On closed-loop equilibrium strategies for mean-field stochastic linear quadratic problems *

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Abstract

This article is concerned with linear quadratic optimal control problems of mean-field stochastic differential equations (MF-SDE) with deterministic coefficients. To treat the time inconsistency of the optimal control problems, linear closed-loop equilibrium strategies are introduced and characterized by variational approach. Our developed methodology drops the delicate convergence procedures in [24]. When the MF-SDE reduces to SDE, our Riccati system coincides with the analogue in [24]. However, these two systems are in general different from each other due to the conditional mean-field terms in the MF-SDE. Eventually, the comparisons with pre-committed optimal strategies, open-loop equilibrium strategies are given in details.

Keywords. Mean-field linear-quadratic optimal control problems, time inconsistency, closed-loop equilibrium strategies, Riccati system.

AMS Mathematics subject classification. 93E20, 49N10, 49N70.

1. Introduction.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with the filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$, on which a one-dimensional standard Wiener process $W(\cdot)$ is defined such that $\mathbb{F}$ is the natural filtration generated by $W(\cdot)$. We define the set of initial pair

$$\mathscr{I} := \Big\{(t,\xi)\Big|\ t\in[0,T],\ \xi \text{ is } \mathcal{F}_t\text{-measurable},\ \mathbb{E}|\xi|^2<\infty\Big\}.$$

For any $(t,\xi)\in\mathscr{I}$, let us first consider the following stochastic differential equation (SDE):

$$\begin{cases} dX(r) = \big[A(r)X(r)+B(r)u(r)\big]dr + \big[C(r)X(r)+D(r)u(r)\big]dW(r), \quad r\in[t,T], \\ X(t)=\xi, \end{cases} \tag{1.1}$$

where $A, C : [0,T]\to\mathbb{R}^{n\times n}$, $B, D : [0,T]\to\mathbb{R}^{n\times m}$ are proper functions. In the above, $X(\cdot):[t,T]\times\Omega\to\mathbb{R}^n$, $u(\cdot):[t,T]\times\Omega\to\mathbb{R}^m$, are called the *state process*, *control process*, respectively. We define the set of *admissible control processes* on $[t,T]$ by

$$\mathscr{U}[t,T] := \Big\{u:[t,T]\times\Omega\to\mathbb{R}^m\big|u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable}, \ \mathbb{E}\int_t^T |u(s)|^2ds<\infty\Big\}.$$

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Under suitable conditions, for any $u(\cdot)\in\mathscr{U}[t,T]$, (1.1) admits a unique solution $X(\cdot)=X(\cdot\,;t,\xi,u(\cdot))$ such that the following cost functional is well-defined,

$$J(u(\cdot);t,X(t))=\frac{1}{2}\mathbb{E}_t\Big\{\int_t^T\big[\,\langle\, Q(s)X(s),X(s)\,\rangle+\langle\, R(s)u(s),u(s)\,\rangle\,\big]ds+\langle\, GX(T),X(T)\,\rangle\Big\},\tag{1.2}$$

with $Q:[0,T]\to\mathbb{R}^{n\times n}$, $R:[0,T]\to\mathbb{R}^{m\times m}$ and $G\in\mathbb{R}^{n\times n}$, $\mathbb{E}_t(\cdot):=\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$. We then state the classical stochastic linear quadratic (SLQ) optimal control problem as follows.

Problem (SLQ) Given $(t,X(t))\in\mathscr{I}$, and (1.1), (1.2), find a $\bar u(\cdot)\in\mathscr{U}[t,T]$ such that

$$J(\bar u(\cdot);t,X(t))=\operatorname*{essinf}_{u(\cdot)\in\mathscr{U}[t,T]}J(u(\cdot);t,X(t)).$$

For given initial pair (t,ξ), $\bar u(\cdot)\in\mathscr{U}[t,T]$ is called an *optimal control*, $\bar X(\cdot)$ is called an *optimal state process*, and $(\bar X(\cdot),\bar u(\cdot))$ is called an *optimal pair*.

If $t=0$, $\xi=x_0\in\mathbb{R}^n$, above (1.1) and (1.2) reduce to

$$\begin{cases} dX(r)=\big[A(r)X(r)+B(r)u(r)\big]dr+\big[C(r)X(r)+D(r)u(r)\big]dW(r),\\ X(0)=x_0,\end{cases}\tag{1.3}$$

and

$$J(u(\cdot);0,x_0)=\frac{1}{2}\mathbb{E}\int_0^T\big[\,\langle\, Q(s)X(s),X(s)\,\rangle+\langle\, R(s)u(s),u(s)\,\rangle\,\big]ds+\frac{1}{2}\mathbb{E}\,\langle\, GX(T),X(T)\,\rangle\,.\tag{1.4}$$

In this case, Problem (SLQ) reduces to the following form.

Problem (SLQ_0) For any $x_0\in\mathbb{R}^n$, find a $\bar u(\cdot)\in\mathscr{U}[0,T]$ such that

$$J(\bar u(\cdot);0,x_0)=\inf_{u(\cdot)\in\mathscr{U}[0,T]}J(u(\cdot);0,x_0).$$

To see the connections between *static* Problem (SLQ_0) and *dynamic* Problem (SLQ), suppose $\bar u(\cdot)$ is an optimal control of the former one, $\bar X(\cdot):=\bar X(\cdot;0,x_0)$ is the associated state process. For any $t\in[0,T)$, it is standard that ([25])

$$J(\bar u(\cdot)\big|_{[t,T]};t,\bar X(t))=\operatorname*{essinf}_{u(\cdot)\in\mathscr{U}[t,T]}J(u(\cdot);t,\bar X(t)).\tag{1.5}$$

In other words, the restriction $\bar u(\cdot)\big|_{[t,T]}$ of $\bar u(\cdot)$ on $[t,T]$ is an optimal control of Problem (SLQ) with corresponding $(t,\bar X(t))\in\mathscr{I}$. In the literature, it is referred as the *time consistency* of Problem (SLQ).

Stochastic linear quadratic problems have been extensively studied in the literature, for which we refer the readers to Bismut [2], Chen et al. [8], Tang [17], Wonham [20], Yong and Zhou [25], and the rich references therein. Moreover, there are lots of extensions for the above SLQ problem. For example, by adding $\mathbb{E}X(\cdot)$, $\mathbb{E}u(\cdot)$, which is named as *mean-field* terms, into the diffusion and drift of (1.3), we arrive at

$$\begin{cases} dX(s)=\big[A(s)X(s)+\widetilde A(s)\mathbb{E}X(s)+B(s)u(s)+\widetilde B(s)\mathbb{E}u(s)\big]ds\\ \qquad\qquad+\big[C(s)X(s)+\widetilde C(s)\mathbb{E}X(s)+D(s)u(s)+\widetilde D(s)\mathbb{E}u(s)\big]dW(s),\quad s\in[0,T],\\ X(0)=x_0.\end{cases}\tag{1.6}$$

Given suitable maps $A,C,\widetilde A,\widetilde C:[0,T]\to\mathbb{R}^{n\times n}$, $B,D,\widetilde B,\widetilde D:[0,T]\to\mathbb{R}^{n\times m}$, under mild conditions, (1.6) admits a unique solution $X(\cdot)$. Such a system is referred as linear mean-field (forward) SDE, which is historically called the McKean-Vlasov SDE. Mean-field SDEs, which were suggested by Kac [13] in 1956 and later initiated by McKean [15] in 1966, can describe particle systems in the mesoscopic level, and are of great importance in applications, see Huang et al [12], Dawson [9], Buckdahn et al [4], and the reference therein. On the other hand, for the above SLQ problem, sometimes one needs to keep the variation of state process $X(\cdot)$ as small as possible (see e.g. the formulation of mean-variance portfolio selection problems). Therefore, a natural form of the cost functional could be the following,

$$
\begin{aligned}
J(u(\cdot);0,x_0) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T & \big[\langle \widetilde{Q}(s)\mathbb{E}X(s), \mathbb{E}X(s)\rangle + \langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle\big]ds \\
& + \langle GX(T), X(T)\rangle + \langle \widetilde{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)]\rangle \Big\},
\end{aligned} \tag{1.7}
$$

for some $\widetilde{Q} : [0,T] \to \mathbb{R}^{n\times n}$, and $\widetilde{G} \in \mathbb{R}^{n\times n}$. Then given state equation (1.6), cost functional (1.7), the associated Problem (SLQ_0) is referred as a special case of *mean-field SLQ problem*. Notice that optimal control problems of mean-field SDEs were received a lot of attention, and they were carried out in Andersson-Djehiche [1], Buckdahn et al. [5], Meyer-Brandis et al. [16], Yong [23], Carmona et al. [7], and so on.

Returning back to the above Problem (SLQ), it is curious to ask: how to formulate a similar *dynamic* mean-field SLQ problem with conditional expectation? Inspired by the comparable roles of $\mathbb{E}_t$, $\mathbb{E}$ in (1.2), (1.4), one could replace expectation operator in (1.7) by conditional expectation operator $\mathbb{E}_\cdot$. Obviously, one can adopt the same trick to mean-field SDE (1.6). To sum up, for any $t \in [0,T)$, in this paper we consider a controlled mean-field SDE

$$
\begin{cases}
dX(s) = \big[A(s)X(s) + \widetilde{A}(s)\mathbb{E}_tX(s) + B(s)u(s) + \widetilde{B}(s)\mathbb{E}_tu(s)\big]ds \\
\qquad\qquad + \big[C(s)X(s) + \widetilde{C}(s)\mathbb{E}_tX(s) + D(s)u(s) + \widetilde{D}(s)\mathbb{E}_tu(s)\big]dW(s), \quad s \in [0,T], \\
X(0) = x_0,
\end{cases} \tag{1.8}
$$

and a *conditional* cost functional

$$
\begin{aligned}
J(u(\cdot);t,X(t)) = \frac{1}{2}\mathbb{E}_t\Big\{ \int_t^T & \Big[\langle Q(s)X(s), X(s)\rangle + 2\langle S(s)X(s), u(s)\rangle + \langle R(s)u(s), u(s)\rangle \\
& + \langle \widetilde{Q}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)]\rangle + 2\langle \widetilde{S}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[u(s)]\rangle \\
& + \langle \widetilde{R}(s)\mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)]\rangle\Big]ds + \langle GX(T), X(T)\rangle + \langle \widetilde{G}\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)]\rangle \Big\}.
\end{aligned} \tag{1.9}
$$

In the following, $\mathbb{E}_tX$, $\mathbb{E}_tu$ appearing in (1.8), (1.9) are referred as the *conditional mean-field terms*. Under some mild conditions, for any $x_0 \in \mathbb{R}^n$, $u(\cdot) \in \mathscr{U}[t,T]$, equation (1.8) admits a unique solution $X(\cdot)$, and the cost functional $J(u(\cdot);t,X(t))$ is well-defined. We pose the mean-field SLQ problem.

Problem (MF-SLQ). Given $t \in [0,T]$, $x_0 \in \mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathscr{U}[t,T]$ such that

$$
J(\bar{u}(\cdot);t,X(t)) = \inf_{u(\cdot)\in\mathscr{U}[t,T]} J(u(\cdot);t,X(t)). \tag{1.10}
$$

Unlike the above Problem (SLQ), the optimal control $\bar{u}(\cdot)$ depends on the parameter t. Therefore, our Problem (MF-SLQ) is time-inconsistent in the sense that the optimal control $\bar{u}(\cdot)$ in general does not satisfy the previous condition (1.5). That is to say, an optimal control obtained for the current initial pair (t,ξ) will barely keep optimal as time goes by. Although the time-consistency of optimal control problem is a good feature, in real world, it is a little too ideal and rarely exists. Instead, most, if not all, practical models that people encounter are not time-consistent. For example, if there exists hyperbolic discounting function, which is a fundamental notion in economics, in the cost functional, the time inconsistency obviously arises (see Wei et al [21].) In our setting, the time inconsistency is caused by the appearance of conditional mean-field terms in both (1.8) and (1.9). To treat the time inconsistent optimal control problems, in the literature people choose to seek time consistent equilibrium control within game-theoretical framework. We first look at the case when (1.8) reduces to a controlled SDE, while the time inconsistency comes from the cost functional. In this scenario, there are mainly two kinds of equilibrium controls: (linear) closed-loop equilibrium controls (CLECs) and open-loop equilibrium controls (OLECs). When the cost functional includes the non-exponential discounting functions or conditional variance of terminal state process, two classes of time inconsistent SLQ problems were investigated in Yong [22], Björk et al [3], respectively. To explicitly represent the corresponding CLECs, they introduced different Riccati systems. On the other hand, Hu et al [10], [11] introduced the notion of OLECs by spike variation. They obtained the existence and uniqueness of the linear closed-loop representations of OLECs by a new Riccati system. Wang [18] discussed CLECs, OLECs, and the closed-loop representation of OLECs in a unified manner, and established the corresponding characterizations, respectively. Now let us return back to the general Problem (MF-SLQ). Yong [24] introduced a class of linear CLECs and a class of Riccati system by the multi-person differential games ideas. They also discussed the solvability of Riccati system under proper conditions. As to the case of OLECs, Wang [19] established the characterizations of OLECs, and the closed-loop representations of OLECs, respectively, which extended the corresponding results in [18].

In this paper, we adopt the variational approach to study CLECs of Problem (MF-SLQ). To represent the CLECs in a linear feedback manner, we introduce closed-loop equilibrium strategies (CLESs) and appropriate closed-loop equilibrium state. If we directly follow the ideas in [18] by using the solution of (1.8) as the equilibrium state, the constructed control process will unavoidably depend on the additional parameter t and is obviously not a candidate for equilibrium control. As a result, we need to properly introduce a new controlled linear SDE as *auxiliary* state equation (e.g. [19], [24]). In contrary to *single* state equation case in [18], a system including a SDE and a conditional SDE is actually required here. This $double\ feature$ also happens to be true for the Riccati system. Actually, in [18] the Riccati system contains *two* backward ODEs, while our corresponding system turns out to be the scenario with $four$ equations. We emphasize that the derived Riccati system in some sense has specific *hierarchical* feature, and several delicate and subtle techniques are thus demanded. We refer to Lemma 3.4 for more details. The main result of this article is the characterization of CLESs, which includes *first-order equilibrium condition* and *second-order equilibrium condition* (see Theorem 2.1). In contrast with [24], our introduced Riccati system differs from theirs, and our obtained second-order equilibrium condition is new. In addition, the complicated convergence arguments, which are indispensable in [24], are dropped here. Some careful comparisons with the pre-committed optimal strategy, open-loop equilibrium controls, and the existing closed-loop equilibrium controls are carried out. Eventually, several interesting facts are revealed as follows.

• Notice that our definition of CLECs via spike variation has similar spirit to that via multi-person differential games in [24]. However, according to our study, the obtained Riccati system in these two articles are quite different, which implies that these two CLECs actually do not equal to each other. However, in some special cases, such as $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$, they actually coincide with each other. This indicates the non-triviality of conditional mean-field terms in (1.8).

• In the same mean-field scenario, our CLECs are different from the closed-loop optimal strategies (CLOS) in [14]. However, when there are no mean-field terms in the state equations (1.8) and the cost functional (1.9), our CLECs are the same as CLOSs. This in some sense reflects the meaningfulness of CLECs.

The article is organized as follows. In Section 2, we introduce some useful spaces and notions, present the main result, and compare our study with existing papers in details. In Section 3, we give the proof of the main result. Section 4 concludes this paper.

2. Preliminary notations

Given $\mathbb{S}^{m\times m}$ the set of symmetric $m\times m$ matrices, we first introduce the following hypothesis.

(H1) For the coefficients in (1.8) and (1.9), suppose

$$\begin{aligned} &A,\ \widetilde{A},\ C,\ \widetilde{C}\in L^\infty(0,T;\mathbb{R}^{n\times n}),\quad B,\ \widetilde{B},\ D,\ \widetilde{D}\in L^\infty(0,T;\mathbb{R}^{n\times m}),\\ &Q,\ \widetilde{Q}\in L^\infty(0,T;\mathbb{S}^{n\times n}),\quad S,\ \widetilde{S}\in L^\infty(0,T;\mathbb{R}^{m\times n}),\quad G,\ \widetilde{G}\in\mathbb{S}^{n\times n},\quad R,\ \widetilde{R}\in L^\infty(0,T;\mathbb{S}^{m\times m}). \end{aligned}$$

In the following, we make the conventions that

$$\begin{aligned} &\mathscr{A}:=A+\widetilde{A},\quad \mathscr{B}:=B+\widetilde{B},\quad \mathscr{C}:=C+\widetilde{C},\quad \mathscr{D}:=D+\widetilde{D},\\ &\mathscr{R}:=R+\widetilde{R},\quad \mathscr{Q}:=Q+\widetilde{Q},\quad \mathscr{G}:=G+\widetilde{G},\quad \mathscr{S}=S+\widetilde{S}. \end{aligned}\tag{2.1}$$

For $0\le s\le t\le T$, $H:=\mathbb{R}^n,\mathbb{R}^{n\times n}$, etc, we define the following spaces.

$$\begin{aligned} &L^2_{\mathcal{F}_t}(\Omega;H):=\Big\{X:\Omega\to H,\mathcal{F}_t\text{-measurable},\ \mathbb{E}|X|^2<\infty,\Big\}\\ &L^\infty(s,t;H):=\Big\{X:[s,t]\to H,\ \text{measurable, essentially bounded,}\Big\}\\ &L^2_{\mathbb{F}}(\Omega;L^1(s,t;H)):=\Big\{X:[s,t]\times\Omega\to H,\ \mathbb{F}\text{-adapted, measurable, }\mathbb{E}\Big[\int_s^t|X(r)|dr\Big]^2<\infty,\Big\}\\ &L^2_{\mathbb{F}}(s,t;H):=\Big\{X:[s,t]\times\Omega\to H;\ \mathbb{F}\text{-adapted, measurable, }\mathbb{E}\int_s^t|X(r)|^2dr<\infty,\Big\}\\ &L^2_{\mathbb{F}}(\Omega;C([s,t];H)):=\Big\{X:[s,t]\times\Omega\to H,\ \mathbb{F}\text{-adapted, measurable, continuous, }\mathbb{E}\sup_{r\in[s,t]}|X(r)|^2<\infty,\Big\}\\ &C_{\mathbb{F}}([s,t];L^2(\Omega;\mathbb{R}^n)):=\Big\{X:[s,t]\times\Omega\to H,\ \mathbb{F}\text{-adapted, measurable, }\sup_{r\in[s,t]}\mathbb{E}|X(r)|^2<\infty.\Big\} \end{aligned}$$

According to the Introduction, the time inconsistency of Problem (MF-SLQ) essentially comes from the t-dependence of optimal controls. In other words, we shall look for some proper equilibrium control process without such reliance. Thanks to the linear-quadratic framework, we also hope the desired control process to be the feedback form.

Before introducing the notions for Problem (MF-SLQ), let us look at the particular case of $\widetilde{A}=\widetilde{B}=\widetilde{C}=\widetilde{D}=0$. In other words, (1.8) reduces to a controlled SDE of

$$\begin{cases} dX(s)=\big[A(s)X(s)+B(s)u(s)\big]ds+\big[C(s)X(s)+D(s)u(s)\big]dW(s),\quad s\in[0,T],\\ X(0)=x_0. \end{cases}\tag{2.2}$$

For any $t\in[0,T)$, $v\in\mathbb{R}^m$, $\varepsilon>0$, and $\Theta\in L^2(0,T;\mathbb{R}^{m\times m})$, we define

$$u(s):=\Theta(s)X(s),\quad u^\varepsilon(s):=\Theta(s)X^\varepsilon(s)+vI_{[t,t+\varepsilon]}(s),\quad s\in[0,T],\tag{2.3}$$

where X^ε satisfies (2.2) associated with u^ε. The following definition comes from [18].

Definition 2.1. $\Theta^*(\cdot)$ *is called a closed-loop equilibrium strategy, if for any* $x\in\mathbb{R}^n$, $t\in[0,T)$, $\varepsilon>0$, $v\in\mathbb{R}^m$,

$$\varliminf_{\varepsilon\to0}\frac{J([\Theta^*X^\varepsilon+vI_{[t,t+\varepsilon]}]\big|_{[t,T]};t,X^*(t))-J\big([\Theta^*X^*]\big|_{[t,T]};t,X^*(t)\big)}{\varepsilon}\ge0.\tag{2.4}$$

$u^*:=\Theta^*X^*$, X^*, *is respectively called a closed-loop equilibrium control, a closed-loop equilibrium state.*

The expression of $u^* = \Theta^* X^*$ tells us two important features. First, there is no t-dependence on u^* which makes it a candidate for equilibrium control. Second, u^* has the linear feedback form of X^* which is the solution of state equation (2.2).

We return back to our Problem (MF-SLQ). Because of the mean-field terms in (1.8), the solution X obviously depends on parameter t. Therefore, if we simply mimic the technique in SDEs case and construct $u := \Theta X$ as above, the resulting u also relies on t and is of course not a suitable candidate for equilibrium control. To overcome this problem, we introduce the following controlled SDE as auxiliary state equation,

$$
\begin{cases} d\mathscr{X}(s) = \big[\mathscr{A}(s)\mathscr{X}(s) + \mathscr{B}(s)u(s)\big]ds + \big[\mathscr{C}(s)\mathscr{X}(s) + \mathscr{D}(s)u(s)\big]dW(s), \quad s \in [0,T], \\ \mathscr{X}(0) = x_0. \end{cases} \tag{2.5}
$$

Consequently, we end up with a system containing (1.8), (2.5).* We look for candidate equilibrium control of the form $u := \Theta \mathscr{X}$, with $\mathscr{X}$ being the candidate equilibrium state. Notice that such a u not only drops the t-dependence, but also at least satisfies: indicating the importance of the mean-field terms in (1.8), covering the controlled SDEs case in e.g. [18]. Eventually, we point out that similar treatments on u, $\mathscr{X}$ also appeared in [24], [19].

If we plug $u := \Theta \mathscr{X}$ into (1.8), (2.5), we then arrive at

$$
\begin{cases} d\mathscr{X}(s) = (\mathscr{A}(s) + \mathscr{B}(s)\Theta(s))\mathscr{X}(s)ds + (\mathscr{C}(s) + \mathscr{D}(s)\Theta(s))\mathscr{X}(s)dW(s), \\ dX(s) = \big[A(s)X(s) + \widetilde{A}(s)\mathbb{E}_t X(s) + B(s)\Theta(s)\mathscr{X}(s) + \widetilde{B}(s)\Theta(s)\mathbb{E}_t\mathscr{X}(s)\big]ds \\ \qquad\quad + \big[C(s)X(s) + \widetilde{C}(s)\mathbb{E}_t X(s) + D(s)\Theta(s)\mathscr{X}(s) + \widetilde{D}(s)\Theta(s)\mathbb{E}_t\mathscr{X}(s)\big]dW(s), \\ \mathscr{X}(0) = X(0) = x_0, \end{cases} \tag{2.6}
$$

where $s,\ t \in [0,T]$. Unlike $X(\cdot)$, $\mathscr{X}(\cdot)$ does not rely on t. Moreover,

$$
\mathbb{P}\big\{\omega \in \Omega; \mathscr{X}(s,\omega) = X(s,\omega),\ \forall s \in [0,t]\big\} = 1, \quad \mathbb{E}_t\mathscr{X} = \mathbb{E}_t X.
$$

In the following, we may omit the reference to the time variable whenever necessary. To define CLESs, we also need the perturbations of $\mathscr{X}$, X, i.e., $\mathscr{X}^\varepsilon$, X^ε that are described by

$$
\begin{cases} d\mathscr{X}^\varepsilon = \big[(\mathscr{A} + \mathscr{B}\Theta)\mathscr{X}^\varepsilon + \mathscr{B}vI_{[t,t+\varepsilon]}\big]ds + \big[(\mathscr{C} + \mathscr{D}\Theta)\mathscr{X}^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]}\big]dW(s), \\ dX^\varepsilon = \big[AX^\varepsilon + \widetilde{A}\mathbb{E}_t X^\varepsilon + B\Theta\mathscr{X}^\varepsilon + \mathscr{B}vI_{[t,t+\varepsilon]} + \widetilde{B}\mathbb{E}_t\Theta\mathscr{X}^\varepsilon\big]ds \\ \qquad\quad + \big[CX^\varepsilon + \widetilde{C}\mathbb{E}_t X^\varepsilon + D\Theta\mathscr{X}^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]} + \widetilde{D}\mathbb{E}_t\Theta\mathscr{X}^\varepsilon\big]dW(s), \\ \mathscr{X}^\varepsilon(0) = X^\varepsilon(0) = x_0. \end{cases} \tag{2.7}
$$

Here $t \in [0,T)$ is fixed, $v \in \mathbb{R}^m$, $\varepsilon > 0$.

Definition 2.2. $\Theta^*(\cdot) \in L^2(0,T;\mathbb{R}^{m\times m})$ *is called a closed-loop equilibrium strategy (CLES) of Problem (MF-SLQ), if for any* $x \in \mathbb{R}^n$, $t \in [0,T)$, $\varepsilon > 0$, $v \in \mathbb{R}^m$,

$$
\varliminf_{\varepsilon\to 0} \frac{J(u_0^\varepsilon(\cdot);t,X^*(t)) - J\big(u^*(\cdot)\big|_{[t,T]};t,X^*(t)\big)}{\varepsilon} \geq 0, \tag{2.8}
$$

where $u^* := \Theta^*\mathscr{X}^*$, $u_0^\varepsilon := \Theta^*\mathscr{X}^\varepsilon + vI_{[t,t+\varepsilon]}$, X^*, X^ε *are respectively the solution of (2.6), (2.7), associated with* (t,x_0,u^*), (t,x_0,u^ε). *Here* $\mathscr{X}^*$ *is called a closed-loop equilibrium state,* u^* *is called a (linear) closed-loop equilibrium control.*

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*We emphasize that the idea of adding a proper equation and then constructing a new system also appeared in [6] where another different topic was carried out.

Now we state the main result of this paper.

Theorem 2.1. *Suppose (H1) holds and (2.1) is given.* $\Theta^*(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$ *is a CLES if and only if*

(i) for almost all $s \in [0,T]$*, the following equality holds,*

$$(\mathscr{R}(s)+\mathscr{D}(s)^\top U_2^*(s)\mathscr{D}(s))\Theta^*(s)+\mathscr{S}(s)+\mathscr{B}(s)^\top U_1^*(s)+\mathscr{D}(s)^\top U_2^*(s)\mathscr{C}(s)=0, \quad a.s. \tag{2.9}$$

(ii) the following inequality holds,

$$\mathscr{R}(s)+\mathscr{D}(s)^\top U_2^*(s)\mathscr{D}(s)\geq 0, \qquad a.e. \ \ s\in[0,T]. \ \ a.s. \tag{2.10}$$

In the above, U_1^**,* U_2^* *satisfy*

$$\begin{cases} dU_1^* = -\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_1^* + U_1^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)+\mathscr{Q} \\ \qquad\quad +\mathscr{S}^\top\Theta^* + \big[\Theta^*\big]^\top\mathscr{S}+\big[\Theta^*\big]^\top\mathscr{R}\Theta^*\Big]ds, \\ dU_2^* = -\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_2^* + U_2^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)-\big[\widetilde{A}+\widetilde{B}\Theta^*\big]^\top U_3^* \\ \qquad\quad -\big[U_3^*\big]^\top\big[\widetilde{A}+\widetilde{B}\Theta^*\big]-(\widetilde{C}+\widetilde{D}\Theta^*)^\top U_3^*(\mathscr{C}+\mathscr{D}\Theta^*)-(\mathscr{C}+\mathscr{D}\Theta^*)^\top\big[U_3^*\big]^\top(\widetilde{C}+\widetilde{D}\Theta^*) \\ \qquad\quad +(\widetilde{C}+\widetilde{D}\Theta^*)^\top U_4^*(\widetilde{C}+\widetilde{D}\Theta^*)+Q+S^\top\Theta^*+\big[\Theta^*\big]^\top S+\big[\Theta^*\big]^\top R\Theta^*\Big]ds, \\ dU_3^* = -\Big[A^\top U_3^*+U_3^*(\mathscr{A}+\mathscr{B}\Theta^*)+C^\top U_3^*(\mathscr{C}+\mathscr{D}\Theta^*)-C^\top U_4^*(\widetilde{C}+\widetilde{D}\Theta^*) \\ \qquad\quad +Q+S^\top\Theta^*-U_4^*(\widetilde{A}+\widetilde{B}\Theta^*)\Big]ds, \\ dU_4^* = -\Big[A^\top U_4^*+U_4^*A+C^\top U_4^*C+Q\Big]ds, \\ U_1^*(T)=G+\widetilde{G}, \quad U_2^*(T)=G, \quad U_3^*(T)=G, \quad U_4^*(T)=G. \end{cases} \tag{2.11}$$

The previous (2.9), (2.10) are called the *first-order, second-order equilibrium conditions* for Problem (MF-SLQ).

Remark 2.1. *Given* $\Theta^* \in L^2(0,T;\mathbb{R}^{m\times n})$*, let us carefully look at the solvability of (2.11). The fourth equation with respect to* U_4^* *happens to be the classical second-order adjoint equation in stochastic LQ optimal control problem. Therefore, the solvability of* $U_4 \in C([0,T];\mathbb{R}^{n\times n})$ *is easy to obtain. Considering* $\Theta^* \in L^2(0,T;\mathbb{R}^{m\times n})$*, we thus derive the solvability of* U_3^**,* U_2^**,* U_1^* *one by one. It is worthy mentioning that* U_1^**,* U_2^**,* U_4^* *are symmetrical, while* U_3^* *is not.*

As to condition (2.10), let us compare it with that in the existing literature. First, if $\widetilde{A}=\widetilde{B}=\widetilde{C}=\widetilde{D}=0$*, then (2.10) reduces to the second-order equilibrium condition in [18]. Second, such conclusion has not been obtained in [24] where the closed-loop equilibrium strategies for Problem (MF-SLQ) were given by multi-person differential games approach. Third, it is also essentially different from the second-order necessary conditions of open-loop equilibrium control for Problem (MF-SLQ) in [19]. We will demonstrate more related details in the next three subsections.*

I. Equilibrium strategy vs optimal strategy

In this part, we make some comparisons with the pre-committed optimal strategy in the existing literature (e.g., [14], [24]).

To begin with, let us recall some notions. For any $t \in [0,T)$, $(\Theta,\widetilde{\Theta}) \in \Big(L^2(t,T;\mathbb{R}^{m\times n})\Big)^2$ is called a closed-loop strategy of Problem (MF-SLQ) on $[t,T]$ if for given $\xi \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the following SDE admits a unique solution $X \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)$

$$
(2.12)\qquad \begin{cases} dX = \Big[[A+B\Theta]X + [\widetilde{A} + B\widetilde{\Theta} + \widetilde{B}(\Theta+\widetilde{\Theta})]\mathbb{E}_t X\Big]ds \\ \qquad\quad +\Big[[C+D\Theta]X + [\widetilde{C} + D\widetilde{\Theta} + \widetilde{D}(\Theta+\widetilde{\Theta})]\mathbb{E}_t X\Big]dW(s), \ s\in[t,T], \\ X(t) = \xi, \end{cases}
$$

such that $u(t,\cdot) := [\Theta X + \widetilde{\Theta}\mathbb{E}_t X] \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. $(\Theta', \widetilde{\Theta}')$ is called a closed-loop optimal strategy if

$$
J(t,\xi;\Theta' X' + \widetilde{\Theta}'\mathbb{E}_t X'(\cdot)) \le J(t,\xi;u(\cdot))
$$

for any $u \in \mathscr{U}[t,T]$. Here X' is the solution to (2.12) associated with $(\Theta', \widetilde{\Theta}')$.

Given $(\Theta, \Lambda) \in \Big(L^2(0,T;\mathbb{R}^{m\times n})\Big)^2$, let us consider the following system on $[0,T]$,

$$
(2.13)\qquad \begin{cases} dP = -\Big[PA + A^\top P + C^\top PC + Q + (PB + C^\top PD + S^\top)\Theta\Big]ds \\ d\Pi = -\Big[\Pi\mathscr{A} + \mathscr{A}^\top \Pi + \mathscr{Q} + \mathscr{C}^\top P\mathscr{C} + [\Pi\mathscr{B} + \mathscr{C}^\top P\mathscr{D} + \mathscr{S}^\top]\Lambda\Big]ds, \\ P(T) = G, \quad \Pi(T) = \widetilde{G} + G. \end{cases}
$$

By [14], we have

Lemma 2.1. *Suppose (H1) holds. Then* $(\Theta', \widetilde{\Theta}') \in \Big(L^2(t,T;\mathbb{R}^{m\times n})\Big)^2$ *is a pair of closed-loop optimal strategy of Problem (MF-SLQ) if and only if*

$$
(2.14)\qquad \begin{aligned} & R + D^\top P' D \ge 0, \quad \mathscr{R} + \mathscr{D}^\top P' \mathscr{D} \ge 0, \\ & (\mathscr{R} + \mathscr{D}^\top P'\mathscr{D})(\Theta' + \widetilde{\Theta}') + \Big[\mathscr{B}^\top \Pi' + \mathscr{D}^\top P'\mathscr{C} + \mathscr{S}\Big] = 0, \\ & (R + D^\top P' D)\Theta' + \Big[B^\top \Pi' + D^\top P' C + S\Big] = 0, \end{aligned}
$$

where (P', Π') *is the solution to (2.13) associated with* $(\Theta', \Theta' + \widetilde{\Theta}')$. *In this case,* $u' := \Theta' X' + \widetilde{\Theta}'\mathbb{E}_t X'$ *is a pre-committed optimal control of Problem (MF-SLQ).*

Let us make some comments on pre-committed optimal strategy $(\Theta', \widetilde{\Theta}')$ and closed-loop equilibrium strategy Θ^*. First, the former strategy contains a pair of functions, while the later is concerned with one function. Second, the pre-committed optimal control u' has the linear feedback form of $(X', \mathbb{E}_t X')$ and a parameter t, while the closed-loop equilibrium control u^* is feedback with respect to $\mathscr{X}^*$. Third, system (2.13) includes two coupled equations while our (2.11) has four equations. Fourth, the previous two inequalities in (2.14) are comparable with our (2.10). Even when $\widetilde{R} = 0$, $\widetilde{D} = 0$ and these two conditions merge into one, it is still different from (2.10). Fifth, it is obvious to see that the last two equalities in (2.14) are comparable, yet different from our (2.9). Eventually, we mention that (2.9), (2.10) will coincide with (2.14) when Problem (MF-SLQ) reduces to classical SLQ problem.

II. Equilibrium strategy: open-loop vs closed-loop

To seek linear open-loop equilibrium controls with closed-loop representation, the author [19] introduced open-loop equilibrium strategy, which is comparable with our closed-loop equilibrium strategy, for Problem (MF-SLQ). In this part, we discuss in details the connections between these two notions.

To begin with, for given $\bar\Theta \in L^2(0,T;\mathbb{R}^{m\times m})$, we introduce $(\bar{\mathscr{X}},\bar X)$,

$$\begin{cases} d\bar{\mathscr{X}}(s) = (\mathscr{A}(s)+\mathscr{B}(s)\bar\Theta(s))\bar{\mathscr{X}}(s)ds + (\mathscr{C}(s)+\mathscr{D}(s)\bar\Theta(s))\bar{\mathscr{X}}(s)dW(s),\\ d\bar X(s) = \big[A(s)\bar X(s)+\widetilde A(s)\mathbb{E}_t\bar X(s)+B(s)\Theta(s)\bar{\mathscr{X}}(s)+\widetilde B(s)\bar\Theta(s)\mathbb{E}_t\bar{\mathscr{X}}(s)\big]ds\\ \qquad\qquad +\big[C(s)\bar X(s)+\widetilde C(s)\mathbb{E}_t\bar X(s)+D(s)\bar\Theta(s)\bar{\mathscr{X}}(s)+\widetilde D(s)\bar\Theta(s)\mathbb{E}_t\bar{\mathscr{X}}(s)\big]dW(s),\\ \bar{\mathscr{X}}(0)=\bar X(0)=x_0. \end{cases} \tag{2.15}$$

We also introduce the following perturbations of $\bar{\mathscr{X}}$, $\bar X$, i.e., $\mathscr{X}^\varepsilon$, X^ε that are described by

$$\begin{cases} d\mathscr{X}^\varepsilon = \big[\mathscr{A}\mathscr{X}^\varepsilon+\mathscr{B}\bar\Theta\bar{\mathscr{X}}+\mathscr{B}vI_{[t,t+\varepsilon]}\big]ds+\big[\mathscr{C}\mathscr{X}^\varepsilon+\mathscr{D}\bar\Theta\bar{\mathscr{X}}+\mathscr{D}vI_{[t,t+\varepsilon]}\big]dW(s),\\ dX^\varepsilon = \big[AX^\varepsilon+\widetilde A\mathbb{E}_tX^\varepsilon+B\bar\Theta\bar{\mathscr{X}}+\mathscr{B}vI_{[t,t+\varepsilon]}+\widetilde B\mathbb{E}_t\bar\Theta\bar{\mathscr{X}}\big]ds\\ \qquad +\big[CX^\varepsilon+\widetilde C\mathbb{E}_tX^\varepsilon+D\bar\Theta\bar{\mathscr{X}}+\mathscr{D}vI_{[t,t+\varepsilon]}+\widetilde D\mathbb{E}_t\bar\Theta\bar{\mathscr{X}}\big]dW(s),\\ \mathscr{X}^\varepsilon(0)=X^\varepsilon(0)=x_0. \end{cases} \tag{2.16}$$

$\bar\Theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times m})$ is called a linear *open-loop equilibrium strategy* (OLES) of Problem (MF-SLQ), if for any $x\in\mathbb{R}^n$, $t\in[0,T)$, $\varepsilon>0$, $v\in\mathbb{R}^m$,

$$\varliminf_{\varepsilon\to0}\frac{J(u^\varepsilon(\cdot);t,\bar X(t))-J\big(\bar u(\cdot)\big|_{[t,T]};t,\bar X(t)\big)}{\varepsilon}\ge0, \tag{2.17}$$

where $\bar u:=\bar\Theta\bar{\mathscr{X}}$, $u^\varepsilon:=\bar\Theta\bar{\mathscr{X}}+vI_{[t,t+\varepsilon]}$, $(\bar{\mathscr{X}},\bar X)$, $(\mathscr{X}^\varepsilon,X^\varepsilon)$ are the solution of (2.15), (2.16), associated with $(t,x_0,\bar u)$, (t,x_0,u^ε), respectively. Here $\bar{\mathscr{X}}$ is called an *open-loop equilibrium state*, $\bar u$ is called an *open-loop equilibrium control* with closed-loop representation.

For later convenience, we introduce

$$\begin{cases} dP_1=-\Big[P_1(\mathscr{A}+\mathscr{B}\Theta)+\mathscr{A}^\top P_1+\mathscr{Q}+\mathscr{C}^\top P_2(\mathscr{C}+\mathscr{D}\Theta)\Big]ds\\ dP_2=-\Big[P_2(\mathscr{A}+\mathscr{B}\Theta)+A^\top P_2+C^\top P_2(\mathscr{C}+\mathscr{D}\Theta)+F_1+F_2\Theta\Big]ds,\\ dP_3=-\Big[P_3A+A^\top P_3+C^\top P_3C-Q\Big]ds,\\ P_1(T)=[G+\widetilde G],\quad P_2(T)=G,\quad P_3(T)=-G, \end{cases} \tag{2.18}$$

where the non-homogeneous terms F_1, F_2 are defined as,

$$F_1:=[Q+C^\top P_3\widetilde C+P_3\widetilde A],\quad F_2:=[C^\top P_3\widetilde D+P_3\widetilde B]. \tag{2.19}$$

The following characterization of open-loop equilibrium operator was given in [19].

Lemma 2.2. *Suppose (H1) holds with $S=\widetilde S=0$. Then $\bar\Theta\in L^2(0,T;\mathbb{R}^{m\times n})$ is an open-loop equilibrium strategy of Problem (MF-SLQ) if and only if*

$$[\mathscr{R}+\mathscr{D}^\top\bar P_2\mathscr{D}]\Theta^*+\mathscr{B}^\top\bar P_1+\mathscr{D}^\top\bar P_2\mathscr{C}=0,\quad \mathscr{R}-\mathscr{D}^\top\bar P_3\mathscr{D}\ge0, \tag{2.20}$$

where $(\bar P_1,\bar P_2,\bar P_3)$ is the solution to system (2.18) associated with $\bar\Theta$.

According to [19], system (2.18) is called the Riccati system to represent equilibrium strategy $\bar\Theta$, and the two conditions in (2.20) are named as first-order, second-order equilibrium conditions, respectively.

To carry out comparisons, we start with several differences.

• By definition, the way of defining u_0^ε in (2.8) is obviously different from that of u^ε in (2.17), which implies that the solution $(\mathscr{X}^\varepsilon,X^\varepsilon)$ to (2.7) also differs from the analogue solution to (2.16).

• In our case, the Riccati system (2.11) is much more complex, and contains four backward ODEs with three of them coupled, while in (2.18) it is simpler, and only includes three equations with two of them depending on each other. Moreover, our U_1^*, U_2^*, U_4^* in (2.11) are symmetrical while the analogue (P_1, P_2) in (2.18) are not.

• As to the first-order equilibrium conditions (2.9) and that in (2.20), even though they have the same structure, they are still different because of Riccati system (2.11) and (2.18).

To summarize, closed-loop equilibrium strategies are in general different from open-loop equilibrium strategies. However, the following example shows that they actually equal for some mean-field SLQ problems.

Example 2.1. *Suppose $m = n = 1$, $G > 0$, $Q(\cdot) \geq 0$,*

$$\widetilde{C} = C = 0, \quad B = D = 1, \quad S = \widetilde{S} = 0, \quad R = \widetilde{R} = 0, \quad Q + \widetilde{Q} = 0, \quad G + \widetilde{G} = 0.$$

• *If $\widetilde{A} = \widetilde{B} = \widetilde{D} = 0$, then the Problem (MF-SLQ) reduces to classical SLQ problem. According to Example 3.2 in [18], there exists a unique open-loop equilibrium strategy $\bar{\Theta}$, a unique closed-loop equilibrium strategy Θ^*, such that $\Theta^* = \bar{\Theta} = 0$. In addition, $U_1^* = \bar{P}_1 = 0$, $U_2^* = \bar{P}_2$ which reduces to*

$$\begin{cases} d\bar{P}_2(s) = -\big[2A(s)\bar{P}_2(s) + Q(s)\big]ds, \quad s \in [0, T], \\ \bar{P}_2(T) = G. \end{cases} \tag{2.21}$$

• *For the general mean-field SLQ problem, it is easy to see there exists a closed-loop equilibrium strategy $\Theta^* = 0$. Moreover, $U_1^* = 0$,*

$$\begin{cases} dU_2^* = -\Big[2\mathscr{A}U_2^* + 2\widetilde{A}U_3^* + Q\Big]ds \\ dU_3^* = -\Big[AU_3^* + \mathscr{A}U_3^* - Q - \widetilde{A}U_4^*\Big]ds \\ dU_4^* = -\Big[2AU_4^* - Q\Big]ds \\ U_2^*(T) = G, \quad U_3^*(T) = -G, \quad U_4^*(T) = -G. \end{cases} \tag{2.22}$$

In contrast, there also exists an open-loop equilibrium strategy $\bar{\Theta} = 0$. Moreover, $\bar{P}_1 = 0$,

$$\begin{cases} d\bar{P}_2 = -\Big[\mathscr{A}\bar{P}_2 + A\bar{P}_2 + Q + \widetilde{A}\bar{P}_3\Big]ds, \\ d\bar{P}_3 = -\Big[2A\bar{P}_3 - Q\Big]ds \\ \bar{P}_2(T) = G, \quad \bar{P}_3(T) = -G. \end{cases} \tag{2.23}$$

Hence we have the following equalities,

$$\Theta^* = \bar{\Theta}, \quad (U_1^*, U_3^*, U_4^*) = (\bar{P}_1, \bar{P}_2, \bar{P}_3).$$

Moreover, if $\widetilde{A} = 0$, then $U_2^* = U_3^* = \bar{P}_2$.

We also give an example where both open-loop equilibrium strategy and closed-loop equilibrium strategy exist uniquely. Moreover, they are equal to each other.

Example 2.2. *Suppose $n = m = 1$, $S = \widetilde{S} = 0$, $A = \widetilde{A} = 0$, $B = \widetilde{B} = 0$, $R = \widetilde{R} = 0$, $Q \geq 0$, $G > 0$.*

• *Let $\widetilde{C} = 0$, $\widetilde{D} = 0$, $C = D = 1$. Then Problem (MF-SLQ) reduces to a SLQ problem. We see that $\Theta^* = -1$ is a closed-loop equilibrium strategy, and U_2^* satisfies*

$$U_2^*(t) = G + \int_t^T Q(s)ds, \quad t \in [0, T]. \tag{2.24}$$

To explain the uniqueness, suppose Θ' is another closed-loop equilibrium strategy. By Theorem 2.1, $U_2'(\Theta'+1)=0$, where

$$\begin{cases} dU_2' = -\Big[(1+\Theta')^2 U_2' + Q\Big]ds, \quad s\in[0,T],\\ U_2'(T)=G. \end{cases}$$

It is easy to see $U_2' = U_2^*$, $\Theta' = -1 = \Theta^*$, which implies the uniqueness of closed-loop equilibrium strategy.

- *Let $C=0$, $D=0$, $\widetilde{C}=\widetilde{D}=1$. Then $\Theta^*=-1$ is a closed-loop equilibrium strategy, and the corresponding $U_2^*(\cdot)$ satisfies (2.24). To discuss the uniqueness, suppose Θ' is another closed-loop equilibrium strategy. According to Theorem 2.1, $U_2'(\Theta'+1)=0$, where U_2' satisfies*

$$\begin{cases} dU_2' = -\Big[2(1+\Theta')^2 U_3' + (1+\Theta')^2 U_4' + Q\Big]ds,\\ dU_3' = Qds,\\ dU_4' = -Qds,\\ U_2'(T)=G, \quad U_3'(T)=-G, \quad U_4'(T)=G. \end{cases}$$

By calculation, $U_3'+U_4'=0$ and $U_4'>0$ imply that

$$U_2'(t) = G + \int_t^T \Big[(1+\Theta'(r))^2 U_3'(r) + Q(r)\Big]dr. \tag{2.25}$$

Since $U_2'(T)=G>0$, $U_2'(\cdot)$ is continuous, hence there exists $\delta>0$ such that for $r\in[T-\delta,T]$, $U_2^*(r)>0$. This indicates that for almost $r\in[T-\delta,T]$, $\Theta'(r)=-1$. Plugging it into (2.25), for any $t\in[T-\delta,T]$, we have $U_2'(t)=U_2^*(t)$. Hence $U_2'(T-\delta)\geq U_2'(T)=G>0$. By induction, we have $U_2'>0$, $\Theta'=-1$. This implies the uniqueness of closed-loop equilibrium strategy.

- *Let $C=0$, $D=0$, $\widetilde{C}=\widetilde{D}=1$. It is easy to see $\bar{\Theta}=-1$ is an open-loop equilibrium strategy, and the corresponding $\bar{P}_2$ equals to U_2^* in (2.24). The uniqueness is also easy to check.*

III. Comparisons with the existing closed-loop equilibrium strategy

In this subsection, we assume that $S=\widetilde{S}=0$ for simplicity. To begin with, let us recall the closed-loop equilibrium strategy introduced [24].

- First, they introduced partition $\Delta := \big\{0=t_0<t_1<\cdots<t_N=T\big\}$ on $[0,T]$, and the following equation

$$\begin{cases} dX^\Delta(s) = \Big[A(s)X^\Delta(s) + B(s)u^\Delta(s) + \widetilde{A}(s)\mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)] + \widetilde{B}(s)\mathbb{E}_{\rho^\Delta(s)}[u^\Delta(s)]\Big]ds\\ \qquad\qquad + \Big[C(s)X^\Delta(s) + D(s)u^\Delta(s) + \widetilde{C}(s)\mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)] + \widetilde{D}(s)\mathbb{E}_{\rho^\Delta(s)}[u^\Delta(s)]\Big]dW(s),\\ X^\Delta(0)=x_0, \end{cases} \tag{2.26}$$

where $\rho^\Delta(\cdot) := \sum_{k=0}^{N-1} t_k I_{[t_k,t_{k+1})}(\cdot)$, and

$$u^\Delta(s) := \Theta^\Delta(s)X^\Delta(s) + \big[\widehat{\Theta}^\Delta(s) - \Theta^\Delta(s)\big]\mathbb{E}_{\rho^\Delta(s)}[X^\Delta(s)], \quad s\in[0,T].$$

Recall that in Subsection I, there is additional parameter t in the optimal state equation and pre-committed optimal control, see e.g., (2.12) and Lemma 2.1. To get rid of this t without breaking the original structure, the author in [24] used the above partition Δ to introduce function ρ^Δ. This also partially explains the appearance of (u^Δ, X^Δ), $(\Theta^\Delta, \widehat{\Theta}^\Delta)$.

• Second, for any bounded $\mathbb{F}_{t_k}$-measurable v, they defined control $u_k^{\Delta,v}$ on $[t_k,T]$ as follows,

$$u_k^{\Delta,v}(s) := vI_{[t_k,t_{k+1})} + \Theta^{\Delta}(s)X_k^{\Delta,v}(s) + \big[\widehat{\Theta}^{\Delta}(s) - \Theta^{\Delta}(s)\big]\mathbb{E}_{\rho^{\Delta}(s)}[X_k^{\Delta,v}(s)], \quad s \in [t_k,T].$$

Here $X_k^{\Delta,v}$ is the solution of (2.26) on $[t_k,T]$ associated with $(u_k^{\Delta,v}, X^{\Delta}(t_k))$. Then the Δ-equilibrium strategy pair $(\Theta^{\Delta}, \widehat{\Theta}^{\Delta})$ is defined by

$$\widetilde{J}(X^{\Delta}(\cdot), u^{\Delta}(\cdot); t_k, X^{\Delta}(t_k)) \leq \widetilde{J}(X^{\Delta,v}(\cdot), u^{\Delta,v}(\cdot); t_k, X^{\Delta}(t_k)),$$

where $t \in [0,T]$, $(X(\cdot), u(\cdot))$ satisfies (2.26),

$$\begin{aligned}
&\widetilde{J}(X(\cdot), u(\cdot); t, X^{\Delta}(t)) \\
&= \frac{1}{2}\mathbb{E}_t\Big\{\int_t^T \Big[\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle + \langle \widetilde{Q}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)]\rangle \\
&\qquad + \langle \widetilde{R}(s)\mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)]\rangle\Big]ds + \langle GX(T), X(T)\rangle + \langle \widetilde{G}\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)]\rangle\Big\}.
\end{aligned}$$

We emphasize that $\widetilde{J}(X(\cdot), u(\cdot); t, X^{\Delta}(t))$ differs from $J(u(\cdot); t, X(t))$ in (1.9), even though they have the same form.

• Third, if there exist $(\Theta, \widehat{\Theta}) \in C([0,T]; \mathbb{R}^{m\times n})$ such that

$$\lim_{\|\Delta\|\to 0}\Big[\|\Theta^{\Delta} - \Theta\| + \|\widehat{\Theta}^{\Delta} - \widehat{\Theta}\|\Big] = 0, \tag{2.27}$$

then $(\Theta, \widehat{\Theta})$ is called the closed-loop equilibrium strategy of Problem (MF-SLQ). For this $\widehat{\Theta}$, they introduced $\widehat{u} := \widehat{\Theta}\widehat{X}$, where

$$\begin{cases} d\widehat{X} = \big[\mathscr{A} + \mathscr{B}\widehat{\Theta}\big]\widehat{X}ds + \big[\mathscr{C} + \mathscr{D}\widehat{\Theta}\big]\widehat{X}dW(s), \\ \widehat{X}(0) = x_0. \end{cases} \tag{2.28}$$

In this case, $\widehat{u}$, $\widehat{X}$, are called closed-loop equilibrium control, closed-loop equilibrium state, respectively.

We first point out several relations between the way of defining the above $(\Theta, \widehat{\Theta})$ and our Θ^*. By the convergence in (2.27), as well as the definition of ρ^{Δ}, we can regard X^{Δ}, u^{Δ}, as the *discrete* version of our $\mathscr{X}$, $u := \Theta\mathscr{X}$, with $\mathscr{X}$ in (2.6). Secondly, to define the closed-loop equilibrium strategy, a class of Δ-equilibrium strategy was introduced in [24] by a new functional $\widetilde{J}$. Notice that this $\widetilde{J}$, which is given with respect to (X,u) in (2.26), differs from our J, which is a functional of (X,u) in (1.8). Thirdly, to obtain closed-loop equilibrium strategy $(\Theta, \widehat{\Theta})$, some complex and delicate convergence arguments are required for $(\Theta^{\Delta}, \widehat{\Theta}^{\Delta})$. In contrast, by our variational approach (e.g. [3], [10], [11], [18], [19]), there is no partition involved, which saves us from the complicated convergence procedures. Eventually, there are technique consistency between [24] and ours. For example, the above perturbation $u_k^{\Delta,v}$ is similar as u_0^{ε} in Definition 2.2. In addition, equation (2.28) is comparable with our (2.6).

Let us make comparisons between [24] and ours from the view of Riccati systems which are used to represent the equilibrium strategy. In [24], they introduce the following

$$\begin{cases} d\widehat{P}_1 = -\Big[(\mathscr{A} + \mathscr{B}\widehat{\Theta})^{\top}\widehat{P}_1 + \widehat{P}_1(\mathscr{A} + \mathscr{B}\widehat{\Theta}) + (\mathscr{C} + \mathscr{D}\widehat{\Theta})^{\top}\widehat{P}_2(\mathscr{C} + \mathscr{D}\widehat{\Theta}) + \mathscr{Q} + \big[\widehat{\Theta}\big]^{\top}\mathscr{R}\widehat{\Theta}\Big]ds, \\ d\widehat{P}_2 = -\Big[(\mathscr{A} + \mathscr{B}\widehat{\Theta})^{\top}\widehat{P}_2 + \widehat{P}_2(\mathscr{A} + \mathscr{B}\widehat{\Theta}) + (\mathscr{C} + \mathscr{D}\widehat{\Theta})^{\top}\widehat{P}_2(\mathscr{C} + \mathscr{D}\widehat{\Theta}) + Q + \big[\widehat{\Theta}\big]^{\top}R\widehat{\Theta}\Big]ds, \\ \widehat{P}_1(T) = (G + \widetilde{G}), \quad \widehat{P}_2(T) = G, \end{cases} \tag{2.29}$$

where the equilibrium strategy $\widehat{\Theta}$ satisfies

$$(\mathscr{R}(s)+\mathscr{D}(s)^\top \widehat{P}_2(s)\mathscr{D}(s))\widehat{\Theta}(s)+\mathscr{B}(s)^\top \widehat{P}_1(s)+\mathscr{D}(s)^\top \widehat{P}_2(s)\mathscr{C}(s)=0. \quad a.s. \tag{2.30}$$

It differs from our system (2.11). In fact, from our system (2.11), the mean-field terms in the state equation (1.8) not only play important roles in U_1^*, U_2^*, but also make the Riccati system further couples with U_3^*, U_4^*. To summarize, for the same MF-SLQ problem, two different approach lead to different Riccati systems and the resulting CLESs. The following example shows another interesting result: by means of variational approach, there exist another MF-SLQ problem that share the same Riccati system as in [24].

Example 2.3. *Given (2.5), suppose the optimal control problem is to find proper $\bar{u}(\cdot)$ satisfying*

$$\mathscr{J}(\bar{u}(\cdot);t,\mathscr{X}(t))=\inf_{u(\cdot)\in\mathscr{U}[t,T]}\mathscr{J}(u(\cdot);t,\mathscr{X}(t)).$$

Here $\mathscr{J}(u(\cdot);t,\mathscr{X}(t))$ has the same form as (1.9) with X replaced by $\mathscr{X}$. For simplicity, we assume that $S=\widetilde{S}=0$. According to [18], the corresponding closed-loop equilibrium strategy Θ^* satisfies

$$(\mathscr{R}(s)+\mathscr{D}(s)^\top U_2^*(s)\mathscr{D}(s))\Theta^*(s)+\mathscr{B}(s)^\top U_1^*(s)+\mathscr{D}(s)^\top U_2^*(s)\mathscr{C}(s)=0, \tag{2.31}$$

where (U_1^*,U_2^*) are described as,

$$\begin{cases} dU_1^*=-\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_1^*+U_1^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)+\mathscr{Q}+\big[\Theta^*\big]^\top\mathscr{R}\Theta^*\Big]ds,\\ dU_2^*=-\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_2^*+U_2^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)+Q+\big[\Theta^*\big]^\top R\Theta^*\Big]ds,\\ U_1^*(T)=(G+\widetilde{G}),\quad U_2^*(T)=G. \end{cases} \tag{2.32}$$

Notice that (2.30), (2.29) are the same as (2.31), (2.32), respectively.

To conclude this part, we present several examples.

Example 2.4. *We give two cases when the Riccati systems (2.29) and (2.11) coincide with each other.*

- *Suppose $S=\widetilde{S}=0$, and the state equation (1.8) reduces to a linear SDE, i.e., $\widetilde{A}=\widetilde{B}=\widetilde{C}=\widetilde{D}=0$. Then both (2.29) and (2.11) become*

$$\begin{cases} dU_1^*=-\Big[(A+B\Theta^*)^\top U_1^*+U_1^*(A+B\Theta^*)+(C+D\Theta^*)^\top U_2^*(C+D\Theta^*)+\mathscr{Q}+\big[\Theta^*\big]^\top\mathscr{R}\Theta^*\Big]ds,\\ dU_2^*=-\Big[(A+B\Theta^*)^\top U_2^*+U_2^*(A+B\Theta^*)+(C+D\Theta^*)^\top U_2^*(C+D\Theta^*)+Q+\big[\Theta^*\big]^\top R\Theta^*\Big]ds,\\ U_1^*(T)=(G+\widetilde{G}),\quad U_2^*(T)=G, \end{cases} \tag{2.33}$$

where Θ^* satisfies

$$(\mathscr{R}(s)+D(s)^\top U_2^*(s)D(s))\Theta^*(s)+B(s)^\top U_1^*(s)+D(s)^\top U_2^*(s)C(s)=0. \tag{2.34}$$

Notice that such system was also obtained in [18].

- *Suppose $G=0$, $Q=0$, $\widetilde{S}=S=0$. Then both (2.29) and (2.11) reduce to*

$$\begin{cases} dU_1^*=-\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_1^*+U_1^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)+\widetilde{Q}+\big[\Theta^*\big]^\top\mathscr{R}\Theta^*\Big]ds,\\ dU_2^*=-\Big[(\mathscr{A}+\mathscr{B}\Theta^*)^\top U_2^*+U_2^*(\mathscr{A}+\mathscr{B}\Theta^*)+(\mathscr{C}+\mathscr{D}\Theta^*)^\top U_2^*(\mathscr{C}+\mathscr{D}\Theta^*)+\big[\Theta^*\big]^\top R\Theta^*\Big]ds,\\ U_1^*(T)=(G+\widetilde{G}),\quad U_2^*(T)=0. \end{cases}$$

The following example shows some interesting features caused by the mean-field terms in (1.8).

Example 2.5. *Suppose $G \geq 0$,*

$$\widetilde{C} + C = 0, \quad \widetilde{D} + D = 0, \quad \widetilde{A} = 0, \quad \widetilde{B} = 0, \quad S = \widetilde{S} = 0, \quad \mathscr{R} > \delta > 0. \tag{2.35}$$

Then (2.9) becomes $\mathscr{R}\Theta^* + B^\top U_1^* = 0$, where

$$\begin{cases} dU_1^* = -\Big[(A + B\Theta^*)^\top U_1^* + U_1(A + B\Theta^*) + \mathscr{Q} + \big[\Theta^*\big]^\top \mathscr{R}\Theta^*\Big]ds, \\ U_1^*(T) = (G + \widetilde{G}). \end{cases} \tag{2.36}$$

- *In contrast with the SDEs case in Example 2.4, because of the special relation between $\widetilde{C}$ and $\widetilde{D}$, the coupling relation between U_1^*, U_2^* as in (2.33) breaks. Moreover, unlike (2.34), U_2^* becomes useless even though $D \neq 0$.*
- *Under condition (2.35), we see that (2.29) also reduces to (2.36), which indicates another coincidence between our work and [24]. Moreover, according to Theorem 4.5 in [24], (2.36) is solvable and closed-loop equilibrium strategy exists.*

Remark 2.2. *For problem (MF-SLQ), the mean-field terms appear in both the cost functional (1.9) and the state equation (1.8). In order to see the interesting role of $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{D}$, let us suppose that $\widetilde{G} = 0$, $\widetilde{R} = 0$, $\widetilde{Q} = 0$, $\widetilde{S} = 0$. For the pre-committed optimal control case, the mean-field terms in (1.8) shall transform the traditional Riccati equation into the Riccati system. For the time consistent equilibrium control case, it is easy to check that such a change also happens for system (2.11). However, it is not the case for (2.29). From this sense, the time inconsistency caused by the mean-field terms in the state equation can be well revealed by our variational approach.*

Remark 2.3. *As shown in [24], the multi-person differential games method works well when the coefficients Q, R in the cost functional depend on both s and t. In contrast, we emphasize that the developed variational techniques can also be used to treat such a case. We hope to report relevant results in the future publications.*

3. Proof of the main result

In this section, we give the proof of Theorem 2.1.

Given (2.6), (2.7), we define $\mathscr{X}_0^\varepsilon := \mathscr{X}^\varepsilon - \mathscr{X}$, $X_0^\varepsilon := X^\varepsilon - X$ on $[0,T]$, which are described as

$$\begin{cases} d\mathscr{X}_0^\varepsilon = \big[(A_\theta + \widetilde{A}_\theta)\mathscr{X}_0^\varepsilon + \mathscr{B}vI_{[t,t+\varepsilon]}\big]ds + \big[(C_\theta + \widetilde{C}_\theta)\mathscr{X}_0^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]}\big]dW(s), \\ dX_0^\varepsilon = \big[AX_0^\varepsilon + \widetilde{A}_\theta\mathbb{E}_tX_0^\varepsilon + B\Theta\mathscr{X}_0^\varepsilon + \mathscr{B}vI_{[t,t+\varepsilon]}\big]ds \\ \qquad\quad +\big[CX_0^\varepsilon + \widetilde{C}_\theta\mathbb{E}_tX_0^\varepsilon + D\Theta\mathscr{X}_0^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]}\big]dW(s), \\ X_0^\varepsilon(0) = \mathscr{X}_0^\varepsilon(0) = 0. \end{cases} \tag{3.1}$$

In the above, we used the fact that $\mathbb{E}_tX_0^\varepsilon = \mathbb{E}_t\mathscr{X}_0^\varepsilon$ and notational convention

$$A_\theta := A + B\Theta, \quad \widetilde{A}_\theta := \widetilde{A} + \widetilde{B}\Theta, \quad C_\theta := C + D\Theta, \quad \widetilde{C}_\theta := \widetilde{C} + \widetilde{D}\Theta. \tag{3.2}$$

Remark 3.1. *Let us point out two useful facts as follows.*

- *If $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$, then $\mathscr{X}_0^\varepsilon \equiv X_0^\varepsilon := X^\varepsilon - X$, and*

$$\begin{cases} d\mathscr{X}_0^\varepsilon = \big[(A + B\Theta)\mathscr{X}_0^\varepsilon + BvI_{[t,t+\varepsilon]}\big]ds + \big[(C + D\Theta)\mathscr{X}_0^\varepsilon + DvI_{[t,t+\varepsilon]}\big]dW(s), \\ \mathscr{X}_0^\varepsilon(0) = 0. \end{cases}$$

It corresponds to the one used in [18].

- *It is easy to see the following estimates,*

$$\mathbb{E}\sup_{r\in[t,t+\varepsilon]}|\mathscr{X}_0^\varepsilon(r)|^2+\mathbb{E}\sup_{r\in[t,t+\varepsilon]}|X_0^\varepsilon(r)|^2\le K\varepsilon.$$

Lemma 3.1. Suppose $\Theta(\cdot)\in L^2(0,T;\mathbb{R}^{m\times m})$. Then

$$J(u^\varepsilon(\cdot);t,X(t))-J(u(\cdot);t,X(t))=J_1(t)+J_2(t)+\mathbb{E}_t\int_t^{t+\varepsilon}\langle(\Theta^\top\mathscr{R}+\mathscr{S}^\top)v,\mathscr{X}_0^\varepsilon\rangle\,ds, \tag{3.3}$$

where

$$\begin{aligned}
J_1(t)&:=\mathbb{E}_t\Big\{\int_t^T\Big[\langle F_1,X_0^\varepsilon\rangle+\langle F_2,\mathscr{X}_0^\varepsilon\rangle+\langle F_3,vI_{[t,t+\varepsilon)}\rangle\Big]ds+\langle GX(T)+\widetilde{G}\mathbb{E}_t[X(T)],X_0^\varepsilon(T)\rangle\Big\},\\
J_2(t)&:=\frac{1}{2}\mathbb{E}_t\int_t^T\Big[\langle F_1^\varepsilon,X_0^\varepsilon\rangle+\langle F_2^\varepsilon,\mathscr{X}_0^\varepsilon\rangle\Big]ds+\frac{1}{2}\mathbb{E}_t\langle GX_0^\varepsilon(T)+\widetilde{G}\mathbb{E}_tX_0^\varepsilon(T),X_0^\varepsilon(T)\rangle,
\end{aligned}$$

and

$$\begin{cases}
F_1\equiv QX+\widetilde{Q}\mathbb{E}_tX+S^\top\Theta\mathscr{X}+\widetilde{S}^\top\Theta\mathbb{E}_t\mathscr{X},\\
F_2\equiv\Theta^\top SX+\Theta^\top\widetilde{S}\mathbb{E}_tX+\Theta^\top R\Theta\mathscr{X}+\Theta^\top\widetilde{R}\Theta\mathbb{E}_t\mathscr{X},\\
F_3\equiv SX+\widetilde{S}\mathbb{E}_tX+R\Theta\mathscr{X}+\widetilde{R}\Theta\mathbb{E}_t\mathscr{X}+\dfrac{1}{2}\mathscr{R}v,\\
F_1^\varepsilon\equiv QX_0^\varepsilon+\widetilde{Q}\mathbb{E}_tX_0^\varepsilon+S^\top\Theta\mathscr{X}_0^\varepsilon+\widetilde{S}^\top\Theta\mathbb{E}_t\mathscr{X}_0^\varepsilon,\\
F_2^\varepsilon\equiv\Theta^\top SX_0^\varepsilon+\Theta^\top\widetilde{S}\mathbb{E}_tX_0^\varepsilon+\Theta^\top R\Theta\mathscr{X}_0^\varepsilon+\Theta^\top\widetilde{R}\Theta\mathbb{E}_t\mathscr{X}_0^\varepsilon.
\end{cases} \tag{3.4}$$

Proof. By the definitions of X, X^ε and X_0^ε, we deal with the terms in the cost functional one by one. First let us treat the term associated with Q,

$$\langle QX^\varepsilon,X^\varepsilon\rangle-\langle QX,X\rangle=2\langle QX,X_0^\varepsilon\rangle+\langle QX_0^\varepsilon,X_0^\varepsilon\rangle.$$

By the definitions of u and u^ε, we have

$$\begin{aligned}
&\langle SX^\varepsilon,u^\varepsilon\rangle-\langle SX,u\rangle\\
&=\langle S^\top\Theta\mathscr{X}_0^\varepsilon,X_0^\varepsilon\rangle+\langle X_0^\varepsilon,S^\top\big[\Theta\mathscr{X}+vI_{[t,t+\varepsilon]}\big]\rangle+\langle\mathscr{X}_0^\varepsilon,\Theta^\top SX\rangle+\langle SX,vI_{[t,t+\varepsilon]}\rangle.
\end{aligned}$$

We also have

$$\begin{cases}
\langle Ru^\varepsilon,u^\varepsilon\rangle-\langle Ru,u\rangle\\
=\langle\Theta^\top R\Theta\mathscr{X}_0^\varepsilon,\mathscr{X}_0^\varepsilon\rangle+\langle R\Theta\mathscr{X}_0^\varepsilon,\Theta\mathscr{X}+vI_{[t,t+\varepsilon]}\rangle+\langle RvI_{[t,t+\varepsilon]},\Theta\mathscr{X}_0^\varepsilon\rangle\\
\quad+\langle RvI_{[t,t+\varepsilon]},\Theta\mathscr{X}+vI_{[t,t+\varepsilon]}\rangle+\langle\Theta^\top R\Theta\mathscr{X},\mathscr{X}_0^\varepsilon\rangle+\langle R\Theta\mathscr{X},vI_{[t,t+\varepsilon]}\rangle\\
=\langle\Theta^\top R\Theta\mathscr{X}_0^\varepsilon,\mathscr{X}_0^\varepsilon\rangle+\langle\mathscr{X}_0^\varepsilon,\Theta^\top R\big[2\Theta\mathscr{X}+vI_{[t,t+\varepsilon]}\big]\rangle\\
\quad+\langle RvI_{[t,t+\varepsilon]},\Theta\mathscr{X}_0^\varepsilon\rangle+\langle R(2\Theta\mathscr{X}+v),vI_{[t,t+\varepsilon]}\rangle.
\end{cases}$$

Similarly one can obtain the terms involving $\bar{Q}$, $\bar{S}$, $\bar{R}$ as,

$$\begin{cases} \langle \widetilde{Q}\mathbb{E}_t X^\varepsilon, \mathbb{E}_t X^\varepsilon \rangle - \langle \widetilde{Q}\mathbb{E}_t X, \mathbb{E}_t X \rangle \\ = 2\langle \widetilde{Q}\mathbb{E}_t X, \mathbb{E}_t X_0^\varepsilon \rangle + \langle \widetilde{Q}\mathbb{E}_t X_0^\varepsilon, \mathbb{E}_t X_0^\varepsilon \rangle, \\ \langle \widetilde{S}\mathbb{E}_t X^\varepsilon, \mathbb{E}_t u^\varepsilon \rangle - \langle \widetilde{S}\mathbb{E}_t X, \mathbb{E}_t u \rangle \\ = \langle \widetilde{S}^\top \Theta \mathbb{E}_t \mathscr{X}_0^\varepsilon, \mathbb{E}_t X_0^\varepsilon \rangle + \langle \mathbb{E}_t X_0^\varepsilon, \widetilde{S}^\top \big[\Theta \mathbb{E}_t \mathscr{X} + vI_{[t,t+\varepsilon]}\big] \rangle \\ \quad + \langle \mathbb{E}_t \mathscr{X}_0^\varepsilon, \Theta^\top \widetilde{S}\mathbb{E}_t X \rangle + \langle \widetilde{S}\mathbb{E}_t X, vI_{[t,t+\varepsilon]} \rangle, \\ \langle \widetilde{R}\mathbb{E}_t u^\varepsilon, \mathbb{E}_t u^\varepsilon \rangle - \langle \widetilde{R}\mathbb{E}_t u, \mathbb{E}_t u \rangle \\ = \langle \Theta^\top \widetilde{R}\Theta \mathbb{E}_t \mathscr{X}_0^\varepsilon, \mathbb{E}_t \mathscr{X}_0^\varepsilon \rangle + \langle \mathbb{E}_t \mathscr{X}_0^\varepsilon, \Theta^\top \widetilde{R}\big[2\Theta \mathbb{E}_t \mathscr{X} + vI_{[t,t+\varepsilon]}\big] \rangle \\ \quad + \langle \widetilde{R}vI_{[t,t+\varepsilon]}, \Theta \mathbb{E}_t \mathscr{X}_0^\varepsilon \rangle + \langle \widetilde{R}(2\Theta \mathbb{E}_t \mathscr{X} + v), vI_{[t,t+\varepsilon]} \rangle. \end{cases}$$

At last we look at the terms associated with G and $\bar{G}$,

$$\begin{cases} \langle GX^\varepsilon(T), X^\varepsilon(T) \rangle - \langle GX(T), X(T) \rangle \\ = 2\langle GX(T), X_0^\varepsilon(T) \rangle + \langle GX_0^\varepsilon(T), X_0^\varepsilon(T) \rangle, \\ \langle \widetilde{G}\mathbb{E}_t X^\varepsilon(T), \mathbb{E}_t X^\varepsilon(T) \rangle - \langle \widetilde{G}\mathbb{E}_t X(T), \mathbb{E}_t X(T) \rangle \\ = 2\langle \widetilde{G}\mathbb{E}_t X(T), \mathbb{E}_t X_0^\varepsilon(T) \rangle + \langle \widetilde{G}\mathbb{E}_t X_0^\varepsilon(T), \mathbb{E}_t X_0^\varepsilon(T) \rangle. \end{cases}$$

To sum up, one then obtains (3.3). □

I. A decoupling method to conditional mean-field FBSDEs

In order to equivalently transform $J_1(t)$ and $J_2(t)$ of Lemma 3.1 into the desired forms, we need to decouple the following forward-backward system

$$\begin{cases} d\mathscr{X} = (H_1\mathscr{X} + H_2)dr + (H_3\mathscr{X} + H_4)dW(r), \\ dX = \big[A_1X + A_2\mathbb{E}_tX + A_3\mathscr{X} + A_4\mathbb{E}_t\mathscr{X} + A_5\big]dr \\ \qquad + \big[B_1X + B_2\mathbb{E}_tX + B_3\mathscr{X} + B_4\mathbb{E}_t\mathscr{X} + B_5\big]dW(r), \\ dY = -\Big[C_1Y + C_2\mathbb{E}_tY + C_3Z + C_4\mathbb{E}_tZ + C_5X + C_6\mathbb{E}_tX \\ \qquad + C_7\mathscr{X} + C_8\mathbb{E}_t\mathscr{X}\Big]dr + ZdW(r), \\ X(0) = x, \quad \mathscr{X}(0) = x', \quad Y(T,t) = D_1X(T) + D_2\mathbb{E}_tX(T). \end{cases} \tag{3.5}$$

In other words, we will represent (Y,Z) by means of $(\mathscr{X}, X)$ and a new system of BSDEs. The involved procedures are inspired by e.g., [10], [23], and the obtained result fully covers Lemma 4.2 in [18].

(H2) For $\mathbb{H} := \mathbb{R}^m,\ \mathbb{R}^n,\ \mathbb{R}^{n\times n}$, etc, let $D_1,\ D_2,\ x,\ x' \in \mathbb{H},\ A_i,\ B_i,\ C_i,\ H_j \in L^2(0,T;\mathbb{H})$, $i = 1,2,3,4$, $j = 1,2$, $H_2,\ H_4,\ A_5,\ B_5 \in L^2_{\mathbb{F}}(0,T;\mathbb{H})$.

For $t \in [0,T]$, $s \in [t,T]$, suppose that

$$Y(s,t) = \mathcal{P}_1(s)X(s) + \mathcal{P}_2(s)\mathbb{E}_tX(s) + \mathbb{E}_t\mathcal{P}_3(s) + \mathcal{P}_4(s) + \mathcal{P}_5(s)\mathscr{X}(s) + \mathcal{P}_6(s)\mathbb{E}_t\mathscr{X}(s), \tag{3.6}$$

where $\mathcal{P}_1(\cdot),\ \mathcal{P}_2(\cdot),\ \mathcal{P}_5(\cdot),\ \mathcal{P}_6(\cdot)$ are deterministic, $\mathcal{P}_3(\cdot),\ \mathcal{P}_4(\cdot)$ are stochastic processes satisfying

$$\begin{cases} d\mathcal{P}_i(s) = \Pi_i(s)ds, \quad i = 1,2,5,6, \quad \mathcal{P}_1(T) = D_1, \quad \mathcal{P}_2(T) = D_2, \quad \mathcal{P}_5(T) = \mathcal{P}_6(T) = 0, \\ d\mathcal{P}_j(s) = \Pi_j(s)ds + \mathcal{L}_j(s)dW(s), \quad j = 3,4, \quad \mathcal{P}_3(T) = 0, \quad \mathcal{P}_4(T) = D_3. \end{cases}$$

Here $\Pi_i(\cdot)$ are to be determined. It is easy to see

$$d\mathbb{E}_t X = \big[(A_1 + A_2)\mathbb{E}_t X + (A_3 + A_4)\mathbb{E}_t \mathscr{X} + A_5\big] dr, \quad d\mathbb{E}_t \mathscr{X} = \big[H_1 \mathbb{E}_t \mathscr{X} + H_2\big] dr.$$

Using Itô's formula, we derive that

$$\begin{cases} d\big[\mathcal{P}_1 X\big] = \Big[\Pi_1 X + \mathcal{P}_1(A_1 X + A_2 \mathbb{E}_t X + A_3 \mathscr{X} + A_4 \mathbb{E}_t \mathscr{X} + A_5)\Big] ds \\ \qquad\qquad\quad + \mathcal{P}_1\big(B_1 X + B_2 \mathbb{E}_t X + B_3 \mathscr{X} + B_4 \mathbb{E}_t \mathscr{X} + B_5\big) dW(s), \\ d\big[\mathcal{P}_2 \mathbb{E}_t X\big] = \Big\{\Pi_2 \mathbb{E}_t X + \mathcal{P}_2\big[(A_1 + A_2)\mathbb{E}_t X + (A_3 + A_4)\mathbb{E}_t \mathscr{X} + A_5\big]\Big\} ds, \\ d\big[\mathcal{P}_5 \mathscr{X}\big] = \Big[\Pi_5 \mathscr{X} + \mathcal{P}_5 H_1 \mathscr{X} + \mathcal{P}_5 H_2\Big] ds + \mathcal{P}_5[H_3 \mathscr{X} + H_4] dW(s), \\ d\big[\mathcal{P}_6 \mathbb{E}_t \mathscr{X}\big] = \Big\{\Pi_6 \mathbb{E}_t \mathscr{X} + \mathcal{P}_6 H_1 \mathbb{E}_t \mathscr{X} + \mathcal{P}_6 H_2\Big\} ds. \end{cases}$$

As a result, we have

$$\begin{aligned} dY = &\Big\{\big[\Pi_1 + \mathcal{P}_1 A_1\big] X + (\mathcal{P}_1 A_2 + \Pi_2 + \mathcal{P}_2(A_1 + A_2))\mathbb{E}_t X + (\mathcal{P}_1 A_3 + \Pi_5 + \mathcal{P}_5 H_1)\mathscr{X} \\ &+ \big[\mathcal{P}_1 A_4 + \mathcal{P}_2(A_3 + A_4) + \mathcal{P}_6 H_1 + \Pi_6\big]\mathbb{E}_t \mathscr{X} \\ &+ \mathbb{E}_t\big[\Pi_3 + \mathcal{P}_2 A_5 + \mathcal{P}_6 H_2\big] + \Pi_4 + \mathcal{P}_1 A_5 + \mathcal{P}_5 H_2\Big\} ds \\ &+ \Big[\mathcal{P}_1 B_1 X + \mathcal{P}_1 B_2 \mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3)\mathscr{X} + \mathcal{P}_1 B_4 \mathbb{E}_t \mathscr{X} + \mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\Big] dW(s). \end{aligned}$$

Consequently, it is necessary to see

$$Z = \mathcal{P}_1 B_1 X + \mathcal{P}_1 B_2 \mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3)\mathscr{X} + \mathcal{P}_1 B_4 \mathbb{E}_t \mathscr{X} + \mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4. \tag{3.7}$$

In this case, from (3.6), (3.7), we see that

$$\begin{cases} \mathbb{E}_t Y = (\mathcal{P}_1 + \mathcal{P}_2)\mathbb{E}_t X + \mathbb{E}_t\big[\mathcal{P}_3 + \mathcal{P}_4\big] + (\mathcal{P}_5 + \mathcal{P}_6)\mathbb{E}_t \mathscr{X}, \\ \mathbb{E}_t Z = (\mathcal{P}_1 B_1 + \mathcal{P}_1 B_2)\mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4)\mathbb{E}_t \mathscr{X} \\ \qquad\quad + \mathbb{E}_t\big[\mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\big]. \end{cases}$$

On the other hand, by the previous representations,

$$\begin{aligned} &-\Big[C_1 Y + C_2 \mathbb{E}_t Y + C_3 Z + C_4 \mathbb{E}_t Z + C_5 X + C_6 \mathbb{E}_t X + C_7 \mathscr{X} + C_8 \mathbb{E}_t \mathscr{X}\Big] \\ &= -C_1\Big\{\mathcal{P}_1 X + \mathcal{P}_2 \mathbb{E}_t X + \mathbb{E}_t \mathcal{P}_3 + \mathcal{P}_4 + \mathcal{P}_5 \mathscr{X} + \mathcal{P}_6 \mathbb{E}_t \mathscr{X}\Big\} \\ &\quad -C_2\Big\{(\mathcal{P}_1 + \mathcal{P}_2)\mathbb{E}_t X + \mathbb{E}_t\big[\mathcal{P}_3 + \mathcal{P}_4\big] + (\mathcal{P}_5 + \mathcal{P}_6)\mathbb{E}_t \mathscr{X}\Big\} \\ &\quad -C_3\Big[\mathcal{P}_1 B_1 X + \mathcal{P}_1 B_2 \mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3)\mathscr{X} + \mathcal{P}_1 B_4 \mathbb{E}_t \mathscr{X} \\ &\quad + \mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\Big] - C_4\Big[(\mathcal{P}_1 B_1 + \mathcal{P}_1 B_2)\mathbb{E}_t X \\ &\quad + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4)\mathbb{E}_t \mathscr{X} + \mathbb{E}_t\big[\mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\big]\Big] \\ &\quad -C_5 X - C_6 \mathbb{E}_t X - C_7 \mathscr{X} - C_8 \mathbb{E}_t \mathscr{X}. \end{aligned}$$

At this moment, we choose $\Pi_i(\cdot)$ in the following ways,

$$\begin{cases} 0 = \Pi_1 + \mathcal{P}_1 A_1 + C_1 \mathcal{P}_1 + C_3 \mathcal{P}_1 B_1 + C_5, \\ 0 = \Pi_2 + \mathcal{P}_1 A_2 + \mathcal{P}_2 (A_1 + A_2) + C_1 \mathcal{P}_2 + C_2 (\mathcal{P}_1 + \mathcal{P}_2) + C_3 \mathcal{P}_1 B_2 + C_6 + C_4 \mathcal{P}_1 (B_1 + B_2), \\ 0 = \Pi_4 + \mathcal{P}_1 A_5 + \mathcal{P}_5 H_2 + C_1 \mathcal{P}_4 + C_3 \big[\mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\big], \\ 0 = \Pi_5 + \mathcal{P}_1 A_3 + \mathcal{P}_5 H_1 + C_1 \mathcal{P}_5 + C_3 (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3) + C_7, \\ 0 = \Pi_6 + \mathcal{P}_1 A_4 + \mathcal{P}_2 (A_3 + A_4) + \mathcal{P}_6 H_1 + C_1 \mathcal{P}_6 + C_2 (\mathcal{P}_5 + \mathcal{P}_6) \\ \qquad + C_3 \mathcal{P}_1 B_4 + C_4 (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4) + C_8, \\ 0 = \Pi_3 + \mathcal{P}_1 A_6 + \mathcal{P}_2 A_5 + \mathcal{P}_6 H_2 + C_1 \mathcal{P}_3 + C_2 (\mathcal{P}_3 + \mathcal{P}_4) + C_4 \big[\mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\big]. \end{cases}$$

Next we make above arguments rigorous. Given (2.1), for $s \in [0,T]$, we consider

$$\begin{cases} d\mathcal{P}_1 = -\Big[\mathcal{P}_1 A_1 + C_1 \mathcal{P}_1 + C_3 \mathcal{P}_1 B_1 + C_5\Big] ds, \\ d\mathcal{P}_2 = -\Big\{\mathcal{P}_2 (A_1 + A_2) + (C_1 + C_2) \mathcal{P}_2 + C_2 \mathcal{P}_1 + \mathcal{P}_1 A_2 + C_3 \mathcal{P}_1 B_2 + C_6 + C_4 \mathcal{P}_1 (B_1 + B_2)\Big\} ds, \\ d\mathcal{P}_3 = -\Big[(C_1 + C_2) \mathcal{P}_3 + \mathcal{P}_2 A_5 + \mathcal{P}_6 H_2 + C_2 \mathcal{P}_4 + C_4 \big[\mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\big]\Big] ds + \mathcal{L}_3 dW(s), \\ d\mathcal{P}_4 = -\Big\{C_1 \mathcal{P}_4 + C_3 \mathcal{L}_4 + \mathcal{P}_1 A_5 + \mathcal{P}_5 H_2 + C_3 \big[\mathcal{P}_1 B_5 + \mathcal{P}_5 H_4\big]\Big\} ds + \mathcal{L}_4 dW(s), \\ d\mathcal{P}_5 = -\Big[\mathcal{P}_5 H_1 + C_1 \mathcal{P}_5 + C_3 \mathcal{P}_5 H_3 + \mathcal{P}_1 A_3 + C_3 \mathcal{P}_1 B_3 + C_7\Big] ds, \\ d\mathcal{P}_6 = -\Big[\mathcal{P}_6 H_1 + (C_1 + C_2) \mathcal{P}_6 + \mathcal{P}_1 A_4 + \mathcal{P}_2 (A_3 + A_4) + C_2 \mathcal{P}_5 \\ \qquad + C_3 \mathcal{P}_1 B_4 + C_4 (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4) + C_8\Big] ds, \\ \mathcal{P}_1(T) = D_1,\ \mathcal{P}_2(T) = D_2,\ \mathcal{P}_3(T) = \mathcal{P}_5(T) = \mathcal{P}_6(T) = 0,\ \mathcal{P}_4(T) = 0. \end{cases} \tag{3.8}$$

Under (H2), it is easy to see

$$\mathcal{P}_1(\cdot),\ \mathcal{P}_2(\cdot),\ \mathcal{P}_5(\cdot),\ \mathcal{P}_6(\cdot) \in C([0,T];\mathbb{R}^{n\times n}),\ (\mathcal{P}_3, \mathcal{L}_3), (\mathcal{P}_4, \mathcal{L}_4) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).$$

At this moment, for $s \in [0,T]$, and $t \in [0,s]$, we define a pair of processes

$$\begin{cases} \mathscr{Y}(s,t) := \Big[\mathcal{P}_1 X + \mathcal{P}_2 \mathbb{E}_t X + \mathbb{E}_t \mathcal{P}_3 + \mathcal{P}_4 + \mathcal{P}_5 \mathscr{X} + \mathcal{P}_6 \mathbb{E}_t \mathscr{X}\Big](s), \\ \mathscr{Z}(s,t) := \Big[\mathcal{P}_1 B_1 X + \mathcal{P}_1 B_2 \mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3) \mathscr{X} + \mathcal{P}_1 B_4 \mathbb{E}_t \mathscr{X} + \mathcal{P}_1 B_5 + \mathcal{L}_4 + \mathcal{P}_5 H_4\Big](s). \end{cases} \tag{3.9}$$

By the results of $\mathcal{P}_i(\cdot)$, we conclude that

$$(\mathscr{Y}_d(\cdot), \mathscr{Z}_d(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n),$$

where $(\mathscr{Y}_d(s), \mathscr{Z}_d(s)) \equiv (\mathscr{Y}(s,s), \mathscr{Z}(s,s))$ with $s \in [0,T]$.

Lemma 3.2. *Suppose $(\mathscr{X}, X, Y, Z)$ is the unique solution of (3.5), $(\mathscr{Y}, \mathscr{Z})$ are defined in (3.9). Then for any $t \in [0,T]$,*

$$\begin{aligned} &\mathbb{P}\Big\{\omega \in \Omega;\ Y(s,t) = \mathscr{Y}(s,t),\quad \forall s \in [t,T]\Big\} = 1, \\ &\mathbb{P}\Big\{\omega \in \Omega;\ Z(s,t) = \mathscr{Z}(s,t)\Big\} = 1,\quad s \in [t,T].\ \ a.e. \end{aligned} \tag{3.10}$$

Proof. Given (3.27), it is easy to see that

$$\begin{cases} \mathbb{E}_t \mathscr{Y} = (\mathcal{P}_1 + \mathcal{P}_2) \mathbb{E}_t X + \mathbb{E}_t[\mathcal{P}_3 + \mathcal{P}_4] + (\mathcal{P}_5 + \mathcal{P}_6) \mathbb{E}_t \mathscr{X}, \\ \mathbb{E}_t \mathscr{Z} = \mathcal{P}_1 (B_1 + B_2) \mathbb{E}_t X + (\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4) \mathbb{E}_t \mathscr{X} + \mathcal{P}_1 \mathbb{E}_t B_5 + \mathbb{E}_t \mathcal{L}_4 + \mathbb{E}_t \mathcal{P}_5 H_4. \end{cases}$$

Using Itô's formula, we know that

$$\begin{cases} d\big[\mathcal{P}_1 X\big] = \Big[-(C_1\mathcal{P}_1 + C_3\mathcal{P}_1 B_1 + C_5)X + \mathcal{P}_1(A_2\mathbb{E}_t X + A_3\mathscr{X} + A_4\mathbb{E}_t\mathscr{X} + A_5)\Big]ds \\ \qquad\qquad + \mathcal{P}_1\big(B_1 X + B_2\mathbb{E}_t X + B_3\mathscr{X} + B_4\mathbb{E}_t\mathscr{X} + B_5\big)dW(s), \\ d\big[\mathcal{P}_2\mathbb{E}_t X\big] = \Big\{-\Big[\mathcal{P}_1 A_2 + C_1\mathcal{P}_2 + C_2(\mathcal{P}_1+\mathcal{P}_2) + C_3\mathcal{P}_1 B_2 + C_6 + C_4\mathcal{P}_1(B_1+B_2)\Big]\mathbb{E}_t X \\ \qquad\qquad + \mathcal{P}_2\big[(A_3+A_4)\mathbb{E}_t\mathscr{X} + \mathbb{E}_t A_5\big]\Big\}ds, \\ d\big[\mathcal{P}_5\mathscr{X}\big] = \Big[-\big[\mathcal{P}_1 A_3 + C_1\mathcal{P}_5 + C_3(\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3)\big]\mathscr{X} + \mathcal{P}_5 H_2\Big]ds + \mathcal{P}_5(H_3\mathscr{X} + H_4)dW(s), \\ d\big[\mathcal{P}_6\mathbb{E}_t\mathscr{X}\big] = \Big\{-\Big[\mathcal{P}_1 A_4 + \mathcal{P}_2(A_3+A_4) + C_1\mathcal{P}_6 + C_2(\mathcal{P}_5+\mathcal{P}_6) + C_3\mathcal{P}_1 B_4 \\ \qquad\qquad + C_4(\mathcal{P}_1 B_3 + \mathcal{P}_5 H_3 + \mathcal{P}_1 B_4)\Big]\mathbb{E}_t\mathscr{X} + \mathcal{P}_6\mathbb{E}_t H_2\Big\}ds. \end{cases}$$

Consequently, after some calculations one has

$$d\mathscr{Y} = -\Big[C_1\mathscr{Y} + C_2\mathbb{E}_t\mathscr{Y} + C_3\mathscr{Z} + C_4\mathbb{E}_t\mathscr{Z} + C_5 X + C_6\mathbb{E}_t X + C_7\mathscr{X} + C_8\mathbb{E}_t\mathscr{X}\Big]dr + \mathscr{Z}dW(r).$$

Then for any $t\in[0,T]$, $(\mathscr{Y},\mathscr{Z})\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ satisfies the backward equation in (3.5). By the uniqueness of BSDEs, we see the conclusion. ∎

II. An equivalent transformation of J_1

To obtain the characterization in Theorem 2.1, we need some equivalent transformation on J_1.

Given processes $\mathscr{F}_1$, $\mathscr{F}_2$ to be determined later, for $i=1,2$, suppose the following BSDEs admit unique pairs of $(Y_i,Z_i)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$, respectively,

$$\begin{cases} dY_i(s,t) = -\mathscr{F}_i(s,t)ds + Z_i(s,t)dW(s), \quad s\in[t,T], \\ Y_1(T,t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T), \quad Y_2(T,t)=0. \end{cases}$$

By Itô's formula,

$$\begin{aligned} d\,\langle\, Y_1, X_0^\varepsilon\,\rangle &= \langle\, -\mathscr{F}_1 + A^\top Y_1 + C^\top Z_1, X_0^\varepsilon\,\rangle + \langle\, \widetilde{A}_\theta^\top Y_1 + \widetilde{C}_\theta^\top Z_1, \mathbb{E}_t X_0^\varepsilon\,\rangle \\ &\quad + \langle\, \Theta^\top(B^\top Y_1 + D^\top Z_1), \mathscr{X}_0^\varepsilon\,\rangle + \langle\, \mathscr{B}^\top Y_1 + \mathscr{D}^\top Z_1, vI_{[t,t+\varepsilon]}\,\rangle \\ &\quad + \Big[\langle\, Z_1, X_0^\varepsilon\,\rangle + \langle\, Y_1, CX_0^\varepsilon + \widetilde{C}_\theta\mathbb{E}_t X_0^\varepsilon + D\Theta\mathscr{X}_0^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]}\,\rangle\Big]dW(s). \end{aligned}$$

Recall $X_0^\varepsilon(t)=0$, we see that

$$\begin{aligned} &\mathbb{E}_t\,\langle\, Y_1(T), X_0^\varepsilon(T)\,\rangle + \mathbb{E}_t\int_t^T \langle\, F_1, X_0^\varepsilon\,\rangle\, ds \\ &= \mathbb{E}_t\int_t^T \langle\, -\mathscr{F}_1 + F_1 + A^\top Y_1 + C^\top Z_1 + \widetilde{A}_\theta^\top\mathbb{E}_t Y_1 + \widetilde{C}_\theta^\top\mathbb{E}_t Z_1, X_0^\varepsilon\,\rangle\, ds \\ &\quad + \mathbb{E}_t\int_t^T \langle\, \Theta^\top(B^\top Y_1 + D^\top Z_1), \mathscr{X}_0^\varepsilon\,\rangle\, ds + \mathbb{E}_t\int_t^{t+\varepsilon}\langle\, \mathscr{B}^\top Y_1 + \mathscr{D}^\top Z_1, v\,\rangle\, ds. \end{aligned}$$

Therefore, we choose

$$\begin{cases} dY_1 = -\Big[F_1 + A^\top Y_1 + C^\top Z_1 + \widetilde{A}_\theta^\top\mathbb{E}_t Y_1 + \widetilde{C}_\theta^\top\mathbb{E}_t Z_1\Big]ds + Z_1 dW(s) \\ Y_1(T,t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T), \end{cases} \tag{3.11}$$

and then

$$
\mathbb{E}_t \langle GX(T)+\widetilde{G}\mathbb{E}_t X(T), X_0^\varepsilon(T)\rangle + \mathbb{E}_t\int_t^T \langle F_1, X_0^\varepsilon\rangle ds
= \mathbb{E}_t\int_t^T \langle \Theta^\top(B^\top Y_1 + D^\top Z_1), \mathscr{X}_0^\varepsilon\rangle ds + \mathbb{E}_t\int_t^{t+\varepsilon}\langle \mathscr{B}^\top Y_1 + \mathscr{D}^\top Z_1, v\rangle ds. \tag{3.12}
$$

Under (H1), for given $\Theta\in L^2(0,T;\mathbb{R}^{m\times n})$, the wellposedness of (3.11) is easy to see. To obtain $\mathscr{F}_2$, let us use Itô's formula as follows,

$$
\begin{aligned}
d\langle Y_2, \mathscr{X}_0^\varepsilon\rangle &= \langle -\mathscr{F}_2 + (\mathscr{A}+\mathscr{B}\Theta)^\top Y_2 + (\mathscr{C}+\mathscr{D}\Theta)^\top Z_2, \mathscr{X}_0^\varepsilon\rangle ds + \langle \mathscr{B}^\top Y_2 + \mathscr{D}^\top Z_2, vI_{[t,t+\varepsilon]}\rangle ds\\
&\quad + \Big[\langle Z_2, \mathscr{X}_0^\varepsilon\rangle + \langle Y_2, (\mathscr{C}+\mathscr{D}\Theta)\mathscr{X}_0^\varepsilon + \mathscr{D}vI_{[t,t+\varepsilon]}\rangle\Big] dW(s).
\end{aligned}
$$

Consequently, recall $Y_2(T,t)=0$, $\mathscr{X}_0^\varepsilon(t)=0$, we have

$$
\begin{aligned}
&\mathbb{E}_t\int_t^T \langle \Theta^\top(B^\top Y_1 + D^\top Z_1) + F_2, \mathscr{X}_0^\varepsilon\rangle ds\\
&= \mathbb{E}_t\int_t^T \langle -\mathscr{F}_2 + (\mathscr{A}+\mathscr{B}\Theta)^\top Y_2 + (\mathscr{C}+\mathscr{D}\Theta)^\top Z_2 + \Theta^\top(B^\top Y_1 + D^\top Z_1) + F_2, \mathscr{X}_0^\varepsilon\rangle ds\\
&\quad + \mathbb{E}_t\int_t^T \langle \mathscr{B}^\top Y_2 + \mathscr{D}^\top Z_2, vI_{[t,t+\varepsilon]}\rangle ds.
\end{aligned}
$$

We thus choose

$$
\begin{cases}
dY_2 = -\Big[(\mathscr{A}+\mathscr{B}\Theta)^\top Y_2 + (\mathscr{C}+\mathscr{D}\Theta)^\top Z_2 + \Theta^\top(B^\top Y_1 + D^\top Z_1) + F_2\Big] ds + Z_2 dW(s),\\
Y_2(T,t) = 0.
\end{cases} \tag{3.13}
$$

Under (H1), for given $\Theta\in L^2(0,T;\mathbb{R}^{m\times n})$, the wellposedness of (3.13) is easy to see. Recall the above (3.12), we have

$$
\begin{aligned}
&\mathbb{E}_t\langle Y_1(T,t), X_0^\varepsilon(T)\rangle + \mathbb{E}_t\int_t^T \langle F_1, X_0^\varepsilon\rangle ds + \mathbb{E}_t\int_t^T\langle F_2, \mathscr{X}_0^\varepsilon\rangle ds\\
&= \mathbb{E}_t\int_t^{t+\varepsilon}\langle \mathscr{B}^\top(Y_1+Y_2) + \mathscr{D}^\top(Z_1+Z_2), v\rangle ds.
\end{aligned}
$$

By the definition of J_1 in Lemma 3.1, for any $t\in[0,T)$, we arrive at

$$
J_1(t) = \mathbb{E}_t\int_t^{t+\varepsilon}\langle F_3 + \mathscr{B}^\top(Y_1+Y_2) + \mathscr{D}^\top(Z_1+Z_2), v\rangle ds. \tag{3.14}
$$

To summarize, we have

Lemma 3.3. *Given $\Theta\in L^2(0,T;\mathbb{R}^{m\times m})$, under (H1), we have (3.14).*

Remark 3.2. *If there is no mean-field terms in (1.8), i.e., $\widetilde{A}=\widetilde{B}=\widetilde{C}=\widetilde{D}=0$, then*

$$
\begin{cases}
dY_1 = -\Big[F_1 + A^\top Y_1 + C^\top Z_1\Big] ds + Z_1 dW(s),\\
dY_2 = -\Big[(A+B\Theta)^\top Y_2 + (C+D\Theta)^\top Z_2 + \Theta^\top(B^\top Y_1 + D^\top Z_1) + F_2\Big] ds + Z_2 dW(s),\\
Y_1(T,t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T), \quad Y_2(T,t) = 0.
\end{cases} \tag{3.15}
$$

Taking $\mathcal{Y} := Y_1 + Y_2$, $\mathcal{Z} := Z_1 + Z_2$, we conclude that

$$
\begin{cases}
d\mathcal{Y} = -\Big[A_\theta^\top \mathcal{Y} + C_\theta^\top \mathcal{Z} + [Q + S^\top\Theta + \Theta^\top S + \Theta^\top R\Theta]X + [\widetilde{Q} + \widetilde{S}^\top\Theta\\
\qquad + \Theta^\top\widetilde{S} + \Theta^\top\widetilde{R}\Theta]\mathbb{E}_t X\Big] dr + \mathcal{Z} dW(r), \quad r\in[t,T],\\
\mathcal{Y}(T,t) = GX(T) + \widetilde{G}\mathbb{E}_t X(T).
\end{cases}
$$

In addition, the following recovers the analogue study in [18]

$$J_1(t) = -\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, B^\top \mathcal{Y} + D^\top \mathcal{Z} + (S + R\Theta)X + (\widetilde{S} + \widetilde{R}\Theta)\mathbb{E}_t X + \frac{1}{2}\mathscr{R}v, v \,\rangle\, ds.$$

We continue to transform $J_1(t)$ of (3.14) into the desired form. To this end, we define

$$\bar{Y}^\top := (Y_1^\top, Y_2^\top) \;\; \bar{Z}^\top := (Z_1^\top, Z_2^\top), \quad \bar{B}^\top := \big(\mathscr{B}^\top, \mathscr{B}^\top\big), \quad \bar{D}^\top := \big(\mathscr{D}^\top, \mathscr{D}^\top\big), \tag{3.16}$$

from which one has

$$J_1(t) = \mathbb{E}_t \int_t^T \langle\, F_3 + \bar{B}^\top \bar{Y} + \bar{D}^\top \bar{Z}, vI_{[t,t+\varepsilon]} \,\rangle\, dr. \tag{3.17}$$

To describe $(\bar{Y}, \bar{Z})$, we first extend $\mathscr{X},\ X,\ x$ into R^{2n}-vectors $\bar{\mathscr{X}},\ \bar{X},\ \bar{x}$, i.e.,

$$\bar{\mathscr{X}}^\top := (\mathscr{X}^\top, 0,), \quad \bar{X}^\top := (X^\top, 0), \quad \bar{x}_0^\top := (x_0^\top, 0). \tag{3.18}$$

By defining $\mathbb{R}^{2n\times 2n}$-valued functions H_1, H_3,

$$H_1 := \mathrm{diag}\big\{A_\theta + \widetilde{A}_\theta, 0\big\}, \quad H_3 := \mathrm{diag}\big\{C_\theta + \widetilde{C}_\theta, 0\big\}, \tag{3.19}$$

from the first equation in (2.6) one has

$$\begin{cases} d\bar{\mathscr{X}}(r) = H_1(r)\bar{\mathscr{X}}(r)dr + H_3(r)\bar{\mathscr{X}}(r)dW(r), \quad r \in [0,T], \\ \bar{\mathscr{X}}(0) = \bar{x}_0. \end{cases} \tag{3.20}$$

To treat $\bar{X}$, we define

$$\begin{aligned} &A_1 := \mathrm{diag}\Big\{A, 0\Big\}, \quad B_1 := \mathrm{diag}\Big\{C, 0\Big\}, \quad A_2 := \mathrm{diag}\Big\{\widetilde{A}, 0\Big\}, \quad B_2 := \mathrm{diag}\Big\{\widetilde{C}, 0\Big\}, \\ &A_3 := \mathrm{diag}\Big\{B\Theta, 0\Big\}, \quad B_3 := \mathrm{diag}\Big\{D\Theta, 0\Big\}, \quad A_4 := \mathrm{diag}\Big\{\widetilde{B}\Theta, 0\Big\}, \quad B_4 := \mathrm{diag}\Big\{\widetilde{D}\Theta, 0\Big\}. \end{aligned} \tag{3.21}$$

Consequently, by the second equation in (2.6), one has

$$\begin{cases} d\bar{X} = \big[A_1\bar{X} + A_2\mathbb{E}_t\bar{X} + A_3\mathscr{X} + A_4\mathbb{E}_t\bar{\mathscr{X}}\big]dr \\ \qquad\quad + \big[B_1\bar{X} + B_2\mathbb{E}_t\bar{X} + B_3\bar{\mathscr{X}} + B_4\mathbb{E}_t\bar{\mathscr{X}}\big]dW(r), \quad r \in [0,T], \\ \bar{X}(0) = \bar{x}_0. \end{cases}$$

Here it is worthy mentioning that

$$H_1 = \sum_{i=1}^4 A_i, \qquad H_3 = \sum_{i=1}^4 B_i. \tag{3.22}$$

At last, to treat $(\bar{Y}, \bar{Z})$, we introduce

$$C_1 := \left\{ \begin{matrix} A^\top, & 0, \\ \Theta^\top B^\top, & \mathscr{A}^\top + \Theta^\top \mathscr{B}^\top, \end{matrix} \right\}, \quad C_3 := \left\{ \begin{matrix} C^\top, & 0, \\ \Theta^\top D^\top, & \mathscr{C}^\top + \Theta^\top \mathscr{D}^\top, \end{matrix} \right\}, \tag{3.23}$$

and

$$
(3.24) \qquad \begin{cases} C_2 := \mathrm{diag}\Big\{\widetilde{A}_\theta^\top, 0\Big\}, \quad C_4 := \mathrm{diag}\Big\{\widetilde{C}_\theta^\top, 0\Big\}, \\ C_5 := \begin{Bmatrix} Q, & 0, \\ \Theta^\top S, & 0, \end{Bmatrix}, \quad C_6 := \begin{Bmatrix} \widetilde{Q}, & 0, \\ \Theta^\top \widetilde{S}, & 0, \end{Bmatrix}, \\ C_7 := \begin{Bmatrix} S^\top \Theta, & 0, \\ \Theta^\top R\Theta, & 0, \end{Bmatrix}, \quad C_8 := \begin{Bmatrix} \widetilde{S}^\top \Theta, & 0, \\ \Theta^\top \widetilde{R}\Theta, & 0, \end{Bmatrix}, \\ \mathcal{G} := \mathrm{diag}\Big\{G, 0\Big\}, \quad \widetilde{\mathcal{G}} := \mathrm{diag}\Big\{\widetilde{G}, 0\Big\}. \end{cases}
$$

Therefore, we have

$$
\begin{cases} d\bar{Y} = -\Big[C_1\bar{Y} + C_2\mathbb{E}_t\bar{Y} + C_3\bar{Z} + C_4\mathbb{E}_t\bar{Z} + C_5\bar{X} + C_6\mathbb{E}_t\bar{X} + C_7\bar{\mathscr{X}} + C_8\mathbb{E}_t\bar{\mathscr{X}}\Big]dr + \bar{Z}dW(r), \\ \bar{Y}(T,t) = \mathcal{G}\bar{X}(T) + \widetilde{\mathcal{G}}\mathbb{E}_t\bar{X}(T). \end{cases}
$$

To sum up, we have obtained a new forward-backward system of $(\bar{\mathscr{X}}, \bar{X}, \bar{Y}, \bar{Z})$,

$$
(3.25) \qquad \begin{cases} d\bar{\mathscr{X}} = H_1\bar{\mathscr{X}}dr + H_3\bar{\mathscr{X}}dW(r), \quad r \in [0,T], \\ d\bar{X} = \big[A_1\bar{X} + A_2\mathbb{E}_t\bar{X} + A_3\mathscr{X} + A_4\mathbb{E}_t\bar{\mathscr{X}}\big]dr \\ \qquad + \big[B_1\bar{X} + B_2\mathbb{E}_t\bar{X} + B_3\bar{\mathscr{X}} + B_4\mathbb{E}_t\bar{\mathscr{X}}\big]dW(r), \\ d\bar{Y} = -\Big[C_1\bar{Y} + C_2\mathbb{E}_t\bar{Y} + C_3\bar{Z} + C_4\mathbb{E}_t\bar{Z} + C_5\bar{X} + C_6\mathbb{E}_t\bar{X} \\ \qquad + C_7\bar{\mathscr{X}} + C_8\mathbb{E}_t\bar{\mathscr{X}}\Big]dr + \bar{Z}dW(r), \\ \bar{X}(0) = \bar{\mathscr{X}}(0) = \bar{x}_0, \quad \bar{Y}(T,t) = (Y(T,t)^\top, 0)^\top. \end{cases}
$$

We apply decoupling technique to (3.25). Inspired by (3.8), we introduce

$$
(3.26) \qquad \begin{cases} d\bar{P}_1 = -\Big[\bar{P}_1A_1 + C_1\bar{P}_1 + C_3\bar{P}_1B_1 + C_5\Big]ds, \\ d\bar{P}_2 = -\Big\{\bar{P}_2(A_1 + A_2) + (C_1 + C_2)\bar{P}_2 + C_2\bar{P}_1 + \bar{P}_1A_2 + C_3\bar{P}_1B_2 + C_6 + C_4\bar{P}_1(B_1 + B_2)\Big\}ds, \\ d\bar{P}_5 = -\Big[\bar{P}_5H_1 + C_1\bar{P}_5 + C_3\bar{P}_5H_3 + \bar{P}_1A_3 + C_3\bar{P}_1B_3 + C_7\Big]ds, \\ d\bar{P}_6 = -\Big[\bar{P}_6H_1 + (C_1 + C_2)\bar{P}_6 + \bar{P}_1A_4 + \bar{P}_2(A_3 + A_4) + C_2\bar{P}_5 \\ \qquad + C_3\bar{P}_1B_4 + C_4(\bar{P}_1B_3 + \bar{P}_5H_3 + \bar{P}_1B_4) + C_8\Big]ds, \\ \bar{P}_1(T) = \mathcal{G}, \quad \bar{P}_2(T) = \widetilde{\mathcal{G}}, \quad \bar{P}_6(T) = \bar{P}_5(T) \equiv 0, \end{cases}
$$

where the coefficients are defined above. Here we use the notation $\bar{P}_5$, $\bar{P}_6$, rather than $\bar{P}_3$, $\bar{P}_4$, to keep consistency with that in (3.8).

The solvability of (3.26) is easy to see. We emphasize that $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_5$, $\bar{P}_6$ are $\mathbb{R}^{5n\times 5n}$-valued. For $s \in [0,T]$, $t \in [0,s]$, we define

$$
(3.27) \qquad \begin{cases} \bar{\mathscr{Y}}(\cdot,t) := \Big[\bar{P}_1\bar{X} + \bar{P}_2\mathbb{E}_t\bar{X} + \bar{P}_5\bar{\mathscr{X}} + \bar{P}_6\mathbb{E}_t\bar{\mathscr{X}}\Big], \\ \bar{\mathscr{Z}}(\cdot,t) := \Big[\bar{P}_1B_1\bar{X} + \bar{P}_1B_2\mathbb{E}_t\bar{X} + (\bar{P}_1B_3 + \bar{P}_5H_3)\bar{\mathscr{X}} + \bar{P}_1B_4\mathbb{E}_t\bar{\mathscr{X}}\Big]. \end{cases}
$$

By the results of $\bar{P}_i(\cdot)$, we conclude that

$$
(\bar{\mathscr{Y}}_d(\cdot), \bar{\mathscr{Z}}_d(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)
$$

where

$$(\bar{\mathscr{Y}}_d(s), \bar{\mathscr{Z}}_d(s)) \equiv (\bar{\mathscr{Y}}(s,s), \bar{\mathscr{Z}}(s,s)), \quad s \in [0,T].$$

According to Lemma 3.2, for any $t \in [0,T)$,

$$\mathbb{P}\Big\{\omega \in \Omega;\ \bar{Y}(s,t) = \bar{\mathscr{Y}}(s,t), \quad \forall s \in [t,T]\Big\} = 1,$$
$$\mathbb{P}\Big\{\omega \in \Omega;\ \bar{Z}(s,t) = \bar{\mathscr{Z}}(s,t)\Big\} = 1, \quad s \in [t,T].\ \ a.e.$$

As a result, we see that

$$\begin{aligned}\bar{B}^\top \bar{Y} + \bar{D}^\top \bar{Z} &= (\bar{B}^\top \bar{P}_1 + \bar{D}^\top \bar{P}_1 B_1)\bar{X} + (\bar{B}^\top \bar{P}_2 + \bar{D}^\top \bar{P}_1 B_2)\mathbb{E}_t \bar{X} \\ &\quad + \big[\bar{B}^\top \bar{P}_5 + \bar{D}^\top (\bar{P}_1 B_3 + \bar{P}_5 H_3)\big]\bar{\mathscr{X}} + \big[\bar{B}^\top \bar{P}_6 + \bar{D}^\top \bar{P}_1 B_4\big]\mathbb{E}_t \bar{\mathscr{X}}.\end{aligned}$$

Since $\mathbb{E}_t \mathscr{X} = \mathbb{E}_t X$, we thus have $\mathbb{E}_t \bar{\mathscr{X}} = \mathbb{E}_t \bar{X}$, and

$$\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \bar{B}^\top \bar{Y} + \bar{D}^\top \bar{Z}, v \,\rangle\, dr = \mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \mathscr{W} \bar{\mathscr{X}}, v \,\rangle\, dr, \tag{3.28}$$

where we see the following by (3.22)

$$\begin{aligned}\mathscr{W} &:= \bar{B}^\top(\bar{P}_1 + \bar{P}_2 + \bar{P}_5 + \bar{P}_6) + \bar{D}^\top \Big[\bar{P}_1(B_1 + B_2 + B_3 + B_4) + \bar{P}_5 H_3\Big] \\ &= \bar{B}^\top(\bar{P}_1 + \bar{P}_2 + \bar{P}_5 + \bar{P}_6) + \bar{D}^\top \big[\bar{P}_1 + \bar{P}_5\big] H_3.\end{aligned} \tag{3.29}$$

For later notational simplicity, we define

$$\begin{cases} M := \bar{P}_1 + \bar{P}_2 + \bar{P}_5 + \bar{P}_6; \quad N := \bar{P}_1 + \bar{P}_5; \\ U_1 := M^{(11)} + M^{(21)}; \quad U_2 := N^{(11)} + N^{(21)}; \quad U_3 := N^{(11)}; \\ U_4 := P_1^{(11)} + P_1^{(21)}; \quad U_5 := P_1^{(11)}. \end{cases} \tag{3.30}$$

We state the desired transformation of $J_1(t)$ as follows.

Lemma 3.4. *Given $\Theta \in L^2(0,T;\mathbb{R}^{m\times m})$, under (H1) we have*

$$J_1(t) = \frac{1}{2}\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \mathscr{R} v, v \,\rangle\, dr + \mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \Big[(\mathscr{R} + \mathscr{D}^\top U_2 \mathscr{D})\Theta + \mathscr{S} + \mathscr{B}^\top U_1 + \mathscr{D}^\top U_2 \mathscr{C}\Big] \mathscr{X}, v \,\rangle\, dr, \tag{3.31}$$

where U_1, U_2 satisfy

(3.32)

$$\begin{cases} dU_1 = -\Big[(A_\theta + \widetilde{A}_\theta)^\top U_1 + U_1(A_\theta + \widetilde{A}_\theta) + (C_\theta + \widetilde{C}_\theta)^\top U_2 (C_\theta + \widetilde{C}_\theta) + \mathscr{Q} + \mathscr{S}^\top \Theta + \Theta^\top \mathscr{S} + \Theta^\top \mathscr{R} \Theta\Big] ds, \\ dU_2 = -\Big[(A_\theta + \widetilde{A}_\theta)^\top U_2 + U_2(A_\theta + \widetilde{A}_\theta) + (C_\theta + \widetilde{C}_\theta)^\top U_2 (C_\theta + \widetilde{C}_\theta) - \widetilde{A}_\theta^\top U_3 - \widetilde{C}_\theta^\top U_3 (C_\theta + \widetilde{C}_\theta) \\ \qquad\quad - U_3^\top \widetilde{A}_\theta - (C_\theta + \widetilde{C}_\theta)^\top U_3^\top \widetilde{C}_\theta + \widetilde{C}_\theta^\top U_5 \widetilde{C}_\theta + Q + S^\top \Theta + \Theta^\top S + \Theta^\top R \Theta\Big] ds, \\ dU_3 = -\Big[A^\top U_3 + U_3(A_\theta + \widetilde{A}_\theta) + C^\top U_3 (C_\theta + \widetilde{C}_\theta) - C^\top U_5 \widetilde{C}_\theta + Q + S^\top \Theta - U_5 \widetilde{A}_\theta\Big] ds, \\ dU_5 = -\Big[A^\top U_5 + U_5 A + C^\top U_5 C + Q\Big] ds, \\ U_1(T) = G + \widetilde{G}, \quad U_2(T) = G, \quad U_3(T) = G, \quad U_5(T) = G. \end{cases}$$

Proof. For readability, we split the proof into several parts.

Step 1. We prove equality (3.31).

From (3.28), (3.30), (3.17), we have

$$J_1(t)=\mathbb{E}_t\int_t^{t+\varepsilon}\langle(\bar B^\top M+\bar D^\top NH_3)\bar{\mathscr{X}}+F_3,v\rangle\,dr. \tag{3.33}$$

Using the definitions of $\bar{\mathscr{X}}$, H_3, and $\bar B$, $\bar D$, we further deduce that

$$\begin{aligned}&\bar B^\top M\bar{\mathscr{X}}=\mathscr{B}^\top\big[M^{(11)}+M^{(21)}\big]\mathscr{X}=\mathscr{B}^\top U_1\mathscr{X},\\&\bar D^\top NH_3\bar{\mathscr{X}}=\mathscr{D}^\top\big[N^{(11)}+N^{(21)}\big](C_\theta+\widetilde C_\theta)\mathscr{X}=\mathscr{D}^\top U_2(C_\theta+\widetilde C_\theta)\mathscr{X},\end{aligned} \tag{3.34}$$

where $M:=\big\{M^{(ij)}\big\}_{1\le i,j\le 2}$, $N:=\big\{N^{(ij)}\big\}_{1\le i,j\le 2}$, $M^{(ij)},N^{(ij)}$ is $\mathbb{R}^{n\times n}$-valued matrix. Recall the definition of F_3, we obtain that

$$\begin{aligned}J_1(t)&=\mathbb{E}_t\int_t^T\langle\,SX+\widetilde S\mathbb{E}_tX+R\Theta\mathscr{X}+\widetilde R\Theta\mathbb{E}_t\mathscr{X}+\frac12\mathscr{R}v,vI_{[t,t+\varepsilon]}\,\rangle\,dr\\&\quad+\mathbb{E}_t\int_t^T\langle\,\big[\mathscr{B}^\top U_1+\mathscr{D}^\top U_2(C_\theta+\widetilde C_\theta)\big]\mathscr{X},vI_{[t,t+\varepsilon]}\,\rangle\,dr\\&=\frac12\mathbb{E}_t\int_t^{t+\varepsilon}\langle\,\mathscr{R}v,v\,\rangle\,dr+\mathbb{E}_t\int_t^{t+\varepsilon}\langle\,\Big[(\mathscr{R}+\mathscr{D}^\top U_2\mathscr{D})\Theta+\mathscr{S}+\mathscr{B}^\top U_1+\mathscr{D}^\top U_2\mathscr{C}\Big]\mathscr{X},v\,\rangle\,dr.\end{aligned} \tag{3.35}$$

Step 2. We derive the equation of U_1 defined in (3.30).

Thanks to (3.26) and the definition of M in (3.30), it is a direct calculation that

$$\begin{cases}dM=-\Big[(C_1+C_2)M+MH_1+(C_3+C_4)NH_3+C_5+C_6+C_7+C_8\Big]ds,\\M(T)=\mathcal{G}+\widetilde{\mathcal{G}}.\end{cases}$$

To obtain the equation of U_1, we first deal with the nonhomogeneous term in the above equation.

By the definitions of $C_1,\ C_2$, we see that

$$\begin{aligned}&\Big[(C_3+C_4)NH_3+C_5+C_6+C_7+C_8\Big]^{(11)}\\&=(C+\widetilde C_\theta)^\top N^{(11)}(C_\theta+\widetilde C_\theta)+\mathscr{Q}+\mathscr{S}^\top\Theta,\\&\Big[(C_3+C_4)NH_3+C_5+C_6+C_7+C_8\Big]^{(21)}\\&=\big[\Theta^\top D^\top N^{(11)}+(\mathscr{C}+\mathscr{D}\Theta)^\top N^{(21)}\big](C_\theta+\widetilde C_\theta)+\Theta^\top\mathscr{S}+\Theta^\top\mathscr{R}\Theta.\end{aligned}$$

Moreover,

$$\begin{aligned}&\Big[(C_1+C_2)M\Big]^{(11)}=(A+\widetilde A_\theta)^\top M^{(11)},\quad\Big[(C_1+C_2)M\Big]^{(21)}=\Theta^\top B^\top M^{(11)}+(\mathscr{A}+\mathscr{B}\Theta)^\top M^{(21)},\\&\Big[MH_1\Big]^{(11)}=M^{(11)}(A_\theta+\widetilde A_\theta),\quad\Big[MH_1\Big]^{(21)}=M^{(21)}(A_\theta+\widetilde A_\theta).\end{aligned}$$

Therefore,

$$\begin{cases}dM^{(11)}=-\Big[(A+\widetilde A_\theta)^\top M^{(11)}+M^{(11)}(A_\theta+\widetilde A_\theta)+(C+\widetilde C_\theta)^\top N^{(11)}(C_\theta+\widetilde C_\theta)+\mathscr{Q}+\mathscr{S}^\top\Theta\Big]ds,\\dM^{(21)}=-\Big[\Theta^\top B^\top M^{(11)}+(\mathscr{A}+\mathscr{B}\Theta)^\top M^{(21)}+M^{(21)}(A_\theta+\widetilde A_\theta)\\\qquad\qquad+\big[\Theta^\top D^\top N^{(11)}+(\mathscr{C}+\mathscr{D}\Theta)^\top N^{(21)}\big](C_\theta+\widetilde C_\theta)+\Theta^\top\mathscr{S}+\Theta^\top\mathscr{R}\Theta\Big]ds.\end{cases}$$

Then our result is led by some direct calculations.

Step 3. Since U_1 depends on U_2, we obtain the equation described by U_2, U_3 defined in (3.30).

Thanks to (3.26) and the definition of N in (3.30), we have

$$\begin{cases} dN = -\Big[C_1N + NH_1 + C_3NH_3 - P_1(A_2+A_4) - C_3P_1(B_2+B_4) + C_5 + C_7\Big]ds, \\ N(T) = \mathcal{G}. \end{cases}$$

By the definition of C_1, H_1, H_3, we have

$$\begin{aligned} &\Big[C_1N\Big]^{(11)} = A^\top N^{(11)}, \quad \Big[C_1N\Big]^{(21)} = \Theta^\top B^\top N^{(11)} + (\mathscr{A}+\mathscr{B}\Theta)^\top N^{(21)}, \\ &\Big[NH_1\Big]^{(i1)} = N^{(i1)}(A_\theta + \widetilde{A}_\theta), \quad i=1,2, \quad \Big[C_3NH_3\Big]^{(11)} = C^\top N^{(11)}(C_\theta+\widetilde{C}_\theta), \\ &\Big[C_3NH_3\Big]^{(21)} = \big[\Theta^\top D^\top N^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top N^{(21)}\big](C_\theta+\widetilde{C}_\theta). \end{aligned}$$

As to the nonhomogeneous term in the equation of N, we have

$$\begin{aligned} &\Big[P_1(A_2+A_4)\Big]^{(i1)} = P_1^{(i1)}\widetilde{A}_\theta, \quad i=1,2, \quad \Big[C_3\bar{P}_1(B_2+B_4)\Big]^{(11)} = C^\top P_1^{(11)}\widetilde{C}_\theta, \\ &\Big[C_3\bar{P}_1(B_2+B_4)\Big]^{(21)} = (\Theta^\top D^\top P_1^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top P_1^{(21)})\widetilde{C}_\theta. \end{aligned}$$

To sum up,

$$\begin{cases} dN^{(11)} = -\Big[A^\top N^{(11)} + N^{(11)}(A_\theta+\widetilde{A}_\theta) + C^\top N^{(11)}(C_\theta+\widetilde{C}_\theta) - P_1^{(11)}\widetilde{A}_\theta \\ \qquad\qquad -C^\top P_1^{(11)}\widetilde{C}_\theta + Q + S^\top\Theta\Big]ds, \\ dN^{(21)} = -\Big[\Theta^\top B^\top N^{(11)} + (\mathscr{A}+\mathscr{B}\Theta)^\top N^{(21)} + N^{(21)}(A_\theta+\widetilde{A}_\theta) \\ \qquad\qquad +(\Theta^\top D^\top N^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top N^{(21)})(C_\theta+\widetilde{C}_\theta) \\ \qquad\qquad -P_1^{(21)}\widetilde{A}_\theta - (\Theta^\top D^\top P_1^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top P_1^{(21)})\widetilde{C}_\theta + \Theta^\top S + \Theta^\top R\Theta\Big]ds. \end{cases}$$

By the definition of U_3 in (3.30), we immediately obtain the third equation of (3.32). As to U_2, we have

$$\begin{cases} dU_2 = -\Big[(A_\theta+\widetilde{A}_\theta)^\top U_2 + U_2(A_\theta+\widetilde{A}_\theta) + (C_\theta+\widetilde{C}_\theta)^\top U_2(C_\theta+\widetilde{C}_\theta) - \widetilde{A}_\theta^\top U_3 - \widetilde{C}_\theta^\top U_3(C_\theta+\widetilde{C}_\theta) \\ \qquad\qquad -U_4\widetilde{A}_\theta - (C_\theta+\widetilde{C}_\theta)^\top U_4\widetilde{C}_\theta + \widetilde{C}_\theta^\top U_5\widetilde{C}_\theta + Q + S^\top\Theta + \Theta^\top S + \Theta^\top R\Theta\Big]ds, \\ U_2(T) = G. \end{cases} \tag{3.36}$$

Step 4. We obtain the equations of U_4, U_5 defined in (3.30).

By the definitions of C_1, A_1, C_3, B_1, we have

$$\begin{aligned} &\Big[C_1P_1\Big]^{(11)} = A^\top P^{(11)}, \quad \Big[C_1P\Big]^{(21)} = \Theta^\top B^\top P^{(11)} + (\mathscr{A}+\mathscr{B}\Theta)^\top P^{(21)}, \\ &\Big[P_1A_1\Big]^{(i1)} = P_1^{(i1)}A, \quad i=1,2, \quad \Big[C_3P_1B_1\Big]^{(11)} = C^\top P_1^{(11)}C, \\ &\Big[C_3P_1B_1\Big]^{(21)} = (\Theta^\top D^\top P_1^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top P_1^{(21)})C. \end{aligned}$$

Therefore, from the first equation in (3.26) we have

$$\begin{cases} dP_1^{(11)} = -\Big[A^\top P_1^{(11)} + P_1^{(11)}A + C^\top P_1^{(11)}C + Q\Big]ds, \\ dP_1^{(21)} = -\Big[\Theta^\top B^\top P_1^{(11)} + (\mathscr{A}+\mathscr{B}\Theta)^\top P_1^{(21)} + P_1^{(21)}A \\ \qquad\qquad +(\Theta^\top D^\top P_1^{(11)} + (\mathscr{C}+\mathscr{D}\Theta)^\top P_1^{(21)})C + \Theta^\top S\Big]ds, \\ P_1^{(11)}(T) = G, \quad P_1^{(21)}(T) = 0. \end{cases}$$

By the definition of U_3 in (3.30), we immediately obtain the fourth equation of (3.32). As to U_4,

$$
\begin{aligned}
& d\Big[P_1^{(11)}+P_1^{(21)}\Big] \\
& = -\Big[A_\theta^\top P_1^{(11)}+(A_\theta+\widetilde{A}_\theta)^\top P_1^{(21)}+Q+(P_1^{(11)}+P_1^{(21)})A+\big[C_\theta^\top P_1^{(11)}+(C_\theta+\widetilde{C}_\theta)P_1^{(21)}\big]C+\Theta^\top S\Big] \\
& = -\Big[(A_\theta+\widetilde{A}_\theta)^\top(P_1^{(11)}+P_1^{(21)})-\widetilde{A}_\theta^\top P_1^{(11)}+(P_1^{(11)}+P_1^{(21)})A+Q+\Theta^\top S \\
& \qquad +(C_\theta+\widetilde{C}_\theta)^\top(P_1^{(11)}+P_1^{(21)})C-\widetilde{C}_\theta^\top P_1^{(11)}C\Big].
\end{aligned}
$$

As a result, we obtain the following equation of U_4

$$
\begin{cases}
dU_4=-\Big[U_4A+(A_\theta+\widetilde{A}_\theta)^\top U_4+(C_\theta+\widetilde{C}_\theta)^\top U_4C-\widetilde{C}_\theta^\top U_5C+Q+\Theta^\top S-\widetilde{A}_\theta^\top U_5\Big]ds, \\
U_4(T)=G.
\end{cases}
$$

By comparison, we know that $U_3=U_4^\top$, and we obtain the equation of U_2 in (3.32) by means of (3.36). □

III. An equivalent transformation of J_2

In this subsection, we are devoted to $J_2(t)$ in Lemma 3.1, the basic ideas of which are similar as above.

To begin with, inspired by the procedures in deriving (3.11), (3.13), for $t\in[0,T]$, we introduce the following system of BSDEs,

$$
\begin{cases}
dY_1^\varepsilon=-\Big[\frac{1}{2}F_1^\varepsilon+A^\top Y_1^\varepsilon+C^\top Z_1^\varepsilon+\widetilde{A}_\theta^\top\mathbb{E}_tY_1^\varepsilon+\widetilde{C}_\theta^\top\mathbb{E}_tZ_1^\varepsilon\Big]ds+Z_1^\varepsilon dW(s), \\
dY_2^\varepsilon=-\Big[(A_\theta+\widetilde{A}_\theta)^\top Y_2^\varepsilon+(C_\theta+\widetilde{C}_\theta)^\top Z_2^\varepsilon+\Theta^\top(B^\top Y_1^\varepsilon+D^\top Z_1^\varepsilon)+\frac{1}{2}F_2^\varepsilon\Big]ds+Z_2^\varepsilon dW(s), \\
Y_1^\varepsilon(T,t)=\frac{1}{2}GX_0^\varepsilon(T)+\frac{1}{2}\widetilde{G}\mathbb{E}_tX_0^\varepsilon(T),\quad Y_2^\varepsilon(T,t)=0,
\end{cases}
\tag{3.37}
$$

where A_θ, $\widetilde{A}_\theta$, C_θ, $\widetilde{C}_\theta$ are defined in (3.2), X_0^ε is in (3.1), $F_1^\varepsilon, F_2^\varepsilon$ are in Lemma 3.1. Similar to Lemma 3.3, we have

Lemma 3.5. *Given $\Theta\in L^2(0,T;\mathbb{R}^{m\times m})$, we have*

$$
J_2(t)=\mathbb{E}_t\int_t^{t+\varepsilon}\langle\,\mathscr{B}^\top(Y_1^\varepsilon+Y_2^\varepsilon)+\mathscr{D}^\top(Z_1^\varepsilon+Z_2^\varepsilon),v\,\rangle\,ds.
\tag{3.38}
$$

For $\mathscr{X}_0^\varepsilon$, X_0^ε in (3.1), $(Y_i^\varepsilon,Z_i^\varepsilon)$ in (3.37), we extend them into R^{2n}-vectors $\bar{\mathscr{X}}_0^\varepsilon$, $\bar{X}_0^\varepsilon$, $\bar{Y}_i^\varepsilon$, $\bar{Z}_i^\varepsilon$, where

$$
\bar{\mathscr{X}}_0^\varepsilon:=\left\{\begin{array}{c}\mathscr{X}_0^\varepsilon\\0\end{array}\right\},\quad \bar{X}_0^\varepsilon:=\left\{\begin{array}{c}X_0^\varepsilon\\0\end{array}\right\},\quad \bar{Y}^\varepsilon:=\left\{\begin{array}{c}Y_1^\varepsilon\\Y_2^\varepsilon\end{array}\right\},\quad \bar{Z}^\varepsilon:=\left\{\begin{array}{c}Z_1^\varepsilon\\Z_2^\varepsilon\end{array}\right\}.
$$

To obtain the equation of $\bar{\mathscr{X}}_0^\varepsilon$, we define $\mathbb{R}^{2n}$-valued functions H_2, H_4 as

$$
H_2:=\left\{\begin{array}{c}\mathscr{B}vI_{[t,t+\varepsilon]}\\0\end{array}\right\},\quad H_4:=\left\{\begin{array}{c}\mathscr{D}vI_{[t,t+\varepsilon]}\\0\end{array}\right\}.
$$

As a result, with H_1, H_3 in (3.19), we have

$$
\begin{cases}
d\bar{\mathscr{X}}_0^\varepsilon=\big[H_1\bar{\mathscr{X}}_0^\varepsilon+H_2\big]dr+\big[H_3\bar{\mathscr{X}}_0^\varepsilon+H_4\big]dW(r),\quad r\in[0,T], \\
\bar{\mathscr{X}}_0^\varepsilon(0)=0.
\end{cases}
$$

To treat $\bar{X}_0^\varepsilon(\cdot)$, we use $\mathbb{R}^{2n\times 2n}$-valued $A_i(\cdot)$, $B_i(\cdot)$, $i=1,2,3,4$ as in (3.21) and define

$$A_5(\cdot)\equiv H_2(\cdot),\quad B_5(\cdot)\equiv H_4(\cdot). \tag{3.39}$$

Consequently, we have

$$\begin{cases} d\bar{X}_0^\varepsilon=\big[A_1\bar{X}_0^\varepsilon+A_2\mathbb{E}_t\bar{X}_0^\varepsilon+A_3\bar{\mathscr{X}}_0^\varepsilon+A_4\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon+A_5\big]dr\\ \qquad\quad +\big[B_1\bar{X}_0^\varepsilon+B_2\mathbb{E}_t\bar{X}_0^\varepsilon+B_3\bar{\mathscr{X}}_0^\varepsilon+B_4\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon+B_5\big]dW(r),\\ \bar{X}_0^\varepsilon(0)=0.\end{cases}$$

To treat $(\bar{Y}^\varepsilon,\bar{Z}^\varepsilon)$, we use $C_i(\cdot)$, $i=1,2,3,4,5,6,7,8$ as in (3.24). Therefore,

$$\begin{cases} d\bar{Y}^\varepsilon=-\Big[C_1\bar{Y}^\varepsilon+C_2\mathbb{E}_t\bar{Y}^\varepsilon+C_3\bar{Z}^\varepsilon+C_4\mathbb{E}_t\bar{Z}^\varepsilon\Big]dr-\dfrac{1}{2}\Big[C_5\bar{X}^\varepsilon+C_6\mathbb{E}_t\bar{X}^\varepsilon\\ \qquad\quad +C_7\bar{\mathscr{X}}^\varepsilon+C_8\mathbb{E}_t\bar{\mathscr{X}}^\varepsilon\Big]dr+\bar{Z}^\varepsilon dW(r),\\ \bar{Y}^\varepsilon(T,t)=\dfrac{1}{2}\mathcal{G}\bar{X}^\varepsilon(T)+\dfrac{1}{2}\widetilde{\mathcal{G}}\mathbb{E}_t\bar{X}^\varepsilon(T).\end{cases}$$

To sum up, we derive the following forward-backward system

$$\begin{cases} d\bar{\mathscr{X}}_0^\varepsilon=\big[H_1\bar{\mathscr{X}}_0^\varepsilon+H_2\big]dr+\big[H_3\bar{\mathscr{X}}_0^\varepsilon+H_4\big]dW(r),\quad r\in[0,T],\\ d\bar{X}_0^\varepsilon=\big[A_1\bar{X}_0^\varepsilon+A_2\mathbb{E}_t\bar{X}_0^\varepsilon+A_3\bar{\mathscr{X}}_0^\varepsilon+A_4\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon+A_5\big]dr\\ \qquad\quad +\big[B_1\bar{X}_0^\varepsilon+B_2\mathbb{E}_t\bar{X}_0^\varepsilon+B_3\bar{\mathscr{X}}_0^\varepsilon+B_4\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon+B_5\big]dW(r),\\ d\bar{Y}^\varepsilon=-\Big[C_1\bar{Y}^\varepsilon+C_2\mathbb{E}_t\bar{Y}^\varepsilon+C_3\bar{Z}^\varepsilon+C_4\mathbb{E}_t\bar{Z}^\varepsilon+\dfrac{1}{2}\big[C_5\bar{X}_0^\varepsilon+C_6\mathbb{E}_t\bar{X}_0^\varepsilon\\ \qquad\quad +C_7\bar{\mathscr{X}}_0^\varepsilon+C_8\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon\big]\Big]dr+\bar{Z}^\varepsilon dW(r),\\ \bar{X}_0^\varepsilon(0)=\bar{\mathscr{X}}_0^\varepsilon(0)=\bar{x}_0,\quad \bar{Y}^\varepsilon(T,t)=(Y_1^\varepsilon(T,t)^\top,0)^\top.\end{cases}$$

Moreover, using the previous defined $\bar{B}^\top$, $\bar{D}^\top$, we have

$$J_2(t)=\mathbb{E}_t\int_t^{t+\varepsilon}\langle\,\bar{B}^\top\bar{Y}^\varepsilon+\bar{D}^\top\bar{Z}^\varepsilon,v\,\rangle\,dr. \tag{3.40}$$

Next we decouple the above forward-backward system of $(\bar{\mathscr{X}}_0^\varepsilon,\bar{X}_0^\varepsilon,\bar{Y}^\varepsilon,\bar{Z}^\varepsilon)$. To do so, we introduce

$$\begin{cases} d\bar{P}_1^\varepsilon=-\Big[\bar{P}_1^\varepsilon A_1+C_1\bar{P}_1^\varepsilon+C_3\bar{P}_1^\varepsilon B_1+\dfrac{1}{2}C_5\Big]ds,\\ d\bar{P}_2^\varepsilon=-\Big\{\bar{P}_2^\varepsilon(A_1+A_2)+(C_1+C_2)\bar{P}_2^\varepsilon+C_2\bar{P}_1^\varepsilon+\bar{P}_1^\varepsilon A_2+C_3\bar{P}_1^\varepsilon B_2+\dfrac{1}{2}C_6+C_4\bar{P}_1^\varepsilon(B_1+B_2)\Big\}ds,\\ d\bar{P}_3^\varepsilon=-\Big[(C_1+C_2)\bar{P}_3^\varepsilon+\bar{P}_2^\varepsilon A_5+\bar{P}_6^\varepsilon H_2+C_2\bar{P}_4^\varepsilon+C_4\big[\bar{P}_1^\varepsilon B_5+\bar{P}_5^\varepsilon H_4\big]\Big]ds,\\ d\bar{P}_4^\varepsilon=-\Big\{C_1\bar{P}_4^\varepsilon+\bar{P}_1^\varepsilon A_5+\bar{P}_5^\varepsilon H_2+C_3\big[\bar{P}_1^\varepsilon B_5+\bar{P}_5^\varepsilon H_4\big]\Big\}ds,\\ d\bar{P}_5^\varepsilon=-\Big[\bar{P}_5^\varepsilon H_1+C_1\bar{P}_5^\varepsilon+C_3\bar{P}_5^\varepsilon H_3+\bar{P}_1^\varepsilon A_3+C_3\bar{P}_1^\varepsilon B_3+\dfrac{1}{2}C_7\Big]ds,\\ d\bar{P}_6^\varepsilon=-\Big[\bar{P}_6^\varepsilon H_1+(C_1+C_2)\bar{P}_6^\varepsilon+\bar{P}_1^\varepsilon A_4+\bar{P}_2^\varepsilon(A_3+A_4)+C_2\bar{P}_5^\varepsilon\\ \qquad\quad +C_3\bar{P}_1^\varepsilon B_4+C_4(\bar{P}_1^\varepsilon B_3+\bar{P}_5^\varepsilon H_3+\bar{P}_1^\varepsilon B_4)+\dfrac{1}{2}C_8\Big]ds,\\ \bar{P}_1^\varepsilon(T)=\dfrac{1}{2}\mathcal{G},\quad \bar{P}_2^\varepsilon(T)=\dfrac{1}{2}\widetilde{\mathcal{G}},\quad \bar{P}_6^\varepsilon(T)=\bar{P}_5^\varepsilon(T)=\bar{P}_3^\varepsilon=\bar{P}_4^\varepsilon=(0,0)^\top.\end{cases} \tag{3.41}$$

In the above, since all the coefficients are deterministic, $\bar{P}_3^\varepsilon$, $\bar{P}_4^\varepsilon$ are described by backward ODEs, but not backward SDEs. In addition, $\bar{P}_1^\varepsilon$, $\bar{P}_2^\varepsilon$, $\bar{P}_5^\varepsilon$, $\bar{P}_6^\varepsilon$ are $\mathbb{R}^{2n\times 2n}$-valued, $\bar{P}_3^\varepsilon$, $\bar{P}_4^\varepsilon$ are $\mathbb{R}^{2n\times n}$-valued, and their solvability is easy to see with $\bar{P}_i^\varepsilon = \frac{1}{2}\bar{P}_i$, $i = 1, 2, 5, 6$. Recall (3.30), we then have

$$
\bar{P}_1^\varepsilon + \bar{P}_2^\varepsilon + \bar{P}_5^\varepsilon + \bar{P}_6^\varepsilon = \frac{1}{2}M, \quad \bar{P}_1^\varepsilon + \bar{P}_5^\varepsilon = \frac{1}{2}N. \tag{3.42}
$$

If we define two $\mathbb{R}^{2n\times m}$-valued functions $\bar{\mathscr{B}}_\varepsilon$, $\bar{\mathscr{D}}_\varepsilon$ as,

$$
\bar{\mathscr{B}}_\varepsilon^\top := \big\{\mathscr{B}^\top I_{[t,t+\varepsilon]}, 0\big\}, \quad \bar{\mathscr{D}}_\varepsilon^\top := \big\{\mathscr{D}^\top I_{[t,t+\varepsilon]}, 0\big\},
$$

we know that $H_2 = \bar{\mathscr{B}}_\varepsilon v$, $H_4 = \bar{\mathscr{D}}_\varepsilon v$, where v is $\mathbb{R}^m$-valued. We introduce a system of ODE

$$
\begin{cases}
d\bar{\mathscr{P}}_4^\varepsilon = -\Big\{C_1\bar{\mathscr{P}}_4^\varepsilon + (\bar{P}_1^\varepsilon + \bar{P}_5^\varepsilon)\bar{\mathscr{B}}_\varepsilon + C_3\big[\bar{P}_1^\varepsilon + \bar{P}_5^\varepsilon\big]\bar{\mathscr{D}}_\varepsilon\Big\}ds, \\
d\bar{\mathscr{P}}_3^\varepsilon = -\Big[(C_1 + C_2)\bar{\mathscr{P}}_3^\varepsilon + (\bar{P}_2^\varepsilon + \bar{P}_6^\varepsilon)\bar{\mathscr{B}}_\varepsilon + C_2\bar{\mathscr{P}}_4^\varepsilon + C_4\big[\bar{P}_1^\varepsilon + \bar{P}_5^\varepsilon\big]\bar{\mathscr{D}}_\varepsilon\Big]ds, \\
\bar{\mathscr{P}}_3^\varepsilon(T) = 0, \quad \bar{\mathscr{P}}_4^\varepsilon(T) = 0,
\end{cases} \tag{3.43}
$$

which of course is well-posed. By the uniqueness of ODEs, $\bar{P}_4^\varepsilon = \bar{\mathscr{P}}_4^\varepsilon v$, $\bar{P}_3^\varepsilon = \bar{\mathscr{P}}_3^\varepsilon v$.

For later usefulness, we define

$$
\mathscr{M}_\varepsilon := \bar{\mathscr{P}}_3^\varepsilon + \bar{\mathscr{P}}_4^\varepsilon, \quad \mathscr{M}_\varepsilon^\top = \big([\mathscr{M}_\varepsilon^{(1)}]^\top, [\mathscr{M}_\varepsilon^{(2)}]^\top\big), \quad V_1^\varepsilon \equiv \mathscr{M}_\varepsilon^{(2)} + \mathscr{M}_\varepsilon^{(1)}. \tag{3.44}
$$

By (3.43) and (3.42), we see that

$$
d\mathscr{M}_\varepsilon = -\Big[(C_1 + C_2)\mathscr{M}_\varepsilon + \frac{1}{2}M\bar{\mathscr{B}}_\varepsilon + \frac{1}{2}(C_3 + C_4)N\bar{\mathscr{D}}_\varepsilon\Big]ds.
$$

According to the definitions of C_1, $\bar{\mathscr{B}}_\varepsilon$, $\bar{\mathscr{D}}_\varepsilon$, we have

$$
\begin{cases}
d\mathscr{M}_\varepsilon^{(1)} = -\Big[(A + \widetilde{A}_\theta)^\top\mathscr{M}_\varepsilon^{(1)} + \frac{1}{2}M^{(11)}\mathscr{B}I_{[t,t+\varepsilon]} + \frac{1}{2}(C + \widetilde{C}_\theta)^\top N^{(11)}\mathscr{D}I_{[t,t+\varepsilon]}\Big]ds, \\
d\mathscr{M}_\varepsilon^{(2)} = -\Big[(B\Theta)^\top\mathscr{M}_\varepsilon^{(1)} + (\mathscr{A} + \mathscr{B}\Theta)^\top\mathscr{M}_\varepsilon^{(2)} + \frac{1}{2}M^{(21)}\mathscr{B}I_{[t,t+\varepsilon]} \\
\qquad\qquad + \frac{1}{2}\big[(D\Theta)^\top N^{(11)} + (\mathscr{C} + \mathscr{D}\Theta)^\top N^{(21)}\big]\mathscr{D}I_{[t,t+\varepsilon]}\Big]ds, \\
\mathscr{M}_\varepsilon^{(1)}(T) = 0, \quad \mathscr{M}_\varepsilon^{(2)}(T) = 0.
\end{cases}
$$

Consequently, for U_1, U_2 defined in (3.34),

$$
\begin{cases}
dV_1^\varepsilon = -\Big[(A_\theta + \widetilde{A}_\theta)^\top V_1^\varepsilon + \frac{1}{2}U_1\mathscr{B}I_{[t,t+\varepsilon]} + \frac{1}{2}(C_\theta + \widetilde{C}_\theta)^\top U_2\mathscr{D}I_{[t,t+\varepsilon]}\Big]ds, \\
V_1^\varepsilon(T) = 0.
\end{cases}
$$

Lemma 3.6. *Given $\Theta \in L^2(0, T; \mathbb{R}^{m\times m})$, and U_2 in (3.32), we have*

$$
J_2(t) = \frac{1}{2}\mathbb{E}_t\int_t^{t+\varepsilon} \langle\, \mathscr{D}^\top U_2\mathscr{D}v, v\,\rangle\, dr + o(\varepsilon). \tag{3.45}
$$

Proof. To begin with, we give a new representation of $J_2(t)$.

For $s \in [0, T]$, $t \in [0, s]$, and $\bar{P}_i^\varepsilon$ in (3.41), we define a pair of processes

$$
\begin{cases}
\bar{\mathscr{Y}}_0^\varepsilon(\cdot, t) := \Big[\bar{P}_1^\varepsilon\bar{X}_0^\varepsilon + \bar{P}_2^\varepsilon\mathbb{E}_t\bar{X}_0^\varepsilon + \mathbb{E}_t\bar{P}_3^\varepsilon + \bar{P}_4^\varepsilon + \bar{P}_5^\varepsilon\bar{\mathscr{X}}_0^\varepsilon + \bar{P}_6^\varepsilon\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon\Big], \\
\bar{\mathscr{Z}}_0^\varepsilon(\cdot, t) := \Big[\bar{P}_1^\varepsilon B_1\bar{X}_0^\varepsilon + \bar{P}_1^\varepsilon B_2\mathbb{E}_t\bar{X}_0^\varepsilon + (\bar{P}_1^\varepsilon B_3 + \bar{P}_5^\varepsilon H_3)\bar{\mathscr{X}}_0^\varepsilon + \bar{P}_1^\varepsilon B_4\mathbb{E}_t\bar{\mathscr{X}}_0^\varepsilon + \bar{P}_1^\varepsilon B_5 + \bar{P}_5^\varepsilon H_4\Big].
\end{cases} \tag{3.46}
$$

We see that

$$(\bar{\mathscr{Y}}_d^\varepsilon(\cdot), \bar{\mathscr{Z}}_d^\varepsilon(\cdot)) \in L_{\mathbb{F}}^2(\Omega; C([0,T];\mathbb{R}^n)) \times L_{\mathbb{F}}^2(0,T;\mathbb{R}^n),$$

where

$$(\bar{\mathscr{Y}}_d^\varepsilon(s), \bar{\mathscr{Z}}_d^\varepsilon(s)) \equiv (\bar{\mathscr{Y}}_0^\varepsilon(s,s), \bar{\mathscr{Z}}_0^\varepsilon(s,s)), \quad s \in [0,T].$$

According to Lemma 3.2, for any $t \in [0,T)$,

$$\begin{aligned}
&\mathbb{P}\Big\{\omega \in \Omega;\ \bar{Y}^\varepsilon(s,t) = \bar{\mathscr{Y}}_0^\varepsilon(s,t), \quad \forall s \in [t,T]\Big\} = 1,\\
&\mathbb{P}\Big\{\omega \in \Omega;\ \bar{Z}^\varepsilon(s,t) = \bar{\mathscr{Z}}_0^\varepsilon(s,t)\Big\} = 1, \quad s \in [t,T].\ \ a.e.
\end{aligned}$$

Since $\mathbb{E}_t \mathscr{X}_0^\varepsilon = \mathbb{E}_t X_0^\varepsilon$, we thus have $\mathbb{E}_t \bar{\mathscr{X}}_0^\varepsilon = \mathbb{E}_t \bar{X}_0^\varepsilon$. Recall the notations in the above (3.44), we conclude that

$$\begin{aligned}
J_2(t) &= \mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \bar{B}^\top \bar{Y}^\varepsilon + \bar{D}^\top \bar{Z}^\varepsilon, v \,\rangle\, dr\\
&= \mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \frac{1}{2}(\bar{B}^\top M + \bar{D}^\top N H_3)\bar{\mathscr{X}}_0^\varepsilon + \Big[\bar{B}^\top \mathscr{M}_\varepsilon + \frac{1}{2}\bar{D}^\top N \bar{\mathscr{D}}_\varepsilon\Big] v, v \,\rangle\, dr\\
&= \mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \frac{1}{2}\big[\mathscr{B}^\top U_1 + \mathscr{D}^\top U_2 (C_\theta + \widetilde{C}_\theta)\big]\bar{\mathscr{X}}_0^\varepsilon + \Big[\mathscr{B}^\top V_1^\varepsilon + \frac{1}{2}\mathscr{D}^\top U_2 \mathscr{D}\Big] v, v \,\rangle\, dr.
\end{aligned} \tag{3.47}$$

Now we treat the right hand of (3.47) one by one. First, by Remark 3.1, we see that

$$\frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \mathscr{B}^\top U_1 \mathscr{X}_0^\varepsilon + \mathscr{D}^\top U_2 (C_\theta + \widetilde{C}_\theta)\mathscr{X}_0^\varepsilon, v \,\rangle\, dr = o(1).$$

As to the second term of V_1^ε, we have $\lim\limits_{\varepsilon \to 0} \sup\limits_{r \in [t, t+\varepsilon]} |V_1^\varepsilon(r)| = 0$. We then conclude that

$$\frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \mathscr{B}^\top V_1^\varepsilon v, v \,\rangle\, dr = o(1).$$

To sum up, our conclusion (3.45) is implied by (3.47). ☐

IV. Proof of the main result

Proof. From Remark 3.1, we have

$$\lim_{\varepsilon \to 0} \Big[\frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \langle (\Theta^\top \mathscr{R} + \mathscr{S}^\top) v, \mathscr{X}_0^\varepsilon \,\rangle\, ds\Big] = 0.$$

Using Lemma 3.1, 3.4, 3.6, Θ^* is closed-loop equilibrium strategies if and only if

$$\lim_{\varepsilon \to 0} \Big[\frac{1}{2\varepsilon} \int_t^{t+\varepsilon} \langle (\mathscr{R} + \mathscr{D}^\top U_2^* \mathscr{D}) v, v \,\rangle\, dr + \frac{1}{\varepsilon}\mathbb{E}_t \int_t^{t+\varepsilon} \langle\, \mathscr{G}_1 \mathscr{X}^*, v \,\rangle\, dr\Big] \geq 0, \tag{3.48}$$

where

$$\mathscr{G}_1 := (\mathscr{R} + \mathscr{D}^\top U_2^* \mathscr{D})\Theta^* + \mathscr{S} + \mathscr{B}^\top U_1^* + \mathscr{D}^\top U_2^* \mathscr{C}.$$

A standard procedure leads to the conclusion in Theorem 2.1. ☐

4. Concluding remarks

In this article, we investigate a class of time inconsistent stochastic linear quadratic problems for mean-field stochastic differential equations in Markovian framework. We introduce the closed-loop equilibrium strategies by variational ideas and establish their characterization by first-order, second-order equilibrium conditions. We also make careful comparisons with various existing notions, namely, pre-committed optimal strategies, open-loop equilibrium strategies, closed-loop equilibrium strategies, and reveal several new phenomena arising here. There are some other interesting topics along this line, such as the solvability of Riccati system under proper conditions, the uniqueness of CLESs, the corresponding study with random coefficients, and the investigation with nonlinear state equation and cost functional. We hope to discuss them in the future publications.

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