BOUNDARY NULL CONTROLLABILITY AS THE LIMIT OF INTERNAL CONTROLLABILITY: THE HEAT CASE

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ABSTRACT. It is well known that for the heat equation with Dirichlet boundary condition both internal and boundary null controllability hold with controls applied to any open subset of the domain and any open subset of the boundary, respectively. The purpose of this paper is to show that for the heat equation the boundary null controllability can be obtained as the limit of distributed null controllability.

Keywords: Internal null controllability, boundary null controllability, Carleman estimates, singular pertubations.

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1. INTRODUCTION

Let $T > 0$ and let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary Γ is regular enough and let ω be a (small) nonempty subset of Ω.

It is well known that for any $y_0 \in L^2(\Omega)$, there exists a pair $(y, f) \in C([0,T]; L^2(\Omega)) \times L^2(\omega \times (0,T))$ which solves the following distributed null controllability problem

$$\left| \begin{array}{lll} y_t - \Delta y = f 1_\omega & \text{in} & Q := \Omega \times (0,T), \\ y = 0 & \text{on} & \Sigma := \Gamma \times (0,T), \\ y(x,0) = y_0(x),\ y(x,T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{1.1}$$

Moreover, if $\Gamma_0 \subset \Gamma$ is a non-empty open subset of the boundary, it is also known that there exists a pair $(y, g) \in \big(C([0,T]; H^{-1}(\Omega)) \cap L^2(Q)\big) \times L^2(\Gamma_0 \times (0,T))$ solution of the following boundary null controllability problem

$$\left| \begin{array}{lll} y_t - \Delta y = 0 & \text{in} & Q, \\ y = g 1_{\Gamma_0} & \text{on} & \Sigma, \\ y(x,0) = y_0(x),\ y(x,T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{1.2}$$

In this paper we work on the following question:

Question: Let $\epsilon > 0$ and let ω_ϵ be an ϵ-neighborhood of Γ_0 which shrinks to Γ_0 as $\epsilon \to 0^+$. Can we find a sequence (y_ϵ, f_ϵ), with *supp* $f_\epsilon \subset \omega_\epsilon$, such that the distributed null control problem (1.1) converges, in some sense, to the boundary null control problem (1.2) as $\epsilon \to 0^+$?

This question of approximating a boundary null controllability problem by a distributed one has been already proved to hold in the case of the wave equation by Fabre in [3]. We also cite [10] for some recent

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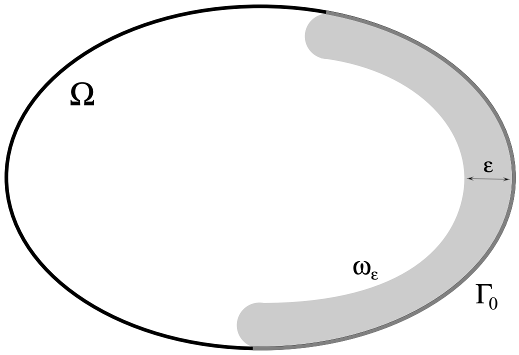

FIGURE 1. The set Γ_0 and its ϵ-neighborhood ω_ϵ.

developments for hyperbolic equations and [12] for the same problem in the context of stabilization. For parabolic equations, there has been a recent paper by Letrouit (see [15]), which considers, in the one-dimensional setting, the case of a control region shrinking to a point which is located within the domain. There, it is shown that, depending on arithmetic properties of the limiting point, one may recover the pointwise controllability for the heat equation with some minimal time. Nevertheless, it is important to say that the result of Letrouit cannot be applied to the case we are considering (even in the one-dimensional case), because here the boundary condition for the limiting problem is non-homogeneous and his estimates do no not allow to pass to the limit when the limiting point is on the boundary. Thus, as far as we know, this is the first time that, for the heat equation, the question of distributed control problem converging to the boundary control one is considered.

Concerning controllability problems for parabolic equations, the first results on this subject go back to Fattorini and Russell in [5, 6], where the moment method was used to prove boundary and internal null controllability in one space dimension. For the multidimensional case, an important work is [17], where the boundary null controllability with controls applied to the whole boundary is proved. Also in the multidimensional case, when the control is applied to a (small) part of the domain or the boundary, we refer to the works of Imanuvilov and Fursikov in [1, 2, 9] and Lebeau and Robbiano in [14]. Hence, we already know that both null control problems (1.1) and (1.2) are solvable and we are only interested in showing that one problem converges to the other. In fact, our main result reads as follows.

Theorem 1.1. *Let $T > 0$, $y_0 \in L^2(\Omega)$ and let Γ_0 be a non-empty open subset of Γ. For any $\epsilon > 0$, let ω_ϵ be a non-empty open neighborhood of Γ_0 given by*

$$\omega_\epsilon = \bigcup_{x\in\Gamma_0} B(x,\epsilon)\cap\Omega.$$

There exists a sequence $(y_\epsilon, f_\epsilon) \in C([0,T];L^2(\Omega)) \times L^2(\omega_\epsilon\times(0,T))$ that solves the distributed null control problem (1.1), such that $y_\epsilon \rightharpoonup y$ weakly in $L^2(Q)$, $f_\epsilon \rightharpoonup g$ weakly in $L^2(0,T;(H^2\cap H_0^1(\Omega))')$ and the pair $(y,g) \in \big(C([0,T];H^{-1}(\Omega))\cap L^2(Q)\big)\times L^2(\Gamma_0\times(0,T))$ solves boundary null control problem (1.2). Moreover, the identity

$$\iint_Q y_\epsilon h\,dxdt = \iint_{\omega_\epsilon\times(0,T)} f_\epsilon z\,dxdt + \int_\Omega y_0(x)z(x,0)\,dx, \tag{1.3}$$

converges to

$$\iint_Q yh\,dxdt = -\int_0^T \int_{\Gamma_0} g\frac{\partial z}{\partial \nu}\,dydt + \int_\Omega y_0(x)z(x,0)\,dx, \tag{1.4}$$

for every $(z^T,h) \in L^2(\Omega)\times L^2(Q)$*, where* z *is the solution of*

$$\begin{cases} -z_t - \Delta z = h & in \quad Q,\\ z = 0 & in \quad \Sigma,\\ z(x,T) = z^T & in \quad \Omega.\end{cases}$$

We prove Theorem 1.1 in Section 5.

Remark 1.2. *In Theorem 1.1, since we lose the boundary conditions in the limit, we only have that* $y_\epsilon \rightharpoonup y$ *weakly in* $L^2(Q)$ *and* $f_\epsilon \rightharpoonup g$ *weakly in* $L^2(0,T;(H^2\cap H_0^1(\Omega))')$.

In order to prove Theorem 1.1, it is important to understand how one can solve problems (1.1) and (1.2). Indeed, a classical argument to solve the control problems (1.1) and (1.2) is to consider the adjoint system

$$\left|\begin{array}{lll} -\varphi_t - \Delta\varphi = h & \text{in} & Q,\\ \varphi = 0 & \text{on} & \Sigma,\\ \varphi(x,T) = \varphi^T(x) & \text{in} & \Omega,\end{array}\right. \tag{1.5}$$

and show that the following inequalities

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C(\epsilon)\left(\iint_{\omega_\epsilon\times(0,T)} |\varphi|^2\,dxdt + \iint_Q |h|^2\,dxdt\right), \tag{1.6}$$

and

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C\left(\iint_{\Gamma_0\times(0,T)} |\frac{\partial\varphi}{\partial\nu}(y,t)|^2\,dydt + \iint_Q |h|^2\,dxdt\right), \tag{1.7}$$

hold for every $\varphi^T \in L^2(\Omega)$ and $h \in L^2(Q)$, respectively.

Inequalities (1.6) and (1.7) are the so-called *internal observability inequality* and *boundary observability inequality*, respectively. The constants C and $C(\epsilon)$ are known as *observability constants.*

Here, to prove Theorem 1.1, we must obtain the optimal observability constant $C(\epsilon)$, with respect to ϵ, when ω_ϵ is an ϵ-neighborhood of Γ_0 and ϵ is sufficiently small. The obtainment of the optimal $C(\epsilon)$ will be a consequence of a sharp Carleman inequality near the boundary (see Theorem 2.1). In fact, we show that $C(\epsilon) = O(\epsilon^{-3})$.

2. A sharp Carleman inequality

In this section, we prove a sharp Carleman inequality near the boundary for system (1.5). More precisely, we show that when the control region is a neighborhood of the boundary of radius ϵ then the observability constant is $O(\epsilon^{-3})$ (see Theorem 2.1 below). This result will be the key point for proving Theorem 1.1.

First, let us introduce several classical weights in the study of Carleman inequalities for parabolic equations (see [7, 9, 16]). The basic weight will be a function $\psi \in C^2(\overline{\Omega})$ verifying

$$|\nabla\psi(x)| > 0,\ \forall x \in \overline{\Omega},$$

$$\frac{\partial\psi}{\partial\nu}(x) \le 0, \quad \forall x \in \Gamma\setminus\Gamma_0.$$

Then, for $\lambda > 0$, we set:

$$\phi(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{t(T-t)};\ \alpha(x,t) = \frac{e^{\lambda(\psi(x)+m_1)} - e^{\lambda(||\psi||_\infty+m_2)}}{t(T-t)}, \tag{2.1}$$

with $m_1 = ||\psi||_\infty + 2$ and $m_2 = ||\psi||_\infty + 3$.

Notice that by the choice of m_1 and m_2 we have that α is negative and that, for any $s > 0$ and any $k \in \mathbb{N}$, terms of the form $e^{2s\alpha}\phi^k$ are bounded uniformly with respect to x and t.

Also, we point out that similar weights as the ones in (2.1) have been introduced in [16], the difference being the time behavior due to the consideration of the Stokes equation.

We also introduce the following notation:

$$I(s,\lambda;\varphi) := s^3\lambda^4 \iint_Q e^{2s\alpha}\phi^3|\varphi|^2 dxdt + s\lambda^2 \iint_Q e^{2s\alpha}\phi|\nabla\varphi|^2 dxdt + s^{-1}\iint_Q e^{2s\alpha}\phi^{-1}\left(|\varphi_t|^2 + \sum_{i,j=1}^N \left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right|^2\right) dxdt, \tag{2.2}$$

where s and λ are positive real numbers and $\varphi = \varphi(x,t)$.

The main objective of this section is to prove the following result.

Theorem 2.1. *Let $0 < \epsilon < 1$ and let ω_ϵ be a non-empty open neighborhood of Γ_0 given by*

$$\omega_\epsilon = \bigcup_{x\in\Gamma_0} B(x,\epsilon)\cap\Omega. \tag{2.3}$$

Given $\varphi^T \in L^2(\Omega)$ and $h \in L^2(Q)$, there exist three constants $\lambda_0 = \lambda_0(\Omega,\Gamma_0)$, $s_0 = s_0(\Omega,\Gamma_0)(T+T^2)$ and $C = C(\Omega,\Gamma_0)$, such that, for every $s \geq s_0$ and any $\lambda \geq \lambda_0$ the following inequality holds

$$I(s,\lambda;\varphi) \leq C\left(\iint_Q e^{2s\alpha}|h|^2\,dxdt + \epsilon^{-3}s^7\lambda^4 \iint_{\omega_\epsilon\times(0,T)} e^{2s\alpha}\phi^7|\varphi|^2\,dxdt\right), \tag{2.4}$$

for every φ solution of (1.5).

Carleman estimates as the one given in Theorem 2.1 are classical tools in Control theory (see [7, 9, 13]). Concerning estimate (2.4), if one follows the standard proofs of distributed Carleman inequalities for the heat equation (for instance [9]), one sees that it gives a constant of order ϵ^{-4}, which is not sufficient for our purpose. Hence, the main novelty in Theorem 2.1 is the order ϵ^{-3} for the observability constant. To prove this result, we start from a boundary Carleman inequality and use a localization argument to get the internal observation with the precise constant.

Remark 2.2. *In Theorem 2.1, the order ϵ^{-3} for the observability constant is optimal. Indeed, consider $\Omega = (0,1)$ and $\varphi(x,t) = e^{-\pi^2 t}\sin(\pi x)$, then one has*

$$\int_0^1 |\sin(x)|^2 dx = \frac{1}{2}$$

and

$$\int_0^T \int_0^\epsilon e^{-2\pi^2 t}|\sin(\pi x)|^2 dxdt = \frac{1}{4\pi^2}\big(1 - e^{-2\pi^2 T}\big)(\epsilon - \frac{\sin(2\pi\epsilon)}{2\pi}) = O(\epsilon^3).$$

In what follows we need the following boundary Carleman estimate.

Theorem 2.3. *Given $\varphi^T \in L^2(\Omega)$ and $h \in L^2(Q)$, there exist three constants $\lambda_0 = \lambda_0(\Omega, \Gamma_0)$, $s_0 = s_0(\Omega, \Gamma_0)(T + T^2)$ and $C = C(\Omega, \Gamma_0)$, such that for every $s \geq s_0$ and any $\lambda \geq \lambda_0$ the following inequality holds*

$$I(s, \lambda; \varphi) \leq C \left(\iint_Q e^{2s\alpha} |h|^2 \, dxdt + s\lambda \iint_{\Gamma_0 \times (0,T)} e^{2s\alpha(y,t)} \phi(y,t) \left| \frac{\partial \varphi}{\partial \nu}(y,t) \right|^2 dydt \right), \tag{2.5}$$

for every φ solution of (1.5).

A proof of Theorem 2.3 can be found in [2]. In there the dependence in time of the weight functions is of order $t^{-2}(T-t)^{-2}$. Nevertheless, by following the proof, it is not difficult to see that one can consider weights like (2.1).

In the proof of Theorem 2.1, we will use the following result of localization near the boundary which is proved in [4] (see also [11], appendix C).

Lemma 2.4. *There exist open sets $U_1, U_2, \dots U_{n_0}$, and $\epsilon_0 > 0$ such that*

$$\overline{\omega}_\epsilon \subset \bigcup_{i=1}^{n_0} U_i, \quad \text{where} \quad \overline{\omega}_\epsilon \quad \text{denotes the closure of} \quad \omega_\epsilon, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{2.6}$$

Let $\nu(y)$ be the unit exterior normal vector to Γ at the point $y \in \Gamma$;

$$\forall x \in \Omega \cap U_i, \quad \exists! (y, z) \in (\Gamma \cap U_i) \times \mathbb{R}_+ \quad \text{such that} \quad x = y - z\nu(y). \tag{2.7}$$

$$\begin{aligned} &\text{The mappings} \quad J_i^{-1} : x \to (y, z) \quad \text{are} \quad C^2\text{-diffeomorphisms from} \quad \Omega \cap U_i \quad \text{on their images,} \\ &\text{which map} \quad \omega_\epsilon \cap U_i \quad \text{into} \quad (\Gamma \cap U_i) \times (0, \epsilon). \end{aligned} \tag{2.8}$$

There exist $\epsilon_0, m > 0$ and $M > 0$ such that

$$\forall z \in [0, \epsilon_0], \quad \forall i \quad \text{we have} \quad m \leq |J_i(y, z)| \leq M. \tag{2.9}$$

Here, $|J_i(y, z)|$ denotes the Jacobian of J_i at the point (y, z), with the map $(y, z) \mapsto |J_i(y, z)|$ being C^1 and

$$|J_i(y, 0)| = 1, \quad \forall y \in \Gamma \cap U_i. \tag{2.10}$$

Finally, if v is a function defined over $\Omega \cap U_i$, we write $\widetilde{v}(y, z) = v(x)$, and then

$$\frac{\partial \widetilde{v}}{\partial z}(y, z) = -\nabla v(x) \cdot \nu(p(x)), \tag{2.11}$$

and

$$\frac{\partial^2 \widetilde{v}}{\partial z^2}(y, z) = \sum_{i,j=1}^N \frac{\partial^2 v(x)}{\partial x_i \partial x_j} \nu_i(p(x)) \nu_j(p(x)) + (\nabla v(x) \cdot \frac{\partial \nu(y)}{\partial y}) \sum_{j=1}^N \frac{\partial p(x)}{\partial x_j}, \tag{2.12}$$

for $v \in H^2(\Omega \cap U_i)$, where $p(x) = y$.

Let us now prove Theorem 2.1.

Proof of Theorem 2.1. From Theorem 2.3 , it is sufficient to prove that

$$s\lambda \iint_{\Gamma_0 \times (0,T)} e^{2s\alpha(y,t)} \phi(y,t) |\frac{\partial \varphi}{\partial \nu}(y,t)|^2 \, dydt \leq C\epsilon^{-3} s^7 \lambda^4 \iint_{\omega_\epsilon \times (0,T)} e^{2s\alpha} \phi^7 |\varphi|^2 \, dxdt + \delta I(s, \lambda; \varphi), \tag{2.13}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

From Lemma 2.4 we have that

$$-\frac{\partial \varphi}{\partial \nu}(y,t) = \frac{d\varphi}{dz}(y - z\nu(y), t)\big|_{z=0} = \frac{d\widetilde{\varphi}}{dz}(y, z, t)|_{z=0}, \quad y \in \Gamma.$$

Also, the following estimate holds

$$s\lambda \iint_{\Gamma_0\times(0,T)} e^{2s\alpha(y,t)}\phi(y,t)\left|\frac{\partial\varphi}{\partial\nu}(y,t)\right|^2 dydt \leq s\lambda\sum_{i=1}^{n_0}\iint_{(\Gamma_0\cap U_i)\times(0,T)} e^{2s\widetilde{\alpha}(y,0,t)}\widetilde{\phi}(y,0,t)\left|\frac{d\widetilde{\varphi}}{dz}(y,z,t)|_{z=0}\right|^2 dydt.$$

Let us introduce a nonnegative function $\theta\in C^3(0,1)$ such that $\theta=1$ in $(0,\frac{\epsilon}{2})$, $\theta(\epsilon)=0$ and

$$\theta_z = O(\epsilon^{-1}),\quad \theta_{zz}=O(\epsilon^{-2}),\quad \theta_{zzz}=O(\epsilon^{-3}).$$

A simple computation now gives

$$\begin{aligned}
&s\lambda e^{2s\widetilde{\alpha}(y,0,t)}\widetilde{\phi}(y,0,t)\left|\frac{d}{dz}\widetilde{\varphi}(y,z,t)|_{z=0}\right|^2 = -s\lambda\int_0^\epsilon \frac{d}{dz}\left(e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta(z)\right)dz\\
&= -s\lambda\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta_z(z)\,dz - 2s^2\lambda\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\alpha}_z(y,z,t)\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta(z)\,dz \qquad (2.14)\\
&- s\lambda\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}_z(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta(z)\,dx - 2s\lambda\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)\widetilde{\varphi}_z(y,z,t)\widetilde{\varphi}_{zz}(y,z,t)\theta(z)\,dz.
\end{aligned}$$

We recall that

$$\widetilde{\alpha}_z(y,z,t) = -\lambda\phi(x,t)\nabla\psi(x)\cdot\nu(p(x)) \text{ and } \widetilde{\phi}_z(y,z,t) = -\lambda\phi(x,t)\nabla\psi(x)\cdot\nu(p(x)),$$

and use the estimate

$$\begin{aligned}
&2s\lambda\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)\widetilde{\varphi}_z(y,z,t)\widetilde{\varphi}_{zz}(y,z,t)\theta(z)\,dz\\
&\quad\leq Cs^3\lambda^2\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^3(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta(z)\,dz + \delta s^{-1}\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^{-1}(y,z,t)|\widetilde{\varphi}_{zz}(y,z,t)|^2\,dz,
\end{aligned}$$

to see that

$$\begin{aligned}
&s\lambda\iint_{\Gamma_0\times(0,T)} e^{2s\alpha(y,t)}\phi(y,t)|\frac{\partial\varphi}{\partial\nu}(y,t)|^2\,dydt\\
&\qquad\leq -\sum_{i=1}^{n_0} s\lambda\int_0^T\int_{\Gamma_0\cap U_i}\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta_z(z)\,dzdydt\\
&\qquad+ C\sum_{i=1}^{n_0} s^3\lambda^2\int_0^T\int_{\Gamma_0\cap U_i}\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^3(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta(z)\,dzdydt\\
&\qquad\quad+\delta\sum_{i=1}^{n_0} s^{-1}\int_0^T\int_{\Gamma_0\cap U_i}\int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^{-1}(y,z,t)|\widetilde{\varphi}_{zz}(y,z,t)|^2\,dzdydt, \qquad (2.15)
\end{aligned}$$

for any $\delta>0$ and some $C=C(\delta)>0$.

The last term on the right-hand side of (2.15) is bounded by $\delta I(s,\lambda,\varphi)$. Hence, we just have to estimate the first two terms. For the first one, integration by parts gives

$$\begin{aligned}
&- s\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\theta_z(z)\,dzdydt \\
&\quad = 2s^2\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\alpha}_z(y,z,t)\widetilde{\phi}(y,z,t)\widetilde{\varphi}(y,z,t)\widetilde{\varphi}_z(y,z,t)\theta_z(z)\,dzdydt \\
&\qquad + s\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)\widetilde{\varphi}_z(y,z,t)\widetilde{\varphi}(y,z,t)\theta_{zz}(z)\,dzdydt \\
&\qquad + s\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}_z(y,z,t)\widetilde{\varphi}(y,z,t)\widetilde{\varphi}_z(y,z,t)\theta_z(z)\,dzdydt \\
&\qquad + s\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)\widetilde{\varphi}(y,z,t)\widetilde{\varphi}_{zz}(y,z,t)\theta_z(z)\,dzdydt = \sum_{i=1}^4 A_i. \qquad (2.16)
\end{aligned}$$

The first term is easily estimated as

$$\begin{aligned}
A_1 \le{}& \delta s\lambda^2 \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\,dzdydt \qquad (2.17)\\
&+ C\epsilon^{-2}s^3\lambda^2 \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^3(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\,dzdydt,
\end{aligned}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

Using integration by parts, we obtain

$$\begin{aligned}
A_2 ={}& -s^2\lambda \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\alpha}_z(y,z,t)\widetilde{\phi}(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\theta_{zz}(z)\,dzdydt \\
&- \frac{s\lambda}{2} \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}_z(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\theta_{zz}(z)\,dzdydt \qquad (2.18)\\
&- \frac{s\lambda}{2} \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\theta_{zzz}(z)\,dzdydt \\
&\le C\epsilon^{-3}s^2\lambda^2 \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}^2(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\,dzdydt,
\end{aligned}$$

where C does not depend on ϵ.

Next, using Young's inequality, one sees that

$$\begin{aligned}
A_3 \le{}& \delta s\lambda^2 \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}_z(y,z,t)|^2\,dzdydt \qquad (2.19)\\
&+ C\epsilon^{-2}s \int_0^T \int_{\Gamma_0\cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)}\widetilde{\phi}(y,z,t)|\widetilde{\varphi}(y,z,t)|^2\,dzdydt,
\end{aligned}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

Finally, for the last term in (2.16), we have

$$A_4 \leq \delta s^{-1} \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^{-1}(y,z,t) |\widetilde{\varphi}_{zz}(y,z,t)|^2 \, dzdydt \\ + C\epsilon^{-2} s^3 \lambda^2 \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^3(y,z,t) |\widetilde{\varphi}(y,z,t)|^2 \, dzdydt, \tag{2.20}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

For the second term on the right-hand side of (2.15), we can proceed in the same way to obtain the following estimate

$$s^3\lambda^2 \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^3(y,z,t) |\widetilde{\varphi}_z(y,z,t)|^2 \theta(z) \, dzdydt \\ \leq C\epsilon^{-2} s^7 \lambda^4 \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^7(y,z,t) |\widetilde{\varphi}(y,z,t)|^2 \, dzdydt \\ + \delta \Bigg(s\lambda^2 \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}(y,z,t) |\widetilde{\varphi}_z(y,z,t)|^2 \, dzdydt \\ + s^{-1} \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^{-1}(y,z,t) |\widetilde{\varphi}_{zz}(y,z,t)|^2 \, dzdydt \Bigg), \tag{2.21}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

Putting (2.16)-(2.21) in (2.15) we obtain

$$s\lambda \iint_{\Gamma_0 \times (0,T)} e^{2s\alpha(y,t)} \phi(y,t) |\frac{\partial \varphi}{\partial \nu}(y,t)|^2 \, dydt \\ \leq C\epsilon^{-3} s^7 \lambda^4 \sum_{i=1}^{n_0} \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^7(y,z,t) |\widetilde{\varphi}(y,z,t)|^2 \, dzdydt \\ + \delta \Bigg(s\lambda^2 \sum_{i=1}^{n_0} \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}(y,z,t) |\widetilde{\varphi}_z(y,z,t)|^2 \, dzdydt \\ + s^{-1} \sum_{i=1}^{n_0} \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}^{-1}(y,z,t) |\widetilde{\varphi}_{zz}(y,z,t)|^2 \, dzdydt \Bigg), \tag{2.22}$$

for any $\delta > 0$ and some $C = C(\delta) > 0$.

Noticing that

$$s\lambda^2 \int_0^T \int_{\Gamma_0 \cap U_i} \int_0^\epsilon e^{2s\widetilde{\alpha}(y,z,t)} \widetilde{\phi}(y,z,t) |\widetilde{\varphi}_z(y,z,t)|^2 \, dzdydt \\ \leq Cs\lambda^2 \int_0^T \int_\Omega e^{2s\alpha(x,t)} \phi(x,t) |\nabla \varphi(x,t)|^2 \, dxdt, \quad i = 1, \ldots, n_0 \tag{2.23}$$

and that

$$\begin{aligned}
&s^{-1}\int_0^T\int_{\Gamma_0\cap U_i}\int_0^{\epsilon} e^{2s\tilde{\alpha}(y,z,t)}\widetilde{\phi}^{-1}(y,z,t)|\widetilde{\varphi}_{zz}(y,z,t)|^2\,dzdydt\\
&\leq Cs^{-1}\sum_{i,j=1}^{N}\int_0^T\int_{\Omega} e^{2s\alpha(x,t)}\phi^{-1}(x,t)\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x,t)\right|^2\,dxdt\\
&+Cs^{-1}\sum_{i=1}^{n_0}\int_0^T\int_{\Omega} e^{2s\alpha(x,t)}\phi^{-1}(x,t)\left|\nabla\varphi(x,t)\right|^2\,dxdt,
\end{aligned} \tag{2.24}$$

from (2.22) we readily obtain (2.13).

□

3. Construction of a sequence of controls

In this section, we construct a sequence of internal controls such that the distributed null control problem (1.1) converges, in the sense given in Theorem 1.1, to the boundary null control problem (1.2). To do this, we must improve the Carleman inequality given in Theorem 2.1. This new Carleman inequality will contain only weight functions that do not vanish at $t = 0$.

We set a function

$$l(t) = \begin{cases} (T^2/4) & \text{if } 0 \leq t \leq T/2,\\ t(T-t) & \text{if } T/2 \leq t \leq T,\end{cases}$$

and define new weight functions to be

$$\beta(x,t) = \frac{e^{\lambda(\psi(x)+m_1)} - e^{\lambda(\|\psi\|_\infty+m_2)}}{l(t)},\ \ \gamma(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{l(t)},$$

$$\widehat{\gamma}(t) = \min_{x\in\overline{\Omega}}\gamma(x,t),\ \ \gamma^*(t) = \max_{x\in\overline{\Omega}}\gamma(x,t),\ \ \beta^*(t) = \max_{x\in\overline{\Omega}}\beta(x,t),\ \ \widehat{\beta} = \min_{x\in\overline{\Omega}}\beta(x,t).$$

Notice that β is negative and that $\alpha \leq \beta$, and for any $s > 0$ and any $k \in \mathbb{N}$, terms of the form $e^{2s\beta}\gamma^k$ are bounded uniformly with respect to x and t.

From now on, we fix s and λ for which Theorem 2.1 holds (giving the Carleman estimate). With these new weights, we have the following observability inequality:

Lemma 3.1. *There exists $C > 0$ such that*

$$\begin{aligned}
&\|\varphi(0)\|^2_{L^2(\Omega)} + \iint_Q e^{2s\beta}\gamma^{-1}\left(|\varphi_t|^2 + \sum_{i,j=1}^{N}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^2\right)dxdt + \iint_Q e^{2s\beta}\gamma|\nabla\varphi|^2\,dxdt\\
&\quad + \iint_Q e^{2s\beta}\gamma^3|\varphi|^2\,dxdt \leq C\left(\epsilon^{-3}\int_0^T\int_{\omega_\epsilon} e^{2s\beta}\gamma^7|\varphi|^2\,dxdt + \iint_Q e^{2s\beta}|h|^2\,dxdt\right),
\end{aligned}$$

for every φ solution of (1.5).

Proof. The proof of this lemma is standard. It combines energy estimates, using a cut-off function, together with the fact that $\alpha \leq \beta$ in Q. A proof is given in [8, Lemma 1] but, for sake of completeness, a detailed proof is given in the appendix.

□

Notice that to prove Lemma 3.1, we strongly use the fact that the weight β does not vanish at $t = 0$.

Now, for each $\epsilon > 0$, we construct a pair $(\widehat{y}_\epsilon, \widehat{f}_\epsilon)$ solution of the distributed control problem (1.1).

Let $P_0 := \{w \in C^2(\overline{Q}), \quad w = 0 \text{ in } \Gamma \times (0,T)\}$ and in P_0 we define the bilinear form

$$a_\epsilon(w_1, w_2) := \iint_Q e^{2s\beta} \mathcal{L}^* w_1 \mathcal{L}^* w_2 dxdt + \frac{1}{\epsilon^3} \iint_{\omega_\epsilon \times (0,T)} e^{2s\beta^*} (\gamma^*)^7 w_1 w_2 dxdt,$$

where $\mathcal{L}^* := \partial_t + \Delta$.

Thanks to the Carleman inequality given in Lemma 3.1, we have that $a_\epsilon : P_0 \times P_0 \longrightarrow \mathbb{R}$ is a symmetric, positive definite bilinear form.

We denote by $P = P(\epsilon)$ the completion of P_0 with respect to the norm associated to $a_\epsilon(\cdot,\cdot)$ (which we denote by $\|\cdot\|_P$). This is a Hilbert space and $a_\epsilon(\cdot,\cdot)$ is a continuous and coercive bilinear form on P.

Consider the linear form $l : P \longrightarrow \mathbb{R}$ given by

$$< l, \varphi > = \int_\Omega y_0 \varphi(0) dx.$$

From Lemma 3.1, we have that

$$\begin{aligned} \big| < l, \varphi > \big| &\leq \|y_0\|_{L^2(\Omega)} \|\varphi(0)\|_{L^2(\Omega)} \\ &\leq C \|y_0\|_{L^2(\Omega)} \|\varphi\|_P, \end{aligned}$$

where C does not depend on ϵ.

By Lax-Milgram theorem, there exists a unique $\widehat{\varphi}_\epsilon \in P$ such that

$$a_\epsilon(\widehat{\varphi}_\epsilon, \varphi) = < l, \varphi >, \quad \forall \varphi \in P.$$

Defining $\widehat{y}_\epsilon = e^{2s\beta} \mathcal{L}^* \widehat{\varphi}_\epsilon$ and $\widehat{f}_\epsilon = -\frac{1}{\epsilon^3} e^{2s\beta^*} (\gamma^*)^7 \widehat{\varphi}_\epsilon$, we have that

$$\iint_Q \widehat{y}_\epsilon \mathcal{L}^* \varphi dxdt - \iint_{\omega_\epsilon \times (0,T)} \widehat{f}_\epsilon \varphi dxdt = \int_\Omega y_0 \varphi(0) dx, \quad \forall \varphi \in P, \tag{3.1}$$

and the following estimate holds

$$\|\widehat{\varphi}_\epsilon\|_P^2 = \iint_Q e^{-2s\beta} \big|\widehat{y}_\epsilon\big|^2 dxdt + \epsilon^3 \iint_{\omega_\epsilon \times (0,T)} e^{-2s\beta^*} (\gamma^*)^{-7} \big|\widehat{f}_\epsilon\big|^2 dxdt \leq C \|y_0\|_{L^2(\Omega)}^2, \tag{3.2}$$

where C does not depend on ϵ.

From (3.1) and (3.2), it follows that $(\widehat{y}_\epsilon, \widehat{f}_\epsilon)$ solves the distributed null control problem (1.1) and the following estimates hold

$$\|e^{-s\beta^*} (\gamma^*)^{-\frac{7}{2}} \widehat{f}_\epsilon\|_{L^2(\omega_\epsilon \times (0,T))}^2 \leq C \epsilon^{-3} \|y_0\|_{L^2(\Omega)}^2, \tag{3.3}$$

and

$$\|e^{-s\beta} \widehat{y}_\epsilon\|_{L^2(Q)}^2 \leq C \|y_0\|_{L^2(\Omega)}^2, \tag{3.4}$$

where C does not depend on ϵ.

4. Passage to the limit

Let $\mathcal{L}^* := \partial_t + \Delta$ and $(\widehat{y}_\epsilon, \widehat{f}_\epsilon) = (e^{2s\beta} \mathcal{L}^* \widehat{\varphi}_\epsilon, -\frac{1}{\epsilon^3} e^{2s\beta^*} (\gamma^*)^7 \widehat{\varphi}_\epsilon)$ the sequence constructed in the previous section.

The following result is a direct consequence of Lemma 3.1.

Proposition 4.1. *We have that $e^{s\widehat{\beta}} \widehat{\gamma}^{-1} \widehat{\varphi}_\epsilon \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ with*

$$\|e^{s\widehat{\beta}} \widehat{\gamma}^{-1} \widehat{\varphi}_\epsilon\|_{L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega))}^2 \leq \|\widehat{\varphi}_\epsilon\|_P^2 \leq C \|y_0\|_2^2,$$

where C does not depend on ϵ.

Moreover, there exists a function φ *such that, up to a subsequence,* $e^{s\widehat{\beta}}\widehat{\gamma}^{-1}\widehat{\varphi}_\epsilon \rightharpoonup e^{s\widehat{\beta}}\widehat{\gamma}^{-1}\varphi$ *weakly in* $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ *and* $e^{s\widehat{\beta}}\widehat{\gamma}^{-1}\frac{\partial\varphi}{\partial\nu}\in L^2(\Gamma_0\times(0,T))$.

Using Proposition 4.1, we prove the following result.

Lemma 4.2. *Let*

$$L_\epsilon : L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\to\mathbb{R}$$

$$v\to\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon(x,t)v(x,t)\,dxdt.$$

Then, L_ϵ *are bounded in* $L^2(0,T;(H^2(\Omega)\cap H_0^1(\Omega))')$ *and converge (up to a subsequence) for the weak topology of this space to*

$$L : L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\to\mathbb{R}$$

$$v\to\frac{1}{3}\iint_{\Gamma_0\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\frac{\partial\varphi}{\partial\nu}(y,t)\frac{\partial v}{\partial\nu}(y,t)\,dydt.$$

Proof. We divide the proof into two steps.

Step 1: Boundedness of L_ϵ

From (3.2), it follows that

$$|L_\epsilon v|\le\frac{C}{\epsilon^{3/2}}\left(\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|v(x,t)|^2\,dxdt\right)^{\frac{1}{2}}\|y_0\|. \tag{4.1}$$

On the other hand we have that

$$\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|v(x,t)|^2\,dxdt\le\frac{1}{\epsilon^3}\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\Gamma\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7|\widetilde{v}(y,z,t)|^2dzdydt. \tag{4.2}$$

Since $v\in L^2(0,T;H^2(\omega_\epsilon\cap U_k))$ and $v=0$ in $\Gamma\cap U_k$, from (2.12) it follows that $\widetilde{v}\in H^2(0,\epsilon;L^2((0,T)\times\Gamma\cap U_k))$ and $\widetilde{v}(y,0,t)=0$ and we can write

$$\widetilde{v}(y,z,t)=z\frac{\partial\widetilde{v}}{\partial z}(y,0,t)+\int_0^z\int_0^s\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\,drds. \tag{4.3}$$

From (4.1)-(4.3), we have the following estimate

$$\begin{aligned}|L_\epsilon v|^2 &\le\frac{C}{\epsilon^3}\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\Gamma\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7\left|z\frac{\partial\widetilde{v}}{\partial z}(y,0,t)+\int_0^z\int_0^s\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\,drds\right|^2dzdydt\\ &\le\frac{C}{\epsilon^3}\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\Gamma\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7z^2\left(\left|\frac{\partial\widetilde{v}}{\partial z}(y,0,t)\right|^2+\int_0^\epsilon\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|^2dr\right)dzdydt\\ &\le C\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\Gamma\cap U_k} e^{2s\beta^*}(\gamma^*)^7\left(\left|\frac{\partial\widetilde{v}}{\partial z}(y,0,t)\right|^2+\int_0^\epsilon\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|^2dr\right)dydt\\ &\le C\|v\|_{L^2(0,T;H^2\cap H_0^1)},\end{aligned} \tag{4.4}$$

where C depends only on Ω and T.

Therefore, L_ϵ is bounded in $L^2(0,T;(H^2\cap H_0^1)')$ and there exists $L\in L^2(0,T;(H^2\cap H_0^1)')$, the weak limit of L_ϵ.

Step 2: Characterization of the limit L

We introduce

$$\widetilde{\omega_\epsilon} = \{x \in \omega_\epsilon \quad \text{such that} \quad \exists (y,z) \in \Gamma_0 \times (0,\epsilon), x = y - z\nu(y)\};$$

so that $\widetilde{\omega_\epsilon} \subset \omega_\epsilon$ and, for $\epsilon > 0$ small enough, $\widetilde{\omega_\epsilon}$ is the set points $x \in \omega_\epsilon$ such that its projection $p(x)$ on the boundary Γ belongs to Γ_0. We have

$$meas[\partial(\omega_\epsilon \setminus \widetilde{\omega_\epsilon}) \cap \Gamma] \to 0 \quad \text{as} \quad \epsilon \to 0,$$

where the measure here denotes the boundary measure.

We write

$$\begin{aligned}
\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon(x,t)v(x,t)\,dxdt &= \frac{1}{\epsilon^3}\iint_{\widetilde{\omega}_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon(x,t)v(x,t)\,dxdt \\
&+ \frac{1}{\epsilon^3}\iint_{\omega_\epsilon\setminus\widetilde{\omega}_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon(x,t)v(x,t)\,dxdt \\
&= M_\epsilon^1(v) + M_\epsilon^2(v).
\end{aligned}$$

Claim 1: $M_\epsilon^2(v) \to 0$ as $\epsilon \to 0^+$.

Proof of Claim 1: Since

$$|M_\epsilon^2(v)|^2 \le \frac{C}{\epsilon^3}\iint_{\omega_\epsilon\setminus\widetilde{\omega}_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|v(x,t)|^2\,dxdt, \tag{4.5}$$

it is sufficient to prove that

$$\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\setminus\widetilde{\omega}_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|v(x,t)|^2\,dxdt \to 0, \quad \text{as} \quad \epsilon \to 0^+.$$

In fact, we have that

$$\begin{aligned}
&\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\setminus\widetilde{\omega}_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|v(x,t)|^2\,dxdt \\
&\le \frac{C}{\epsilon^3}\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\partial(\omega_\epsilon\setminus\widetilde{\omega}_\epsilon)\cap\Gamma\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7\left(z^2\left|\frac{\partial\widetilde{v}}{\partial z}(y,0,t)\right|^2 + \left|\int_0^z\int_0^s\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|drds\right|^2\right)dzdydt \\
&\qquad\le C\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\partial(\omega_\epsilon\setminus\widetilde{\omega}_\epsilon)\cap\Gamma\cap U_k} e^{2s\beta^*}(\gamma^*)^7\left|\frac{\partial\widetilde{v}}{\partial z}(y,0,t)\right|^2 dydt \\
&\qquad+ \frac{C}{\epsilon^3}\sum_{k=1}^{n_0}\int_0^T\!\!\int_{\partial(\omega_\epsilon\setminus\widetilde{\omega}_\epsilon)\cap\Gamma\cap U_k} e^{2s\beta^*}(\gamma^*)^7\int_0^\epsilon\left|\int_0^z\int_0^s\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|drds\right|^2 dzdydt.
\end{aligned} \tag{4.6}$$

Using Hölder inequality, we readily see that

$$\left(\int_0^z\int_0^s\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|drds\right)^2 \le z\int_0^z\left(s\int_0^s\left|\frac{\partial^2\widetilde{v}}{\partial z^2}(y,r,t)\right|^2 dr\right)ds,$$

and then

$$\int_0^\epsilon \left| \int_0^z \int_0^s \left| \frac{\partial^2 \widetilde{v}}{\partial z^2}(y,r,t) \right| drds \right|^2 dz \leq \int_0^\epsilon z \int_0^\epsilon \left(s \int_0^\epsilon \left| \frac{\partial^2 \widetilde{v}}{\partial z^2}(y,r,t) \right|^2 dr \right) ds\, dz$$
$$\leq C\epsilon^4 \|v\|^2_{L^2(H^2\cap H^1_0)}. \quad (4.7)$$

Putting (4.7) in (4.6), we get that

$$\frac{1}{\epsilon^3} \iint_{\omega_\epsilon \setminus \widetilde{\omega}_\epsilon \times (0,T)} e^{2s\beta^*}(\gamma^*)^7 |v(x,t)|^2 \, dxdt$$
$$\leq C \int_0^T \int_{\partial(\omega_\epsilon \setminus \widetilde{\omega}_\epsilon)\cap\Gamma} e^{2s\beta^*}(\gamma^*)^7 \left| \frac{\partial v}{\partial \nu}(y,t) \right|^2 dydt + \epsilon \|v\|^2_{L^2(H^2\cap H^1_0)}, \quad (4.8)$$

for some $C > 0$ which does not depend on ϵ. This proves the claim. □

Claim 2: $M^1_\epsilon(v) \to \dfrac{1}{3} \displaystyle\iint_{\Gamma_0\times(0,T)} \frac{\partial \varphi}{\partial \nu} \frac{\partial v}{\partial \nu} dydt$ *as* $\epsilon \to 0^+$.

Let $\{\theta_i\}_{i=1}^{n_0}$ a partition of unity associated to $\{U_i\}_{i=1}^{n_0}$. Defining $v_k = \theta_k v$, we have that $\widetilde{v}_k \in C^1([0,\epsilon]; L^2((0,T)\times\Gamma_0\cap U_k))$ and

$$\widetilde{v}_k(y,z,t) = z\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t) + zV^k(z), \quad \text{with} \quad \lim_{z\to 0} V^k(y,z,t) = 0.$$

Thus, we write

$$M^1_\epsilon(v) = \frac{1}{\epsilon^3} \sum_{k=1}^{n_0} \iint_{\widetilde{\omega}_\epsilon \cap U_k \times (0,T)} e^{2s\beta^*}(\gamma^*)^7 \theta_k(x) \widehat{\varphi}_\epsilon(x,t) v(x,t)\, dxdt$$
$$= \frac{1}{\epsilon^3} \sum_{k=1}^{n_0} \int_0^T \int_{\Gamma_0\cap U_k} \int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7 \widetilde{\widehat{\varphi}_\epsilon}(y,z,t) \left(z\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t) + zV^k(z) \right) |J_k(y,z)| \, dzdydt$$
$$= \sum_{k=1}^{n_0} (A^k_\epsilon(v) + B^k_\epsilon(v)), \quad (4.9)$$

where

$$A^k_\epsilon(v) = \frac{1}{\epsilon^3} \int_0^T \int_{\Gamma_0\cap U_k} \int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7 \widetilde{\widehat{\varphi}_\epsilon}(y,z,t) z \frac{\partial \widetilde{v}_k}{\partial z}(y,0,t) |J_k(y,z)|\, dzdydt$$

and

$$B^k_\epsilon(v) = \frac{1}{\epsilon^3} \int_0^T \int_{\Gamma_0\cap U_k} \int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7 \widetilde{\widehat{\varphi}_\epsilon}(y,z,t) zV^k(y,z,t) |J_k(y,z)|\, dzdydt.$$

Now, a simple computation gives

$$\sum_{k=1}^{n_0} |B^k_\epsilon(v)|$$
$$\leq C \left(\epsilon^{-3} \iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7 |\widehat{\varphi}_\epsilon(x,t)|^2 \, dxdt \right)^{\frac{1}{2}} \left(\epsilon^{-3} \sum_{k=1}^{n_0} \int_0^T \int_{\Gamma_0\cap U_k} \int_0^\epsilon z^2 |V^k(y,z,t)|^2 \, dzdydt \right)^{\frac{1}{2}}$$
$$\leq C \|\widehat{\varphi}_\epsilon\|_P \left(\sum_{k=1}^{n_0} \|V^k\|_{L^\infty_z(0,\epsilon;L^2(\Gamma_0\times(0,T)))} \right)^{\frac{1}{2}}, \quad (4.10)$$

where C does not depend on ϵ. Hence, it follows that

$$\sum_{k=1}^{n_0} |B_\epsilon^k(v)| \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$

To prove the convergence of A_ϵ^k, we use that

$$|J_k(y,z)| = |J_k(y,0)| + \int_0^z \frac{\partial}{\partial s}(|J_k(y,s)|)\, ds = 1 + \int_0^z \frac{\partial}{\partial s}(|J_k(y,s)|)\, ds$$

and that

$$\widetilde{\widehat{\varphi}}_\epsilon(y,z,t) = \int_0^z \frac{\partial}{\partial s}\widetilde{\widehat{\varphi}}_\epsilon(y,s,t)\, ds, \quad t > 0,$$

to write

$$\begin{aligned} A_\epsilon^k(v) &= \frac{1}{\epsilon^3}\int_0^T\!\!\int_{\Gamma_0\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7 \widetilde{\widehat{\varphi}}_\epsilon(y,z,t) z \frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\, dzdydt \\ &\quad + \frac{1}{\epsilon^3}\int_0^T\!\!\int_{\Gamma_0\cap U_k}\int_0^\epsilon e^{2s\beta^*}(\gamma^*)^7 \widetilde{\widehat{\varphi}}_\epsilon(y,z,t) z \frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\int_0^z \frac{\partial}{\partial s}(|J_k(y,s)|)\, ds\, dzdydt \\ &= A_\epsilon^{k,1}(v) + A_\epsilon^{k,2}(v). \end{aligned} \tag{4.11}$$

Since $\int_0^z \frac{\partial}{\partial s}(|J_k(y,s)|) = O(z)$, we estimate $A_\epsilon^{k,2}(v)$ as follows

$$\begin{aligned} &A_\epsilon^{k,2}(v) \\ &\leq \frac{C}{\epsilon^3}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k}\int_0^\epsilon e^{4s\beta^*}(\gamma^*)^{14}\left|\int_0^z \frac{\partial}{\partial s}\widetilde{\widehat{\varphi}}_\epsilon(y,s,t)\, ds\right|^2 dzdydt\right)^{\frac12}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k}\int_0^\epsilon z^4 \left|\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\right|^2 dzdydt\right)^{\frac12} \\ &\leq \frac{C}{\epsilon^3}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k} e^{4s\beta^*}(\gamma^*)^{14}\int_0^\epsilon \epsilon\left(\int_0^\epsilon \left|\frac{\partial}{\partial s}\widetilde{\widehat{\varphi}}_\epsilon(y,s,t)\right|^2 ds\right) dzdydt\right)^{\frac12}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k}\int_0^\epsilon z^4 \left|\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\right|^2 dzdydt\right)^{\frac12} \\ &\quad \leq \frac{C}{\epsilon^3}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k} \epsilon^2 e^{4s\beta^*}(\gamma^*)^{14}\int_0^\epsilon \left|\frac{\partial}{\partial z}\widetilde{\widehat{\varphi}}_\epsilon(y,z,t)\right|^2 dzdydt\right)^{\frac12}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k} \epsilon^5 \left|\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\right|^2 dydt\right)^{\frac12} \\ &\leq C\sqrt{\epsilon}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k} e^{4s\beta^*}(\gamma^*)^{14}\int_0^\epsilon \left|\frac{\partial}{\partial z}\widetilde{\widehat{\varphi}}_\epsilon(y,z,t)\right|^2 dzdydt\right)^{\frac12}\left(\int_0^T\!\!\int_{\Gamma_0\cap U_k} \left|\frac{\partial \widetilde{v}_k}{\partial z}(y,0,t)\right|^2 dydt\right)^{\frac12}. \end{aligned} \tag{4.12}$$

Now, from the choice of m_1 and m_2 and $\lambda > 1$ we have

$$(\gamma^*)^{14} e^{4s\beta^*} \leq C\gamma e^{2s\beta}$$

and from the Carleman inequality in Lemma 3.1, we get

$$A_\epsilon^{k,2}(v) \leq C\sqrt{\epsilon}\|\widehat{\varphi}_\epsilon\|_P \left\|\frac{\partial v_k}{\partial \nu}\right\|_{L^2(\Gamma_0\cap U_k\times(0,T))} \leq C\sqrt{\epsilon}\|\widehat{\varphi}_\epsilon\|_P \|v\|_{L^2(0,T;H^2(\Omega)\cap H_0^1)}, \tag{4.13}$$

which implies that $A_\epsilon^{k,2}(v) \to 0$ as $\epsilon \to 0^+$.

For $A_\epsilon^{k,1}(v)$, we have

$$
\begin{aligned}
A_\epsilon^{k,1}(v) &= \frac{1}{\epsilon^3}\int_0^T\int_{\Gamma_0\cap U_k}\int_0^\epsilon ze^{2s\beta^*}(\gamma^*)^7\int_0^z\left(\frac{\partial}{\partial s}\widetilde{\widehat{\varphi}_\epsilon}(y,s,t)-\frac{\partial}{\partial s}\widetilde{\widehat{\varphi}_\epsilon}(y,0,t)\right)\frac{\partial}{\partial z}\widetilde{v_k}(y,0,t)\,dsdzdydt\\
&+\frac{1}{\epsilon^3}\int_0^T\int_{\Gamma_0\cap U_k}\int_0^\epsilon ze^{2s\beta^*}(\gamma^*)^7\int_0^z\frac{\partial}{\partial z}\widetilde{\widehat{\varphi}_\epsilon}(y,0,t)\frac{\partial}{\partial z}\widetilde{v_k}(y,0,t)\,dsdzdydt\\
&= I_1+I_2.
\end{aligned}
$$

For the first term, we have that

$$
\begin{aligned}
I_1 &= \frac{1}{\epsilon^3}\int_0^T\int_{\Gamma_0\cap U_k}e^{2s\beta^*}(\gamma^*)^7\frac{\partial}{\partial z}\widetilde{v_k}(y,0,t)\int_0^\epsilon z\int_0^z\left(\int_0^s\frac{\partial^2}{\partial\eta^2}\widetilde{\widehat{\varphi}_\epsilon}(y,\eta,t)d\eta\right)dsdzdydt\\
&\le C\epsilon^{-3}\left(\int_0^T\int_{\Gamma_0\cap U_k}e^{4s\beta^*-2s\widehat{\beta}}(\gamma^*)^8\left|\frac{\partial}{\partial z}\widetilde{v_k}(y,0,t)\right|^2dydt\right)^{1/2}\\
&\quad\times\left(\int_0^T\int_{\Gamma_0\cap U_k}\left[e^{s\widehat{\beta}}\widehat{\gamma}^{-1/2}\int_0^\epsilon z\int_0^z\left(\int_0^s\frac{\partial^2}{\partial\eta^2}\widetilde{\widehat{\varphi}_\epsilon}(y,\eta,t)d\eta\right)dsdz\right]^2dydt\right)^{1/2}.
\end{aligned}
$$

Moreover, using Hölder inequality and (4.7), we get

$$
\begin{aligned}
&\int_0^T\int_{\Gamma_0\cap U_k}\left[e^{s\widehat{\beta}}\widehat{\gamma}^{-1/2}\int_0^\epsilon z\int_0^z\left(\int_0^s\frac{\partial^2}{\partial\eta^2}\widetilde{\widehat{\varphi}_\epsilon}(y,\eta,t)d\eta\right)dsdz\right]^2dydt\\
&\le\frac{\epsilon^3}{3}\int_0^T\int_{\Gamma_0\cap U_k}e^{2s\widehat{\beta}}\widehat{\gamma}^{-1}\int_0^\epsilon z\int_0^z s\int_0^s\left|\frac{\partial^2\widetilde{\widehat{\varphi}_\epsilon}}{\partial\eta^2}(y,\eta,t)\right|^2d\eta dsdz\,dydt\\
&\le C\epsilon^7\|\widehat{\varphi}_\epsilon\|_P^2,
\end{aligned}
$$

which gives that $I_1\to 0$ as $\epsilon\to 0^+$.

To finish the proof, we recall that $e^{s\widehat{\beta}}\widehat{\gamma}^{-\frac12}\frac{\partial\widehat{\varphi_\epsilon}}{\partial\nu}$ converges weakly to $e^{s\widehat{\beta}}\widehat{\gamma}^{-\frac12}\frac{\partial\widehat{\varphi}}{\partial\nu}$ in $L^2(\Gamma_0\times(0,T))$. From (4.9) and the previous estimates, we conclude that

$$
\begin{aligned}
\lim_{\epsilon\to 0}M_\epsilon^1(v) &= \lim_{\epsilon\to 0}\sum_{i=1}^{n_0}A_\epsilon^{k,1}(v)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\sum_{i=1}^{n_0}\int_0^T\int_{\Gamma_0\cap U_k}\int_0^\epsilon ze^{2s\beta^*}(\gamma^*)^7\int_0^z\frac{\partial}{\partial s}\widetilde{\widehat{\varphi}_\epsilon}(y,0,t)\frac{\partial}{\partial s}\widetilde{v_k}(y,0,t)\,dsdzdydt\\
&= \lim_{\epsilon\to 0}\frac{1}{3}\sum_{i=1}^{n_0}\int_0^T\int_{\Gamma_0\cap U_k}e^{2s\beta^*}(\gamma^*)^7\frac{\partial}{\partial z}\widetilde{\widehat{\varphi}_\epsilon}(y,0,t)\frac{\partial}{\partial z}\widetilde{v_k}(y,0,t)\,dydt\\
&= \lim_{\epsilon\to 0}\frac{1}{3}\int_0^T\int_{\Gamma_0}e^{2s\beta^*}(\gamma^*)^7\frac{\partial}{\partial\nu}\widehat{\varphi}_\epsilon(y,t)\frac{\partial}{\partial\nu}v(y,t)\,dydt\\
&= \frac{1}{3}\int_0^T\int_{\Gamma_0}e^{2s\beta^*}(\gamma^*)^7\frac{\partial}{\partial\nu}\widehat{\varphi}(y,t)\frac{\partial}{\partial\nu}v(y,t)\,dydt,
\end{aligned}
$$

and the proof of Lemma 4.2 is complete. □

5. Proof of Theorem 1.1

In this section we explain in which sense the distributed null control problem (1.1) converges to the boundary null control problem (1.2).

The following result is a direct consequence of Lemma 4.2.

Proposition 5.1. *Assume that*

$$\iint_Q e^{2s\beta}|\mathcal{L}^*\varphi_\epsilon|^2 dxdt + \frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7|\varphi_\epsilon|^2 dxdt \le C,$$

where C does not depend on ϵ. Then, the linear form $G_\epsilon : L^2(\Omega)\times L^2(Q)\to\mathbb{R}$ given by

$$(z^T,h)\mapsto \frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\varphi_\epsilon(x,t)z(x,t)\,dxdt,$$

for every $(z^T,h)\in L^2(\Omega)\times L^2(Q)$, where z is the solution of

$$\begin{cases} -z_t-\Delta z = h & \text{in} \quad Q,\\ z=0 & \text{on} \quad \Sigma,\\ z(x,T)=z^T & \text{in} \quad \Omega,\end{cases} \tag{5.1}$$

converges weakly to $G : L^2(\Omega)\times L^2(Q)\to\mathbb{R}$ given by

$$G(z^T,h)=\frac{1}{3}\iint_{\Gamma_0\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\frac{\partial\varphi}{\partial\nu}(y,t)\frac{\partial z}{\partial\nu}(y,t)\,dydt,$$

with $e^{s\widehat{\beta}}\widehat{\gamma}^{-1}\dfrac{\partial\varphi}{\partial\nu}\in L^2(\Gamma_0\times(0,T))$.

Let us now prove the main result of this paper.

Proof of Theorem 1.1. Let now $(\widehat{y}_\epsilon,\widehat{f}_\epsilon) = (e^{2s\beta}\mathcal{L}^*\widehat{\varphi}_\epsilon, -\frac{1}{\epsilon^3}e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon)$ be a solution of the distributed null controllability problem (1.1), that is to say

$$\left|\begin{array}{lll} \widehat{y}_{\epsilon\,t}-\Delta\widehat{y}_\epsilon = -\dfrac{1}{\epsilon^3}e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon 1_{\omega_\epsilon} & \text{in} & Q,\\ \widehat{y}_\epsilon = 0 & \text{on} & \Sigma,\\ \widehat{y}_\epsilon(x,0)=y_0(x) & \text{in} & \Omega,\\ \widehat{y}_\epsilon(x,T)=0 & \text{in} & \Omega.\end{array}\right. \tag{5.2}$$

We recall that

$$\|e^{-s\beta}\widehat{y}_\epsilon\|^2_{L^2(Q)} \le C\|y_0\|^2_{L^2(\Omega)}, \tag{5.3}$$

where C does not depend on ϵ.

It is not difficult to see that

$$\iint_Q \widehat{y}_\epsilon h\,dxdt = -\frac{1}{\epsilon^3}\iint_{\omega_\epsilon\times(0,T)} e^{2s\beta^*}(\gamma^*)^7\widehat{\varphi}_\epsilon z\,dxdt + \int_\Omega y_0(x)z(x,0)\,dx, \tag{5.4}$$

for every $(z^T,h)\in L^2(\Omega)\times L^2(Q)$, where z is the associated solution to (5.1).

From Proposition 5.1 and estimate (5.3), it follows that

$$\iint_Q yh\,dxdt = -\frac{1}{3}\int_0^T\int_{\Gamma_0} e^{2s\beta^*}(\gamma^*)^7\frac{\partial\varphi}{\partial\nu}\frac{\partial z}{\partial\nu}\,dxdt + \int_\Omega y_0(x)z(x,0)\,dx, \tag{5.5}$$

where $e^{-s\beta}\widehat{y}\in L^2(Q)$ and $e^{s\widehat{\beta}}\widehat{\gamma}^{-1}\frac{\partial\varphi}{\partial\nu}\in L^2(\Gamma_0\times(0,T))$.

We finish noticing that identity (5.5) implies that y is the solution of

$$\begin{cases} y_t - \Delta y = 0 & \text{in} \quad Q \\ y = \frac{1}{3} e^{2s\beta^*} (\gamma^*)^7 \dfrac{\partial \varphi}{\partial \nu} 1_{\Gamma_0} & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad \Omega, \\ y(x,T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{5.6}$$

which means that $(y, \dfrac{1}{3} e^{2s\beta^*} (\gamma^*)^7 \dfrac{\partial \varphi}{\partial \nu})$ solves the boundary null control problem (1.2). □

Appendix A. Proof of Lemma 3.1

Proof of Lemma 3.1. We begin noticing that, for s fixed, we have

$$\iint_Q e^{2s\alpha} |h|^2 \, dxdt + \epsilon^{-3} \iint_{\omega_\epsilon \times (0,T)} e^{2s\alpha} \phi^7 |\varphi|^2 \, dxdt$$
$$\leq C \left(\epsilon^{-3} \int_0^T \int_{\omega_\epsilon} e^{2s\beta} \gamma^7 |\varphi|^2 \, dxdt + \iint_Q e^{2s\beta} |h|^2 \, dxdt \right),$$

because $e^{2s\alpha}\phi^7$ is uniformly bounded on $[0, T/2]$, $e^{2s\beta}\gamma^7$ is bounded from below on $[0, T/2]$ by a positive constant, $\alpha = \beta$ and $\phi = \gamma$ on $(T/2, T)$ and $\alpha \leq \beta$ in Q.

Also from the fact that $\alpha = \beta$ and $\phi = \gamma$ on $(T/2, T)$, it follows that

$$\int_{T/2}^T \int_\Omega e^{2s\beta} \gamma^{-1} \left(|\varphi_t|^2 + \sum_{i,j=1}^N \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right|^2 \right) dxdt + \int_{T/2}^T \int_\Omega e^{2s\beta} \gamma |\nabla \varphi|^2 \, dxdt + \int_{T/2}^T \int_\Omega e^{2s\beta} \gamma^3 |\varphi|^2 \, dxdt$$
$$= \int_{T/2}^T \int_\Omega e^{2s\alpha} \phi^{-1} \left(|\varphi_t|^2 + \sum_{i,j=1}^N \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right|^2 \right) dxdt + \int_{T/2}^T \int_\Omega e^{2s\alpha} \phi |\nabla \varphi|^2 dxdt + \int_{T/2}^T \int_\Omega e^{2s\alpha} \phi^3 |\varphi|^2 dxdt$$
$$\leq C \left(\epsilon^{-3} \int_0^T \int_{\omega_\epsilon} e^{2s\beta} \gamma^7 |\varphi|^2 \, dxdt + \iint_Q e^{2s\beta} |h|^2 \, dxdt \right). \tag{A.1}$$

Now, we introduce a cut-off function $\theta \in C^\infty([0,T])$ such that $0 \leq \theta \leq 1$ with $\theta = 1$ on $[0, T/2]$ and $\theta = 0$ on $[3T/4, T]$. Denoting by $z = \theta\varphi$, we have the equation

$$\begin{cases} -z_t - \Delta z = \theta h - \theta' \varphi & \text{in} \quad Q, \\ z = 0 & \text{in} \quad \Sigma, \\ z(x,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

Using the Carleman estimate given in Theorem 2.1, we have

$$\|\theta' \varphi\|^2_{L^2(Q)} \leq C \int_{T/2}^{3T/4} \int_\Omega |\varphi|^2 dxdt \leq C \int_{T/2}^{3T/4} \int_\Omega e^{2s\alpha} \phi^3 |\varphi|^2 dxdt$$
$$\leq C \left(\epsilon^{-3} \int_0^T \int_{\omega_\epsilon} e^{2s\beta} \gamma^7 |\varphi|^2 \, dxdt + \iint_Q e^{2s\beta} |h|^2 \, dxdt \right),$$

since the function $e^{2s\alpha}\phi^3$ is bounded from below on $(T/2, 3T/4)$ by a positive constant.

Also, as θ vanishes on $(3T/4, T)$, we have

$$\|\theta h\|^2_{L^2(Q)} \leq C\left(\epsilon^{-3}\int_0^T\int_{\omega_\epsilon} e^{2s\beta}\gamma^7|\varphi|^2\,dxdt + \iint_Q e^{2s\beta}|h|^2\,dxdt\right).$$

From energy estimates, regularity results for the heat equation with Dirichlet boundary condition, and the fact that $\theta = 1$ on $[0, T/2]$, we obtain

$$\begin{aligned}\|\varphi(0)\|^2_{L^2(\Omega)} &+ \|\varphi\|^2_{L^2(0,T/2;H^2(\Omega))} + \|\varphi_t\|^2_{L^2(0,T/2;L^2(\Omega))}\\ &\leq \|z(0)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(0,T;H^2(\Omega))} + \|z_t\|^2_{L^2(0,T:L^2(\Omega))}\\ &\leq C(\|\theta h\|^2_{L^2(Q)} + \|\theta'\varphi\|^2_{L^2(Q)}).\end{aligned}$$

Hence, since the weights of the form $e^{2s\beta}\gamma^k$ are uniformly bounded in $[0, T/2]$, we obtain the following estimate on $(0, T/2)$:

$$\begin{aligned}\|\varphi(0)\|^2_{L^2(\Omega)} &+ \int_0^{T/2}\int_\Omega e^{2s\beta}\gamma^{-1}\left(|\varphi_t|^2 + \sum_{i,j=1}^N\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^2\right)dxdt\\ &+ \int_0^{T/2}\int_\Omega e^{2s\beta}\gamma|\nabla\varphi|^2\,dxdt + \int_0^{T/2}\int_\Omega e^{2s\beta}\gamma^3|\varphi|^2\,dxdt\\ &\leq C\left(\epsilon^{-3}\int_0^T\int_{\omega_\epsilon} e^{2s\beta}\gamma^7|\varphi|^2\,dxdt + \iint_Q e^{2s\beta}|h|^2\,dxdt\right)\end{aligned} \tag{A.2}$$

Finally, gathering estimates A.1 and A.2, we obtain

$$\begin{aligned}\|\varphi(0)\|^2_{L^2(\Omega)} &+ \iint_Q e^{2s\beta}\gamma^{-1}\left(|\varphi_t|^2 + \sum_{i,j=1}^N\left|\frac{\partial^2\varphi}{\partial x_i\partial x_j}\right|^2\right)dxdt+\\ &+ \iint_Q e^{2s\beta}\gamma|\nabla\varphi|^2\,dxdt + \iint_Q e^{2s\beta}\gamma^3|\varphi|^2\,dxdt\\ &\leq C\left(\epsilon^{-3}\int_0^T\int_{\omega_\epsilon} e^{2s\beta}\gamma^7|\varphi|^2\,dxdt + \iint_Q e^{2s\beta}|h|^2\,dxdt\right),\end{aligned}$$

which proves Lemma 3.1.

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