f \quad \text{in } B_{2R},$$

and $u \geq 0$ in $\mathbb{R}^n$, where $f \in L^\infty(\Omega) \cap C(\Omega)$. Then

$$\left(\frac{1}{|B_R|} \int_{B_R} u^{p_0} \right)^{1/p_0} \leq C \left\{ \inf_{B_R} u + R^{2\alpha} \|f\|_{L^\infty(B_{2R})} \right\},$$

where p_0 and C are positive universal constants depending on parameters of the problem, in particular it depends of C_0 such that $c^+ R^{2s-1} \leq C_0$.

Proof. From [33, Lemma 4.3], a version in ball of [40, Theorem 6.1] it follows if $v \in C(\bar{B}_2)$ is a viscosity solution of

$$\mathcal{M}^+_{\mathcal{L}}(v) \geq \tilde{f} \quad \text{in } B_2, \tag{4.1}$$

and $v \geq 0$ in $\mathbb{R}^n$, there exist C and p_0 depending on the parameters of the problem, in particular depending on c^+ or a bound for it, such that

$$\left(\frac{1}{|B_1|} \int_{B_1} v^{p_0} \right)^{1/p_0} \leq C \left\{ \inf_{B_1} u + \|\tilde{f}\|_{L^\infty(B_2)} \right\}. \tag{4.2}$$

Hence if $u \in C(\bar{B}_{2R})$ satisfies

$$\mathcal{M}^+_{\mathcal{L}}(u) \geq f \quad \text{in } B_{2R},$$

then $v(x) = u(Rx)$ satisfies (4.1) with $\tilde{f} = R^{2s} f$, c^+ replaced by $c^+ R^{2s-1}$ and c^+ can be replace by C_0. Therefore, we deduce the results by rewriting (4.2) in terms of u. □

Proposition 4.2 (Improvement of ABP)**.** *Let Ω be an open domain with $R(\Omega) < +\infty$. Suppose that u is a bounded viscosity solution of*

$$\mathcal{M}^+_{\mathcal{L}}(u) \geq f \quad \text{in } \Omega,$$

with $u \leq 0$ in Ω^c and $f \in L^\infty(\Omega) \cap C(\Omega)$. Then, there exist $C(c) > 0$ (not depending on u) such that

$$\sup_{\Omega} u \leq C(R(\Omega))^{2s} \|f\|_{L^\infty(\Omega)}.$$

Remark 4.3. From here, with the same as in the proof of (ii) of Theorem 4.8 in [13] we have the following lower bound for λ_1^+:

$$\lambda_1^+(\Omega) \geq C R(\Omega)^{-2s}.$$

Proof. Here we follow closely [12] (see also [13]). Notice that it is direct to see that u^+ is a viscosity solution of

$$\mathcal{M}^+_{\mathcal{L}}(u^+) \geq -\|f\|_{L^\infty(\Omega)} \quad \text{in } \Omega, \tag{4.3}$$

with $u^+ = 0$ in Ω^c.

Now we assume first that Ω is bounded. Then the supremum of u is achieved, so there exists $\tilde{x} \in \Omega$ such that

$$M := \sup_{\Omega} u = u(\tilde{x}).$$

To simplify the notation, we write $R := R(\Omega)$ and $B_R := B_R(\tilde{x})$. We know that

$$\frac{\text{meas}(B_R \setminus \Omega)}{\text{meas}(B_R)} \geq \sigma.$$

Consider the function

$$w = M - u^+.$$

We claim that w is a viscosity solution of

$$\mathcal{M}^-_{\mathcal{L}}(w) \leq \|f\|_{L^\infty(\Omega)} \quad \textit{in } \mathbb{R}^n.$$

Indeed, let φ be a test function in some neighborhood V with $\varphi(x_0) = w(x_0)$ and $\varphi < w$ in $V \setminus \{x_0\}$. Then if we define v as

$$v(x) = w(x) \quad \text{if } x \in \mathbb{R}^n \setminus V \quad \text{and} \quad v(x) = \varphi(x) \quad \text{if } x \in V.$$

If $w(x_0) = v(x_0) = M$ then v has a global maximum at x_0 so

$$\mathcal{M}^-_{\mathcal{L}}(v) \leq 0 \leq \|f\|_{L^\infty(\Omega)}.$$

If $w(x_0) = v(x_0) < M$ then $x_0 \in \Omega$. If we define $\tilde{\varphi} := M - \varphi > M - w = u^+$ in $V \setminus \{x_0\}$ and $u^+(x_0) = \tilde{\varphi}(x_0)$. So, $z(x) := M - v(x)$ is a test function for u^+ that is a viscosity solution of (4.3) therefore

$$\mathcal{M}^+_{\mathcal{L}}(-v) = \mathcal{M}^+_{\mathcal{L}}(z) \geq -\|f\|_{L^\infty(\Omega)}.$$

Thus,

$$\mathcal{M}^-_{\mathcal{L}}(v) \leq \|f\|_{L^\infty(\Omega)}.$$

Finally, Theorem 4.1 applied to v in B_{2R} gives that

$$\begin{aligned}
(\sigma)^{1/p} M &\leq \left(\frac{\text{meas}(B_R \setminus \Omega)}{\text{meas}(B_R)}\right)^{1/p} M \\
&\leq \left(\frac{1}{\text{meas}(B_R)} \int_{B_R \setminus \Omega} v^p\right)^{1/p} \\
&\leq \left(\frac{1}{\text{meas}(B_R)} \int_{B_R} v^p\right)^{1/p} \\
&\leq C\{\inf_{B_R} v + R^{2\alpha} \|f\|_{L^\infty(B_{2R})}\} \\
&= C R^{2\alpha} \|f\|_{L^\infty(B_{2R} \cap \Omega)},
\end{aligned}$$

where $p := p_0 > 0$. This proves the desired inequalities.

In case that Ω is unbounded, the proof is the same with minor changes. We define $M := \sup_\Omega u$ and we take, for any $\eta > 0$, a point x_0 such that $M - \eta \leq u(x_0)$. We now have that $v(x_0) \leq \eta$. We proceed as before and get desired estimate by letting $\eta \to 0$.

□

5. Proof of main results

This section is devoted to prove Theorems 1.1 and 1.2. We first use Proposition 4.2 to get the following relevant sweeping lemma in the spirit of [11, Theorem 2.2].

Theorem 5.1. *Suppose $u, v \in C(\bar{\Omega}) \cap L^1(\omega_s)$ $\lambda \geq 0$ and $f \in C(\Omega)$ satisfying*

$$\begin{cases} Iu + \lambda u \leq f & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u \geq 0 & \textit{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad \textit{resp.} \qquad \begin{cases} Iu + \lambda u \geq f & \textit{in } \Omega, \\ u < 0 & \textit{in } \Omega, \\ u \leq 0 & \textit{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

and

$$\begin{cases} Iv + \lambda v \geq f & \text{in } \Omega, \\ v \leq 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ v(x_0) > u(x_0), & \end{cases} \quad \text{resp.} \quad \begin{cases} Iv + \lambda v \leq f & \text{in } \Omega, \\ v \geq 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ v(x_0) < u(x_0), & \end{cases}$$

for some point $x_0 \in \Omega$ and $f \leq 0$ (resp. $f \geq 0$). Then $u \equiv tv$ for some $t > 0$.

Proof. Let u, v be two functions satisfying the first set of inequalities in Theorem 5.1. Take a compact set $K \subset \Omega$ such that $|\Omega \setminus K| \leq \varepsilon_1$, where with ε_1 is such that by Proposition 4.2 the maximum principle is valid for $\mathcal{M}^+_{\mathcal{L}} \cdot + \lambda \cdot$ in $\Omega \setminus K$. This follows from Proposition 4.2 with $f := \lambda u$. Set now $z_t = v - tu$. If t is large enough, $z_t < 0$ in K. For $t \geq 1$, we have

$$\mathcal{M}^+_{\mathcal{L}} z_t + \lambda z_t \geq Iv - tIu + \lambda z_t = (1-t)f \geq 0 \quad \text{in } \Omega$$

and $z_t \leq 0$ in $\mathbb{R}^n \setminus (\Omega \setminus K)$, then we get $z_t \leq 0$ in $\Omega \setminus K$ and thus $z_t \leq 0$ in Ω. So, by the strong maximum principle, either $z_t \equiv 0$ in Ω in which case we are done, or $z_t < 0$ in Ω. We define

$$\tau = \inf\{t \mid z_t < 0 \text{ in } \Omega\}.$$

Since $v(x_0) > u(x_0)$ we have $\tau > 1$. Now we repeat the same argument for z_τ. So, either $z_\tau \equiv 0$ in Ω in which case we are done or $z_\tau < 0$ in Ω. In this case there exists $\eta > 0$ such that $z_{\tau-\eta} < 0$ in K. Now we repeat again the same argument for $z_{\tau-\eta}$, which yields a contradiction with the definition of τ.

If the inequalities satisfied by u, v are reversed (second set of inequalities in Theorem 5.1), we consider the function $tu - v$ and the same argument. □

Remark 5.2. Here we remark that if $f \equiv 0$ in Theorem 5.1, we just need $v(x_0) > 0$ instead of $v(x_0) > u(x_0)$ (resp. $v(x_0) < 0$ instead of $v(x_0) < u(x_0)$). See also [36, Theorem 4.2] for local case.

A consequence of Theorem 5.1 is an upper bound of the principal half-eigenvalue in terms of thickness of the domain. For each $\rho \in \mathbb{R}$, we define the nonlinear operator G_ρ by

$$G_\rho(u) = -Iu - \rho u.$$

We say the operator G_ρ satisfies the maximum principle in Ω if, whenever $v \in LSC(\Omega) \cap L^1(\omega_s)$ is a solution of $G_\rho v \leq 0$ in Ω with $v \leq 0$ in $\mathbb{R}^n \setminus \Omega$, we have $v \leq 0$ in Ω. Similarly, we say that the operator G_ρ satisfies the minimum principle in Ω if, whenever $v \in LSC(\Omega) \cap L^1(\omega_s)$ is a solution of $G_\rho v \geq 0$ in Ω with $v \geq 0$ in $\mathbb{R}^n \setminus \Omega$, we have $v \geq 0$ in Ω.

Define constants

$$\mu^+(I, \Omega) = \sup\{\rho : G_\rho \text{ satisfies the maximum principle in } \Omega\},$$

and

$$\mu^-(I, \Omega) = \sup\{\rho : G_\rho \text{ satisfies the minimum principle in } \Omega\}.$$

We will eventually show that $\lambda^\pm(I, \Omega) = \mu^\pm(I, \Omega)$. The following lemma is the first step in this direction.

Lemma 5.3. *We have*

$$\lambda^\pm(I, \Omega) \leq \mu^\pm(I, \Omega) < \infty.$$

Proof. Here we follow the argument as in [2, Lemma 3.7]. We show that

$$\lambda^+(I,\Omega) \le \mu^+(I,\Omega).$$

Suppose on the contrary that $\mu^+(I,\Omega) < \rho_1 < \rho_2 < \lambda^+(I,\Omega)$. Then we may select a function v_1 satisfying

$$-Iv_1 \le \rho_1 v_1 \quad \text{in } \Omega$$

and such that $v_1 \le 0$ in $\mathbb{R}^n \setminus \Omega$ and $v_1 > 0$ somewhere in Ω. We can also select v_2 such that $v_2 > 0$ in Ω, $v_2 \ge 0$ in $\mathbb{R}^n \setminus \Omega$ and v_2 satisfies

$$-Iv_2 \ge \rho_2 v_2 \quad \text{in } \Omega.$$

Since $\rho_1 v_2 < \rho_2 v_2$, we may apply Theorem 5.1 to deduce $v_2 = tv_1$ for some $t > 0$. This implies that

$$\rho_1 t v_1 \ge -I(tv_1) = -Iv_2 \ge \rho_2 v_2 = \rho_2 t v_1,$$

and since there exists $x_0 \in \Omega$ such that $v_1(x_0) > 0$ we obtain that $\rho_1 \ge \rho_2$, a contradiction. Hence, $\lambda^+(I,\Omega) \le \mu^+(I,\Omega)$. By a similar argument, we can obtain $\lambda^-(I,\Omega) \le \mu^-(I,\Omega)$.

Finally, we prove that the operator G_ρ does not satisfy the minimum principle in Ω for all large ρ. Choosing a continuous function $h \le 0$, $h \not\equiv 0$ with compact support in Ω. By Theorem 2.6, there exists a unique solution of the following problem

$$\begin{cases} -Iv = h & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

According to the comparison principle, $v \le 0$ in Ω. Since $h \not\equiv 0$, we have $v \not\equiv 0$. Hence, $v < 0$ in Ω by the strong maximum principle. Since h has compact support in Ω, we may select a constant $\rho_0 > 0$ such that $\rho_0 v \le h$. Therefore, v satisfies

$$-Iv \ge \rho_0 v \quad \text{in } \Omega$$

and so evidently the operator G_ρ does not satisfy the minimum principle in Ω, for any $\rho \ge \rho_0$. Thus $\lambda^-(I,\Omega) \le \rho_0$. By a similar argument, we have that $\lambda^+(I,\Omega) < \infty$. □

Next, we prove Theorem 1.2 by using Theorem 5.1.

Proof of Theorem 1.2. Suppose $u_1 := u$ satisfies (1.2). Then, we apply Theorem 5.1 with $u = \phi_1^+$ and $v = u_1$ and Remark 5.2. □

The proof of Theorem 1.1 follows by using the Krein-Rutman Theorem. In order to give the proof we introduce some notation and definitions. We set the space

$$X := \{f \in C(\mathbb{R}^n) \,:\, f = 0 \text{ in } \mathbb{R}^n \setminus \Omega\},$$

and we denote K the closed convex cone in X with vertex 0

$$K := \{f \in X \,:\, f \ge 0 \text{ in } \Omega\}.$$

The cone K induces an ordering $\preceq$ on X as follows: given $f, g \in X$ we say that

$$f \preceq g \iff g - f \in K.$$

Given $f \in L^\infty(\mathbb{R}^n)$, let u be a viscosity solution of

$$\begin{cases} -Iu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.1}$$

Since I is invertible, we define the solution operator T as

$$T(f) := I^{-1}(-f) = u.$$

Proof of Theorem 1.1. We check that the hypothesis of Theorem A.1 are fulfilled.

The operator T is *positively* $1-$*homogeneous.* Given $t > 0$, we have that $T(tf) = u$ where u is a viscosity solution of $-I(u) = tf$ in Ω, $u = 0$ in Ω^c. Since I is a $1-$homogeneous operator it holds that $f = -I(t^{-1}u)$, from where follows that $tT(f) = u$.

From Proposition 2.7 it follows that T is a *continuous operator* on X. Moreover, by using the Hölder regularity up the boundary of I given in Theorem 3.6 and the Arzelá-Ascoli theorem, it follows that T is a *compact operator* on X.

The order $\preceq$ *is increasing.* Given $f, g \in X$ such that $f \preceq g$, let u and v be viscosity solutions of $-Iu = f$, $-Iv = g$ in Ω and $u = v = 0$ in Ω^c. By definition of the order, we get that $-I(u) = f \geq g = -I(v)$ in Ω, and $u = v = 0$ in Ω^c. Hence, by using the Comparison principle given in Lemma 3.4, it follows that $u \leq v$ in $\mathbb{R}^n$, from where $T(f) \preceq T(g)$.

Moreover, the order $\preceq$ *is strictly increasing.* If now $f \neq g$ are functions such that $f \prec g$, by definition of the order, and by using Theorem 2.5, we obtain that

$$-\mathcal{M}^-_{\mathcal{L}}(v-u) \geq g - f > 0 \quad \text{ in } \Omega.$$

Applying the Strong Maximum Principle stated in Theorem 2.8 it follows that $v - u > 0$ in Ω, from where $T(f) \prec T(g)$.

Finally, the (H) *Condition* below in this context means that there exist a non-zero function $f_0 \in K$, here we take f_0 with compact support in Ω and we can take M large such that $f_0 \preceq MT(f_0)$. This conditions are possible since $T(f_0) > 0$ in Ω by applying the Strong Maximum Principle, and f_0 has compact support in Ω.

Consequently, there exists a positive eigenfunction $f \in K$ of T with corresponding eigenvalue μ, which, for $\mu \neq 0$, satisfies $T(f) = \mu f$ if and only if $-I(f) = \lambda^* f$ for $\lambda^* = \frac{1}{\mu}$.

It is now immediate from the definitions of $\mu^+(I,\Omega)$ and $\lambda^+(I,\Omega)$ that $\mu^+(I,\Omega) \leq \lambda^* \leq \lambda^+(I,\Omega)$, and therefore $\lambda^* = \mu^+(I,\Omega) = \lambda^+(I,\Omega)$ by Lemma 5.3. By a similar argument, we know $\lambda^-(I,\Omega)$ is also the eigenvalue of operator $-I$. We complete the proof by using Theorem 3.6 to get the regularity of the eigenfunction.

□

6. An application: Decay estimates for the evolution equation

In this section we are interested in the asymptotic behavior as $t \to \infty$ of the solutions of a evolution-type equation involving the operator I defined in (1.1). In order to state our results, it is convenient to define the notion of viscosity solution in this context.

We denote the cylinder of radius r, height τ and center $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ by $C_{r,\tau}(x,t) := B_r(x) \times (t - \tau, \tau)$.

In this settings, we define the space of lower and upper semicontinuous functions as follows.

Definition 6.1. $LSC((t_1, t_2] \to L^1(\omega_s))$ consists of all measurable functions $u \colon \mathbb{R}^n \times (t_1, t_2] \to \mathbb{R}$ such that for every $t \in (t_1, t_2]$,

i) $\|u(\cdot,t)^-\|_{L^1(\omega_s)} < \infty$,

ii) $\lim_{\tau\to 0} \|(u(\cdot,t) - u(\cdot,t-\tau))^+\|_{L^1(\omega_s)} = 0$.

Similarly, $u \in USC((t_1,t_2] \to L^1(\omega_s))$ if $-u \in LSC((t_1,t_2] \to L^1(\omega_s))$. We finally denote $C((t_1,t_2] \to L^1(\omega_s)) = LSC((t_1,t_2] \to L^1(\omega_s)) \cap USC((t_1,t_2] \to L^1(\omega_s))$.

A lower semicontinuous test function is a pair $(\varphi, C_{r,\tau}(x,t))$ such that

$$\varphi \in C^{1,1}_x C^1_t(C_{r,\tau}(x,t)) \cap LSC((t-\tau,\tau] \to L^1(\omega_s)).$$

Similarly, $(\varphi, C_{r,\tau}(x,t))$ is an upper semicontinuous test function if the pair $(-\varphi, C_{r,\tau}(x,t))$ is a lower semicontinuous test function.

Definition 6.2. Given an elliptic operator I, a function $u \in LSC(\Omega \times (t_1,t_2]) \cap LSC((t_1,t_2] \to L^1(\omega_s))$ is said to be a *viscosity super solution* to $u_t \geq Iu$ in $\Omega\times(t_1,t_2]$, if for every lower semicontinuous test function $(\varphi, C_{r,\tau}(x,t))$ and $(x,t) \in \Omega \times (t_1,t_2]$, whatever

i) $\varphi(x,t) = u(x,t)$ and
ii) $\varphi(y,s) \leq u(y,s)$ for $(y,s) \in \mathbb{R}^n \times (t-\tau,t]$,

we have that $\varphi_t(x,t) \geq I\varphi(x,t)$.

The definition of u being a *viscosity sub solution* to $u_t \leq Iu$ in $\Omega\times(t_1\times t_2]$ is done similarly to the definition of super solution replacing LSC by USC and reversing the last two inequalities. Finally, a *viscosity solution* to $u_t = Iu$ in $\Omega \times (t_1,t_2]$ is a function which is a super and sub solution simultaneously.

Let u be a viscosity solution to the parabolic equation

$$\begin{cases} u_t = -Iu & \text{in } \Omega \times (0,\infty), \\ u(x,0) = h_0(x) & \text{in } \Omega \times \{0\}, \\ u(x,t) = 0 & \text{in } \partial\Omega \times (0,\infty). \end{cases} \tag{6.1}$$

This type of equation is studied in [17] and [18], see also reference therein.

We are interested in the asymptotic behavior, as $t \to \infty$, of the solution $h(x,t)$ of (6.1). Based on results of the local heat equation, one expects h to decay to zero exponentially and that the rate of decay and the extinction profile are somehow connected with the principal eigenvalue λ and the eigenfunction v given in Theorem 1.1, i.e.,

$$\begin{cases} -Iv = \lambda v & \text{in } \tilde{\Omega} \\ v = 0 & \text{in } \mathbb{R}^n \setminus \tilde{\Omega}. \end{cases} \tag{6.2}$$

for a $\tilde{\Omega}$ such that Ω is compactly contain in $\tilde{\Omega}$.

Proposition 6.3. *Let h, v, λ, Ω and $\tilde{\Omega}$ be as above. We have that the*

$$h \leq e^{-\lambda t} v$$

and

$$\sup_{\Omega\times(0,\infty)} \frac{h(x,t)}{v(x)e^{-\lambda t}} \leq \sup_{\Omega} \frac{h_0^+(x)}{v(x)},$$

where $h_0^+ = \max\{h_0, 0\}$ denotes the positive part of h_0.

Proof. By replacing h_0 with its positive part if necessary, we may assume that the initial data h_0 is non-negative. We define the function $w(x) = e^{-\lambda t} v(x)$. Therefore, since v is an eigenfunction of (6.2) we get that

$$(w)_t = -\lambda v(x) e^{-\lambda} = -Iv \cdot e^{-\lambda t} = -Iw, \tag{6.3}$$

in Ω in the viscosity sense, where we have used that I is $1-$homogeneous.

Observe that the eigenfunction v is non-negative in $\tilde{\Omega}$ but moreover, by the Strong maximum principle (Theorem 2.8) it is in fact positive in $\tilde{\Omega}$. Since Ω is compactly contained in $\tilde{\Omega}$ we get that $v > 0$ in Ω. Hence we can normalize v so that $h_0 \leq v$ and $\sup_\Omega \frac{h_0^+(x)}{v(x)} = 1$. Using now that h is a viscosity solution of (6.1) and w of (6.3) we can use the comparison principle (see Theorem 3.7 in [17]) to deducce that $h \leq w$, giving the result. □

Corollary 6.4. *Let h be a viscosity solution of* (6.1) *with $h_0 \in C(\bar{\Omega})$ and $h_0 \geq 0$. Then*

$$\sup_\Omega |h(x,t)| = o(e^{-\lambda t}) \qquad \textit{for all } \lambda < \lambda_1(\tilde{\Omega})$$

being λ_1 the principal eigenvalue of (6.2) *for any $\tilde{\Omega}$ such that Ω is compactly contain in $\tilde{\Omega}$*

Appendix A. the Krein–Rutman Theorem

Let X be a real Banach space. Let K be a closed convex cone in X with vertex 0, i.e.,

- $0 \in K$,
- $x \in K, t \in \mathbb{R}^+$ then $tx \in K$,
- $x, y \in K$ then $x + y \in K$.

We further assume that

$$K \cap -K = \{0\}.$$

The cone K induces an ordering $\preceq$ on X as follows. Given $x, y \in X$ we say that

$$x \preceq y \iff y - x \in K.$$

The ordering $\preceq$ is said to be *strict* if $x \preceq y$ and $x \neq y$ and this will be denoted by $x \prec y$. A mapping $T : X \to X$ is said to be *increasing* if $x \preceq y \Rightarrow T(x) \preceq T(y)$, and it is said to be *strictly increasing* if $x \prec y$ implies $Tx \prec Ty$. The mapping is said to be *compact* if it takes bounded subsets of X into relatively compact subsets of X. We say that the mapping is *positively* $1-$*homogeneous* if it satisfies the relation $T(tx) = tT(x)$ for all $x \in X$ and $t \in \mathbb{R}^+$.

Theorem A.1 (Krein-Rutmann for non-linear operator, [32], [1])**.** *Let $T : X \to X$ be an increasing, positively $1-$homogeneous compact continuous operator(non-linear) on X for which there exists a non-zero $u \in K$ and $M > 0$ such that*

$$\textbf{(H)} \qquad u \preceq MTu.$$

Then, T has a non-zero eigenvector $x_0 \in K$.

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