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# A STOCHASTIC MAXIMUM PRINCIPLE FOR SWITCHING DIFFUSIONS USING CONDITIONAL MEAN-FIELDS WITH APPLICATIONS TO CONTROL PROBLEMS *

Son L. Nguyen[1], Dung T. Nguyen[2] and George Yin[3]

**Abstract**. This paper obtains a maximum principle for switching diffusions with mean-field interactions. The motivation stems from a wide range of applications for networked control systems in which large-scale systems are encountered and mean-field interactions are involved. Because of the complexity due to the switching, little has been done for the associate control problems with mean-field interactions. The main ingredient of this work is the use of conditional mean-fields, which is distinct from the existing literature. Using the maximum principle, optimal controls of linear quadratic Gaussian controls with mean-field interactions for switching diffusions are carried out. Numerical examples are also provided for demonstration.

**Résumé**. ...

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## 1. Introduction

This work focuses on obtaining maximum principles for regime-switching diffusions with mean-field interactions. It is an effort to blend two research areas of recent interests together. One of them is mean-field-type controls and mean-field games, whereas the other is switching diffusions. Our effort stems from the need of dealing with large, complex, and networked control systems. We aim to bring about certain salient features of the systems leading to the solution of certain difficult problems, for example, certain mean-field control problems.

In the new era, the advent of sciences and technology provides us with unprecedented opportunity for dealing with "big data" and rather complex systems. Meanwhile, taking into consideration of various modeling perspectives, we face the challenges of treating large-scale systems. At the forefront, reducing computational complexity has become a pressing issue. One class of such networked systems is the mean-field models, which

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[1] Department of Mathematics, University of Puerto Rico, Rio Piedras campus, San Juan, PR 00936, USA, sonluu.nguyen@upr.edu.

[2] Department of Applied Mathematics, Faculty of Applied Science, University of Technology - Vietnam National University, Ho Chi Minh city, Vietnam, dungnt@hcmut.edu.vn.

[3] Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, gyin@wayne.edu.

were originated from statistical mechanics and physics (for instance, in the derivation of Boltzmann or Vlasov equations in kinetic gas theory) concerning the interactions of many particles. To reduce the complexity of interactions owing to a large number of particles (or many bodies in the language of statistical physics), all interactions to a particle are replaced by a single averaged interaction. Studying the limits of mean-field models has been a long-standing problem and presents many technical challenges. Some of the questions were concerned with characterization of the limit of the empirical probability distribution of the systems when the size of the systems tend to infinity, the fluctuations and large deviations of the systems around the limit. Perhaps, the first work along this line was due to McKean [22]. Subsequently, the problems were investigated in different contexts by many researchers in [3, 6–9, 32, 33] and references therein. In 1980s, Dawson established the foundation by obtaining law of large numbers, central limit theorem, as well as phase transition, etc. [6], and Dawson and Gärtnet proved large deviations results in [7]; see also related work of Gärtner [9]. One may find a systematic introduction to the topic and related problems in [32]. In recent years, such systems gained resurgent interests and enjoyed a wide range applications from cyber-physical systems, to platoon control of autonomous vehicles, finance, intelligent power systems, battery management, and many others. Especially, in the areas of control and optimization, there have been great interests in considering mean-field games and mean-field control problems; see for example, [1, 11–14, 16–18, 24, 30] among others.

Along another line, it has been recognized that traditional models using continuous processes alone represented by solutions to deterministic differential equations and stochastic differential equations are often inadequate in many real-world applications. Arising from control engineering, queueing networks, manufacturing and production planning, parameter estimation, multi-agent systems, filtering of dynamic systems, ecological and biological systems, and finance and economics, etc., numerous complex systems contain both continuous dynamics and discrete events. Because of the demand, switching diffusions (also known as hybrid switching diffusions) have drawn growing and resurgent attention. A switching diffusion is a two-component process $\big(X_t, \alpha_t\big)_{t\geq 0}$ in which the continuous component $X_t$ evolves according to the diffusion process whose drift and diffusion coefficients depend on the state of $\alpha_t$, whereas $\alpha_t$ takes values in a set consisting of isolated points. In their comprehensive treatment of hybrid switching diffusions, Mao and Yuan [21] focused on $\alpha(\cdot)$ being a continuous-time and homogeneous Markov chain independent of the Brownian motion and the generator of the Markov chain being a constant matrix. Realizing the need, treating the two components jointly, Yin and Zhu [40] extended the study to the Markov process $\big(X_t, \alpha_t\big)_{t\geq 0}$ by allowing the generator of $\alpha(\cdot)$ to depend on the current state $X_t$. In the past decades, such switch diffusions have been studied extensively. Many results have been obtained. To date, switching diffusions have been used a wide variety of applications.

Naturally, the studies on mean-field models for switching diffusions have also been examined; see for example [37]. Moreover, treating mean-field games with many particles or multi-agents and random switching was considered in Wang and Zhang [35], in which it requires to have a Markov chain for each particle. Thus, there are as many switching processes as that of particles and all of these Markov chains are independent. In the very recent work [43], Zhang, Sun, and Xiong studied a mean-field control problem for a general model including both switching and jump. In the dynamics of this control problem, the term $\mathbb{E}(X_t)$ is used to represent the mean-field term of the controlled jump-diffusion system with switching. Even though it was not explicitly mentioned in the reference, the system considered there is the limit of finite population of weakly interacting jump-diffusion systems with Markovian switching. In addition, in the prelimit of the weakly interacting systems, all the particles contain independent switching processes or Markov chains. This model, therefore, is in some sense similar to what was used in [35].

Nevertheless, for many applications such as in consensus problems [39], wireless communications [38], social networks [10], and financial market [2], there is only one switching process through which all the particles are correlated. This switching process serves as a modulating force. Such a case is more relevant, better reflects the reality, and is appropriate for networked systems, but cannot be handled by the works mentioned previously. The main reason is that in the limit of the mean-field term is not represented by an expected value or deterministic function but a random process, namely, a time-dependent conditional mean.

Although mean-field games and mean-field-type of controls have received much needed attention in recent years, the study for mean-field control and games with regime-switching is still largely in an early stage. In fact, not much effort has been devoted to such problems. One of the exception is [36], in which a LQG social optimal problem with mean-field term was considered. The effort was to approximate the mean-field term $x_t^{(N)}$ in the finite population model. By taking $N$ large enough, the Brownian motion is averaged out and the limit becomes a switched ODE than a switching diffusion. It would be worth noting that the process is freezed to approximate the mean-field term in the associated limit control problem. It only plays a role as a random coefficient in the limit control problem and is not affected by the control.

In contrast to the existing work, this paper is devoted to the study of maximum principle for switching diffusions with mean-field interactions. The difficulties as alluded to in the previous paragraph involve not only the mean-field interactions, but also the "correlation" due to the modulating Markov chain. The key approach that we are taking is the use of conditional mean. It is often regarded that a maximum principle is largely of theoretical value. In this paper, we show that the desired result, in fact, leads to computable control strategies. For simplicity and ease of notation, we only consider the problem in one-dimensional setting. However, the results can be easily extended to the multidimensional case without much difficulty but with much more complex notation.

We note that in [4], mean-field control problems with partial observation is treated. The idea of converting a partially observable system into a fully observable system is used. Then naturally, a conditional mean-field is used with the conditioning on the observations. In [5], a probabilistic approach for mean-field games with major and minor players was given. A conditional mean-field with condition on the major player is used. Similar models leading to the use of conditional mean fields such as systems with a common noise and leader-follower problems can be found in [27, 31]. In this paper, we treat mean-field controls of switching diffusions. The treatment of maximum principle also uses the backward stochastic differential equation approach as in [26]. The main difference from [4], [5], [27], and [31] is that the conditional mean field comes from the switching diffusion itself. As demonstrated in [25], the limit of the corresponding empirical measures is not deterministic but a random measure that depends on the history of the Markovian switching process. In fact, we showed that the limit distribution is a conditional law depending the Markov chain, which turns out to be the unique solution of a stochastic McKean-Vlasov equation. This dictates our setup in this paper and the use of conditional mean field.

The rest of the paper is arranged as follows. Section 2 begins with the precise formulation of the problem. Section 3 obtains necessary and sufficient conditions for optimality. Section 4 is devoted to a network of linear quadratic regulators modulated by a random switching process. Our goal is to obtain the optimal control of this problem. Also presented in this section is a numerical example, which demonstrates from another angle that our approach is computationally feasible. Finally, Section 5 makes some final remarks. An appendix collecting some technical results are provided at the end of the paper.

## 2. Formulation

We work with a finite time horizon $[0, T]$ for some $T > 0$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})_{t\geq 0}$ be a filtered probability space satisfying the usual conditions. To get the main ideas across, throughout the paper, we choose to work with scalar system. This is no loss of generality. The same approach applies to vector-valued systems. The notation, however, is much more complex. It seems to be a better idea to present the main features of the problem rather than getting bogged down with complex notations. To proceed, let $(W_t)_{t\geq 0}$ be a 1-dimensional standard Brownian motion and $(\alpha_t)_{t\geq 0}$ a continuous-time Markov chain defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Assume the Markov chain $(\alpha_t)_{t\geq 0}$ takes values in a finite state space $\mathcal{M} = \{1, 2, \dots, m_0\}$ with a generator $Q = \big(q_{i_0,j_0}\big)_{i_0,j_0\in\mathcal{M}}$ satisfying $q_{i_0 j_0} \geq 0$ for $i_0 \neq j_0 \in \mathcal{M}$ and $\sum_{j_0\in\mathcal{M}} q_{i_0 j_0} = 0$ for each $i_0 \in \mathcal{M}$. For $t > 0$, denote $\mathcal{F}_t^{\alpha} = \sigma\big\{\alpha_s : 0 \leq s \leq t\big\}$, $\mathcal{F}_t^{W} = \sigma\big\{W_s : 0 \leq s \leq t\big\}$, and $\mathcal{F}_t^{W,\alpha} = \sigma\big\{W_s, \alpha_s : 0 \leq s \leq t\big\}$. For simplicity, we can assume $\mathcal{F}_t = \mathcal{F}_t^{W,\alpha}$. Consider $b(\cdot,\cdot,\cdot,\cdot,\cdot) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$, $\sigma(\cdot,\cdot,\cdot,\cdot,\cdot) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{M} \to \mathbb{R}$, and $\psi(\cdot), \varphi(\cdot), \phi(\cdot), \eta(\cdot) : \mathbb{R} \to \mathbb{R}$. Let $U$, the action space, be a non-empty and convex subset of $\mathbb{R}$, and $\mathcal{U}$ be the class of measurable, $\mathcal{F}_t$-adapted, and square integrable processes $u(\cdot,\cdot):[0,T]\times\Omega\to U$. We call $\mathcal{U}$ the set of admissible controls.

For each $u(\cdot)\in\mathcal{U}$, we consider the following stochastic differential equation

$$\begin{aligned} dX_t &= b\Big(t, X_t, \mathbb{E}\big(\psi(X_t)\big|\mathcal{F}^{\alpha}_{t-}\big), u_t, \alpha_{t-}\Big)dt + \sigma\Big(t, X_t, \mathbb{E}\big(\varphi(X_t)\big|\mathcal{F}^{\alpha}_{t-}\big), u_t, \alpha_{t-}\Big)dW_t, \\ X_0 &= x_0, \end{aligned} \tag{2.1}$$

where $x_0$ is a real number. This mean-field SDE is obtained as the mean-square limit as $n\to\infty$ of a system of interacting particles of the form

$$dX^{i,n}_t = b\Big(t, X^{i,n}_t, \frac{1}{n}\sum_{i=1}^{n}\psi(X^{i,n}_t), u_t, \alpha_{t-}\Big)dt + \sigma\Big(t, X^{i,n}_t, \frac{1}{n}\sum_{i=1}^{n}\varphi(X^{i,n}_t), u_t, \alpha_{t-}\Big)dW^i_t,$$

where $\big(W^i(\cdot), i\ge 1\big)$ is a collection of independent standard Brownian motions. Note that for more generality, we consider the mean-field term as nonlinear functions of the state with the use of $\psi(\cdot)$ and $\varphi(\cdot)$, respectively. Moreover, in (2.1), the conditional expectations $\mathbb{E}\big(\psi\big(X_t\big)\big|\mathcal{F}^{\alpha}_{t-}\big)$ and $\mathbb{E}\big(\varphi\big(X_t\big)\big|\mathcal{F}^{\alpha}_{t-}\big)$ appear instead of the expectations $\mathbb{E}\big(\psi(X_t)\big)$ and $\mathbb{E}\big(\varphi(X_t)\big)$ because of the effect of the common switching process $(\alpha_t)_{t\ge 0}$. Because all the particles depend on the history of this process, their average (mean-field term) must depend on the history of $\alpha(\cdot)$; see [25]. Note also that in (2.1), in lieu of the usual formulation using the mean of $X_t$, we used the conditional expectations $\mathbb{E}(\psi(X_t)|\mathcal{F}^{\alpha}_{t-})$, $\mathbb{E}(\varphi(X_t)|\mathcal{F}^{\alpha}_{t-})$, $\mathbb{E}(\phi(X_t)|\mathcal{F}^{\alpha}_{t-})$, $\mathbb{E}(\eta(X_t)|\mathcal{F}^{\alpha}_{t-})$, which is more general. This is because we often need to consider the conditional moments beyond the usual consideration of mean (e.g., second moment, variance, as well as higher moments) in such applications as cyber-physical systems and social network modeling. The boundedness essentially ensures the finite moment, which can be done by using the usual truncated moments analogous to the treatment of diffusion processes.

Our control problem consists of minimizing the cost functional given by

$$J(u) = \mathbb{E}\Bigg[\int_0^T h\Big(t, X_t, \mathbb{E}\big(\phi(X_t)\big|\mathcal{F}^{\alpha}_{t-}\big), u_t, \alpha_{t-}\Big)dt + g\Big(X_T, \mathbb{E}\big(\eta(X_T)\big|\mathcal{F}^{\alpha}_{T-}\big), \alpha_T\Big)\Bigg]. \tag{2.2}$$

This cost functional is also of mean-field type, but unlike [35, 43], in lieu of a deterministic function $\mathbb{E}(X_t)$, the conditional mean-field terms are used because of the appearance of $\mathbb{E}\big(\phi\big(X_t\big)\big|\mathcal{F}^{\alpha}_{t-}\big)$ and $\mathbb{E}\big(\eta\big(X_T\big)\big|\mathcal{F}^{\alpha}_{T-}\big)$. As explained above, these mean-field terms are the conditional expectations with respect to the history of the switching process because in our finite population model, the switching process $\alpha_t$ is common to all of the particles, and thus affects all of them.

Throughout the paper we make the following assumptions

**Assumption A.**

(A1) The functions $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, and $\eta(\cdot)$ are continuously differentiable; $g(\cdot,\cdot,i_0)$ is continuously differentiable with respect to $(x,y)$; $b(\cdot,\cdot,\cdot,\cdot,i_0), \sigma(\cdot,\cdot,\cdot,\cdot,i_0)$, and $h(\cdot,\cdot,\cdot,\cdot,i_0)$ are continuous in $t$ and continuously differentiable with respect to $(x,y,u)$.

(A2) In (A1), for each $t$ and $i_0$, all derivatives of $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, $g(\cdot,\cdot,i_0)$, $b(t,\cdot,\cdot,\cdot,i_0)$, $\sigma(t,\cdot,\cdot,\cdot,i_0)$, and $h(t,\cdot,\cdot,\cdot,i_0)$ with respect to $x$, $y$, and $u$ are Lipschitz continuous and bounded.

The existence and uniqueness of (2.1) is given in the following lemma. The proof is relegate to the appendix.

**Lemma 2.1.** *Under Assumption* (A)*, for each admissible control* $u(\cdot)\in\mathcal{U}$*, equation* (2.1) *has a unique solution.*

To proceed, associated with each pair $(i_0,j_0)\in\mathcal{M}\times\mathcal{M}$, $i_0\ne j_0$, define

$$\big[M_{i_0j_0}\big](t) = \sum_{0\le s\le t}\mathbb{1}\big(\alpha(s_-)=i_0\big)\mathbb{1}\big(\alpha(s)=j_0\big), \qquad \big\langle M_{i_0j_0}\big\rangle(t) = \int_0^t q_{i_0j_0}\mathbb{1}\big(\alpha(s_-)=i_0\big)ds, \tag{2.3}$$

where $\mathbb{1}$ denotes the usual zero-one indicator function. It follows from Lemma IV.21.12 in [29] that the process $M_{i_0j_0}(t)$, $0 \le t \le T$, defined by

$$M_{i_0j_0}(t) = \big[M_{i_0j_0}\big](t) - \big\langle M_{i_0j_0}\big\rangle(t) \tag{2.4}$$

is a purely discontinuous and square integrable martingale with respect to $\mathcal{F}_t^\alpha$, which is null at the origin. The processes $[M_{i_0j_0}](t)$ and $\langle M_{i_0j_0}\rangle(t)$ are its optional and predictable quadratic variations, respectively. For convenience, we define

$$M_{i_0i_0}(t) = \big[M_{i_0i_0}\big](t) = \big\langle M_{i_0i_0}\big\rangle(t) = 0, \quad \text{for each} \;\; i_0 \in \mathcal{M} \;\; \text{and} \;\; t \ge 0.$$

From the definition of optional quadratic covariations (see [20]), we have the following orthogonality relations:

$$\big[M_{i_0j_0}, W\big] = 0, \quad \big[M_{i_0j_0}, M_{i_1j_1}\big] = 0 \text{ when } (i_0, j_0) \ne (i_1, j_1). \tag{2.5}$$

With the setup given above, our objective is to obtain a maximum principle.

# 3. Necessary and Sufficient Conditions for Optimality

## 3.1. Taylor Expansions

For an admissible control $u(\cdot)$, denote the corresponding trajectory of (2.1) by $X^u(\cdot)$. In particular, if $\bar{u}(\cdot)$ is an optimal control, then $X^{\bar{u}}(\cdot)$ is the associated optimal trajectory. Define the perturbed control as follow

$$u_t^\theta = \bar{u}_t + \theta\big(v_t - \bar{u}_t\big), \quad 0 \le t \le T, \tag{3.1}$$

where $\theta \ge 0$ and $v(\cdot)$ is an arbitrary admissible control in $\mathcal{U}$. For simplicity, we shall use $X^\theta(\cdot)$ to denote the trajectory corresponding to the control $u^\theta(\cdot)$. Denote

$$\begin{aligned}
\bar{b}(t) &= b\Big(t, X_t^{\bar{u}}, \mathbb{E}\big(\psi\big(X_t^{\bar{u}}\big)\big|\mathcal{F}_{t-}^\alpha\big), \bar{u}_t, \alpha_{t-}\Big),\\
\bar{\sigma}(t) &= \sigma\Big(t, X_t^{\bar{u}}, \mathbb{E}\big(\varphi\big(X_t^{\bar{u}}\big)\big|\mathcal{F}_{t-}^\alpha\big), \bar{u}_t, \alpha_{t-}\Big),\\
\bar{h}(t) &= h\Big(t, X_t^{\bar{u}}, \mathbb{E}\big(\phi\big(X_t^{\bar{u}}\big)\big|\mathcal{F}_{t-}^\alpha\big), \bar{u}_t, \alpha_{t-}\Big),\\
\bar{g}(t) &= g\Big(X_t^{\bar{u}}, \mathbb{E}\big(\eta\big(X_t^{\bar{u}}\big)\big|\mathcal{F}_{t-}^\alpha\big), \alpha_t\Big),\\
\bar{\phi}(t) &= \phi\big(X_t^{\bar{u}}\big),\; \bar{\varphi}(t) = \varphi\big(X_t^{\bar{u}}\big),\; \bar{\psi}(t) = \psi\big(X_t^{\bar{u}}\big),\; \bar{\eta}(t) = \eta\big(X_t^{\bar{u}}\big).
\end{aligned}$$

To proceed, we first consider a sensitivity result since we are interested in how close the trajectories $X^\theta(\cdot)$ and $X^{\bar{u}}(\cdot)$ to each other. In what follows, we use $b_x$, $b_y$, and $b_u$ to denote the partial derivatives of $b$ with respective to $x$, $y$, and $u$, respectively. Similar notation will be used for other functions such as $\sigma, g, \phi, \psi, \bar{b}, \bar{\sigma}, \bar{g}, \bar{\phi}, \bar{\psi}, \ldots$. We have the following lemma, which implies $X_t^{\bar{u}}$ is the uniform mean square limit of $X_t^\theta$ as $\theta \to 0$.

**Lemma 3.1.** *Let $X^\theta(\cdot)$ and $X^{\bar{u}}(\cdot)$ be the trajectories of* (2.1) *corresponding to controls $u^\theta(\cdot)$ and $\bar{u}(\cdot)$, respectively. Then there exists a constant $K$ such that for all $\theta > 0$,*

$$\mathbb{E}\left[\sup_{0 \le t \le T} \big|X_t^\theta - X_t^{\bar{u}}\big|^2\right] \le K\theta^2.$$

*Proof.* By the Burkholder-Davis-Gundy inequality, Assumption (A), and (3.1), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le T}\big|X_s^\theta - X_s^{\bar u}\big|^2\Big]\\
&\le 2T\mathbb{E}\bigg[\int_0^t \Big|b\Big(s, X_s^\theta, \mathbb{E}\big(\psi(X_s^\theta)\big|\mathcal{F}_{s-}^\alpha\big), u_s^\theta, \alpha_{s-}\Big) - b\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi(X_s^{\bar u})\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s, \alpha_{s-}\Big)\Big|^2 ds\bigg]\\
&\quad + 8T\mathbb{E}\bigg[\int_0^t \Big|\sigma\Big(s, X_s^\theta, \mathbb{E}\big(\psi(X_s^\theta)\big|\mathcal{F}_{s-}^\alpha\big), u_s^\theta, \alpha_{s-}\Big) - \sigma\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi(X_s^{\bar u})\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s, \alpha_{s-}\Big)\Big|^2 ds\bigg]\\
&\le K\mathbb{E}\bigg[\int_0^t \Big(\big|X_s^\theta - X_s^{\bar u}\big|^2 + \mathbb{E}\Big(\big|\psi(X_s^\theta) - \psi(X_s^{\bar u})\big|^2\Big|\mathcal{F}_{s-}^\alpha\Big) + \big|u_s^\theta - \bar u_s\big|^2\Big) ds\bigg]\\
&\le K\mathbb{E}\bigg[\int_0^t \Big(\big|X_s^\theta - X_s^{\bar u}\big|^2 + \theta^2\big|v_s - \bar u_s\big|^2\Big) ds\bigg], \quad 0\le t\le T.
\end{aligned}$$

In virtue of Gronwall's inequality, we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|X_t^\theta - X_t^{\bar u}\big|^2\Big] \le \theta^2 K e^{KT}\mathbb{E}\bigg[\int_0^T \big|v_t - \bar u_t\big|^2 dt\bigg],$$

which leads to the desired inequality. □

Next, we will explore the "higher order" sensitivity. The following lemma shows that $z(\cdot)$, the solution given by (3.2), is nothing but the uniform mean square derivative of $X^\theta(\cdot)$ with respect to $\theta$ at $\theta = 0$.

**Lemma 3.2.** *Let $z(\cdot)$ be the solution to the following linear equation*

$$\begin{aligned}
dz_t &= \Big[\bar b_x(t)z_t + \bar b_y(t)\mathbb{E}\Big(\bar\psi_x(t)z_t\Big|\mathcal{F}_{t-}^\alpha\Big) + \bar b_u(t)\big(v_t - \bar u_t\big)\Big]dt\\
&\quad + \Big[\bar\sigma_x(t)z_t + \bar\sigma_y(t)\mathbb{E}\Big(\bar\varphi_x(t)z_t\Big|\mathcal{F}_{t-}^\alpha\Big) + \bar\sigma_u(t)\big(v_t - \bar u_t\big)\Big]dW_t,\\
z_0 &= 0.
\end{aligned}\tag{3.2}$$

*Then*

$$\lim_{\theta\to 0^+}\mathbb{E}\bigg[\sup_{0\le s\le T}\bigg|\frac{X_s^\theta - X_s^{\bar u}}{\theta} - z_s\bigg|^2\bigg] = 0.$$

*Proof.* According to Assumption (A), all the coefficients of (3.2) are bounded. Thus, by Gronwall's inequality, it can be shown that (3.2) has a unique solution $z(\cdot)$ satisfying

$$\mathbb{E}\Big(\sup_{0\le s\le T}|z_s|^2\Big) < \infty.$$

For $t\ge 0$, we put

$$y_t^\theta = \frac{X_t^\theta - X_t^{\bar u}}{\theta} - z_t.$$

It follows that $X_t^\theta = X_t^{\bar u} + \theta(y_t^\theta + z_t)$ and $y_0^\theta = 0$. In view of Lemma 3.1,

$$\mathbb{E}\Big(\sup_{0\le s\le T}|y_s^\theta|^2\Big) < \infty \ \text{ for each } \ \theta > 0.$$

In addition,

$$
\begin{aligned}
y_t^\theta = & \Bigg\{ \frac{1}{\theta} \int_0^t \Big[ b\Big(s, X_s^{\bar u} + \theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + \theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + \theta(v_s - \bar u_s), \alpha_{s-}\Big) - \bar b(s)\Big] ds \\
& - \int_0^t \Big[ \bar b_x(s) z_s + \bar b_y(s)\mathbb{E}\big(\bar\psi_x(s) z_s\big|\mathcal{F}_{s-}^\alpha\big) + \bar b_u(s)\big(v_s - \bar u_s\big)\Big] ds \Bigg\} \\
& + \Bigg\{ \frac{1}{\theta} \int_0^t \Big[ \sigma\Big(s, X_s^{\bar u} + \theta(y_s^\theta + z_s), \mathbb{E}\big(\varphi\big(X_s^{\bar u} + \theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + \theta(v_s - \bar u_s), \alpha_{s-}\Big) - \bar\sigma(s)\Big] dW_s \\
& - \int_0^t \Big[ \bar\sigma_x(s) z_s + \bar\sigma_y(s)\mathbb{E}\big(\bar\varphi_x(s) z_s\big|\mathcal{F}_{s-}^\alpha\big) + \bar\sigma_u(s)\big(v_s - \bar u_s\big)\Big] dW_s \Bigg\} \\
=: & \, I^\theta(t) + J^\theta(t).
\end{aligned}
$$

To estimate $I^\theta(t)$, we observe that

$$
\begin{aligned}
& \frac{d}{dr} b\Big(\cdot, X_s^{\bar u} + r\theta(y_s^\theta + z_s), \cdot, \cdot, \cdot\Big) = b_x\Big(\cdot, X_s^{\bar u} + r\theta(y_s^\theta + z_s), \cdot, \cdot, \cdot\Big)\theta(y_s^\theta + z_s), \\
& \frac{d}{dr} b\Big(\cdot, \cdot, \mathbb{E}\big(\psi\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \cdot, \cdot\Big) \\
& = b_y\Big(\cdot, \cdot, \mathbb{E}\big(\psi\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \cdot, \cdot\Big)\mathbb{E}\Big(\psi_x\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\theta(y_s^\theta + z_s)\Big|\mathcal{F}_{s-}^\alpha\Big), \\
& \frac{d}{dr} b\Big(\cdot, \cdot, \cdot, \bar u_s + r\theta(v_s - \bar u_s), \cdot\Big) = b_u\Big(\cdot, \cdot, \cdot, \bar u_s + r\theta(v_s - \bar u_s), \cdot\Big)\theta(v_s - \bar u_s).
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
& \frac{1}{\theta} \int_0^t \Big[ b\Big(s, X_s^{\bar u} + \theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + \theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + \theta(v_s - \bar u_s), \alpha_{s-}\Big) - \bar b(s)\Big] ds \\
& = \int_0^t \int_0^1 b_x\Big(s, X_s^{\bar u} + r\theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + r\theta(v_s - \bar u_s), \alpha_{s-}\Big)(y_s^\theta + z_s) dr ds \\
& + \int_0^t \int_0^1 b_y\Big(s, X_s^{\bar u} + r\theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + r\theta(v_s - \bar u_s), \alpha_{s-}\Big) \\
& \quad \times \mathbb{E}\Big(\psi_x\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)(y_s^\theta + z_s)\Big|\mathcal{F}_{s-}^\alpha\Big) dr ds \\
& + \int_0^t \int_0^1 b_u\Big(s, X_s^{\bar u} + r\theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + r\theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + r\theta(v_s - \bar u_s), \alpha_{s-}\Big)(v_s - \bar u_s) dr ds.
\end{aligned}
$$

For notational convenience, denote $X_t^{r,\theta} = X_t^{\bar u} + r\theta(y_t^\theta + z_t)$ and $u_t^{r,\theta} = \bar u_t + r\theta(v_t - \bar u_t)$. We have

$$\begin{aligned}
I^\theta(t) &= \frac{1}{\theta}\int_0^t \Big[b\Big(s, X_s^{\bar u} + \theta(y_s^\theta + z_s), \mathbb{E}\big(\psi\big(X_s^{\bar u} + \theta(y_s^\theta + z_s)\big)\big|\mathcal{F}_{s-}^\alpha\big), \bar u_s + \theta(v_s - \bar u_s), \alpha_{s-}\Big) - \bar b(s)\Big]ds \\
&\quad - \int_0^t \Big[\bar b_x(s) z_s + \bar b_y(s)\mathbb{E}\big(\bar\psi_x(s) z_s\big|\mathcal{F}_{s-}^\alpha\big) + \bar b_u(s)\big(v_s - \bar u_s\big)\Big]ds \\
&= \int_0^t\int_0^1 b_x\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) y_s^\theta drds \\
&\quad + \int_0^t\int_0^1 b_y\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big)\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) y_s^\theta\Big|\mathcal{F}_{s-}^\alpha\Big) drds \\
&\quad + \int_0^t\int_0^1 \Big[b_x\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) - \bar b_x(s)\Big] z_s drds \\
&\quad + \int_0^t\int_0^1 \Big[b_y\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big)\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) \\
&\qquad\qquad - \bar b_y(s)\mathbb{E}\big(\psi_x\big(X_s^{\bar u}\big) z_s\big|\mathcal{F}_{s-}^\alpha\big)\Big] drds \\
&\quad + \int_0^t\int_0^1 \Big[b_u\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) - \bar b_u(s)\Big](v_s - \bar u_s) drds \\
&= I_1^\theta(t) + I_2^\theta(t) + I_3^\theta(t) + I_4^\theta(t) + I_5^\theta(t).
\end{aligned}$$

To proceed, we prove that the forth term $I_4^\theta(t)$ in the right-hand side tends to 0 in $L^2[0,T]$ as $\theta \to 0$. Then, in a similar way, it follows that all the last three terms $I_3^\theta(t)$, $I_4^\theta(t)$, and $I_5^\theta(t)$ tend to 0 in $L^2[0,T]$ as $\theta \to 0$. We have

$$\begin{aligned}
I_4^\theta(t) &:= \int_0^t\int_0^1 \Big[b_y\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big)\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) \\
&\qquad - \bar b_y(s)\mathbb{E}\Big(\psi_x\big(X_s^{\bar u}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big)\Big] drds \\
&= \int_0^t\int_0^1 \Big[b_y\Big(s, X_s^{r,\theta}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) \\
&\qquad - b_y\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big)\Big]\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) drds \\
&\quad + \int_0^t\int_0^1 \Big[b_y\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi\big(X_s^{r,\theta}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) \\
&\qquad - b_y\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi\big(X_s^{\bar u}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big)\Big]\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) drds \\
&\quad + \int_0^t\int_0^1 \Big[b_y\Big(s, X_s^{\bar u}, \mathbb{E}\big(\psi\big(X_s^{\bar u}\big)\big|\mathcal{F}_{s-}^\alpha\big), u_s^{r,\theta}, \alpha_{s-}\Big) - \bar b_y(s)\Big]\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) drds \\
&\quad + \int_0^t\int_0^1 \bar b_y(s)\Big[\mathbb{E}\Big(\psi_x\big(X_s^{r,\theta}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big) - \mathbb{E}\Big(\psi_x\big(X_s^{\bar u}\big) z_s\Big|\mathcal{F}_{s-}^\alpha\Big)\Big] drds \\
&=: I_{4,1}^\theta(t) + I_{4,2}^\theta(t) + I_{4,3}^\theta(t) + I_{4,4}^\theta(t).
\end{aligned}$$

Since the derivatives $\psi_x$ and $b_y$ are all Lipschitz and bounded, by the Cauchy-Schwarz inequality, there is a constant $K$ such that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}\big|I^\theta_{4,2}(t)\big|^2\Big)\\
&\le \left(\mathbb{E}\int_0^T\int_0^1\Big|b_y\Big(s,X^{\bar u}_s,\mathbb{E}\big(\psi(X^{r,\theta}_s)\big|\mathcal{F}^\alpha_{s-}\big),u^{r,\theta}_s,\alpha_{s-}\Big)-b_y\Big(s,X^{\bar u}_s,\mathbb{E}\big(\psi(X^{\bar u}_s)\big|\mathcal{F}^\alpha_{s-}\big),u^{r,\theta}_s,\alpha_{s-}\Big)\Big|^2drds\right)\\
&\qquad\times\left(\mathbb{E}\int_0^T\int_0^1\Big|\mathbb{E}\Big(\psi_x(X^{r,\theta}_s)z_s\Big|\mathcal{F}^\alpha_{s-}\Big)\Big|^2drds\right)\\
&\le K\left(\mathbb{E}\int_0^T\int_0^1\Big|\mathbb{E}\Big(\psi(X^{r,\theta}_s)-\psi(X^{\bar u}_s)\Big|\mathcal{F}^\alpha_{s-}\Big)\Big|^2drds\right)\left(\mathbb{E}\int_0^T\int_0^1\Big|\mathbb{E}\Big(z_s\Big|\mathcal{F}^\alpha_{s-}\Big)\Big|^2drds\right)\\
&\le K\left(\mathbb{E}\int_0^T\int_0^1\mathbb{E}\Big(\big|X^{r,\theta}_s-X^{\bar u}_s\big|^2\Big|\mathcal{F}^\alpha_{s-}\Big)drds\right)\left(\mathbb{E}\int_0^T\int_0^1\mathbb{E}\Big(|z_s|^2\Big|\mathcal{F}^\alpha_{s-}\Big)drds\right)\\
&=K\left(\int_0^T\int_0^1\mathbb{E}\Big|r\theta\big(y^\theta_t+z_t\big)\Big|^2drdt\right)\left(\int_0^T\int_0^1\mathbb{E}|z_t|^2drdt\right).
\end{aligned}$$

Likewise, we can derive similar estimates for $I^\theta_{4,1}(T)$, $I^\theta_{4,3}(T)$, and $I^\theta_{4,4}(T)$, and obtain the following inequality

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}\big|I^\theta_4(t)\big|^2\Big)\le K&\left(\int_0^T\int_0^1\mathbb{E}\big|r\theta\big(y^\theta_t+z_t\big)\big|^2drdt\right)\left(\int_0^T\mathbb{E}\big|z_t\big|^2dt\right)\\
&+K\left(\int_0^T\int_0^1\mathbb{E}\big|r\theta v_t\big|^2drdt\right)\left(\int_0^T\mathbb{E}|z_t|^2dt\right)=:\rho(\theta).
\end{aligned}$$

Similarly, we can obtain the same estimates for $I^\theta_3(t)$ and $I^\theta_5(t)$. In addition, by the boundedness of $b_x$, $b_y$, and $\psi_x$, we can prove that

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big|I^\theta_1(t)\big|^2+\sup_{0\le t\le T}\big|I^\theta_2(t)\big|^2\Big)\le K\left[\int_0^T\mathbb{E}\Big(\sup_{0\le s\le t}\big|y^\theta_s\big|^2\Big)dt\right].$$

Combining these inequalities yields

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big|I^\theta(t)\big|^2\Big)\le K\left[\int_0^T\mathbb{E}\Big(\sup_{0\le s\le t}\big|y^\theta_s\big|^2\Big)dt+\rho(\theta)\right].$$

Next, by using the Burkholder-Davis-Gundy inequality and similar estimates for the diffusion terms in $J^\theta(t)$, we arrive at

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big|y^\theta(t)\big|^2\Big)\le 2\mathbb{E}\Big(\sup_{0\le t\le T}\big|I^\theta(t)\big|^2+\big|J^\theta(t)\big|^2\Big)\le K\left[\int_0^T\mathbb{E}\Big(\sup_{0\le s\le t}\big|y^\theta_s\big|^2\Big)dt+\rho(\theta)\right].$$

According to Gronwall's inequality, $\mathbb{E}\Big(\sup_{0\le t\le T}\big|y^\theta(t)\big|^2\Big)\le K\rho(\theta)e^{KT}$. Since $\lim_{\theta\to0^+}\rho(\theta)=0$ according to Lemma 3.1, we obtain

$$\lim_{\theta\to0^+}\mathbb{E}\Big(\sup_{0\le t\le T}\big|y^\theta(t)\big|^2\Big)=0.$$

This completes the proof. □

Continuing on our investigation of sensitivity, the next lemma is concerned with the sensitivity of the cost functional $J(\cdot)$ with respect to the parameter $\theta$. It gives the Gateaux derivative of $J(\cdot)$.

**Lemma 3.3.** *The Gateaux derivative of the cost functional $J(\cdot)$ is given by*

$$\begin{aligned}\frac{d}{d\theta}J\big(\bar u+\theta(v-\bar u)\big)\bigg|_{\theta=0}&=\mathbb{E}\bigg\{\int_0^T\Big[\bar h_x(t)z_t+\bar h_y(t)\mathbb{E}\big(\bar\phi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)+\bar h_u(t)(v_t-\bar u_t)\Big]dt\bigg\}\\&\quad+\mathbb{E}\Big[\bar g_x(T)z_T+\bar g_y(T)\mathbb{E}\big(\bar\eta_x(T)z_T\big|\mathcal{F}^\alpha_{T-}\big)\Big].\end{aligned}\tag{3.3}$$

*Proof.* We have

$$\begin{aligned}&\frac{d}{d\theta}\mathbb{E}\Big[g\Big(X^\theta_T,\mathbb{E}\big(\eta(X^\theta_T)\big|\mathcal{F}^\alpha_{T-}\big),\alpha_T\Big)\Big]\bigg|_{\theta=0}\\&=\lim_{\theta\to0}\mathbb{E}\left[\frac{g\Big(X^\theta_T,\mathbb{E}\big(\eta(X^\theta_T)\big|\mathcal{F}^\alpha_{T-}\big),\alpha_T\Big)-g\Big(\bar X_T,\mathbb{E}\big(\eta(\bar X_T)\big|\mathcal{F}^\alpha_{T-}\big),\alpha_T\Big)}{\theta}\right]\\&=\lim_{\theta\to0}\bigg\{\mathbb{E}\int_0^1g_x\Big(\bar X_T+r\big(X^\theta_T-\bar X_T\big),\mathbb{E}\Big[\eta\Big(\bar X_T+r\big(X^\theta_T-\bar X_T\big)\Big)\Big|\mathcal{F}^\alpha_{T-}\Big],\alpha_T\Big)\frac{X^\theta_T-\bar X_T}{\theta}dr\\&\quad+\mathbb{E}\int_0^1g_y\Big(\bar X_T+r\big(X^\theta_T-\bar X_T\big),\mathbb{E}\Big[\eta\Big(\bar X_T+r\big(X^\theta_T-\bar X_T\big)\Big)\Big|\mathcal{F}^\alpha_{T-}\Big],\alpha_T\Big)\\&\qquad\times\mathbb{E}\Big[\eta_x\Big(\bar X_T+r\big(X^\theta_T-\bar X_T\big)\Big)\frac{X^\theta_T-\bar X_T}{\theta}\Big|\mathcal{F}^\alpha_{T-}\Big]dr\bigg\}\\&=\mathbb{E}\Big[\bar g_x(T)z_T+\bar g_y(T)\mathbb{E}\big(\bar\eta_x(T)z_T\big|\mathcal{F}^\alpha_{T-}\big)\Big].\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{d}{d\theta}\mathbb{E}\bigg[\int_0^Th\Big(t,X^\theta_t,\mathbb{E}\big(\phi(X^\theta_t)\big|\mathcal{F}^\alpha_{t-}\big),u^\theta_t,\alpha_{t-}\Big)dt\bigg]\bigg|_{\theta=0}\\&=\mathbb{E}\bigg\{\int_0^T\Big[\bar h_x(t)z_t+\bar h_y(t)\mathbb{E}\big(\bar\phi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)+\bar h_u(t)(v_t-\bar u_t)\Big]dt\bigg\}.\end{aligned}$$

A combination of the above equations and (2.2) yields (3.3). □

## 3.2. BSDEs with Conditional Mean-Field and Markovian Switching and Duality

In this section, we investigate the adjoint equation associate with the control problem, which is a stochastic equation with a terminal condition. In order to do so, we first introduce the backward stochastic differential equations with conditional mean-field and Markovian switching and provide some mild conditions for the existence and uniqueness of solutions. Although we only consider the one-dimensional BSDE, the result still works for multi-dimensional case under similar conditions.

To proceed, we denote

$$\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})=\Big\{\varphi:[0,T]\times\Omega\to\mathbb{R},\ \mathcal{F}\text{-adapted càdlàg process, }\mathbb{E}\Big[\sup_{0\le t\le T}|\varphi_t|^2\Big]<\infty\Big\},\tag{3.4}$$

and

$$\mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}) = \Big\{\psi : [0,T]\times\Omega \to \mathbb{R},\ \mathcal{F}\text{-progressively measurable process}\Big\}, \tag{3.5}$$

$$\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) = \Big\{\psi \in \mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}) : \|\psi\|_2^2 = \mathbb{E}\Big[\int_0^T |\psi_t|^2 dt\Big] < \infty\Big\}. \tag{3.6}$$

In addition, denote

$$\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R}) = \Big\{\lambda = \big(\lambda_{i_0j_0} : i_0, j_0 \in \mathcal{M}\big) \text{ such that } \lambda_{i_0j_0} \in \mathcal{L}^0_{\mathcal{F}}(0,T;\mathbb{R}),\ \lambda_{i_0i_0} \equiv 0$$
$$\text{for } i_0, j_0 \in \mathcal{M}, \text{ and } \sum_{i_0,j_0\in\mathcal{M}} \mathbb{E}\Big[\int_0^T \big|\lambda_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\Big] < \infty\Big\}. \tag{3.7}$$

Recall that $M_{i_0j_0}(\cdot)$, $i_0, j_0 \in \mathcal{M}$, are martingales associate with the Markov chain $\alpha(\cdot)$. For a collection of $\mathcal{F}$-progressively measurable functions $\lambda(\cdot) = \big(\lambda_{i_0j_0}(\cdot)\big)_{i_0,j_0\in\mathcal{M}}$, $t \geq 0$, we denote

$$\int_0^t \lambda_s \bullet dM_s = \sum_{i_0,j_0\in\mathcal{M}} \int_0^t \lambda_{i_0j_0}(s) dM_{i_0j_0}(s), \qquad \lambda_t \bullet dM_t = \sum_{i_0,j_0\in\mathcal{M}} \lambda_{i_0j_0}(t) dM_{i_0j_0}(t).$$

Let $F(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot) : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$ and $\Psi(\cdot), \Phi(\cdot) : \mathbb{R} \to \mathbb{R}$ be functions. Let $\xi$ be an $\mathcal{F}_T$-measurable random variable. A triple of functions $\big(Y(\cdot), Z(\cdot), \Lambda(\cdot)\big) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$ is called a solution of the BSDE

$$\begin{cases} dY_t = F\Big(t, Y_t, Z_t, \mathbb{E}\big(\Psi\big(Y_t\big)\big|\mathcal{F}^{\alpha}_{t-}\big), \mathbb{E}\big(\Phi\big(Z_t\big)\big|\mathcal{F}^{\alpha}_{t-}\big)\Big)dt + Z_t dW_t + \Lambda_t \bullet dM_t, \quad 0 \leq t \leq T \\ Y_T = \xi, \end{cases} \tag{3.8}$$

if they satisfy the following equation

$$Y_t = \xi - \int_t^T F\Big(s, Y_s, Z_s, \mathbb{E}\big(\Psi\big(Y_s\big)\big|\mathcal{F}^{\alpha}_{s-}\big), \mathbb{E}\big(\Phi\big(Z_s\big)\big|\mathcal{F}^{\alpha}_{s-}\big)\Big)ds - \int_t^T Z_s dW_s - \int_t^T \Lambda_s \bullet dM_s \tag{3.9}$$

for $0 \leq t \leq T$. Note that (3.8) is a BSDE with conditional mean-field coupling terms and Markovian switching. Similar BSDE with conditional mean-field term with respect to Brownian motion can be found in [5, 31]. To prove the existence and uniqueness of the solution of (3.8) we make the following assumption.

**Assumption B.**

(B1) $\mathbb{E}|\xi|^2 < \infty$.

(B2) $F(t, Y, Z, y, z)$ is $\mathcal{F}$-progressively measurable for each $(Y, Z, y, z)$ and $F(t, 0, 0, 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$.

(B3) There exists a constant $K > 0$ such that for $t \in [0,T]$ and $Y, Z, Y', Z', y, z, y', z' \in \mathbb{R}$,

$$\Big|F\big(t, Y, Z, y, z\big) - F\big(t, Y', Z', y', z'\big)\Big| \leq K\Big(|Y - Y'| + |Z - Z'| + |y - y'| + |z - z'|\Big) \quad a.s.$$

and

$$\big|\Psi(y) - \Psi(y')\big| \vee \big|\Phi(z) - \Phi(z')\big| \leq K\Big(|y - y'| + |z - z'|\Big).$$

We have the following theorem, which establishes the existence and uniqueness of solution for the backward stochastic differential equation of interest. To keep a better flow of presentation, the proof of Theorem 3.4 is relegated to the appendix.

**Theorem 3.4.** *Under Assumption* (B), *the backward stochastic differential equation* (3.8) *has a unique solution* $\big(Y(\cdot),Z(\cdot),\Lambda(\cdot)\big)\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$.

We are now in a position to introduce the adjoint equation associate to our control problem. Consider the following BSDE for $\big(\bar p(\cdot),\bar q(\cdot),\bar\lambda(\cdot)\big)$,

$$\begin{aligned}
d\bar p_t &= -\Big[\bar b_x(t)\bar p_t+\bar\sigma_x(t)\bar q_t+\bar h_x(t)\Big]dt\\
&\quad-\Big[\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\psi_x(t)+\mathbb{E}\big(\bar\sigma_y(t)\bar q_t\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\varphi_x(t)+\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\phi_x(t)\Big]dt\\
&\quad+\bar q_t dW_t+\bar\lambda_t\bullet dM_t, \qquad (3.10)\\
\bar p_T &= \bar g_x(T)+\mathbb{E}\big(\bar g_y(T)\big|\mathcal{F}^{\alpha}_{T-}\big)\bar\eta_x(T). \qquad (3.11)
\end{aligned}$$

In view of Theorem 3.4 and Assumption (A), this backward equation has a unique solution $\big(\bar p(\cdot),\bar q(\cdot),\bar\lambda(\cdot)\big)\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$. Consequently, we have the following inequality

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\bar p_t|^2\Big]+\mathbb{E}\Big[\int_0^T|\bar q_t|^2dt\Big]+\sum_{i_0,j_0\in\mathcal{M}}\mathbb{E}\Big[\int_0^T\big|\bar\lambda_{i_0j_0}(t)\big|^2d\big[M_{i_0j_0}\big](t)\Big]<\infty,$$

where $\bar\lambda_t=\big(\bar\lambda_{i_0j_0}(t)\big)_{i_0,j_0\in\mathcal{M}}$.

**Lemma 3.5.** *The following identity holds true*

$$\mathbb{E}\big(\bar p_Tz_T\big)=\mathbb{E}\Bigg[\int_0^T\Big(\bar p_t\bar b_u(t)\big(v_t-\bar u_t\big)-\bar h_x(t)z_t-\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\phi_x(t)z_t+\bar q_t\bar\sigma_u(t)\big(v_t-\bar u_t\big)\Big)dt\Bigg].$$

*Proof.* By the Itô formula for semimartingale (see, for example, Theorem I.4.57 in [15]), we have

$$\begin{aligned}
\bar p_Tz_T &= \bar p_0z_0+\int_0^T z_{t-}d\bar p_t+\int_0^T\bar p_{t-}dz_t+\int_0^t d\langle\bar p_t,z_t\rangle\\
&=\int_0^T z_td\bar p_t+\int_0^T\bar p_tdz_t+\int_0^t d\langle\bar p_t,z_t\rangle\\
&=\int_0^T\Big[\bar p_t\bar b_x(t)z_t+\bar p_t\bar b_y(t)\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^{\alpha}_{t-}\big)+\bar p_t\bar b_u(t)\big(v_t-\bar u_t\big)\Big]dt\\
&\quad+\int_0^T\Big[-\bar b_x(t)\bar p_tz_t-\bar\sigma_x(t)\bar q_tz_t-\bar h_x(t)z_t\Big]dt\\
&\quad+\int_0^T\Big[-\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\psi_x(t)z_t-\mathbb{E}\big(\bar\sigma_y(t)\bar q_t\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\varphi_x(t)z_t-\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^{\alpha}_{t-}\big)\bar\phi_x(t)z_t\Big]dt\\
&\quad+\int_0^T\Big[\bar q_t\bar\sigma_x(t)z_t+\bar q_t\bar\sigma_y(t)\mathbb{E}\big(\bar\varphi_x(t)z_t\big|\mathcal{F}^{\alpha}_{t-}\big)+\bar q_t\bar\sigma_u(t)\big(v_t-\bar u_t\big)\Big]dt+N_T,
\end{aligned}$$

where $\big(N_t\big)_{0\le t\le T}$ is a continuous-time square-integrable martingale. Taking the expectation in both sides and then using the identity $\mathbb{E}(X)=\mathbb{E}\big[\mathbb{E}(X|\mathcal{F}^\alpha_{T-})\big]$ together with Lemma A.1(ii), we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\bar p_t\bar b_y(t)\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)-\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\psi_x(t)z_t\Big)\\
&=\mathbb{E}\Big[\mathbb{E}\Big(\bar p_t\bar b_y(t)\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)-\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\psi_x(t)z_t\Big|\mathcal{F}^\alpha_{T-}\Big)\Big]\\
&=\mathbb{E}\Big[\mathbb{E}\Big(\bar p_t\bar b_y(t)\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)-\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\psi_x(t)z_t\Big|\mathcal{F}^\alpha_{t-}\Big)\Big]\\
&=\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)-\mathbb{E}\big(\bar\psi_x(t)z_t\big|\mathcal{F}^\alpha_{t-}\big)\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\\
&=0.
\end{aligned}$$

Similarly, we have

$$\mathbb{E}\big[\bar p_Tz_T\big]=\mathbb{E}\int_0^T\Big[\bar p_t\bar b_u(t)\big(v_t-\bar u_t\big)-\bar h_x(t)z_t-\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^\alpha_{t-}\big)\bar\phi_x(t)z_t+\bar q_t\bar\sigma_u(t)\big(v_t-\bar u_t\big)\Big]dt.$$

This completes the proof. □

For $(t,\hat x,u,p,q,i_0)\in[0,T]\times\mathbb{R}^4\times U\times\mathbb{R}\times\mathbb{R}\times\mathcal{M}$, define the Hamiltonian

$$\begin{aligned}
H(t,\hat x,u,p,q,i_0)=&h\big(t,x,y_1,u,i_0\big)+b\big(t,x,y_2,u,i_0\big)p\\
&+\sigma\big(t,x,y_3,u,i_0\big)q,
\end{aligned}$$

where $\hat x=(x,y_1,y_2,y_3)\in\mathbb{R}^4$. For simplicity, for a random variable $x$, $H(t,x,u,p,q,i_0)$ will be used instead of $H\big(t,x,\mathbb{E}(\phi(x)|\mathcal{F}^\alpha_{t-}),\mathbb{E}(\psi(x)|\mathcal{F}^\alpha_{t-}),\mathbb{E}(\varphi(x)|\mathcal{F}^\alpha_{t-}),u,p,q,i_0\big)$ with a little abuse of notation. That is,

$$\begin{aligned}
H(t,x,u,p,q,i_0)=&h\big(t,x,\mathbb{E}(\phi(x)|\mathcal{F}^\alpha_{t-}),u,i_0\big)+b\big(t,x,\mathbb{E}(\psi(x)|\mathcal{F}^\alpha_{t-}),u,i_0\big)p\\
&+\sigma\big(t,x,\mathbb{E}(\varphi(x)|\mathcal{F}^\alpha_{t-}),u,i_0\big)q.
\end{aligned}\tag{3.12}$$

We next establish the link between the Gateaux derivative of the cost functional with the Hamiltonian $H$.

**Lemma 3.6.** *The Gateaux derivative of the cost functional can be expressed in terms of the Hamiltonian H in the following way*

$$\begin{aligned}
\frac{d}{d\theta}J\big(\bar u+\theta(v-\bar u)\big)\bigg|_{\theta=0}&=\mathbb{E}\bigg(\int_0^T\Big(\bar h_u(t)(v_t-\bar u_t)+\bar p_t\bar b_u(t)(v_t-\bar u_t)+\bar q_t\bar\sigma_u(t)(v_t-\bar u_t)\Big)dt\bigg)\\
&=\mathbb{E}\bigg(\int_0^T\frac{d}{du}H\big(t,\bar X_t,\bar u_t,\bar p_t,\bar q_t,\alpha_{t-}\big)(v_t-\bar u_t)dt\bigg).
\end{aligned}\tag{3.13}$$

*Proof.* In view of (3.11), we have

$$\begin{aligned}
\mathbb{E}\big(\bar p_Tz_T\big)&=\mathbb{E}\mathbb{E}\Big[\bar g_x(T)z_T+\mathbb{E}\big(\bar g_y(T)\big|\mathcal{F}^\alpha_{T-}\big)\bar\eta_x(T)z_T\Big|\mathcal{F}^\alpha_{T-}\Big]\\
&=\mathbb{E}\Big[\bar g_x(T)z_T+\mathbb{E}\big(\bar g_y(T)\big|\mathcal{F}^\alpha_{T-}\big)\mathbb{E}\big(\bar\eta_x(T)z_T\big|\mathcal{F}^\alpha_{T-}\big)\Big]\\
&=\mathbb{E}\mathbb{E}\Big[\bar g_x(T)z_T+\bar g_y(T)\mathbb{E}\big(\bar\eta_x(T)z_T\big|\mathcal{F}^\alpha_{T-}\big)\Big|\mathcal{F}^\alpha_{T-}\Big]\\
&=\mathbb{E}\Big[\bar g_x(T)z_T+\bar g_y(T)\mathbb{E}\big(\bar\eta_x(T)z_T\big|\mathcal{F}^\alpha_{T-}\big)\Big].
\end{aligned}$$

Combining the equation above with Lemma 3.3 and Lemma 3.5, we obtain the desired identities. □

### 3.3. Necessary and Sufficient Conditions for Optimality

The results of this section characterize the optimal control by providing necessary and sufficient conditions. Our first result in this section is a necessary condition for optimality.

**Theorem 3.7.** *Under Assumption* (A), *if* $\bar{u}(\cdot)$ *is an optimal control with state trajectory* $\bar{X}(\cdot)$, *then there exists a triple* $\big(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\lambda}(\cdot)\big)$ *of adapted processes that satisfies the BSDE* (3.10) *and* (3.11) *such that*

$$\frac{d}{du} H\big(t, \bar{X}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \alpha_{t-}\big)(v - \bar{u}_t) \geq 0, \quad dtd\mathbb{P}\text{-a.s. on } [0,T] \times \Omega \text{ for any } v \in U.$$

*Proof.* Because of the convexity of $U$, for any $v(\cdot) \in \mathcal{U}$ and $\theta \in [0,1]$, the perturbation $u^\theta = \bar{u} + \theta(v - \bar{u}) \in \mathcal{U}$. Since $\bar{u}(\cdot)$ is optimal, by Lemma 3.6, we have

$$\frac{d}{d\theta} J\big(\bar{u} + \theta(v - \bar{u})\big)\Big|_{\theta=0} = \mathbb{E}\bigg( \int_0^T \frac{d}{du} H\big(t, \bar{X}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \alpha_{t-}\big)(v_t - \bar{u}_t) dt \bigg) \geq 0.$$

This can be reduced to

$$\frac{d}{du} H\big(t, \bar{X}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t, \alpha_{t-}\big)(v - \bar{u}_t) \geq 0 \quad dtd\mathbb{P}\text{-a.s. on } [0,T] \times \Omega \text{ for any } v \in U.$$

The proof is thus concluded. □

Next, we present a sufficient condition for optimality. To proceed, we make the following additional assumptions.

(A3) The functions $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, and $\eta(\cdot)$ are convex, the function $g(\cdot,\cdot,\cdot)$ is convex in $(x,y)$, and the Hamiltonian $H(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is convex in $(\hat{x}, u)$.
(A4) The functions $b_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, $\sigma_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, $h_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, and $g_y(\cdot,\cdot,\cdot)$ are nonnegative.

We have the following theorem.

**Theorem 3.8.** *Assume that assumptions* (A1)-(A4) *hold. Let* $\bar{u}(\cdot)$ *be a control in* $\mathcal{U}$ *with the corresponding state trajectory* $\bar{X}(\cdot)$. *Let* $\big(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\lambda}(\cdot)\big)$ *be the solution to the adjoint equation. If*

$$H(t, \bar{x}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t) = \inf_{v \in U} H(t, \bar{x}_t, v, \bar{p}_t, \bar{q}_t), \quad dtd\mathbb{P}\text{-a.s. on } [0,T] \times \Omega, \tag{3.14}$$

*then* $\bar{u}(\cdot)$ *is an optimal control.*

*Proof.* Let $u(\cdot)$ be an arbitrary admissible control in $\mathcal{U}$ with the corresponding state trajectory $X^u(\cdot)$. Denote $\phi(t) = \phi\big(X_t^u\big)$, $\varphi(t) = \varphi\big(X_t^u\big)$, $\psi(t) = \psi\big(X_t^u\big)$, $\eta(t) = \eta\big(X_t^u\big)$, and

$$\begin{aligned} b(t) &= b\Big(t, X_t^u, \mathbb{E}\big(\psi\big(X_t^u\big)\big|\mathcal{F}_{t-}^\alpha\big), u_t, \alpha_{t-}\Big), \quad \sigma(t) = \sigma\Big(t, X_t^u, \mathbb{E}\big(\varphi\big(X_t^u\big)\big|\mathcal{F}_{t-}^\alpha\big), u_t, \alpha_{t-}\Big), \\ h(t) &= h\Big(t, X_t^u, \mathbb{E}\big(\phi\big(X_t^u\big)\big|\mathcal{F}_{t-}^\alpha\big), u_t, \alpha_{t-}\Big), \quad g(T) = g\Big(X_T^u, \mathbb{E}\big(\eta\big(X_T^u\big)\big|\mathcal{F}_{T-}^\alpha\big), \alpha_T\Big). \end{aligned}$$

In addition, denote

$$\bar{H}(t) = H(t, \bar{X}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t) \ \text{ and } \ H(t) = H(t, X_t^u, u_t, \bar{p}_t, \bar{q}_t).$$

By the convexity of $g$ and $\eta$ and (3.11), we have

$$\begin{aligned}\mathbb{E}\Big[\bar g(T)-g(T)\Big] &= \mathbb{E}\Big[g\Big(\bar X_T,\mathbb{E}\big(\eta(\bar X_T)\big|\mathcal{F}^\alpha_{T-}\big),\alpha_T\Big)-g\Big(X^u_T,\mathbb{E}\big(\eta(X^u_T)\big|\mathcal{F}^\alpha_{T-}\big),\alpha_T\Big)\Big]\\ &\le \mathbb{E}\Big\{\bar g_x(T)\big(\bar X_T-X^u_T\big)+\bar g_y(T)\mathbb{E}\Big[\eta\big(\bar X_T\big)-\eta\big(X^u_T\big)\Big|\mathcal{F}^\alpha_{T-}\Big]\Big\}\\ &\le \mathbb{E}\Big\{\bar g_x(T)\big(\bar X_T-X^u_T\big)+\bar g_y(T)\mathbb{E}\Big[\bar\eta_x(T)\big(\bar X_T-X^u_T\big)\Big|\mathcal{F}^\alpha_{T-}\Big]\Big\}\\ &= \mathbb{E}\Big[\bar p_T\big(\bar X_T-X^u_T\big)\Big]. \end{aligned}\tag{3.15}$$

By the Itô formula and the facts that

$$\bar H(t)=\bar h(t)+\bar b(t)\bar p_t+\bar\sigma(t)\bar q_t \quad\text{and}\quad H(t)=h(t)+b(t)\bar p_t+\sigma(t)\bar q_t,$$

$$\begin{aligned}&\mathbb{E}\Big[\bar p_T\big(\bar X_T-X^u_T\big)\Big]\\ &=\mathbb{E}\Big[\int_0^T\big(\bar X_t-X^u_t\big)d\bar p_t+\int_0^T\bar p_t d\big(\bar X_t-X^u_t\big)+\int_0^T\bar q_t\big(\bar\sigma(t)-\sigma(t)\big)dt\Big]\\ &=-\mathbb{E}\Big\{\int_0^T\big(\bar X_t-X^u_t\big)\Big[\bar b_x(t)\bar p_t+\bar\sigma_x(t)\bar q_t+\bar h_x(t)+\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\psi_x(t)+\mathbb{E}\big(\bar\sigma_y(t)\bar q_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\varphi_x(t)\\ &\qquad+\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^\alpha_{t-}\big)\bar\phi_x(t)\Big]dt\Big\}+\mathbb{E}\Big[\int_0^T\bar p_t\big(\bar b(t)-b(t)\big)dt\Big]+\mathbb{E}\Big[\int_0^T\bar q_t\big(\bar\sigma(t)-\sigma(t)\big)dt\Big]\\ &=-\mathbb{E}\Big\{\int_0^T\big(\bar X_t-X^u_t\big)\Big[\bar b_x(t)\bar p_t+\mathbb{E}\big(\bar b_y(t)\bar p_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\psi_x(t)+\bar\sigma_x(t)\bar q_t+\mathbb{E}\big(\bar\sigma_y(t)\bar q_t\big|\mathcal{F}^\alpha_{t-}\big)\bar\varphi_x(t)+\bar h_x(t)\\ &\qquad+\mathbb{E}\big(\bar h_y(t)\big|\mathcal{F}^\alpha_{t-}\big)\bar\phi_x(t)\Big]dt\Big\}+\mathbb{E}\Big[\int_0^T\big(\bar H(t)-H(t)\big)dt\Big]-\mathbb{E}\Big[\int_0^T\big(\bar h(t)-h(t)\big)dt\Big].\end{aligned}$$

To proceed, we use assumptions (A3) and (A4) to obtain

$$\begin{aligned}&\bar H(t)-H(t)\\ &\le \bar H_x(t)\big(\bar X_t-X^u_t\big)+\bar h_y(t)\mathbb{E}\Big(\bar\phi(t)-\phi(t)\Big|\mathcal{F}^\alpha_{t-}\Big)+\bar b_y(t)\mathbb{E}\Big(\bar\psi(t)-\psi(t)\Big|\mathcal{F}^\alpha_{t-}\Big)\bar p_t\\ &\qquad+\bar\sigma_y(t)\mathbb{E}\Big(\bar\varphi(t)-\varphi(t)\Big|\mathcal{F}^\alpha_{t-}\Big)\bar q_t+\bar H_u(t)\big(\bar u_t-u_t\big)\\ &\le \bar H_x(t)\big(\bar X_t-X^u_t\big)+\bar h_y(t)\mathbb{E}\Big[\bar\phi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]+\bar b_y(t)\mathbb{E}\Big[\bar\psi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]\bar p_t\\ &\qquad+\bar\sigma_y(t)\mathbb{E}\Big[\bar\varphi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]\bar q_t+\bar H_u(t)\big(\bar u_t-u_t\big)\\ &\le \bar H_x(t)\big(\bar X_t-X^u_t\big)+\bar h_y(t)\mathbb{E}\Big[\bar\phi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]+\bar b_y(t)\mathbb{E}\Big[\bar\psi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]\bar p_t\\ &\qquad+\bar\sigma_y(t)\mathbb{E}\Big[\bar\varphi_x(t)\big(\bar X_t-X^u_t\big)\Big|\mathcal{F}^\alpha_{t-}\Big]\bar q_t.\end{aligned}$$

Note that in the last inequality, we have used $\bar{H}_u(t)\big(\bar{u}_t - u_t\big) \le 0$, which is a consequence of the minimality condition. Combining the inequalities, we arrive at

$$
\begin{aligned}
&J(\bar{u}) - J(u) \\
&= \mathbb{E}\int_0^T \Big(\bar{h}(t) - h(t)\Big)dt + \mathbb{E}\Big(\bar{g}(T) - g(T)\Big) \\
&\le \mathbb{E}\Big[\int_0^T \big(\bar{H}(t) - H(t)\big)dt\Big] - \mathbb{E}\bigg\{\int_0^T \big(\bar{X}_t - X_t^u\big)\Big[\bar{b}_x(t)\bar{p}_t + \mathbb{E}\big(\bar{b}_y(t)\bar{p}_t\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\psi}_x(t) + \bar{\sigma}_x(t)\bar{q}_t \\
&\qquad + \mathbb{E}\big(\bar{\sigma}_y(t)\bar{q}_t\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\varphi}_x(t) + \bar{h}_x(t) + \mathbb{E}\big(\bar{h}_y(t)\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\phi}_x(t)\Big]dt\bigg\} \\
&= \mathbb{E}\Big[\int_0^T \big(\bar{H}(t) - H(t)\big)dt\Big] - \mathbb{E}\bigg\{\int_0^T \big(\bar{X}_t - X_t^u\big)\Big[\bar{H}_x(t) + \mathbb{E}\big(\bar{b}_y(t)\bar{p}_t\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\psi}_x(t) \\
&\qquad + \mathbb{E}\big(\bar{\sigma}_y(t)\bar{q}_t\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\varphi}_x(t) + \mathbb{E}\big(\bar{h}_y(t)\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{\phi}_x(t)\Big]dt\bigg\} \\
&= \mathbb{E}\Big[\int_0^T \big(\bar{H}(t) - H(t)\big)dt\Big] - \mathbb{E}\bigg\{\int_0^T \big(\bar{X}_t - X_t^u\big)\Big[\bar{H}_x(t) + \bar{b}_y(t)\mathbb{E}\big(\bar{\psi}_x(t)\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{p}_t \\
&\qquad + \bar{\sigma}_y(t)\mathbb{E}\big(\bar{\varphi}_x(t)\big|\mathcal{F}_{t-}^{\alpha}\big)\bar{q}_t + \bar{h}_y(t)\mathbb{E}\big(\bar{\phi}_x(t)\big|\mathcal{F}_{t-}^{\alpha}\big)\Big]dt\bigg\} \le 0.
\end{aligned}
$$

This shows that $\bar{u}(\cdot)$ is optimal, which concludes the proof. □

## 4. Application: A Linear-Quadratic Control Problem

A linear quadratic regulator problem is one of the main building blocks of control and systems theory. Likewise, linear quadratic controls with mean-field terms is one of the basic models for mean-field games. In this section, we examine linear quadratic controls with both mean-field interactions and Markovian modulation. It turns out that the solution of this problem crucially depends on the maximum principle developed in the previous section.

### 4.1. **A Linear-Quadratic Control Problem**

Using the maximum principle obtained, we solve a linear quadratic stochastic control problem with switching and mean-field interactions. The SDE is now linear in the continuous variable $x_0$ as follows

$$
\begin{aligned}
dX_t &= \Big[A(\alpha_{t-})X_t + \hat{A}(\alpha_{t-})\mathbb{E}\big(X_t\big|\mathcal{F}_{t-}^{\alpha}\big) + B(\alpha_{t-})u_t\Big]dt \\
&\qquad + \Big[C(\alpha_{t-})X_t + \hat{C}(\alpha_{t-})\mathbb{E}\big(X_t\big|\mathcal{F}_{t-}^{\alpha}\big) + D(\alpha_{t-})u_t\Big]dW_t, \\
X_0 &= x_0.
\end{aligned} \tag{4.1}
$$

The cost functional is of the quadratic form in the continuous variable $x_0$

$$
J(u) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(R(\alpha_{t-})X_t^2 + N(\alpha_{t-})u_t^2\Big)dt + S(\alpha_T)X_T^2\Big]. \tag{4.2}
$$

Here, $A(i_0), \hat{A}(i_0), B(i_0), C(i_0), \hat{C}(i_0), D(i_0) \in \mathbb{R}$ and $R(i_0), N(i_0), S(i_0) > 0$ for each $i_0 \in \mathcal{M}$. Note that minimizing $J(u)$ is a strictly convex optimization problem with $J(u) \to \infty$ as $\|u\|_2^2 \to \infty$. For this problem, an optimal control exists and is unique. As a result, (A4) used in the general result is not needed here.

With the preparation of the last section, the adjoint equation reads

$$\begin{aligned} dp_t &= -\Big[A(\alpha_{t-})p_t + C(\alpha_{t-})q_t + R(\alpha_{t-})X_t + \hat{A}(\alpha_{t-})\mathbb{E}\big(p_t\big|\mathcal{F}^\alpha_{t-}\big) + \hat{C}(\alpha_{t-})\mathbb{E}\big(q_t\big|\mathcal{F}^\alpha_{t-}\big)\Big]dt \\ &\quad + q_t dW_t + \lambda_t \bullet dM_t, \\ p_T &= S(\alpha_T)X_T. \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H(t,x,u,p,q,i_0) =& \frac{1}{2}\Big[R(i_0)x^2 + N(i_0)u^2\Big] + \Big[A(i_0)x + \hat{A}(i_0)\mathbb{E}\big(x\big|\mathcal{F}^\alpha_{t-}\big) + B(i_0)u\Big]p \\ &+ \Big[C(i_0)x + \hat{C}(i_0)\mathbb{E}\big(x\big|\mathcal{F}^\alpha_{t-}\big) + D(i_0)u\Big]q. \end{aligned}$$

It follows from the necessary condition for optimality that

$$N(\alpha_{t-})u_t = -\Big[B(\alpha_{t-})p_t + D(\alpha_{t-})q_t\Big]. \tag{4.3}$$

Denote

$$\hat{X}_t = \mathbb{E}\big(X_t\big|\mathcal{F}^\alpha_{t-}\big), \quad \hat{p}_t = \mathbb{E}\big(p_t\big|\mathcal{F}^\alpha_{t-}\big), \quad \hat{q}_t = \mathbb{E}\big(q_t\big|\mathcal{F}^\alpha_{t-}\big).$$

In addition, for simplicity, put $A_t = A(\alpha_{t-})$ and $\hat{A}_t = \hat{A}(\alpha_{t-})$. The functions $B_t$, $C_t$, $\hat{C}_t$, $D_t$, and $R_t, N_t$ are defined in the same way. The optimal control system then takes the feedback form

$$\begin{aligned} dX_t &= \left[A_tX_t + \hat{A}_t\hat{X}_t - \frac{B_t^2}{N_t}p_t - \frac{B_tD_t}{N_t}q_t\right]dt + \left[C_tX_t + \hat{C}_t\hat{X}_t - \frac{D_tB_t}{N_t}p_t - \frac{D_t^2}{N_t}q_t\right]dW_t, \\ dp_t &= -\left[A_tp_t + C_tq_t + R_tX_t + \hat{A}_t\hat{p}_t + \hat{C}_t\hat{q}_t\right]dt + q_tdW_t + \lambda_t \bullet dM_t, \\ X_0 &= x_0, \\ p_T &= S(\alpha_T)X_T, \end{aligned}$$

which is a fully coupled conditional mean-field forward-backward SDE with Markovian switching (cf. the terminology used in [31]). In order to solve this system, we put

$$p_t = \nu(t,\alpha_t)X_t + \gamma(t,\alpha_t)\mathbb{E}(X_t|\mathcal{F}^\alpha_{t-}) \tag{4.4}$$

for some functions $\nu(\cdot,\cdot), \gamma(\cdot,\cdot) : [0,T] \times \mathcal{M} \to \mathbb{R}$ differentiable in $t$ to be determined. For each $i_0 \in \mathcal{M}$ and $t \geq 0$, denote $\nu'(t,i_0) = \frac{d}{dt}\nu(t,i_0)$ and $\gamma'(t,i_0) = \frac{d}{dt}\gamma(t,i_0)$. We have

$$d\nu(t,\alpha_t) = \Big[\nu'(t,\alpha_t) + Q\nu(t)(\alpha_t)\Big]dt + \sum_{i_0,j_0 \in \mathcal{M}} \big(\nu(t,j_0) - \nu(t,i_0)\big)dM_{i_0j_0}(t),$$

where $Q\nu(t)(i_0) = \sum_{j_0 \in \mathcal{M}} q_{i_0j_0}\big(\nu(t,j_0) - \nu(t,i_0)\big)$. A similar equation holds for $\gamma(t,\alpha_t)$.

Since $\mathbb{E}\big(X_t\big|\mathcal{F}^{\alpha}_{(t+\delta)_-}\big)=\mathbb{E}\big(X_t\big|\mathcal{F}^{\alpha}_{t-}\big)$, in view of Lemma A.1(ii),

$$\begin{aligned}
&\mathbb{E}\Big(X_{t+\delta}\Big|\mathcal{F}^{\alpha}_{(t+\delta)_-}\Big)-\mathbb{E}\big(X_t\big|\mathcal{F}^{\alpha}_{t-}\big)\\
&=\mathbb{E}\Big(X_{t+\delta}-X_t\Big|\mathcal{F}^{\alpha}_{(t+\delta)_-}\Big)\\
&=\mathbb{E}\bigg\{\int_t^{t+\delta}\Big[A_sX_s+\hat{A}_s\mathbb{E}\big(X_s\big|\mathcal{F}^{\alpha}_{s_-}\big)-\frac{B_s^2}{N_s}p_s-\frac{B_sD_s}{N_s}q_s\Big]ds\bigg|\mathcal{F}^{\alpha}_{(t+\delta)_-}\bigg\}\\
&=\int_t^{t+\delta}\Big[\big(A_s+\hat{A}_s\big)\hat{X}_s-\frac{B_s^2}{N_s}\hat{p}_s-\frac{B_sD_s}{N_s}\hat{q}_s\Big]ds.
\end{aligned}$$

This implies

$$d\hat{X}_t=\Big[\big(A_t+\hat{A}_t\big)\hat{X}_t-\frac{B_t^2}{N_t}\hat{p}_t-\frac{B_tD_t}{N_t}\hat{q}_t\Big]dt. \tag{4.5}$$

Denote $\nu_t=\nu(t,\alpha_t)$, $\nu'_t=\nu'(t,\alpha_t)$, $\gamma_t=\gamma(t,\alpha_t)$, and $\gamma'_t=\gamma'(t,\alpha_t)$. Since $p_t=\nu_tX_t+\gamma_t\hat{X}_t$, $u_t=-\frac{B_t}{N_t}(\lambda_tX_t+\gamma_t\hat{X}_t)-\frac{D_t}{N_t}q_t$. On one hand, we have

$$dp_t=-\Big[A_tp_t+C_tq_t+R_tX_t+\hat{A}_t\big(\nu_t+\gamma_t\big)\hat{X}_t+\hat{C}_t\hat{q}_t\Big]dt+q_tdW_t+\lambda_t\bullet dM_t. \tag{4.6}$$

On the other hand, by the Itô formula,

$$\begin{aligned}
dp_t&=d\big(\nu_tX_t+\gamma_t\hat{X}_t\big)\\
&=X_td\nu_t+\nu_tdX_t+\hat{X}_td\gamma_t+\gamma_td\hat{X}_t\\
&=\Big[\nu'_t+Q\nu_t(\alpha_t)\Big]X_tdt+X_t\sum_{i_0,j_0\in\mathcal{M}}\big(\nu(t,j_0)-\nu(t,i_0)\big)dM_{i_0j_0}(t)\\
&\quad+\nu_t\Big[A_tX_t+\hat{A}_t\hat{X}_t-\frac{B_t^2}{N_t}\big(\nu_tX_t+\gamma_t\hat{X}_t\big)-\frac{B_tD_t}{N_t}q_t\Big]dt\\
&\quad+\nu_t\Big[C_tX_t+\hat{C}_t\hat{X}_t-\frac{B_tD_t}{N_t}\big(\nu_tX_t+\gamma_t\hat{X}_t\big)-\frac{D_t^2}{N_t}q_t\Big]dW_t\\
&\quad+\Big[\gamma'_t+Q\gamma_t(\alpha_t)\Big]\hat{X}_tdt+\hat{X}_t\sum_{i_0,j_0\in\mathcal{M}}\big(\gamma(t,j_0)-\gamma(t,i_0)\big)dM_{i_0j_0}(t)\\
&\quad+\gamma_t\Big[\Big(A_t+\hat{A}_t\Big)\hat{X}_t-\frac{B_t^2}{N_t}\big(\nu_t+\gamma_t\big)\hat{X}_t-\frac{B_tD_t}{N_t}\hat{q}_t\Big]dt.
\end{aligned}\tag{4.7}$$

By equalizing the coefficients of $dW_t$ and $dM_{i_0j_0}(t)$ in (4.6) and (4.7), we obtain

$$q_t=\frac{\nu_t\big(N_tC_t-\nu_tB_tD_t\big)}{N_t+\nu_tD_t^2}X_t+\frac{\nu_t\big(N_t\hat{C}_t-\gamma_tB_tD_t\big)}{N_t+\nu_tD_t^2}\hat{X}_t, \tag{4.8}$$

and

$$\lambda_{i_0j_0}(t)=X_t\Big(\nu(t,j_0)-\nu(t,i_0)\Big)+\hat{X}_t\Big(\gamma(t,j_0)-\gamma(t,i_0)\Big),\quad i_0,j_0\in\mathcal{M}. \tag{4.9}$$

According to (4.8) and Lemma A.1, we obtain

$$\hat{q}_t=\frac{\nu_t\Big[N_t\big(C_t+\hat{C}_t\big)-(\nu_t+\gamma_t)B_tD_t\Big]}{N_t+\nu_tD_t^2}\hat{X}_t,\quad \text{a.s.} \tag{4.10}$$

Next, equating the coefficients of $dt$ in (4.6) and (4.7) yields

$$\begin{aligned}
&-\Big[A_t p_t + C_t q_t + R_t X_t + \hat{A}_t(\nu_t+\gamma_t)\hat{X}_t + \hat{C}_t\hat{q}_t\Big]\\
&= \Big[\nu_t' + Q\nu_t(\alpha_t)\Big]X_t + \nu_t\Big[A_t X_t + \hat{A}_t\hat{X}_t - \frac{B_t^2}{N_t}\big(\nu_t X_t + \gamma_t\hat{X}_t\big) - \frac{B_t D_t}{N_t}q_t\Big] + \Big[\gamma_t' + Q\gamma_t(\alpha_t)\Big]\hat{X}_t\\
&\quad + \gamma_t\Big[\Big(A_t + \hat{A}_t\Big)\hat{X}_t - \frac{B_t^2}{N_t}(\nu_t+\gamma_t)\hat{X}_t - \frac{B_t D_t}{N_t}\hat{q}_t\Big]. \qquad (4.11)
\end{aligned}$$

To proceed, by writing $p_t$, $q_t$, and $\hat{q}_t$ in terms of $X_t$ and $\hat{X}_t$ as in (4.4), (4.8), and (4.10), we arrive at the following equation for $X_t$ and $\hat{X}_t$ from (4.11):

$$\begin{aligned}
&- X_t\left[\nu_t A_t + R_t + \frac{\nu_t C_t\big(N_t C_t - \nu_t B_t D_t\big)}{N_t + \nu_t D_t^2}\right]\\
&\quad - \hat{X}_t\left\{\gamma_t A_t + (\nu_t+\gamma_t)\hat{A}_t + \frac{\nu_t C_t\big(N_t\hat{C}_t - \gamma_t B_t D_t\big)}{N_t+\nu_t D_t^2} + \frac{\nu_t\hat{C}_t\big[N_t\big(C_t+\hat{C}_t\big) - (\nu_t+\gamma_t)B_t D_t\big]}{N_t+\nu_t D_t^2}\right\}\\
&= X_t\left\{\nu_t' + Q\nu_t(\alpha_t) + \nu_t A_t - \frac{\nu_t^2 B_t^2}{N_t} - \frac{\nu_t^2 B_t D_t\big(N_t C_t - \nu_t B_t D_t\big)}{N_t\big(N_t+\nu_t D_t^2\big)}\right\}\\
&\quad + \hat{X}_t\left\{\gamma_t' + Q\gamma_t(\alpha_t) + \nu_t\hat{A}_t - \frac{\nu_t\gamma_t B_t^2}{N_t} - \frac{\nu_t^2 B_t D_t\big(N_t\hat{C}_t - \gamma_t B_t D_t\big)}{N_t\big(N_t+\nu_t D_t^2\big)}\right.\\
&\qquad\left. + \gamma_t\big(A_t+\hat{A}_t\big) - \frac{\nu_t\gamma_t B_t^2}{N_t} - \frac{\nu_t\gamma_t B_t D_t\big[N_t\big(C_t+\hat{C}_t\big) - (\nu_t+\gamma_t)B_t D_t\big]}{N_t\big(N_t+\nu_t D_t^2\big)}\right\}.
\end{aligned}$$

By equating the coefficients of $X_t$ and $\hat{X}_t$ we obtain the following equations for $\lambda_t$ and $\gamma_t$.

$$\begin{cases}
\nu_t' + \big(2A_t + C_t^2\big)\nu_t - \dfrac{\big(B_t + D_t C_t\big)^2\nu_t^2}{N_t+\nu_t D_t^2} + R_t + Q\nu_t(\alpha_t) = 0,\\
\nu_T = S(\alpha_T),
\end{cases} \qquad (4.12)$$

and

$$\begin{cases}
\gamma_t' + 2\big(A_t+\hat{A}_t\big)\gamma_t + \big(2C_t\hat{C}_t + \hat{C}_t^2 + 2\hat{A}_t\big)\nu_t\\
\qquad -\dfrac{B_t\gamma_t + \hat{C}_t D_t\nu_t}{N_t + D_t^2\nu_t}\Big[\Big(2B_t + 2C_t D_t + \hat{C}_t D_t\Big)\nu_t + B\gamma_t\Big] + Q\gamma_t(\alpha_t) = 0,\\
\gamma_T = 0.
\end{cases} \qquad (4.13)$$

Note that (4.12) is a Riccati equation that can be rewritten as follow

$$\begin{cases}
\nu'(t,i_0) + \big(2A(i_0) + C^2(i_0)\big)\nu(t,i_0) - \dfrac{\big(B(i_0) + D(i_0)C(i_0)\big)^2\nu^2(t,i_0)}{N(i_0) + \nu(t,i_0)D(i_0)^2}\\
\qquad + R(i_0) + \sum_{j_0\in\mathcal{M}} q_{i_0 j_0}\big(\nu(t,j_0) - \nu(t,i_0)\big) = 0,\\
\nu(T,i_0) = S(i_0).
\end{cases} \qquad (4.14)$$

Since $R(i_0), S(i_0) > 0$ for each $i_0 \in \mathcal{M}$, by [42, Lemma 1], (4.14) always has a unique positive solution $\nu(t,i_0)$.

It is worth mentioning that $\hat{X}(\cdot)$ can be determined separately as the solution of a linear switched differential equation. Indeed, it follows from (4.5), (4.4), and (4.10) that

$$d\hat{X}_t = \left[\big(A_t + \hat{A}_t\big) - \frac{(\nu_t + \gamma_t)B_t^2 + \nu_t B_t D_t\big(C_t + \hat{C}_t\big)}{N_t + \nu_t D_t^2}\right]\hat{X}_t dt, \quad \hat{X}_0 = x_0. \tag{4.15}$$

Thus, we have proved the following theorem.

**Theorem 4.1.** *The optimal control $u(\cdot) \in \mathcal{U}$ for the linear quadratic control problem is given in the feedback form by*

$$u_t = -\frac{\Big[B_t + C_t D_t\Big]\nu_t X_t + \Big[B_t\gamma_t + \hat{C}_t D_t \nu_t\Big]\hat{X}_t}{N_t + D_t^2\nu_t}, \tag{4.16}$$

*where $\nu(\cdot)$, $\gamma(\cdot)$, and $\hat{X}(\cdot)$ are solutions to* (4.12), (4.13), *and* (4.15), *respectively.*

## 4.2. Numerical Examples

In this section, we demonstrate the use of the maximum principle solving the LQG problem with Markov switching and mean-field interactions by providing some numerical examples. For simplicity, let us consider equation (4.1) in which the Markov chain takes two possible values 1 and 2 (i.e., $\mathcal{M} = \{1, 2\}$) with the generator of the Markov chain being

$$Q = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$$

and the initial condition $x = 5$. For illustration purpose, assume the finite time horizon is given with $T = 2$ and that the coefficients of the dynamic equation are given below

$$\begin{array}{llllll} A(1) = 2, & \hat{A}(1) = 1, & B(1) = 2, & C(1) = 1, & \hat{C}(1) = 2, & D(1) = 2, \\ A(2) = 5, & \hat{A}(2) = 5, & B(2) = 4, & C(2) = 2, & \hat{C}(2) = 3, & D(2) = 1. \end{array}$$

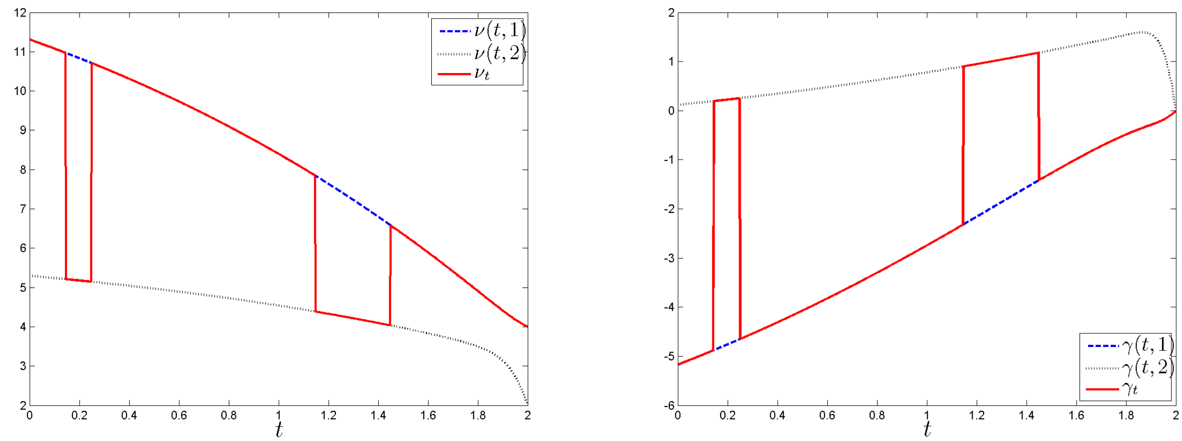
We consider the cost function defined by equation (4.2) with

$$\begin{array}{lll} N(1) = 1, & R(1) = 2, & S(1) = 4, \\ N(2) = 4, & R(2) = 5, & S(2) = 2. \end{array}$$

Using Euler's method with the step size $h = 2 \cdot 10^{-3}$ and Markov chain $\alpha_t$, we can obtain trajectories of $\nu_t$ and $\gamma_t$, which are solutions to equations (4.14) and (4.13), respectively. Their graphs are shown in Figure 1. In Figure 1a, the dashed line is the graph of $\nu(t, 1)$, the dotted line is the graph of $\nu(t, 2)$, and the solid line is the graph of $\nu_t = \nu(t, \alpha_t)$ whose values are switched between the dashed line ($\nu(t, 1)$) and the dotted line ($\nu(t, 2)$). Similarly, in Figure 1b, the dashed line is the graph of $\gamma(t, 1)$, the dotted line is the graph of $\gamma(t, 2)$, and the solid line is the graph of $\gamma_t = \gamma(t, \alpha_t)$ whose values are also switched between the dashed line ($\gamma(t, 1)$) and the dotted line ($\gamma(t, 2)$).

Next, in order to study the behavior of the solution to equation (4.1) and the corresponding mean-field control problems, we first generate 10000 independent Brownian motions and then use Euler's method to achieve approximations of $X_t$ and $\hat{X}_t = \mathbb{E}\big(X_t \big| \mathcal{F}_{t-}^{\alpha}\big)$. Their graphs are shown in Figure 2a, in which the more fluctuating function is $X_t$ and the smoother function is $\hat{X}_t$. It is not surprising that $\hat{X}_t$ is more stable.

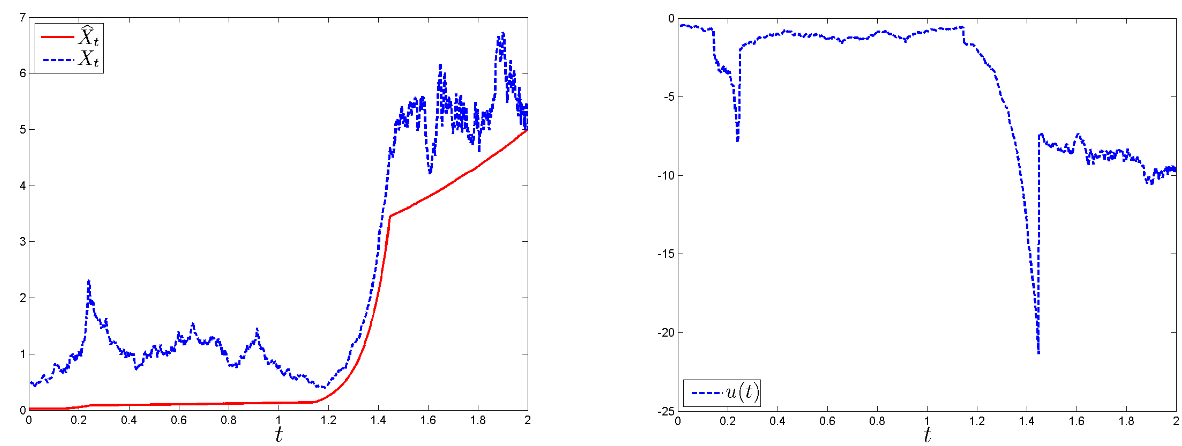
Finally, we can use equation (4.16) in Theorem 4.1 and the approximations of $\nu_t, \gamma_t, X_t$, and $\hat{X}_t$ obtained above to derive the optimal control $u_t$ whose graph is shown in Figure 2b.




(A) A trajectory of solution $\nu_t$ of equation (4.14)

(B) A trajectory of solution $\gamma_t$ of equation (4.13)

Figure 1. Simulations for $\nu_t$ and $\gamma_t$




(A) Solutions $X_t$ and $\hat{X}_t$ of the control problems

(B) The optimal control $u_t$

Figure 2. Simulations for $X_t, \hat{X}_t$, and $u_t$

## 5. Final Remarks

This paper obtained a maximum principle for controlled switching diffusions with mean-field interactions. The consideration of the models is largely because of the current needs of handling networked control systems. The results cannot be obtained using known results in the literature. The difficulties are due to nonlinearity, the addition of the random switching process, and the inclusion of the mean-field interactions. A key idea and insight of the paper is the use of conditional distributions and the use of a law of large numbers for such processes [25]. An example considering LQG for switching diffusions with mean-field interactions is provided. The optimal control is obtained by using the maximum principle. Hopefully, our approach will open up new avenues for networked systems and for treatment of mean-field control and games for systems with random switching.

## Appendix A. Appendix

### A.1. **A Lemma on Conditional Expectation**

**Lemma A.1.** *We have the following assertions.*

(i) *For any $\mathcal{F}_T$-measurable random variable $X$ satisfying $\mathbb{E}|X| < \infty$, and $t \le T$, $\mathbb{E}(X|\mathcal{F}^\alpha_t) = \mathbb{E}(X|\mathcal{F}^\alpha_{t-})$ a.s.*

(ii) *For any $t \in (0, T]$, and $X(\cdot)$ satisfying* (2.1), $\mathbb{E}(X_t|\mathcal{F}^\alpha_{T-}) = \mathbb{E}(X_t|\mathcal{F}^\alpha_{t-})$.

*Proof.* (i) Since $P\big(\alpha(t) \neq \alpha(t_-)\big) = 0$ for any $t > 0$, one has

$$\mathbb{E}\Big\{\big[\mathbb{1}\big(\alpha(t) = i_0\big) - \mathbb{1}\big(\alpha(t_-) = i_0\big)\big]\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t-}\right)\Big\} = 0 \quad \text{for any } i_0 \in \mathcal{M}.$$

This leads to

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}\big(\alpha(t) = i_0\big)\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t-}\right)\Big] &= \mathbb{E}\Big[\mathbb{1}\big(\alpha(t_-) = i_0\big)\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t-}\right)\Big]\\ &= \mathbb{E}\Big[\mathbb{1}\big(\alpha(t_-) = i_0\big)\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t}\right)\Big],\end{aligned}$$

which implies $\mathbb{1}\big(\alpha(t) = i_0\big)\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t-}\right) = \mathbb{1}\big(\alpha(t_-) = i_0\big)\,\mathbb{E}\left(X\big|\mathcal{F}^\alpha_{t}\right)$ for each $i_0 \in \mathcal{M}$. Thus $\mathbb{E}(X|\mathcal{F}^\alpha_t) = \mathbb{E}(X|\mathcal{F}^\alpha_{t-})$ almost surely.

(ii) It suffices to show that $\mathbb{E}\Big[\mathbb{1}\big(\alpha(t+h) = i_0\big)\,\mathbb{E}\left(X_t\big|\mathcal{F}^\alpha_{t-}\right)\Big] = \mathbb{E}\Big[\mathbb{1}\big(\alpha(t+h) = i_0\big)X_t\Big]$ for any $h \in (0, T-t)$ and $i_0 \in \mathcal{M}$. We have

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}\big(\alpha(t+h) = i_0\big)\,\mathbb{E}\left(X_t\big|\mathcal{F}^\alpha_{t-}\right)\Big] &= \mathbb{E}\Big[X_t\,\mathbb{E}\left(\mathbb{1}\big(\alpha(t+h) = i_0\big)\big|\mathcal{F}^\alpha_{t-}\right)\Big]\\ &= \mathbb{E}\Big[X_t\,\mathbb{E}\left(\mathbb{1}\big(\alpha(t+h) = i_0\big)\big|\mathcal{F}^\alpha_{t-} \vee \mathcal{F}^W_t\right)\Big]\\ &= \mathbb{E}\Big[\mathbb{1}\big(\alpha(t+h) = i_0\big)X_t\Big].\end{aligned}$$

This completes the proof. □

### A.2. **Proof of Lemma** 2.1

It suffices to prove the existence as the uniqueness is trivial. For two processes $Y^1(\cdot), Y^2(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R})$, we denote

$$D_t(Y^1, Y^2) = \Big(\mathbb{E}\sup_{0\le s\le t}\big|Y^1_s - Y^2_s\big|^2\Big)^{1/2}, \qquad 0 \le t \le T.$$

Under Assumption (A) we can define the mapping $\Pi : \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}) \to \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R})$, which maps each $y(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R})$ to the unique solution $Y(\cdot)$ of the following equation

$$dY_t = b\Big(t, Y_t, \mathbb{E}\big(\psi(y_t)\big|\mathcal{F}^\alpha_{t-}\big), u_t, \alpha_{t-}\Big)dt + \sigma\Big(t, Y_t, \mathbb{E}\big(\varphi(y_t)\big|\mathcal{F}^\alpha_{t-}\big), u_t, \alpha_{t-}\Big)dW_t, \quad Y_0 = x_0.$$

By Assumption (A), the Burkholder-Davis-Gundy inequality, and the Gronwall inequality, we can show that there exists a constant $C$ such that

$$\Big[D_t\big(\Pi(y^1), \Pi(y^2)\big)\Big]^2 \le C\int_0^t \Big(D_s(y^1, y^2)\Big)^2 ds \quad \forall\, y^1, y^2 \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}), 0 \le t \le T.$$

Next, we define $X^0(\cdot) \equiv x_0$, $X^{n+1} = \Pi(X^n) = \Pi^{n+1}(X^0)$ for $n \geq 0$. Then it follows from the above inequality that

$$\Big[D_T\big(X^n, X^{n+1}\big)\Big]^2 \leq \frac{CT}{n!}\Big[D_T\big(X^0, X^1\big)\Big]^2.$$

This shows that $\big(X^n(\cdot)\big)_{n\geq 0}$ is a Cauchy sequence in the complete metric space $\big(\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}), D_T\big)$. Therefore, this sequence converges to some process $X(\cdot)$ in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ which is a fixed point of $\Pi$. As a consequence, $X(\cdot)$ is the solution of (2.1). This completes the proof.

**Remark A.2.** The above proof uses an argument in [32]; see also [23]. An alternative proof can be carried out by freezing the sample paths of the switching process as in the proof of [25, Theorem 4.6].

## A.3. **Proof of Theorem** 3.4

According to [19, Theorem 5.15], for $(y,z,\nu) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$, there exists a unique solution $(Y,Z,\Lambda) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$ to the following BSDE

$$Y_t = \xi - \int_t^T F\Big(s, Y_s, Z_s, \mathbb{E}\big(\Psi\big(y_s\big)\big|\mathcal{F}^\alpha_{s-}\big), \mathbb{E}\big(\Phi\big(z_s\big)\big|\mathcal{F}^\alpha_{s-}\big)\Big)ds - \int_t^T Z_s dW_s - \int_t^T \Lambda_s \bullet dM_s.$$

Thus, we can define the mapping $\Upsilon : \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R}) \to \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$ which maps $(y,z,\lambda)$ to $(Y,Z,\Lambda)$.

We will show that $\Upsilon$ is a contraction mapping. Let $(y^i, z^i, \lambda^i) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $(Y^i, Z^i, \Lambda^i) = \Upsilon(y^i, z^i, \nu^i)$ for $i = 1,2$. Denote $(\hat{y}, \hat{z}, \hat{\lambda}) = (y^1 - y^2, z^1 - z^2, \lambda^1 - \lambda^2)$, $(\widehat{Y}, \widehat{Z}, \widehat{\Lambda}) = (Y^1 - Y^2, Z^1 - Z^2, \Lambda^1 - \Lambda^2)$, and

$$\widehat{F}(t) = F\Big(t, Y^1_t, Z^1_t, \mathbb{E}\big(\Psi\big(y^1_t\big)\big|\mathcal{F}^\alpha_{t-}\big), \mathbb{E}\big(\Phi\big(z^1_t\big)\big|\mathcal{F}^\alpha_{t-}\big)\Big) - F\Big(t, Y^2_t, Z^2_t, \mathbb{E}\big(\Psi\big(y^2_t\big)\big|\mathcal{F}^\alpha_{t-}\big), \mathbb{E}\big(\Phi\big(z^2_t\big)\big|\mathcal{F}^\alpha_{t-}\big)\Big).$$

Take $\beta > 0$ to be chosen later and apply generalized Itô formula (see [28, Theorem 18]) to $e^{\beta t}\big|\widehat{Y}\big|^2$ on $[0,T]$, we obtain

$$\begin{aligned}\big|\widehat{Y}(0)\big|^2 = &- \int_0^T e^{\beta t}\Big(\beta\big|\widehat{Y}(t)\big|^2 - 2\widehat{Y}(t)\widehat{F}(t)\Big)dt - \int_0^T e^{\beta t}\big|\widehat{Z}(t)\big|^2 dt \\ &- \sum_{i_0,j_0=1}^{m_0} \int_0^T e^{\beta t}\big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t) - 2\int_0^T e^{\beta t}\widehat{Y}(t)\widehat{Z}(t)dW(t) \\ &- 2\sum_{i_0,j_0=1}^{m_0} \int_0^T e^{\beta t}\widehat{Y}(t)\widehat{\Lambda}_{i_0j_0}(t)dM_{i_0j_0}(t). \end{aligned} \tag{A.1}$$

In view of the Cauchy-Schwarz inequality,

$$\mathbb{E}\bigg[\bigg(\int_0^T e^{2\beta t}\big|\widehat{Y}(t)\big|^2\big|\widehat{Z}(t)\big|^2 dt\bigg)^{1/2}\bigg] \leq \frac{e^{\beta T}}{2}\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\widehat{Y}(t)\big|^2 + \int_0^T \big|\widehat{Z}(t)\big|^2 dt\bigg] < \infty.$$

Similarly,

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_0^T e^{2\beta t}\big|\widehat{Y}(t)\big|^2\big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg)^{1/2}\bigg] &\leq \frac{e^{\beta T}}{2}\mathbb{E}\bigg[\sup_{0\leq t\leq T}\big|\widehat{Y}(t)\big|^2 + \int_0^T \big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg] \\ &< \infty.\end{aligned}$$

Hence, by the Burkholder-Davis-Gundy inequality, the stochastic integrals

$$\int_0^t e^{\beta s}\widehat{Y}(s)\widehat{Z}(s)dW(s) \ \text{ and } \ \int_0^t e^{\beta s}\widehat{Y}(s)\widehat{\Lambda}_{i_0j_0}(s)dM_{i_0j_0}(s),\ i_0,j_0\in\mathcal{M},\ 0\le t\le T$$

are martingales. Taking expectation in (A.1), we obtain

$$\begin{aligned}
&\mathbb{E}\big|\widehat{Y}(0)\big|^2+\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(\beta\big|\widehat{Y}(t)\big|^2+\big|\widehat{Z}(t)\big|^2\Big)dt+\sum_{i_0,j_0=1}^{m_0}\int_0^T e^{\beta t}\big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg]\\
&=2\mathbb{E}\bigg[\int_0^T e^{\beta t}\widehat{Y}(t)\widehat{F}(t)dt\bigg]\\
&\le 2K\mathbb{E}\bigg\{\int_0^T e^{\beta t}\big|\widehat{Y}(t)\big|\Big[\big|\widehat{Y}(t)\big|+\big|\widehat{Z}(t)\big|+K\Big(\mathbb{E}\big(|\hat{y}(t)|\big|\mathcal{F}^{\alpha}_{t-}\big)+\mathbb{E}\big(|\hat{z}(t)|\big|\mathcal{F}^{\alpha}_{t-}\big)\Big)\Big]dt\bigg\}\\
&\le\big(2K+8K^2+16K^4\big)\mathbb{E}\bigg[\int_0^T e^{\beta t}\big|\widehat{Y}(t)\big|^2dt\bigg]\\
&\qquad+\frac{1}{4}\mathbb{E}\bigg\{\int_0^T e^{\beta t}\Big\{\big|\widehat{Z}(t)\big|^2+\Big[\mathbb{E}\big(|\hat{y}(t)|\big|\mathcal{F}^{\alpha}_{t-}\big)\Big]^2+\Big[\mathbb{E}\big(|\hat{z}(t)|\big|\mathcal{F}^{\alpha}_{t-}\big)\Big]^2\Big\}dt\bigg\}\\
&\le\big(2K+8K^2+16K^4\big)\mathbb{E}\bigg[\int_0^T e^{\beta t}\big|\widehat{Y}(t)\big|^2dt\bigg]+\frac{1}{4}\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(\big|\widehat{Z}(t)\big|^2+|\hat{y}(t)|^2+|\hat{z}(t)|^2\Big)dt\bigg].
\end{aligned}$$

Taking $\beta=1+2K+8K^2+16K^4$ in the above inequalities leads to

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(\big|\widehat{Y}(t)\big|^2+\big|\widehat{Z}(t)\big|^2\Big)dt+\sum_{i_0,j_0=1}^{m_0}\int_0^T e^{\beta t}\big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg]\\
&\le\frac{1}{3}\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(|\hat{y}(t)|^2+|\hat{z}(t)|^2\Big)dt\bigg]\\
&\le\frac{1}{3}\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(|\hat{y}(t)|^2+|\hat{z}(t)|^2\Big)dt+\sum_{i_0,j_0=1}^{m_0}\int_0^T e^{\beta t}\big|\hat{\lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg].
\end{aligned}$$

Since $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\times\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R})$ is a Banach space endowed with the norm

$$\|(Y,Z,\Lambda)\|_\beta=\mathbb{E}\bigg[\int_0^T e^{\beta t}\Big(\big|\widehat{Y}(t)\big|^2+\big|\widehat{Z}(t)\big|^2\Big)dt+\sum_{i_0,j_0=1}^{m_0}\int_0^T e^{\beta t}\big|\widehat{\Lambda}_{i_0j_0}(t)\big|^2 d\big[M_{i_0j_0}\big](t)\bigg]^{1/2},$$

it follows that $\big\|\big(\widehat{Y},\widehat{Z},\widehat{\Lambda}\big)\big\|_\beta\le(1/3)\big\|\big(\hat{y},\hat{z},\hat{\lambda}\big)\big\|_\beta$ or equivalently,

$$\big\|\Upsilon(y^1,z^1,\lambda^1)-\Upsilon(y^2,z^2,\lambda^2)\big\|_\beta\le\frac{1}{3}\big\|(y^1,z^1,\lambda^1)-(y^2,z^2,\lambda^2)\big\|_\beta.$$

Hence, $\Upsilon$ is a contraction mapping and consequently has a unique fixed point that is the solution to the desired BSDE. □

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