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LOCAL STATE OBSERVATION FOR STOCHASTIC HYPERBOLIC EQUATIONS *, **

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Abstract. In this paper, we solve a local state observation problem for stochastic hyperbolic equations without boundary conditions, which is reduced to a local unique continuation property for these equations. This result is proved by a global Carleman estimate. As far as we know, this is the first result in this topic.

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1. Introduction

Let $T>0$ and $G\subset\mathbb{R}^n$ $(n\in\mathbb{N})$ be a bounded domain. Throughout this paper, we will use C to denote a generic positive constant depending only on T and G, which may change from line to line. Put $Q=G\times(0,T)$.

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ with $\mathbb{F}\overset{\triangle}{=}\{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined. Assume that H is a Fréchet space. Let $L^2_{\mathbb{F}}(0,T;H)$ be the Fréchet space consisting all H-valued $\mathbb{F}$-adapted process $X(\cdot)$ such that $\mathbb{E}|X(\cdot)|^2_{L^2(0,T;H)}<+\infty$, on which the canonical quasi-norm is endowed. By $L^\infty_{\mathbb{F}}(0,T;H)$ we denote the Fréchet space of all H-valued $\mathbb{F}$-adapted bounded processes equipped with the canonical quasi-norm and by $L^2_{\mathbb{F}}(\Omega;C([0,T];H))$ the Fréchet space of all H-valued $\mathbb{F}$-adapted continuous precesses $X(\cdot)$ with $\mathbb{E}|X(\cdot)|^2_{C([0,T];H)}<+\infty$, equipped with the canonical quasi-norm.

This paper is devoted to the study of a local state observation problem for the following stochastic hyperbolic equation:

$$\sigma dz_t-\Delta z dt=\big(b_1 z_t+b_2\cdot\nabla z+b_3 z\big)dt+b_4 z dW(t)\qquad\text{in } Q,\tag{1.1}$$

where $\sigma\in C^1(\overline{Q})$ is positive and

$$b_1\in L^\infty_{\mathbb{F}}(0,T;L^\infty_{loc}(G)),\qquad b_2\in L^\infty_{\mathbb{F}}(0,T;L^\infty_{loc}(G;\mathbb{R}^n)),$$
$$b_3\in L^\infty_{\mathbb{F}}(0,T;L^n_{loc}(G)),\qquad b_4\in L^\infty_{\mathbb{F}}(0,T;L^\infty_{loc}(G)).$$

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Put

$$\mathbb{H}_T \triangleq L^2_{\mathbb{F}}(\Omega; C([0,T]; H^1_{loc}(G))) \cap L^2_{\mathbb{F}}(\Omega; C^1([0,T]; L^2_{loc}(G))). \tag{1.2}$$

Definition 1.1. *We call $z \in \mathbb{H}_T$ a solution of equation (1.1) if for each $t \in [0,T]$, $G' \subset\subset G$ and $\eta \in H^1_0(G')$, it holds that*

$$\begin{aligned} &\int_{G'} \sigma z_t(t,x)\eta(x)dx - \int_{G'} \sigma z_t(0,x)\eta(x)dx - \int_0^t \int_{G'} \sigma_t(t,x) z_t(t,x)\eta(x)dx \\ &= \int_0^t \int_{G'} \Big[-\nabla z(s,x)\cdot\nabla\eta(x) + \big(b_1 z_t + b_2\cdot\nabla z + b_3 z\big)\eta(x)\Big] dxds + \int_0^t\int_{G'} b_4 z\eta(x)dxdW(s), \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{1.3}$$

Let $S \subset\subset G$ be a C^2-hypersurface. Let $x_0 \in S\setminus\partial G$ and suppose that S divides the ball $B_\rho(x_0) \subset G$, centered at x_0 and with radius ρ, into two parts $\mathcal{D}^+_\rho$ and $\mathcal{D}^-_\rho$. Denote as usual by $\nu(x)$ the unit normal vector to S at x inward to $\mathcal{D}^+_\rho$.

Let y be a solution to equation (1.1) and $\varepsilon > 0$ a fixed small number. More precisely, this paper is devoted to the following local state observation problem:

(Pu) Can y in $\mathcal{D}^+_\rho \times (\varepsilon, T-\varepsilon)$ be uniquely determined by the values of y in $\mathcal{D}^-_\rho \times (0,T)$?

Remark 1.1. *In the formulation of Problem* **(Pu)***, we introduce a number ε. Generally speaking, ε depends on σ and ρ and cannot be zero. Indeed, due to the finite speed of the propagation of the solution to (1.1), one need to wait for some time to observe the solution; that is to say, one loses some information for time. The number ε characterizes such loss.*

In other words, Problem **(Pu)** concerns that whether the state in one side of S uniquely determines the state in the other side. Clearly, it is equivalent to the following unique continuation problem:

(Pu1) Can we conclude that $y = 0$ in $\mathcal{D}^+_\rho \times (\varepsilon, T-\varepsilon)$, provided that $y = 0$ in $\mathcal{D}^-_\rho \times (0,T)$?

State observation problems for deterministic PDEs are studied extensively in the literature. Generally speaking, a state observation result is a statement in the following sense:

Let u be a solution of a PDE and two regions $\mathcal{M}_1 \subset \mathcal{M}_2$. Then u is determined uniquely in $\mathcal{M}_2$ by the observation on $\mathcal{M}_1$.

Problem **(Pu)** is a natural generalization of the unique continuation problems under the stochastic setting, i.e., $\mathcal{M}_1 = \mathcal{D}^-_\rho$ and $\mathcal{M}_2 = B_\rho(x_0)$.

State observation problem has a close connection with the unique continuation problem. In many cases, such as Problem **(Pu)** and Problem **(Pu1)**, these two kinds of problems are equivalent.

There is a long history for the study of unique continuation property (UCP for short) for deterministic PDEs. Classical results date back to Cauchy-Kovalevskaya theorem and Holmgren's uniqueness theorem. These two results need the coefficients of the PDE analytic to get the UCP. In 1939, T. Carleman introduced in [4] a new method, which was based on weighted estimates, to prove UCP for two dimensional elliptic equations with L^∞ coefficients. This method, which is called "Carleman estimates method" nowadays, turns out to be quite powerful and has been developed extensively in the literature and becomes the most useful tool to solve state observation problems and obtain UCP for PDEs (e.g. [5,7,8,22,23,27]). In particular, unique continuation results for solutions of hyperbolic equations across hypersurfaces were studied by many authors (e.g. [6, 9, 18, 19, 21]).

Compared with the deterministic PDEs, there are very few results concerning UCP for stochastic PDEs. To our best knowledge, [24] is the first result for state observation problems of stochastic PDEs, in which the author shows that a solution to a stochastic parabolic equation can be determined by an observation in any subdomain and this result was improved in [10, 15] where some quantitative results were obtained. The first result of state observation problems for stochastic hyperbolic equations was obtained in [25]. Some improvements were made in [14, 16]. The results in [14] and [25] concerned the global state observation problem for stochastic hyperbolic equations with a homogeneous Dirichlet boundary condition, i.e., they concluded that the solution to a stochastic hyperbolic equation can be determined by an observation in a large enough subdomain. In this paper, we focus on the local state observation problem for stochastic hyperbolic equations without boundary condition, that is, can a solution be determined locally?

To present the main result of this paper, let us first introduce the following notion.

Definition 1.2. *Let $x_0 \in S$ and $K > 0$. S is said to satisfy the outer paraboloid condition with K at x_0 if there exists a neighborhood $\mathcal{V}$ of x_0 and a paraboloid $\mathcal{P}$ tangential to S at x_0 and $\mathcal{P} \cap \mathcal{V} \subset \mathcal{D}_\rho^-$ with $\mathcal{P}$ congruent to $x_1 = K\sum_{j=2}^{n} x_j^2$.*

The main result in this paper is the following one.

Theorem 1.1. *Let $x_0 \in S\setminus\partial S$ such that $\dfrac{\partial\sigma(x_0,T/2)}{\partial\nu} < 0$, and let S satisfy the outer paraboloid condition with*

$$K < \frac{-\frac{\partial\sigma}{\partial\nu}(x_0,T/2)}{4(|\sigma|_{L^\infty(B_\rho(x_0,T/2))}+1)}. \tag{1.4}$$

Then for any $z \in \mathbb{H}_T$ which solves equation (1.1) satisfying that

$$z = 0 \quad \text{on } \mathcal{D}_\rho^- \times (0,T),\ \mathbb{P}\text{-a.s.}, \tag{1.5}$$

there is a neighborhood $\mathcal{U}$ of x_0 and $\varepsilon \in (0,T/2)$ such that

$$z = 0 \quad \text{in } \big(\mathcal{U} \cap \mathcal{D}_\rho^+\big) \times (\varepsilon, T-\varepsilon),\ \mathbb{P}\text{-a.s.} \tag{1.6}$$

Remark 1.2. *Theorem 1.1 concludes that the value of z on $\big(\mathcal{U} \cap \mathcal{D}_\rho^+\big) \times (\varepsilon, T-\varepsilon)$ can be determined by the observation on $\mathcal{D}_\rho^- \times (0,T)$, $\mathbb{P}$-a.s., which answer the state observation problem* **(Pu)**.

Theorem 1.1 is an immediate corollary of the following result.

Theorem 1.2. *Let $x_0 \in S\setminus\partial S$ such that $\dfrac{\partial\sigma(x_0,T/2)}{\partial\nu} < 0$, and let S satisfy the outer paraboloid condition with*

$$K < \frac{-\frac{\partial\sigma}{\partial\nu}(x_0,T/2)}{4(|\sigma|_{L^\infty(B_\rho(x_0,T/2))}+1)}. \tag{1.7}$$

Let $z \in \mathbb{H}_T$ be a solution of the equation (1.1) satisfying that

$$z = \frac{\partial z}{\partial\nu} = 0 \quad \text{on } (0,T)\times S,\ \ \mathbb{P}\text{-a.s.} \tag{1.8}$$

Then, there is a neighborhood $\mathcal{U}$ of x_0 and $\varepsilon \in (0,T/2)$ such that

$$z = 0 \quad \text{in } (\mathcal{U} \cap \mathcal{D}_\rho^+) \times (\varepsilon, T-\varepsilon),\ \ \mathbb{P}\text{-a.s.} \tag{1.9}$$

Remark 1.3. *In Theorem 1.2, we assume that $\dfrac{\partial\sigma(x_0,T/2)}{\partial\nu} < 0$. It is related to the propagation of wave. This is a reasonable assumption since the UCP may not hold if it is not fulfilled (e.g. [1]). It can be regarded as a kind of pseudoconvex condition (e.g. [7, Chapter XXVII]).*

Remark 1.4. *If S is a hyperplane, then Condition 1.7 always satisfies since we can take $K = 0$.*

Remark 1.5. *From Theorem 1.2, one can get many classical UCP results for deterministic hyperbolic equations (e.g. [6, 17, 21]).*

Remark 1.6. *It is interesting to consider the state observation problem for more general stochastic hyperbolic equations, that is, the term* Δz *being replaced by* $\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(b^{ij}\frac{\partial z}{\partial x_i}\Big)$ *with* $b^{ij}(\cdot)\in L^2_{\mathbb{F}}(\Omega;\ C^1([0,T];W^{2,\infty}(G)))$ $(i,j=1,\cdots,n)$*. However, we do not know how to handle such case now. Indeed, as far as we know, even for deterministic hyperbolic equation, when* Δz *being replaced by* $\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a^{ij}\frac{\partial z}{\partial x_i}\Big)$ *with* $a^{ij}(\cdot)\in C^1([0,T];W^{2,\infty}(G)))$ $(i,j=1,\cdots,n)$*, the corresponding unique continuation problem is not solved. In our opinion, the main difficulty to solve this problem is how to construct a suitable weight function to establishing a Carleman estimate like Theorem 4.1.*

Similar to the deterministic settings, we shall use the stochastic versions of Carleman estimate for stochastic hyperbolic equations to establish our estimate. Carleman estimates for stochastic PDEs are studied extensively in recent years (see [3, 11–14, 20, 26] and the references therein). Carleman estimate for stochastic hyperbolic equations was first obtained in [25]. Compared with the result in [25], we need to handle a more complex case (see Section 2 for more details).

The rest of this paper is organized as follows. In Section 2, we derive a point-wise estimate for stochastic hyperbolic operator, which is crucial for us to establish the desired Carleman estimate in this paper. In Section 3, we explain the choice of weight function in the Carleman estimate. Section 4 is devoted to the proof of a Carleman estimate while Section 5 is addressed to the proof of the main result.

2. A point-wise estimate for stochastic hyperbolic operator

We introduce the following point-wise Carleman estimate for stochastic hyperbolic operators. This estimate has its own independent interest and will be of particular importance in the proof for the main result. In the rest of this paper, for simplicity, we use the notation $y_j=\dfrac{\partial y}{\partial x_j}$, $j=1,\cdots,n$ for the partial derivative of a function y with respect to x_j, where x_j is the j-th coordinate of a generic point $x=(x_1,\cdots,x_n)$ in $\mathbb{R}^n$.

Lemma 2.1. *Let* $\ell,\Psi\in C^2((0,T)\times\mathbb{R}^n)$*. Assume* u *is an* $H^2_{loc}(\mathbb{R}^n)$*-valued* $\mathbb{F}$*-adapted process such that* u_t *is an* $L^2(\mathbb{R}^n)$*-valued semimartingale. Set* $\theta=e^\ell$ *and* $v=\theta u$*. Then, for a.e.* $x\in\mathbb{R}^n$ *and* $\mathbb{P}$*-a.s.* $\omega\in\Omega$*,*

$$
\begin{aligned}
&\theta\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^{n}b^{ij}\ell_i v_j+\Psi v\Big)\Big[\sigma du_t-\sum_{i,j=1}^{n}(b^{ij}u_i)_j dt\Big]\\
&+\sum_{i,j=1}^{n}\Big[\sum_{i',j'=1}^{n}\Big(2b^{ij}b^{i'j'}\ell_{i'}v_i v_{j'}-b^{ij}b^{i'j'}\ell_i v_{i'}v_{j'}\Big)-2b^{ij}\ell_t v_i v_t+\sigma b^{ij}\ell_i v_t^2\\
&\qquad+\Psi b^{ij}v_i v-\Big(A\ell_i+\frac{1}{2}\Psi_i\Big)b^{ij}v^2\Big]_j dt\\
&+d\Big[\sigma\sum_{i,j=1}^{n}b^{ij}\ell_t v_i v_j-2\sigma\sum_{i,j=1}^{n}b^{ij}\ell_i v_j v_t+\sigma^2\ell_t v_t^2-\sigma\Psi v_t v+\Big(\sigma A\ell_t+\frac{1}{2}(\sigma\Psi)_t\Big)v^2\Big] \qquad (2.1)\\
=&\Big\{\Big[(\sigma^2\ell_t)_t+\sum_{i,j=1}^{n}(\sigma b^{ij}\ell_i)_j-\sigma\Psi\Big]v_t^2-2\sum_{i,j=1}^{n}[(\sigma b^{ij}\ell_j)_t+b^{ij}(\sigma\ell_t)_j]v_i v_t\\
&+\sum_{i,j=1}^{n}\Big[(\sigma b^{ij}\ell_t)_t+\sum_{i',j'=1}^{n}\Big(2b^{ij'}(b^{i'j}\ell_{i'})_{j'}-(b^{ij}b^{i'j'}\ell_{i'})_{j'}\Big)+\Psi b^{ij}\Big]v_i v_j
\end{aligned}
$$

$$+Bv^2+\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j+\Psi v\Big)^2\Big\}dt+\sigma^2\theta^2\ell_t(du_t)^2,$$

where $(du_t)^2$ *denotes the quadratic variation process of* u_t*, and* A *and* B *are stated as follows:*

$$\begin{cases} A\overset{\triangle}{=}\sigma(\ell_t^2-\ell_{tt})-\displaystyle\sum_{i,j=1}^n(b^{ij}\ell_i\ell_j-b^{ij}_j\ell_i-b^{ij}\ell_{ij})-\Psi,\\ B\overset{\triangle}{=}A\Psi+(\sigma A\ell_t)_t-\displaystyle\sum_{i,j}(Ab^{ij}\ell_i)_j+\frac{1}{2}\Big[(\sigma\Psi)_{tt}-\sum_{i,j=1}^n(b^{ij}\Psi_i)_j\Big].\end{cases}\tag{2.2}$$

Remark 2.1. *When* $\sigma=1$*, the equality* (2.1) *had been established in [25]. The computation for the general* σ *is more complex. One needs to handle the terms concerning* σ *carefully.*

Proof. By $v(t,x)=\theta(t,x)u(t,x)$, we have

$$u_t=\theta^{-1}(v_t-\ell_t v),u_j=\theta^{-1}(v_j-\ell_j v)$$

for $j=1,2,\ldots,n$. Then, for that θ is deterministic, we have

$$\sigma du_t=\sigma d\big[\theta^{-1}(v_t-\ell_t v)\big]=\sigma\theta^{-1}\Big[dv_t-2\ell_t v_t dt+(\ell_t^2-\ell_{tt})v dt\Big].\tag{2.3}$$

Moreover, we find that

$$\begin{aligned}\sum_{i,j=1}^n(b^{ij}u_i)_j&=\sum_{i,j=1}^n\big[b^{ij}\theta^{-1}(v_i-\ell_i v)\big]_j\\ &=\theta^{-1}\sum_{i,j=1}^n\big[(b^{ij}v_i)_j-2b^{ij}\ell_i v_j+(b^{ij}\ell_i\ell_j-b^{ij}_j\ell_i-b^{ij}\ell_{ij})v\big].\end{aligned}\tag{2.4}$$

As an immediate result of (2.3) and (2.4), we have that

$$\begin{aligned}&\sigma du_t-\sum_{i,j=1}^n(b^{ij}u_i)_j dt\\ =\;&\theta^{-1}\Big\{\Big[\sigma dv_t-\sum_{i,j=1}^n(b^{ij}v_i)_j dt\Big]+\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j\Big)dt\\ &+\Big[\sigma(\ell_t^2-\ell_{tt})-\sum_{i,j=1}^n(b^{ij}\ell_i\ell_j-b^{ij}_j\ell_i-b^{ij}\ell_{ij})\Big]v dt\Big\}.\end{aligned}\tag{2.5}$$

Therefore, by (2.5) and the definition of A in (2.2), we get

$$\begin{aligned}
&\theta\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j+\Psi v\Big)\Big[\sigma du_t-\sum_{i,j=1}^n (b^{ij}u_i)_j dt\Big]\\
&=\Big(-2\sigma^2\ell_t v_t+2\sigma\sum_{i,j=1}^n b^{ij}\ell_i v_j+\sigma\Psi v\Big)dv_t\\
&\quad+\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j+\Psi v\Big)\Big[-\sum_{i,j=1}^n (b^{ij}v_i)_j+Av\Big]dt\\
&\quad+\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j+\Psi v\Big)^2 dt.
\end{aligned}\tag{2.6}$$

Let us continue to analyze the first two terms in the right-hand side of (2.6).

For the first term in the right-hand side of (2.6), we find that

$$-2\sigma^2\ell_t v_t dv_t=d(-\sigma^2\ell_t v_t^2)+\sigma^2\ell_t(dv_t)^2+(\sigma^2\ell_t)_t v_t^2 dt,$$

$$2\sigma\sum_{i,j=1}^n b^{ij}\ell_i v_j dv_t=d\Big(2\sigma v_t\sum_{i,j=1}^n b^{ij}\ell_i v_j\Big)-2\sum_{i,j=1}^n(\sigma b^{ij}\ell_i)_t v_j v_t dt-2\sigma\sum_{i,j=1}^n b^{ij}\ell_i v_{jt}v_t dt,$$

and

$$\sigma\Psi v dv_t=d(\Psi\sigma v v_t)-(\sigma\Psi)_t v v_t dt-\sigma\Psi v_t^2 dt.$$

Consequently, we get that

$$\begin{aligned}
&\Big(-2\sigma\ell_t v_t+2\sum_{i,j=1}^n b^{ij}\ell_i v_j+\Psi v\Big)\sigma dv_t\\
&=d\Big[-\sigma^2\ell_t v_t^2+2\sigma v_t\sum_{i,j=1}^n b^{ij}\ell_i v_j+\sigma\Psi v v_t-\frac{1}{2}(\sigma\Psi)_t v^2\Big]\\
&\quad-\Big\{\sum_{i,j=1}^n(\sigma b^{ij}\ell_i v_t^2)_j-\Big[(\sigma\ell_t)_t+\sum_{i,j=1}^n(\sigma b^{ij}\ell_i)_j-\sigma\Psi\Big]v_t^2\\
&\qquad+2\sum_{i,j=1}^n(\sigma b^{ij}\ell_i)_t v_j v_t-\frac{1}{2}(\sigma\Psi)_{tt}v^2\Big\}dt+\sigma^2\ell_t(dv_t)^2.
\end{aligned}\tag{2.7}$$

In a similar manner, for the second term in the right-hand side of (2.6), we find that

$$\begin{aligned}
&-2\sigma\ell_t v_t\Big[-\sum_{i,j=1}^n(b^{ij}v_i)_j+Av\Big]\\
&=2\Big[\sum_{i,j=1}^n(\sigma b^{ij}\ell_t v_i v_t)_j-\sum_{i,j=1}^n b^{ij}(\sigma\ell_t)_j v_i v_t\Big]+\sum_{i,j=1}^n(\sigma b^{ij}\ell_t)_t v_i v_j\\
&\quad-\Big(\sigma\sum_{i,j=1}^n b^{ij}\ell_t v_i v_j+\sigma A\ell_t v^2\Big)_t+(\sigma A\ell_t)_t v^2,
\end{aligned}\tag{2.8}$$

$$
\begin{aligned}
&2\sum_{i,j=1}^{n} b^{ij}\ell_i v_j\Big[-\sum_{i,j=1}^{n}(b^{ij}v_i)_j+Av\Big]\\
&=-\sum_{i,j=1}^{n}\Big[\sum_{i',j'=1}^{n}\Big(2b^{ij}b^{i'j'}\ell_{i'}v_iv_{j'}-b^{ij}b^{i'j'}\ell_iv_{i'}v_{j'}\Big)-Ab^{ij}\ell_iv^2\Big]_j\\
&\quad+\sum_{i,j,i',j'=1}^{n}\Big[2b^{ij'}(b^{i'j}\ell_{i'})_{j'}-(b^{ij}b^{i'j'}\ell_{i'})_{j'}\Big]v_iv_j-\sum_{i,j=1}^{n}(Ab^{ij}\ell_i)_jv^2,
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
&\Psi v\Big[-\sum_{i,j=1}^{n}(b^{ij}v_i)_j+Av\Big]\\
&=-\sum_{i,j=1}^{n}\Big(\Psi b^{ij}vv_i-\frac{1}{2}\Psi_ib^{ij}v^2\Big)_j+\Psi\sum_{i,j=1}^{n}b^{ij}v_iv_j+\Big[-\frac{1}{2}\sum_{i,j=1}^{n}(b^{ij}\Psi_i)_j+A\Psi\Big]v^2.
\end{aligned}
\tag{2.10}
$$

Finally, from (2.6) to (2.10), we arrive at the desired equality (2.1). □

3. Choice of the weight function

In this section, we explain the choice of the weight function which will be used to establish our global Carleman estimate. Although such kind of functions are already used in [2], we give full details for the sake of completion and the convenience of readers.

The weight function is given as follows:

$$
\varphi(x,t)=hx_1+\frac{1}{2}\sum_{j=2}^{n}x_j^2+\frac{1}{2}\Big(t-\frac{T}{2}\Big)^2+\frac{1}{2}\tau,
\tag{3.1}
$$

where h and τ are suitable parameters, whose precise meanings will be explained in the sequel.

Without loss of generality, we assume that

$$
0=(0,\cdots,0)\in S\setminus\partial S
$$

and

$$
\nu(0)=(1,0,\cdots,0).
$$

For some $r>0$, for that S is C^2, we can parameterize S in the neighborhood of the origin by

$$
x_1=\gamma(x_2,\cdots,x_n),\quad |x_2|^2+\cdots+|x_n|^2<r.
\tag{3.2}
$$

For notational brevity, denote

$$
a(x,t)=\frac{\partial\sigma}{\partial\nu}.
$$

Hereafter, we set

$$
\begin{cases}
B_r\Big(0,\dfrac{T}{2}\Big)=\Big\{(x,t):\,(x,t)\in\mathbb{R}^{n+1},\,|x|^2+\Big(t-\dfrac{T}{2}\Big)^2<r^2\Big\},\\
B_r(0)=\{x:\,x\in\mathbb{R}^n,\,|x|<r\}.
\end{cases}
\tag{3.3}
$$

By (1.7), we have that

$$\begin{cases} -\alpha_0 = a\Big(0, \dfrac{T}{2}\Big) < 0, \\ K < \dfrac{\alpha_0}{4(|\sigma|_{L^\infty(B_r(0,T/2))} + 1)}, \\ -K\displaystyle\sum_{j=2}^n x_j^2 < \gamma(x_2, \cdots, x_n), \quad \text{if } \displaystyle\sum_{j=2}^n x_j^2 < r. \end{cases} \tag{3.4}$$

Let

$$M_1 = \max\left\{|\sigma|_{C^1(B_r(0,0))}, 1\right\}. \tag{3.5}$$

Denote

$$\mathcal{D}_r^- = \{x : x \in B_r(0),\ x_1 < \gamma(x_2, \cdots, x_n)\}, \quad \mathcal{D}_r^+ = B_r(0) \setminus \overline{D_r^-}.$$

For any $\alpha \in (0, \alpha_0)$, in accordance with the continuity of $a(x,t)$ and the first inequality in (3.4), it is clear that there exists a $\delta_0 > 0$ small enough such that $0 < \delta_0 < \min\{1, r^2\}$, which would be specified later, and

$$a(x,t) < -\alpha \ \text{ if } \ |x|^2 + \Big(t - \frac{T}{2}\Big)^2 \le \delta_0. \tag{3.6}$$

Letting $M_0 = |\sigma|_{L^\infty(B_r(0,T/2))}$, by the second inequality in (3.4), we can always choose $K > 0$ so large that

$$K < \frac{1}{2h} < \frac{\alpha}{4(M_0+1)}. \tag{3.7}$$

Following immediately from (3.7), we have that

$$1 - 2hK > 0, \quad h\alpha - 2(M_0+1) > 0. \tag{3.8}$$

For K and h such chosen, we will further take $\tau \in (0,1)$ so small that

$$\Big|\max\Big\{\frac{K}{1-2hK}, \frac{1}{2h}\Big\}\Big|^2 \tau^2 + \frac{2\tau}{1-2hK} \le \delta_0. \tag{3.9}$$

For convenience of notations, by denoting $\mu_0(\tau)$ the term in the left hand side of (3.9) and letting $\mathcal{A}_0 = \min\{\sigma, 1\}$, we further assume that

$$\begin{cases} h^2\mathcal{A}_0 > 2hM_1\sqrt{\mu_0(\tau)} + 2M_1\mu_0(\tau), \\ \alpha h > 2(M_1^2 + M_1)\sqrt{\mu_0(\tau)} - (M_0^2 + nM_0) - (n-1). \end{cases} \tag{3.10}$$

For any positive number μ with $2\mu > \tau$, let

$$Q_\mu = \Big\{(x,t) \in \mathbb{R}^{n+1} \Big| \, x_1 > \gamma(x_2, x_3, \cdots, x_n),\ \sum_{j=2}^N x_j^2 < \delta_0, \varphi(x,t) < \mu\Big\}. \tag{3.11}$$

The set Q_μ defined above is not empty. To show this, it is only to prove that the defining conditon $\varphi(x,t) < \mu$ is compatible with the first defining condition, i.e., $x_1 > \gamma(x_2, x_3, \cdots, x_n)$. By assumption, we know that

$$\gamma(x_2, x_3, \cdots, x_n) > -K\sum_{j=2}^n x_j^2,$$

then together with the first inequality in (3.8), we have that

$$\begin{aligned}\varphi(x,t) &\geq -hK\sum_{j=2}^{n}x_j^2+\frac{1}{2}\sum_{j=2}^{n}x_j^2+\frac{1}{2}\left(t-\frac{T}{2}\right)^2+\frac{1}{2}\tau\\ &= \left(\frac{1}{2}-Kh\right)\sum_{j=2}^{n}x_j^2+\frac{1}{2}\left(t-\frac{T}{2}\right)^2+\frac{1}{2}\tau\\ &> \frac{\tau}{2}.\end{aligned}$$

Noting that $(x,t)\in Q_\mu$ implies $\varphi(x,t)<\mu$, together with $2\mu>\tau$, we see by definition that $Q_\mu\neq\emptyset$ as desired.

In what follows, we will show that how to determine the number δ_0 appearing in (3.9). Let $(x,t)\in\overline{Q}_\tau$. Here Q_τ is defined as in (3.11) for $2\tau>\tau$. From the definition of Q_τ and noting that

$$\gamma(x_2,x_3,\cdots,x_n)>-K\sum_{j=2}^{n}x_j^2,$$

we find that

$$x_1\leq-\frac{\tau}{2h}\sum_{j=2}^{n}x_j^2-\frac{1}{2h}\left(t-\frac{T}{2}\right)^2+\frac{\tau}{2h}\leq\frac{\tau}{2h}. \tag{3.12}$$

On the other hand, by

$$-K\sum_{j=2}^{n}x_j^2\leq x_1,$$

we obtain that

$$-Kh\sum_{j=2}^{n}x_j^2+\frac{1}{2}\sum_{j=2}^{n}x_j^2+\frac{1}{2}\left(t-\frac{T}{2}\right)^2+\frac{1}{2}\tau\leq\tau.$$

Thus

$$\sum_{j=2}^{n}x_j^2<\frac{\tau}{1-2Kh}.$$

We then get that

$$-x_1\leq K\sum_{j=2}^{n}x_j^2<\frac{K\tau}{1-2Kh}. \tag{3.13}$$

Combining (3.12) and (3.13), we arrive at

$$|x_1|\leq\max\left\{\frac{K}{1-2hK},\ \frac{1}{2h}\right\}\tau. \tag{3.14}$$

Thus, by the restriction imposed on $\varphi(x,t)$ in the definition of Q_τ and (3.13), we find that

$$\begin{aligned}\tau\ &>\varphi(x,t)=hx_1+\frac{1}{2}\sum_{j=2}^{n}x_j^2+\frac{1}{2}\left(t-\frac{T}{2}\right)^2+\frac{\tau}{2}\\ &>-\frac{Kh\tau}{1-2Kh}+\frac{1}{2}\sum_{j=2}^{n}x_j^2+\frac{1}{2}\left(t-\frac{T}{2}\right)^2+\frac{\tau}{2}.\end{aligned} \tag{3.15}$$

This gives that

$$\Big(t-\frac{T}{2}\Big)^2<\frac{2Kh\tau}{1-2Kh}+\tau=\frac{\tau}{1-2Kh}. \tag{3.16}$$

Correspondingly, we have that

$$\begin{aligned}|x|^2+\Big(t-\frac{T}{2}\Big)^2 &= x_1^2+\sum_{j=2}^n x_j^2+\Big(t-\frac{T}{2}\Big)^2\\ &\le \Big|\max\Big\{\frac{K}{1-2Kh},\frac{1}{2h}\Big\}\Big|^2\tau^2+\frac{2\tau}{1-2Kh}.\end{aligned}$$

Returning back to (3.6), by (3.13), (3.14) and (3.16), we choose the δ_0 in the following style:

$$\delta_0>\mu_0(\tau)=\Big|\max\Big\{\frac{K}{1-2Kh},\frac{1}{2h}\Big\}\Big|^2\tau^2+\frac{2\tau}{1-2Kh}. \tag{3.17}$$

4. A global Carleman estimate

This section is devoted to establishing a global Carleman estimate for the stochastic hyperbolic operator presented in Section 1, based on the point-wise Carleman estimate given in Section 2. It will be shown that it is the key to the proof of the main result.

We have the following global Carleman estimate.

Theorem 4.1. *Let u be an $H^2_{loc}(\mathbb{R}^n)$-valued $\mathbb{F}$-adapted process such that u_t is an $L^2(\mathbb{R}^n)$-valued semimartingale. If u is supported in Q_τ, then there exist a constant $C>0$ and an $s_0>0$ depending on σ,τ such that for all $s\ge s_0$, we have that*

$$\begin{aligned}&\mathbb{E}\int_{Q_\tau}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)\big(\sigma du_t-\Delta u dt\big)dx\\ &\ge C\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt\\ &\quad+\mathbb{E}\int_{Q_\tau}\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)^2dxdt+C\mathbb{E}\int_{Q_\tau}\sigma^2\theta^2\ell_t(du_t)^2dx.\end{aligned} \tag{4.1}$$

Proof. We apply the result of Lemma 2.1 to show our key Carleman estimate. Let $(b^{ij})_{1\le i,j\le n}$ be the unit matrix of nth order and let $\Psi=0$ in (2.1). Then we find that

$$\begin{aligned}&\theta\,(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v)\,(\,\sigma du_t-\Delta u dt)\\ &+\nabla\cdot\big[2(\nabla v\cdot\nabla\ell)\nabla v-|\nabla v|^2\nabla\ell-2\ell_t v_t\nabla v+\sigma v_t^2\nabla\ell-A\nabla\ell\, v^2\big]dt\\ &+d\big(\sigma\ell_t|\nabla v|^2-2\sigma\nabla\ell\cdot\nabla v v_t+\sigma^2\ell_t v_t^2+\sigma A\ell_t v^2\big)\\ =&\ \big\{\big[(\sigma^2\ell_t)_t+\nabla\cdot(\sigma\nabla\ell)\big]v_t^2-2\big[(\sigma\nabla\ell)_t+\nabla(\sigma\ell_t)\big]\cdot\nabla v\, v_t+\big[(\sigma\ell_t)_t+\Delta\ell\big]|\nabla v|^2\\ &+Bv^2+\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)^2\big\}dt+\sigma^2\theta^2\ell_t(du_t)^2.\end{aligned} \tag{4.2}$$

It is easy to show that A and B are stated respectively as follows:

$$\begin{cases}A=\sigma(\ell_t^2-\ell_{tt})-(|\nabla\ell|^2-\Delta\ell),\\ B=(\sigma A\ell_t)_t-\nabla\cdot(A\nabla\ell).\end{cases} \tag{4.3}$$

Now let $\ell = s\varphi^{-\lambda}$ with φ the weight function given by (3.1). Then, some simple computations show that

$$\begin{cases} \ell_t &= -s\lambda\varphi_t\varphi^{-n-1} = -s\lambda\Big(t-\dfrac{T}{2}\Big)\varphi^{-\lambda-1}, \\ \ell_{tt} &= s\lambda(\lambda+1)\Big(t-\dfrac{T}{2}\Big)^2\varphi^{-\lambda-2} - s\lambda\varphi^{-\lambda-1}, \\ \nabla\ell &= -s\lambda\varphi^{-\lambda-1}\nabla\varphi, \\ \Delta\ell &= s\lambda(\lambda+1)\varphi^{-\lambda-2}|\nabla\varphi|^2 - s\lambda\varphi^{-\lambda-1}\Delta\varphi, \\ \nabla\ell_t &= s\lambda(\lambda+1)\varphi^{-\lambda-2}\Big(t-\dfrac{T}{2}\Big)\nabla\varphi. \end{cases} \tag{4.4}$$

We begin to analyze the terms in the right hand side of (4.2) one by one. The first one reads

$$\begin{aligned} & \big[(\sigma^2\ell_t)_t + \nabla\cdot(\sigma\nabla\ell)\big]v_t^2 \\ = \;& \big(2\sigma\sigma_t\ell_t + \sigma^2\ell_{tt} + \nabla\sigma\cdot\nabla\ell + \sigma\Delta\ell\big)v_t^2 \\ = \;& \Big[2\sigma\sigma_t\ell_t + \sigma^2\ell_{tt} - s\lambda(\nabla\sigma\cdot\nabla\varphi + \sigma\Delta\varphi)\varphi^{-\lambda-1} + s\lambda(\lambda+1)\sigma|\nabla\varphi|^2\varphi^{-\lambda-2}\Big]v_t^2 \\ = \;& -s\lambda\varphi^{-\lambda-1}\Big[2\sigma\sigma_t\left(t-\frac{T}{2}\right) + \sigma^2 + (\nabla\sigma\cdot\nabla\varphi + \sigma\Delta\varphi)\Big]v_t^2 \\ & + s\lambda(\lambda+1)\varphi^{-\lambda-2}\Big[\sigma^2\Big(t-\frac{T}{2}\Big)^2 + \sigma|\nabla\varphi|^2\Big]v_t^2 \\ \geq \;& -s\lambda\varphi^{-\lambda-1}\Big\{ha + 2(M_1^2+M_1)\sqrt{\mu_0(\tau)} + \big[M_0^2 + (n-1)M_0\big]\Big\}v_t^2 \\ & + s\lambda(\lambda+1)h^2\sigma\varphi^{-\lambda-2}v_t^2 \\ \geq \;& s\lambda\varphi^{-\lambda-1}\Big\{h\alpha - 2(M_1^2+M_1)\sqrt{\mu_0(\tau)} - \big[M_0^2 + (n-1)M_0\big]\Big\}v_t^2 \\ & + h^2\sigma s\lambda(\lambda+1)\varphi^{-\lambda-2}v_t^2. \end{aligned} \tag{4.5}$$

Likewise, the second term in the right hand side of (4.2) reads

$$\begin{aligned} & -2\big[(\sigma\nabla\ell)_t + \nabla(\sigma\ell_t)\big]\cdot\nabla v\, v_t \\ = \;& -2\big(\sigma_t\nabla\ell + \sigma\nabla\ell_t + \ell_t\nabla\sigma + \sigma\nabla\ell_t\big)\cdot\nabla v\, v_t \\ = \;& \Big\{2s\lambda\varphi^{-\lambda-1}\Big[\sigma_t\nabla\varphi + \Big(t-\frac{T}{2}\Big)\nabla\sigma\Big] - 2s\lambda(\lambda+1)\varphi^{-\lambda-2}\sigma t\nabla\varphi\Big\}\cdot\nabla v\, v_t \\ = \;& 2s\lambda\varphi^{-\lambda-2}\Big\{\Big[\sigma_t\nabla\varphi + \Big(t-\frac{T}{2}\Big)\nabla\sigma\Big]\varphi - (\lambda+1)\sigma\Big(t-\frac{T}{2}\Big)\nabla\varphi\Big\}\cdot\nabla v v_t \\ \geq \;& -s\lambda\varphi^{-\lambda-2}\big(M_1h + 2M_1\sqrt{\mu_0(\tau)}\big)\tau\big(|\nabla v|^2 + v_t^2\big) \\ & - s\lambda(\lambda+1)\varphi^{-\lambda-2}\big(hM_1\sqrt{\mu_0(\tau)} + M_1\mu_0(\tau)\big)\big(|\nabla v|^2 + v_t^2\big). \end{aligned} \tag{4.6}$$

Thus, there exists a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ it holds that

$$\begin{aligned} & -2\big[(\sigma\nabla\ell)_t + \nabla(\sigma\ell_t)\big]\cdot\nabla v\, v_t \\ & \geq -2s\lambda(\lambda+1)\varphi^{-\lambda-2}\big(hM_1\sqrt{\mu_0(\tau)} + M_1\mu_0(\tau)\big)\big(|\nabla v|^2 + v_t^2\big). \end{aligned} \tag{4.7}$$

Treating the third term in the right hand side of (4.2) in the same fashion, we obtain

$$\begin{aligned} & \big[(\sigma\ell_t)_t + \Delta\ell\big]|\nabla v|^2 \\ = \ & \big(\sigma_t\ell_t + \sigma\ell_{tt} + \Delta\ell\big)|\nabla v|^2 \\ = \ & -s\lambda\varphi^{-n-1}\Big[\sigma_t\Big(t-\frac{T}{2}\Big) + \sigma + \Delta\varphi\Big]|\nabla v|^2 \\ & \quad + s\lambda(\lambda+1)\varphi^{-\lambda-2}\Big[\sigma\Big(t-\frac{T}{2}\Big)^2 + |\nabla\varphi|^2\Big]|\nabla v|^2 \\ \geq \ & -s\lambda\varphi^{-\lambda-1}\Big(M_1\sqrt{\mu_0(\tau)} + M_0 + (n-1)\Big)|\nabla v|^2 + h^2 s\lambda(\lambda+1)\varphi^{-\lambda-2}|\nabla v|^2. \end{aligned} \tag{4.8}$$

Following (4.5), (4.7), (4.8) and noticing (3.10), we find that

$$\begin{aligned} & \big[(\sigma^2\ell_t)_t + \nabla\cdot(\sigma\nabla\ell)\big]v_t^2 - 2[(\sigma\nabla\ell)_t + \nabla(\sigma\ell_t)]\cdot\nabla v\, v_t + \big[(\sigma\ell_t)_t + \Delta\ell\big]|\nabla v|^2 \\ & \geq Cs\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2 + v_t^2) \end{aligned} \tag{4.9}$$

for all $\lambda > \lambda_0$.

Next, note that in our case $A = \sigma(\ell_t^2 - \ell_{tt}) - (|\nabla\ell|^2 - \Delta\ell)$. Then it is easy to show that

$$\begin{aligned} A = \ & s^2\lambda^2\varphi^{-2\lambda-2}\Big[\sigma\Big(t-\frac{T}{2}\Big)^2 - |\nabla\varphi|^2\Big] + s\lambda(\lambda+1)\varphi^{-\lambda-2}\Big[|\nabla\varphi|^2 - \sigma\Big(t-\frac{T}{2}\Big)^2\Big] \\ & + s\lambda\varphi^{-\lambda-1}\big[\sigma - (n-1)\big]. \end{aligned} \tag{4.10}$$

Thus, under some simple but a little more bothersome calculations, it holds that

$$\begin{aligned} B & = (\sigma A\ell_t)_t - \nabla\cdot(A\nabla\ell) \\ & = \sigma_t A\ell_t + \sigma A_t\ell_t + \sigma A\ell_{tt} - \nabla A\cdot\nabla\ell - A\Delta\ell \\ & = 3s^3\lambda^2(\lambda+1)^2\Big(t-\frac{T}{2}\Big)^2\Big[\Big(t-\frac{T}{2}\Big)^2 - |\nabla\varphi|^2\Big]\varphi^{-3\lambda-4} \\ & \quad + 3s^3\lambda^2(\lambda+1)^2|\nabla\varphi|^2\Big[|\nabla\varphi|^2 - \Big(t-\frac{T}{2}\Big)^2\Big]\varphi^{-3\lambda-4} \\ & \quad + O(s^3\lambda^3\varphi^{-3\lambda-3}) + O(s^2\lambda^4\varphi^{-3\lambda-4}). \end{aligned} \tag{4.11}$$

It is easy to see that there exist an $\lambda_1 > 0$ and $s_0 > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_0$,

$$Bv^2 \geq Cs^3\lambda^4\varphi^{-3\lambda-4}v^2. \tag{4.12}$$

Next, integrating (4.2) over Q_τ and taking mathematical expectation, we obtain that

$$\begin{aligned}&\mathbb{E}\int_{Q_\tau}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)\big(\sigma du_t-\Delta u dt\big)dx\\ \geq\, & C\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt\\ &+\mathbb{E}\int_{Q_\tau}\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)^2dxdt+C\mathbb{E}\int_{Q_\tau}\sigma^2\theta^2\ell_t(du_t)^2dx.\end{aligned}\tag{4.13}$$

Thus we complete the proof. □

5. Proofs of Theorem 1.2

This section is dedicated to the proof of Theorem 1.2.

Proof. Without loss of generality, we assume that

$$x_0=(0,0,\cdots,0),\quad \nu(x_0)=(1,0,\cdots,0)$$

and S is parameterized as in Section 3 near 0. Also, K,δ_0,h,τ are all given as in Section 3. By the definition of $\varphi(x,t)$ and Q_μ, for any $\mu\in(0,\tau]$, the boundary Γ_μ of Q_μ is composed of the following three parts:

$$\begin{cases}\Gamma_\mu^1=\Big\{(x,t)\in\mathbb{R}^{n+1}\Big|x_1=\gamma(x_2,x_3,\cdots,x_n),\ \sum\limits_{j=2}^n x_j^2<\delta_0,\varphi(x,t)<\mu\Big\},\\ \Gamma_\mu^2=\Big\{(x,t)\in\mathbb{R}^{n+1}\Big|x_1>\gamma(x_2,x_3,\cdots,x_n),\ \sum\limits_{j=2}^n x_j^2<\delta_0,\varphi(x,t)=\mu\Big\},\\ \Gamma_\mu^3=\Big\{(x,t)\in\mathbb{R}^{n+1}\Big|x_1>\gamma(x_2,x_3,\cdots,x_n),\ \sum\limits_{j=2}^n x_j^2=\delta_0,\varphi(x,t)<\mu\Big\},\end{cases}\tag{5.1}$$

i.e., $\Gamma_\mu=\Gamma_\mu^1\cup\Gamma_\mu^2\cup\Gamma_\mu^3$. Next, we show that in fact $\Gamma_\mu^3=\emptyset$. Based on the conditions

$$x_1>\gamma(x_2,x_3,\cdots,x_n)$$

and

$$\gamma(x_2,x_3,\cdots,x_n)>-K\sum_{j=2}^n x_j^2$$

and the definition of φ, it follows that

$$(1-2Kh)\sum_{j=2}^n x_j^2+\Big(t-\frac{T}{2}\Big)^2<2\Big[hx_1+\sum_{j=2}^n x_j^2+\Big(t-\frac{T}{2}\Big)^2\Big]=2\varphi-\tau<2\mu-\tau<\tau.\tag{5.2}$$

Also, note that Γ_μ^3 is subordinated to $\sum\limits_{j=2}^n x_j^2=\delta_0$. Thus, from (5.2), it follows that

$$\delta_0<\frac{\tau}{1-2Kh},$$

a contradiction to

$$\delta_0 > \frac{\tau}{1-2Kh}$$

introduced in (3.17). As a direct result, we conclude that $\Gamma_\mu = \Gamma_\mu^1 \cup \Gamma_\mu^2$.

Moreover, it is clear that

$$\Gamma_\mu^1 \cup \Gamma_\mu^2 \subset \overline{Q_\tau}.$$

Define

$$t_0 = \sqrt{\frac{\tau}{1-2Kh}}\ .$$

Then by (3.16), it follows

$$\begin{cases} \Gamma_\mu^1 \subset \{x \mid x_1 = \gamma(x_2, x_3, \cdots, x_n)\} \times \{t \mid \ |t - T/2| \le t_0\}, \\ \Gamma_\mu^2 \subset \{x \mid \varphi(x,t) = \mu\}, \quad \mu \in (0,\tau]. \end{cases} \tag{5.3}$$

It is clear that

$$\Gamma_\mu^j \subset \Gamma_\tau^j, \quad j = 1, 2.$$

To apply the result of Theorem 4.1 to the present case, we adopt the truncation method. For convenience in the later statement, denote $Q_\tau = Q_1$. Fixing a arbitrarily small number $\widetilde{\tau} \in (0, \frac{\tau}{8})$, let

$$Q_{k+1} = \{(t,x) | \varphi(x,t) < \tau - k\widetilde{\tau}, k = 1,2,3\}.$$

Then it is easy to show that that

$$Q_4 \subset Q_3 \subset Q_2 \subset Q_1.$$

Introduce a truncation function $\chi \in C_0^\infty(Q_2)$ in the following manner

$$\chi \in [0,1] \quad \text{and} \quad \chi = 1 \quad \text{in} \quad Q_3.$$

Let z be the solution of (1.1). Let $\Phi = \chi z$. Then a little bothersome calculation gives that

$$\begin{cases} \sigma d\Phi_t - \Delta\Phi dt = (b_1\Phi_t + b_2 \cdot \nabla\Phi + b_3\Phi)dt + fdt + b_4\Phi dW(t) & \text{in } Q_\tau, \\ \Phi = 0, \ \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on } \Gamma_\tau. \end{cases} \tag{5.4}$$

Here, we denote by

$$f = \sigma\chi_{tt}z + 2a\chi_t z_t - 2\nabla\chi \cdot \nabla z - z\Delta\chi - b_1\chi_t z - b_2 \cdot z\nabla\chi.$$

From the definition of χ, f is clearly supported in $Q_2 \setminus \overline{Q}_3$.

Let

$$F = b_1\Phi_t + b_2 \cdot \nabla\Phi + b_3\Phi + f.$$

In stead of u by Φ in (4.1), we have that

$$\begin{aligned} &\mathbb{E}\int_{Q_\tau} \theta\big(-2\sigma\ell_t v_t + 2\nabla\ell \cdot \nabla v\big)\big(\sigma d\Phi_t - \Delta\Phi dt\big)dx \\ \ge\ & C\mathbb{E}\int_{Q_\tau} \Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2 + v_t^2) + s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt \\ &+\mathbb{E}\int_{Q_\tau} \big(-2\sigma\ell_t v_t + 2\nabla\ell \cdot \nabla v\big)^2 dxdt + C\mathbb{E}\int_{Q_\tau} \sigma^2\theta^2\ell_t(d\Phi_t)^2 dx. \end{aligned} \tag{5.5}$$

Due to the elementary property of the Itô integral, it is clear that

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)\big(\sigma d\Phi_t-\Delta\Phi dt\big)dx\\
&=\mathbb{E}\int_{Q_\tau}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)Fdxdt\\
&\quad+\mathbb{E}\int_{Q_\tau}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)b_4\Phi dW(t)dx\\
&\leq\mathbb{E}\int_{Q_\tau}\theta^2F^2dxdt+\mathbb{E}\int_{Q_\varepsilon}\theta\big(-2\sigma\ell_t v_t+2\nabla\ell\cdot\nabla v\big)^2dxdt.
\end{aligned}$$

Thus, we show that

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt+\mathbb{E}\int_{Q_\tau}\sigma^2\theta^2\ell_t(d\Phi_t)^2dx\\
&\leq C\mathbb{E}\int_{Q_\tau}\theta^2F^2dxdt.
\end{aligned}$$

Let us now do some estimate for the right hand side of the above inequality.

$$\mathbb{E}\int_{Q_\tau}\theta^2F^2dxdt\leq 2\mathbb{E}\int_{Q_\tau}\Big(b_1\Phi_t+b_2\cdot\nabla\Phi+b_3\Phi\Big)^2dxdt+2\mathbb{E}\int_{Q_\tau}|f|^2dxdt. \tag{5.6}$$

Note that f is supported in $Q_2\setminus\overline{Q}_3$. Hence

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\theta^2|f|^2dxdt\\
&=\mathbb{E}\int_{Q_\tau}\theta^2|\sigma\chi_{tt}z+2\sigma\chi_t z_t-2\nabla\chi\cdot\nabla z-z\Delta\chi-b_1\chi_t z-b_2\cdot z\nabla\chi|^2dxdt\\
&\leq C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned} \tag{5.7}$$

Thus, we achieve that

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\theta^2F^2dxdt\\
&\leq C\mathbb{E}\int_{Q_1}\theta^2\Big(\Phi_t^2+|\nabla\Phi|^2+\Phi^2\Big)dxdt+\ C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned} \tag{5.8}$$

And then

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt+\mathbb{E}\int_{Q_\varepsilon}\sigma^2\theta^2\ell_t(d\Phi_t)^2dx\\
&\leq C\mathbb{E}\int_{Q_1}\theta^2\Big(\Phi_t^2+|\nabla\Phi|^2+\Phi^2\Big)dxdt+\ C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned} \tag{5.9}$$

Recalling that

$$(d\Phi_t)^2=b_4^2\Phi^2dt,\quad \ell_t=-s\lambda\Big(t-\frac{T}{2}\Big)\varphi^{-\lambda-1},$$

we have

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt\\
&\le C\mathbb{E}\int_{Q_1}\theta^2\Big(\Phi_t^2+|\nabla\Phi|^2+\Phi^2\Big)dxdt+\mathbb{E}\int_{Q_\tau}\theta^2\sigma^2 s\lambda t\varphi^{-\lambda-1}b_4^2\Phi^2dxdt\\
&\quad+C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned}\tag{5.10}$$

Then for s and λ large enough, it follows that

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla v|^2+v_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}v^2\Big]dxdt\\
&\le C\mathbb{E}\int_{Q_\tau}\theta^2\Big(\Phi_t^2+|\nabla\Phi|^2+\Phi^2\Big)dxdt+\ C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned}\tag{5.11}$$

Now, we find that

$$|\nabla v|^2+v_t^2\ge C\theta^2\Big(s^2\lambda^2\varphi^{-2\lambda-2}\Phi^2+|\nabla\Phi|^2+\Phi_t^2\Big).$$

Thus for large s and λ, it follows that

$$\begin{aligned}
&\mathbb{E}\int_{Q_\tau}\theta^2\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla\Phi|^2+\Phi_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}\Phi^2\Big]dxdt\\
&\le C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned}\tag{5.12}$$

Recall that and $\Phi=z$ in $Q_3\subset Q_\tau$. It is easy to show that

$$\begin{aligned}
&\mathbb{E}\int_{Q_3}\theta^2\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla z|^2+z_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}z^2\Big]dxdt\\
&\le C\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\theta^2\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt.
\end{aligned}\tag{5.13}$$

Note that

$$\varphi(x,t)<\tau-3\widetilde{\tau}\ \text{ for }\ (x,t)\in Q_4,$$

then

$$\theta(x,t)=e^{s\varphi(x,t)^{-\lambda}}>e^{s(\tau-3\widetilde{\tau})^{-\lambda}}\ \text{ for }\ (x,t)\in Q_4.$$

Moreover, since

$$\tau-2\widetilde{\tau}<\varphi(x,t)<\tau-\widetilde{\tau}\ \text{ for }\ (x,t)\in Q_2\setminus\overline{Q}_3,$$

then

$$e^{s(\tau-\widetilde{\tau})^{-\lambda}}<\theta(x,t)<e^{s(\tau-2\widetilde{\tau})^{-\lambda}}\ \text{ for }\ (x,t)\in Q_2\setminus\overline{Q}_3.$$

Therefore

$$\begin{aligned}
&\mathbb{E}\int_{Q_4}\Big[s\lambda^2\varphi^{-\lambda-2}(|\nabla z|^2+z_t^2)+s^3\lambda^4\varphi^{-3\lambda-4}z^2\Big]dxdt\\
&\le Ce^{2[s(\tau-2\widetilde{\tau})^{-\lambda}-s(\tau-3\widetilde{\tau})^{-\lambda}]}\mathbb{E}\int_{Q_2\setminus\overline{Q}_3}\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt\\
&\le Ce^{2[s(\tau-2\widetilde{\tau})^{-\lambda}-s(\tau-3\widetilde{\tau})^{-\lambda}]}\mathbb{E}\int_{Q_\tau}\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt
\end{aligned}\tag{5.14}$$

For brevity, by letting

$$\overline{\mu}=2[(\tau-2\widetilde{\tau})^{-\lambda}-(\tau-3\widetilde{\tau})^{-\lambda}],$$

we can get that

$$\mathbb{E}\int_{Q_4}(|\nabla z|^2+z_t^2+z^2)dxdt \le Ce^{s\overline{\mu}}\mathbb{E}\int_{Q_\tau}\Big(z_t^2+|\nabla z|^2+z^2\Big)dxdt. \qquad (5.15)$$

For that $\overline{\mu}<0$, so if we let $s\to+\infty$, we find $z=0$ in Q_4. Taking Q_4 the desired region, we complete the proof. □

References

[1] S. Alinhac, *Non-unicité du problème de Cauchy. Ann. of Math. (2)* **117**(1983), 77–108.

[2] A. Amirov, *Integral Geometry and Inverse Problems for Kinetic Equations,* Doctoral dissertation, Novosivbirsk (1988).

[3] V. Barbu, A. Răscanu and G. Tessitore, *Carleman estimate and cotrollability of linear stochastic heat equatons, Appl. Math. Optim.,* **47**(2003), 97–120.

[4] T. Carleman, *Sur un probléme d'unicité pour les systèmes d' équations aux derivées partielles à deux variables independantes, Ark. Mat. Astr. Fys. 26B,* **17**(1939), 1–9.

[5] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations. Amer. J. Math.* **80**(1958), 16–36.

[6] L. Hörmander, *On the uniqueness of the Cauchy problem under partial analyticity assumptions. Geometrical optics and related topics (Cortona, 1996),* 179–219, Progr. Nonlinear Differential Equations Appl., 32, Birkhäuser Boston, Boston, MA, 1997.

[7] L. Hörmander, *The analysis of linear partial differential operators. IV. Fourier integral operators.* Classics in Mathematics. Springer-Verlag, Berlin, 2009.

[8] C. E. Kenig, *Carleman estimates, uniform Sobolev inequalities for second-order differential operators, and unique continuation theorems. Proceedings of the International Congress of Mathematicians,* Vol. 1, 2, 948–960, Amer. Math. Soc., Providence, RI, 1987.

[9] C. E. Kenig, A. Ruiz and C. D. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. Duke Math. J.* **55** (1987), 329–347.

[10] H. Li and Q. Lü, *A quantitative boundary unique continuation for stochastic parabolic equations. J. Math. Anal. Appl.* **402**(2013), 518–526.

[11] X. Liu, *Global Carleman estimate for stochastic parabolic equations, and its application. ESAIM Control Optim. Calc. Var.* **20** (2014), 823–839.

[12] Q. Lü, *Carleman estimate for stochastic parabolic equations and inverse stochastic parabolic problems.* Inverse Problems **28**(2012), no. 4, 045008.

[13] Q. Lü, *Observability estimate for stochastic Schrödinger equations and its applications. SIAM J. Control Optim.* **51**(2013), no. 1, 121–144.

[14] Q. Lü, *Observability estimate and state observation problems for stochastic hyperbolic equations.* Inverse Problems **29**(2013), 095011, 22 pp.

[15] Q. Lü and Z. Yin, *Unique continuation for stochastic heat equations. ESAIM Control Optim. Calc. Var.* **21**(2015), 378–398.

[16] Q. Lü and X. Zhang, *Global uniqueness for an inverse stochastic hyperbolic problem with three unknowns. Comm. Pure Appl. Math.* **68** (2015), 948–963.

[17] L. Robbiano, *Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. Comm. Partial Differential Equations* **16** (1991), 789–800.

[18] L. Robbiano and C. Zuily, *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. Invent. Math.* **131** (1998), 493–539.

[19] C. D. Sogge, Uniqueness in Cauchy problems for hyperbolic differential operators. Trans. Amer. Math. Soc. 333 (1992), 821–833.

[20] S. Tang and X. Zhang, *Null controllability for forward and backward stochastic parabolic equations, SIAM J. Control Optim.,* **48**(2009), 2191–2216.

[21] D. Tataru, *Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. Comm. Partial Differential Equations* **20** (1995), 855–884.

[22] D. Tataru, *Unique continuation problems for partial differential equations. Geometric methods in inverse problems and PDE control,* 239–255, IMA Vol. Math. Appl., 137, Springer, New York, 2004.

[23] D. Tataru, *Carleman estimates, unique continuation and applications.* http:// www.math.berkeley.edu/ tataru/papers/ ucpnotes.ps

[24] X. Zhang, *Unique continuation for stochastic parabolic equations. Differential Integral Equations.* 21 (2008), 81–93.

[25] X. Zhang, *Carleman and observability estimates for stochastic wave equations, SIAM J. Math. Anal.,* **40**(2008), 851–868.

[26] X. Zhang, *A unified controllability/observability theory for some stochastic and deterministic partial differential equations. Proceedings of the International Congress of Mathematicians.* Volume IV, 3008–3034, Hindustan Book Agency, New Delhi, 2010.

[27] C. Zuily, *Uniqueness and nonuniqueness in the Cauchy problem.* Progress in Mathematics, 33. Birkhäuser Boston, Inc., Boston, MA, 1983.