Minimal time sliding mode control for evolution equations in Hilbert spaces

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Abstract. This work is concerned with the time optimal control problem for evolution equations in Hilbert spaces. The attention is focused on the maximum principle for the time optimal controllers having the dimension smaller that of the state system, in particular for minimal time sliding mode controllers, which is one of the novelties of this paper. We provide the characterization of the controllers by the optimality conditions determined for some general cases. The proofs rely on a set of hypotheses meant to cover a large class of applications. Examples of control problems governed by parabolic equations with potential and drift terms, porous media equation or reaction-diffusion systems with linear and nonlinear perturbations, describing real world processes, are presented at the end.

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1 Problem presentation

The purpose of this paper is to study the time optimal control for a family of evolution equations in Hilbert spaces. In time optimal control the optimality criterion is the elapsed time. Here, by the time optimal control problem we mean to search for a constrained internal controller able to drive the trajectory of the solution from an initial state to a given target set in the shortest time, while controlling over the complete timespan.

Minimum time control problems have been initiated by Fattorini in the paper [12] and developed later in the monograph [13]. A list of only few titles dealing with this subject, in special for problems governed by parabolic type equations includes [16], [17], [18], [20], [21], [23]. In what concerns problems governed by abstract evolution equations, we cite [2] and the monographs [3], [4], [7]. In [2] the existence and uniqueness of a viscosity solution was provided for the Bellman equation associated with the time-optimal control problem for a semilinear evolution equation in a Hilbert space, while in [5] the time optimal control was studied for the Navier-Stokes equations. The existence of the optimal time control for a phase-field system was proved in [20] for a regular double-well potential, by using the Carleman inequality and the maximum principle was established by using two controls acting in subsets of the space domain. The asymptotic behavior of the solutions of a class of abstract parabolic time optimal control problems when the generators converge, in an appropriate sense, to a given strictly negative operator was studied in [19]. For a large class of problems and aspects related to this subject we refer the reader to the recent monograph [22].

From the perspective of applications, many processes in engineering, physics, biology, medicine, environmental sciences, ecology require solutions relying on time optimal control problems. The theoretical results in this paper aim to cover models governed by parabolic equations with potential and drift terms and various reaction-diffusion systems with linear or nonlinear perturbations, or nonlocal control problems, presented in the last section.

Especially of interest in applications is to control a system using a controller whose dimension is smaller than that of the state system. In this case the initial datum is steered not into a point, but into a linear manifold of the state space, situation which is relevant for the sliding mode control (for

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some references see e.g., [8], [9], [11]). The solution to such a problem, which is more challenging from the mathematical point of view, is a central point in our theoretical approach.

We prove the existence of the time optimal control and the first order necessary conditions of optimality in relation with the evolution equation on a Hilbert space H,

$$y'(t) + Ay(t) = Bu(t),\ t > 0, \tag{1.1}$$

$$y(0) = y_0. \tag{1.2}$$

Here, A is a nonlinear and unbounded operator over a Hilbert space H, B is a linear operator from a Banach space U to H, u is a controller constrained to belong to U and y is the solution to (1.1) corresponding to the initial datum y_0 and controlled by Bu. The following minimization problem is studied:

$$\text{Minimize } \left\{J(T,u) := T;\ (T,u) \in \mathcal{U}_{ad},\ Py(T) = Py^{tar}\right\} \tag{$\mathcal{P}$}$$

where y is the solution to (1.1)-(1.2) and

$$\mathcal{U}_{ad} := \{(T,u);\ T \in \mathbb{R},\ T > 0,\ u \in L^\infty(0,T;U),\ \|u(t)\|_U \le \rho \text{ a.e. } t > 0\}. \tag{1.3}$$

As it will be further explained, P is an algebraic projection of the solution from H to H or to a subspace of it, y^{tar} is a prescribed target for the state and ρ is a positive constant at our choice.

Relying on certain hypotheses ensuring the well-posedness of (1.1)-(1.2) and on the hypothesis of a not empty admissible set $\mathcal{U}_{ad}$, the existence of optimal controls is proved. A maximum principle is first provided for an intermediate approximating problem. This generates a sequence of approximating optimal solutions which converges to a precise optimal pair to ($\mathcal{P}$), whose characterization is a central point of the paper.

The results are more relevant in the case of states with many components. That is why, for the sake of a clearer explanation and for a simpler notation, let us first assume that the state y in (1.1)-(1.2) has two components, $y = (y_1, y_2) \in H = H_1 \times H_2$ and $u = (u_1, u_2) \in U = U_1 \times U_2$, where H_i are Hilbert spaces and U_i are Banach spaces, $i = 1, 2$.

We focus on two problems. The more challenging case is to steer, by the action of only one control $Bu = (B_1u_1, 0)$, only the first component $y_1(t)$ of the state $y(t)$ from its initial value into a manifold $\mathcal{S}$, within a minimal time T^*. In this case, the target manifold is $y_1 = y_1^{\text{target}}$. This action may be realized using effectively one controller acting in the first equation. Thus, the state is forced to reach the manifold $\mathcal{S} = \{y;\ y_1 = y_1^{\text{target}}\}$ on which it may continue to slide, for $t \ge T^*$, possibly under supplementary conditions and by performing a controller slight modification after the time T^*. This turns out to be in fact the sliding mode control and it will be detailed for a reaction-diffusion model in Section 6, Example 3.

Another possibility is to control both state components, forcing them to reach a prescribed point target $y^{tar} := (y_1^{\text{target}}, y_2^{\text{target}})$, by employing two controllers, with $Bu = (B_1u_1, B_2u_2)$.

Because the intention is to simultaneously prove the objectives stated before, these are formalized by means of the minimization problem ($\mathcal{P}$) involving a mapping $P \in L(H,H)$ covering each of the following situations:

(i) $P(y_1, y_2) = (y_1, y_2) \in H_1 \times H_2$, $B(u_1, u_2) = (B_1u_1, B_2u_2)$, $y^{tar} = (y_1^{\text{target}}, y_2^{\text{target}}) = Py^{tar}$, in the case when both state components are controlled by two controllers;

(ii) $P(y_1, y_2) = (y_1, 0)$, $y_1 \in H_1$, $y_2 \in H_2$, $B(u_1, u_2) = (B_1u_1, 0)$, $Py^{tar} := (y_1^{\text{target}}, 0)$, in the situation when only the first component is controlled by one controller.

With this notation, (1.1) can be rewritten

$$y_1'(t) + (Ay(t))_1 = B_1u_1(t),\ y_2'(t) + (Ay(t))_2 = B_2u_2(t),\ \text{a.e. } t > 0$$

in the case (i) and

$$y_1'(t) + (Ay(t))_1 = B_1u_1(t),\ y_2'(t) + (Ay(t))_2 = 0,\ \text{a.e. } t > 0$$

in the case (ii), where $Ay(t) = ((Ay(t))_1, (Ay(t))_2)$.

Actually, in case (ii), P is an algebraic projection of H into $H_1 \times \{0\} \subset H$, mapping $y = (y_1, y_2)$ into its first component y_1 which is the only one controlled in this case. For the second component of Py corresponding to the second equation which is not controlled we set the value zero, in order to ensure a compact notation consistent in the calculations (e.g., of the type $\|Pv\|_H \leq C \|Bv\|_H$) with the value $Bu_2 = 0$.

In both cases we agree to use the same notation y^{tar} in order to allow a compact writing. In the first case, y^{tar} contains the targets for each state component. In the second case, the essential role is played by the first component of Py^{tar} while the second component of y^{tar} plays no role. We can set the latter zero, even if this is not a target, because its action, as well as that of the second component of Py will be cancelled in the calculations by the second zero component of Bu.

This explanation can be extended to the case with the state y having n components, either when all n components are controlled by n controllers, or when only k trajectories $(y_1, ..., y_k)$ are led into $(y_1^{\text{target}}, ..., y_k^{\text{target}})$, by using k controllers, via $Bu = (B_1 u_1, ...B_k u_k, 0, ..., 0)$. We note that we can change the notation, indicating the vector $(y_1, ..., y_k)$ of the first k components still by y_1 and the vector $(y_{k+1}, ..., y_n)$ by y_2 and we can use a similar notation for u and B. So, the general case can be reduced to that with two state components. To conclude, for the writing simplicity, we shall refer in the sequel to the case with two state components.

The paper is organized as follows. The theoretical results rely on a set of hypotheses, $(a_1) - (a_6)$, (b_1), (c_1), listed in Section 2. For the passing to the limit proof in Theorem 5.5, Section 5.3, there are necessary some technical assumptions $(d_1) - (d_5)$, including the hypothesis (2.14). This is essential for the characterization of the controller if only one state is controlled ($P \neq I$, $Bu = (B_1 u_1, 0)$), which may be the more relevant in applications. This is one of the novelty of this paper, besides the results characterizing the controller for evolution equations with some general nonlinear operators. Section 3 includes some results of existence, beginning with the well-posedness of the state system (1.1)-(1.2), in Theorem 3.2. The existence of the minimum time is provided in Theorem 3.3. In Section 4, we employ an approximating problem $(\mathcal{P}_\varepsilon)$ indexed along a small parameter ε occurring in some penalization terms of the functional J. After giving a basic result in Theorem 4.1 for the existence of a solution to $(\mathcal{P}_\varepsilon)$, the convergence of $(\mathcal{P}_\varepsilon)$ to $(\mathcal{P})$ is proved in Theorem 4.2. This result is strong by asserting that if one fix an optimal pair (T^*, u^*) in $(\mathcal{P})$, the sequence of optimal pairs in $(\mathcal{P}_\varepsilon)$ tends exactly to (T^*, u^*). The necessary conditions of optimality for $(\mathcal{P}_\varepsilon)$ are determined in Proposition 5.4 at the end of an extremely technical procedure, while in Theorem 5.5 the necessary conditions of optimality for $(\mathcal{P})$ are obtained as the limit of the previous ones, as $\varepsilon \to 0$, after sharp estimates for the approximating solution. A particular case for U, usually encountered, is treated in Corollary 5.6. Applications of these results, including a detailed example of minimum time sliding mode control, are presented in the last section. In the Appendix we provide some definitions and general results necessary in the paper.

2 Functional framework and basic hypotheses

Functional framework. Let V_i and H_i, $i = 1, 2$, be Hilbert spaces and consider the standard triplet $V_i \subset H_i \equiv H_i^* \subset V_i^*$, with compact embeddings, where V_i^* is the dual of V_i. Let U_i, $i = 1, 2$, be Banach spaces with the duals U_i^* uniformly convex, implying that U_i^* and U_i are reflexive (see e.g., [6], p. 2). Let us denote

$$V = V_1 \times V_2,\ H = H_1 \times H_2,\ V^* = V_1^* \times V_2^*,\ U = U_1 \times U_2,\ U^* = U_1^* \times U_2^*.$$

We recall that the operator P was defined in the introduction (see $(i) - (ii)$), but as a matter of fact we can define it on any space $X = X_1 \times X_2$, where X_i can be V_i, H_i, V_i^*. Also, we use the same symbol P for $X_i = U_i$. Thus,

$$P : X = X_1 \times X_2 \to X,\ P \in L(X, X),$$

and it is defined as

$$Py = (y_1, y_2) \text{ or } Py = (y_1, 0),\ \text{ for } y = (y_1, y_2) \in X.$$

It can be easily seen $P^2 = P$ and $\|Py\|_X \leq \|y\|_X$.

Let $A : V \to V^*$. We denote by $A_H : D(A_H) \subset H \to H$ the *restriction* of A on H defined by $A_H y = Ay$ for $y \in D(A_H) = \{y \in V;\ Ay \in H\}$. In the sequel, $\Gamma : V \to V^*$ is the duality mapping from V to V^* and Γ_H is the restriction of Γ to H (see 7.15).

Let $A \in C^1(V, V^*)$. The Gâteaux derivative of A is the linear operator $A'(y) : V \to V^*$ defined by

$$A'(y)z = \lim_{\lambda \to 0} \frac{A(y + \lambda z) - Ay}{\lambda} \text{ strongly in } V^*, \text{ for all } y, z \in V$$

and $A'_H(y)$ is the restriction of $A'(y)$ to H (see (7.1)-(7.2)). Properties of these operators are given in the Appendix.

Notation. Let X and Y be Banach spaces. By $L^p(0, T; X)$ we denote the space of p-summable functions from $(0, T)$ to X, for $1 \le p \le \infty$. $W^{1,p}(0, T; X) = \{f;\ f : [0, T] \to X$, absolutely continuous, $f, df/dt \in L^p(0, T; X)\}$. $C(X, Y)$ and $C^1(X, Y)$ are the spaces of continuous and differentiable Gâteaux, respectively, operators from X to Y. $L(X, Y)$ is the space of linear continuous operators from X to Y. We denote the scalar product and norm in the space X by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$, respectively.
We shall denote by C, C_i, α_i, γ_i, $i = 0, 1, 2, ...$ positive constants that may change from line to line.

Some other notation and definitions related to the hypotheses below can be found in the Appendix.

Hypotheses $(a_1), (a_2), (b_1), (c_1)$

(a_1) $A : V \to V^*$ is demicontinuous, $A0 = 0$,

$$\langle Ay - A\overline{y}, y - \overline{y}\rangle_{V^*,V} \ge \alpha_1 \|y - \overline{y}\|_V^2 - \alpha_2 \|y - \overline{y}\|_H^2, \text{ for all } y, \overline{y} \in V,\ \alpha_1 > 0, \tag{2.1}$$

$$A \text{ is bounded on bounded subsets of } V, \tag{2.2}$$

$$D(A_H) = D(\Gamma_H) := D_H. \tag{2.3}$$

(a_2)

$$(A_H y, \Gamma_H y)_H \ge \alpha_3 \|\Gamma_H y\|_H^2 - \alpha_4 \|y\|_V^2, \text{ for all } y \in D_H. \tag{2.4}$$

(b_1)

$$B \in L(U, H).$$

(c_1) For each $y^{tar} \in H$, and ρ positive large enough, there exists $T_* > 0$ and $u \in L^\infty(0, T; U)$ with $\sup_{t \ge 0} \|u(t)\|_U \le \rho$, such that $Py(T_*) = Py^{tar}$, where y is the solution to (1.1)-(1.2).

Hypotheses (a_1), (a_2), (b_1) are necessary to prove the state system well-posedness and the existence of the solution to $(\mathcal{P})$, in Section 3. The minimization problem $(\mathcal{P})$ is relevant if the set $\mathcal{U}_{ad}$ is not empty. Hypothesis (c_1) ensures that $\mathcal{U}_{ad} \ne \varnothing$ and it is used in the proof of the control existence. We specify that the proof of the controllability of (1.1)-(1.2) is beyond the objective of this paper. However, for the reader convenience, the existence of a least a pair (T, u) in the admissible set, or equivalently an example of proving the controllability of (1.1)-(1.2) in some cases, is given in Appendix, Proposition 7.1. Next, in the Examples, the reliability of (c_1) is commented in each case.

Hypotheses $(a_3) - (a_6)$

(a_3) $A \in C^1(V; V^*)$ and $A_H \in C^1(D_H; H)$.

(a_4) $A'(y)$ and $A'_H(y)$ defined by (7.1) and (7.2) respectively, satisfy

$$\|A'(y)z\|_{V^*} \le C \|z\|_V (1 + C_1 \|y\|_V^\kappa), \text{ for all } y, z \in V, \text{ some } \kappa \in \mathbb{R},\ \kappa \ge 0, \tag{2.5}$$

$$\|A'_H(y)z\|_H \le C \|z\|_{D_H} (1 + C_1 \|y\|_V^\kappa), \text{ for all } y, z \in D_H,\ \kappa \ge 0. \tag{2.6}$$

(a_5) A' is strongly continuous from V to $L_s(V, V^*)$, (see (7.3)) and $A'_H(y)$ is strongly continuous from V to $L(D_H, H)$, namely

$$\begin{aligned} \|A'_H(y_n)\psi - A'_H(y)\psi\|_H &\to 0 \text{ for all } \psi \in D_H,\ y_n, y \in D_H, \\ \text{as } y_n &\to y \text{ strongly in } V. \end{aligned} \tag{2.7}$$

(a_6) The adjoint operator $(A'(y))^* : V \to V^*$ satisfies the condition

$$\langle (A'(y))^* z, \Gamma_\nu z\rangle_{V^*,V} \ge \gamma_1 \|\Gamma_\nu z\|_H^2 - \gamma_2 \|z\|_V^2 (1 + \gamma_3 \|y\|_V^l) - \gamma_4, \tag{2.8}$$

for all $y, z \in V$, some $l \geq 0$, $\gamma_1, \gamma_2 > 0$, where Γ_ν, $\nu > 0$, is the Yosida approximation of Γ, see (7.17).

Hypotheses $(a_3) - (a_6)$ are necessary for the proof of the existence of the system in variations, the adjoint system and the determination of the approximating optimality conditions.

Hypotheses $(d_1) - (d_5)$

Assume that $U \neq H$ and that there exists $\alpha \in (0,1)$ such that

(d_1)

$$\left\| P\Gamma_H^{-\alpha/2} v \right\|_H \leq C \left\| B^* v \right\|_{U^*}, \ v \in H. \tag{2.9}$$

(d_2) $(A'_H(y))^*$ satisfies the relations

$$((A'_H(y))^* v, \Gamma_H^{-\alpha} v)_H \geq C_1 \left\| \Gamma_H^{(1-\alpha)/2} v \right\|_H^2 - C_2 \left\| v \right\|_H^2 (1 + C_3 \left\| y \right\|_V^l), \tag{2.10}$$

for all $y, v \in D_H$, some $l \geq 0$, and

$$((A'_H(y))^* v, \Gamma_H^{-1} v)_H \geq C_1 \left\| v \right\|_H^2 - C_2 \left\| v \right\|_{V^*}^2 (1 + C_3 \left\| y \right\|_V^l), \tag{2.11}$$

(d_3)

$$\left\| Pv \right\|_{V^*} \leq C^* \left\| B^* v \right\|_{U^*}, \text{ for } v \in H. \tag{2.12}$$

(d_4) Let $Py = (y_1, 0)$ and let ρ be sufficiently large. For each $u \in L^\infty(0, \infty; U)$, with $\left\| u(t) \right\|_U \leq \rho$ there exists $\widehat{z}$, possibly depending on u, such that

$$\widehat{y}(t) = (y_1^{\text{target}}, \widehat{z}) \tag{2.13}$$

satisfies

$$\langle Ay(t) - A\widehat{y}, P(y(t) - \widehat{y}) \rangle_{V^*, V} \geq -C_3 \left\| P(y(t) - \widehat{y}) \right\|_H^2, \tag{2.14}$$

for all $y \in V$, and $t \in (0, T_* + \delta)$, with the choice

$$\rho > \rho_1, \ \rho_1 := C^* \left\| A_H \widehat{y} \right\|_V. \tag{2.15}$$

(d_5) Let $P = I$, assume

$$\rho > \rho_1, \ \rho_1 := C^* \left\| A_H y^{tar} \right\|_V, \tag{2.16}$$

and relation (2.14), where $\widehat{y}$ is replaced by $y^{tar} = (y_1^{\text{target}}, y_2^{\text{target}})$.

We specify that C^* in (2.15) and (2.16) is exactly the constant C^* occurring in (2.12), depending on the domain Ω and B, T_* is the time specified in the controllability hypothesis (c_1), δ is arbitrary and $y(t)$ is the solution to (1.1)-(1.2) corresponding to u.

Assumption (2.14) is a basic statement in the proof of the characterization of the controller in the case when only one state is controlled by one controller. This is the case when the state is allowed to reach a sliding manifold.

We also note that if $B_1 = B_2 = I$ or $B_1 = I$ and $B_2 = 0$ and the spaces are such that $V \subset U$ or $H \subset U$, then (2.12) is automatically satisfied. The case $U = H$ will be treated in Corollary 5.6.

Immediate consequences of the previous hypotheses are:

The operator $\lambda I + A$, for λ positive large enough, is coercive and A_H is quasi m-accretive on $H \times H$, implied by (a_1).

The operator $A'(y)$ satisfies the estimate

$$\langle A'(y)z, z \rangle_{V^*, V} \geq \alpha_1 \left\| z \right\|_V^2 - \alpha_2 \left\| y \right\|_H^2, \text{ for all } z \in V. \tag{2.17}$$

The operator $A'(y)|_H = A'_H(y)$ is quasi m-accretive for each $y \in V$ and

$$(A'_H(y)z, z)_H \geq \alpha_1 \left\| z \right\|_V^2 - \alpha_2 \left\| y \right\|_H^2, \text{ for all } y, \ z \in D_H. \tag{2.18}$$

By (2.5) we have

$$\left\| A'(y) \right\|_{L(V, V^*)} \leq C(1 + C_1 \left\| y \right\|_V^\kappa). \tag{2.19}$$

3 Existence results

In this section we provide the proofs of the existence of the solution to the state system and of a solution to the minimization problem $(\mathcal{P})$. All over in this section, we assume (a_1), (a_2), (b_1), (c_1). Let

$$X_T = C([0,T];H) \cap W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;D_H). \tag{3.1}$$

Definition 3.1. A strong solution to the Cauchy problem (1.1)-(1.2) is a continuous function $y : [0,T] \to H$, which is a.e. differentiable and satisfies (1.1) a.e. $t \in (0,T)$ and (1.2).

Theorem 3.2. *Let $T > 0$, $u \in L^2(0,T;U)$, $y_0 \in V$. Then, (1.1)-(1.2) has a unique strong solution $y \in X_T$, satisfying*

$$\begin{aligned} & \|y(t)\|_V^2 + \int_0^t \|A_H y(\tau)\|_H^2 \, d\tau + \int_0^t \|y(\tau)\|_{D_H}^2 \, d\tau + \int_0^t \|y'(\tau)\|_H^2 \, d\tau \\ \leq \;& C \left(\|y_0\|_V^2 + \rho^2 T \right) e^{CT} := C_T, \text{ for all } t \in [0,T], \end{aligned} \tag{3.2}$$

with C a positive constant. Moreover, for two solutions y and $\overline{y}$ corresponding to u and $\overline{u}$ we have

$$\|(y-\overline{y})(t)\|_H^2 + \int_0^t \|(y-\overline{y})(\tau)\|_V^2 \, d\tau \leq C \left(\|y_0 - \overline{y_0}\|_H^2 + \|u - \overline{u}\|_{L^2(0,T;U)}^2 \right), \text{ for all } t \in [0,T]. \tag{3.3}$$

Finally, if $u_n \in L^\infty(0,T;U)$, $u_n \to u$ weak-star in $L^\infty(0,T;U)$, then the solution y_n corresponding to u_n tends to y, the solution corresponding to u, namely

$$\begin{aligned} y_n \;&\to\; y \text{ weakly in } W^{1,2}(0,T;H) \cap L^2(0,T;D_H), \\ &\quad\;\; \text{weak-star in } L^\infty(0,T;V), \text{ strongly in } L^2(0,T;H), \\ y_n(t) \;&\to\; y(t) \text{ strongly in } H, \text{ uniformly in } [0,T]. \end{aligned} \tag{3.4}$$

Proof. We recall that A_H is quasi m-accretive on $H \times H$. Assume first that the right-hand side of (1.1), $f = Bu$ is in $W^{1,1}(0,T;H)$ and $y_0 \in D_H = D(A_H)$. In this case we obtain a unique solution $y \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D_H)$ (see e.g., [6], p. 151, Theorem 4.9), implying that $A_H y \in L^\infty(0,T;H)$. A first estimate is obtained by testing the equation (1.1) by $y(t)$ and integrating it over $(0,t)$,

$$\begin{aligned} & \frac{1}{2} \|y(t)\|_H^2 + \alpha_1 \int_0^t \|y(\tau)\|_V^2 \, d\tau \\ \leq \;& \frac{1}{2} \|y_0\|_H^2 + \alpha_2 \int_0^t \|y(\tau)\|_H^2 \, d\tau + \int_0^t \|Bu(\tau)\|_H \|y(\tau)\|_H \, d\tau, \end{aligned}$$

which yields

$$\|y(t)\|_H^2 + \int_0^t \|y(\tau)\|_V^2 \, d\tau \leq C \left(\|y_0\|_H^2 + \int_0^T \|u(t)\|_U^2 \, dt \right) e^{Ct}. \tag{3.5}$$

Then, we multiply (1.1) in H by $\Gamma_H y(t)$, use (2.4) and integrate over $(0,t)$, obtaining

$$\begin{aligned} & \left\| \Gamma_H^{1/2} y(t) \right\|_H^2 + \alpha_3 \int_0^t \|\Gamma_H y(\tau)\|_H^2 \, d\tau \\ \leq \;& \left\| \Gamma_H^{1/2} y_0 \right\|_H^2 + \int_0^t \|Bu(\tau)\|_H \|\Gamma_H y(\tau)\|_H \, d\tau + \alpha_4 \int_0^t \|y(\tau)\|_V^2 \, d\tau \\ \leq \;& \|y_0\|_V^2 + \frac{1}{2} \int_0^t \|\Gamma_H y(\tau)\|_H^2 \, d\tau + C_1 \int_0^T \|u(t)\|_U^2 \, dt + C_2 \int_0^t \|y(\tau)\|_V^2 \, d\tau. \end{aligned}$$

Using (3.5) we get

$$\|y(t)\|_V^2 + \int_0^t \|\Gamma_H y(\tau)\|_H^2 \, d\tau \leq C \left(\|y_0\|_V^2 + \int_0^T \|u(t)\|_U^2 \, dt \right) e^{Ct} \leq C_T, \tag{3.6}$$

for all $t \in [0,T]$, $T > 0$. We note that C_T is continuous and increasing with respect to T, but it can vary from line to line via the constant C. This implies that

$$\int_0^T \|y(\tau)\|_{D_H}^2 \, d\tau \leq C_T \text{ and } \int_0^T \|A_H y(\tau)\|_H^2 \, d\tau \leq C_T. \tag{3.7}$$

By comparison with (1.1) we deduce that $\int_0^T \|y'(\tau)\|_H^2 \, d\tau \leq C_T$. By gathering all estimates we obtain (3.2).

To prove (3.3) we consider two solutions corresponding to (y_0, u) and $(\overline{y_0}, \overline{u})$, write the difference of the equations for these solutions, test it by $(y - \overline{y})(t)$, integrate over $(0,t)$ and apply the Gronwall lemma.

We proceed further by a density argument. We take $y_0^n \in D_H$ and $u_n \in W^{1,1}(0,T;U)$ such that $y_0^n \to y_0$ strongly in V and $u_n \to u$ strongly in $L^2(0,T;U)$, the latter implying $Bu_n \to Bu$ strongly in $L^2(0,T;H)$. It follows that the solution to (1.1) with Bu_n instead of Bu and with the initial datum y_0^n has a unique strong solution $y_n \in W^{1,\infty}(0,T;H) \cap L^\infty(0,T;D_H)$, satisfying (3.2) and (3.3). From here, it follows that $y_n \to y$ strongly in $C([0,T];H) \cap L^2(0,T;V)$, as $n \to \infty$, and the estimate (3.3) for y_n is preserved at limit. The right-hand side of (3.2) is bounded and so $A_H y_n \to A_H y$ weakly in $L^2(0,T;H)$, since A_H is strongly-weakly closed, and $y_n' \to y'$ weakly in $L^2(0,T;H)$. The estimate (3.2) is preserved at limit by the lower weakly continuity of the norms.

Let $u_n \in L^\infty(0,T;U)$, $u_n \to u$ weak-star in $L^\infty(0,T;U)$. Then, (1.1)-(1.2) has a unique solution y_n satisfying (3.2). Since the estimates are uniform, on a subsequence we get the convergences in the first line of (3.4). The strongly convergence follows by the Aubin-Lions lemma and the last one by the Arzelà-Ascoli theorem. Passing to the limit in (1.1)-(1.2) written for y_n we get (1.1)-(1.2) corresponding to y. $\square$

We observe that by (3.4) we deduce that $y \in C_w([0,T];V)$, that is, except for a subset of zero measure, y is a weak continuous function from $[0,T]$ in V. We recall that $y \in C_w([0,T];V)$, if when $t_n \to t$, as $n \to \infty$, it follows that $y(t_n) \to y(t)$ weakly in V. Indeed, in the proof of Theorem 3.2 we have for $t_n \to t$, that $y(t_n) \to y(t)$ strongly in H. On the other hand, $\|y(t_n)\|_V \leq C_T$ and so $y(t_n) \to \xi$ weakly in V. But the limit is unique and so $\xi = y(t) \in V$, for all $t \geq 0$.

Similarly, we deduce that $Ay \in C_w([0,T];V^*)$. By (2.2), $\|Ay(t_n)\|_{V^*}$ is bounded, since $\|y(t_n)\|_V \leq C_T$ and so $Ay(t_n) \to \zeta$ weakly in V^*. On the other hand, $Ay(t_n) \to Ay(t)$ strongly in D_H^*, the dual of D_H because $y(t_n) \to y(t)$ in H. Thus, $\zeta = Ay(t) \in V^*$, for all $t \geq 0$.

Now we prove the existence of the minimum in $(\mathcal{P})$. Recall that $y^{tar} := (y_1^{\text{target}}, y_2^{\text{target}})$ if $P = I$ and $y^{tar} := (y_1^{\text{target}}, z)$, $z \in H$ if $P \neq I$.

Theorem 3.3. *Let*

$$y_0 \in V, \ Py^{tar} \in P(H), \ Py_0 \neq Py^{tar}.$$

Then, problem $(\mathcal{P})$ has at least one solution (T^*, u^*) with the corresponding state $y^{T^*,u^*} := y^*$.

Proof. We recall that we have assumed (c_1) asserting that the admissible set $\mathcal{U}_{ad} \neq \varnothing$. The functional $J(T,u) = T$ is nonnegative, hence it has an infimum. We denote $\inf J(T,u) = T^* \geq 0$. Let us consider a minimizing sequence (T_n, u_n),

$$T_n > 0, \ u_n \in L^\infty(0,\infty;U), \ \sup_{t \in (0,\infty)} \|u_n(t)\|_U \leq \rho, \ Py^{T_n,u_n}(T_n) = Py^{tar},$$

where y^{T_n,u_n} is the solution to the state system corresponding to (T_n, u_n), such that

$$T^* \leq J(T_n, u_n) = T_n \leq T^* + \frac{1}{n}, \ n \geq 1. \tag{3.8}$$

On a subsequence it follows that

$$u_n \to u^* \text{ weak-star in } L^\infty(0,T;U), \ Bu_n \to Bu^* \text{ weak-star in } L^\infty(0,T;H), \text{ for all } T > 0. \tag{3.9}$$

We see that $T_n \to T^*$ and passing to the limit in (3.8) we get that $J(T^*, u^*) = T^*$. The state system corresponding to each $T > 0$ and u_n has a unique solution y^{T,u_n} satisfying (3.2). In particular, this is true for $T = T_n$ and $T = T^*$. We note that the restriction of the solution y^{T_n,u_n} to $(0, T^*)$ is in fact the solution y^{T^*,u_n}. We have by (3.2)

$$\left\|y^{T^*,u_n}\right\|^2_{X_{T^*}} + \left\|A_H y^{T^*,u_n}\right\|^2_{L^2(0,T^*;H)} \le \left\|y^{T_n,u_n}\right\|^2_{X_{T_n}} + \left\|A_H y^{T_n,u_n}\right\|^2_{L^2(0,T_n;H)} \le C_{T_n} \le C_{T^*+1},$$

where C_T depends continuously and increasingly on T (see (3.2)). Therefore, by selecting a subsequence and recalling (3.4) we have

$$\begin{aligned} y^{T^*,u_n} \quad &\to \quad y^* := y^{T^*,u^*} \text{ weakly in } W^{1,2}(0,T^*;H) \cap L^2(0,T^*;D_H), \\ &\qquad \text{weak-star in } L^\infty(0,T^*;V) \text{ and strongly in } L^2(0,T^*;H), \qquad (3.10) \\ &\qquad A_H y^{T^*,u_n} \to A_H y^* \text{ weakly in } L^2(0,T^*;H), \end{aligned}$$

since A_H is strongly-weakly closed. By Ascoli-Arzelà theorem we still get

$$y^{T^*,u_n}(t) \to y^*(t) \text{ strongly in } H, \text{ uniformly in } t \in [0,T^*]. \qquad (3.11)$$

Also, by the last assertion in Theorem 3.2 we infer that y^* is the solution to the state system corresponding to (T^*, u^*). We show next the convergence of y^{T_n,u_n} to y^* as $n \to \infty$. For any $v \in H$ we have

$$\begin{aligned} &\left|\int_0^{T_n} (y^{T_n,u_n}(t), v)_H dt - \int_0^{T^*} (y^*(t), v)_H dt\right| \le \left|\int_0^{T_n} (y^{T_n,u_n}(t), v)_H dt - \int_0^{T^*} (y^{T_n,u_n}(t), v)_H dt\right|_H \\ &+ \left|\int_0^{T^*} (y^{T_n,u_n}(t), v)_H dt - \int_0^{T^*} (y^*(t), v)_H dt\right| \\ = \ &\left|\int_{T^*}^{T_n} (y^{T_n,u_n}(t), v)_H dt\right| + \left|\int_0^{T^*} (y^{T_n,u_n}(t) - y^*(t), v)_H dt\right| \\ \le \ &C \left\|y^{T_n,u_n}\right\|_{L^\infty(0,T_n;H)} |T_n - T^*| + \left|\int_0^{T^*} \left(y^{T^*,u_n}(t) - y^*(t), v\right)_H dt\right| \to 0, \end{aligned}$$

because $\left\|y^{T_n,u_n}(t)\right\|_{L^\infty(0,T_n;H)} \le C_{T_n} \le C_{T^*+1}$ and by (3.4). We took into account that y^{T_n,u_n} restricted to $(0, T^*)$ coincides with y^{T^*,u_n}. In a similar way, we can prove the weak convergences of the other sequences, that is $(y^{T_n,u_n})' \to (y^*)'$, $A_H y^{T_n,u_n} \to A_H y^*$ weakly in $L^2(0,T^*;H)$ and that $y^{T_n,u_n} \to y^*$ strongly in $L^2(0,T^*;H)$.

It remains to prove that $Py^*(T^*) = Py^{tar}$. We have

$$\begin{aligned} &\left\|P(y^{T_n,u_n}(T_n) - y^*(T^*))\right\|_H \le \left\|P(y^{T_n,u_n}(T_n) - y^{T_n,u_n}(T^*))\right\|_H + \left\|P(y^{T_n,u_n}(T^*) - y^*(T^*))\right\|_H \\ = \ &\left\|\int_{T^*}^{T_n} (Py^{T_n,u_n})' dt\right\| + \left\|P(y^{T^*,u_n}(T^*) - y^*(T^*))\right\|_H \\ \le \ &\sqrt{T_n - T^*} \left\|P(y^{T_n,u_n})'\right\|_{L^2(0,T_n;H)} + \left\|P(y^{T^*,u_n}(T^*) - y^*(T^*))\right\|_H \end{aligned}$$

which tend to zero since $\left\|P(y^{T_n,u_n})'\right\|_{L^2(0,T_n;H)} \le C_{T^*+1}$ and by (3.4). Hence

$$\lim_{n\to\infty} Py^{T_n,u_n}(T_n) = Py^{tar} = Py^*(T^*).$$

From here, we also deduce that $T^* > 0$. Otherwise, we would have $Py^*(T^*) = Py^*(0) = Py_0$, that is $Py_0 = Py^{tar}$ which contradicts the hypothesis that $Py_0 \neq Py^{tar}$. Thus, we have obtained $T^* > 0$, u^* with the restriction $\|u^*\|_{L^\infty(0,T^*;U)} \le \rho$, and $J(T^*, u^*) = T^*$. We have got that T^* is the unique infimum time at which $Py^*(T^*) = Py^{tar}$. This ends the proof. □

4 The approximating problem

Let ε be positive and consider the problem

$$\text{Minimize } \{J_\varepsilon(T,u) \;\; T>0,\; u\in L^\infty(0,\infty;U),\; ess \sup_{t\in(0,\infty)} \|u(t)\|_U \le \rho\}, \qquad (\mathcal{P}_\varepsilon)$$

subject to (1.1)-(1.2), where

$$\begin{aligned} J_\varepsilon(T,u) &= T + \frac{1}{2\varepsilon}\left\|Py(T)-Py^{tar}\right\|_H^2 \qquad (4.1)\\ &\quad + \frac{\varepsilon}{2}\int_0^T \|Pu(t)\|_U^2\,dt + \frac{1}{2}\int_0^T \left\|\int_0^t P(u(\tau)-u^*(\tau))d\tau\right\|_U^2 dt. \end{aligned}$$

Theorem 4.1. *Let $y_0\in V$, $Py^{tar}\in P(H)$, $Py_0 \neq Py^{tar}$. Then, problem $(\mathcal{P}_\varepsilon)$ has at least a solution $(T_\varepsilon^*, u_\varepsilon^*)$, with the corresponding state $y^{T_\varepsilon^*,u_\varepsilon^*} := y_\varepsilon^*$.*

Proof. Since $J_\varepsilon(T,u)$ is nonnegative, there exists $d_\varepsilon = \inf J_\varepsilon(T,u)$ and it is positive. Indeed, we note that if $J_\varepsilon(T,u)=0$, each term should be equal with 0. This implies that in the second term of J_ε, $P(y(T=0)) - Py^{tar} = 0$ which is a contradiction with the fact that $Py_0 \neq Py^{tar}$. We conclude that the optimal T_ε^* must be positive.
We consider a minimizing sequence $(T_\varepsilon^n, u_\varepsilon^n)$ with $T_\varepsilon^n > 0$ and $\|u_\varepsilon^n(t)\|_U \le \rho$, satisfying

$$d_\varepsilon \le J_\varepsilon(T_\varepsilon^n, u_\varepsilon^n) \le d_\varepsilon + \frac{1}{n},\; n\ge 1. \qquad (4.2)$$

Hence, $T_\varepsilon^n \to T_\varepsilon^*$, as $n\to\infty$. Then, for any $\delta>0$ there exists n_δ such that $T_\varepsilon^n \ge T_\varepsilon^* - \delta$, with δ arbitrarily small, for $n\ge n_\delta$. On a subsequence

$$u_\varepsilon^n \to u_\varepsilon^*,\; Pu_\varepsilon^n \to Pu_\varepsilon^* \text{ weak-star in } L^\infty(0,T;U),\; Bu_\varepsilon^n \to Bu_\varepsilon^* \text{ weak-star in } L^\infty(0,T;H),$$

for all $T>0$. Then, the state system corresponding to any $T > d_\varepsilon + 1$ and u_ε^n has a unique continuous solution satisfying (3.2) on $(0,T)$, and it tends, as $n\to\infty$, to the solution corresponding to (T, u_ε^*). In particular, this happens for $T = T_\varepsilon^* - \delta$, with δ arbitrary small. Then, on a subsequence denoted still by n, we have

$$\begin{aligned} &y^{T_\varepsilon^*-\delta,u_\varepsilon^n} \to y^{T_\varepsilon^*-\delta,u_\varepsilon^*} \text{ strongly in } L^2(0,T_\varepsilon^*-\delta;H), \qquad (4.3)\\ &\text{weakly in } W^{1,2}(0,T_\varepsilon^*-\delta;H)\cap L^2(0,T_\varepsilon^*-\delta;D_H), \text{ and weak-star in } L^\infty(0,T_\varepsilon^*-\delta;V), \end{aligned}$$

$$A_H y^{T_\varepsilon^n-\delta,u_\varepsilon^n} \to A_H y_\varepsilon^* \text{ weakly in } L^2(0,T_\varepsilon^*-\delta;H), \qquad (4.4)$$

$$y^{T_\varepsilon^*-\delta,u_\varepsilon^n}(t) \to y_\varepsilon^*(t) \text{ strongly in } H, \text{ uniformly in } t\in[0,T_\varepsilon^*-\delta]. \qquad (4.5)$$

Next, we proceed in a similar way as in Theorem 3.3 to show that $y^{T_\varepsilon^n,u_\varepsilon^n} \to y^{T_\varepsilon^*,u_\varepsilon^*} := y_\varepsilon^*$ in the corresponding spaces and that

$$Py^{T_\varepsilon^n,u_n}(T_\varepsilon^n) \to Py^{T_\varepsilon^*,u_\varepsilon^*}(T_\varepsilon^*) \text{ strongly in } H. \qquad (4.6)$$

These imply that y_ε^* is the solution to the state system corresponding to $(T_\varepsilon^*, u_\varepsilon^*)$.

Let us denote

$$h_\varepsilon^n(t) = \int_0^t P(u_\varepsilon^n - u^*)(\tau)d\tau,\; t\ge 0.$$

Taking $\psi\in U^*$ we have

$$\begin{aligned} \left\langle \int_0^t P(u_\varepsilon^n-u^*)(\tau)d\tau, \psi\right\rangle_{U,U^*} &= \int_0^t \langle P(u_\varepsilon^n-u^*)(\tau),\psi\rangle_{U,U^*}\,d\tau \to \\ \int_0^t \langle P(u_\varepsilon^*-u^*)(\tau),\psi\rangle_{U,U^*}\,d\tau &= \left\langle \int_0^t P(u_\varepsilon^*-u^*)(\tau)d\tau,\psi\right\rangle_{U,U^*},\text{ for all } t\ge 0. \end{aligned}$$

So, we get that $h_\varepsilon^n(t) \to h_\varepsilon^*(t)$ weakly in U, for all $t \geq 0$ and

$$h_\varepsilon^*(t) = \int_0^t P(u_\varepsilon^* - u^*)(\tau)d\tau. \tag{4.7}$$

Passing to the limit in (4.2) we get on the basis of the previous convergences and of the weakly lower semicontinuity of the norms, that $J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) = d_\varepsilon$, that is $(T_\varepsilon^*, u_\varepsilon^*)$ is an optimal controller in (P_ε). □

Theorem 4.2. *Assume $y_0 \in V$, $Py^{tar} \in P(H)$, $Py_0 \neq Py^{tar}$. Let (T^*, u^*, y^*) be optimal in $(\mathcal{P})$ and $(T_\varepsilon^*, u_\varepsilon^*, y_\varepsilon^*)$ be optimal in $(\mathcal{P}_\varepsilon)$. Then,*

$$T_\varepsilon^* \to T^*,\ u_\varepsilon^* \to u^* \text{ weak-star in } L^\infty(0,T^*;U),\ Bu_\varepsilon^* \to Bu^* \text{ weak-star in } L^\infty(0,T^*;H), \tag{4.8}$$

$$\begin{aligned} y_\varepsilon^* &\to y^* \text{ strongly in } L^2(0,T^*;H), \\ &\quad \text{weakly in } W^{1,2}(0,T^*;H) \text{ and weak-star in } L^\infty(0,T^*;V), \end{aligned} \tag{4.9}$$

$$Ay_\varepsilon^* \to Ay^* \text{ weakly in } L^2(0,T^*;H), \tag{4.10}$$

$$y_\varepsilon^*(T^*) \to y^*(T^*) \text{ strongly in } H,\ Py^*(T^*) = P^{tar}. \tag{4.11}$$

Proof. Let $(T_\varepsilon^*, u_\varepsilon^*, y_\varepsilon^*)$ be optimal in (P_ε). Then,

$$\begin{aligned} & J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) = T_\varepsilon^* + \frac{1}{2\varepsilon}\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 \\ & +\frac{\varepsilon}{2}\int_0^{T_\varepsilon^*} \|Pu_\varepsilon^*(t)\|_U^2\, dt + \frac{1}{2}\int_0^{T_\varepsilon^*} \left\|\int_0^t P(u_\varepsilon^*(\tau) - u^*(\tau))d\tau\right\|_U^2 dt \\ \leq\ & J_\varepsilon(T,u) = T + \frac{1}{2\varepsilon}\left\|Py(T) - Py^{tar}\right\|_H^2 \\ & +\frac{\varepsilon}{2}\int_0^T \|Pu(t)\|_U^2\, dt + \frac{1}{2}\int_0^T \left\|\int_0^t P(u(\tau) - u^*(\tau))d\tau\right\|_U^2 dt, \end{aligned} \tag{4.12}$$

for any $T > 0$ and $u \in L^\infty(0,\infty;U)$, $\|u(t)\|_U \leq \rho$ a.e. $t > 0$, where y_ε^* is the solution to the state system corresponding to $(T_\varepsilon^*, u_\varepsilon^*)$ and y is the solution to the state system corresponding to (T,u). Let us set in (4.12), $T = T^*$ and $u = u^*$, an optimal controller in (P). Thus, the second and the last term on the right-hand side of (4.12) are zero and

$$\begin{aligned} & J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) = T_\varepsilon^* + \frac{1}{2\varepsilon}\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 + \frac{\varepsilon}{2}\int_0^{T_\varepsilon^*} \|Pu_\varepsilon^*(t)\|_U^2\, dt \\ & +\frac{1}{2}\int_0^{T_\varepsilon^*} \left\|\int_0^t P(u_\varepsilon^*(\tau) - u^*(\tau))d\tau\right\|_U^2 dt \leq T^* + \frac{\varepsilon}{2}\int_0^{T^*} \|Pu^*(t)\|_U^2\, dt. \end{aligned} \tag{4.13}$$

Then, $T_\varepsilon^* \to T^{**}$ and $T_\varepsilon^* \geq T^{**} - \delta$, with δ arbitrarily small, and selecting a subsequence indicated still by ε, we have

$$\begin{aligned} u_\varepsilon^* &\to u^{**} \text{ weak-star in } L^\infty(0,T;U),\ \|u^{**}(t)\|_U \leq \rho, \\ Bu_\varepsilon^* &\to Bu^{**} \text{ weak-star in } L^2(0,T;H),\ \text{for all } T > 0. \end{aligned}$$

The solution y_ε^* satisfies the estimates

$$\|y_\varepsilon^*\|_{X_{T_\varepsilon^*}}^2 + \|A_H y_\varepsilon^*\|_{L^2(0,T_\varepsilon^*;H)}^2 \leq C_{T_\varepsilon^*}, \tag{4.14}$$

where $X_T := C([0,T],H) \cap L^\infty(0,T;V) \cap W^{1,2}(0,T;H)$, and $y_\varepsilon^* \to y^{**}$ in the spaces $X_{T^{**}-\delta}$ defined on $(0, T^{**}-\delta)$, for δ arbitrary. As in the previous proof we show that all the convergences (4.3)-(4.5) take place also in the spaces defined on $(0, T^{**})$. Also, we have

$$\|y_\varepsilon^*(T_\varepsilon^*) - y^{**}(T^{**}-\delta)\|_H \leq \sqrt{T_\varepsilon^* - T^{**} + \delta}\, \|(y_\varepsilon^*)'\|_{L^2(0,T_\varepsilon^*;H)} + \|y_\varepsilon^*(T_\varepsilon^* - \delta) - y^{**}(T^{**}-\delta)\|_H\,,$$

implying that $\|y_\varepsilon^*(T_\varepsilon^*) - y^{**}(T^{**})\|_H \to 0$, strongly in H, as $\varepsilon \to 0$. By (4.13) we have

$$\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 \leq 2\varepsilon T^* + \varepsilon^2 \int_0^{T^*} \|Pu^*(t)\|_U^2 \, dt$$

and so $Py_\varepsilon^*(T_\varepsilon^*) \to Py^{tar}$ strongly in H, implying the relation $Py^{**}(T^{**}) = Py^{tar}$. Again by (4.13),

$$T_\varepsilon^* \leq J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) \leq T^* + \frac{\varepsilon}{2} \int_0^{T^*} \|Pu^*(t)\|_U^2 \, dt,$$

whence we get at limit that $T^{**} \leq T^*$. Now T^{**} and u^{**} satisfy the restrictions required in problem (P), that is $T^{**} > 0$, $\|u^{**}(t)\|_U \leq \rho$, and $Py^{**}(T^{**}) = Py^{tar}$, and since T^* is the infimum in (P) it follows that $T^{**} = T^*$. Recalling (4.7) we define

$$E_\varepsilon(T) := \int_0^T \left\|\int_0^t P(u_\varepsilon^* - u^*)(\tau)d\tau\right\|_U^2 dt = \int_0^T \|h_\varepsilon^*(t)\|_U^2 \, dt, \text{ for all } T > 0.$$

We have by (4.13) that

$$T_\varepsilon^* + E_\varepsilon(T_\varepsilon^*) \leq J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) \leq T^* + \frac{\varepsilon}{2} \int_0^{T^*} \|Pu^*(t)\|_U^2 \, dt$$

and so $T^{**} + \limsup_{\varepsilon \to 0} E_\varepsilon(T_\varepsilon^*) \leq T^*$, hence

$$\limsup_{\varepsilon \to 0} \int_0^{T_\varepsilon^*} \|h_\varepsilon^*(t)\|_U^2 \, dt = \limsup_{\varepsilon \to 0} \int_0^{T_\varepsilon^*} \left\|\int_0^t P(u_\varepsilon^* - u^*)(\tau)d\tau\right\|_U^2 dt = 0. \tag{4.15}$$

Therefore,

$$h_\varepsilon^* \to 0 \text{ strongly in } L^2(0, T^*; U), \text{ as } \varepsilon \to 0. \tag{4.16}$$

On the other hand, we know that $u_\varepsilon^* \to u^{**}$ weakly in $L^2(0, T^*; U)$, so that

$$\int_0^t \langle P(u^{**} - u^*)(\tau), \psi \rangle_{U,U^*} \, d\tau = 0, \text{ for all } t \in (0, T^*), \ \psi \in U^*,$$

implying that $u^{**} = u^*$ on $(0, T^*)$.

For a later use we prove that

$$h_\varepsilon^*(T_\varepsilon^*) \to 0 \text{ strongly in } U. \tag{4.17}$$

We write

$$h_\varepsilon^*(T_\varepsilon^*) - h_\varepsilon^*(t) = \int_t^{T_\varepsilon^*} P(u_\varepsilon^* - u^*)(s)ds, \text{ for all } t \in [0, T_\varepsilon^*].$$

Then,

$$\|h_\varepsilon^*(T_\varepsilon^*)\|_U \leq \|h_\varepsilon^*(t)\|_U + \int_t^{T_\varepsilon^*} \|P(u_\varepsilon^* - u^*)(s)\|_U \, ds \leq \|h_\varepsilon^*(t)\|_U + 2\rho(T_\varepsilon^* - t).$$

Let us take $t \in [T_\varepsilon^* - \tau, T_\varepsilon^*]$ with $\tau > \varepsilon$, and integrate the previous inequality along with t in this interval. We have

$$\tau \|h_\varepsilon^*(T_\varepsilon^*)\|_U \leq \int_{T_\varepsilon^* - \tau}^{T_\varepsilon^*} \|h_\varepsilon^*(t)\|_U \, dt + 2\rho\tau^2 \leq \int_0^{T_\varepsilon^*} \|h_\varepsilon^*(t)\|_U \, dt + 2\rho\tau^2 \leq \sqrt{T_\varepsilon^*} \left(\int_0^{T_\varepsilon^*} \|h_\varepsilon^*(t)\|_U^2 \, dt\right)^{1/2} + 2\rho\tau^2.$$

Let us make ε goes to zero and get by (4.15) that

$$\limsup_{\varepsilon \to 0} \|h_\varepsilon^*(T_\varepsilon^*)\|_U \leq 2\rho\tau,$$

which yields (4.17), since τ is arbitrary. On the basis of (4.13) we write that

$$T_\varepsilon^* + \frac{1}{2\varepsilon}\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 \le J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) \le T^* + \frac{\varepsilon}{2}\int_0^{T^*} \|Pu^*(t)\|_U^2\, dt$$

whence

$$\limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon}\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 = 0. \tag{4.18}$$

We conclude that $\lim_{\varepsilon\to 0} J_\varepsilon(T_\varepsilon^{**}, u_\varepsilon^{**}) = T^{**} = J(T^{**}, u^{**})$ and so (T^{**}, u^{**}) is optimal in (P). But, T^* is also optimal and unique and it follows $T^{**} = T^*$ and $u^{**} = u^*$ a.e. on $(0, T^*)$. Eventually, we also have obtained (4.9)-(4.11), as claimed. □

5 The maximum principle

In this section, besides (a_1), (a_2), (b_1), (c_1), we assume $(a_3) - (a_6)$.

5.1 The system of first order variations and the dual system

Let us introduce the Cauchy problem

$$\begin{aligned} Y'(t) + A'(y_\varepsilon^*(t))Y(t) &= Bv(t), \text{ a.e. } t > 0, \\ Y(0) &= 0, \end{aligned} \tag{5.1}$$

where $v \in L^\infty(0,\infty;U)$, $\|v(t)\|_U \le C$, a.e. $t \ge 0$.

Proposition 5.1. *Problem* (5.1) *has a unique solution*

$$Y \in C([0,T];H) \cap W^{1,2}(0,T;V^*) \cap L^2(0,T;V), \text{ for all } T > 0. \tag{5.2}$$

Proof. We recall that $A'(y_\varepsilon^*(t))$ is continuous from V to V^*, for all $t \ge 0$, and has the properties (2.17) and (2.5), $\|A'(y_\varepsilon^*(t))z\|_{V^*} \le C\|z\|_V$, due to $\|y_\varepsilon^*(t)\|_V \le C_T$, by (3.2). Then, the result claimed in the statement is ensured by the Lions theorem. □

Let $(T_\varepsilon^*, u_\varepsilon^*)$ be an optimal controller in $(\mathcal{P}_\varepsilon)$. For $\lambda > 0$, we set

$$u_\varepsilon^\lambda = Pu_\varepsilon^* + \lambda v, \text{ where } v = P(\overline{u} - u_\varepsilon^*),\ \ \|\overline{u}\|_{L^\infty(0,\infty;U)} \le \rho. \tag{5.3}$$

In this way we can give variations to both controllers, if $P = I$, or to the first component in the case $P \ne I$. We define $Y_\lambda = \frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}$, where y_ε^λ is the solution to the state system (1.1)-(1.2) corresponding to u_ε^λ and T_ε^*.

Proposition 5.2. *Let Y be the solution to* (5.1) *and let $T > 0$. We have*

$$\lim_{\lambda\to 0} \frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda} = Y \text{ strongly in } C([0,T];H) \cap L^2(0,T;V), \text{ as } \lambda \to 0, \tag{5.4}$$

which ensures that (5.1) *is just the system of first order variations related to* (1.1)-(1.2).

Proof. Let us define

$$\zeta_\lambda = \frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda} - Y.$$

We write the equation for y_ε^λ subtract the equation for y_ε^*, divide by λ and subtract the equation (5.1). The equation verified by ζ_λ reads

$$\begin{aligned} \zeta_\lambda' + A'(y_\varepsilon^*)\zeta_\lambda + \frac{Ay_\varepsilon^\lambda - Ay_\varepsilon^*}{\lambda} - A'(y_\varepsilon^*)\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda} &= 0, \\ \zeta_\lambda(0) &= 0. \end{aligned} \tag{5.5}$$

Now, we can represent the third term as

$$Ay_\varepsilon^\lambda - Ay_\varepsilon^* = \int_0^1 (A'(\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(y_\varepsilon^\lambda - y_\varepsilon^*)d\nu \tag{5.6}$$

and so, the equation becomes

$$\zeta_\lambda' + A'(y_\varepsilon^*)\zeta_\lambda + \int_0^1 (A'(\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*) - A'(y_\varepsilon^*))\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}d\nu = 0. \tag{5.7}$$

We test (5.7) by $\zeta_\lambda(t)$, integrate with respect to t, and get, by (2.17) and (2.5) that

$$\begin{aligned}
&\frac{1}{2}\|\zeta_\lambda(t)\|_H^2 + \alpha_1\int_0^t \|\zeta_\lambda(\tau)\|_V^2 d\tau \le \alpha_2\int_0^t \|\zeta_\lambda(\tau)\|_H^2 d\tau \\
&+\int_0^1\int_0^t \left|\left\langle (A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(\tau)) - A'(y_\varepsilon^*(\tau)))\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}(\tau), \zeta_\lambda(\tau)\right\rangle_{V^*,V}\right| d\tau d\nu \\
\le\ & \alpha_2\int_0^t \|\zeta_\lambda(\tau)\|_H^2 d\tau + \int_0^1\int_0^T \left\|A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(\tau)) - A'(y_\varepsilon^*(\tau))\right\|_{L(V,V^*)} \left\|\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}(\tau)\right\|_V d\tau.
\end{aligned}$$

By Gronwall's lemma we obtain

$$\begin{aligned}
&\|\zeta_\lambda(t)\|_H^2 + \int_0^t \|\zeta_\lambda(\tau)\|_V^2 d\tau \qquad (5.8)\\
\le\ & C_{T_\varepsilon^*}\int_0^1 \left\{\left(\int_0^T \left\|A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(\tau)) - A'(y_\varepsilon^*(\tau))\right\|_{L(V,V^*)}^2 d\tau\right)^{1/2}\right. \\
&\left.\times\left(\int_0^T \left\|\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}(\tau)\right\|_V^2 d\tau\right)^{1/2} d\tau\right\} d\nu.
\end{aligned}$$

We recall that by (3.2), we have $\|y_\varepsilon^*(t)\|_V \le e^{CT}\left(\|y_0\|_V^2 + \rho^2 T\right) \le C_T$, for all $t\in[0,T]$, and

$$\left\|y_\varepsilon^\lambda(t)\right\|_V \le C\left(\|y_0\|_V^2 + \int_0^T \left\|u^\lambda(t)\right\|_U^2 dt\right)e^{CT} \le e^{CT}\left(\|y_0\|_V^2 + \rho^2 T\right) \le C_T.$$

Here, C_T may change from line to line. Moreover, by (2.19)

$$\|A'(y_\varepsilon^*(\tau))\|_{L(V,V^*)} \le (1 + C\|y_\varepsilon^*(\tau)\|_V^\kappa) \le C_T, \text{ and}$$

$$\left\|A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(\tau))\right\|_{L(V,V^*)} \le (1 + C\left\|(\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(\tau)\right\|_V^\kappa) \le C_T.$$

We also recall (3.3) which yields, for all $t\in[0,T_\varepsilon^*]$, that

$$\left\|(y_\varepsilon^\lambda - y_\varepsilon^*)(t)\right\|_H^2 + \int_0^t \left\|(y_\varepsilon^\lambda - y_\varepsilon^*)(\tau)\right\|_V^2 d\tau \le C\left\|u_\varepsilon^\lambda - u_\varepsilon^*\right\|_{L^2(0,T;U)}^2 \le CT\lambda^2\rho^2 \tag{5.9}$$

and so

$$\int_0^t \left\|\frac{y_\varepsilon^\lambda - y_\varepsilon^*}{\lambda}(\tau)\right\|_V^2 d\tau \le CT\rho^2, \text{ for all } t\in[0,T].$$

Therefore,

$$y_\varepsilon^\lambda \to y_\varepsilon^* \text{ strongly in } C([0,T];H)\cap L^2(0,T;V), \text{ as } \lambda\to 0 \tag{5.10}$$

and

$$\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^* \to y_\varepsilon^* \text{ strongly in } C([0,T];H)\cap L^2(0,T;V), \text{ as } \lambda\to 0, \tag{5.11}$$

for ν fixed, implying that

$$(\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(t) \to y_\varepsilon^*(t) \text{ strongly in } V, \text{ a.e. } t \in (0,T). \tag{5.12}$$

This yields that

$$A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(t)) \to A'(y_\varepsilon^*(t)) \text{ strongly in } L(V,V^*), \text{ a.e. } t \in (0,T)$$

by (a_4) and (7.3). We denote $f_\lambda(t) := \left\| A'((\nu y_\varepsilon^\lambda + (1-\nu)y_\varepsilon^*)(t)) - A'(y_\varepsilon^*(t)) \right\|_{L(V,V^*)}^2$ and infer that $f_\lambda(t) \to 0$ a.e. $t \in (0,T)$, and that $|f_\lambda(t)| \le C$. This implies, by the Lebesgue dominated convergence theorem, that $f_\lambda \to 0$ in $L^2(0,T)$. Thus, by (5.8)

$$\zeta_\lambda \to 0 \text{ strongly in } C([0,T];H) \cap L^2(0,T;V), \text{ as } \lambda \to 0.$$

This proves (5.4). □

Now, we introduce the adjoint system

$$-p_\varepsilon'(t) + (A'(y_\varepsilon^*(t)))^* p_\varepsilon(t) = 0, \text{ a.e. } t \in (0,T_\varepsilon^*), \tag{5.13}$$

$$p_\varepsilon(T_\varepsilon^*) = \frac{1}{\varepsilon}(Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}). \tag{5.14}$$

Proposition 5.3. *Let $Py^{tar} \in P(D_H)$ and assume* (2.8)*. Then, for each $\varepsilon > 0$, problem* (5.13)-(5.14) *has a unique solution*

$$p_\varepsilon \in C([0,T_\varepsilon^*];H) \cap W^{1,2}(0,T_\varepsilon^*;H) \cap L^2(0,T_\varepsilon^*;D_H). \tag{5.15}$$

Proof. By Theorem 3.2 we deduce that $p_\varepsilon(T_\varepsilon^*) \in D_H$, since $P(y_\varepsilon^*(T_\varepsilon^*) - y^{tar}) \in P(D_H)$. If $P \ne I$, the second component of p_ε is $0 \in D_H$. We use the transformation $t \to T_\varepsilon^* - t$ and so (5.13)-(5.14) transforms into a forward equation. The operator $(A'(y_\varepsilon^*(t)))^*$ is continuous from V to V^* for all $t \ge 0$ and satisfies the properties of Lions theorem, such that we can deduce, as in Proposition 5.1, that (5.13)-(5.14) has a unique solution

$$p_\varepsilon \in C([0,T_\varepsilon^*];H) \cap W^{1,2}(0,T_\varepsilon^*;V^*) \cap L^2(0,T_\varepsilon^*;V). \tag{5.16}$$

A first estimate is obtained by testing (5.13) by $p_\varepsilon(t)$ and integrating over (t,T_ε^*). Using (2.17), this yields

$$\|p_\varepsilon(t)\|_H^2 + \int_0^t \|p_\varepsilon(\tau)\|_V^2 \, d\tau \le C \|p_\varepsilon(T_\varepsilon^*)\|_H^2, \text{ for all } t \in [0,T_\varepsilon^*]. \tag{5.17}$$

To prove the additional regularity we multiply (5.13) by $\Gamma_\nu p_\varepsilon(t)$, integrate over (t,T_ε^*) and use (2.8). We have

$$\begin{aligned} & \frac{1}{2}\langle p_\varepsilon(t), \Gamma_\nu p_\varepsilon(t)\rangle_{V^*,V} + \gamma_1 \int_0^t \|\Gamma_\nu p_\varepsilon(\tau)\|_H^2 \, d\tau \\ \le\; & \frac{1}{2}\langle p_\varepsilon(T_\varepsilon^*), \Gamma_\nu p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} + \gamma_2 \int_0^t \|p_\varepsilon(\tau)\|_V^2 (1 + \gamma_3 \|y_\varepsilon^*(\tau)\|_V^l) d\tau. \end{aligned}$$

Taking into account (5.17) and (3.2), that is $\|y_\varepsilon^*(\tau)\|_V \le C_{T_\varepsilon^*} < C_{T^*+1}$, we obtain

$$\begin{aligned} & \left\|\Gamma_\nu^{1/2} p_\varepsilon(t)\right\|_H^2 + \int_0^t \|\Gamma_\nu p_\varepsilon(\tau)\|_H^2 \, d\tau \\ \le\; & C\left(\|p_\varepsilon(T_\varepsilon^*)\|_H \|\Gamma_H p_\varepsilon(T_\varepsilon^*)\|_H + 1\right) \le C\left(\|p_\varepsilon(T_\varepsilon^*)\|_H \|p_\varepsilon(T_\varepsilon^*)\|_{D_H} + 1\right), \end{aligned} \tag{5.18}$$

for all $t \in [0,T_\varepsilon^*]$, since $\|\Gamma_H p_\varepsilon(\tau)\|_H = \|p_\varepsilon(\tau)\|_{D_H}$.

Here, we used the relation $\|\Gamma_\nu p_\varepsilon(T_\varepsilon^*)\|_H \leq \|\Gamma_H p_\varepsilon(T_\varepsilon^*)\|_H$ for $p_\varepsilon(T_\varepsilon^*) \in D_H$. Now, we can pass to the limit as $\nu \to 0$ and obtain

$$\Gamma_\nu p_\varepsilon \to \Gamma_H p \text{ weakly in } L^2(0,T_\varepsilon^*;H), \text{ weak-star in } L^\infty(0,T_\varepsilon^*;V),$$

and so by (5.18) we get

$$\|p_\varepsilon(t)\|_V^2 + \int_0^t \|\Gamma_H p_\varepsilon(\tau)\|_H^2\, d\tau \leq C\left(\|p_\varepsilon(T_\varepsilon^*)\|_H \|p_\varepsilon(T_\varepsilon^*)\|_{D_H} + 1\right), \text{ for all } t \in [0,T_\varepsilon^*].$$

Thus, $p_\varepsilon \in L^2(0,T_\varepsilon^*;D_H) \cap L^\infty(0,T_\varepsilon^*;V)$. For a.a. $t \in (0,T_\varepsilon^*)$ we still have

$$\|(A_H'(y_\varepsilon^*(t)))^* p_\varepsilon(t)\|_H \leq \|p_\varepsilon(t)\|_{D_H} (1 + C\|y_\varepsilon^*(t)\|_V^\kappa).$$

By (5.13) it follows that $p_\varepsilon' \in L^2(0,T_\varepsilon^*;H)$ and so (5.15) is proved. □

5.2 Approximating optimality conditions

Let us introduce the sets

$$K = \{w \in U;\ \|w\|_U \leq \rho\},\ \mathcal{K}_T = \{z \in L^2(0,T;U);\ z(t) \in K \text{ a.e. } t \in (0,T)\},$$

and denote the normal cone to K at w by

$$N_K(w) = \{\zeta \in U^*;\ \langle \zeta, w - \overline{w}\rangle_{U^*,U} \geq 0, \text{ for all } \overline{w} \in K\}, \tag{5.19}$$

and the normal cone to $\mathcal{K}_T$ at ω by

$$N_{\mathcal{K}_T}(\omega) = \left\{\chi \in L^2(0,T;U^*);\ \int_0^T \langle \chi(t), (\omega - \overline{\omega})(t)\rangle_{U^*,U}\, dt \geq 0, \text{ for all } \overline{\omega} \in \mathcal{K}_T\right\}. \tag{5.20}$$

We recall (see e.g., [4]) that $\chi \in N_{\mathcal{K}_T}(\omega)$ iff $\chi(t) \in N_K(\omega(t))$ a.e. $t \in (0,T)$.
We denote by $F : U \to U^*$ the duality mapping of U (see (7.4) in the Appendix) and recall that $h_\varepsilon^*(t)$ was defined in (4.7), $h_\varepsilon^*(t) = \int_0^t P(u_\varepsilon^* - u^*)(\tau)d\tau$.

Proposition 5.4. *Assume*

$$y_0 \in V,\ Py^{tar} \in P(D_H),\ Py_0 \neq Py^{tar}. \tag{5.21}$$

Let $(T_\varepsilon^*, u_\varepsilon^*)$ *be an optimal control in* (P_ε) *with the optimal state* y_ε^*. *Then,*

$$Pu_\varepsilon^*(t) = -(\varepsilon F + N_K)^{-1}\left(B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau\right), \text{ for all } t \in [0,T_\varepsilon^*], \tag{5.22}$$

and

$$\begin{aligned} &\rho\left\|B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau + \varepsilon F(Pu_\varepsilon^*(t))\right\|_{U^*} + (A_H y_\varepsilon^*(t), p_\varepsilon(t))_H + \\ &+ \int_t^{T_\varepsilon^*} (Pu_\varepsilon^*(\tau), F(h_\varepsilon^*(\tau)))_{U,U^*}\, d\tau + \frac{\varepsilon}{2}\|Pu_\varepsilon^*(t)\|_U^2 \\ = &\ 1 + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2,\ t \in [0,T_\varepsilon^*], \end{aligned} \tag{5.23}$$

where p_ε *is the solution to the adjoint equation* (5.13)-(5.14). *Moreover,* $t \to u_\varepsilon^*(t)$ *turns out to be continuous on* $[0,T_\varepsilon^*]$.

Proof. Let $(T_\varepsilon^*, u_\varepsilon^*)$ be an optimal controller in (P_ε). We shall compute separate variations with respect to T_ε^* and u_ε^*. By the condition of optimality for u_ε^* we have

$$J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) \le J_\varepsilon(T, u), \text{ for all } u(t) \in K, \ \|u(t)\|_U \le \rho, \text{ a.e. } t \ge 0.$$

In particular, replacing u by $u_\varepsilon^\lambda = Pu_\varepsilon^* + \lambda v$ with $v = P(\overline{u} - u_\varepsilon^*)$, $\overline{u}(t) \in K$, and performing some calculations, recalling (7.11) we get

$$\begin{aligned}&\left(\frac{1}{\varepsilon}(Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}), PY(T_\varepsilon^*)\right)_H + \varepsilon \int_0^{T_\varepsilon^*} \langle F(Pu_\varepsilon^*(t)), v(t)\rangle_{U^*,U}\, dt \\ &+ \int_0^{T_\varepsilon^*} \left\langle F(h_\varepsilon^*(t)), \int_0^t v(s)ds \right\rangle_{U^*,U} dt \ge 0.\end{aligned} \tag{5.24}$$

Observing that

$$\begin{aligned}&\int_0^{T_\varepsilon^*} \left\langle F(h_\varepsilon^*(t)), \int_0^t v(s)ds \right\rangle_{U^*,U} dt = \int_0^{T_\varepsilon^*} \int_0^t \langle F(h_\varepsilon^*(t)), v(s)\rangle_{U^*,U}\, dsdt \\ = &\int_0^{T_\varepsilon^*} \int_s^{T_\varepsilon^*} \langle F(h_\varepsilon^*(t)), v(s)\rangle_{U^*,U}\, dtds = \int_0^{T_\varepsilon^*} \left\langle \int_s^{T_\varepsilon^*} F(h_\varepsilon^*(t))dt, v(s) \right\rangle_{U^*,U} ds\end{aligned} \tag{5.25}$$

we obtain

$$\left(\frac{1}{\varepsilon}(Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}), Y(T_\varepsilon^*)\right)_H + \int_0^{T_\varepsilon^*} \left\langle \varepsilon F(Pu_\varepsilon^*(t)) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau, v(t) \right\rangle_{U^*,U} dt \ge 0. \tag{5.26}$$

Here we used that $P^2 = P$ and the fact that $(Pw, P\overline{w})_H$ is the same with $(Pw, \overline{w})$ when $Pw = (w_1, 0)$. We test (5.1) by $p_\varepsilon(t)$ and integrate over $(0, T_\varepsilon^*)$. By a straightforward calculation we obtain

$$\begin{aligned}&\int_\Omega (p_\varepsilon(T_\varepsilon^*)Y(T_\varepsilon^*) - p_\varepsilon(0)Y(0))\, dx + \int_0^{T_\varepsilon^*} \langle(-p_\varepsilon' + (A'(y_\varepsilon^*))^* p_\varepsilon)(t), Y(t)\rangle_{V^*,V}\, dt \\ = &\int_0^{T_\varepsilon^*} (Bv(t), p_\varepsilon(t))_H dt.\end{aligned}$$

Using again the adjoint system, this equation reduces to

$$\left(\frac{1}{\varepsilon}P(y_\varepsilon^*(T_\varepsilon^*) - Py^{tar}), Y(T_\varepsilon^*)\right)_H = \int_0^{T_\varepsilon^*} \langle B^* p_\varepsilon(t), v(t)\rangle_{U^*,U}\, dt. \tag{5.27}$$

We recall that $v = P(\overline{u} - u_\varepsilon^*)$. Replacing the left-hand side of (5.27) into (5.26) we deduce that

$$\int_0^{T_\varepsilon^*} (B^* p_\varepsilon(t), v(t))_{U^*,U}\, dt + \int_0^{T_\varepsilon^*} \left(\varepsilon F(Pu_\varepsilon^*(t)) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau, v(t) \right)_{U^*,U} dt \ge 0, \tag{5.28}$$

for all $\overline{u}(t) \in K$, that is $\overline{u}(t) \in U$, $\|\overline{u}(t)\|_U \le \rho$ a.e. $t \ge 0$. This yields

$$\int_0^{T_\varepsilon^*} \left\langle -B^* p_\varepsilon(t) - \varepsilon F(u_\varepsilon^*(t)) - \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau, Pu_\varepsilon^*(t) - P\overline{u}(t) \right\rangle_{U^*,U} dt \ge 0,$$

for all $\overline{u}(t) \in K$, a.e. $t \in (0, T_\varepsilon^*)$, and implies, by (5.19), that

$$z_\varepsilon(t) := -B^* p_\varepsilon(t) - \varepsilon F(Pu_\varepsilon^*(t)) - \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau \in N_K(Pu_\varepsilon^*(t)), \text{ a.e. } t \in (0, T_\varepsilon^*), \tag{5.29}$$

or, equivalently,

$$Pu_\varepsilon^*(t) = (\varepsilon F + N_K)^{-1}\left(-B^* p_\varepsilon(t) - \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau\right), \text{ a.e. } t \in (0, T_\varepsilon^*). \tag{5.30}$$

Moreover, relation (5.30) implies that $t \to u_\varepsilon^*(t)$ is continuous, because $(\varepsilon F + N_K)^{-1}$ is single-valued and Lipschitz continuous, the integral is continuous and p_ε belongs to $C([0, T_\varepsilon^*]; H)$, so that (5.30) is true for all $t \in [0, T_\varepsilon^*]$. We also note that

$$F(h_\varepsilon^*(\tau)) = F\left(\int_0^\tau P(u_\varepsilon^* - u^*)(s)ds\right) = PF\left(\int_0^\tau P(u_\varepsilon^* - u^*)(s)ds\right).$$

We recall by (4.14) that $A_H y_\varepsilon^*(t) \in H$, $(y_\varepsilon^*)'(t) \in H$ a.e. $t \in (0, T_\varepsilon^*)$. Also, $y_\varepsilon^* \in C_w([0, T_\varepsilon^*]; V)$ and $Ay \in C_w([0, T_\varepsilon^*]; V^*)$ (see the observation after Theorem 3.2). By the state equation we have $(y_\varepsilon^*)' = -Ay_\varepsilon^* + Bu_\varepsilon^* \in C_w([0, T_\varepsilon^*]; V^*)$. Indeed, $y_\varepsilon^*(T_\varepsilon^*) \in V$, so $Ay_\varepsilon^*(T_\varepsilon^*) \in V^*$ and $Bu_\varepsilon^*(T_\varepsilon^*) \in H$. Thus, $(y_\varepsilon^*)'(T_\varepsilon^*) \in V^*$, for all $\varepsilon > 0$. Recalling (5.14) and that $Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar} \in P(D_H)$, we deduce that $p_\varepsilon(T_\varepsilon^*) \in D_H$, for all $\varepsilon > 0$.

Next, we keep u_ε^* fixed and give variations to T_ε^*. Since T_ε^* realizes the minimum in (P_ε) we can write

$$J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) \leq J_\varepsilon(T_\varepsilon^* + \lambda, u_\varepsilon^*), \;\; \lambda > 0,$$

that is,

$$\begin{aligned}
& J_\varepsilon(T_\varepsilon^*, u_\varepsilon^*) = T_\varepsilon^* + \frac{1}{2\varepsilon}\left\|P(y_\varepsilon^{T_\varepsilon^*, u_\varepsilon^*}(T_\varepsilon^*) - Py^{tar})\right\|_H^2 \\
& + \frac{\varepsilon}{2}\int_0^{T_\varepsilon^*} \|Pu_\varepsilon^*(t)\|_U^2 \,dt + \frac{1}{2}\int_0^{T_\varepsilon^*}\left\|\int_0^t P(u_\varepsilon^*(\tau) - u^*(\tau))d\tau\right\|_U^2 dt \\
\leq \;\; & J_\varepsilon(T_\varepsilon^* + \lambda, u_\varepsilon^*) = T_\varepsilon^* + \lambda + \frac{1}{2\varepsilon}\left\|P(y_\varepsilon^{T_\varepsilon^*+\lambda, u_\varepsilon^*}(T_\varepsilon^* + \lambda) - Py^{tar})\right\|_H^2 + \frac{\varepsilon}{2}\int_0^{T_\varepsilon^*+\lambda} \|Pu_\varepsilon^*(t)\|_U^2 \,dt \\
& + \frac{1}{2}\int_0^{T_\varepsilon^*+\lambda}\left\|\int_0^t P(u_\varepsilon^*(\tau) - u^*(\tau))d\tau\right\|_U^2 dt.
\end{aligned}$$

In these calculations we took into account that u_ε^* and the solution to the approximating state are continuous with respect to $t \in [0, \infty)$. Then, the solution $y_\varepsilon^{T_\varepsilon^*+\lambda, u_\varepsilon^*}(t)$ calculated for $t \in (0, T_\varepsilon^* + \lambda)$ and u_ε^*, restricted to $(0, T_\varepsilon^*)$ coincides with $y_\varepsilon^{T_\varepsilon^*, u_\varepsilon^*}(t)$ the solution calculated on $(0, T_\varepsilon^*)$, which was denoted by $y_\varepsilon^*(t)$. Performing some calculations we get

$$1 + \frac{1}{\varepsilon}\left\langle (y_\varepsilon^*)'(T_\varepsilon^*), Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*, V} + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \frac{\varepsilon}{2}\|Pu_\varepsilon^*(T_\varepsilon^*)\|_U^2 \geq 0.$$

Doing the same for $T_\varepsilon^* - \lambda$ and observing that the solution $y_\varepsilon^{T_\varepsilon^*, u_\varepsilon^*}(t)$, calculated for $t \in (0, T_\varepsilon^*)$ and u_ε^*, restricted to $(0, T_\varepsilon^* - \lambda)$ is in fact $y_\varepsilon^{T_\varepsilon^*-\lambda, u_\varepsilon^*}(t)$ the solution calculated on $(0, T_\varepsilon^* - \lambda)$, we get the reverse inequality. Finally, we obtain

$$1 + \frac{1}{\varepsilon}\left\langle (y_\varepsilon^*)'(T_\varepsilon^*), Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*, V} + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \frac{\varepsilon}{2}\|Pu_\varepsilon^*(T_\varepsilon^*)\|_U^2 = 0. \tag{5.31}$$

Then, using the state system (1.1) for y_ε^* and the final conditions of the adjoint system, we can express the term

$$\frac{1}{\varepsilon}\left\langle (y_\varepsilon^*)'(T_\varepsilon^*), Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*, V} = \left\langle Bu_\varepsilon^*(T_\varepsilon^*) - Ay_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\right\rangle_{V^*, V}.$$

Plugging this in (5.31), we obtain

$$1 + \langle u_\varepsilon^*(T_\varepsilon^*), B^* p_\varepsilon(T_\varepsilon^*)\rangle_{U, U^*} - \langle AX_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\rangle_{V^*, V} + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \frac{\varepsilon}{2}\|Pu_\varepsilon^*(T_\varepsilon^*)\|_U^2 = 0. \tag{5.32}$$

We replace $Pu_\varepsilon^*(T_\varepsilon^*)$ from (5.30),

$$Pu_\varepsilon^*(T_\varepsilon^*) = (\varepsilon F + N_K)^{-1}(-B^* p_\varepsilon(T_\varepsilon^*)),$$

which can be still written

$$\varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*)) + z_\varepsilon^*(T_\varepsilon^*) = -B^* p_\varepsilon(T_\varepsilon^*), \text{ where } z_\varepsilon^*(T_\varepsilon^*) \in N_K(Pu_\varepsilon^*(T_\varepsilon^*)). \tag{5.33}$$

By using this and (7.9) we obtain for the second term in (5.32)

$$\begin{aligned} \langle u_\varepsilon^*(T_\varepsilon^*), B^* p_\varepsilon(T_\varepsilon^*)\rangle_{U,U^*} &= -\langle u_\varepsilon^*(T_\varepsilon^*), \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*)) + z_\varepsilon^*(T_\varepsilon^*)\rangle_{U,U^*} \\ &= -\varepsilon \left\| Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 - \rho \left\| z_\varepsilon^*(T_\varepsilon^*)\right\|_{U^*} \\ &= -\varepsilon \left\| Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 - \rho \left\| B^* p_\varepsilon(T_\varepsilon^*) + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*}. \end{aligned}$$

Therefore, (5.32) becomes

$$\begin{aligned} &1 - \rho \left\| B^* p_\varepsilon(T_\varepsilon^*) + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*} - \varepsilon \left\| Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 - \langle Ay_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} \\ &+\frac{1}{2}\left\| h_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 + \frac{\varepsilon}{2}\left\| Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 = 0, \end{aligned}$$

which finally can be written

$$\rho \left\| B^* p_\varepsilon(T_\varepsilon^*) + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*} + \langle Ay_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} + \frac{\varepsilon}{2}\left\| Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 = 1 + \frac{1}{2}\left\| h_\varepsilon^*(T_\varepsilon^*)\right\|_U^2. \tag{5.34}$$

The next calculation can be performed due to the supplementary regularity of p_ε, that is $p_\varepsilon' \in L^2(0, T_\varepsilon^*; H)$, given by (5.15). We multiply scalarly the state equation by $p_\varepsilon'(t)$, add with the adjoint equation multiplied by $(y_\varepsilon^*)'(t)$, getting

$$(A_H y_\varepsilon^*(t), p_\varepsilon'(t))_H + ((A_H'(y_\varepsilon^*(t)))^* p_\varepsilon(t), (y_\varepsilon^*)'(t))_H = (Bu_\varepsilon^*(t), p_\varepsilon'(t))_H,$$

a.e. $t \in (0, T_\varepsilon^*)$, that reduces to

$$(A_H y_\varepsilon^*(t), p_\varepsilon(t))_H' = \langle u_\varepsilon^*(t), B^* p_\varepsilon'(t)\rangle_{U,U^*}, \text{ a.e. } t \in (0, T_\varepsilon^*).$$

We integrate on (t, T_ε^*) and obtain

$$\begin{aligned} (A_H y_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*))_H - (A_H y_\varepsilon^*(t), p_\varepsilon(t))_H &= \int_t^{T_\varepsilon^*} \langle u_\varepsilon^*(s), B^* p_\varepsilon'(s)\rangle_{U,U^*} ds \qquad (5.35) \\ = \int_t^{T_\varepsilon^*} \left\langle Pu_\varepsilon^*(s), \left(B^* p_\varepsilon(s) + \int_s^{T_\varepsilon^*} F(h_\varepsilon^*(\tau)) d\tau\right)'\right\rangle_{U,U^*} ds \\ - \int_t^{T_\varepsilon^*} \left\langle Pu_\varepsilon^*(s), \left(\int_s^{T_\varepsilon^*} F(h_\varepsilon^*(\tau)) d\tau\right)'\right\rangle_{U,U^*} ds, \end{aligned}$$

since $\langle u_\varepsilon^*(s), Pk(s)\rangle_{U,U^*} = \langle Pu_\varepsilon^*(s), Pk(s)\rangle_{U,U^*}$ where $k(s)$ is either $p_\varepsilon(s)$ or $\int_s^{T_\varepsilon^*} F(h_\varepsilon^*(\tau)) d\tau$, both containing P in their expressions. Denoting

$$\zeta_\varepsilon(t) = -B^* p_\varepsilon(t) - \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau)) d\tau$$

we see by (5.30) that $\zeta_\varepsilon(t) = (\varepsilon F + N_K)(Pu_\varepsilon^*(t))$. By (7.14),

$$\begin{aligned} j_\varepsilon^*(\zeta_\varepsilon(t)) &= \frac{\varepsilon}{2}\left\| Pu_\varepsilon^*(t)\right\|_U^2 + \rho \left\| z_\varepsilon(t)\right\|_{U^*} \qquad (5.36) \\ &= \frac{\varepsilon}{2}\left\| Pu_\varepsilon^*(t)\right\|_U^2 + \rho \left\| B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau)) d\tau + \varepsilon F(Pu_\varepsilon^*(t))\right\|_{U^*}, \ t \in [0, T_\varepsilon^*]. \end{aligned}$$

Thus, we can express (5.30) as

$$Pu_\varepsilon^*(t) = (\varepsilon F + N_K)^{-1}(\zeta_\varepsilon) = (\partial j_\varepsilon)^{-1}(\zeta_\varepsilon(t)) = \partial j_\varepsilon^*(\zeta_\varepsilon(t)), \tag{5.37}$$

(see (7.10)-(7.14)). Then, the integrand of the first term on the right-hand side in (5.35) becomes

$$\begin{aligned} & \left\langle Pu_\varepsilon^*(t), \left(B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau\right)'\right\rangle_{U,U^*} \\ = & -\left\langle Pu_\varepsilon^*(t), \left(-B^* p_\varepsilon(t) - \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau\right)'\right\rangle_{U,U^*} \\ = & -\left(\partial j_\varepsilon^*(\zeta_\varepsilon(t)), \zeta_\varepsilon'(t)\right)_{U^*} = -\frac{dj_\varepsilon^*}{dt}\left(\zeta_\varepsilon(t)\right). \end{aligned}$$

Plugging this in (5.35) we get

$$\begin{aligned} & \left(A_H y_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\right)_H + j_\varepsilon^*\left(\zeta_\varepsilon(T_\varepsilon^*)\right) \\ = & \left(A_H y_\varepsilon^*(t), p_\varepsilon(t)\right)_H + j_\varepsilon^*\left(\zeta_\varepsilon(t)\right) + \int_t^{T_\varepsilon^*} \left(Pu_\varepsilon^*(s), F(h_\varepsilon^*(s))\right)_{U,U^*} ds, \end{aligned} \tag{5.38}$$

for all $t \in [0, T_\varepsilon^*]$. By comparison with (5.34), we obtain

$$\begin{aligned} & (A_H y_\varepsilon^*(t), p_\varepsilon(t))_H + j_\varepsilon^*\left(\zeta_\varepsilon(t)\right) + \int_t^{T_\varepsilon^*} \left(Pu_\varepsilon^*(s), F(h_\varepsilon^*(s))\right)_{U,U^*} ds \\ = & 1 + \frac{1}{2}\left\|h_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 - \rho\left\|B^* p_\varepsilon(T_\varepsilon^*) + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*} - \frac{\varepsilon}{2}\left\|Pu_\varepsilon^*(T_\varepsilon^*)\right\|_H^2 + j_\varepsilon^*\left(\zeta_\varepsilon(T_\varepsilon^*)\right). \end{aligned}$$

Recalling (5.36), this yields

$$\begin{aligned} & (A_H y_\varepsilon^*(t), p_\varepsilon(t))_H + \frac{\varepsilon}{2}\left\|Pu_\varepsilon^*(t)\right\|_U^2 + \rho\left\|B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau + \varepsilon F(Pu_\varepsilon^*(t))\right\|_{U^*} \\ & + \int_t^{T_\varepsilon^*} \left(Pu_\varepsilon^*(s), F(h_\varepsilon^*(s))\right)_{U,U^*} ds \\ = & 1 + \frac{1}{2}\left\|h_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 - \rho\left\|B^* p_\varepsilon(T_\varepsilon^*) + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*} - \frac{\varepsilon}{2}\left\|Pu_\varepsilon^*(T_\varepsilon^*)\right\|_H^2 \\ & + \frac{\varepsilon}{2}\left\|Pu_\varepsilon^*(T_\varepsilon^*)\right\|_U^2 + \rho\left\|B^* p_\varepsilon(T_\varepsilon^*) + \int_{T_\varepsilon^*}^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau + \varepsilon F(Pu_\varepsilon^*(T_\varepsilon^*))\right\|_{U^*} \end{aligned}$$

and so we obtain (5.23), as claimed. □

5.3 Optimality conditions for (P)

In order to ensure the passing to the limit in the approximating optimality conditions (5.22)-(5.23) we complete the hypotheses $(a_1) - (a_6)$, (b_1), (c_1) with $(d_1) - (d_5)$.

Theorem 5.5. *Let*

$$y_0 \in V,\ Py^{tar} \in P(D_H),\ Py_0 \neq Py^{tar}. \tag{5.39}$$

Let (T^*, u^*, y^*) be an optimal pair in (P). Then, the first order necessary conditions of optimality are

$$Pu^*(t) \in (N_K)^{-1}\left(-B^* p(t)\right),\ a.e.\ t \in (0, T^*), \tag{5.40}$$

$$\rho\left\|B^* p(t)\right\|_{U^*} + (A_H y^*(t), p(t))_H = 1,\ a.e.\ t \in (0, T^*), \tag{5.41}$$

where y^* *is the solution to the state system* (1.1)-(1.2) *corresponding to* (T^*, u^*), *and* p *is a solution to*

$$-p'(t) + (A'(y^*(t)))^* p(t) = 0, \ \text{a.e. } t \in (0, T^*), \tag{5.42}$$

$$p(T^*) \in V^*. \tag{5.43}$$

Proof. First, we prove that

$$\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} \le C, \tag{5.44}$$

with C independent of ε.

Let us begin with the case $P(y_1, y_2) = (y_1, 0)$, $B(u_1, u_2) = (u_1, 0)$. We recall that in this case $y^{tar} = (y_1^{\text{target}}, z)$, $\forall z \in H$, and $Py^{tar} = (y_1^{\text{target}}, 0)$. We start from (5.34) and express the third term on the left-hand side as

$$\begin{aligned} \langle Ay_\varepsilon^*(T_\varepsilon^*), p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} &= \frac{1}{\varepsilon}\left\langle Ay_\varepsilon^*(T_\varepsilon^*), Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*,V} \\ = \ \frac{1}{\varepsilon}\left\langle Ay_\varepsilon^*(T_\varepsilon^*) - A\widehat{y}, Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*,V} &+ \frac{1}{\varepsilon}\left\langle A\widehat{y}, Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\rangle_{V^*,V}, \end{aligned}$$

where $\widehat{y} = (y_1^{\text{target}}, \widehat{z})$ set by (2.13), satisfying (2.14), with the choice (2.15). We note that

$$Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar} = (Py_\varepsilon^*(T_\varepsilon^*) - y_1^{\text{target}}, 0) = Py_\varepsilon(T_\varepsilon^*) - P\widehat{y}. \tag{5.45}$$

Then, by (5.34), we can write

$$\begin{aligned} &\rho\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} + \langle Ay_\varepsilon^*(T_\varepsilon^*) - A\widehat{y}, p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} \qquad (5.46)\\ \le \ & 1 + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon\rho\|F(Pu_\varepsilon^*(T_\varepsilon^*))\|_{U^*} + |(A_H\widehat{y}, p_\varepsilon(T_\varepsilon^*))_H| \\ \le \ & 1 + \|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon\rho^2 + \left|\langle Pp_\varepsilon(T_\varepsilon^*), A_H\widehat{y}\rangle_{V^*,V}\right| \le 1 + \|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon\rho^2 + \|Pp_\varepsilon(T_\varepsilon^*)\|_{V^*}\|A_H\widehat{y}\|_V. \end{aligned}$$

Here, we took into account that

$$p_\varepsilon(T_\varepsilon^*) = \frac{1}{\varepsilon}(Py_\varepsilon(T_\varepsilon^*) - Py^{tar}) = \frac{1}{\varepsilon}P(Py_\varepsilon(T_\varepsilon^*) - Py^{tar}) = Pp_\varepsilon(T_\varepsilon^*). \tag{5.47}$$

Now, we use (2.14) which is assumed to take place for $t \in (0, T_* + \delta)$, with T_* the time specified in the controllability hypothesis (c_1), with $T_* \ge T^*$. Recall that $T_\varepsilon^* \to T^*$. Hence, for ε sufficiently small, $T_\varepsilon^* \in (0, T^* + \delta) \subset (0, T_* + \delta)$, with δ arbitrary small and it follows that relation (2.14) can take place also for $t = T_\varepsilon^*$, that is

$$\begin{aligned} \langle Ay_\varepsilon^*(T_\varepsilon^*) - A\widehat{y}, p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} &= \frac{1}{\varepsilon}\langle Ay_\varepsilon^*(T_\varepsilon^*) - A\widehat{y}, P(y_\varepsilon(T_\varepsilon^*) - \widehat{y})\rangle_{V^*,V} \\ \ge \ -\frac{C_3}{\varepsilon}\|Py_\varepsilon(T_\varepsilon^*) - P\widehat{y}\|_{P(H)}^2 &= -\frac{C_3}{\varepsilon}\left\|Py_\varepsilon(T_\varepsilon^*) - y_1^{\text{target}}\right\|_{P(H)}^2. \end{aligned}$$

Here we used (5.45). Then,

$$\begin{aligned} \rho\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} &\le 1 + \|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon C\rho^2 + \|Pp_\varepsilon(T_\varepsilon^*)\|_{V^*}\|A_H\widehat{y}\|_V + C_3\|p_\varepsilon(T_\varepsilon^*)\|_H^2 \\ \le \ 1 + \|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon\rho^2 &+ C^*\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*}\|A_H\widehat{y}\|_V - \frac{C_3}{\varepsilon}\left\|Py_\varepsilon(T_\varepsilon^*) - Py^{tar}\right\|_H^2, \end{aligned}$$

where we took into account (2.12). We recall (4.18),

$$\limsup_{\varepsilon\to 0}\frac{1}{2\varepsilon}\left\|Py_\varepsilon^*(T_\varepsilon^*) - Py^{tar}\right\|_H^2 = 0,$$

and so, by (2.15), we can write

$$\rho\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} < 1 + C + \rho_1\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*},$$

because $\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon C\rho^2 \to 0$. The convergence of the first term is due to (4.17). This yields

$$(\rho - \rho_1)\,\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} < 1 + C,$$

and choosing $\rho > \rho_1$ we finally get (5.44).

As a matter of fact, in the proof of (5.44) the second component $\widehat{z}$ of $\widehat{y}$ can be generally set as the second component of the approximating state solution.

If $P = I$, we proceed in the same way, and use that $y^{tar} = (y_1^{\text{target}}, y_2^{\text{target}})$ and (d_5) and take Ay^{tar} instead of $A\widehat{y}$. We have

$$\begin{aligned} &\left\langle Ay_\varepsilon^*(T_\varepsilon^*) - Ay^{tar}, p_\varepsilon(T_\varepsilon^*)\right\rangle_{V^*,V} = \frac{1}{\varepsilon}\left\langle Ay_\varepsilon^*(T_\varepsilon^*) - Ay^{tar}, y_\varepsilon(T_\varepsilon^*) - y^{tar}\right\rangle_{V^*,V} \\ \geq\;& -\frac{C_3}{\varepsilon}\left\|y_\varepsilon(T_\varepsilon^*) - y^{tar}\right\|_H^2 = -\frac{C_3}{\varepsilon}\left\|Py_\varepsilon(T_\varepsilon^*) - Py^{tar}\right\|_H^2 \end{aligned}$$

which tends to zero by (4.18). Here, $Py_\varepsilon(T_\varepsilon^*) - Py^{tar}$ has both nonzero components.

We recall the adjoint system given by (5.13)-(5.14). Since the final data is bounded in U^*, according to (5.44), we expect to obtain at limit a solution with a weaker regularity. We are going to obtain some uniform estimates for the solution p_ε.
A first estimate is obtained by multiplying scalarly (5.13) by $\Gamma_H^{-1}p_\varepsilon(t)$ and integrating from t to T_ε^*

$$\frac{1}{2}\|p_\varepsilon(t)\|_{V^*}^2 + \int_t^{T_\varepsilon^*}\left((A_H'(y_\varepsilon^*(\tau)))^* p_\varepsilon(\tau), \Gamma_H^{-1}p_\varepsilon(\tau)\right)_H d\tau = \frac{1}{2}\|p_\varepsilon(T_\varepsilon^*)\|_{V^*}^2,$$

where the first term on the right-hand side was obtained by using the properties of the duality mapping (7.15). According to (2.11) and (2.12) for $v = p_\varepsilon(T_\varepsilon^*) \in V^*$, we successively get

$$\begin{aligned} &\frac{1}{2}\|p_\varepsilon(t)\|_{V^*}^2 + C_1\int_t^{T_\varepsilon^*}\|p_\varepsilon(\tau)\|_H^2 d\tau \leq \frac{1}{2}\|p_\varepsilon(T_\varepsilon^*)\|_{V^*}^2 + C_2\int_t^{T_\varepsilon^*}\|p_\varepsilon(\tau)\|_{V^*}^2(1 + C_3\|y_\varepsilon^*(\tau)\|_V^l)d\tau \\ \leq\;& \frac{1}{2}\|B^*p_\varepsilon(T_\varepsilon^*)\|_{U^*}^2 + C_4\int_t^{T_\varepsilon^*}\|p_\varepsilon(\tau)\|_{V^*}^2 d\tau. \end{aligned}$$

Here, we used (3.2). By Gronwall lemma and (5.44) we obtain

$$\|p_\varepsilon(t)\|_{V^*}^2 + \int_t^{T_\varepsilon^*}\|p_\varepsilon(\tau)\|_H^2 d\tau \leq C, \text{ for all } t \in [0, T_\varepsilon^*], \tag{5.48}$$

independently on ε.

Next, we multiply scalarly (5.13) by $\Gamma_H^{-\alpha}p_\varepsilon(t)$ (where α is chosen by (2.9) and (2.10)) and integrate from t to T_ε^*. Applying (2.10) and (2.9) we obtain

$$\begin{aligned} &\frac{1}{2}\left\|\Gamma^{-\alpha/2}p_\varepsilon(t)\right\|_H^2 + C_1\int_t^{T_\varepsilon^*}\left\|\Gamma_H^{(1-\alpha)/2}p_\varepsilon(\tau)\right\|_H^2 d\tau \\ \leq\;& C_2\int_t^{T_\varepsilon^*}\|p_\varepsilon(\tau)\|_H^2(1 + C_3\|y_\varepsilon^*(\tau)\|_V^l) + \frac{1}{2}\|B^*p_\varepsilon(T_\varepsilon^*)\|_{U^*}^2. \end{aligned} \tag{5.49}$$

Recalling (5.48) we obtain $\left\|\Gamma_H^{(1-\alpha)/2}p_\varepsilon\right\|_{L^2(0,T_\varepsilon^*;H)} \leq C$, that is

$$\|p_\varepsilon\|_{L^2\left(0,T_\varepsilon^*;D\left(\Gamma_H^{(1-\alpha)/2}\right)\right)} \leq C. \tag{5.50}$$

To use further these estimate we have to modify the functional framework in the following sense. We extend the operator $(A_H'(y_\varepsilon^*(t)))^*$ to H, for all $t \in [0, T_\varepsilon^*]$, namely we define $\tilde{A}_H'(y_\varepsilon^*(t)) : H \subset D_H^* \to D_H^*$ by

$$\left\langle \tilde{A}_H'(y_\varepsilon^*(t))v, \psi\right\rangle_{D_H^*,D_H} = (v, (A_H'(y_\varepsilon^*(t)))\psi)_H, \text{ for } v \in H,\ \psi \in D_H, \text{ for all } t \in [0, T_\varepsilon^*].$$

The norm on D_H^* is defined by $\|\theta\|_{D_H^*} = \left\|(A_H'(y_\varepsilon^*(t)))^{-1}\theta\right\|_H$. We have, by (2.6)

$$\begin{aligned}
&\left|\left\langle \tilde{A}_H'((y_\varepsilon^*(t))v, \psi\right\rangle_{D_H^*, D_H}\right| = |(v, A_H'((y_\varepsilon^*(t))\psi)_H| \leq \|v\|_H \|A_H'((y_\varepsilon^*(t))\psi\|_H \\
\leq\ & C\|v\|_H \|\psi\|_{D_H} (1 + \|y_\varepsilon^*(t)\|_V^\kappa) \leq C\|v\|_H \|\psi\|_{D_H},
\end{aligned}$$

which yields for $v = p_\varepsilon$,

$$\left\|\tilde{A}_H'(y_\varepsilon^*(t))p_\varepsilon\right\|_{L^2(0,T_\varepsilon^*;D_H^*)} \leq C\|p_\varepsilon\|_{L^2(0,T_\varepsilon^*;H)} \leq C,$$

since $\|y_\varepsilon^*(t)\|_V \leq C$. By comparison in the adjoint equation (5.13) we obtain

$$\|p_\varepsilon'\|_{L^2(0,T_\varepsilon^*;D_H^*)} \leq C. \tag{5.51}$$

We recall that $T_\varepsilon^* \to T^*$ and so $T^* - \delta \leq T_\varepsilon^*$, with δ arbitrary, so that the estimates are true also on $(0, T^* - \delta)$. By (5.48) and the latter, selecting a subsequence, denoted still by ε, we have

$$p_\varepsilon \to p \text{ weakly in } L^2(0, T^* - \delta; H), \text{ weak-star in } L^\infty(0, T^* - \delta; V^*),$$

$$p_\varepsilon' \to p' \text{ weakly in } L^2(0, T^* - \delta; D_H^*).$$

Because δ is arbitrary, the previous convergences take place on $\cap_{\delta>0}(0, T^* - \delta) = (0, T^*)$.

Since $D\left(\Gamma_H^{(1-\alpha)/2}\right)$ is compact in H and $H \subset D_H^*$, we have by Aubin-Lions lemma that

$$p_\varepsilon \to p \text{ strongly in } L^2(0, T^*; H). \tag{5.52}$$

Then, using the convergence $y_\varepsilon^*(t) \to y^*(t)$ strongly in V a.e. t, and the continuity (2.7), we have

$$\begin{aligned}
&\left\langle \tilde{A}_H'(y_\varepsilon^*)p_\varepsilon(t) - \tilde{A}_H'(y^*)p(t), \psi(t)\right\rangle_{D_H^*, D_H} \\
=\ & \left\langle (\tilde{A}_H'(y_\varepsilon^*) - \tilde{A}_H'(y^*))p_\varepsilon(t), \psi(t)\right\rangle_{D_H^*, D_H} + \left\langle (\tilde{A}_H'(y^*)(p_\varepsilon(t) - p(t)), \psi(t)\right\rangle_{D_H^*, D_H} \\
=\ & \langle p_\varepsilon(t), (A_H'(y_\varepsilon^*) - A_H'(y^*))\psi(t)\rangle_H + \langle p_\varepsilon(t) - p(t), A_H'(y^*)\psi(t)\rangle_H,
\end{aligned}$$

for $\psi \in L^2(0, T_\varepsilon^*; D_H)$, which implies by the previous convergences that

$$\tilde{A}_H'(y_\varepsilon^*)p_\varepsilon \to \tilde{A}_H'(y^*)p \text{ weakly in } L^2(0, T^*; D_H^*), \text{ as } \varepsilon \to 0.$$

We also have

$$B^*p_\varepsilon \to B^*p \text{ strongly in } L^2(0, T^*; U^*). \tag{5.53}$$

By these convergences we obtain (5.42) in the sense of distributions and a.e.

We go back now to (5.22), and recall (5.29) which can be equivalently written

$$N_{\mathcal{K}_{T_\varepsilon^*}}(Pu_\varepsilon^*) \ni z_\varepsilon = -B^*p_\varepsilon - \varepsilon F(Pu_\varepsilon^*) - \int_\cdot^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau.$$

We pass to the limit as $\varepsilon \to 0$ and have

$$u_\varepsilon^* \to u^* \text{ weakly in } L^2(0, T^*; U), \ B^*p_\varepsilon \to B^*p \text{ strongly in } L^2(0, T^*; U^*),$$

$$F(h_\varepsilon^*) \to 0 \text{ strongly in } L^2(0, T^*; U^*),$$

by (4.16), and

$$\int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau \to 0, \text{ strongly in } U^*, \text{ for all } t \in (0, T_\varepsilon^*).$$

Therefore,

$$z_\varepsilon = -B^* p_\varepsilon - \varepsilon F(Pu_\varepsilon^*) - \int_\cdot^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau \to -B^* p \text{ strongly in } L^2(0,T^*;U^*).$$

But $N_{\mathcal{K}_{T^*}}$ is maximal monotone from $L^2(0,T^*;U)$ to $L^2(0,T^*;U^*)$, that is weakly-strongly closed and since $Pu_\varepsilon^* \to Pu^*$ weakly in $L^2(0,T^*;U)$, we get $-B^*p \in N_{\mathcal{K}_{T^*}}(Pu^*)$, or equivalently (5.40).

Finally, we have to pass to the limit in (5.23). For this, we integrate (5.23) from s to s', $0 < s < s' < T^*$ and get

$$\begin{aligned} &\rho \int_s^{s'} \left\| B^* p_\varepsilon(t) + \int_t^{T_\varepsilon^*} F(h_\varepsilon^*(\tau))d\tau + \varepsilon F(Pu_\varepsilon^*(t)) \right\|_{U^*} dt + \int_s^{s'} (Ay_\varepsilon^*(t), p_\varepsilon(t))_H \, dt \\ &+ \int_s^{s'} \int_t^{T_\varepsilon^*} \langle Pu_\varepsilon^*(\tau), F(h_\varepsilon^*(\tau))\rangle_{U,U^*} \, d\tau dt + \frac{\varepsilon}{2} \int_s^{s'} \|Pu_\varepsilon^*(t)\|_H^2 \, dt \\ = \; & (s'-s)\left(1 + \frac{1}{2}\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2\right). \end{aligned}$$

We recall that $Ay_\varepsilon^* \to Ay^*$ weakly in $L^2(0,T^*;H)$ and note that

$$\int_t^{T_\varepsilon^*} \langle Pu_\varepsilon^*(\tau), F(h_\varepsilon^*(\tau))\rangle_{U,U^*} \, d\tau \to 0, \text{ for all } t \in (0,T_\varepsilon^*).$$

Finally, $\|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 \to 0$, by (4.17). We pass to the limit as ε goes to 0 and get

$$\int_s^{s'} \{\rho \|B^* p(t)\|_{U^*} + (Ay^*(t), p(t))_H\} \, dt = (s'-s).$$

Dividing by $(s'-s)$ and passing to the limit as $s \to s'$ we obtain (5.41), for a.e. $t \in (0,T^*)$. □

In the case when $U = U^* = H$ we have a particular result for which we assume the hypotheses $(a_1) - (a_6)$ and replace $(d_1) - (d_5)$ by simpler ones.

Corollary 5.6. *Let* $U = U^* = H$ *and assume* (5.39), (2.14), *and*

$$((A'_H(y))^* v, v)_H \geq C_1 \|v\|_H^2 - C_2 \|v\|_V^2 (1 + C_3 \|y\|_V^l), \text{ for all } y, v \in V, \; l \geq 0, \tag{5.54}$$

$$\|Pv\|_H \leq C^* \|B^* v\|_H, \text{ for } v \in H, \tag{5.55}$$

$$\rho > \rho_1, \; \rho_1 := C^* \|A_H \widehat{y}\|_H, \tag{5.56}$$

(*instead of* (2.12)). *Then,* (5.40)-(5.42) *take place and* $p(T^*) \in H$.

Proof. We resume the proof of the estimate for $\|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*}$ in Theorem 5.5 and have now in (5.46)

$$\begin{aligned} &\rho \|B^* p_\varepsilon(T_\varepsilon^*)\|_{U^*} + \langle Ay_\varepsilon^*(T_\varepsilon^*) - A\widehat{y}, p_\varepsilon(T_\varepsilon^*)\rangle_{V^*,V} \\ \leq \; & 1 + \|h_\varepsilon^*(T_\varepsilon^*)\|_U^2 + \varepsilon\rho^2 + |(A_H \widehat{y}, p_\varepsilon(T_\varepsilon^*))_H| \\ \leq \; & 1 + C + \|Pp_\varepsilon(T_\varepsilon^*)\|_H \|A_H \widehat{y}\|_H < C_1 + \rho_1 \|B^* p_\varepsilon(T_\varepsilon^*)\|_H. \end{aligned}$$

Since $U^* = H$ we get $\|B^* p_\varepsilon(T_\varepsilon^*)\|_H \leq C$, which will ensure a more regular solution for p. We multiply (5.13) by $p_\varepsilon(t)$, integrate from t to T_ε^* and use (5.54) to obtain

$$\|p_\varepsilon(t)\|_H^2 + \int_t^{T_\varepsilon^*} \|p_\varepsilon(\tau)\|_V^2 \, d\tau \leq C, \text{ for all } t \in [0, T_\varepsilon^*]. \tag{5.57}$$

Then,

$$\int_0^{T_\varepsilon^*} \|(A'_H(y_\varepsilon^*))^* p_\varepsilon(t)\|_{V^*}^2 \, dt \leq \int_0^{T_\varepsilon^*} \|p_\varepsilon(t)\|_V^2 (1 + C_3 \|y_\varepsilon^*(t)\|_V^l) dt \leq C$$

and by (5.13) we infer that

$$\int_0^{T_\varepsilon^*} \|p_\varepsilon'(t)\|_{V^*}^2 \, dt \le C.$$

On a subsequence we obtain

$$\begin{array}{rcl} p_\varepsilon & \to & p \text{ weakly in } L^2(0,T^*;V) \cap W^{1,2}(0,T^*;V^*), \text{ weak-star in } L^\infty(0,T^*;H), \\ & & \text{strongly in } L^2(0,T^*;H), \end{array}$$

where p turns out to be the solution to (5.42). The rest of the proof can be led as in Theorem 5.5. □

Remark 5.7. Consider the case $P = I$ and $y^{tar} = 0$ and assume that, for each $\widetilde{y} \in C([0,T];H) \cap L^2(0,T;V)$, the linearized problem

$$Y'(t) + A'(\widetilde{y})Y(t) = Bv(t), \text{ a.e. } t > 0, \ Y(0) = Y_0$$

is exactly null controllable in the following sense: for each $Y_0 \in H$ with $\|Y_0\|_H \le 1$, there is $v \in L^2(0,T;U)$, with $\|v\|_{L^2(0,T;U)} \le \gamma$, such that $Y(T) = 0$. Then, Theorem 5.5 remains true, without assuming $(d_1) - (d_5)$. Here there is the argument. By the above controllability hypothesis, we get for the dual equation, $-p'(t) + (A'(\widetilde{y}))^* p(t) = 0$, the following observability inequality:

$$\|p(0)\|_H \le \gamma \left(\int_0^{T_\varepsilon^*} \|B^* p(t)\|_{U^*}^2 \, dt \right)^{1/2}$$

and therefore

$$\|p(t)\|_H \le \gamma \left(\int_t^{T_\varepsilon^*} \|B^* p(\tau)\|_{U^*}^2 \, d\tau \right)^{1/2}.$$

Then, substituting in (5.23) we get $\|p_\varepsilon(t)\|_H \le \gamma_1 \int_t^{T_\varepsilon^*} \|B^* p_\varepsilon(\tau)\|_U^2 \, d\tau + C(\varepsilon) + C \le C_t$, for all $t \in [0, T_\varepsilon^*]$ and then $\int_0^{T_\varepsilon^* - \delta} \|p_\varepsilon(t)\|_H^2 \, dt \le C_\delta$, $\delta > 0$. Thus, we may pass to the limit in (5.13)-(5.14) to get (5.40)-(5.42).

6 Examples

We particularize our results to some equations and systems modelling various processes in physical applications. Let Ω be an open bounded subset of $\mathbb{R}^d$, $d \le 3$, with a sufficient regular boundary $\partial\Omega$ and let ν be the outward normal to $\partial\Omega$. Let $L^r(\Omega)$ be the space of r-summable functions, $y : \Omega \to \mathbb{R}$, with the norm $\|y\|_r = \left(\int_\Omega |y|^r \, dx \right)^{1/r}$, $1 \le r < \infty$, and $\|y\|_\infty = ess\sup_{x \in \Omega} |y(x)|$ for $r = \infty$. The spaces $H^r(\Omega) := W^{1,r}(\Omega)$, with $1 \le r < \infty$, and $H_0^1(\Omega)$ are the standard Sobolev spaces, and $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

Example 1. Diffusion equation with a potential and drift term. Let us consider the problem

$$\begin{aligned} y_t - \Delta y + \beta(y) + a_1 y - \nabla \cdot (by) &= u, \text{ in } (0,\infty) \times \Omega, \\ y(0) &= y_0, \text{ in } \Omega, \\ \frac{\partial y}{\partial \nu} + \gamma y &= 0, \text{ on } (0,\infty) \times \partial\Omega, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \beta &: \mathbb{R} \to \mathbb{R}, \ \beta \in C^1(\mathbb{R}), \ \beta(0) = 0, \\ 0 &< a_0 \le \beta'(r) \le L(|r|^\kappa + 1), \text{ for all } r \in \mathbb{R}, \ \kappa \in [0,2], \end{aligned} \tag{6.2}$$

$$a_1 \in L^\infty(\Omega), \ b \in (W^{1,\infty}(\Omega))^3, \ b \cdot \nu = 0 \text{ on } \partial\Omega, \ \gamma \in L^\infty(\partial\Omega), \ \gamma \ge \gamma_0 > 0 \text{ a.e.} \tag{6.3}$$

This problem characterizes the evolution of a diffusion process under the influence of a potential β and of a drift term $\nabla \cdot (by)$. For $\beta = 0$ the model can describe the diffusion with transport of a substance in a fluid. If $b = 0$, $\beta(y) = y^3$ and $a_1 = -1$ we note that this is the Allen-Cahn equation describing the phase transitions of a material, which can exists in different phases, under the influence of a double-well potential. Such a problem with different assumptions for β was treated in [3], Section 6.1.4.

We study problem $(\mathcal{P})$ for $u \in L^\infty(0,\infty; L^2(\Omega))$ with $\|u(t)\|_H \le \rho$ a.e. $t \ge 0$.

Proposition 6.1. *Let $y_0 \in V$, $y^{\text{target}} \in D_H$, $d = -\Delta y^{\text{target}} + \beta(y^{\text{target}}) + a_1 y^{\text{target}} - \nabla \cdot (by^{\text{target}}) \in L^2(\Omega)$ and $\rho > \|d\|_{L^2(\Omega)}$. Then, there exists (T^*, u^*) solution to $(\mathcal{P})$ satisfying (5.40)-(5.41), where p solves*

$$\begin{aligned} p_t - \Delta p + \beta'(y)p + a_1 p + b \cdot \nabla p &= 0, \quad \text{in } (0,\infty) \times \Omega, \\ \frac{\partial p}{\partial \nu} + \gamma p &= 0, \quad \text{on } (0,\infty) \times \partial\Omega, \\ p(T^*) &\in H. \end{aligned}$$

Proof. Let us set:

$$H = L^2(\Omega),\ V = H^1(\Omega),\ V^* = (H^1(\Omega))^*,\ D_H = \left\{ y \in H^2(\Omega);\ \frac{\partial y}{\partial \nu} + \gamma y = 0 \text{ on } \partial\Omega \right\},$$

$$\Gamma : V \to V^*,\ \langle \Gamma y, \psi \rangle_{V^*,V} = \int_\Omega \nabla y \cdot \nabla \psi dx + \int_{\partial\Omega} \gamma(x) y \psi d\sigma, \text{ for } \psi \in V,$$

$$\Gamma_H : D_H \subset H \to H,\ \Gamma_H = -\Delta,$$

$$U = H,\ B = I,\ A : V \to V^*,$$

$$\langle Ay, \psi \rangle_{V^*,V} = \int_\Omega (\nabla y + by) \cdot \nabla \psi dx + \int_\Omega (\beta(y) + a_1 y)\psi dx + \int_{\partial\Omega} \gamma(x) y \psi d\sigma,$$

$$A_H : D_H \subset H \to H,\ A_H y = -\Delta y + \beta(y) + a_1 y - \nabla \cdot (by).$$

We shall check first the hypotheses $(a_1) - (c_1)$ in Section 3. Since β is maximal monotone we have

$$\begin{aligned} \langle Ay - A\overline{y}, y - \overline{y} \rangle_{V^*,V} &\ge \int_\Omega (|\nabla(y - \overline{y})|^2 + (\beta(y) - \beta(\overline{y}))(y - \overline{y}))dx + \int_\Omega a_1(x)(y - \overline{y})^2 dx \\ &+ \int_\Omega b(y - \overline{y}) \cdot \nabla(y - \overline{y})dx + \int_{\partial\Omega} \gamma(x)(y - \overline{y})^2 d\sigma \ge C_1 \|y - \overline{y}\|_V^2 - C_2 \|y - \overline{y}\|_H^2,\ y, \overline{y} \in V, \end{aligned}$$

implying that $\lambda I + A$ is coercive for λ large. Here, we used the trace theorem, $\|y\|_{L^2(\partial\Omega)} \le C_{tr} \|y\|_V$, with C_{tr} is a constant.

By (6.2) it follows that $|\beta(r)| \le C |r|^{\kappa+1} + |r|$, and so, for $\kappa \in [0,2]$ we have that

$$\|\beta(y)\|_H \le C \int_\Omega \left(|y|^{2(\kappa+1)} + |y|^2 \right) dx \le C_1 \left(\|y\|_V^{2(\kappa+1)} + \|y\|_V^2 \right) \cdot$$

Let $y_n \to y$ strongly in V. Since $\|\beta(y_n)\|_H \le C$ it follows that $\beta(y_n) \to \beta(y)$ weakly in H because β is strongly-weakly closed. Moreover, we have $\beta(y_n) \to \beta(y)$ a.e. on Ω, and so $\beta(y_n) \to \beta(y)$ strongly in H, by Vitali's theorem. Therefore, it follows that A is continuous from V to V^*. Then,

$$\begin{aligned} &\langle Ay, \psi \rangle_{V^*,V} \le C \|y\|_V \|\psi\|_V + \langle \beta(y) + a_1 y, \psi \rangle_{V^*,V} - \langle \nabla \cdot (by), \psi \rangle_{V^*,V} \\ \le\ & C \|y\|_V \|\psi\|_V + \|\beta(y)\|_{V^*} \|\psi\|_V + \|by\|_2 \|\nabla \psi\|_2 \le C \left(\|y\|_V + \|y\|_V^{k+1} + \|b\|_\infty \|y\|_H \right) \|\psi\|_V, \end{aligned}$$

hence $\|Ay\|_{V^*}$ is bounded on bounded subsets.

Relation (2.4) is immediately verified, because

$$\begin{aligned} &(-\Delta y + \beta(y) + a_1 y - \nabla \cdot (by), -\Delta y)_H \ge \|\Delta y\|_H^2 - \|a_1\|_\infty \|\nabla y\|_H^2 - \|-\Delta y\|_H \|\nabla \cdot (by)\|_H \\ \ge\ & \frac{1}{2} \|\Gamma_H y\|_H^2 - \|a_1\|_\infty \|\nabla y\|_H^2 - \sum_{i=1}^3 \|b_i y\|_V^2 \ge \frac{1}{2} \|\Gamma_H y\|_H^2 - C \|y\|_V^2, \end{aligned}$$

since $(\beta(y), -\Delta y)_H \geq 0$ by the monotonicity of β.

The controllability (c_1) follows by Proposition 7.1.

Next we verify $(a_3)-(a_5)$ in Section 5.1. We introduce $A'(y): V \to V^*$,

$$\langle A'(y)z, \psi\rangle_{V^*,V} = \int_\Omega (\nabla z + bz)\cdot \nabla\psi dx + \int_\Omega (\beta'(y)z + a_1 z)\psi dx + \int_{\partial\Omega} \gamma(x) z\psi d\sigma,$$

then $A'_H(y): D_H \to H$, $A'_H(y)z = -\Delta z + \beta'(y)z + a_1 z - \nabla\cdot(bz)$, and

$$(A'_H(y))^* z = -\Delta z + \beta'(y)z + a_1 z - b\cdot\nabla z \text{ with } \frac{\partial z}{\partial\nu} + \gamma z = 0 \text{ on } \partial\Omega,$$

and provide first some estimates. Using the Hölder inequality we have

$$I_1^2 = \|\beta'(y)z\|_H^2 \leq C\int_\Omega (|y|^{2\kappa}+1)\,|z|^2\,dx \leq C\left(\int_\Omega |y|^{2\kappa q}\,dx\right)^{1/q}\left(\int_\Omega |z|^{2q'}\,dx\right)^{1/q'} + C_1\|z\|_H^2$$
$$\leq \quad C\left(\|y\|_{2\kappa q}^{2\kappa}\|z\|_{2q'}^2 + \|z\|_H^2\right), \text{ for } y, z\in D_H,$$

where $1/q+1/q'=1$. Now, we recall the embedding $W^{s,m}(\Omega)\subset L^r(\Omega)$, where $d>sm$, $m\leq r\leq \frac{dm}{d-sm}$ (see [1], p. 217, Theorem 7.57) and apply it for $m=2$, $r=2q'$, $s=1-\alpha$, for $\alpha\in(0,1)$ to get

$$H^1(\Omega)\subset W^{1-\alpha,2}(\Omega) = H^{1-\alpha}(\Omega)\subset L^{2q'}(\Omega),$$

with $q'>1$. Then, $\|y\|_{2\kappa q}\leq C\|y\|_V$ if $2\kappa q\leq 6$. Thus, we obtain

$$I_1 = \|\beta'(y)z\|_H \leq C\left(\|y\|_V^\kappa \|z\|_{H^{1-\alpha}(\Omega)} + \|z\|_H\right) \leq C\|z\|_V\left(\|y\|_V^\kappa + 1\right). \tag{6.4}$$

To this end we must have $3>2(1-\alpha)$ which is satisfied for $\alpha\in[0,1]$ and

$$2\leq 2q' \leq \frac{6}{3-2(1-\alpha)} \text{ implying } q'\leq \frac{3}{1+2\alpha}.$$

In particular, these are true for $\kappa\leq 2$, $q'\geq 3$. Then

$$\|A'(y)z\|_{V^*} \leq C_1\|z\|_V + \|\beta'(y)z\|_{V^*} \leq C_2\|z\|_V(1+C\|y\|_V^\kappa), \text{ for } y,z\in V,$$
$$\|A'_H(y)z\|_H \leq C_1\|z\|_{D_H}(1+C\|y\|_V^\kappa), \text{ for } y,z\in D_H.$$

Moreover, $y\to A'(y)z$ is continuous from V to $L(V,V^*)$. Indeed, let $y_n\in V$, $y_n\to y$ strongly in V. Then, as before, $\beta'(y_n)\to\beta'(y)$ strongly in H. Therefore,

$$\|A'(y_n)z - A'(y)z\|_{V^*} = \int_\Omega (\beta'(y_n)-\beta'(y))z\psi dx \to 0.$$

Similarly, let y_n, $y\in D_H$, $y_n\to y$ strongly in V and $z\in D_H\subset C(\overline{\Omega})$. Then,

$$\|A'(y_n)z - A'(y)z\|_H = \|(\beta'(y_n)-\beta'(y))z\|_H \to 0.$$

To prove hypothesis (a_6), equivalently (2.8), we calculate

$$\begin{aligned}
&\langle -\Delta z + \beta'(y)z + a_1 z - b\cdot\nabla z, \Gamma_\nu z\rangle_{V^*,V}\\
\geq\ & \|\Gamma_\nu z\|_H^2 - \|\beta'(y)z\|_H\|\Gamma_\nu z\|_H - \|a_1\|_\infty\|z\|_H\|\Gamma_\nu z\|_H - \|b\cdot\nabla z\|_H\|\Gamma_\nu z\|_H\\
\geq\ & C\|\Gamma_\nu z\|_H^2 - C_2(\|y\|_V^\kappa+1)\|z\|_V^2, \text{ for } z\in V.
\end{aligned}$$

Here, we used the last inequality in (6.4) and the following relations

$$\langle \Gamma z, \Gamma_\nu z\rangle_{V^*,V} \geq \|\Gamma_\nu z\|_H^2,\ z\in V, \tag{6.5}$$

$$\|\Gamma_H z\|_H \geq \|\Gamma_\nu z\|_H\,,\ z \in D_H, \tag{6.6}$$

$$\|z\|_{D_H} = \|\Gamma_H z\|_H\,. \tag{6.7}$$

Since $U = H$ it remains to check the hypotheses (5.54),

$$((A'_H(y))^* z, z)_H \geq \|\nabla z\|_H^2 - \|a_1\|_\infty \|z\|_H^2 - \|b \cdot \nabla z\|_H \|z\|_H \geq C_1 \|z\|_V^2 - C_2 \|z\|_H^2\,,$$

and (5.55) which is automatically verified with C^*, for ρ large enough. Thus, Corollary 5.6 can be applied. □

We remark, that in virtue of Remark 5.7, Proposition 6.1 applies to equation (6.1) with an internal controller

$$y_t - \Delta y + \beta(y) + a_1 y - \nabla \cdot (by) = 1_{\Omega_0} u, \text{ in } (0,\infty) \times \Omega,$$

where Ω_0 is an open subset of Ω and 1_{Ω_0} is the characteristic function of Ω_0. Indeed, by [15], the corresponding linearized system is exactly null controllable.

Example 2. Porous media equation. Let us consider the porous media equation

$$\begin{aligned} y_t - \Delta\beta(y) &= u, \text{ in } (0,\infty) \times \Omega, \\ y &= 0 \text{ on } (0,\infty) \times \partial\Omega, \\ y(0) &= y_0, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \beta &: \mathbb{R} \to \mathbb{R},\ \beta \in C^2(\mathbb{R}),\ \beta(0) = 0, \\ 0 &< a_0 \leq \beta'(r) \leq c_1 |r|^\kappa + c_2, \text{ for } r \in \mathbb{R},\ c_1, c_2 > 0,\ 0 \leq \kappa < 1. \end{aligned} \tag{6.9}$$

The hypothesis for κ places the equation in the slow diffusion case. We study problem $(\mathcal{P})$ for $u \in L^\infty(0,\infty; H^{-1}(\Omega))$, $\|u(t)\|_{H^{-1}(\Omega)} \leq \rho$, a.e. $t \geq 0$.

Proposition 6.2. *Let $y^{\text{target}} \in H_0^1(\Omega)$, $\Delta\beta(y^{\text{target}}) \in H^{-1}(\Omega)$, $y_0 \in H_0^1(\Omega)$, $\int_0^{y_0} \beta(s)ds \in L^1(\Omega)$. Then, there exists T^*, u^* and y^* solution to $(\mathcal{P})$ satisfying* (5.40)-(5.41)*, where U, H, V are chosen below and $p \in C_w([0,T^*]; H^{-1}(\Omega)) \cap L^2(0,T^*; L^2(\Omega))$ is the solution to*

$$\begin{aligned} -p_t - \Delta(\beta(y^*)p) &= 0, \textit{ in } (0,T^*) \times \Omega, \\ p &= 0 \textit{ on } (0,T^*) \times \partial\Omega, \\ p(T^*) &\in H^{-1}(\Omega). \end{aligned} \tag{6.10}$$

Proof. The proof is led in three steps. First, we prove an intermediate result for β having the properties

$$0 < a_0 \leq \beta'(r) \leq M_1,\ |\beta''(r)| \leq M_2, \text{ for all } r \in \mathbb{R}. \tag{6.11}$$

Then, we consider (6.8) by replacing β by the Yosida approximation β_ν which has the properties (6.11) and obtain the minimum time controllability for the approximating solution y_ν. Third, we pass to the limit as $\nu \to 0$. To this end, we choose

$$D_H = H_0^1(\Omega),\ V = L^2(\Omega),\ H = H^{-1}(\Omega) \equiv (H^{-1}(\Omega))^*,\ V^* = (L^2(\Omega))^*,$$

where $(L^2(\Omega))^*$ is the dual of $L^2(\Omega)$ in the pairing with $H^{-1}(\Omega)$ as pivot space. Moreover,

$$P = I,\ B = I,\ U = H^{-1}(\Omega) = U^* \text{ and } \Gamma_H : D_H \subset H \to H,\ \Gamma_H = -\Delta.$$

We define the operator $A : V \to V^*$ by

$$\langle Ay, \psi \rangle_{V^*,V} = (\beta(y), \psi)_V,\ \text{for } y, \psi \in V = L^2(\Omega),$$

and $A_H : D_H \subset H \to H$ by $A_H y = -\Delta\beta(y)$.

The norm on $V^* = (L^2(\Omega))^*$ is given by $(\theta, \theta)_{V^*} = \|\psi\|_{L^2(\Omega)}$, where $\theta = A\psi$.

The controllability (c_1) follows by Proposition 7.1. We begin to check the hypotheses of Corollary 5.6. First,

$$\langle Ay - A\overline{y}, y - \overline{y}\rangle_{V^*,V} = (\beta(y) - \beta(\overline{y}), y - \overline{y})_V \geq a_0 \|y - \overline{y}\|_V^2,$$

which implies the coercivity, too. Then,

$$(A_H y, \Gamma_H y)_H = \langle -\Delta\beta(y), y\rangle_{H,D_H} = \int_\Omega \beta'(y) |\nabla y|^2 \, dx \geq a_0 \|y\|_{D_H}^2.$$

We have $A'_H(y)z = -\Delta(\beta'(y)z)$ and $(A'(y))^* z = -\Delta(\beta'(y)z)$, where $\beta'(y)p \in V = L^2(\Omega)$. Next,

$$\|A'(y)z\|_{V^*} = \|\beta'(y)z\|_V \leq M_1 \|z\|_V, \text{ for } y, z \in V = L^2(\Omega),$$

and if $y_n \to y$ strongly in $V = L^2(\Omega)$, we have

$$\|A'(y_n)z - A'(y)z\|_{V^*} = \|(\beta'(y_n) - \beta'(y))z\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

This follows by Lebesgue dominated convergence theorem since $\beta'(y_n)z \to \beta'(y)z$ a.e. on Ω and $|(\beta'(y_n) - \beta'(y))z|_{L^2(\Omega)} \leq 2M_1 |z|$. Then, since we can write $\beta(r) = a_0 r + \beta'_1(r)$ we have

$$\langle A'(y)z, \Gamma_\nu(z)\rangle_{V^*,V} \geq \|\Gamma_\nu z\|_H^2 - (\beta'_1(y)z, \Gamma_\nu z)_{L^2(\Omega)} \geq C_1 \|\Gamma_\nu z\|_H^2 - C_2 \|z\|_{L^2(\Omega)}^2.$$

Finally, we have to check (5.54), that is

$$(-\Delta(\beta'(y)z), z)_H = \langle -\Delta(\beta'(y)z), z\rangle_{V^*,V} = \int_\Omega \beta'(y) z^2 dx \geq a_0 \|z\|_V^2,$$

while (5.55) which is automatically verified. Thus, we get a minimum time and a controller satisfying the thesis of Corollary 5.6.

In the second step we replace β by β_ν in (6.8). Both β'_ν and β''_ν are bounded by constants C_ν, for each $\nu > 0$. On the basis of the previous result we obtain that there exists T^*_ν, u^*_ν and y^*_ν satisfying

$$Pu^*_\nu(t) \in (N_K)^{-1}(-B^* p_\nu(t)), \text{ a.e. } t \in (0, T^*_\nu), \tag{6.12}$$

$$\rho \|p_\nu(t)\|_{U^*} + \int_\Omega \beta_\nu(y^*_\nu(t) p_\nu(t) dx = 1, \text{ a.e. } t \in (0, T^*_\nu), \tag{6.13}$$

where y^*_ν is the solution to the approximating state system (6.8) (with β_ν), corresponding to (T^*_ν, u^*_ν), and p_ν is a solution to

$$-p'_\nu(t) - \Delta(\beta'_\nu(y^*_\nu) p_\nu) = 0, \text{ a.e. } t \in (0, T^*_\nu), \tag{6.14}$$

$$\|p_\nu(T^*_\nu)\|_{H^{-1}(\Omega)} \leq C. \tag{6.15}$$

A first estimate for y^*_ν reads

$$\|y^*_\nu\|_{L^\infty(0,T^*_\nu;L^2(\Omega)) \cap L^2(0,T^*_\nu;H^1_0(\Omega))} \leq C, \tag{6.16}$$

where C denote several constants. By multiplying the approximating equation (6.8) by $\beta_\nu(y^*_\nu(t))$ and integrating on $(0, t)$ we obtain

$$\int_\Omega j_\nu(y^*_\nu(t)) dx + \int_0^t \|\nabla\beta_\nu(y^*_\nu(\tau))\|_{L^2(\Omega)}^2 d\tau \leq \int_\Omega j_\nu(y_0) dx + \int_0^t \|u_\nu(\tau)\|_{H^{-1}(\Omega)} \|\beta_\nu(y^*_\nu(\tau))\|_{H^1_0(\Omega)} d\tau,$$

where $\partial j_\nu(r) = \beta_\nu(r)$ and $\partial j(r) = \beta(r)$ for all $r \in \mathbb{R}$. This implies

$$\int_\Omega j_\nu(y^*_\nu(t)) dx + \int_0^t \|\nabla\beta_\nu(y^*_\nu(\tau))\|_{L^2(\Omega)}^2 d\tau \leq C\left(\int_\Omega j(y_0) dx + T^*_\nu \rho^2\right). \tag{6.17}$$

Since $j(r) = \int_0^r \beta(s)ds$ and $j(y_0) \in L^1(\Omega)$ it follows that the right-hand side in (6.17) is bounded independently of ν. This yields

$$\|\beta_\nu(y^*_\nu)\|_{L^2(0,T^*_\nu;H^1_0((\Omega))} \le C. \tag{6.18}$$

Then, we multiply (6.14) by $p_\nu(t)$ and integrate over (t, T^*_ν), getting

$$\|p_\nu\|_{L^\infty(0,T^*_\nu;H^{-1}(\Omega))\cap L^2(0,T^*_\nu;L^2(\Omega))} \le C. \tag{6.19}$$

Next, we determine an estimate for $A'(y^*_\nu)p_\nu$ and begin by computing

$$\begin{aligned}\int_\Omega |\beta'_\nu(y^*_\nu(t)p_\nu(t))|^q\,dx &\le C_1\left(\int_\Omega |p_\nu(t)|^2\,dx\right)^{q/2}\left(\left(\int_\Omega |y^*_\nu(t)|^{\kappa q'}\,dx\right)^{1/q'}+1\right)\\ &\le C_1\,\|p_\nu(t)\|^q_{L^2(\Omega)}\,(\|y_\nu(t)\|^\kappa_{\kappa q'}+1),\ \text{for a.e. } t,\end{aligned}$$

where $\frac{1}{q'} = 1 - \frac{q}{2}$, that is $q' = \frac{2}{2-q}$, for $1 < q < 2$ and $\kappa q' = \frac{2\kappa q}{2-q} \le 2$, meaning that $q \le \frac{2}{\kappa+1}$, which is true if $\kappa < 1$. Therefore, by (6.16) and (6.19) we obtain

$$\|\beta'_\nu(y^*_\nu)p_\nu\|_{L^q(0,T^*_\nu;L^q(\Omega))} \le C_2,\ 1 < q < 2. \tag{6.20}$$

This implies that

$$\|\Delta\beta'_\nu(y^*_\nu)p_\nu\|_{L^q(0,T^*_\nu;X)} + \|p'_\nu\|_{L^q(0,T^*_\nu;X)} \le C_3 \tag{6.21}$$

where X is the image of $L^q(\Omega)$ by the operator $-\Delta$. More precisely, X is the completion of $L^q(\Omega)$ in the norm $|||w|||_X = \left\|A^{-1}w\right\|_{L^q(\Omega)}$. Moreover, applying the same argument as in Theorem 5.5 we can deduce that $T^*_\nu \to T^*$, and on a subsequence, it follows that

$$y^*_\nu \to y^* \text{ weakly in } W^{1,2}(0,T^*;H^{-1}(\Omega)) \cap L^2(0,T^*;H^1_0(\Omega)),\ \text{strongly in } L^2(0,T^*;L^2(\Omega)),$$

$$\beta_\nu(y^*_\nu) \to \beta(y^*) \text{ strongly in } L^2(0,T^*;L^2(\Omega)),$$

since $y_\nu \to y$ strongly, $\beta_\nu(y^*_\nu) \to \eta$ weakly in $L^2(0,T^*_\nu;L^2(\Omega))$ and β is strongly-weakly closed. Then,

$$\begin{aligned}p_\nu \quad \to \quad & p \text{ weakly in } W^{1,2}(0,T^*;X) \cap L^2(0,T^*;L^2(\Omega)),\\ & \text{weak-star in } L^\infty(0,T^*;H^{-1}(\Omega)),\ \text{strongly in } L^2(0,T^*;H^{-1}(\Omega)),\end{aligned} \tag{6.22}$$

and

$$\beta'_\nu(y^*_\nu)p_\nu \to \zeta = \beta'(y^*)p \text{ weakly in } L^q(0,T^*;L^q(\Omega)). \tag{6.23}$$

Indeed, $\beta'_\nu(y^*_\nu) \to \beta'(y^*)$ a.e., $\int_0^{T^*_\nu} \|\beta'_\nu(y^*_\nu(t))\|_{L^2(\Omega)}\,dt \le \int_0^{T^*_\nu} \|y^*_\nu(t)\|^2_{H^1_0(\Omega)}\,dt \le C$ and so

$$\beta'_\nu(y^*_\nu) \to \beta'(y^*) \text{ weakly in } L^2(0,T^*;L^2(\Omega)).$$

Then, by (6.22)

$$\beta'_\nu(y^*_\nu)p_\nu \to \beta'(y^*)p \text{ weakly in } L^1(0,T^*;L^1(\Omega))$$

and choosing $\varphi \in C_0^\infty((0,T)\times\Omega)$ with $T > T^*$ we have

$$\int_0^{T^*_\nu}\int_\Omega (\beta'_\nu(y^*_\nu)p_\nu - \beta'(y^*)p)\varphi dxdt \to 0,$$

which yields that $\zeta = \beta'(y^*)p$ a.e. Thus, (6.23) holds true. Now, we can pass to the limit in (6.14) and (6.15) to get (6.10) and in (6.13) to deduce

$$\rho\,\|B^*p(t)\|_{H^{-1}(\Omega)} + (Ay^*(t),p(t))_{H^{-1}(\Omega)} = 1,\ \text{a.e. } t \in (0,T^*).$$

Finally, we pass to the limit in (6.12), written as $-B^*p_\nu \in N_{\mathcal{K}_{T^*}}(Pu^*_\nu)$, taking into account that $Pu^*_\nu \to Pu^*$ weakly in $L^2(0,T^*;H^{-1}(\Omega))$, $-B^*p_\nu \to -B^*p$ strongly in $L^2(0,T^*;H^{-1}(\Omega))$, and $N_{\mathcal{K}_{T^*}}$ is weakly-strongly closed. Here, $\mathcal{K}_{T^*} = \{w \in L^2(0,T^*;H^{-1}(\Omega));\ \|w(t)\|_{H^{-1}(\Omega)} \le \rho \text{ a.e. } t\}$. □

Example 3. Sliding mode control for reaction-diffusion systems with nonlinear perturbations. Let us consider the system

$$\begin{aligned} y_t - D_1\Delta y + f(y,z) &= u, && \text{in } (0,\infty)\times\Omega, \\ z_t - D_2\Delta z + g(y,z) &= 0, && \text{in } (0,\infty)\times\Omega, \\ \frac{\partial y}{\partial \nu} &= \frac{\partial z}{\partial \nu} = 0, && \text{on } (0,\infty)\times\partial\Omega, \\ y(0) &= y_0,\ z(0) = z_0, && \text{in } \Omega. \end{aligned} \tag{6.24}$$

For certain expressions of f and g, equation (6.24) can model different reaction-diffusion processes, as for instance the diffusion, in a habitat Ω, of two populations with the densities y and z, interacting between them according to the laws expressed by f and g.

In some situations, (2.14) can be satisfied and so one can control the first component of the state y, with one controller, letting z uncontrolled. In this example we shall focus on the situation when $V \subset U$ and prove the minimum time sliding mode control for this system.

Case I. Let us consider that f, g are generally nonlinear, $f,\ g \in C^2(\mathbb{R}\times\mathbb{R})$, such that

$$\sup_{(r_1,r_2)\in\mathbb{R}\times\mathbb{R}} (|\nabla f(r_1,r_2)| + |\nabla g(r_1,r_2)|) \le M \tag{6.25}$$

and $D_i > 0$, $i = 1, 2$. We study problem $(\mathcal{P})$ with $U = L^4(\Omega)$. We set

$$\begin{aligned} H &= L^2(\Omega)\times L^2(\Omega),\ V = H^1(\Omega)\times H^1(\Omega),\ V^* = (H^1(\Omega))^*\times(H^1(\Omega))^*, \\ D &= \left\{ w\in H^2(\Omega);\ \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega \right\},\ D_H = D\times D, \\ U &= (L^4(\Omega), L^4(\Omega)),\ U^* = (L^{4/3}(\Omega), L^{4/3}(\Omega))),\ (B(u,u_2) = (u,0), \\ P(y,z) &= (y,0),\ y^{tar} = (y_1^{\text{target}}, z_1),\ \forall z_1 \in L^2(\Omega), \end{aligned}$$

$$\begin{aligned} A\ &:\ V\to V^*,\ \langle A(y,z),(\psi_1,\psi_2)\rangle_{V^*,V} = D_1\int_\Omega \nabla y\cdot\nabla\psi_1 dx + D_2\int_\Omega \nabla z\cdot\nabla\psi_2 dx \\ &+ \int_\Omega (f(y,z)\psi_1 + g(y,z)\psi_2)dx,\ \text{for } (y,z)\in V, \end{aligned}$$

and $A_H : D_H \subset H \to H$,

$$A_H w = \begin{bmatrix} -D_1\Delta y + f(y,z) \\ -D_2\Delta z + g(y,z) \end{bmatrix},\ w = (y,z).$$

In this case,

$$\mathbf{\Gamma}_H : D_H \subset H \to H,\ \mathbf{\Gamma}_H(y,z) = ((I-\Delta)y, (I-\Delta)z) = (\Gamma_H y, \Gamma_H z),$$

with the homogeneous Neumann boundary condition. The controllability (c_1) can follow as in [8]. Namely, first it is proved that there exists a controller $u \in L^\infty((0,\infty)\times\Omega)$ and T_*, such that for ρ large enough $y(T_*) = y_1^{\text{target}}$, where T_* depends on ρ. This controller belongs also to $L^\infty(0,\infty;L^4(\Omega))$ but the controllability follows with a different ρ calculated from a relation between the norms in $L^4(\Omega)$ and $L^\infty(\Omega)$. Moreover, the time T_* is smaller as ρ is greater.

Let $K = \{u \in U = L^4(\Omega)\times L^4(\Omega);\ \|u\|_U \le \rho\}$.

Proposition 6.3. *Assume* (6.25) *and* $y_0 \in H^1(\Omega)$, $y_1^{\text{target}} \in D$, $\Delta y_1^{\text{target}} \in H^1(\Omega)$. *Then, there exists* T^*, u^* *solution to* $(\mathcal{P})$ *satisfying* (5.40)-(5.41),

$$Pu^*(t) \in (N_K)^{-1}(-p(t)),\ \text{a.e. } t\in(0,T^*),$$

$$\rho\,\|p(t)\|_{U^*} + (A_H y^*(t), v(t))_H = 1,\ \text{a.e. } t\in(0,T^*),$$

where $v=(p,q)$ *is the solution to*

$$\begin{array}{rcll} -p_t - D_1\Delta p + f_y(y^*,z^*)p + g_y(y^*,z^*)q &=& 0, & \textit{in } (0,\infty)\times\Omega, \\ -q_t - D_2\Delta q + f_z(y^*,z^*)p + g_z(y^*,z^*)q &=& 0,, & \textit{in } (0,\infty)\times\Omega, \\ \dfrac{\partial p}{\partial\nu} &=& \dfrac{\partial q}{\partial\nu} = 0, & \textit{on } (0,\infty)\times\partial\Omega, \\ p(T^*) &\in& V^*,\ q(T^*)=0,\ \textit{in } \Omega. & \end{array}$$

Moreover, $y(t) = y_1^{\text{target}}$ *for* $t > T^*$.

Proof. It is obvious that A is continuous, monotone and coercive and (2.1)-(2.3), (a_2) are satisfied. We have for $w := (y,z)$ and $v := (p,q)$

$$A'_H(y,z) = \begin{bmatrix} -D_1\Delta + f_y(y,z) & f_z(y,z) \\ g_y(y,z) & -D_2\Delta + g_z(y,z) \end{bmatrix},$$

$$(A'_H(y,z))^* = \begin{bmatrix} -D_1\Delta + f_y(y,z) & g_y(y,z) \\ f_z(y,z) & -D_2\Delta + g_z(y,z) \end{bmatrix},$$

where f_y, f_z, g_y, g_z denote the partial derivatives of f and g with respect to their arguments and they belong to $L^\infty(\mathbb{R}\times\mathbb{R})$. Then,

$$(A_H(y,z), \mathbf{\Gamma}_H(y,z))_H \geq C(\|\Gamma_H y\|_2^2 + \|\Gamma_H z\|_2^2 - C_1\|(y,z)\|_V^2$$

and it is easy to see that $(a_3)-(a_5)$ are satisfied. Regarding (a_6) or (2.8) we have for $v=(p,q)$,

$$\langle (A'_H(y,z))^* v, \mathbf{\Gamma}_\nu v\rangle_{V^*,V} \geq \|\mathbf{\Gamma}_\nu v\|_H^2 - \|v\|_H \|\mathbf{\Gamma}_\nu v\|_H \geq C\|\mathbf{\Gamma}_\nu v\|_H^2 - C_1\|v\|_V^2,$$

where $\mathbf{\Gamma}_\nu(p,q) = (\Gamma_\nu p, \Gamma_\nu q)$.

Next, we verify the hypotheses in Section 5.3. Relation (2.12) is satisfied because

$$\|Pv\|_{V^*} = \|p\|_{(H^1(\Omega))^*} \leq C^*\|p\|_{4/3} = C^*\|B^*p\|_{U^*},$$

since $H^1(\Omega)\subset L^4(\Omega)$ for $d\leq 3$. Also,

$$\left\|P\mathbf{\Gamma}_H^{-\alpha/2}v\right\|_H = \left\|\Gamma_H^{-\alpha/2}p\right\|_H \leq C\|p\|_{H^{-\alpha}(\Omega)} \leq C_1\|p\|_{4/3} = C_1\|B^*p\|_{U^*}, \text{ for } v\in H,$$

$$\begin{array}{rcl} \left(A'_H(y)v, \mathbf{\Gamma}_H^{-1}v\right)_H &\geq& C\|v\|_H^2 - \|f_y(y,z)p\|_H \left\|\Gamma_H^{-1}p\right\|_H - \|g_z(y,z)q\|_H\left\|\Gamma_H^{-1}q\right\|_H \\ &\geq& C\|v\|_H^2 - M\|v\|_H\|v\|_{V^*} \geq C_1\|v\|_H^2 - C_2\|v\|_{V^*}^2 \end{array}$$

and

$$\left(A'_H(y)v, \mathbf{\Gamma}_H^{-\alpha}v\right)_H \geq C\left\|\mathbf{\Gamma}_H^{(1-\alpha)/2}v\right\|_H^2 - C_1\|v\|_H^2,$$

where $\mathbf{\Gamma}_H^{-\alpha}(y,z) = (\Gamma_H^{-\alpha}y, \Gamma_H^{-\alpha}z)$. Here we used the estimate

$$\left\|\Gamma_H^{-\alpha/2}p\right\|_H = \|p\|_{H^{-\alpha}(\Omega)} \leq \|v\|_H$$

since $L^2(\Omega)\subset H^{-\alpha}(\Omega)$. Moreover, $D(\Gamma_H^{-\alpha/2}) = H^{-\alpha}(\Omega)$, according to the characterization of the domains of the fractionary powers of $-\Delta$ given in [14], Theorem 2, for $\alpha<1$.

Now, we have to check (2.15) and (2.14). To this end, we assume that $\Delta y_1^{\text{target}}\in H^1(\Omega)$ and choose $\widehat{z}$ to be exactly the second component z of the solution to (6.24). First, we prove that it has the necessary regularity. We recall (3.2) and (3.6), that is

$$\|(y,z)(t)\|_V^2 + \|(y,z)\|_{L^2(0,T;H)\cap L^2(0,T;D_H)} \leq C\left(\|(y_0,z_0)\|_V^2 + \int_0^t \|u(\tau)\|_U^2\,d\tau\right)e^{Ct},$$

for all $t \in [0,T]$, $T > 0$. Since $y_t \in L^2(0,T;H)$ we expect to have a more regular z. We consider the equation

$$w_t - D_2 \Delta w + g_z(y,z)w = -g_y(y,z)y_t, \tag{6.26}$$

with $\frac{\partial w}{\partial \nu} = 0$ and $w(0) = z_t(0) = D_2\Delta z_0 - g(y_0, z_0) \in P(V) = H^1(\Omega)$. This is computed directly from the equation for z, observing that by the hypotheses for g and the initial data we have $g(y_0 z_0) \in (I-P)(V) = H^1(\Omega)$. We note that (6.26) represents also the equation for z_t, obtained by formally differentiating the equation in z. Since in (6.26) all coefficients on the left-hand side are in $L^\infty(\Omega)$ and $g_y(y,z)y_t \in L^2(0,T;H)$, it follows that it has an unique solution

$$w \in C_w([0,T];V) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap W^{1,2}(0,T;H)$$

and so we can deduce that $w = z_t$ and that z_t belongs to the same spaces. Moreover, by multiplying (6.26) by w_t we obtain, by some calculations similar to those in Theorem 3.2, that

$$\|z_t(t)\|_{C_w([0,T];V)\cap L^2(0,T;W)\cap W^{1,2}(0,T;H)} \leq C_T^1, \text{ for all } t \in [0,T],$$

where this constant depends also on $z_t(0) \in P(V)$, namely on $\|\Delta z_0\|_V$,

$$C_T^1 := C\left(\|(y_0\|^2_{H^1(\Omega)} + \|\Delta z_0\|_{H^1(\Omega)} + T\rho^2\right)e^{CT}.$$

Going back to the equation in z we have

$$-D_2\Delta z(t) + g(y(t), z(t)) = -z_t(t) \in V,$$

because $\|g(y(t),z(t))\|_{H^1(\Omega)} \leq C_T$. Indeed, e.g., $\|g_y(y(t),z(t))\nabla y(t)\|_{H^1(\Omega)} \leq MC_T$, by (3.2). Next, in the same way we see that if $\Delta y_1^{\text{target}} \in H^1(\Omega)$, then $f(y_1^{\text{target}}, z(t)) \in H^1(\Omega)$ and so $-D_1\Delta y_1^{\text{target}} + f(y_1^{\text{target}}, z(t)) \in H^1(\Omega)$. Moreover,

$$\begin{aligned} &\left\|-D_1\Delta y_1^{\text{target}} + f(y(t),z(t))\right\|_{H^1(\Omega)} + \|-D_2\Delta z(t) + g(y(t),z(t))z(t)\|_{H^1(\Omega)} \\ \leq\ & C_T + C_T^1 \leq C(C_0 + \sqrt{T}\rho)e^{CT},\ t \geq 0. \end{aligned}$$

Here, $C_0 = \|y_0\|_{H^1(\Omega)} + \|\Delta z_0\|_V$. This implies that

$$\left\|A(y_1^{\text{target}}, \widehat{z})\right\|_V \leq C(C_0 + \sqrt{T}\rho)e^{CT}.$$

In order to satisfy (2.15) we have to impose that

$$\rho > C^* C(C_0 + \sqrt{T}\rho)e^{CT}. \tag{6.27}$$

We can check that if

$$\sqrt{T}e^{CT} < \frac{1}{C} \tag{6.28}$$

then

$$\rho > \frac{CC_0C^*e^{CT}}{1 - C\sqrt{T}e^{CT}} \tag{6.29}$$

and consequently (6.27) are satisfied. We note that, for any positive constant C, the equation $\sqrt{T}e^{CT} = \frac{1}{C}$ has a unique solution T_{**} and so any $T \in [0, T_{**})$ verifies (6.28). We can choose ρ sufficiently large, such that the time T_* in hypothesis (c_1) becomes smaller enough, such that to remain in $(0, T_{**})$. We have to check (2.14), that is

$$\begin{aligned} &\left\langle A(y(t),z(t)) - A(y_1^{\text{target}}, \widehat{z}), y - y_1^{\text{target}}\right\rangle_{V^*,V}, \\ =\ & D_1\left\|\nabla(y(t) - y_1^{\text{target}})\right\|_{L^2(\Omega)} + \int_\Omega f(y(t),z(t)) - f(y_1^{\text{target}}, \widehat{z}))(y(t) - y_1^{\text{target}})dx \\ \geq\ & D_1\left\|\nabla(y(t) - y_1^{\text{target}})\right\|_{L^2(\Omega)} - L_f\left\|y(t) - y_1^{\text{target}}\right\|^2_{L^2(\Omega)}, \end{aligned}$$

which is true for any $t \geq 0$, in particular for $t \in (0, T_* + \delta)$.

Finally, we prove that $Py(t) = y_1^{\text{target}}$ for $t \geq T^*$. Let us denote the solution to (6.24) for $t \geq T^*$ by $(\widetilde{y}(t), \widetilde{z}(t))$. Then, it satisfies

$$\begin{aligned} \widetilde{y}_t - D_1\Delta\widetilde{y} + f(\widetilde{y}, \widetilde{z}) &= u, \qquad \text{in } (T^*, \infty) \times \Omega, \\ \widetilde{z}_t - D_2\Delta\widetilde{z} + g(\widetilde{y}, \widetilde{z}) &= 0, \qquad \text{in } (T^*, \infty) \times \Omega, \end{aligned} \tag{6.30}$$

with homogeneous Neumann boundary conditions and the initial data at $t = T^*$, $\widetilde{y}(T^*) = y_1^{\text{target}}$, $\widetilde{z}(T^*) = z(T^*)$. The second equation with $\widetilde{y} = y_1^{\text{target}}$ has a unique solution well defined. If u is replaced in (6.30) by

$$u(t) = \begin{cases} u^*(t), & t \leq T^* \\ \widetilde{u}(t), & t > T^* \end{cases}$$

where $\widetilde{u}(t) = -D_1\Delta y_1^{\text{target}} + f(y_1^{\text{target}}, \widetilde{z}(t))$, then $(\widetilde{y}(t), \widetilde{z}(t)) = (y_1^{\text{target}}, \widetilde{z}(t))$ verifies the first equation and this proves that the solution slides on the manifold y_1^{target} for all $t \geq 0$. Thus Theorem 5.5 can be applied to obtain the conclusion of the Proposition 6.3. □

Case II. We can also put into evidence a particular case in which the choice of $\widehat{z}$ is independent of u, T and the system solution. Let us assume that $y_1^{\text{target}} = 0$, and

$$f(0, z) = 0 \text{ for all } z, \ \ f(y, z)y \geq 0 \text{ for all } (y, z) \in H. \tag{6.31}$$

We have to check (2.14). We set $\widehat{y} = (0, \widehat{z})$, where $\widehat{z}$ in this case can be taken any value such that $A(0, \widehat{z}) \in V$, in particular $\widehat{z} = 0$. We have

$$\langle A(y, z) - A(0, \widehat{z}), y\rangle_{V^*,V} = (-D_1\Delta y + f(y, z), y))_H \geq D_1 \|\nabla y\|_H^2 \geq 0.$$

A particular situation is $f(y, z) = yf_2(z)$, with $f_2(z)$ Lipschitz and positive, for example $f_2(z) = \frac{z^2}{1+z^2}$.

Case III. Reaction-diffusion systems with linear perturbations. Let us consider (6.24) with $f(y, z) = a_1y + b_1z$ and $g(y, z) = a_2y + b_2z$. The functional framework is the same as in the precedent example and all hypotheses are satisfied. We shall check only (d_4), by setting $\widehat{y} = (y_1^{\text{target}}, z)$, where z is the solution to (6.24) corresponding to T and u. We have

$$\begin{aligned} &\langle A(y, z) - A(\overline{y}, \widehat{z}), y - \overline{y}\rangle_{V^*,V}, \\ = \ &(-D_1\Delta(y - \overline{y}) + a_1(y - \overline{y}) + b_1(z - z), (y - \overline{y})))_H \geq C_1 \|(y - \overline{y})\|_V^2 \geq 0. \end{aligned}$$

Case IV. FitzHugh-Nagumo reaction-diffusion model. For $f(r_1, r_2) = \alpha_0r_1 + r_2$, $g(r_1, r_2) = -\sigma r_1 + \gamma r_2$ and $D_2 = 0$, the system (6.24) becomes the well-known FitzHugh-Nagumo model (studied e.g. in [16]). In this case, the hypotheses are verified with the choice

$$\begin{aligned} H &= L^2(\Omega) \times L^2(\Omega), \ V = H^1(\Omega) \times L^2(\Omega), \ V^* = (H^1(\Omega))^* \times L^2(\Omega), \\ D_H &= \left\{ y \in H^2(\Omega); \ \frac{\partial y}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} \times L^2(\Omega). \end{aligned}$$

Example 4. Phase field systems. Let us consider the phase-field system of Caginalp type, for the phase function φ and the energy σ written in the following form (see e.g., [8])

$$\sigma_t - k\Delta\sigma + kl\Delta\varphi = f + u, \ \text{in } (0, \infty) \times \Omega,$$

$$\varphi_t - \nu\Delta\varphi + \beta(\varphi) + \pi(\varphi) = \gamma\sigma - \gamma l\varphi, \ \text{in } (0, \infty) \times \Omega,$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0, \ \text{in } (0, \infty) \times \partial\Omega,$$

$$\varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0, \ \text{in } \Omega$$

and study $(\mathcal{P})$ with $U=(L^2(\Omega),L^2(\Omega))$, $Pu=(u,0)$, $B(u,u_2)=(u,0)$ and the rest of the spaces as in Example 3. Here $\beta(r)=r^3$ and $\pi(r)=-r$ for $r\in\mathbb{R}$, the function $\beta+\pi$ representing the double-well potential. The controllability hypothesis (c_1) can be proved in a similar way with the proof of [8] for the case $u\in L^\infty(0,T;L^2(\Omega))$, $\|u(t)\|_2\le\rho$ a.e. t. The regularity of the second state component φ is proved as in the precedent example and so $\widehat{z}:=\varphi(t)$ which is the appropriate choice for checking (2.15) and (2.14). We mention that the proof of the minimum time for the Caginalp system with the singular logarithmic potential is considered in [10].

Example 5. Diffusion with nonlocal controllers. We note that the theory works too if B is a nonlocal operator. This is the case when $B_iu_i\neq u_i$. Let us consider, for instance Example 1, where $B:L^2(\Omega_1)\to L^2(\Omega)$ is defined by

$$(Bu)(z)=\int_{\Omega_1}K(x,z)u(z)dz,\ x\in\Omega$$

and $K\in L^2(\Omega\times\Omega_1)$. If the kernel K is such that

$$\|B^*v\|_{L^2(\Omega_1)}\ge\gamma\|v\|_{L^2(\Omega)},\ \text{for } v\in L^2(\Omega),$$

it follows that the controllability assumption holds and all conditions are satisfied. Here, $B^*:L^2(\Omega)\to L^2(\Omega_1)$ is defined by $B^*v(x)=\int_\Omega K(x,z)v(x)dx$, because

$$\begin{aligned}(Bu,v)_{L^2(\Omega)} &= \int_\Omega v(x)\int_{\Omega_1}K(x,z)u(z)dzdx\\ &= \int_{\Omega_1}u(z)\int_\Omega K(x,z)v(x)dxdz=(u,B^*v)_{L^2(\Omega_1)}.\end{aligned}$$

7 Appendix

Some definitions and results related to operators in Hilbert spaces. Let H, V be Hilbert spaces, V^* the dual of V, $V\subset H\subset V^*$ with compact injections. Let $A:V\to V^*$.

The operator A is *demicontinuous* if $y_n\to y$ strongly in V implies $Ay_n\to Ay$ weakly in V^*, as $n\to\infty$.

Let $\langle\cdot,\cdot\rangle_{V^*,V}$ denote the pairing between V^* and V. The operator A is *coercive* if

$$\lim_{\|y\|_V\to\infty}\frac{\langle Ay,y-y^0\rangle_{V^*,V}}{\|y\|_V}=+\infty,\ \text{for some } y^0\in V.$$

Let A be an operator on the Hilbert space H. It is called *m-accretive* if it is *accretive*,

$$(Ay-A\overline{y},y-\overline{y})_H\ge 0,\ \text{for all } y,\overline{y}\in D(A)$$

and m-accretive if $R(I+A)=H$, where I is the identity operator and R is the range. The operator A is *quasi m-accretive* if $(\lambda I+A)$ is m-accretive for λ sufficiently large.

The operator $A_H:D(A_H)\subset H\to H$ is the *restriction* of A on H defined as $A_Hy=Ay$ for $y\in D(A_H)=\{y\in V;\ Ay\in H\}$.

Let $A:V\to V^*$ be single-valued, monotone, demicontinuous and coercive. Then, it follows that it is *surjective* (see e.g. [6], p. 36, Corollary 2.2) and A_H is m-accretive on $H\times H$.

Let $A\in C^1(V,V^*)$. The Gâteaux derivative of A is the linear operator $A'(y):V\to V^*$ defined by

$$A'(y)z=\lim_{\lambda\to 0}\frac{A(y+\lambda z)-Ay}{\lambda}\ \text{strongly in } V^*,\ \text{for all } y,z\in V. \tag{7.1}$$

If $A_H\in C^1(D(A_H),H)$ we similarly define $(A_H)'(y):D(A_H)'=D(A_H)\subset H\to H$ by

$$(A_H)'(y)z=\lim_{\lambda\to 0}\frac{A_H(y+\lambda z)-A_Hy}{\lambda}\ \text{strongly in } H,\ \text{for all } y,z\in D(A_H) \tag{7.2}$$

and observe that $(A_H)'(y) = (A')_H(y)$, for $y \in D(A_H)$, so that in the paper we use the notation $A'_H(y)$.

Let X and Y be Banach spaces. The operator $G : X \to Y$ is said to be *strongly continuous* from X to $L_s(X,Y)$ if for $y_n \to y$ strongly in X, as $n \to \infty$ it follows

$$\|G(y_n)\psi - G(y)\psi\|_Y \to 0 \text{ for all } \psi \in X. \tag{7.3}$$

Duality mapping. Let U be Banach spaces with the dual U^* uniformly convex, implying that U^* and U is reflexive (see e.g., [6], p. 2). Also, it follows that the norm in U is Gâteaux differentiable.

Let $F : U \to U^*$ be the duality mapping of U, which is single valued and continuous (see e.g., [6], p. 2, Theorem 1.2). We recall that

$$\langle Fu, u\rangle_{U^*,U} = \|u\|_U^2, \ \ \|Fu\|_{U^*} = \|u\|_U. \tag{7.4}$$

Let $K = \{u \in U; \|u\|_U \le \rho\}$, let I_K be the indicator function of K and define

$$j : U \to \mathbb{R}, \ j(u) = I_K(u). \tag{7.5}$$

Then,

$$\partial j(u) = \partial I_K(u) = N_K(u) = \begin{cases} \{\lambda Fu; \lambda > 0\} & \|u\|_U = \rho \\ 0, & \|u\|_U < \rho \\ \varnothing, & \|u\|_U > \rho \end{cases} \tag{7.6}$$

where $\partial j : U \to U^*$ is the subdifferential of j, N_K is the normal cone to K in U^* and $\lambda > 0$. The first line in (7.6) should be understood in the multivalued sense.

The conjugate of j is $j^* : U^* \to \mathbb{R}$,

$$j^*(z) = \sup_{u\in K}\left\{\langle z, v\rangle_{U^*,U} - j(v)\right\} = \sup\left\{\langle z, v\rangle_{U^*,U};\ \|v\|_U \le \rho\right\} = \rho\,\|z\|_{U^*}. \tag{7.7}$$

Then,

$$\partial j^*(z) = (\partial j)^{-1}(z) = N_K^{-1}(z) = \rho\frac{F^{-1}z}{\|z\|_{U^*}}, \ z \in U^*. \tag{7.8}$$

Since U is reflexive, F^{-1} is just the duality mapping of U^* and so $D(F^{-1}) = U^*$.
If $z \in N_K(u)$, then $u \in N_K^{-1}(z)$ and we have

$$\langle z, u\rangle_{U^*,U} = \rho\,\|z\|_{U^*}. \tag{7.9}$$

Let ε be positive and define

$$j_\varepsilon(u) = \frac{\varepsilon}{2}\,\|u\|_U^2 + I_K(u). \tag{7.10}$$

We recall that the subdifferential

$$\partial\left(\frac{1}{2}\,\|u\|_U^2\right) = Fu, \tag{7.11}$$

whence

$$\partial j_\varepsilon(u) = \varepsilon Fu + N_K(u), \text{ for all } u \in K. \tag{7.12}$$

Then,

$$\begin{aligned} j_\varepsilon^*(\zeta) &= \sup_{v\in K}\left\{\langle \zeta, v\rangle_{U^*,U} - j_\varepsilon(v)\right\} = -\inf_{v\in K}\left\{I_K(v) + \frac{\varepsilon}{2}\,\|v\|_U^2 - \langle \zeta, v\rangle_{U^*,U}\right\} \\ &= \langle \varepsilon Fv_\varepsilon + z_\varepsilon, v_\varepsilon\rangle_{U^*,U} - \frac{\varepsilon}{2}\,\|v_\varepsilon\|_U^2 = \frac{\varepsilon}{2}\,\|v_\varepsilon\|_U^2 + \langle z_\varepsilon, v_\varepsilon\rangle_{U^*,U}. \end{aligned}$$

We specify that in the lines before the infimum is realized at v_ε which is the solution to the equation $\varepsilon Fv_\varepsilon + N_K(v_\varepsilon) \ni \zeta$, that is $\varepsilon Fv_\varepsilon + z_\varepsilon = \zeta$, where $z_\varepsilon \in N_K(v_\varepsilon)$. Therefore,

$$j_\varepsilon^*(\zeta) = \langle \varepsilon Fv_\varepsilon + z_\varepsilon, v_\varepsilon\rangle_{U^*,U} - \frac{\varepsilon}{2}\,\|v_\varepsilon\|_U^2 = \frac{\varepsilon}{2}\,\|v_\varepsilon\|_U^2 + \langle z_\varepsilon, v_\varepsilon\rangle_{U^*,U},$$

implying by (7.9) that

$$j_\varepsilon^*(\zeta) = \frac{\varepsilon}{2}\left\|v_\varepsilon\right\|_U^2 + \rho\left\|z_\varepsilon\right\|_{U^*},\ z_\varepsilon \in N_K(v_\varepsilon).$$

Finally, if $u \in (\varepsilon F + N_K)^{-1}(\zeta)$, it follows that

$$(\partial j_\varepsilon)^{-1}(\zeta) = (\varepsilon F + N_K)^{-1}(\zeta) = \partial j_\varepsilon^*(\zeta) \tag{7.13}$$

and

$$j_\varepsilon^*(\zeta) = \frac{\varepsilon}{2}\left\|u_\varepsilon\right\|_U^2 + \rho\left\|z_\varepsilon\right\|_{U^*},\ z_\varepsilon \in N_K(u_\varepsilon). \tag{7.14}$$

The canonical isomorphism Assume now that H and V are Hilbert spaces, and V has the dual V^*. The *duality mapping*, which we denote by $\Gamma : V \to V^*$ is the *canonical isomorphism* of V onto V^* (see e.g., [6], p. 1). We have

$$\Gamma \in L(V, V^*),\ \ \langle \Gamma v, v\rangle_{V^*,V} = \left\|v\right\|_V^2,\ \ \left\|\Gamma v\right\|_{V^*} = \left\|v\right\|_V. \tag{7.15}$$

In addition, Γ_H, the *restriction* of Γ to H, is m-accretive on $H \times H$, with the *linear domain denoted* D_H which is densely, continuously and compactly embedded in V,

$$D(\Gamma_H) := D_H \subset V. \tag{7.16}$$

For $\nu > 0$, we denote by Γ_ν the Yosida approximation of Γ_H, that is

$$\Gamma_\nu y = \frac{1}{\nu}(I - (I + \nu\Gamma_H)^{-1})y = \Gamma_H(I + \nu\Gamma_H)^{-1}y,\ y \in H. \tag{7.17}$$

Comments on the hypothesis of controllability. Hypothesis (c_1) ensures that the admissible set for problem $(\mathcal{P})$ is not empty. For example, in the case of Caginalp phase field models the proof of the controllability was provided in [8]. Further, we shall argue for the reliability of such an hypothesis, giving a brief proof of the controllability of (1.1)-(1.2) in some cases. First, let us set

$$u(t) = -\rho\,\mathrm{Sign}\,(B^*P(y(t) - y^{tar})),$$

where Sign: $U^* \to 2^{U^*}$ is defined by

$$\mathrm{Sign}\,v = \begin{cases} \frac{v}{\|v\|_{U^*}}, & y \neq 0 \\ B(0,\rho), & y = 0. \end{cases} \tag{7.18}$$

Here, $B(0, \rho)$ is the ball of center 0 and radius ρ in U^*. It is well known that $v \to \mathrm{Sign}\,v$ is m-accretive on U^*.

Let us consider the problem

$$\begin{aligned} y'(t) + Ay(t)) &\ni -\rho B\mathrm{Sign}\,(B^*P(y(t) - y^{tar})),\ \text{a.e. } t \in (0,T) \\ y(0) &= y_0. \end{aligned} \tag{7.19}$$

We refer to the case when one state component is controlled by one controller, that is

$$Py = (y_1, 0),\ U = U_1 \times U_2,\ U_1 = U_1^*,\ B = (B_1, 0),\ B_1 : U_1 \to H_1,$$

and assume

$$R(B_1) = H_1. \tag{7.20}$$

(When $P = I$, $Bu = (B_1u_1, B_2u_2)$, we impose the condition $R(B) = H$. The proof is the same, by replacing H_1 by H.)

Hypothesis (7.20) implies, by the Banach closed range theorem (see [24], p. 208, Corollary 1) that $(B_1^*)^{-1}$ is continuous from U_1 to H_1 and $B_1^{-1} \in L(H_1, U_1)$. This means $\left\|(B_1^*)^{-1}z\right\|_{H_1} \leq C\left\|z\right\|_{U_1}$ for $z \in U_1$, or, equivalently

$$\|w\|_{H_1} \leq C\left\|B_1^* w\right\|_{U_1}, \text{ for } w \in H_1. \tag{7.21}$$

We also assume that

$$(A_H y - A_H y^{tar}, P(y - y^{tar}))_H \geq -C_1 \left\|P(y - y^{tar})\right\|_H^2, \text{ for all } y \in D_H,\ Py^{tar} \in P(D_H). \tag{7.22}$$

It is clear that when $B_1 = I$, then $U_1 = H_1$. Otherwise, we have the situation in Example 5.

Proposition 7.1. *Let y_0, $Py^{tar} \in P(D_H)$ and let (a_1), (a_2), (b_1),* (7.20), (7.22) *and*

$$\rho > \left\|A_H y^{tar}\right\|_H + C_1\left\|P(y_0 - y^{tar})\right\|_H$$

hold. Then, there exists $T_* \in (0, T)$ such that, for ρ large enough, $Py(T_*) = Py^{tar}$, where y is the solution to (7.19).

Proof. The operator $B\mathrm{Sign}\,(B^* Pv)$ is m-accretive on H_1. Indeed, for $v, \bar{v} \in H_1$ we have

$$(B\mathrm{Sign}\,(B^* Pv) - B\mathrm{Sign}\,(B^* P\overline{v}), v - \overline{v})_{H_1} = (\mathrm{Sign}\,(B^* Pv) - \mathrm{Sign}\,(B^* P\overline{v}), B^*(v - \overline{v}))_{H_1} \geq 0,$$

because $B^* Pv = B^* v$, $w \rightarrow \mathrm{Sign}\, w$ is m-accretive and $P^2 = P$. For the m-accretivity let us consider the equation

$$y + \rho B\mathrm{Sign}\,(B^* P(y - y^{tar})) = f \in H_1 \tag{7.23}$$

which, by denoting $z = y - y^{tar}$, is equivalent with $z + \rho B\mathrm{Sign}\,(B^* z) = f - y^{tar}$. We set $B^* z = v \in U_1$ and get

$$B^{-1}(B^*)^{-1}v + \rho \mathrm{Sign}\, v = B^{-1} f_1 \in U_1. \tag{7.24}$$

Denoting $G = B^{-1}(B^*)^{-1}$ we see that $G \in L(U_1, U_1)$ and $(Gv, v)_{U_1} = \left\|(B^*)^{-1}v\right\|_{H_1}^2 \geq \|v\|_{U_1}^2$ (because B^* is continuous, see (7.21)). Now, $\mathrm{Sign}\, w$ is m-accretive on $U_1 \times U_1$, hence $R(G{+}\mathrm{Sign}) = U_1$ (see e.g., [6], p. 44, Corollary 2.6).

Then, we prove that (7.19) has a unique solution.

Since A_H is quasi m-accretive and $S = \rho B\mathrm{Sign}\,(B^* P(y(t) - y^{tar}))$ is m-accretive with $D(S) = H_1$ and $D_H \cap \overset{\circ}{D}(S) = D_H \neq \varnothing$, it follows by that $A_H + S$ is quasi m-accretive on $H \times H$ (see [6], p. 43, Theorem 2.6). Therefore, (7.19) has a unique solution $y \in L^\infty(0, T; D_H) \cap W^{1,\infty}(0, T; H)$ satisfying estimate (3.2) (see the proof of Theorem 3.2).

Now, we can justify the controllability assertion. Let us write (7.19) in the equivalent form

$$(y - y^{tar})_t + A_H y - A_H y^{tar} + \rho B\mathrm{Sign}\,(B^* P(y - y^{tar})) \ni -A_H y^{tar}$$

and multiply it by $P(y(t) - y^{tar})$. By (7.22) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|P(y(t) - y^{tar})\right\|_H^2 - C_1\left\|P(y(t) - y^{tar})\right\|_H^2 + \rho\left(B\mathrm{Sign}\,(B^* P(y(t) - y^{tar})), P(y(t) - y^{tar})\right)_H \\ \leq\ & \left\|A_H y^{tar}\right\|_H \left\|P(y(t) - y^{tar})\right\|_H. \end{aligned}$$

Further, we obtain

$$\begin{aligned} &\left\|P(y(t) - y^{tar})\right\|_H \frac{d}{dt}\left\|P(y(t) - y^{tar})\right\|_H + \rho\left\|B^* P(y(t) - y^{tar})\right\|_{U_1^*} \\ \leq\ & \left\|A_H y^{tar}\right\|_H \left\|P(y(t) - y^{tar})\right\|_H + C_1\left\|P(y(t) - y^{tar})\right\|_H^2. \end{aligned}$$

Next, we use (7.21) for $w = P(y(t) - y^{tar})$ which implies $\|B^* P(y(t) - y^{tar})\|_{U_1} \geq C\left\|P(y(t) - y^{tar})\right\|_{H_1}$ and so

$$\frac{d}{dt}\left\|P(y(t) - y^{tar})\right\|_H - C_1\left\|P(y(t) - y^{tar})\right\|_H + \rho \leq \left\|A_H y^{tar}\right\|_H.$$

For $\rho > \|A_H y^{tar}\|_H$, this yields

$$\left\|P(y(t) - y^{tar})\right\|_H < e^{C_1 t} \left\|P(y_0 - y^{tar})\right\|_H - \frac{(\rho - \|A_H y^{tar}\|_H)}{C_1}(e^{C_1 t} - 1).$$

Finally, we obtain that $t \to \|P(y(t) - y^{tar})\|_H$ is strictly decreasing, vanishes at $t = T_*$ below and the previous relation takes place for

$$t \leq T_* = \ln \frac{\rho - \|A_H y^{tar}\|_H}{\rho - (\|A_H y^{tar}\|_H + C_1 \|P(y_0 - y^{tar})\|_H)},$$

for $\rho > \|A_H y^{tar}\|_H + C_1 \|P(y_0 - y^{tar})\|_H$. We also observe that T^* decreases as ρ increases and in fact $T_* \to 0$ as $\rho \to \infty$. This end the proof. □

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References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] V. Barbu, The dynamic programming equation for the time-optimal control problem in infinite dimensions, SIAM J. Control Optim. 29, 445-456, 1991.

[3] V. Barbu, Analysis and Control of Nonlinear Infinite Dimensional Systems, Academic Press, Boston, 1993.

[4] V. Barbu, Mathematical Methods in Optimization of Differential Systems, Kluwer Academic Publishers, Dordrecht, 1994.

[5] V. Barbu, The time optimal control of Navier-Stokes equations, Systems and Control Letters, 30, 93-100, 1997.

[6] V. Barbu, Nonlinear Differential Equations of Monotone Types in Banach Spaces, Springer, New York, 2010.

[7] V. Barbu, Controllability and Stabilization of Parabolic Equations, Birkhäuser Basel, 2018.

[8] V. Barbu, P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, Sliding mode control for a nonlinear phase-field system, SIAM J. Control Optim. 55, 2108-2133, 2017.

[9] C. Cavaterra, D. Enăchescu, G. Marinoschi, Sliding mode control of the Hodgkin–Huxley mathematical model, Evol. Equ. Control Theory, 8, 883-902, 2019.

[10] P. Colli, G. Gilardi, G. Marinoschi, Sliding mode control in a minimal time for a phase-field system with a singular logarithmic potential, in preparation.

[11] P. Colli, G. Gilardi, G. Marinoschi, E. Rocca, Sliding mode control for phase field system related to tumor growth, Appl. Math.Optimiz. 79, 647–670, 2019.

[12] H.O. Fattorini, Time optimal control of solutions of operational differential equations, SIAM J. Control 2, 54–59, 1964.

[13] H. O. Fattorini, Infinite dimensional linear control systems, the time optimal and norm optimal problems, North-Holland Mathematics Studies, 201, Elsevier Science, B. V., Amsterdam, 2005.

[14] D. Fujiwara, Concrete characterization of the domains o fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad. 43, 82-86, 1967.

[15] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Seoul National University, 1996.

[16] K. Kunisch, L. Wang, Time optimal controls of the linear Fitzhugh–Nagumo equation with pointwise control constraints, J. Math. Anal. Appl. 395, 114–130, 2012.

[17] K. Kunisch, L. Wang, Time optimal control of the heat equation with pointwise control constraint, ESAIM: COCV 19, 460–485, 2013.

[18] S. Micu, I. Roventa, M. Tucsnak, Time optimal boundary controls for the heat equation, J. Funct. Anal. 263, 25–49, 2012.

[19] M. Tucsnak, G. Wang, C.-T. Wu, Perturbations of time optimal control problems for a class of abstract parabolic systems, SIAM J. Control Optim. 54, 2965–2991, 2016.

[20] G. Wang, L.J. Wang, The optimal time control of a phase-field system, SIAM J. Control Optim. 42, 1483–1508, 2003.

[21] G. Wang, L.J. Wang, The "bang–bang" principle of time optimal controls for the heat equation with internal controls, Systems Control Lett. 56, 709–713, 2007.

[22] G. Wang, L. Wang, Y. Xu, Y. Zhang, Time Optimal Control of Evolution Equations, Birkhäuser, 2018.

[23] G. Wang, E. Zuazua, On the equivalence of minimal time and minimal norm controls for the internally controlled heat equations. SIAM J. Control Optim. 50, 2938–2958, 2012.

[24] K. Yosida, Functional Analysis. Sixth ed., Springer-Verlag, Berlin 1980.