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ASYMPTOTIC STABILITY OF THE EXACT BOUNDARY CONTROLLABILITY OF NODAL PROFILE FOR QUASILINEAR HYPERBOLIC SYSTEMS *

LIBIN WANG[1] AND KE WANG[2]

Abstract. In this paper, we consider the asymptotic stability of the exact boundary controllability of nodal profile for quasilinear hyperbolic systems. We will prove that if the nodal profile and the given boundary function possess an exponential or polynomial decaying property, then the boundary control function and the solution to the corresponding mixed initial-boundary value problem will possess the same decaying property.

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1. Introduction

The exact boundary controllability of hyperbolic systems is of great importance in both theory and applications. A complete theory on the local exact boundary controllability for 1-D quasilinear hyperbolic systems has been established by means of a constructive method with modular structure (see [8, 12–14]). In the meantime, the global exact boundary controllability for quasilinear hyperbolic systems has been also studied (see [4, 5, 10, 18, 21]).

Motivated by the practical application of gas transport through a pipeline network, M. Gugat et al. proposed a new kind of exact boundary controllability in [6]. Their initiative was almost immediately generalized to general 1-D first order quasilinear hyperbolic systems with general nonlinear boundary conditions, and was called the exact boundary controllability of nodal profile (see [9, 15]). Some related results about the exact boundary controllability of nodal profile can be found in [2, 3]. Differently from the usual exact boundary controllability, the exact boundary controllability of nodal profile requires that the value of solution satisfies the given profiles on one or several nodes for $t \geq T$ by means of boundary controls. However, in these results, the nodal profiles are given only on a finite time interval $[T, \bar{T}]$, where $\bar{T}$ is an arbitrarily given number. In order to consider the asymptotic stability of the solution to the exact boundary controllability of nodal profile, we have to ask whether the same results can be obtained when the nodal profiles are given on an infinite time interval $[T, +\infty)$. In this paper, we will give an affirmative answer to this question in the case of a single node, and show that for

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[1] School of Mathematical Sciences, Fudan University; Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University; Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, China. lbwang@fudan.edu.cn
[2] Department of Mathematics, Donghua University; Institute for Nonlinear Sciences, Donghua University, Shanghai 201620, China. kwang@dhu.edu.cn

the given profiles satisfying certain decaying property as $t \to +\infty$, the employed boundary control function and the solution to the corresponding mixed initial-boundary value problem possess the same decaying property as the nodal profiles.

We will still use the constructive method with modular structure suggested in [8] and [15] to deal with this exact boundary controllability of nodal profile on a semibounded time interval. For this purpose, in the present situation, we need not only the existence and uniqueness of semi-global classical solution to the mixed initial-boundary value problem with two boundaries, but also the existence and uniqueness of semi-global classical solution to the one-sided mixed initial-boundary value problem on a semibounded initial axis. The organization of this paper is as follows: in Section 2 we will give the precise definition of the exact boundary controllability of nodal profile on a semibounded time interval at a boundary node, and present the main results. In Section 3 we will prove the existence and uniqueness of semi-global classical solution to the one-sided mixed initial-boundary value problem on a semibounded initial axis under different hypotheses on the boundary functions and the given nodal profiles. Then the main results will be proved in Sections 4. In Section 5 an application of the main results will be given to the Saint-Venant system for unsteady flows on a single open canal.

2. Definition and main results

Consider the following 1-D first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{2.1}$$

where t is the time variable, x is the spatial variable, $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u)$ is a given $n \times n$ matrix with C^1 elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a C^1 vector function of u and

$$F(0) = 0. \tag{2.2}$$

By (2.2), $u = 0$ is an equilibrium of system (2.1).

By hyperbolicity, for any given u on the domain under consideration, the matrix $A(u)$ possesses n real eigenvalues and a complete set of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\textit{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{2.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\textit{resp. } \det|r_{ij}(u)| \neq 0). \tag{2.4}$$

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n) \tag{2.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \cdots, n), \tag{2.6}$$

where δ_{ij} stands for Kronecker's symbol. We suppose that all $\lambda_i(u)$ and $l_i(u)$(resp. $r_i(u)$) $(i = 1, \cdots, n)$ are also C^1.

Suppose that on the domain under consideration there are no zero eigenvalues:

$$\lambda_r(u) < 0 < \lambda_s(u) \quad (r = 1, \cdots, m;\ s = m + 1, \cdots, n). \tag{2.7}$$

Let

$$v_i = l_i(u)u \quad (i = 1, \cdots, n). \tag{2.8}$$

Noting (2.7), for the forward problem, we give the following general nonlinear boundary conditions

$$x = 0: \quad v_s = G_s(\alpha_s(t), v_1, \cdots, v_m) + H_s(t) \quad (s = m+1, \cdots, n), \tag{2.9}$$

$$x = L: \quad v_r = G_r(\alpha_r(t), v_{m+1}, \cdots, v_n) + H_r(t) \quad (r = 1, \cdots, m), \tag{2.10}$$

where G_i, α_i and H_i $(i = 1, \cdots, n)$ are all C^1 functions with respect to their respective arguments and $\alpha_i(t)$ $(i = 1, \cdots, n)$ may be vector functions. Without loss of generality, we may suppose that

$$G_i(\alpha_i(t), 0, \cdots, 0) \equiv 0 \quad (i = 1, \cdots, n). \tag{2.11}$$

Moreover, the initial condition is prescribed as

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L, \tag{2.12}$$

where L is the length of the spatial interval, and $\varphi(x)$ is a C^1 vector function.

Definition 2.1. For any given C^1 initial data $\varphi(x)$, any given C^1 boundary functions $H_r(t)$ $(r = 1, \cdots, m)$ and $\alpha_i(t)$ $(i = 1, \cdots, n)$, satisfying the conditions of C^1 compatibility at the point $(t, x) = (0, L)$, for any given C^1 vector function $\bar{\bar{u}}(t)$, if there exist $T > 0$ and C^1 boundary controls $H_s(t)$ $(s = m+1, \cdots, n)$ such that the C^1 solution $u = u(t, x)$ to the mixed initial-boundary value problem (2.1), (2.12) and (2.9)-(2.10) fits exactly $\bar{\bar{u}}(t)$ on $x = L$ for $t \ge T$, then we have the exact boundary controllability of nodal profile on the boundary node $x = L$.

Remark 2.2. The conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ or $(0, L)$ for the mixed initial-boundary value problem (2.1), (2.12) and (2.9)-(2.10) can be obtained in a similar way to Remark 1.2 in [15].

Remark 2.3. When the exact boundary controllability of nodal profile on $x = L$ can be realized only in the case that $\varphi(x)$, $H_r(t)$ $(r = 1, \cdots, m)$, $\alpha_i(t)$ $(i = 1, \cdots, n)$ and $\bar{\bar{u}}(t)$ are suitably small in C^1 norm, it is called the local exact boundary controllability of nodal profile; otherwise, the global exact boundary controllability of nodal profile. In this paper, we consider only the local exact boundary controllability of nodal profile.

Remark 2.4. In Definition 2.1, when $t \ge T$, the value of solution $u = \bar{\bar{u}}(t)$ on $x = L$ should satisfy the boundary condition (2.10), in which $v_i = \bar{\bar{v}}_i(t) \overset{\text{def.}}{=} l_i(\bar{\bar{u}}(t))\bar{\bar{u}}(t)$ $(i = 1, \cdots, n)$. Hence, the requirement that the solution $u = u(t, x)$ fits exactly the given value $\bar{\bar{u}}(t)$ on $x = L$ for $t \ge T$ is equivalent to ask that v_s $(s = m+1, \cdots, n)$ fit exactly the given values $\bar{\bar{v}}_s(t)$ $(s = m+1, \cdots, n)$ on $x = L$ for $t \ge T$, then the value of v_r $(r = 1, \cdots, m)$ on $x = L$ for $t \ge T$ can be determined by the boundary condition (2.10) as follows

$$v_r = \bar{\bar{v}}_r(t) \overset{\text{def.}}{=} G_r(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) + H_r(t) \quad (r = 1, \cdots, m). \tag{2.13}$$

We first consider the general case that the initial data, boundary functions and the nodal profile only possess suitably small C^1 norm. Precisely, we have the following theorem, which will be proved in Section 4.1.

Theorem 2.5. *Let*

$$T > L \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(0)}. \tag{2.14}$$

For any given initial data $\varphi(x)$, boundary functions $H_r(t)$ $(r = 1, \cdots, m)$ and $\alpha_i(t)$ $(i = 1, \cdots, n)$ with suitably small C^1 norm $\|\varphi\|_{C^1[0,L]}$, $\|H_r\|_{C^1[0,+\infty)}$ $(r = 1, \cdots, m)$ and $\|\alpha_i\|_{C^1[0,+\infty)}$ $(i = 1, \cdots, n)$, respectively, satisfying *the conditions of* C^1 *compatibility at the point* $(t,x) = (0,L)$*, suppose that the given values* $\bar{\bar{v}}_s(t)$ $(s = m+1,\cdots,n)$ *on* $x = L$ *for* $t \geq T$ *possess suitably small* C^1 *norms* $\|\bar{\bar{v}}_s\|_{C^1[T,+\infty)}$ $(s = m+1,\cdots,n)$*, then there exist boundary controls* $H_s(t)$ $(s = m+1,\cdots,n)$ *with small* C^1 *norms* $\|H_s\|_{C^1[0,+\infty)}$ $(s = m+1,\cdots,n)$*, such that the mixed initial-boundary value problem* (2.1)*,* (2.12) *and* (2.9)*-*(2.10) *admits a unique* C^1 *solution* $u = u(t,x)$ *with small* C^1 *norm on the domain* $R = \{(t,x)|t \geq 0, 0 \leq x \leq L\}$*, which fits exactly the given values* $v_s = \bar{\bar{v}}_s(t)$ $(s = m+1,\cdots,n)$ *on* $x = L$ *for* $t \geq T$*, namely, the given value* $u = \bar{\bar{u}}(t)$*, on the boundary node* $x = L$ *for* $t \geq T$*, where* $\bar{\bar{u}}(t)$ *is defined by the following implicit relationship*

$$\bar{\bar{u}}(t) = \sum_{i=1}^{n} \bar{\bar{v}}_i(t) r_i(\bar{\bar{u}}(t)), \tag{2.15}$$

in which

$$\bar{\bar{v}}_r(t) = G_r(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) + H_r(t) \quad (r = 1,\cdots,m) \tag{2.16}$$

satisfying (2.11)*.*

Remark 2.6. Using (2.11) and (2.16), one easily gets, thanks to the implicit function theorem, there exists $\rho > 0$ such that, if the C^1 norms of $H_r(t)$ $(r = 1,\ldots,m)$ and $\bar{\bar{v}}_s(t)$ $(s = m+1,\ldots,n)$ are sufficiently small, then there is a unique $\bar{\bar{u}}(t) \in C^1$ such that (2.15) holds and its C^1 norm is less than ρ.

We next consider the effect of the exponential or polynomial decaying property possessed by the nodal profile and the given boundary functions on the boundary controls and the solution to the corresponding mixed initial-boundary value problem. For the case with the exponential decaying property, we have

Theorem 2.7. *Suppose that* T *satisfies* (2.14)*. Suppose furthermore that the initial data* $\varphi(x)$ *possesses a suitably small* C^1 *norm* $\|\varphi\|_{C^1[0,L]}$*, the boundary functions* $H_r(t)$ $(r = 1,\cdots,m)$ *and* $\alpha_i(t)$ $(i = 1,\cdots,n)$ *satisfy*

$$\max_{\substack{r=1,\cdots,m\\ i=1,\cdots,n}} \sup_{t\geq 0} \{e^{at}(|H_r(t)| + |H_r'(t)| + |\alpha_i(t)| + |\alpha_i'(t)|)\} \ll 1, \tag{2.17}$$

in which a *is a positive constant. Moreover, the conditions of* C^1 *compatibility at the point* $(t,x) = (0,L)$ *are assumed to be satisfied. Then, for the given nodal values* $\bar{\bar{v}}_s(t)$ $(s = m+1,\cdots,n)$ *on* $x = L$ *for* $t \geq T$*, satisfying*

$$\max_{s=m+1,\cdots,n} \sup_{t\geq T} \{e^{at}(|\bar{\bar{v}}_s(t)| + |\bar{\bar{v}}_s'(t)|)\} \ll 1, \tag{2.18}$$

there exist boundary controls $H_s(t)$ $(s = m+1,\cdots,n)$ *with*

$$e^{at}(|H_s(t)| + |H_s'(t)|) \leq K_1 \quad (s = m+1,\cdots,n), \quad \forall t \geq 0, \tag{2.19}$$

where K_1 *is a positive constant independent of* t*, such that the mixed initial-boundary value problem* (2.1)*,* (2.12) *and* (2.9)*-*(2.10) *admits a unique* C^1 *solution* $u = u(t,x)$ *on the domain* $R = \{(t,x)|t \geq 0, 0 \leq x \leq L\}$*, satisfying*

$$e^{at}(|u(t,x)| + |u_t(t,x)|) \leq K_2, \quad \forall t \geq 0,\ 0 \leq x \leq L, \tag{2.20}$$

where K_2 *is a positive constant independent of* t *and* x*, and fitting exactly the given values* $v_s = \bar{\bar{v}}_s(t)$ $(s = m+1,\cdots,n)$*, namely, the given value* $u = \bar{\bar{u}}(t)$ *(see* (2.15)*), on the boundary node* $x = L$ *for* $t \geq T$*.*

For the case with the polynomial decaying property, we have

Theorem 2.8. *Suppose that T satisfies* (2.14)*, Suppose furthermore that the initial data $\varphi(x)$ possesses a suitably small C^1 norm $\|\varphi\|_{C^1[0,L]}$, the boundary functions $H_r(t)$ $(r=1,\cdots,m)$ and $\alpha_i(t)$ $(i=1,\cdots,n)$ satisfy*

$$\max_{\substack{r=1,\cdots,m\\ i=1,\cdots,n}}\sup_{t\geq 0}\{(1+t)^{1+\mu}(|H_r(t)|+|H_r'(t)|+|\alpha_i(t)|+|\alpha_i'(t)|)\}\ll 1, \tag{2.21}$$

in which μ is a nonnegative constant. Moreover, the conditions of C^1 compatibility at the point $(t,x)=(0,L)$ are assumed to be satisfied. Then, for the given nodal values $\bar{\bar{v}}_s(t)$ $(s=m+1,\cdots,n)$ on $x=L$ for $t\geq T$, satisfying

$$\max_{s=m+1,\cdots,n}\sup_{t\geq T}\{(1+t)^{1+\mu}(|\bar{\bar{v}}_s(t)|+|\bar{\bar{v}}_s'(t)|)\}\ll 1, \tag{2.22}$$

there exist boundary controls $H_s(t)$ $(s=m+1,\cdots,n)$ with

$$(1+t)^{1+\mu}(|H_s(t)|+|H_s'(t)|)\leq K_3 \quad (s=m+1,\cdots,n), \quad \forall t\geq 0, \tag{2.23}$$

where K_3 is a positive constant independent of t, such that the mixed initial-boundary value problem (2.1), (2.12) *and* (2.9)-(2.10) *admits a unique C^1 solution $u=u(t,x)$ on the domain $R=\{(t,x)|t\geq 0, 0\leq x\leq L\}$, satisfying*

$$(1+t)^{1+\mu}(|u(t,x)|+|u_t(t,x)|)\leq K_4, \quad \forall t\geq 0,\ 0\leq x\leq L, \tag{2.24}$$

where K_4 is a positive constant independent of t and x, and fitting exactly the given values $v_s=\bar{\bar{v}}_s(t)$ $(s=m+1,\cdots,n)$, namely, the given value $u=\bar{\bar{u}}(t)$ (see (2.15)*), on the boundary node $x=L$ for $t\geq T$.*

The proof of Theorem 2.7 and Theorem 2.8 will be given in Section 4.2 and Section 4.3, respectively.

Remark 2.9. In Theorem 2.8, if $1+\mu$ in (2.21) and (2.22) is replaced by a positive constant ω, the corresponding result still holds. This means that for the case with the polynomial decaying property, the decay can be slower than $(1+t)^{-1}$.

Remark 2.10. When the nodal profiles are given at the boundary node $x=0$, the exact boundary controllability of nodal profile can be similarly defined. In this case, if the boundary functions $H_s(t)$ $(s=m+1,\cdots,n)$ and the nodal values $\bar{\bar{v}}_r(t)$ $(r=1,\cdots,m)$ on $x=0$ for $t\geq T$ satisfy the corresponding conditions as in Theorem 2.5, Theorem 2.7 or Theorem 2.8, there exist boundary controls $H_r(t)$ $(r=1,\cdots,m)$ such that the similar results hold, provided that (2.14) is replaced by

$$T>L\max_{r=1,\cdots,m}\frac{1}{|\lambda_r(0)|}. \tag{2.25}$$

Remark 2.11. If A and F also depend on x but u in the system (2.1) and $F(x,u)$ satisfies

$$F(x,0)=0, \tag{2.26}$$

under assumption that $\lambda_i(x,u)$, $l_i(x,u)$ (resp. $r_i(x,u)$) and $f_i(x,u)$ $(i=1,\cdots,n)$ are C^1 functions with finite C^1 norms, the corresponding results to Theorems 2.5, 2.7 and 2.8 can be obtained. Based on them, we can prove the asymptotic stability of any given C^1 stationary state for all initial data satisfying the conditions of C^1 compatibility in a C^1-neighborhood of the stationary state and for any given nodal profiles in a sufficiently small C^1-neighborhood of boundary data corresponding to the stationary state (see [6]). This is an interesting and important problem, we will discuss further it in another paper, specially for the application to the Saint-Venant equations.

3. Semi-global C^1 solutions to the one-sided mixed initial-boundary value problem on a semibounded initial axis

In order to prove Theorems 2.5, 2.7 and 2.8, it is necessary to discuss the existence and uniqueness of semi-global C^1 solution to the one-sided mixed initial-boundary value problem on a semibounded initial axis. Some related results can be found in [11] and [16].

3.1. One-sided mixed initial-boundary value problem for the general initial data with small C^1 norm

In this section, we consider the semi-global classical solution to the one-sided mixed initial-boundary value problem for system (2.1) with the initial condition

$$t = 0: \quad u = \varphi(x), \quad x \geq 0 \tag{3.1}$$

and the boundary conditions (2.9). Here, we still suppose that (2.7) and (2.11) for $s = m+1, \cdots, n$ are satisfied. We have the following

Theorem 3.1. *Suppose that l_i (resp. r_i), λ_i, f_i, G_s, H_s, α_s ($i = 1, \cdots, n; s = m+1, \cdots, n$) and φ are all C^1 functions with respect to their arguments, and the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are satisfied. Then, for a preassigned and possibly quite large $T_0 > 0$, the one-sided mixed initial-boundary value problem (2.1), (2.9) and (3.1) admits a unique C^1 solution $u = u(t, x)$ with small C^1 norm on the domain $R(T_0) = \{(t, x)|\ 0 \leq t \leq T_0, 0 \leq x < +\infty\}$, provided that $\|\varphi\|_{C^1[0,+\infty)}$ and $\|H_s\|_{C^1[0,T_0]}$ are suitably small (depending on T_0).*

In order to prove Theorem 3.1, we need the local existence and uniqueness of C^1 solution to the one-sided mixed initial-boundary value problem (2.1), (2.9) and (3.1). For this purpose, taking a positive constant λ satisfying

$$\lambda > \max_{s=m+1,\cdots,n} \lambda_s(0), \tag{3.2}$$

according to Theorem 4.1 and Remark 4.3 of Chapter 1 in [19], there exists a suitably small $\delta > 0$ such that the initial problem (2.1) and (3.1) admits a unique C^1 solution $u = u_1(t, x)$ on the domain $R_1(\delta) = \{(t, x)|x \geq \lambda t,\ 0 \leq t \leq \delta\}$. Then, noting that the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are satisfied, according to Theorem 2.1 and Remark 2.4 of Chapter 4 in [19], there exists a suitably small $\delta^* \in (0, \delta]$ such that the general boundary value problem (2.1), (2.9) and

$$x = \lambda t: \ u = u_1(t, \lambda t) \tag{3.3}$$

admits a unique C^1 solution $u = u_2(t, x)$ on the domain $R_2(\delta^*) = \{(t, x)|0 \leq x \leq \lambda t,\ 0 \leq t \leq \delta^*\}$. Noting again that the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$ are satisfied and $x = \lambda t$ is not the characteristic curve, hence,

$$u(t, x) = \begin{cases} u_1(t, x), & x \geq \lambda t,\ 0 \leq t \leq \delta^*, \\ u_2(t, x), & 0 \leq x \leq \lambda t,\ 0 \leq t \leq \delta^* \end{cases} \tag{3.4}$$

is a unique C^1 solution to the one-sided mixed initial-boundary value problem (2.1), (2.9) and (3.1). Based on the local existence and uniqueness of C^1 solution, in order to prove Theorem 3.1, it is only necessary to prove the following

Lemma 3.2. *Under the hypotheses of Theorem 3.1, for a preassigned and possibly quite large $T_0 > 0$, if $\|\varphi\|_{C^1[0,+\infty)}$ and $\|H_s\|_{C^1[0,T_0]}$ are suitably small (depending on T_0), then, for any C^1 solution $u = u(t, x)$ to the* *one-sided mixed initial-boundary value problem* (2.1)*,* (2.9) *and* (3.1) *on the domain* $R(T) = \{(t,x)|0 \le t \le T, x \ge 0\}$ *with* $0 < T \le T_0$*, we have the following uniform a priori estimate:*

$$\|u(t,\cdot)\|_1 \triangleq \|u(t,\cdot)\|_0 + \|u_x(t,\cdot)\|_0 \le C(T_0), \quad \forall t \in [0,T], \tag{3.5}$$

where $C(T_0)$ *is a sufficiently small positive constant independent of* T *but possibly depending on* T_0*.*

Proof. Let

$$w_i = l_i(u)u_x \quad (i = 1, \cdots, n). \tag{3.6}$$

By (2.5), we have

$$u = \sum_{i=1}^{n} v_i r_i(u) \tag{3.7}$$

and

$$u_x = \sum_{i=1}^{n} w_i r_i(u). \tag{3.8}$$

Noting (2.6), in order to estimate u and u_x, it suffices to estimate $v_i(t,x)$ and $w_i(t,x)$ $(i = 1, \cdots, n)$ on $R(T)$, namely, estimate $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n)$.

By the formulas on the decomposition of waves(see [7, 11, 16, 17, 20]) and noting (2.2) and (3.7), we have

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k + \sum_{j=1}^{n} \tilde{\beta}_{ij}(u) v_j \quad (i = 1, \cdots, n) \tag{3.9}$$

and

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j=1}^{n} \tilde{\gamma}_{ij}(u) w_j \quad (i = 1, \cdots, n), \tag{3.10}$$

where

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{3.11}$$

denotes the directional derivative with respect to t along the ith characteristic,

$$\beta_{ijk}(u) = \big(\lambda_k(u) - \lambda_i(u)\big) l_i(u) \nabla r_j(u) r_k(u), \tag{3.12}$$

$$\gamma_{ijk}(u) = \frac{1}{2}\Big\{\big(\lambda_j(u) - \lambda_k(u)\big) l_i(u)\nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u)\delta_{ik} + (j|k)\Big\}, \tag{3.13}$$

$$\tilde{\beta}_{ij}(u) = \sum_{p,k=1} l_{ip}(u) r_{kj}(u) \int_0^1 \frac{\partial f_p(\tau u)}{\partial u_k} d\tau - l_i(u)\nabla r_j(u) F(u), \tag{3.14}$$

$$\tilde{\gamma}_{ij}(u) = l_i(u)\nabla F(u) r_j(u) - l_i(u)\nabla r_j(u) F(u), \tag{3.15}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms.

For the time being we assume that on the domain $R(T)$,

$$|v(t,x)| \le \eta_0, \quad |w(t,x)| \le \eta_1, \tag{3.16}$$

where η_0 and η_1 are suitably small positive constants. At the end of the proof, we will explain the validity of hypothesis (3.16).

Then, noting (2.6) and (3.7), we have

$$|u(t,x)| \le n\eta_0, \quad \forall (t,x) \in R(T). \tag{3.17}$$

Let

$$V_0(t) = \sup_{x\ge 0} |v(t,x)|, \quad W_0(t) = \sup_{x\ge 0} |w(t,x)|. \tag{3.18}$$

(1) For each $r = 1, \cdots, m$, passing through any given point $(t,x) \in R(T)$, we draw down the r-th characteristic $C_r : \xi = \xi_r(\tau; t, x)$. Noting (2.7), this characteristic must intersect the x-axis at a point $(0, x_{0r})$. Integrating the r-th equation in (3.9) along C_r from 0 to t yields

$$v_r(t,x) = v_r(0,x_{0r}) + \int_0^t \Big(\sum_{j,k=1}^n \beta_{rjk}(u)v_j w_k + \sum_{j=1}^n \tilde{\beta}_{rj}(u)v_j \Big)(\tau, \xi_r(\tau;t,x))d\tau. \tag{3.19}$$

Then, noting (3.16)-(3.17), we have

$$|v_r(t,x)| \le |v_r(0,x_{0r})| + c_1 \int_0^t V_0(\tau)d\tau. \tag{3.20}$$

Here and hereafter, c_i $(i = 1, 2, \cdots)$ denote positive constants possibly depending on η_0, η_1 and T_0.

(2) For each $s = m+1, \cdots, n$, passing through any given point $(t,x) \in R(T)$, we draw down the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$. Noting (2.7), there are two possibilities:

(i) The s-th characteristic C_s intersects the x-axis at a point $(0, x_{0s})$. Similarly to (3.20), integrating the s-th equation in (3.9) along C_s from 0 to t gives

$$|v_s(t,x)| \le |v_s(0,x_{0s})| + c_2 \int_0^t V_0(\tau)d\tau. \tag{3.21}$$

(ii) The s-th characteristic C_s intersects the t-axis at a point $(t_{0s}, 0)$. Integrating the s-th equation in (3.9) along C_s from t_{0s} to t yields

$$v_s(t,x) = v_s(t_{0s},0) + \int_{t_{0s}}^t \Big(\sum_{j,k=1}^n \beta_{sjk}(u)v_j w_k + \sum_{j=1}^n \tilde{\beta}_{sj}(u)v_j \Big)(\tau, \xi_s(\tau;t,x))d\tau. \tag{3.22}$$

Noting (2.11), by the boundary conditions (2.9) we have

$$v_s(t_{0s},0) = \sum_{r=1}^m \bar{G}_{sr}(t_{0s})v_r(t_{0s},0) + H_s(t_{0s}), \tag{3.23}$$

in which

$$\bar{G}_{sr}(t_{0s}) = \int_0^1 \frac{\partial G_s}{\partial v_r}(\alpha_s(t_{0s}), \tau v_1(t_{0s},0), \cdots, \tau v_m(t_{0s},0))d\tau. \tag{3.24}$$

Noting (3.20), by (3.22)-(3.23) we have

$$|v_s(t,x)| \le \kappa_1 \sum_{r=1}^{m} |v_r(0,x_{0r})| + |H_s(t_{0s})| + c_3 \int_0^t V_0(\tau)d\tau, \tag{3.25}$$

henceforth κ_i $(i = 1, 2, \cdots)$ denote positive constants depending only on T_0. Combining (3.20)-(3.21) and (3.25), we obtain

$$V_0(t) \le \kappa_2 \|v(0,\cdot)\|_0 + \|H\|_{C^0[0,T_0]} + c_4 \int_0^t V_0(\tau)d\tau, \tag{3.26}$$

where $H = (H_{m+1}, \cdots, H_n)$. Then, by means of Gronwall's inequality, we have

$$V_0(t) \le \kappa_3 \max\{\|v(0,\cdot)\|_0, \|H\|_{C^0[0,T_0]}\}, \quad \forall t \in [0,T]. \tag{3.27}$$

Noting (2.8) and (3.1), we get

$$V_0(t) \le \kappa_4 \max\{\|\varphi\|_{C^0[0,+\infty)}, \|H\|_{C^0[0,T_0]}\}, \quad \forall t \in [0,T]. \tag{3.28}$$

Then, by (3.7), we finally obtain

$$|u(t,x)| \le \kappa_5 \max\{\|\varphi\|_{C^0[0,+\infty)}, \|H\|_{C^0[0,T_0]}\}, \quad \forall (t,x) \in R(T). \tag{3.29}$$

We next estimate $W_0(t)$.

(I) Similarly to (1), for each $r = 1, \cdots, m$, passing through any given point $(t,x) \in R(T)$, we draw down the r-th characteristic $C_r : \xi = \xi_r(\tau; t, x)$ which intersects the x-axis at a point $(0, x_{0r})$. Integrating the r-th equation in (3.10) along C_r from 0 to t yields

$$w_r(t,x) = w_r(0,x_{0r}) + \int_0^t \Big(\sum_{j,k=1}^{n} \gamma_{rjk}(u) w_j w_k + \sum_{j=1}^{n} \tilde{\gamma}_{rj}(u) w_j \Big)(\tau, \xi_r(\tau; t, x)) d\tau. \tag{3.30}$$

Hence, we have

$$|w_r(t,x)| \le |w_r(0,x_{0r})| + c_5 \int_0^t W_0(\tau)d\tau. \tag{3.31}$$

(II) For any given point $(t,x) \in R(T)$, we draw down the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$ $(s = m+1, \cdots, n)$. As before, there are two possibilities:

(i) The s-th characteristic C_s intersects the x-axis at a point $(0, x_{0s})$. Integrating the s-th equation in (3.10) along C_s from 0 to t, similarly to (3.31), we have

$$|w_s(t,x)| \le |w_s(0,x_{0s})| + c_6 \int_0^t W_0(\tau)d\tau. \tag{3.32}$$

(ii) The s-th characteristic C_s intersects the t-axis at a point $(t_{0s}, 0)$. Integrating the s-th equation in (3.10) along C_s from t_{0s} to t yields

$$w_s(t,x) = w_s(t_{0s},0) + \int_{t_{0s}}^t \Big(\sum_{j,k=1}^{n} \gamma_{sjk}(u) w_j w_k + \sum_{j=1}^{n} \tilde{\gamma}_{sj}(u) w_j \Big)(\tau, \xi_s(\tau; t, x)) d\tau. \tag{3.33}$$

Hence, we have

$$|w_s(t,x)| \le |w_s(t_{0s},0)| + c_7 \int_{t_{0s}}^{t} W_0(\tau)d\tau. \tag{3.34}$$

In order to estimate $|w_s(t_{0s},0)|$, differentiating the boundary conditions (2.9) with respect to t gives

$$x=0: \quad \frac{\partial v_s}{\partial t} = \frac{\partial G_s}{\partial \alpha_s}(\alpha_s(t), v_1, \cdots, v_m)\alpha_s'(t) + \sum_{r=1}^{m} \frac{\partial G_s}{\partial v_r}(\alpha_s(t), v_1, \cdots, v_m)\frac{\partial v_r}{\partial t} + H_s'(t) \quad (s=m+1,\cdots,n). \tag{3.35}$$

By (2.8) and noting (2.1)-(2.2) and (3.7)-(3.8), we have

$$\begin{aligned}
\frac{\partial v_i}{\partial t} &= l_i(u)u_t + u_t^T \nabla l_i(u) u \\
&= l_i(u)\Big(- A(u)u_x + F(u)\Big) + \Big(- A(u)u_x + F(u)\Big)^T \nabla l_i(u) u \\
&= -\lambda_i(u)w_i + l_i(u)F(u) + \Big(- A(u)\sum_{k=1}^{n} w_k r_k(u) + F(u)\Big)^T \nabla l_i(u) \sum_{j=1}^{n} v_j r_j(u) \\
&= -\lambda_i(u)w_i - \sum_{j,k=1}^{n} \Big(\lambda_k r_k^T(u)\nabla l_i(u) r_j(u)\Big) v_j w_k + \sum_{j=1}^{n} \Big(l_i(u)\int_0^1 \nabla F(\tau u)d\tau r_j(u)\Big) v_j \\
&\quad + \sum_{j,k=1}^{n} \Big(r_k^T(u) \int_0^1 \nabla F^T(\tau u) d\tau \nabla l_i(u) r_j(u)\Big) v_j v_k.
\end{aligned} \tag{3.36}$$

Then, noting (2.7) and (3.16), for $\eta_0 > 0$ small enough, by (3.35)-(3.36) we get

$$\begin{aligned}
x=0: \ w_s &= \sum_{r=1}^{m} f_{sr}^{(1)}(t,u) w_r + \sum_{j=1}^{n} f_{sj}^{(2)}(t,u) v_j \\
&\quad + \sum_{\bar{s}=m+1}^{n} f_{s\bar{s}}^{(3)}(t,u) \frac{\partial G_{\bar{s}}}{\partial \alpha_{\bar{s}}}(\alpha_{\bar{s}}(t), v_1, \cdots, v_m)\alpha_{\bar{s}}'(t) + \sum_{\bar{s}=m+1}^{n} f_{s\bar{s}}^{(4)}(t,u) H_{\bar{s}}'(t),
\end{aligned} \tag{3.37}$$

where $f_{sr}^{(1)}(t,u)$, $f_{sj}^{(2)}(t,u)$, $f_{s\bar{s}}^{(3)}(t,u)$ and $f_{s\bar{s}}^{(4)}(t,u)$ are continuous functions of t and u. Noting (2.11), we have

$$\frac{\partial G_s}{\partial \alpha_s}(\alpha_s(t), 0, \cdots, 0)\alpha_s'(t) \equiv 0. \tag{3.38}$$

Then, from (3.37) it follows that

$$x=0: \ w_s = \sum_{r=1}^{m} f_{sr}^{(1)}(t,u) w_r + \bar{f}_s(t,u) + \sum_{\bar{s}=m+1}^{n} f_{s\bar{s}}^{(4)}(t,u) H_{\bar{s}}'(t), \tag{3.39}$$

where $\bar{f}_s(t,u)$ is a continuous function of t and u, moreover,

$$d(u) = \sup_{\substack{0\le t\le T_0 \\ s=m+1,\cdots,n}} |\bar{f}_s(t,u)| \to 0, \quad as\ |u| \to 0. \tag{3.40}$$

Hence, by (3.39) we obtain

$$|w_s(t_{0s},0)| \le \kappa_6 \sum_{r=1}^{m} |w_r(t_{0s},0)| + \kappa_7\big(d(u) + \|H'\|_{C^0[0,T_0]}\big). \tag{3.41}$$

Thus, noting (3.31), by (3.34) and (3.41) we have

$$|w_s(t,x)| \le \kappa_8 \|w(0,\cdot)\|_0 + \kappa_7(d(u) + \|H'\|_{C^0[0,T_0]}) + c_8 \int_0^t W_0(\tau)d\tau. \tag{3.42}$$

Then, combining (3.31) and (3.42) and noting (3.1) and (3.6), by Gronwall's inequality we get

$$W_0(t) \le \kappa_9 \max\{\|\varphi\|_{C^1[0,+\infty)}, d(u) + \|H\|_{C^1[0,T_0]}\}, \quad \forall t \in [0,T]. \tag{3.43}$$

Noting (3.8) and (3.40), (3.5) follows from (3.29) and (3.43).

Thus, by (3.28)-(3.29), (3.40) and (3.43), it is easy to see that $V_0(t)$ and $W_0(t)$ are small enough on $[0,T]$ $(0 < T \le T_0)$ when $\|\varphi\|_{C^1[0,+\infty)}$ and $\|H\|_{C^1[0,T_0]}$ are sufficiently small. Hence, hypothesis (3.16) is reasonable.

The proof of Lemma 3.2 is complete.

□

3.2. One-sided mixed initial-boundary value problem for the initial data with the exponential decaying property

In this subsection, we consider the effect of the exponential decaying property possessed by the initial data on the solution to the one-sided mixed initial-boundary value problem. The hypotheses of the regularity of all the given functions and the C^1 conditions of compatibility at the point $(0,0)$ are the same as Theorem 3.1.

Let

$$\theta = \sup_{x\ge 0} \Big\{ e^{ax}\big(|\varphi(x)| + |\varphi'(x)|\big) \Big\}, \tag{3.44}$$

where $a > 0$ is a positive constant.

We have the following

Theorem 3.3. *For a preassigned and possibly quite large $T_0 > 0$, if θ and $\|H_s\|_{C^1[0,T_0]}$ are suitably small (depending on T_0), then the one-sided mixed initial-boundary value problem* (2.1), (2.9) *and* (3.1) *admits a unique C^1 solution $u = u(t,x)$ on the domain $R(T_0) = \{(t,x)|\ 0 \le t \le T_0, x \ge 0\}$, moreover, we have*

$$e^{ax}\Big(|u(t,x)| + |u_x(t,x)|\Big) \le C(T_0), \quad \forall t \in [0,T_0],\ x \ge 0, \tag{3.45}$$

where $C(T_0)$ is a sufficiently small positive constant possibly depending on T_0.

To prove Theorem 3.3, it is sufficient to prove the following

Lemma 3.4. *Under the hypotheses of Theorem 3.3, for any C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem* (2.1), (2.9) *and* (3.1) *on the domain $R(T) = \{(t,x)|0 \le t \le T, 0 \le x < +\infty\}$ with $0 < T \le T_0$, we have the following uniform a priori estimate:*

$$e^{ax}\Big(|u(t,x)| + |u_x(t,x)|\Big) \le C(T_0), \quad \forall t \in [0,T],\ x \ge 0, \tag{3.46}$$

where $C(T_0)$ is a sufficiently small positive constant independent of T but possibly depending on T_0.

Proof. The proof of this lemma is similar to that of Lemma 3.2. In what follows, we only point out some essentially different points.

Let

$$V(t) = \sup_{x\geq 0}\{e^{ax}|v(t,x)|\}, \quad W(t) = \sup_{x\geq 0}\{e^{ax}|w(t,x)|\}. \tag{3.47}$$

For the time being we assume that on the domain $R(T)$ we have

$$|v(t,x)| \leq \delta_0, \quad |w(t,x)| \leq \delta_1, \tag{3.48}$$

where δ_0 and δ_1 are suitably small positive constant. At the end of the proof, we will explain the validity of hypothesis (3.48).

By (3.9) and (3.10) we have

$$\frac{d(e^{ax}v_i)}{d_i t} = e^{ax}\Big(a\lambda_i(u)v_i + \sum_{j,k=1}^{n}\beta_{ijk}(u)v_j w_k + \sum_{j=1}^{n}\tilde{\beta}_{ij}(u)v_j\Big) \tag{3.49}$$

and

$$\frac{d(e^{ax}w_i)}{d_i t} = e^{ax}\Big(a\lambda_i(u)w_i + \sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j w_k + \sum_{j=1}^{n}\tilde{\gamma}_{ij}(u)w_j\Big). \tag{3.50}$$

In the present situation, for $r = 1, \cdots, m$, integrating the r-th equation in (3.49) along the r-th characteristic $C_r : \xi = \xi_r(\tau; t, x)$ passing through $(t, x) \in R(T)$ from 0 to t, instead of (3.19), we have

$$e^{ax}v_r(t,x) = e^{ax_{0r}}v_r(0,x_{0r}) + \int_0^t e^{a\xi_r}\Big(a\lambda_r(u)v_r + \sum_{j,k=1}^{n}\beta_{rjk}(u)v_j w_k + \sum_{j=1}^{n}\tilde{\beta}_{rj}(u)v_j\Big)(\tau,\xi_r(\tau;t,x))d\tau, \tag{3.51}$$

then, noting (3.44) and (3.48), we get

$$e^{ax}|v_r(t,x)| \leq \kappa_{10}\theta + c_9\int_0^t V(\tau)d\tau. \tag{3.52}$$

For $s = m+1, \cdots, n$, if the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$ passing through $(t, x) \in R(T)$ intersects the x-axis at a point $(0, x_{0s})$, integrating the s-th equation in (3.49) along C_s from 0 to t, similarly to (3.52), we have

$$e^{ax}|v_s(t,x)| \leq \kappa_{11}\theta + c_{10}\int_0^t V(\tau)d\tau. \tag{3.53}$$

If the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$ passing through $(t, x) \in R(T)$ intersects the t-axis at a point $(t_{0s}, 0)$, integrating the s-th equation in (3.49) along C_s from t_{0s} to t, instead of (3.22), we have

$$e^{ax}v_s(t,x) = v_s(t_{0s},0) + \int_{t_{0s}}^t e^{a\xi_s}\Big(a\lambda_s(u)v_s + \sum_{j,k=1}^{n}\beta_{sjk}(u)v_j w_k + \sum_{j=1}^{n}\tilde{\beta}_{sj}(u)v_j\Big)(\tau,\xi_s(\tau;t,x))d\tau. \tag{3.54}$$

Since we have (3.23) on the boundary $x = 0$, noting (3.48), we get from (3.54) that

$$e^{ax}|v_s(t,x)| \leq \kappa_{12}\sum_{r=1}^{m}|v_r(t_{0s},0)| + |H_s(t_{0s})| + c_{11}\int_{t_{0s}}^t V(\tau)d\tau. \tag{3.55}$$

Then, noting (3.52), we obtain

$$e^{ax}|v_s(t,x)| \le \kappa_{13}\theta + \|H_s\|_0 + c_{12}\int_0^t V(\tau)d\tau. \tag{3.56}$$

Combining (3.52)-(3.53) and (3.56) yields

$$V(t) \le \kappa_{14}\theta + \|H_s\|_0 + c_{13}\int_0^t V(\tau)d\tau. \tag{3.57}$$

Then, by means of Gronwall's inequality, we get

$$V(t) \le \kappa_{15}\max\{\theta, \|H\|_{C^0[0,T_0]}\}. \tag{3.58}$$

We now estimate $W(t)$.

In the present situation, for $r = 1, \cdots, m$, integrating the r-th equation in (3.50) along the r-th characteristic $C_r : \xi = \xi_r(\tau; t, x)$ passing through $(t, x) \in R(T)$ from 0 to t, instead of (3.30), we have

$$e^{ax}w_r(t,x) = e^{ax_{0r}}w_r(0,x_{0r}) + \int_0^t e^{a\xi_r}\Big(a\lambda_r(u)w_r + \sum_{j,k=1}^n \gamma_{rjk}(u)w_jw_k + \sum_{j=1}^n \tilde{\gamma}_{rj}(u)w_j\Big)(\tau,\xi_r(\tau;t,x))d\tau. \tag{3.59}$$

Then, noting (3.44) and (3.48), we get

$$e^{ax}|w_r(t,x)| \le \kappa_{15}\theta + c_{14}\int_0^t W(\tau)d\tau. \tag{3.60}$$

For $s = m+1, \cdots, n$, if the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$ passing through $(t, x) \in R(T)$ intersects the x-axis at a point $(0, x_{0s})$, integrating the s-th equation in (3.50) along C_s from 0 to t, similarly to (3.60), we have

$$e^{ax}|w_s(t,x)| \le \kappa_{16}\theta + c_{15}\int_0^t W(\tau)d\tau. \tag{3.61}$$

If the s-th characteristic $C_s : \xi = \xi_s(\tau; t, x)$ passing through $(t, x) \in R(T)$ intersects the t-axis at a point $(t_{0s}, 0)$, integrating the s-th equation in (3.50) along C_s from t_{0s} to t, instead of (3.33), we have

$$e^{ax}w_s(t,x) = w_s(t_{0s},0) + \int_{t_{0s}}^t e^{a\xi_s}\Big(a\lambda_s(u)w_s + \sum_{j,k=1}^n \gamma_{sjk}(u)w_jw_k + \sum_{j=1}^n \tilde{\gamma}_{sj}(u)w_j\Big)(\tau,\xi_s(\tau;t,x))d\tau. \tag{3.62}$$

Noting (3.41), (3.48) and (3.52), we get from (3.62) that

$$e^{ax}|w_s(t,x)| \le \kappa_{17}\big(\theta + d(u) + \|H'\|_{C^0[0,T_0]}\big) + c_{16}\int_0^t W(\tau)d\tau. \tag{3.63}$$

Combining (3.60)-(3.61) and (3.63) gives

$$W(t) \le \kappa_{18}\big(\theta + d(u) + \|H'\|_{C^0[0,T_0]}\big) + c_{17}\int_0^t W(\tau)d\tau. \tag{3.64}$$

Then, by means of Gronwall's inequality, we get

$$W(t) \le \kappa_{19}\max\{\theta, d(u) + \|H\|_{C^1[0,T_0]}\}. \tag{3.65}$$

Noting (3.7)-(3.8) and (3.40), (3.46) follows from (3.58) and (3.65).

Thus, by (3.40), (3.58) and (3.65), it is easy to see that $V(t)$ and $W(t)$ are small enough on $[0,T]$ $(0 < T \le T_0)$ when θ and $\|H\|_{C^1[0,T_0]}$ are sufficiently small. Hence, hypothesis (3.48) is reasonable.

The proof of Lemma 3.4 is complete. □

3.3. One-sided mixed initial-boundary value problem for the initial data with the polynomial decaying property

In this section, we consider the effect of the polynomial decaying property possessed by the initial data on the solution to the one-sided mixed initial-boundary value problem. The hypotheses of the regularity of all the given functions and the C^1 conditions of compatibility at the point $(0,0)$ are the same as Theorem 3.1.

Let

$$\bar{\theta} = \sup_{x\ge 0}\Big\{(1+x)^{1+\mu}\big(|\varphi(x)| + |\varphi'(x)|\big)\Big\}, \tag{3.66}$$

where μ is a nonnegative constant.

We have the following

Theorem 3.5. *For a preassigned and possibly quite large $T_0 > 0$, if $\bar{\theta}$ and $\|H_s\|_{C^1[0,T_0]}$ are suitably small (depending on T_0), then the one-sided mixed initial-boundary value problem* (2.1)*,* (2.9) *and* (3.1) *admits a unique C^1 solution $u = u(t,x)$ on the domain $R(T_0) = \{(t,x)|\ 0 \le t \le T_0, x \ge 0\}$, moreover, we have*

$$(1+x)^{1+\mu}\Big(|u(t,x)| + |u_x(t,x)|\Big) \le C(T_0), \quad \forall t \in [0,T_0],\ x \ge 0, \tag{3.67}$$

where $C(T_0)$ is a sufficiently small positive constant possibly depending on T_0.

To prove Theorem 3.5, it is sufficient to prove the following

Lemma 3.6. *Under the hypotheses of Theorem 3.5, for any C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem* (2.1)*,* (2.9) *and* (3.1) *on the domain $R(T) = \{(t,x)|0 \le t \le T, x \ge 0\}$ with $0 < T \le T_0$, we have the following uniform a priori estimate:*

$$(1+x)^{1+\mu}\Big(|u(t,x)| + |u_x(t,x)|\Big) \le C(T_0), \quad \forall t \in [0,T],\ x \ge 0, \tag{3.68}$$

where $C(T_0)$ is a sufficiently small positive constant independent of T but possibly depending on T_0.

Proof. To prove Lemma 3.6, in the present situation we replace (3.49)-(3.50) by the following

$$\frac{d((1+x)^{1+\mu}v_i)}{d_i t} = (1+x)^{1+\mu}\Big((1+\mu)\lambda_i(u)(1+x)^{-1}v_i + \sum_{j,k=1}^{n}\beta_{ijk}(u)v_j w_k + \sum_{j=1}^{n}\tilde{\beta}_{ij}(u)v_j\Big) \tag{3.69}$$

and

$$\frac{d((1+x)^{1+\mu}w_i)}{d_i t} = (1+x)^{1+\mu}\Big((1+\mu)\lambda_i(u)(1+x)^{-1}w_i + \sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j w_k + \sum_{j=1}^{n}\tilde{\gamma}_{ij}(u)w_j\Big). \tag{3.70}$$

Then, this lemma can be proved similarly to Lemma 3.4. We omit the details here. □

4. The proof of main theorems

4.1. Proof of Theorem 2.5

In this subsection, we still use the constructive method with modular structure suggested in [8] and [10] to prove Theorem 2.5. The proof is divided into the following several steps.

Step 1. From (2.14), it is easy to see that there exists an $\varepsilon_0 > 0$ so small that

$$T_1 < T, \tag{4.1}$$

where

$$T_1 = L \max_{|u|\le\varepsilon_0} \max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(u)}. \tag{4.2}$$

On the domain $R(T_1) = \{(t,x)|0 \le t \le T_1, 0 \le x \le L\}$, we solve a forward mixed initial-boundary value problem for the system (2.1) with the initial condition (2.12), the boundary condition (2.10) on $x = L$ and the following artificial boundary conditions on $x = 0$:

$$x = 0: \quad v_s = g_s(t) \quad (s = m+1, \cdots, n), \tag{4.3}$$

where $g_s(t)$ $(s = m+1, \cdots, n)$ are arbitrarily given C^1 functions with suitably small $C^1[0, T_1]$ norm, satisfying the conditions of C^1 compatibility at the point $(t, x) = (0, 0)$.

By the result about the semi-global C^1 solution to the mixed initial-boundary value problem (see [11]), this forward problem admits a unique C^1 solution $u = u_f(t, x)$ with small C^1 norm on $R(T_1)$. In particular,

$$|u_f(t,x)| \le \varepsilon_0, \quad \forall (t,x) \in R(T_1). \tag{4.4}$$

Thus, we can determine the value of $u_f(t, x)$ on $x = L$, denoted as $\bar{u}(t)$ $(0 \le t \le T_1)$ and its $C^1[0, T_1]$ norm is small.

Step 2. Noting (4.1), there exists $u(t) \in C^1$ on the time interval $[0, +\infty)$, such that

$$u(t) = \begin{cases} \bar{u}(t), & 0 \le t \le T_1, \\ \bar{\bar{u}}(t), & t \ge T, \end{cases} \tag{4.5}$$

which satisfies the boundary condition (2.10) on $t \ge 0$, and

$$\sup_{t\ge 0}\{(|u(t)| + |u'(t)|)\} \ll 1, \tag{4.6}$$

where $\bar{\bar{u}}(t)$ is given by

$$\bar{\bar{u}}(t) = \sum_{i=1}^{n} \bar{\bar{v}}_i(t) r_i(\bar{\bar{u}}(t)) \tag{4.7}$$

with

$$\bar{\bar{v}}_r(t) = G_r(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) + H_r(t) \quad (r = 1, \cdots, m).$$

In order to guarantee that (4.6) holds on the whole time interval $t \ge 0$, we only need to show that $\bar{\bar{u}}(t)$ satisfies

$$\|\bar{\bar{u}}\|_{C^1[T,+\infty)} \ll 1. \tag{4.8}$$

For the time being we assume that on the time interval $[T,+\infty)$, we have

$$|\bar{\bar{u}}(t)| \le \vartheta_0, \quad |\bar{\bar{u}}'(t)| \le \vartheta_1, \tag{4.9}$$

where ϑ_0 and ϑ_1 are suitably small positive constant. We will explain the validity of hypothesis (4.9).

By the boundary condition (2.10) and noting (2.11), for $t \ge T$ we have

$$\begin{aligned}\bar{\bar{v}}_r(t) &= G_r(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) + H_r(t)\\ &= G_r(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t)) - G_r(\alpha_r(t), 0, \cdots, 0) + H_r(t)\\ &= \sum_{s=m+1}^{n} \frac{\partial G_r}{\partial v_s}(\alpha_r(t), \nu_r\bar{\bar{v}}_{m+1}(t), \cdots, \nu_r\bar{\bar{v}}_n(t))\bar{\bar{v}}_s(t) + H_r(t) \ (\nu_r \in (0,1);\ r = 1, \cdots, m),\end{aligned} \tag{4.10}$$

where ν_r depends on t, and

$$\bar{\bar{v}}_r'(t) = \frac{\partial G_r}{\partial \alpha_r}(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t))\alpha_r'(t) + \sum_{s=m+1}^{n} \frac{\partial G_r}{\partial v_s}(\alpha_r(t), \bar{\bar{v}}_{m+1}(t), \cdots, \bar{\bar{v}}_n(t))\bar{\bar{v}}_s'(t) + H_r'(t) \ (r = 1, \cdots, m). \tag{4.11}$$

Then, noting (2.6) and (4.9), for $t \ge T$ we get

$$\begin{aligned}|\bar{\bar{u}}(t)| &= \Big|\sum_{i=1}^{n} \bar{\bar{v}}_i(t) r_i(\bar{\bar{u}}(t))\Big|\\ &\le \sum_{r=1}^{m} |\bar{\bar{v}}_r(t)| + \sum_{s=m+1}^{n} |\bar{\bar{v}}_s(t)|\\ &\le c_{26}\Big(\sum_{s=m+1}^{n} \|\bar{\bar{v}}_s\|_{C^0[T,+\infty)} + \sum_{r=1}^{m} \|H_r\|_{C^0[T,+\infty)}\Big)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}|\bar{\bar{u}}'(t)| =& \Big|\sum_{i=1}^{n} \bar{\bar{v}}_i'(t) r_i(\bar{\bar{u}}(t)) + \sum_{i=1}^{n} \bar{\bar{v}}_i(t)\Big(\sum_{j=1}^{n} \frac{\partial r_i}{\partial u_j}\bar{\bar{u}}_j'(t)\Big)\Big|\\ \le& \sum_{r=1}^{m} |\bar{\bar{v}}_r'(t)| + \sum_{s=m+1}^{n} |\bar{\bar{v}}_s'(t)| + \sum_{r=1}^{m} |\bar{\bar{v}}_r(t)| \Big|\sum_{j=1}^{n} \frac{\partial r_r}{\partial u_j}\bar{\bar{u}}_j'(t)\Big| + \sum_{s=m+1}^{n} |\bar{\bar{v}}_s(t)| \Big|\sum_{j=1}^{n} \frac{\partial r_s}{\partial u_j}\bar{\bar{u}}_j'(t)\Big|\\ \le& c_{27}\Big(\sum_{s=m+1}^{n} \|\bar{\bar{v}}_s\|_{C^1[T,+\infty)} + \sum_{r=1}^{m} \big(\|H_r\|_{C^1[T,+\infty)} + \|\alpha_r'\|_{C^0[T,+\infty)}|\big)\Big),\end{aligned} \tag{4.13}$$

where we also used that $r_i(u)$ $(i = 1, \cdots, n)$ are C^1 functions of u. Thus, noting that $\|H_r\|_{C^1[0,+\infty)}$, $\|\alpha_r\|_{C^1[0,+\infty)}$ and $\|\bar{\bar{v}}_s\|_{C^1[T,+\infty)}$ are suitably small, we get (4.8) immediately. Moreover, we also obtain that hypothesis (4.9) is reasonable.

Step 3. Noting (2.7), we exchange the role of t and x, and solve a leftward one-sided mixed initial-boundary value problem on $R = \{(t,x)|t \ge 0, 0 \le x \le L\}$ for the system (2.1) with the initial condition $u = u(t)$ $(t \ge 0)$ on $x = L$ and the boundary condition on $t = 0$, reduced from the original initial condition (2.12), namely, we solve the following one-sided mixed problem:

$$\begin{cases} \dfrac{\partial u}{\partial x} + A^{-1}(u)\dfrac{\partial u}{\partial t} = A^{-1}(u)F(u), & (4.14a)\\ x = L: \ u = u(t), \quad t \geq 0, & (4.14b)\\ t = 0: \ v_r = v_r(x) = l_r(\varphi(x))\varphi(x) \quad (r = 1, \cdots, m), \quad 0 \leq x \leq L. & (4.14c)\end{cases}$$

According to Theorem 3.1, this leftward problem admits a unique C^1 solution $u = u(t, x)$ with small C^1 norm on R. In particular, we have

$$|u(t, x)| + |u_t(t, x)| \leq \varepsilon_0, \quad \forall t \geq 0, \ 0 \leq x \leq L. \tag{4.15}$$

This C^1 solution $u = u(t, x)$ satisfies (4.14a) and (4.14b), hence it does the system (2.1) and the boundary condition (2.10) on $x = L$. Moreover, $u = u(t, x)$ also satisfies the initial condition (2.12) at $t = 0$ as we are going to show it. In fact, both $u = u(t, x)$ and $u = u_f(t, x)$ are C^1 solutions to the one-sided mixed initial-boundary value problem for the system (4.14a) with the initial condition

$$x = L : u = \bar{u}(t), \quad 0 \leq t \leq T_1 \tag{4.16}$$

and the boundary condition

$$t = 0: \ v_r = v_r(x) = l_r(\varphi(x))\varphi(x) \quad (r = 1, \cdots, m), \quad 0 \leq x \leq L. \tag{4.17}$$

The C^1 solution to the one-sided leftward mixed initial-boundary problem (4.14a) and (4.16)-(4.17) is unique on the maximum determinate domain

$$\bar{\Omega} = \{(t, x) | 0 \leq t \leq t(x), 0 \leq x \leq L\}, \tag{4.18}$$

where $t = t(x)$ denotes the downmost characteristic passing through the point $(x, t) = (L, T_1)$:

$$\begin{cases} t'(x) = \displaystyle\max_{s=m+1,\cdots,n} \frac{1}{\lambda_s(u(t(x), x))}, \\ t(L) = T_1. \end{cases} \tag{4.19}$$

Noting (4.2), (4.4) and (4.15), the triangular domain

$$\Omega = \{(t, x) | 0 \leq t \leq \frac{T_1}{L}x, 0 \leq x \leq L\} \tag{4.20}$$

must be included in the maximum determinate domain $\bar{\Omega}$ of this one-sided mixed problem. Hence, the interval $0 \leq x \leq L$ on the initial axis $t = 0$ is contained in the maximum determinate domain $\bar{\Omega}$ of this one-sided mixed problem. Thus, $u = u(t, x)$ coincides with $u = u_f(t, x)$ on the interval $\{t = 0, 0 \leq x \leq L\}$, then $u = u(t, x)$ satisfies the initial condition (2.12) at $t = 0$ (see [8, 15]).

Step 4. Substituting $u = u(t, x)$ into the boundary condition (2.9) on $x = 0$, we get the desired boundary controls $H_s(t)$ $(s = m + 1, \cdots, n)$ for $t \geq 0$, moreover, since by (2.11) we have

$$\begin{aligned} H_s(t) &= v_s(t) - G_s(\alpha_s(t), v_1(t), \cdots, v_m(t)) \\ &= v_s(t) - \Big(G_s(\alpha_s(t), v_1(t), \cdots, v_m(t)) - G_s(\alpha_1(t), 0, \cdots, 0)\Big) \\ &= v_s(t) - \sum_{r=1}^{m} \frac{\partial G_s}{\partial v_r}(\alpha_s(t), \nu_s v_1(t), \cdots, \nu_s v_m(t)) v_r(t) \ (\nu_s \in (0, 1)), \end{aligned} \tag{4.21}$$

where ν_s depends on t, and

$$H_s'(t) = v_s'(t) - \frac{\partial G_s}{\partial \alpha_s}\big(\alpha_s(t), v_1(t), \cdots, v_m(t)\big)\alpha_s'(t) - \sum_{r=1}^{m} \frac{\partial G_s}{\partial v_r}\big(\alpha_s(t), v_1(t), \cdots, v_m(t)\big)v_r'(t), \tag{4.22}$$

for $t \geq 0$ we get

$$|H_s(t)| \leq c_{28}\|u(\cdot, 0)\|_{C^0[0,+\infty)} \tag{4.23}$$

and

$$|H_s'(t)| \leq c_{29}\Big(\|u(\cdot, 0)\|_{C^1[0,+\infty)} + \|\alpha_s'\|_{C^0[0,+\infty)}\Big). \tag{4.24}$$

Thus, noting (4.15) and that $\|\alpha_s\|_{C^1[0,\infty)}$ is small, we obtain that $\|H_s\|_{C^1[0,\infty)}$ is also small.
The proof of Theorem 2.5 is finished.

4.2. **Proof of Theorem 2.7**

The proof of Theorem 2.7 is similar to that of Theorem 2.5. In what follows, we only point out some essentially different points.

In the present situation, instead of (4.6), we should ask $u(t)$ to satisfy

$$\sup_{t\geq 0}\{e^{at}(|u(t)| + |u'(t)|)\} \ll 1. \tag{4.25}$$

In order to do so, we only need to show that $\bar{\bar{u}}(t)$ satisfies

$$\sup_{t\geq T}\{e^{at}(|\bar{\bar{u}}(t)| + |\bar{\bar{u}}'(t)|)\} \ll 1. \tag{4.26}$$

By (4.10) we have

$$e^{at}|\bar{\bar{u}}(t)| \leq c_{30}e^{at}\Big(\sum_{s=m+1}^{n}|\bar{\bar{v}}_s(t)| + \sum_{r=1}^{m}|H_r(t)|\Big). \tag{4.27}$$

On the other hand, by (4.11) we have

$$e^{at}|\bar{\bar{u}}'(t)| \leq c_{31}e^{at}\Big(\sum_{s=m+1}^{n}\big(|\bar{\bar{v}}_s(t)| + |\bar{\bar{v}}_s'(t)|\big) + \sum_{r=1}^{m}\big(|H_r(t)| + |H_r'(t)| + |\alpha_r'(t)|\big)\Big). \tag{4.28}$$

Then, by (2.17) and (2.18), (4.26) follows from (4.27)-(4.28).

In the present situation, according to Theorem 3.3, the leftward one-sided mixed initial-boundary value problem (4.14) admits a unique C^1 solution $u = u(t, x)$ on R, satisfying

$$e^{at}(|u(t, x)| + |u_t(t, x)|) \leq c_{32}, \quad \forall t \geq 0,\ 0 \leq x \leq L. \tag{4.29}$$

Finally, substituting $u = u(t, x)$ into the boundary condition (2.9) on $x = 0$, we get the desired boundary controls $H_s(t)$ $(s = m + 1, \cdots, n)$ for $t \geq 0$, moreover, noting (4.21)-(4.22), we have

$$e^{at}|H_s(t)| \leq c_{33}e^{at}\Big(|v_s(t)| + \sum_{r=1}^{m}|v_r(t)|\Big) \leq c_{34} \tag{4.30}$$

and

$$e^{at}|H_s'(t)| \le c_{35}e^{at}\Big(|v_s'(t)| + |\alpha_s'(t)| + \sum_{r=1}^{m}|v_r'(t)|\Big) \le c_{36}. \tag{4.31}$$

The proof of Theorem 2.7 is finished.

4.3. **Proof of Theorem 2.8**

In order to prove Theorem 2.8, in the present situation, instead of (4.6), we should ask $u(t)$ to satisfy

$$\sup_{t\ge 0}\{(1+t)^{1+\mu}(|u(t)| + |u'(t)|)\} \ll 1. \tag{4.32}$$

Then, this theorem can be proved similarly to Theorem 2.7. We omit the details here.

5. Application

In this section, we consider the exact boundary controllability of nodal profile on a semibounded interval for Saint-Venant system for unsteady flows on a single open canal.

Suppose that the canal is horizontal and frictionless, the corresponding system can be written as (see [1,15])

$$\begin{cases} \frac{\partial A}{\partial t} + \frac{\partial (AV)}{\partial x} = 0, \\ \frac{\partial V}{\partial t} + \frac{\partial S}{\partial x} = 0, \end{cases} \quad t \ge 0, \quad 0 \le x \le L, \tag{5.1}$$

where L is the length of the canal, $A = A(t,x)$ is the area of the cross section occupied by the water at x at time t, $V = V(t,x)$ is the average velocity over the cross section and

$$S = \frac{1}{2}V^2 + gh(A) + gY_b, \tag{5.2}$$

where g is the gravity constant, constant Y_b denotes the altitude of the bed of canal, and

$$h = h(A) \tag{5.3}$$

is the depth of the water, $h(A)$ being a suitably smooth function of A, such that

$$h'(A) > 0. \tag{5.4}$$

For system (5.1), we consider a subcritical equilibrium state (A_0, V_0) which, by definition, satisfies

$$|V_0| < \sqrt{gA_0h'(A_0)}, \tag{5.5}$$

It is easy to see that, in a neighbourhood of any given subcritical equilibrium state $(A,V) = (A_0,V_0)$, (5.1) is a strictly hyperbolic system with two distinct real eigenvalues

$$\lambda_1 = V - \sqrt{gAh'(A)} < 0 < \lambda_2 = V + \sqrt{gAh'(A)} \tag{5.6}$$

and the corresponding left eigenvectors can be taken as

$$l_1 = \Big(-\sqrt{\frac{gh'(A)}{A}}, 1\Big), \quad l_2 = \Big(\sqrt{\frac{gh'(A)}{A}}, 1\Big). \tag{5.7}$$

We consider that the mixed initial-boundary value problem for system (5.1) with the initial condition

$$t = 0 : \quad (A, V) = (A_0 + \widetilde{A}_0(x), V_0 + \widetilde{V}_0(x)), \quad 0 \le x \le L \tag{5.8}$$

and the flux conditions

$$x = 0 : \quad AV = A_0V_0 + q_2(t), \tag{5.9}$$
$$x = L : \quad AV = A_0V_0 + q_1(t). \tag{5.10}$$

We introduce the Riemann invariants (r, s):

$$\begin{cases} r = \frac{1}{2}(V - V_0 - G(A)), \\ s = \frac{1}{2}(V - V_0 + G(A)), \end{cases} \tag{5.11}$$

where

$$G(A) = \int_{A_0}^{A} \sqrt{\frac{gh'(\tau)}{\tau}} d\tau. \tag{5.12}$$

Then, we have

$$\begin{cases} V = r + s + V_0, \\ A = H(s - r) > 0, \end{cases} \tag{5.13}$$

in which H is the inverse function of G and

$$H(0) = A_0 \tag{5.14}$$

and

$$H'(0) = \sqrt{\frac{A_0}{gh'(A_0)}} > 0. \tag{5.15}$$

It is easy to see that

$$(A, V) = (A_0, V_0) \Leftrightarrow (r, s) = (0, 0). \tag{5.16}$$

In a neighbourhood of (A_0, V_0), system (5.1) can be equivalently rewritten as

$$\begin{cases} \frac{\partial r}{\partial t} + \lambda_1(r, s)\frac{\partial r}{\partial x} = 0, \\ \frac{\partial s}{\partial t} + \lambda_2(r, s)\frac{\partial s}{\partial x} = 0, \end{cases} \tag{5.17}$$

where

$$\begin{cases} \lambda_1(r, s) = r + s + V_0 - \sqrt{gH(s - r)h'(H(s - r))} < 0, \\ \lambda_2(r, s) = r + s + V_0 + \sqrt{gH(s - r)h'(H(s - r))} > 0. \end{cases} \tag{5.18}$$

The boundary conditions (5.9) and (5.10) can be rewritten as

$$x = 0: \quad Q(r, s, q_2(t)) \overset{\text{def.}}{=} (r + s + V_0)H(s - r) - A_0V_0 - q_2(t) = 0, \tag{5.19}$$

$$x = L: \quad T(r, s, q_1(t)) \overset{\text{def.}}{=} (r + s + V_0)H(s - r) - A_0V_0 - q_1(t) = 0. \tag{5.20}$$

Since

$$\frac{\partial Q}{\partial s}\Big|_{(0,0)} = \sqrt{\frac{A_0}{gh'(A_0)}}\Big(\sqrt{gA_0h'(A_0)} + V_0\Big) > 0, \tag{5.21}$$

$$\frac{\partial T}{\partial r}\Big|_{(0,0)} = \sqrt{\frac{A_0}{gh'(A_0)}}\Big(\sqrt{gA_0h'(A_0)} - V_0\Big) < 0, \tag{5.22}$$

the boundary conditions (5.19) and (5.20) can be equivalently rewritten as

$$x = 0: \quad s = G_2(q_2(t), r), \tag{5.23}$$

$$x = L: \quad r = G_1(q_1(t), s) \tag{5.24}$$

with

$$G_2(0, 0) = G_1(0, 0) = 0, \tag{5.25}$$

provided that r and s are small enough. Thus, we have

$$x = 0: \quad s = \bar{G}_2(q_2(t), r) + H_2(t), \tag{5.26}$$

$$x = L: \quad r = \bar{G}_1(q_1(t), s) + H_1(t), \tag{5.27}$$

in which

$$\bar{G}_2(q_2(t), r) = G_2(q_2(t), r) - G_2(q_2(t), 0), \quad H_2(t) = G_2(q_2(t), 0) \tag{5.28}$$

and

$$\bar{G}_1(q_1(t), r) = G_1(q_1(t), r) - G_1(q_1(t), 0), \quad H_1(t) = G_1(q_1(t), 0). \tag{5.29}$$

Then, based on Theorems 2.5, 2.7 and 2.8, we have

Theorem 5.1. *Let*

$$T > \frac{L}{\lambda_2(A_0, V_0)}. \tag{5.30}$$

For any given initial data $(A_0 + \widetilde{A}_0(x), V_0 + \widetilde{V}_0(x))$ with small C^1 norm $\|(\widetilde{A}_0, \widetilde{V}_0)\|_{C^1[0,L]}$ and for any given boundary function $q_1(t)$ with small C^1 norm $\|q_1\|_{C^1[0,+\infty)}$, such that the conditions of C^1 compatibility are satisfied at the point $(t, x) = (0, L)$, if we give the nodal profiles $(\bar{\bar{A}}(t), \bar{\bar{V}}(t))$, being the value of (A, V), at the node $x = L$ on the time interval $[T, +\infty)$ with small norm $\|(\bar{\bar{A}}(\cdot) - A_0, \bar{\bar{V}}(\cdot) - V_0)\|_{C^1[T,+\infty)}$, satisfying the boundary condition (5.10), namely, $\bar{\bar{A}}(t)\bar{\bar{V}}(t) \equiv A_0V_0 + q_1(t)$ on $[T, +\infty)$, then there exists a boundary control $q_2(t)$ with small C^1 norm $\|q_2\|_{C^1[0,+\infty)}$ on the interval $[0, +\infty)$, such that the corresponding mixed initial-boundary value problem (5.1) and (5.8)-(5.10) admits a unique C^1 solution $(A, V) = (A(t, x), V(t, x))$

with small C^1 norm $\|(A(\cdot,\cdot)-A_0, V(\cdot,\cdot)-V_0)\|_{C^1(R)}$ on the domain $R=\{(t,x)|t\geq 0, 0\leq x\leq L\}$. Moreover, at the boundary node $x=L$ the solution exactly satisfies the given nodal profiles

$$A=\bar{\bar{A}}(t), \quad V=\bar{\bar{V}}(t), \quad t\geq T. \tag{5.31}$$

Theorem 5.2. *Under the condition* (5.30)*, for any given initial data $(A_0+\widetilde{A}_0(x), V_0+\widetilde{V}_0(x))$ with small C^1 norm $\|(\widetilde{A}_0,\widetilde{V}_0)\|_{C^1[0,L]}$ and for any given boundary function $q_1(t)$ with*

$$\sup_{t\geq 0}\{e^{at}(|q_1(t)|+|q_1'(t)|)\}\ll 1, \tag{5.32}$$

such that the conditions of C^1 compatibility are satisfied at the point $(t,x)=(0,L)$, if we give the nodal profiles $(\bar{\bar{A}}(t),\bar{\bar{V}}(t))$, being the value of (A,V), at the node $x=L$ on the time interval $[T,+\infty)$, which satisfies

$$\sup_{t\geq T}\{e^{at}(|\bar{\bar{A}}(t)-A_0|+|\bar{\bar{V}}(t)-V_0|+|\bar{\bar{A}}'(t)|+|\bar{\bar{V}}'(t)|)\}\ll 1 \tag{5.33}$$

and the boundary condition (5.10)*, namely, $\bar{\bar{A}}(t)\bar{\bar{V}}(t)\equiv A_0V_0+q_1(t)$ on $[T,+\infty)$, then there exists a boundary control $q_2(t)$ on the interval $[0,+\infty)$ with*

$$\sup_{t\geq 0}\{e^{at}(|q_2(t)|+|q_2'(t)|)\}\leq \bar{K}_1, \tag{5.34}$$

such that the corresponding mixed initial-boundary value problem (5.1) *and* (5.8)-(5.10) *admits a unique C^1 solution $(A,V)=(A(t,x),V(t,x))$ satisfying*

$$e^{at}\Big(|(A(t,x)-A_0, V(t,x)-V_0)|+|(A_t(t,x),V_t(t,x))|\Big)\leq \bar{K}_2, \quad \forall t\geq 0,\ 0\leq x\leq L, \tag{5.35}$$

where $\bar{K}_i$ $(i=1,2)$ are positive constants. Moreover, at the boundary node $x=L$ the solution exactly satisfies the given nodal profiles

$$A=\bar{\bar{A}}(t), \quad V=\bar{\bar{V}}(t), \quad t\geq T. \tag{5.36}$$

Theorem 5.3. *Under the condition* (5.30)*, for any given initial data $(A_0+\widetilde{A}_0(x), V_0+\widetilde{V}_0(x))$ with small C^1 norm $\|(\widetilde{A}_0,\widetilde{V}_0)\|_{C^1[0,L]}$ and for any given boundary function $q_1(t)$ with*

$$\sup_{t\geq 0}\{(1+t)^{1+\mu}(|q_1(t)|+|q_1'(t)|)\}\ll 1, \tag{5.37}$$

such that the conditions of C^1 compatibility are satisfied at the point $(t,x)=(0,L)$, if we give the nodal profiles $(\bar{\bar{A}}(t),\bar{\bar{V}}(t))$, being the value of (A,V), at the node $x=L$ on the time interval $[T,+\infty)$, which satisfies

$$\sup_{t\geq T}\{(1+t)^{1+\mu}(|\bar{\bar{A}}(t)-A_0|+|\bar{\bar{V}}(t)-V_0|+|\bar{\bar{A}}'(t)|+|\bar{\bar{V}}'(t)|)\}\ll 1 \tag{5.38}$$

and the boundary condition (5.10)*, namely, $\bar{\bar{A}}(t)\bar{\bar{V}}(t)\equiv A_0V_0+q_1(t)$ on $[T,+\infty)$, then there exists a boundary control $q_2(t)$ on the interval $[0,+\infty)$ with*

$$\sup_{t\geq 0}\{(1+t)^{1+\mu}(|q_2(t)|+|q_2'(t)|)\}\leq \bar{K}_3, \tag{5.39}$$

such that the corresponding mixed initial-boundary value problem (5.1) *and* (5.8)-(5.10) *admits a unique* C^1 *solution* $(A, V) = (A(t,x), V(t,x))$ *satisfying*

$$(1+t)^{1+\mu}\Big(|(A(t,x)-A_0, V(t,x)-V_0)| + |(A_t(t,x), V_t(t,x))|\Big) \le \bar{K}_4, \quad \forall t \ge 0,\ 0 \le x \le L, \tag{5.40}$$

where $\bar{K}_i$ $(i = 3, 4)$ *are positive constants. Moreover, at the boundary node* $x = L$ *the solution exactly satisfies the given nodal profiles*

$$A = \bar{\bar{A}}(t), \quad V = \bar{\bar{V}}(t), \quad t \ge T. \tag{5.41}$$

Remark 5.4. Noting that the boundary condition (5.26) is equivalent to (5.9) or (5.19), in Theorems 5.1, 5.2 and 5.3, the desired boundary control $q_2(t)$ can be determined by means of the boundary condition (5.9) or (5.19) instead of using directly the boundary conditions (5.26) and (5.28).

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