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NULL-CONTROLLABILITY OF PERTURBED POROUS MEDIUM GAS FLOW *

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Abstract. In this work, we investigate the null-controllability of a nonlinear degenerate parabolic equation, which is the equation satisfied by a perturbation around the self-similar solution of the porous medium equation in Lagrangian-like coordinates. We prove a local null-controllability result for a regularized version of the nonlinear problem, in which singular terms have been removed from the nonlinearity. We use spectral techniques and the source-term method to deal with the linearized problem and the conclusion follows by virtue of a Banach fixed-point argument. The spectral techniques are also used to prove a null-controllability result for the linearized thin-film equation, a degenerate fourth order analog of the problem under consideration.

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1. Introduction

Due to their relevance in physics and engineering, much attention has been devoted in the scientific literature to fluid systems involving the evolution of a free moving boundary. We refer for example to [31] for models regarding the density of a gas penetrating a solid rock, and to [5, 37] for models on the evolution of thin liquid films in wetting and spreading phenomena. These examples appear in physical and industrial processes such as oil recovery, membranes in biophysics, and spin coating of microchips. Despite occurring in such diverse scientific fields, the mathematical modeling of these mechanisms is quite similar and understanding the control-theoretical aspects thereof is of high importance for applications.

An example of a simplified, applicable model is the *porous medium equation*

$$\partial_t h - \partial_z^2(h^m) = 0 \tag{1.1}$$

where $m > 1$. The state $h(t, z)$ may represent the density distribution of a gas flowing in a porous medium, or the height of a thin liquid film deposited onto a solid substrate. By developing the diffusion term, it is readily seen that equation (1.1) degenerates when the state h approaches zero. Thus, any solution with compactly supported initial datum retains the compact support in any finite time. In physical terms, the diffusing gas does not reach any point in space instantaneously, but rather propagates with finite speed. This property results

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in the fact that the porous medium equation is indeed a free boundary problem, the free boundary being given by $\partial\{h > 0\}$. In terms of thin films (see Section 5 for the related thin-film equation), it represents the interface separating the liquid, surrounding air and the adjacent solid, as in Figure 1.

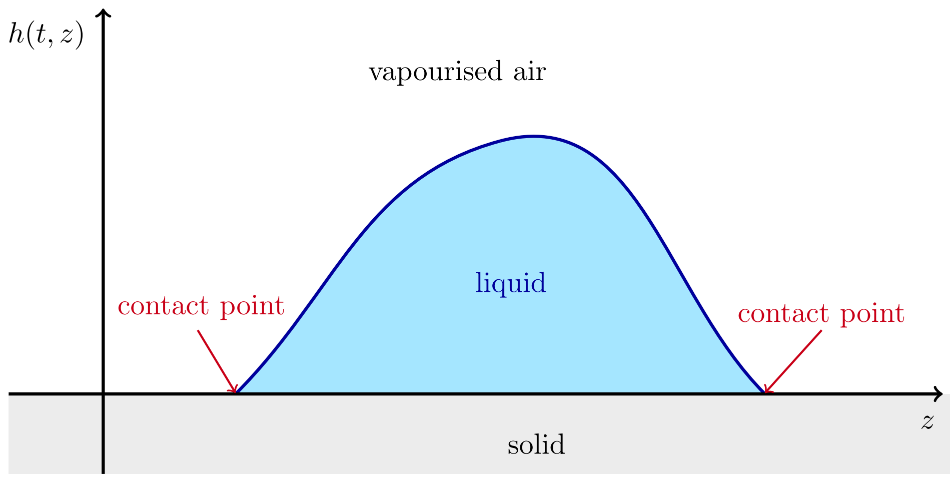

FIGURE 1. The free boundary represents the contact points where the three phases of gas, solid and liquid connect.

While the analytical properties of (1.1) are well understood (particularly in the one dimensional case, see [39]), the literature on its control-theoretical aspects is rather scarce. In view of the known asymptotic behavior of the free boundary problem for large times (see [39, Chapter 18]) and the desired positivity of the state, a natural question which arises is whether one may control the state $h(t, z)$, as well as its interface, to the self-similar Barenblatt trajectory

$$h_B(t, z) = (t+1)^{-\frac{1}{m+1}} \left(1 - \frac{m-1}{2m(m+1)} \frac{z^2}{(t+1)^{\frac{2}{m+1}}}\right)^{\frac{1}{m-1}} \qquad \text{in } \{h_B > 0\}$$

in a given finite time $T > 0$ by means of an additional forcing control term. To the best of our knowledge, this kind of exact-controllability to trajectories question has not been addressed in the existing literature on the porous medium equation.

An important difficulty when tackling this question is the moving time-dependent support of the solution and the target Barenblatt trajectory. As the two are defined in different domains, perturbations of the form $h_B + y$ around Barenblatt are difficult to define in view of linearizing, a key step in proving controllability. Due to the slightly complex form of the Barenblatt, it is more convenient to look at the equation satisfied by the *pressure* $v = \frac{m}{m-1} h^{m-1}$ in self-similar coordinates, namely

$$\begin{cases} \partial_t v - v\partial_z^2 v - (\sigma+1)((\partial_z v)^2 + x\partial_z v) - v = 0 & \text{in } \{v > 0\} \\ v(0, z) = v_0(z) & \text{in } \{v_0 > 0\}, \end{cases} \tag{1.2}$$

(see [34, Section 1.2]) where $\sigma = -\frac{m-2}{m-1} > -1$. In this case, the Barenblatt solution is stationary and supported in the unit interval:

$$\rho(z) = \frac{1}{2}\left(1 - z^2\right) \qquad \text{for } z \in (-1, 1). \tag{1.3}$$

The motivation behind our work is thus to know if one can steer the state $v(t,z)$ and its interface to the stationary Barenblatt solution $\rho(z)$ in a given time $T>0$, by means of an additional forcing control term in the equation.

To overcome the difficulty of the moving domain, a Lagrangian-like change of variables (thus depending on the solution, and called *von Mises transformation*) may be applied, mapping the moving support of the solution onto the support of the Barenblatt profile, now the interval $(-1,1)$. The change of coordinates depends on the solution (and thus its smallness and regularity), and in these new variables the Barenblatt reduces to the constant 1. Since the transformed solution and Barenblatt are defined in the same fixed domain, it will be possible to consider perturbations around the latter. This transformation was introduced by Koch [22], who uses it to show the smoothness of the free boundary and of the pressure up to the interface in any space dimension (see also the work of Kienzler [21]). It is subsequently adapted and used by Seis [34] for quantifying the self-similar asymptotics of the equation close to Barenblatt by using the spectrum of the linearized operator and invariant manifolds. In all of the above-cited works, the authors consider compactly supported, Hölder continuous initial pressures v_0, with non-vanishing gradient. This last condition ensures avoidance of the *waiting-time phenomenon*, namely the existence of a positive time $T^*>0$ up to which the free boundary is stationary, see [39, Chapter 14].

1.1. **Problem formulation**

After the von Mises transform and after cosidering perturbations around the transformed Barenblatt, we are brought to consider the control problem for the transformed perturbation equation (see [34, Section 3]):

$$\begin{cases}\partial_t y-\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y)=\mathcal{N}(y)+u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1)\\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1)=0 & \text{in } (0,T)\\ y(0,x)=y_0(x) & \text{in } (-1,1),\end{cases} \tag{1.4}$$

where $T>0$ and $\sigma>-1$, and the nonlinearity $\mathcal{N}(y)=\mathcal{N}(y,\partial_x y)$ is of the form

$$\mathcal{N}(y)=\rho F(y,\partial_x y)-\rho^{-\sigma}\partial_x(\rho^{\sigma+1}xF(y,\partial_x y)),\quad F(p,q)=\frac{q^2}{1+p+xq},\quad p,q\in\mathbb{R}. \tag{1.5}$$

The distributed control $u=u(t,x)$ appearing in (1.4) actuates inside an open, non-empty subset $\omega=(a,b)\subsetneq(-1,1)$. The solution $y(t,x)$ is a perturbation around the Barenblatt in the new variables (see Remark A.1). Consequently, the null-controllability of (1.4) would heuristically correspond to the exact-controllability of the pressure $v(t,z)$ and its free boundary of a controlled version of (1.2) to the original Barenblatt $\rho(z)$, after reverting the von Mises transformation. As said above, $m=\frac{\sigma+2}{\sigma+1}>-1$.

Hereinafter, we will investigate the null-controllability of (1.4), namely the possibility of steering the solution y to 0 at time T by means of the control u. Considering the full nonlinear problem (1.4) requires high regularity of the trajectory, and thus of the control. Due to the peculiar functional setting detailed below, ensuring this regularity is not straightforward. Hence, in this work, we will prove a local null-controllability result for a regularized version of the nonlinear problem (1.4), in which the singular terms appearing in the denominator of (1.5) have been removed.

1.2. **Functional setting**

Recalling the definition of the degenerate coefficient ρ in (1.3), for $k\geq 0$ we consider spaces

$$\mathcal{H}^k:=\{f\in L^1_{\text{loc}}(-1,1)\colon \|f\|_{\mathcal{H}^k}<\infty\},$$

where $\|f\|^2_{\mathcal{H}^k} := \langle f, f\rangle_{\mathcal{H}^k}$ is the norm induced from the inner product

$$\langle f_1, f_2\rangle_{\mathcal{H}^k} := \sum_{j=0}^{k} \int_{-1}^{1} \rho^{\sigma+j}(\partial_x^j f_1)(\partial_x^j f_2)\,\mathrm{d}x.$$

As $\rho^\sigma \in L^1(-1,1)$ whenever $\sigma > -1$, the measure $\rho^\sigma\,\mathrm{d}x$ is a Radon measure, it is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$ and possesses the same null-sets. For any $k \geq 0$, $\mathcal{H}^k$ are separable Hilbert spaces of which $C^\infty([-1,1])$ are dense subsets according to [33, Lemma 2], [34, Section 4.2]. Additionally, on any $\omega \subsetneq (-1,1)$ they coincide with the unweighted Sobolev spaces $H^k(\omega)$, $k \geq 0$.

1.3. **The main results**

While the nonlinearity in (1.4) is essentially quadratic in a neighborhood of the origin, the denominator may be singular and applying a fixed-point argument using only the weighted Sobolev space theory is not straightforward. To mend this issue, in this paper we concentrate on a truncated version of the nonlinearity. Namely, we multiply the nonlinear terms by a smooth cut-off function which vanishes at points where y and/or $\partial_x y$ are large; the truncated equation would thus be linear at such points.

Let $\chi : [0,\infty) \to [0,1]$ be a smooth cut-off function, supported on $[0,4)$ with $\chi(x) \equiv 1$ on $[0,1]$. Let $0 < \varepsilon, \delta < 1$ satisfying $4(\varepsilon+\delta) < 1$ be fixed. For $p, q \in \mathbb{R}$, and recalling the definition of F in (1.5), we define

$$F_{\varepsilon,\delta}(p,q) = \chi\left(\frac{p^2}{\delta^2}\right)\chi\left(\frac{q^2}{\varepsilon^2}\right)F(p,q). \tag{1.6}$$

We will henceforth only be interested in Problem (1.4) wherein $\mathcal{N}$ is replaced by $\rho F_{\varepsilon,\delta}$, namely

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = \rho F_{\varepsilon,\delta}(y, \partial_x y) + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1). \end{cases} \tag{1.7}$$

We recall, as per (1.3), that $\rho(x) = \frac{1}{2}(1-x^2)$. The main result we show in this work is the following.

Theorem 1.1. *Let $T > 0$, let $\omega \subsetneq (-1,1)$ be an open, non-empty interval, and let $\sigma \in (-1,0)$. Then there exists $r > 0$ such that for every $y_0 \in \mathcal{H}^1$ satisfying $\|y_0\|_{\mathcal{H}^1} \leq r$, there exists a control $u \in L^2(0,T;L^2(\omega))$ for which the unique solution $y \in L^2(0,T;\mathcal{H}^2) \cap C^0([0,T];\mathcal{H}^1)$ of (1.7) satisfies $y(0,\cdot) = y_0$ and $y(T,\cdot) = 0$.*

Remark 1.1. *While being a first step in this direction, Theorem 1.1 is not sufficient to deduce a local controllability result to the Barenblatt trajectory for an associated distributed control problem of the free boundary problem (1.2). If (1.7) is null-controllable with the nonlinearity $\mathcal{N}(y) = \rho F(y) - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}xF(y))$ as in (1.4), then one could deduce such a result. To achieve this, one would need to remove the cut-off factor $\chi(p^2/\delta^2)\chi(q^2/\varepsilon^2)$, and add the high order nonlinear term. The cut-off is identically 1 whenever the solution is of sufficiently small $C^{0,1}([0,T]\times[0,1])$-norm, and this regularity is also sufficient to revert the von Mises transformation. However, Theorem 1.1 does not provide this regularity. Nonetheless, it is the best result that can be obtained by means of an only $L^2(L^2)$-regular control. See Remark A.1 for more details.*

Looking at (1.7), it is natural to first study the null-controllability of the corresponding linear problem, where the nonlinear term is replaced by a source term:

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = f + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1). \end{cases} \tag{1.8}$$

The nonlinear term would be seen as a small perturbation, and be dealt with by means of a fixed-point argument. The latter argument will rely on the particular structure of the nonlinearity, which is now non-singular and essentially quadratic due to the cut-off factor.

Remark 1.2. *The requirement $\sigma \in (-1,0)$ only appears when estimating the nonlinear term in the weighted spaces (see Section 4). The null-controllability and well-posedness of the linearized problem* (1.8) *holds true for any $\sigma > -1$, as seen below. We recall that σ is related to the nonlinearity exponent of the porous medium equation by $m = \frac{\sigma+2}{\sigma+1}$.*

To prove the null-controllability of Problem (1.8), we will make use of the so-called *source-term method*, first introduced by Liu, Takahashi & Tucsnak [28]. Roughly speaking, the strategy involves first showing the null-controllability of the homogeneous problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1), \end{cases} \tag{1.9}$$

and the null-controllability of Problem (1.8) follows provided the source term f vanishes with appropriate decay as $t \nearrow T$. More specifically, the decay of the source term should be quick enough near the final time compared to the control cost in small time. The null-controllability of problem (1.9) is done by combining duality and spectral techniques, making use of the results obtained in the works of Seis [33, 34]. Namely, we prove the following result.

Theorem 1.2. *Let $T > 0$, $\omega \subsetneq (-1,1)$ be an open, non-empty interval, and $\sigma > -1$. Then, for any $y_0 \in \mathcal{H}^0$, there exists a control $u \in L^2(0,T;L^2(\omega))$ such that the unique solution $y \in L^2(0,T;\mathcal{H}^1)\cap C^0([0,T];\mathcal{H}^0)$ of* (1.9) *satisfies $y(0,\cdot) = y_0$ and $y(T,\cdot) = 0$.*

1.4. **State of the art**

In [13], Coron, Diáz, Drici & Mignazzini prove the null-controllability of the porous medium equation set on $(0,1)$ using Dirichlet boundary controls on both ends as well as a scalar forcing control. A control on one end can also be used as long as the other boundary condition is a Neumann one. The authors' strategy follows the *return method* to avoid the appearance of a free boundary, namely, the construction of an adequate non-trivial time-only dependent trajectory, starting and ending at 0, around which the problem is linearized. By a scaling argument, global null-controllability is achieved in arbitrarily small time, and the method guarantees non-negativity of the controls, and thus of the state for positive initial data. This differs from the original motivation behind our work, which was to control the pressure and its free boundary to the non-trivial Barenblatt profile (instead of the null-state). We also refer to the works of Liu & Gao [26, 27] for nonnegativity preserving approximate controllability results for the multi-dimensional porous medium equation set on a bounded domain by means of a distributed control.

Null-controllability results for one-dimensional parabolic equations which degenerate at the boundary such as

$$\partial_t y - \partial_x(x^\alpha \partial_x y) = u\mathbf{1}_\omega \quad \text{in } (0,T)\times(0,1),$$

where $\alpha \in [0,2)$ are shown in the works of Alabau-Boussouira, Cannarsa, Martinez & Vancostenoble [1,9,10] by using Carleman inequalities with degeneracy-adapted weights. In general, one distinguishes the *weak* ($\alpha \in [0,1)$) and *strong* ($\alpha \in [1,2)$) degeneracies, as the functional setting and boundary conditions are different for both cases. The case $\alpha \geq 2$ is excluded as null-controllability does not hold (only *regional* results are true, see [8]). We also refer to the monograph [11] for results on two dimensional problems of the above kind. The question of boundary null-controllability has also been addressed. For instance, Gueye [19] combines the transmutation method and spectral techniques for a weakly degenerate problem, and Moyano [30] makes use of the flatness method for a strongly degenerate problem.

These studies have been extended in the works of Cannarsa, Fragnelli & Rocchetti [6, 7, 16] to degenerate parabolic problems in non-divergence form (more alike (1.4)), such as

$$\partial_t y - a(x)\partial_x^2 y + b(x)\partial_x y + c(t,x)y = u\mathbf{1}_\omega \quad \text{in } (0,T)\times(0,1)$$

where $a \in C^0([0,1])$ may degenerate at $x = 0$ and $x = 1$. Therein, pure homogeneous Dirichlet and Neumann boundary conditions are considered, and null-controllability results are obtained by Carleman inequalities.

Our work may be seen as a further contribution to the controllability theory of linear degenerate parabolic equations. Indeed, while the differential operator in (1.9) may be rewritten as $-\rho\partial_x^2 y + (\sigma+1)x\partial_x y$, the weighted Neumann boundary conditions, which are the natural ones from the calculus of variations point of view, have not been considered in the above-cited papers on problems in non-divergence form. In particular, we do not consider the same weight and functional framework as in [7, 16], since $\frac{b}{a} = \frac{2(\sigma+1)x}{1-x^2} \notin L^1(-1,1)$ in our case. While we use spectral techniques, up to the best of our knowledge, a Carleman inequality for our functional setting is lacking.

Finally, we mention that our strategy for proving the null-controllability of the linearized problem (1.9) can also be applied to obtain a null-controllability result (see Section 5) for the thin-film equation linearized around its self-similar solution, which is a fourth-order degenerate parabolic equation. Up to the best of our knowledge, this has not been tackled in the literature.

1.5. **Scope**

We present the functional properties of the governing differential operator in Section 2. In Section 3, we use its explicit spectrum for proving Theorem 1.2. An adaptation of the source-term method allows us to deduce the null-controllability of the linearized problem (1.8) (Theorem 3.1). In Section 4, we conclude the proof of Theorem 1.1 by means of a Banach fixed-point argument. Finally, in Section 5 we apply the linear controllability theory from Section 3 to deduce the null-controllability of the linearized thin-film equation, a fourth-order analog of (1.9).

1.6. **Notation**

Whenever the dependence on parameters of a constant is not specified, we will write $f \lesssim_S g$ whenever a constant $C \geq 1$, depending only on the set of parameters S, exists such that $f \leq Cg$.

2. The linear degenerate operator

This section is dedicated to a study of the functional and spectral properties of the linear operator $\mathcal{A} = -\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x)$, which will be shown to be self-adjoint and with compact resolvents when viewed as an unbounded operator on the weighted Lebesgue space $\mathcal{H}^0$. The arguments will follow standard theory, starting by noting that symmetry holds as

$$\int_{-1}^{1} \rho^\sigma (\mathcal{A}f_1) f_2 \,\mathrm{d}x = \int_{-1}^{1} \rho^{\sigma+1}(\partial_x f_1)(\partial_x f_2)\,\mathrm{d}x \tag{2.1}$$

for all $f_1, f_2 \in C^\infty([-1,1])$ via integration by parts. To accurately characterize the domain of $\mathcal{A}$ we present some embedding results for the weighted Sobolev spaces $\mathcal{H}^k$.

2.1. **Embeddings for weighted Sobolev spaces**

The following two useful lemmas are taken from the work of Gnann [18]. For the sake of completeness, we provide short proofs in Appendix B.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$. Then*

$$\|(1-x^2)^\alpha f\|^2_{C^0([-1,1])} \lesssim_\alpha \int_{-1}^1 (1-x^2)^{2\alpha-1} f^2 \,\mathrm{d}x + \int_{-1}^1 (1-x^2)^{2\alpha+1} (\partial_x f)^2 \,\mathrm{d}x \tag{2.2}$$

holds for all $f \in C^\infty([-1,1])$.

The following Lemma is as a Hardy-like inequality for the spaces $\mathcal{H}^k$.

Lemma 2.2. *Suppose $\alpha > -1$ and $\beta \in \mathbb{R}$. Then there exists $C = C(\alpha, \beta) > 0$ such that*

$$\int_{-1}^1 (1-x^2)^\alpha f^2 \,\mathrm{d}x \le C \int_{-1}^1 \left((1-x^2)^\beta f^2 + (1-x^2)^{\alpha+2} (\partial_x f)^2\right) \mathrm{d}x \tag{2.3}$$

holds for all $f \in C^\infty([-1,1])$. The constant $C(\alpha,\beta)$ diverges as $\alpha \searrow -1$.

Remark 2.1. *We highlight that an inequality such as*

$$\int_{-1}^1 (1-x^2)^\alpha f^2 \,\mathrm{d}x \lesssim_\alpha \int_{-1}^1 (1-x^2)^{\alpha+2} (\partial_x f)^2 \,\mathrm{d}x$$

is not true, as any nonzero constant is a counterexample.

We combine the two previous lemmas to deduce the following result, which may also be seen as a weighted trace estimate.

Lemma 2.3. *Let $k \ge 1$, $\ell \ge 0$ and $\alpha \ge \frac{\sigma+1+\ell-k}{2}$ with $\alpha > 0$. Then there exists $C = C(k,\alpha) > 0$ such that*

$$\|(1-x^2)^\alpha \partial_x^\ell f\|_{C^0([-1,1])} \le C \|f\|_{\mathcal{H}^{k+\ell}}$$

holds for all $f \in C^\infty([-1,1])$.

Proof. We may replace f by its derivatives, in view of the definition of the $\mathcal{H}^k$-norms. It is thus sufficient to prove the statement for $\ell = 0$. The latter fact follows by successive applications of (2.3) to (2.2), with the effect of

$$\|(1-x^2)^\alpha f\|^2_{C^0([-1,1])} \lesssim_{\alpha,k} \sum_{j=0}^k \int_{-1}^1 (1-x^2)^{2\alpha+2k-1} (\partial_x^j f)^2 \,\mathrm{d}x. \tag{2.4}$$

As $(1-x^2) \le 1$ in $[-1,1]$ and as, by definition,

$$\|f\|^2_{\mathcal{H}^k} := \sum_{j=0}^k \int_{-1}^1 (1-x^2)^{\sigma+j} (\partial_x^j f)^2 \,\mathrm{d}x,$$

we see that it one picks $\alpha \ge \frac{\sigma+1-k}{2}$ with $\alpha > 0$ in (2.4), then $2\alpha + 2k - 1 \ge \sigma + j$ for all $0 \le j \le k$. Thus

$$\sum_{j=0}^k \int_{-1}^1 (1-x^2)^{2\alpha+2k-1} (\partial_x^j f)^2 \,\mathrm{d}x \le \|f\|^2_{\mathcal{H}^k},$$

which in view of (2.4) allows us to conclude. □

The inequality above fails when $k = \sigma + 1 + \ell$ due to the failure of the underlying Hardy inequality. Let us now illustrate (in the particular case of $\mathcal{H}^2$, and recall the definition of ρ in (1.3)) why the previous Lemma may be seen as a weighted trace estimate.

Lemma 2.4 (Boundary conditions)**.** *Let $\sigma > -1$. Then $(\rho^{\sigma+1}\partial_x f)(\pm 1) = 0$ for any $f \in \mathcal{H}^2$.*

Proof. By Lemma 2.3, $\rho^{\sigma+1}\partial_x f$ is continuous on $[-1,1]$. Thus there exists $\lambda \in \mathbb{R}$ such that $(\rho^{\sigma+1}\partial_x f)(x) \to \lambda$ as $x \to \pm 1$. Should $\lambda \neq 0$, then by continuity

$$\rho(x)^\sigma (\partial_x f(x))^2 \geq \frac{\lambda^2}{4\rho(x)^{\sigma+2}}$$

for x near ± 1. As $1/\rho^{\sigma+2} \notin L^1(-1,1)$ whenever $\sigma > -1$, the above inequality along with Lemma 2.2 contradict $f \in \mathcal{H}^2$. □

2.2. Spectrum of the linear operator

We henceforth fix $\sigma > -1$. We summarize the main functional and spectral properties of $\mathcal{A} = -\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x)$. We begin by the following a priori estimate.

Lemma 2.5. *Let $f \in \mathcal{H}^0$ be given and let $u \in \mathcal{H}^1$ be a weak solution to $\mathcal{A}u = f$. There exist $C_1 > 0$ and $C_2 > 0$ such that*

$$(\sigma+1)^2 \int_{-1}^1 \rho^\sigma x^2 (\partial_x u)^2 \,\mathrm{d}x \leq C_1 \int_{-1}^1 \rho^\sigma f^2 \,\mathrm{d}x,$$

and

$$\int_{-1}^1 \rho^{\sigma+2} (\partial_x^2 u)^2 \,\mathrm{d}x \leq C_2 \int_{-1}^1 \rho^\sigma f^2 \,\mathrm{d}x.$$

Proof. Let $\{f_k\}_{k=0}^\infty \subset C^\infty([-1,1])$ be a sequence converging to $f \in \mathcal{H}^0$, and let u_k be a weak solution to $\mathcal{A}u_k = f_k$ for $k \geq 0$. By [23], $u_k \in C^\infty([-1,1])$ and we may thus work with smooth functions to conclude. For notational simplicity, we remove the subscripts k in what follows. We multiply $\mathcal{A}u = f$ by $(\sigma+1)\rho^\sigma x \partial_x u$ and integrate:

$$-(\sigma+1)\int_{-1}^1 \partial_x(\rho^{\sigma+1}\partial_x u) x \partial_x u \,\mathrm{d}x = (\sigma+1)\int_{-1}^1 \rho^\sigma f x \partial_x u \,\mathrm{d}x. \tag{2.5}$$

Integration by parts allows us to rewrite the left-most term as

$$-\int_{-1}^1 \partial_x(\rho^{\sigma+1}\partial_x u) x \partial_x u \,\mathrm{d}x = \int_{-1}^1 \rho^{\sigma+1}(\partial_x u)^2 \,\mathrm{d}x + \int_{-1}^1 \rho^{\sigma+1} x (\partial_x u)(\partial_x^2 u) \,\mathrm{d}x.$$

We now integrate the right-most term by parts to deduce

$$\begin{aligned}\int_{-1}^1 \rho^{\sigma+1} x (\partial_x u)(\partial_x^2 u) \,\mathrm{d}x = (\sigma+1)\int_{-1}^1 \rho^\sigma x (\partial_x u)^2 \,\mathrm{d}x - \int_{-1}^1 \rho^{\sigma+1} x (\partial_x^2 u)(\partial_x u) \,\mathrm{d}x \\ - \int_{-1}^1 \rho^{\sigma+1}(\partial_x u)^2 \,\mathrm{d}x.\end{aligned}$$

Hence,

$$-\int_{-1}^1 \partial_x(\rho^{\sigma+1}\partial_x u) x \partial_x u \,\mathrm{d}x = \frac{1}{2}\int_{-1}^1 \rho^{\sigma+1}(\partial_x u)^2 \,\mathrm{d}x + \frac{\sigma+1}{2}\int_{-1}^1 \rho^\sigma x^2 (\partial_x u)^2 \,\mathrm{d}x. \tag{2.6}$$

As $\sigma > -1$, plugging (2.6) in (2.5) and applying the Young inequality to deduce

$$(\sigma+1)^2 \int_{-1}^1 \rho^\sigma x^2 (\partial_x u)^2 \,\mathrm{d}x \leq \epsilon(\sigma+1)^2 \int_{-1}^1 \rho^\sigma x^2 (\partial_x u)^2 \,\mathrm{d}x + \frac{1}{4\epsilon}\int_{-1}^1 \rho^\sigma f^2 \,\mathrm{d}x$$

for all $\epsilon > 0$. Choosing $\epsilon < 1$ yields the desired conclusion.

For the second estimate, using $(a-b)^2 \leq 2a^2 + 2b^2$ we see that

$$\int_{-1}^{1} \rho^{\sigma+2}(\partial_x^2 u)^2 \,\mathrm{d}x = \int_{-1}^{1} \rho^{\sigma}\left(f - (\sigma+1)x\partial_x u\right)^2 \,\mathrm{d}x \leq 2\int_{-1}^{1} \rho^{\sigma} f^2 \,\mathrm{d}x + 2(\sigma+1)^2 \int_{-1}^{1} \rho^{\sigma} x^2 (\partial_x u)^2 \,\mathrm{d}x.$$

We may thus conclude using the first estimate in the statement. □

Proposition 2.1. *The operator $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}^0$ is self-adjoint, nonnegative, and has compact resolvents.*

Proof. Let us first recall that any symmetric, densely defined operator on a Hilbert space $\mathcal{H}$ is closable, meaning the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ is again the graph of a linear, symmetric operator. Identity (2.1) shows that $\mathcal{A}_{|C^\infty([-1,1])}$ is a symmetric, densely defined operator on the Hilbert space $\mathcal{H}^0$. Let us denote the closure of this operator by $\mathcal{A}$, with domain $\mathcal{D}(\mathcal{A})$. Our goal is to show that $\mathcal{A}$ is the unique self-adjoint extension of $\mathcal{A}_{|C^\infty([-1,1])}$, with domain $\mathcal{D}(\mathcal{A}) = \mathcal{H}^2$.

A standard approximation argument yields $\mathcal{H}^2 \subset \mathcal{D}(\mathcal{A})$. We will show that $\mathbf{A} := \mathcal{A}_{|\mathcal{H}^2}$ is a self-adjoint operator by proving $\mathbf{A}^* \subset \mathbf{A}$. The chain $\mathbf{A} \subset \mathcal{A} \subset \mathcal{A}^* \subset \mathbf{A}^*$ would then imply that $\mathcal{A} = \mathbf{A}$ is self-adjoint, and that any other self-adjoint extension of $\mathcal{A}_{|C^\infty([-1,1])}$ would be jammed in-between $\mathcal{A}$ and $\mathcal{A}^*$ in the above inclusions, and hence coincide with $\mathcal{A}$.

Let $\mathbf{L} := \mathbf{A} + \mathrm{Id}$. The desired inclusion $\mathbf{A}^* \subset \mathbf{A}$ would follow by showing $\mathbf{L}^* \subset \mathbf{L}$. The latter requires us to show that if $u \in \mathcal{H}^0$ is such that $u \in \mathcal{D}(\mathbf{L}^*)$, then $u \in \mathcal{H}^2$.

To this end, we begin by observing that for $f \in \mathcal{H}^0$, the Poisson problem

$$\begin{cases} -\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x w) + w = f & \text{in } (-1,1) \\ (\rho^{\sigma+1}\partial_x w)(\pm 1) = 0, \end{cases}$$

has a unique weak solution $w \in \mathcal{H}^1$ by Lax-Milgram, and $w \in \mathcal{H}^2$ by Lemma 2.5. The operator $\mathbf{L}$ is hence boundedly invertible. Let $u \in \mathcal{D}(\mathbf{L}^*)$. Thus there exists $f \in \mathcal{H}^0$ such that

$$\langle u, \mathbf{L}v\rangle_{\mathcal{H}^0} = \langle f, v\rangle_{\mathcal{H}^0} \quad \text{for all } v \in \mathcal{H}^2. \tag{2.7}$$

For this f, let w denote the weak solution to the Poisson problem above; w also satisfies (2.7) after integration by parts. Thus, taking the difference and considering the test function $v = \mathbf{L}^{-1}(u - w) \in \mathcal{H}^2$, we conclude that $u \in \mathcal{H}^2$.

Finally, by the estimate $\|u\|_{\mathcal{H}^1} \leq \|f\|_{\mathcal{H}^0}$ and the compact embedding $\mathcal{H}^1 \hookrightarrow \mathcal{H}^0$ (see [33, Lemma 4]), it is seen that $(\mathcal{A} + \mathrm{Id})^{-1}$ is compact, and we obtain the desired conclusion. □

By well-known results, we deduce that $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}^0$ has a purely discrete spectrum consisting of an increasing sequence of nonnegative eigenvalues $\{\lambda_k\}_{k=0}^\infty$ with $\lim_{k\to\infty} \lambda_k = \infty$, and an associated sequence of eigenfunctions that form an orthonormal basis of $\mathcal{H}^0$. In order to use spectral techniques for studying the null-controllability of problem (1.8), we need knowledge of the explicit spectrum of $\mathcal{A}$. The definition of the eigenfunctions involves the rising factorials (also called Pochhammer symbols):

$$(s)_j = s(s+1)\dots(s+j-1) \quad \text{for } j \in \mathbb{N} \quad \text{and } (s)_0 = 1 \quad \text{for } s \in \mathbb{R}.$$

For fixed $a, b, c, x \in \mathbb{C}$, we define the hypergeometric series ${}_2F_1(a, b; c; x)$ by

$${}_2F_1(a, b, c, x) := \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} x^j,$$

provided c is not an integer ≤ 0. The series is convergent if $|x| < 1$, and terminates if $a \in \mathbb{Z}$ and becomes a polynomial (see [36, Chapter IV]). We also recall the standard integral definition of the Gamma function $\Gamma(z)$ for $z \in (0, \infty)$:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t.$$

Γ is a monotone increasing function on $(0, \infty)$, and this integral form shows that $\Gamma(1) = 1$ and that $z\Gamma(z) = \Gamma(z+1)$ holds for all $z \in (0, \infty)$.

The following, albeit reformulated result is shown by Seis [33, Theorem 1] and [34, Proposition 6.1] (in any dimension), following ideas from Denzler & McCann [14]. In the one-dimensional case, it may also be found in a previous work of Angenent [2].

Theorem 2.1. *The spectrum of $\mathcal{A}$ consists of simple nonnegative eigenvalues $\{\lambda_k\}_{k=0}^\infty$, given by*

$$\lambda_k = \frac{k^2}{2} + \frac{k}{2}(1 + 2\sigma)$$

for $k \geq 0$. The corresponding eigenfunctions $\{\varphi_k\}_{k=0}^\infty$ are of the form

$$\varphi_k(x) = {}_2F_1\left(-\frac{k}{2},\, \sigma + \frac{k}{2} + \frac{1}{2},\, \frac{1}{2},\, x^2\right) \qquad \text{if } k \text{ is even}$$

and

$$\varphi_k(x) = {}_2F_1\left(-\frac{k-1}{2},\, \sigma + \frac{k}{2} + 1,\, \frac{3}{2},\, x^2\right) x \qquad \text{if } k \text{ is odd}$$

for $x \in (-1, 1)$. In particular, $\lambda_0 = 0$ with associated eigenfunction $\varphi_1(x) = 1$ since constants are in the domain of $\mathcal{A}$.

Let us comment the results of [33, 34] in the specific one-dimensional case we are treating here. A key observation is that the operator $\mathcal{A}$ commutes with the parity operator $\mathbb{P}$ defined by $(\mathbb{P}f)(x) = f(-x)$, which has two eigenvalues: ± 1. One may identify the restriction of $\mathcal{A}$ to even functions with $\ell = 0$, and odd functions with $\ell = 1$, and then aims to simultaneously diagonalize both operators. In [33], Seis computes the spectrum by relating the derived eigenvalue problem to a second-order Fuchsian ODE with three regular singular points. A point spectrum $\{\lambda_{\ell\kappa}\}_{\kappa=0}^\infty$ is obtained, which for convenience we merge here by setting $2\kappa = k$ for $\ell = 0$ and $2\kappa + 1 = k$ for $\ell = 1$.

On another note, as mentioned above the series defining the eigenfunctions φ_k terminates since $-\frac{k}{2} \in \mathbb{Z}$ when k is even (similarly $-\frac{k-1}{2} \in \mathbb{Z}$ when k is odd). Thus, φ_k are polynomials of degree k. It is more advantageous to represent the eigenfunctions in terms of classical orthogonal polynomials, for which explicit norm relations and asymptotic behavior are known. We may in fact relate the eigenfunctions to Jacobi polynomials $P_\ell^{(\alpha,\beta)}(\cdot)$, as:

$${}_2F_1(-\ell, \alpha + \beta + \ell + 1, \alpha + 1, x) = \frac{\ell!}{(\alpha+1)_\ell} P_\ell^{(\alpha,\beta)}(1 - 2x) \quad x \in (-1, 1),$$

for $\alpha, \beta > -1$ and $\ell \geq 0$, see [36, Chapter IV] for instance. The Jacobi polynomials are orthogonal in $L^2(-1, 1)$ with respect to the weight $(1-x)^\alpha (1+x)^\beta$:

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_\ell^{(\alpha,\beta)}(x)^2 \, \mathrm{d}x = \frac{2^{\alpha+\beta+1}}{2\ell + \alpha + \beta + 1} \frac{\Gamma(\ell + \alpha + 1)\Gamma(\ell + \beta + 1)}{\Gamma(\ell + \alpha + \beta + 1)\ell!}, \tag{2.8}$$

see [36, Chapter IV, Section 4.1], which holds for $\alpha, \beta > -1$. Using this, relatively straightforward computations yield the normalized eigenfunctions of the form

$$\overline{\varphi}_k(\cdot) = c_k \varphi_k(\cdot) \tag{2.9}$$

as per the following result.

Lemma 2.6. *Let $k \geq 0$, and let φ_k be the k-th eigenfunction of $\mathcal{A}$. Then*

$$\|\varphi_{2\ell}\|^2_{\mathcal{H}^0} = 2^{-\sigma}\Big(\frac{1}{2}\Big)_\ell^{-2} \frac{\ell!\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)}{(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}$$

if $k = 2\ell$ is even, and

$$\|\varphi_{2\ell+1}\|^2_{\mathcal{H}^0} = 2^{-\sigma}\Big(\frac{3}{2}\Big)_\ell^{-2} \frac{\ell!\Gamma(\ell+\frac{3}{2})\Gamma(\ell+\sigma+1)}{(2\ell+\sigma+\frac{3}{2})\Gamma(\ell+\sigma+\frac{3}{2})}$$

if $k = 2\ell+1$ is odd.

Proof. Let $k = 2\ell$ be even. We write

$$2^{-\sigma}\int_{-1}^1 (1-x^2)^\sigma \varphi_{2\ell}^2 \,\mathrm{d}x = 2^{-\sigma}\Big(\frac{1}{2}\Big)_\ell^{-2} (\ell!)^2 \int_{-1}^1 (1-x^2)^\sigma P_\ell^{(-\frac{1}{2},\sigma)}(1-2x^2)\,\mathrm{d}x.$$

A simple change of variables yields

$$\int_{-1}^1 (1-x^2)^\sigma P_\ell^{(-\frac{1}{2},\sigma)}(1-2x^2)\,\mathrm{d}x = 2^{-\sigma-\frac{1}{2}}\int_{-1}^1 (1-z)^{-\frac{1}{2}}(1+z)^\sigma P_\ell^{(-\frac{1}{2},\sigma)}(z)\,\mathrm{d}z.$$

Using the orthonormality relation (2.8), we obtain

$$\int_{-1}^1 (1-z)^{-\frac{1}{2}}(1+z)^\sigma P_\ell^{(-\frac{1}{2},\sigma)}(z)\,\mathrm{d}z = \frac{2^{\sigma+\frac{1}{2}}}{2\ell+\sigma+\frac{1}{2}}\frac{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)}{\ell!\Gamma(\ell+\sigma+\frac{1}{2})}.$$

We deduce

$$\|\varphi_{2\ell}\|^2_{\mathcal{H}^0} = 2^{-\sigma}\Big(\frac{1}{2}\Big)_\ell^{-2} \frac{\ell!\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)}{(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}.$$

The case when k is odd follows from an analogous computation. □

3. Null-controllability of the linearized problem

Before proceeding with the proofs of the controllability results for the linearized problems, let us argue the well-posedness of

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = f & \text{in } (0,T)\times(-1,1) \\ \big(\rho^{\sigma+1}\partial_x y\big)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1), \end{cases} \tag{3.1}$$

where $T > 0$ and f is an arbitrary source term. The following result holds.

Proposition 3.1. *For every $y_0 \in \mathcal{H}^0$ and $f \in L^2(0,T;\mathcal{H}^0)$, there exists a unique weak solution*

$$y \in L^2(0,T;\mathcal{H}^1) \cap C^0([0,T];\mathcal{H}^0)$$

to Problem (3.1) satisfying the estimate

$$\|y\|_{C^0([0,T];\mathcal{H}^0)} + \|y\|_{L^2(0,T;\mathcal{H}^1)} \leq C_T\big(\|f\|_{L^2(0,T;\mathcal{H}^0)} + \|y_0\|_{\mathcal{H}^0}\big) \tag{3.2}$$

for some $C_T > 0$. If moreover $y_0 \in \mathcal{H}^1$, then y is a strong solution enjoying maximal regularity

$$y \in L^2(0,T;\mathcal{H}^2) \cap H^1(0,T;\mathcal{H}^0) \cap C^0([0,T];\mathcal{H}^1)$$

along with the estimate

$$\|y\|_{C^0([0,T];\mathcal{H}^1)} + \|y\|_{L^2(0,T;\mathcal{H}^2)} \leq C_T\big(\|f\|_{L^2(0,T;\mathcal{H}^0)} + \|y_0\|_{\mathcal{H}^1}\big) \tag{3.3}$$

for some $C_T > 0$.

Proof. The statement follows from well-known semigroup theory results (see for instance [4, Part II, Chapter 1, Section 3]). Indeed, Proposition 2.1 along with [4, Theorem 2.12, Section 2] imply that the self-adjoint operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup in $\mathcal{H}^0$.

We remark that the semigroup theory results make use of the fact that $\mathcal{H}^1$ is the $(\frac{1}{2}, 2)$-interpolation space of $\mathcal{D}(\mathcal{A}) = \mathcal{H}^2$ and $\mathcal{H}^0$. A proof of this may be found in [18, Lemma 3.6] and also [17, Lemma 1.7].

The constant C_T in estimates (3.2), (3.3) depends on T due to the fact that first eigenfunction of $\mathcal{A}$ is associated with the eigenvalue 0. Thus, the contribution of this first mode to the $L^2(0,T;\mathcal{H}^1)$-norm of y is not bounded as $T \to \infty$. □

As discussed in the introduction, the null-controllability of Problem (1.8) requires first proving Theorem 1.2, regarding the null-controllability of the homogeneous problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ \big(\rho^{\sigma+1}\partial_x y\big)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1). \end{cases} \tag{3.4}$$

3.1. **The homogeneous problem**

The main objective in what follows is to provide a proof to Theorem 1.2. Let us begin with a short review of some well-known notions on the null-controllability of linear systems. Let $\mathcal{H}$ and U be two Hilbert spaces. Consider the linear control system

$$\begin{cases} y' = Ay + Bu & \text{in } (0,T) \\ y(0,\cdot) = y_0 \in \mathcal{H}, \end{cases} \tag{3.5}$$

where $A : \mathcal{D}(A) \to \mathcal{H}$ is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on $\mathcal{H}$ and $B \in \mathcal{L}(U, \mathcal{H})$. If (3.5) is null-controllable in time $T > 0$ then the set

$$\mathcal{U}_{T,y_0} = \{u \in L^2(0,T;U) \colon y(T,\cdot) = 0\}$$

is non-empty. The quantity

$$\kappa(T) := \sup_{\|y_0\|_{\mathcal{H}}=1} \inf_{u\in\mathcal{U}_{T,y_0}} \|u\|_{L^2(0,T;U)}$$

is called the *control cost* in time T. It is known that if (3.5) is null-controllable in any time $T > 0$, then $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and non-increasing, and $\lim_{T\searrow 0^+} \kappa(T) = \infty$. This namely implies that for every function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(t) < \gamma(t)$ for every $t > 0$, for every $T > 0$ there exists a control driving the solution of (3.5) to rest im time T such that

$$\|u\|_{L^2(0,T;U)} \leq \gamma(T)\|y_0\|_{\mathcal{H}}.$$

Let us consider the adjoint problem

$$\begin{cases} -\zeta' = A^*\zeta & \text{in } (0,T) \\ \zeta(T,\cdot) = \zeta_T \in \mathcal{H}. \end{cases} \tag{3.6}$$

The following result is relatively standard and may be found in [28, Propositon 2.2], and originates from the work of Fattorini & Russell [15]. Due to a minimal change in the assumptions with respect to [28], we give a short proof below.

Lemma 3.1. *Assume that A is a negative operator*[1]*, with an orthonormal basis of eigenfunctions $\{\varphi_k\}_{k=0}^\infty$ and corresponding decreasing sequence of eigenvalues $\{-\lambda_k\}_{k=0}^\infty$ which satisfy*

$$\inf_{k\geq 0}(\lambda_{k+1}-\lambda_k) = s > 0 \tag{3.7}$$

$$\lambda_k = rk^2 + \mathcal{O}(k) \qquad \text{as } k\to\infty \tag{3.8}$$

for some $r>0$. Assume U is a separable Hilbert space and that there exists $\mu>0$ such that

$$\|B^*\varphi_k\|_U \geq \mu \tag{3.9}$$

for all $k\geq 0$. Then there exists a constant $C_{\text{obs}} = C_{\text{obs}}(T) > 0$ such that the observability inequality

$$\|\zeta(0,\cdot)\|_{\mathcal{H}}^2 \leq C_{\text{obs}}^2 \int_0^T \|B^*\zeta\|_U^2 \, \mathrm{d}t \tag{3.10}$$

holds for any $\zeta_T\in\mathcal{H}$, where ζ is the corresponding solution to (3.6).

Proof. We may write the Fourier decomposition of ζ as

$$\zeta(t,x) = \sum_{k=0}^\infty e^{-\lambda_k(T-t)}\langle\zeta_T,\varphi_k\rangle_{\mathcal{H}}\,\varphi_k(x). \tag{3.11}$$

Since U is separable, it has an orthonormal basis $\{\psi_j\}_{j=0}^\infty$, which combined with identity (3.11) and the time-shift $T-t\longmapsto t$ gives

$$\int_0^T \|B^*\zeta\|_U^2\,\mathrm{d}t = \sum_{j=0}^\infty\int_0^T\left|\sum_{k=0}^\infty e^{-\lambda_k t}\langle\zeta_T,\varphi_k\rangle_{\mathcal{H}}\langle B^*\varphi_k,\psi_j\rangle_U\right|^2 \mathrm{d}t \tag{3.12}$$

for $T>0$ and $\zeta_T\in\mathcal{H}$. Now, making use of assumptions (3.7), (3.8), we deduce from [32, Theorem 1] that there exists $C(T)>0$ such that $\lim_{T\searrow 0^+} C(T)=\infty$ and

$$C(T)\int_0^T\left|\sum_{k=0}^\infty a_k e^{-\lambda_k t}\right|^2\mathrm{d}t \geq \sum_{k=0}^\infty |a_k|^2 e^{-2\lambda_k T}$$

for all $T>0$ and $\{a_k\}_{k=0}^\infty\in\ell^2(\mathbb{N})$. Applying this estimate in (3.12) gives

$$C(T)\int_0^T \|B^*\zeta\|_U^2\,\mathrm{d}t \geq \sum_{j=0}^\infty\sum_{k=0}^\infty e^{-2\lambda_k T}\langle\zeta_T,\varphi_k\rangle_{\mathcal{H}}^2\langle B^*\varphi_k,\psi_j\rangle_U^2$$

for $T>0$ and $\zeta_T\in\mathcal{H}$. This last estimate along with assumption (3.9) yields

$$C(T)\int_0^T \|B^*\zeta\|_U^2\,\mathrm{d}t \geq \mu^2\|\zeta(0,\cdot)\|_{\mathcal{H}}^2$$

for $\zeta_T\in\mathcal{H}$. The observability inequality (3.10) thus holds with $C_{\text{obs}}(T)=\sqrt{\frac{C(T)}{\mu^2}}$. ∎

[1] Meaning $\langle Ay,y\rangle_{\mathcal{H}}\leq 0$ for $y\in\mathcal{D}(A)$.

Remark 3.1 (Decay of the control cost)**.** *When the operator A is strictly negative*[2] *(and thus $\lambda_0 > 0$), then there exist $M_1, M_2 > 0$ such that*

$$\kappa(T) < M_1 e^{\frac{M_2}{T}} \qquad \text{for all } T > 0.$$

This will not hold in our case since $\lambda_0 = 0$ is an eigenvalue. The cost may be shown to be of the same exponential form for small time (see [32, Section 5.2]), but in the long time limit $T \to \infty$, it is rather of the order of a constant. The control cost plays a role in choosing the explicit time-weights in the source-term method, as seen below. For a thorough study, we refer to Tenenbaum & Tucsnak [38].

In our framework, we take $\mathcal{H} = \mathcal{H}^0$ and $U = \mathcal{H}^0(\omega) = L^2(\omega, \rho^\sigma dx)$. The control operator $B \in \mathcal{L}(U, \mathcal{H})$ is given by $Bu = u\mathbf{1}_\omega$, where $\omega = (a, b) \subsetneq (-1, 1)$ is non-empty. Hence, $B^*u = u|_\omega$.

Using Lemma 3.1 and the spectral results from Subsection 2.2, we are now in a position to prove the following result.

Lemma 3.2. *Let $\sigma > -1$. The eigenvalues $\{\lambda_k\}_{k=0}^\infty$ and associated normalized eigenfunctions $\{\overline{\varphi}_k\}_{k=0}^\infty$ of the negative operator $\mathcal{A} : \mathcal{H}^2 \to \mathcal{H}^0$ satisfy conditions* (3.7)-(3.9).

Proof. Due to their form, it is readily seen that the eigenvalues given in Theorem 2.1 satisfy (3.8). The separation condition (3.7) follows from a simple computation:

$$\lambda_{k+1} - \lambda_k \geq k + \sigma + 1 \geq \sigma + 1 \qquad \text{for any } k \geq 0.$$

The main issue will be to show that the normalized eigenfunctions satisfy condition (3.9). Recall that they write

$$\overline{\varphi}_{2\ell}(x) = c_{2\ell} P_\ell^{(-\frac{1}{2}, \sigma)}(1 - 2x^2)$$

when $k = 2\ell$ is even and

$$\overline{\varphi}_{2\ell+1}(x) = c_{2\ell+1} P_\ell^{(\frac{1}{2}, \sigma)}(1 - 2x^2)$$

when $k = 2\ell + 1$ is odd, for $\ell \geq 0$, where

$$c_{2\ell}^2 = \frac{2^\sigma \ell!(2\ell + \sigma + \frac{1}{2})\Gamma(\ell + \sigma + \frac{1}{2})}{\Gamma(\ell + \frac{1}{2})\Gamma(\ell + \sigma + 1)}, \quad c_{2\ell+1}^2 = \frac{2^\sigma k!(2\ell + \sigma + \frac{3}{2})\Gamma(\ell + \sigma + \frac{3}{2})}{\Gamma(\ell + \frac{3}{2})\Gamma(\ell + \sigma + 1)},$$

(see Lemma 2.6 and (2.9)). In view of the fact that $B^*u = u|_\omega$ and since $L^2(a, b)$ and $\mathcal{H}^0(a, b)$ are topologically equivalent, (3.9) may be rewritten as

$$\int_a^b \overline{\varphi}_k^2 \,\mathrm{d}x \geq \mu$$

for some $\mu = \mu(a, b, \sigma) > 0$ independent of k. Now, for any fixed $k \geq 0$, since these eigenfunctions are nonzero solutions of a second order differential equation (they are nonzero polynomials), we have

$$\int_a^b \overline{\varphi}_k^2 \,\mathrm{d}x \geq \mu$$

for some $\mu = \mu(k, a, b, \sigma) > 0$. We are thus going to study the behavior of this quantity as $k \to \infty$.

For technical purposes, let us first assume that $0 \notin (a, b)$. We will add 0 in (a, b) a posteriori, after observing that the asymptotic lower bound does not blow up as $a \to 0$ or $b \to 0$.

Let us assume that $a > 0$ (the cases $b < 0$ and $a < 0$, $b > 0$, $0 \notin (a, b)$ follow similar arguments). Let $k = 2\ell$ with $\ell \geq 0$ be even. We have

$$\int_a^b \overline{\varphi}_{2\ell}^2 \,\mathrm{d}x = c_{2\ell}^2 \int_a^b P_\ell^{(-\frac{1}{2}, \sigma)}(1 - 2x^2)^2 \,\mathrm{d}x. \tag{3.13}$$

[2]Meaning there exists $\alpha > 0$ such that $\langle Ay, y\rangle_{\mathcal{H}} \leq -\alpha\|y\|_{\mathcal{H}}^2$ for all $y \in \mathcal{D}(A)$.

We look to reformulate the integral on the right-hand in view of using the following asymptotic formula:

$$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{\pi}}\sin^{-\alpha-\frac{1}{2}}\left(\frac{\theta}{2}\right)\cos^{-\beta-\frac{1}{2}}\left(\frac{\theta}{2}\right)\cos(\theta\psi(n)-\phi)\right) + \mathcal{O}(n^{-\frac{3}{2}}), \tag{3.14}$$

for $n \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $\theta \in (0, \pi)$, where

$$\psi(n) = \left(n + \frac{1}{2}(\alpha+\beta+1)\right) \quad \text{and} \quad \phi = \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right),$$

see [36, Chapter VIII, Theorem 8.21.8] for instance. Performing the change of variable $\cos\theta = 1-2x^2$, whence $\mathrm{d}x = 2^{-\frac{3}{2}}\sqrt{1+\cos\theta}\,\mathrm{d}\theta$, gives

$$\int_a^b P_\ell^{(-\frac{1}{2},\sigma)}(1-2x^2)^2\,\mathrm{d}x = 2^{-\frac{3}{2}}\int_{\gamma_1}^{\gamma_2} P_\ell^{(-\frac{1}{2},\sigma)}(\cos\theta)^2\sqrt{1+\cos\theta}\,\mathrm{d}\theta,$$

where $\gamma_1 = \arccos(1-2a^2)$ and $\gamma_2 = \arccos(1-2b^2)$, thus now $(\gamma_1,\gamma_2) \subset (0,\pi)$. We may use (3.14), which combined with the above identity gives

$$\begin{aligned}\int_a^b P_\ell^{(-\frac{1}{2},\sigma)}(1-2x^2)^2\,\mathrm{d}x = 2^{-\frac{3}{2}}&\int_{\gamma_1}^{\gamma_2}\frac{1}{\ell\pi}\frac{\cos^2(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4}))}{\cos(\frac{\theta}{2})^{2\sigma+1}}\sqrt{1+\cos\theta}\,\mathrm{d}\theta\\ &+\frac{2^{-\frac{1}{2}}}{\sqrt{\pi}}\int_{\gamma_1}^{\gamma_2}\mathcal{O}\left(\frac{1}{\ell^2}\right)\frac{\cos\left(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4})\right)}{\cos(\frac{\theta}{2})^{\sigma+\frac{1}{2}}}\sqrt{1+\cos\theta}\,\mathrm{d}\theta\\ &+\int_{\gamma_1}^{\gamma_2}\mathcal{O}\Big(\frac{1}{\ell^3}\Big)\sqrt{1+\cos\theta}\,\mathrm{d}\theta\end{aligned}$$

as $\ell \to \infty$. Let us take a closer look at the right-hand side integrals. Using elementary trigonometric relations,

$$\begin{aligned}\int_{\gamma_1}^{\gamma_2}\frac{\cos^2(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4}))}{\cos(\frac{\theta}{2})^{2\sigma+1}}\sqrt{1+\cos\theta}\,\mathrm{d}\theta &= 2^{\sigma+\frac{1}{2}}\int_{\gamma_1}^{\gamma_2}\frac{\cos^2(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4}))}{(1+\cos\theta)^\sigma}\,\mathrm{d}\theta\\ &= 2^{\sigma-\frac{1}{2}}\int_{\gamma_1}^{\gamma_2}\frac{\left(1+\cos(\theta(2\ell+\sigma+\frac{1}{2}))\right)}{(1+\cos\theta)^\sigma}\,\mathrm{d}\theta.\end{aligned}$$

Similarly,

$$\mathcal{O}\left(\frac{1}{\ell^2}\right)\int_{\gamma_1}^{\gamma_2}\frac{\cos\left(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4})\right)}{\cos(\frac{\theta}{2})^{\sigma+\frac{1}{2}}}\sqrt{1+\cos\theta}\,\mathrm{d}\theta = \mathcal{O}\left(\frac{1}{\ell^2}\right)\int_{\gamma_1}^{\gamma_2}\eta(\theta)\frac{\cos\left(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4})\right)}{(1+\cos\theta)^{\frac{2\sigma-1}{4}}}\,\mathrm{d}\theta$$

where $\eta(\theta) = \operatorname{sgn}\left(\pi+\theta+4\pi\lfloor\frac{\pi-\theta}{4\pi}\rfloor\right) \in \{-1,1\}$. Putting together the three previous identities, we obtain

$$\begin{aligned}\int_a^b P_\ell^{(-\frac{1}{2},\sigma)}(1-2x^2)^2\,\mathrm{d}x = \frac{2^{\sigma-2}}{\ell\pi}&\int_{\gamma_1}^{\gamma_2}\frac{\left(1+\cos(\theta(2\ell+\sigma+\frac{1}{2}))\right)}{(1+\cos\theta)^\sigma}\,\mathrm{d}\theta\\ &+\mathcal{O}\left(\frac{1}{\ell^2}\right)\int_{\gamma_1}^{\gamma_2}\eta(\theta)\frac{\cos\left(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4})\right)}{(1+\cos\theta)^{\frac{2\sigma-1}{4}}}\,\mathrm{d}\theta\\ &+\mathcal{O}\Big(\frac{1}{\ell^3}\Big)\end{aligned}$$

as $\ell \to \infty$. Going back to (3.13), we now have

$$\begin{aligned}\int_a^b \overline{\varphi}_{2\ell}^2 \,\mathrm{d}x = \frac{(\ell-1)!(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)} \frac{2^{2\sigma-2}}{\pi} \int_{\gamma_1}^{\gamma_2} \frac{\left(1+\cos(\theta(2\ell+\sigma+\frac{1}{2}))\right)}{(1+\cos\theta)^\sigma} \,\mathrm{d}\theta \\ + \frac{\ell!(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)} \left(\mathcal{O}\left(\frac{1}{\ell^2}\right) \int_{\gamma_1}^{\gamma_2} \eta(\theta) \frac{\cos\left(\theta(\ell+\frac{\sigma}{2}+\frac{1}{4})\right)}{(1+\cos\theta)^{\frac{2\sigma-1}{4}}} \,\mathrm{d}\theta + \mathcal{O}\Big(\frac{1}{\ell^3}\Big) \right)\end{aligned} \tag{3.15}$$

as $\ell \to \infty$. Making use of the relations $(\ell-1)! = \Gamma(\ell)$, $z\Gamma(z) = \Gamma(z+1)$ for $z \in \mathbb{C}$ as well as $\frac{\Gamma(\ell+\alpha)}{\Gamma(\ell+\beta)} \sim \ell^{\alpha-\beta}$ (a consequence of Stirling's formula), we obtain

$$\begin{aligned}\frac{(\ell-1)!}{\Gamma(\ell+\frac{1}{2})} \frac{(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\sigma+1)} &= \frac{\Gamma(\ell)}{\Gamma(\ell+\frac{1}{2})} \left(\ell \frac{\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\sigma+1)} + \frac{(\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\sigma+1)} \right) \\ &= \ell^{-\frac{1}{2}} \left(\ell \frac{\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\sigma+1)} + \frac{\Gamma(\ell+\sigma+\frac{3}{2})}{\Gamma(\ell+\sigma+1)} \right) \\ &\sim 2,\end{aligned} \tag{3.16}$$

and similarly

$$\frac{\ell!(2\ell+\sigma+\frac{1}{2})\Gamma(\ell+\sigma+\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\sigma+1)} \sim 2\ell \tag{3.17}$$

as $\ell \to \infty$. Moreover, recall that for any interval $I \subseteq \mathbb{R}$, the sequence $\{\cos(n\cdot)\}_{n\in\mathbb{N}}$ converges weakly-$*$ to 0 in $L^\infty(I)$ as $n \to \infty$ (an application of the Riemann-Lebesgue Lemma), meaning

$$\int_I \cos(nx)\phi(x) \,\mathrm{d}x \xrightarrow[n\to\infty]{} 0 \quad \text{for all } \phi \in L^1(I). \tag{3.18}$$

Since $(1+\cos(\cdot))^{-\sigma} \in L^1(\gamma_1,\gamma_2)$ and also $\eta(\theta)(1+\cos\theta)^{-\frac{2\sigma-1}{4}} \in L^1(\gamma_1,\gamma_2)$, using (3.16), (3.17) and (3.18) in (3.15), we deduce that

$$\int_a^b \overline{\varphi}_{2\ell}^2 \,\mathrm{d}x \longrightarrow \frac{2^{2\sigma-1}}{\pi} \int_{\gamma_1}^{\gamma_2} \frac{1}{(1+\cos\theta)^\sigma} \,\mathrm{d}\theta$$

as $\ell \to \infty$. A straightforward computation yields

$$\int_{\gamma_1}^{\gamma_2} \frac{1}{(1+\cos\theta)^\sigma} \,\mathrm{d}\theta = 2^{1-\sigma} \int_a^b \frac{1}{(1-x^2)^{\sigma+\frac{1}{2}}} \,\mathrm{d}x,$$

thus we may conclude that

$$\int_a^b \overline{\varphi}_{2\ell}^2 \,\mathrm{d}x \longrightarrow \frac{2^\sigma}{\pi} \int_a^b \frac{1}{(1-x^2)^{\sigma+\frac{1}{2}}} \,\mathrm{d}x \tag{3.19}$$

as $\ell \to \infty$.

The arguments differ very little when $k = 2\ell+1$ is odd, so we provide less detail. We have

$$\int_a^b \overline{\varphi}_{2\ell+1}^2 \,\mathrm{d}x = \frac{2^\sigma \ell!(2\ell+\sigma+\frac{3}{2})\Gamma(\ell+\sigma+\frac{3}{2})}{\Gamma(\ell+\frac{3}{2})\Gamma(\ell+\sigma+1)} \int_a^b x^2 P_\ell^{(\frac{1}{2},\sigma)}(1-2x^2)^2 \,\mathrm{d}x.$$

Applying the same change of variable as in the above computation yields

$$\int_a^b x^2 P_\ell^{(\frac{1}{2},\sigma)}(1-2x^2)^2 \,\mathrm{d}x = \frac{1}{2} \int_{\gamma_1}^{\gamma_2} P_\ell^{(\frac{1}{2},\sigma)}(1-2x^2)^2 (1-\cos\theta)\sqrt{1+\cos\theta} \,\mathrm{d}\theta.$$

By virtue of the asymptotic formula (3.14) and elementary trigonometric identities, we now have

$$\begin{aligned}\int_a^b x^2 P_\ell^{(\frac{1}{2},\sigma)}(1-2x^2)^2\,\mathrm{d}x &= \frac{1}{2\ell\pi}\int_{\gamma_1}^{\gamma_2}\frac{\cos^2(\theta(\ell+\frac{\sigma}{2}+\frac{3}{4})-\frac{\pi}{2})}{\sin^2(\frac{\theta}{2})\cos(\frac{\theta}{2})^{2\sigma+1}}(1-\cos\theta)\sqrt{1+\cos\theta}\,\mathrm{d}\theta\\ &\quad+\mathcal{O}\left(\frac{1}{\ell^2}\right)\int_{\gamma_1}^{\gamma_2}\frac{\cos(\theta(\ell+\frac{\sigma}{2}+\frac{3}{4})-\frac{\pi}{2})}{\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})^{\sigma+\frac{1}{2}}}(1-\cos\theta)\sqrt{1+\cos\theta}\,\mathrm{d}\theta\\ &\quad+\mathcal{O}\Big(\frac{1}{\ell^3}\Big)\\ &=\frac{2^\sigma}{\ell\pi}\int_{\gamma_1}^{\gamma_2}\frac{1+\cos(\theta(2\ell+\sigma+\frac{3}{2})-\pi)}{(1+\cos\theta)^\sigma}\,\mathrm{d}\theta\\ &\quad+\mathcal{O}\left(\frac{1}{\ell^2}\right)\int_{\gamma_1}^{\gamma_2}\sin\left(\theta\left(\ell+\frac{\sigma}{2}+\frac{3}{4}\right)\right)\frac{\sqrt{1-\cos\theta}}{(1+\cos\theta)^{\frac{2\sigma-1}{4}}}\eta(\theta)\,\mathrm{d}\theta\\ &\quad+\mathcal{O}\Big(\frac{1}{\ell^3}\Big)\end{aligned}$$

as $\ell\to\infty$, where $\eta(\theta)\in\{-1,1\}$. Using the parity and periodicity of the cosine, we see that the computations reduce to an almost identical scenario as when k is even, and we may use (3.16), (3.17) and (3.18)[3] to deduce

$$\int_{-1}^1 \overline{\varphi}_{2\ell+1}^2\,\mathrm{d}x \longrightarrow \frac{2^{\sigma+1}}{\pi}\int_a^b \frac{1}{(1-x^2)^{\sigma+\frac{1}{2}}}\,\mathrm{d}x \tag{3.20}$$

as $\ell\to\infty$. As the limit bound in (3.19), (3.20) does not blow up as $a\to 0$, we may conclude the proof. □

We are now in a position to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. The conclusion follows from a well-known adaptation of the HUM method (Hilbert Uniqueness Method, [25, Chapitre 2]). We give brief details for the sake of completeness.

For fixed $\varepsilon>0$, let us introduce the functional

$$J_{\varepsilon,\mathrm{obs}}(\zeta_T)=\frac{1}{2}\int_0^T\int_\omega \rho^\sigma|\zeta|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{-1}^1\rho^\sigma y_0\zeta(0,\cdot)\,\mathrm{d}x+\varepsilon\|\zeta_T\|_{\mathcal{H}^0}$$

for every $\zeta_T\in\mathcal{H}^0$, where ζ is the unique solution to the adjoint problem

$$\begin{cases}\partial_t\zeta+\rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x\zeta)=0 & \text{in } (0,T)\times(-1,1)\\ (\rho^{\sigma+1}\partial_x\zeta)(t,\pm1)=0 & \text{in } (0,T)\\ \zeta(T,x)=\zeta_T(x) & \text{in } (-1,1).\end{cases}\tag{3.21}$$

$J_{\varepsilon,\mathrm{obs}}$ can be shown to be strictly convex, continuous and coercive on $\mathcal{H}^0$ by virtue of the observability inequality (3.10) (which holds for solutions of (3.21) by Lemma 3.2). $J_{\varepsilon,\mathrm{obs}}$ thus has a unique minimizer $\zeta_T^\varepsilon\in\mathcal{H}^0$ by the direct method. Following common practice, we introduce the control $u_\varepsilon=\zeta_\varepsilon\mathbf{1}_\omega$ where ζ_ε is the solution to (3.21) corresponding to ζ_T^ε. We denote by y_ε the solution to (3.4) associated to the control u_ε. Differentiating $J_{\varepsilon,\mathrm{obs}}$ at ζ_T^ε yields the Euler-Lagrange equation

$$\int_0^T\int_\omega\rho^\sigma\zeta_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t+\langle y_0,\varphi(0,\cdot)\rangle_{\mathcal{H}^0}+\varepsilon\left\langle\frac{\zeta_T^\varepsilon}{\|\zeta_T^\varepsilon\|_{\mathcal{H}^0}},\varphi_T\right\rangle_{\mathcal{H}^0}=0 \tag{3.22}$$

[3] Which also holds when $\cos(n\cdot)$ is replaced by $\sin(n\cdot)$.

for every $\varphi_T \in \mathcal{H}^0$. (3.22) along with the observability inequality (3.10) for $\varphi_T = \zeta_T^\varepsilon$ give

$$\|u_\varepsilon\|_{L^2(0,T;L^2(\omega))} \lesssim C_{\text{obs}} \|y_0\|_{\mathcal{H}^0}$$

uniformly in $\varepsilon > 0$. On the other hand, by duality

$$\int_0^T \int_\omega \rho^\sigma \zeta_\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t = \langle y_\varepsilon(T,\cdot), \varphi_T \rangle_{\mathcal{H}^0} - \langle y_0, \varphi(0,\cdot) \rangle_{\mathcal{H}^0}, \tag{3.23}$$

which combined with (3.22) yields

$$\|y_\varepsilon(T,\cdot)\|_{\mathcal{H}^0} \leq \varepsilon. \tag{3.24}$$

Since $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\omega))$, it converges weakly (along a subsequence) to some $u \in L^2(0,T;L^2(\omega))$. Using analytic semigroup estimates such as [4, Theorem 2.12, Section 2] we deduce that along subsequences

$$y_\varepsilon \rightharpoonup y \qquad \text{weakly in } L^2(0,T;\mathcal{H}^1) \cap H^1(0,T;(\mathcal{H}^1)^*)$$

as $\varepsilon \to 0$, where y is the solution to (3.4) with control u. By Aubin-Lions, this gives strong convergence of $\{y_\varepsilon(t,\cdot)\}_{\varepsilon>0}$ to $y(t,\cdot)$ in $\mathcal{H}^0$ for all $t \in [0,T]$, whence $y(T,\cdot) = 0$ in view of (3.24). □

3.2. **Controllability in spite of a source term**

The null-controllability of Problem (1.8) will follow by combining Theorem 1.2 with the source-term method. We review the latter in what follows.

Let $\gamma : (0,\infty) \to [0,\infty)$ be a continuous, non-increasing function satisfying $\lim_{t\to 0} \gamma(t) = \infty$ and

$$\kappa(t) < \gamma(t) \qquad \text{for all } t > 0.$$

We moreover assume that for some $M_1, M_2 > 0$,

$$\gamma(T) = M_1 e^{\frac{M_2}{T}} \qquad \text{for } T \ll 1. \tag{3.25}$$

One may for instance consider the observability constant $t \longmapsto C_{\text{obs}}(t)$ in (3.10), which satisfies the above assumptions, as per [32, Theorem 1 & Section 5.2] (see also [38]). We recall that in our case, condition (3.25) does not hold when the time horizon is large, as $\lambda_0 = 0$ (see Remark 3.1).

Let $T > 0$, $q \in (1, \sqrt{2})$ and $p > 0$ such that $2p > (1+p)q^2$ be fixed. We now consider the continuous, non-increasing function $\theta_{\mathcal{F}} : [0,T] \to [0,\infty)$ defined by

$$\theta_{\mathcal{F}}(t) = \gamma\Big(\frac{q-1}{q^2}(T-t)\Big)^{-(p+1)} \qquad \text{for } t \in [0,T]. \tag{3.26}$$

As $p > 0$, it is easily seen that $\theta_{\mathcal{F}}(T) = 0$. We then consider the continuous and non-increasing function $\theta_0 : \Big[T(1-q^{-2}), T\Big] \to [0,\infty)$ defined by

$$\theta_0(t) = \theta_{\mathcal{F}}\big(q^2(t-T)+T\big)\gamma\big((q-1)(T-t)\big) \qquad \text{for } t \in \Big[T(1-q^{-2}), T\Big], \tag{3.27}$$

which also satisfies $\theta_0(T) = 0$. We extend θ_0 (and use the same notation) to a continuous, non-increasing function on $[0,T]$ by setting

$$\theta_0(t) = \theta_0\Big(T\big(1-q^{-2}\big)\Big) \qquad \text{for } t \in \Big[0, T\big(1-q^{-2}\big)\Big].$$

When dealing with the nonlinear problem, it will be important that the above-defined weights satisfy the condition

$$\frac{\theta_0^2}{\theta_{\mathcal{F}}} \in C^0([0,T]). \tag{3.28}$$

This is accomplished by the choice of $q > 1$ and $p > 0$ above. Indeed, notice that the only obstruction for having (3.28) is the behavior of this quotient near $t = T$, as one may see that

$$\frac{\theta_0^2}{\theta_{\mathcal{F}}}(t) = \frac{\gamma\Big(\frac{q-1}{q^2}(T-t)\Big)^{(p+1)}}{\gamma((q-1)(T-t))^{2p}}.$$

Thus, in view of condition (3.25), the choice $q \in (1, \sqrt{2})$ and $2p > (1+p)q^2$ has the effect of guaranteeing (3.28).

Remark 3.2. *Should $\lambda_0 > 0$, one may for instance consider the explicit weights*

$$\theta_{\mathcal{F}}(t) = e^{-\frac{\alpha}{(T-t)^2}}, \quad \theta_0(t) = M_1 e^{\frac{M_2}{(q-1)(T-t)} - \frac{\alpha}{q^4(T-t)^2}}$$

as in [28], where α, q are appropriately chosen for the fixed-point argument.

To the time-weight functions $\theta_0, \theta_{\mathcal{F}}$, we associate the time-weighted Hilbert spaces

$$\begin{aligned} \mathcal{F} &= \left\{ f \in L^2(0,T;\mathcal{H}^0) \colon \frac{f}{\theta_{\mathcal{F}}} \in L^2(0,T;\mathcal{H}^0) \right\}, \\ \mathcal{U} &= \left\{ u \in L^2(0,T;L^2(\omega)) \colon \frac{u}{\theta_0} \in L^2(0,T;L^2(\omega)) \right\}. \end{aligned} \tag{3.29}$$

The following Theorem is originally shown in [28, Proposition 2.3] (see also [24] and [3] for subsequent adaptations). We assume higher regularity for the initial datum a priori, and thus for the controlled trajectory, having in mind the fixed-point argument. For the sake of completeness, we give a proof below, and the proof follows the same time-splitting scheme of [28].

Theorem 3.1. *Let $T > 0$. There exists a constant $C = C(T) > 0$ and a continuous linear map $\mathfrak{L} : \mathcal{H}^1 \times \mathcal{F} \to \mathcal{U}$ such that for any $y_0 \in \mathcal{H}^1$ and any $f \in \mathcal{F}$, the solution y of (1.8) with control $u = \mathfrak{L}(y_0, f)$ satisfies*

$$\left\| \frac{y}{\theta_0} \right\|_{C^0([0,T];\mathcal{H}^1)} + \left\| \frac{y}{\theta_0} \right\|_{L^2(0,T;\mathcal{H}^2)} + \|u\|_{\mathcal{U}} \leq C(\|f\|_{\mathcal{F}} + \|y_0\|_{\mathcal{H}^1}). \tag{3.30}$$

In particular, since θ_0 is a continuous function satisfying $\theta_0(T) = 0$, the above relation yields $y(T, \cdot) = 0$.

Proof. For $k \in \mathbb{N}$, we define $T_k := T(1 - q^{-k})$: On one hand, we set $a_0 := y_0$ and, for $k \in \mathbb{N}$, we define $a_{k+1} := y_f(T_{k+1}^-, \cdot)$ where y_f is the solution to

$$\begin{cases} \partial_t y_f + \mathcal{A} y_f = f & \text{on } (T_k, T_{k+1}) \\ y_f(T_k^+, \cdot) = 0. \end{cases}$$

From the energy estimate (3.2) in Proposition 3.1, we have

$$\|a_{k+1}\|_{\mathcal{H}^1} \leq \|y_f\|_{C^0([T_k,T_{k+1}];\mathcal{H}^1)} \leq C_T \|f\|_{L^2(T_k,T_{k+1};\mathcal{H}^0)}. \tag{3.31}$$

On the other hand, for $k \in \mathbb{N}$ we consider the homogeneous control system

$$\begin{cases} \partial_t y_u + \mathcal{A} y_u = u_k \mathbf{1}_\omega & \text{on } (T_k, T_{k+1}) \\ y(T_k^+, \cdot) = a_k, \end{cases}$$

where $u_k \in L^2(T_k, T_{k+1}; L^2(\omega))$ is such that $y_u(T_{k+1}^+, \cdot) = 0$ and

$$\|u_k\|^2_{L^2(T_k,T_{k+1};L^2(\omega))} \leq \gamma^2(T_{k+1} - T_k)\|a_k\|^2_{\mathcal{H}^0}. \tag{3.32}$$

Such a u_k exists for any $k \in \mathbb{N}$ by virtue of Theorem 1.2. Now remark that by definition of the weights, one has

$$\theta_0(T_{k+2}) = \theta_{\mathcal{F}}(T_k)\gamma(T_{k+2} - T_{k+1}) \tag{3.33}$$

for $k \in \mathbb{N}$. Now combining (3.32), (3.31), and the fact that $\theta_{\mathcal{F}}$ is a non-increasing function, we obtain

$$\begin{aligned}\|u_{k+1}\|^2_{L^2(T_{k+1},T_{k+2};L^2(\omega))} &\leq \gamma^2(T_{k+2} - T_{k+1})\|a_{k+1}\|^2_{\mathcal{H}^0} \\ &\leq C_T^2\gamma^2\left((q-1)\frac{T}{q^{k+2}}\right)\theta^2_{\mathcal{F}}(T_k)\left\|\frac{f}{\theta_{\mathcal{F}}}\right\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^0)}\end{aligned}$$

for any $k \in \mathbb{N}$. In view of the definition of θ_0 and the relation (3.33), we deduce that

$$\|u_{k+1}\|^2_{L^2(T_{k+1},T_{k+2};L^2(\omega))} \leq C_T^2\theta_0^2(T_{k+2})\left\|\frac{f}{\theta_{\mathcal{F}}}\right\|^2_{L^2(T_k,T_{k+1},\mathcal{H}^0)}.$$

Finally, since θ_0 is a non-increasing function, there exists a constant $C = C(T) > 0$ such that

$$\left\|\frac{u_{k+1}}{\theta_0}\right\|^2_{L^2(T_{k+1},T_{k+2};L^2(\omega))} \leq C\left\|\frac{f}{\theta_{\mathcal{F}}}\right\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^0)} \tag{3.34}$$

for all $k \in \mathbb{N}$. We can now patch the controls u_k for $k \in \mathbb{N}$ all together by defining

$$u := \sum_{k=0}^{\infty} u_k \mathbf{1}_{[T_k,T_{k+1})}.$$

In particular, combining estimates (3.34) and (3.32) (with $k = 0$) yields a constant $C = C(T) > 0$ such that

$$\left\|\frac{u}{\theta_0}\right\|_{L^2(0,T;L^2(\omega))} \leq C\left(\left\|\frac{f}{\theta_{\mathcal{F}}}\right\|_{L^2(0,T;\mathcal{H}^0)} + \|y_0\|_{\mathcal{H}^1}\right),$$

for any $y_0 \in \mathcal{H}^1$ and any $f \in \mathcal{F}$. The state y can also be reconstructed by concatenation, namely $y = y_f + y_u$, continuous at each junction by construction. Indeed,

$$y(T_k^-, \cdot) = y_f(T_k^-, \cdot) + y_u(T_k^-, \cdot) = a_k = y_f(T_k^+, \cdot) + y_u(T_k^+, \cdot) = y(T_k^+, \cdot),$$

and so y satisfies (1.8). We now look to estimate the state y. We use the energy estimate (3.2) from Proposition 3.1 on each time interval to obtain

$$\|y_f\|^2_{C^0([T_k,T_{k+1}];\mathcal{H}^1)} + \|y_f\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^2)} \leq C_T^2\|f\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^0)} \tag{3.35}$$

and

$$\begin{aligned}\|y_u\|^2_{C^0([T_k,T_{k+1}];\mathcal{H}^1)} &+ \|y_u\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^2)} \\ &\leq C_T^2\|a_k\|^2_{\mathcal{H}^1} + C_T^2\|u_k\|_{L^2(T_k,T_{k+1};L^2(\omega))}\end{aligned} \tag{3.36}$$

for $k \in \mathbb{N}$. Proceeding similarly as for estimating the control, we may deduce

$$\|y\|^2_{C^0([T_k,T_{k+1}];\mathcal{H}^1)} + \|y\|^2_{L^2(T_k,T_{k+1};\mathcal{H}^2)} \leq C_T^2 \theta_0^2(T_{k+1}) \left\| \frac{f}{\theta_{\mathcal{F}}} \right\|^2_{L^2(T_{k-1},T_{k+1};\mathcal{H}^0)}$$

for $k \geq 1$, and since θ_0 is a non-increasing function, using (3.32), (3.35), (3.36) (all for $k = 0$) we deduce

$$\left\| \frac{y}{\theta_0} \right\|_{C^0([0,T];\mathcal{H}^1)} + \left\| \frac{y}{\theta_0} \right\|_{L^2(0,T;\mathcal{H}^2)} \leq C \left(\left\| \frac{f}{\theta_{\mathcal{F}}} \right\|_{L^2(0,T;\mathcal{H}^0)} + \|y_0\|_{\mathcal{H}^1} \right).$$

In view of the above estimate and (3.34), we may conclude. □

4. The fixed-point argument

We look to conclude the proof of Theorem 1.1 by virtue of a Banach fixed point argument. Let $X_T := L^2(0,T;\mathcal{H}^2) \cap C^0([0,T];\mathcal{H}^1)$, and consider the time-weighted space

$$\mathcal{Y} := \left\{ y \in X_T : \frac{y}{\theta_0} \in X_T \right\},$$

which is endowed with the Hilbert norm

$$\|y\|^2_{\mathcal{Y}} := \int_0^T \theta_0^{-2}(t) \|y(t,\cdot)\|^2_{X_T} \,\mathrm{d}t.$$

We recall that the weights $\theta_0, \theta_{\mathcal{F}}$ are defined in (3.27) and (3.26) respectively. Let us also denote

$$M := \sup_{t\in[0,T]} \frac{\theta_0^2}{\theta_{\mathcal{F}}}(t),$$

which is finite due to (3.28), and consider the radius

$$r := \min \left\{ \frac{1}{2C(T)}, \frac{1}{8C(T)MC_\sigma} \right\}, \tag{4.1}$$

where $C(T) > 0$ is the constant appearing in the control-continuity estimate (3.30) and $C_\sigma > 0$ appears in the embedding given by Lemma 2.3 (see (4.3) below). We also consider the ball

$$\mathcal{Y}_r := \left\{ y \in \mathcal{Y} : \|y\|_{\mathcal{Y}} \leq r \right\}$$

Given $y_0 \in \mathcal{H}^1$, we may construct the nonlinear map $\mathcal{N} : \mathcal{Y}_r \to \mathcal{Y}_r$ by setting

$$\mathcal{N}(\overline{y}) := y,$$

where y is the solution to the controlled problem

$$\begin{cases} \partial_t y - \rho^{-\sigma}\partial_x(\rho^{\sigma+1}\partial_x y) = \rho F_{\varepsilon,\delta}(\overline{y}, \partial_x \overline{y}) + u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho^{\sigma+1}\partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1), \end{cases}$$

where we recall (see (1.6)) that $F_{\varepsilon,\delta}(p,q) = \frac{q^2}{1+p+xq}$ when $p^2 < \delta^2, q^2 < \varepsilon^2$, and $F_{\varepsilon,\delta} = 0$ whenever $p^2 \geq 4\delta^2$ or $q^2 \geq 4\varepsilon^2$. Also, we assumed that $4(\varepsilon + \delta) < 1$. We are now in a position to prove our main result.

Proof of Theorem 1.1. For the sake of cohesion, we split the proof in three steps.

First step. Fix $y_0 \in \mathcal{H}^1$. We first look to show that the map $\mathcal{N}$ is well-defined and leaves $\mathcal{Y}_r$ invariant. Given $\overline{y} \in \mathcal{Y}_r$ we consider the source term

$$\overline{f} := \rho F_{\varepsilon,\delta}(\overline{y}, \partial_x \overline{y}).$$

Let us first demonstrate that $\overline{f} \in \mathcal{F}$, with $\mathcal{F}$ being defined in (3.29). Recall that

$$F_{\varepsilon,\delta}(\overline{y}, \partial_x \overline{y}) = \chi\left(\frac{\overline{y}^2}{\delta^2}\right)\chi\left(\frac{(\partial_x \overline{y})^2}{\varepsilon^2}\right)\frac{(\partial_x \overline{y})^2}{1+\overline{y}+x\partial_x\overline{y}},$$

where $\chi : [0,\infty) \to [0,1]$ is a smooth cut-off supported on $[0,4)$ with $\chi(x) \equiv 1$ on $[0,1]$. Since $F_{\varepsilon,\delta} \not\equiv 0$ if and only if $|\overline{y}| \leq 2\delta$ and $|\partial_x \overline{y}| \leq 2\varepsilon$ where $4(\varepsilon+\delta) < 1$ (and thus $2(\varepsilon+\delta) < \frac{1}{2}$), using the triangle inequality we have

$$\begin{aligned} |F_{\varepsilon,\delta}(\overline{y}, \partial_x \overline{y})| \leq \frac{(\partial_x \overline{y})^2}{|1+\overline{y}+x\partial_x\overline{y}|} \leq \frac{(\partial_x \overline{y})^2}{1-|\overline{y}|-|\partial_x\overline{y}|} &\leq \frac{(\partial_x \overline{y})^2}{1-2(\varepsilon+\delta)} \\ &\leq 2(\partial_x\overline{y})^2. \end{aligned}$$

Whence,

$$\begin{aligned} \int_0^T\int_{-1}^1 \rho^\sigma \frac{\overline{f}^2}{\theta_{\mathcal{F}}^2}\,\mathrm{d}x\,\mathrm{d}t &\leq 4\int_0^T\int_{-1}^1 \rho^{\sigma+2}\frac{\theta_0^4}{\theta_{\mathcal{F}}^2}\frac{(\partial_x\overline{y})^4}{\theta_0^4}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq 4M^2\int_0^T\int_{-1}^1 \rho^{\sigma+2}\frac{(\partial_x\overline{y})^4}{\theta_0^4}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.2}$$

We now recall that from Lemma 2.3, the embedding

$$\left\|\rho^{\frac{1}{2}}\partial_x\overline{y}\right\|_{C^0([-1,1])} \leq C_\sigma \|\overline{y}\|_{\mathcal{H}^2} \tag{4.3}$$

holds for some $C_\sigma > 0$ whenever $\sigma \in (-1,0)$. As moreover $\overline{y}\,\theta_0^{-1} \in C^0([0,T];\mathcal{H}^1)$, going back to (4.4) we may apply estimate (4.3) to the effect of

$$\begin{aligned} \int_0^T\int_{-1}^1 \rho^{\sigma+2}\frac{(\partial_x\overline{y})^4}{\theta_0^4}\,\mathrm{d}x\,\mathrm{d}t &\leq C_\sigma^2\int_0^T \theta_0^{-2}\|\overline{y}\|^2_{\mathcal{H}^2}\int_{-1}^1\rho^{\sigma+1}\frac{(\partial_x\overline{y})^2}{\theta_0^2}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C_\sigma^2\left(\int_0^T\theta_0^{-2}\|\overline{y}\|^2_{\mathcal{H}^2}\,\mathrm{d}t\right)\left(\sup_{t\in[0,T]}\int_{-1}^1\rho^{\sigma+1}\frac{(\partial_x\overline{y})^2}{\theta_0^2}\,\mathrm{d}x\right) \\ &\leq C_\sigma^2\left(\int_0^T\theta_0^{-2}\|\overline{y}(t,\cdot)\|^2_{X_T}\,\mathrm{d}t\right)^2. \end{aligned} \tag{4.4}$$

Combining estimates (4.2) and (4.4), we deduce

$$\left\|\frac{\overline{f}}{\theta_{\mathcal{F}}}\right\|_{L^2(0,T;\mathcal{H}^0)} \leq 2MC_\sigma\|\overline{y}\|^2_{\mathcal{Y}}, \tag{4.5}$$

and so $\overline{f} \in \mathcal{F}$. Now, let $u := \mathfrak{L}(y_0, \overline{f})$, which is well-defined by Theorem 3.1, and consider the corresponding controlled trajectory $y \in \mathcal{Y}$. We aim to show that $y \in \mathcal{Y}_r$. From the control-continuity estimate (3.30) we have

$$\|y\|_{\mathcal{Y}} \leq C(T)\left(\left\|\frac{\overline{f}}{\theta_{\mathcal{F}}}\right\|_{L^2(0,T;\mathcal{H}^0)} + \|y_0\|_{\mathcal{H}^1}\right).$$

Inequality (4.5) leads us to

$$\|y\|_{\mathcal{Y}} \leq C(T)\Big(2M_0 C_\sigma \|\overline{y}\|_{\mathcal{Y}}^2 + \|y_0\|_{\mathcal{H}^1}\Big).$$

In view of (4.1), choosing $\|y_0\|_{\mathcal{H}^1} \leq r$ leads us to conclude that $y \in \mathcal{Y}_r$.

Second step. Let us now demonstrate that the map $\mathcal{N} : \mathcal{Y}_r \to \mathcal{Y}_r$ is strictly contractive. Observe that for $x \in (-1,1)$, $(p_i, q_i) \in \mathbb{R}$ satisfying $p_i^2 < \delta^2 < 1$ and $q_i^2 < \varepsilon^2 < 1$, $i = 1,2$, one has

$$\begin{aligned} |F_{\varepsilon,\delta}(p_1,q_1) - F_{\varepsilon,\delta}(p_2,q_2)| &\leq 4\big(q_1^2(1+p_2+xq_2) - q_2^2(1+p_1+xq_1)\big) \\ &\leq 4\big((1+p_1+q_1)(q_1^2-q_2^2) + q_1^2(p_2-p_1) + q_1^2(q_2-q_1)\big) \\ &\leq 6(q_1^2-q_2^2) + 4q_1(p_2-p_1) + 4q_1(q_2-q_1). \end{aligned} \tag{4.6}$$

Hence, using estimates (3.30), (4.6) and arguing as in Step 1, we may see that

$$\begin{aligned} \Big\|\mathcal{N}(y_1) - \mathcal{N}(y_2)\Big\|_{\mathcal{Y}} &\leq C(T)\Big\|\rho\Big(F_{\varepsilon,\delta}(y_1,\partial_x y_1) - F_{\varepsilon,\delta}(y_2,\partial_x y_2)\Big)\Big\|_{\mathcal{F}} \\ &\leq 4C(T)C_\sigma M r\,\big\|y_1 - y_2\big\|_{\mathcal{Y}}. \end{aligned}$$

In view of (4.1), we deduce that $\mathcal{N}$ is a strict contraction.

Third step. Thanks to the Banach fixed point theorem, for any $y_0 \in \mathcal{H}^1$ satisfying $\|y_0\|_{\mathcal{H}^1} \leq r$, the nonlinear operator $\mathcal{N} : \mathcal{Y}_r \to \mathcal{Y}_r$ admits a unique fixed point $y \in \mathcal{Y}_r$. We may thus conclude the proof of Theorem 1.1. ☐

5. Null-controllability of the linearized thin-film equation

We give brief arguments as to see that the controllability study in Section 1.2 may also be applied to the one-dimensional thin-film equation linearized around its self-similar profile, derived in [29, 35]. The *thin-film equation*

$$\partial_t h + \partial_z(h^n \partial_z^3 h) = 0 \quad \text{ in } \{h > 0\}$$

where $n \in (0,3)$ represents a more realistic model for the evolution of a liquid film over a solid substrate in a regime known as lubrication approximation. Much like its second order counterpart, the PME (1.1), it is a free boundary problem whenever the initial datum is compactly supported (a physical phenomenon known as *droplets*). We refer to [17, 18, 35] and the references therein for an overview of the well-posedness results, self-similar asymptotics and the role of boundary conditions.

For $n = 1$ (known as linear mobility regime), McCann & Seis [29, 35] replicate the ideas used for the PME in [14, 33, 34] to compute the spectrum of the full linearization of the thin-film equation around its own self-similar (Smyth-Hill) solution. Namely, after an analog rescaling and von Mises transformation, the control problem for the equation linearized around the self-similar solution is of the form

$$\begin{cases} \partial_t y + \mathcal{A}^2 y + \mathcal{A} y = u\mathbf{1}_\omega & \text{in } (0,T)\times(-1,1) \\ (\rho y)(t,\pm 1) = (\rho^2 \partial_x y)(t,\pm 1) = 0 & \text{in } (0,T) \\ y(0,x) = y_0(x) & \text{in } (-1,1). \end{cases} \tag{5.1}$$

where $T > 0$ and $\mathcal{A} = -\rho^{-1}\partial_x(\rho^2\partial_x)$ is the operator governing the linearized porous medium equation (3.4) with $\sigma = 1$. Replicating the linear theory from Section 2, we may deduce that the operator $\mathcal{L} = \mathcal{A}(\mathcal{A} + \mathrm{Id})$ is self-adjoint, non-negative with domain $\mathcal{D}(\mathcal{L}) = \mathcal{H}^4$, and has compact resolvents. Both boundary conditions are automatically satisfied by arguing as in Lemma 2.4. The operator thus generates an analytic semigroup on $\mathcal{H}^0$, which implies the following result (see [35, Lemma 3]).

Proposition 5.1. *Let $T > 0$. For any $y_0 \in \mathcal{H}^0$ and $u \in L^2(0,T;\mathcal{H}^0)$, there exists a unique solution $y \in L^2(0,T;\mathcal{H}^2) \cap C^0([0,T];\mathcal{H}^0)$ to (5.1).*

As done in Section 3, we use the explicit spectrum of the linearized operator $\mathcal{A}(\mathcal{A}+\mathrm{Id})$, given in [29], to demonstrate the null-controllability of (5.1). This is in essence an immediate consequence of Theorem 2.1.

Corollary 5.1 ([29], Corollary 3)**.** *The spectrum of $\mathcal{L}:\mathcal{H}^4\to\mathcal{H}^0$ consists of simple nonnegative eigenvalues $\{\mu_k\}_{k=0}^{\infty}$, given by*

$$\mu_k = \lambda_k^2 + \lambda_k$$

for $k \geq 0$, where λ_k denote the eigenvalues of $\mathcal{A}$ from Theorem 2.1. Moreover, $\mathcal{L}$ and $\mathcal{A}$ have the same normalized eigenfunctions $\{\overline{\varphi}_k\}_{k=0}^{\infty}$, which generate an orthonormal basis of $\mathcal{H}^0$.

As the eigenfunctions of $\mathcal{L}=\mathcal{A}(\mathcal{A}+\mathrm{Id})$ and $\mathcal{A}$ coincide, and the control operator B is the same as in Section 3, we may deduce the following null-controllability result for Problem (5.1).

Theorem 5.1. *Let $T>0$, $\omega \subsetneq (-1,1)$ be an open, non-empty interval, and $\sigma=1$. Then, for any $y_0\in\mathcal{H}^0$, there exists a control $u\in L^2(0,T;L^2(\omega))$ such that the unique solution $y\in L^2(0,T;\mathcal{H}^2)\cap C^0([0,T];\mathcal{H}^0)$ of (5.1) satisfies $y(0,\cdot)=y_0$ and $y(T,\cdot)=0$.*

Proof. In view of Lemmas 3.1 and 3.2, and since the eigenfunctions of $\mathcal{A}$ and $\mathcal{L}$ coincide, we only need to investigate the eigenvalues $\{\mu_k\}_{k=0}^{\infty}$ of the latter operator. Due to their form, it is readily seen the eigenvalues satisfy both the growth condition (3.8) and separation condition (3.7). We may thus conclude by using the HUM method as for the proof of Theorem 1.2. □

6. Conclusion and perspectives

In this work, we addressed the null controllability a one-dimensional degenerate parabolic equation, which represents the problem satisfied by a perturbation around the Barenblatt profile of the free boundary porous medium equation after fixing the moving domain. We proved a local null-controllability result for this perturbed problem with a regularized version of the original nonlinearity. This allowed us to make use of only $L^2(0,T;L^2(\omega))$-regular controls for the fixed-point argument. The linear controllability theory was also applied for proving a null-controllability result for the linearized thin-film equation.

Let us present some directions on how our results may be extended, as well as some related open problems.

6.1. The full nonlinearity and free boundary problem

In this work we only addressed the case where the nonlinearity

$$\mathcal{N}(y)=\rho F(y,\partial_x y)-\rho^{-\sigma}\partial_x(\rho^{\sigma+1}xF(y,\partial_x y)),\quad F(y,\partial_x y)=\frac{(\partial_x y)^2}{1+y+x\partial_x y}$$

is truncated as in (1.6) (and without the higher order term $-\rho^{-\sigma}\partial_x(\rho^{\sigma+1}xF_{\varepsilon,\delta})$). To derive a local null-controllability result for the full nonlinear perturbation equation (1.4), one would need to remove the cut-off, i.e. to ensure that $F_{\varepsilon,\delta}\equiv F$. To ensure this condition, higher regularity of the controlled trajectory y and consequently of the control u is needed. Namely, y should be $C^{0,1}([0,T]\times[-1,1])$ and have small-enough norm. This will be considered in a future work. The Lipschitz regularity of the controlled trajectory is also required for reversing the von Mises transformation in view of obtaining the local controllability to the Barenblatt trajectory for the free boundary problem (see Remark A.1).

6.2. The multi-dimensional case

The null-controllability of perturbation equation in arbitrary dimension is also worth investigating. In the linearized regime, it would read

$$\begin{cases}\partial_t y-\rho^{-\sigma}\nabla\cdot(\rho^{\sigma+1}\nabla y)=u\mathbf{1}_\omega & \text{in } (0,T)\times B_1\\ (\rho^{\sigma+1}\partial_n y)(t,\pm 1)=0 & \text{in } (0,T),\end{cases}$$

where B_1 is the open unit ball, $\rho(x) = \frac{1}{2}(1-|x|^2)$ and $\omega \subsetneq B_1$ is open and non-empty. The well-posedness follows from similar arguments as in the one-dimensional case, and is also argued in [34]. The spectral/Fourier techniques we used in this work are however restricted to the one-dimensional case. Thus, proving the desired observability inequality would likely require a Carleman estimate in the weighted $\mathcal{H}^k$ spaces. To the best of our knowledge, this has not been addressed in the literature.

6.3. **The thin-film equation**

The preceding questions are also open in the case of the (perturbed) thin-film equation of Section 5, for which we have only addressed the null-controllability of the one-dimensional, linearized equation.

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Appendix A. Transformations

The von Mises change of variables of [34, Section 2] (see also [22, Section 5.4], [21]) transforms the free boundary problem (1.2) (in any dimension) into the degenerate-parabolic equation (1.4) (with $u \equiv 0$, in any dimension). It has the effect of fixing the moving domain to the reference domain which is the open unit ball B_1. For $t \geq 0$, the transformation of the spatial coordinates reads

$$x := \frac{z}{\sqrt{2v(t,z) + |z|^2}}, \tag{A.1}$$

where $z \in \{v(t,\cdot) > 0\}$. Hence $x \in B_1$, and the transformation reduces to the identity map when $v(t,z)$ is the Barenblatt $\rho(z)$. The unknown in the new variables is defined as

$$w(t,x) := \sqrt{2v(t,z) + |z|^2},$$

so that in these new variables, the Barenblatt reduces to the constant 1. As the interest is to linearize around the Barenblatt, perturbations of the form $w(t,x) = 1 + y(t,x)$ are considered. In other words,

$$1 + y(t,x) := \sqrt{2v(t,z) + |z|^2}. \tag{A.2}$$

The inverse change of variables reads

$$z = (y(t,x)+1)x, \qquad v(t,z) - \rho(z) = y(t,x) + \frac{1}{2}y(t,x)^2 \tag{A.3}$$

for $t \geq 0$ and $x \in B_1$. The transformation (A.1), (A.2) and (A.3) is rigorously justified in [34, Section 3], and the transformation preserves the smallness of the data.

Remark A.1. *If one may apply the above transformation given a null-controlled trajectory y of* (1.4) *(thus provided $\|y\|_{C^{0,1}([0,T]\times[-1,1])} < 1$, see [34, Lemma 3.2]), then $y(T,\cdot) = 0$ would imply $v(T,\cdot) = \rho(\cdot)$, along with equality of the interfaces, as originally desired. The control in the free boundary problem would a priori be actuating inside a moving subregion (due to the fact that the new spatial variable z depends on the state y). However since the results are local, it may be possible to exhibit a time-independent subregion in the new variables (see for instance [12, Lemma 2.10]). Finally, as one may rewrite $v(t,z) = \rho(x)(1+y(t,x))^2$ in* (A.3)*, the transformation would moreover guarantee the non-negativity of the controlled trajectory v.*

Appendix B. Auxiliary estimates

Herein, we provide short proofs of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. Set $x = \tanh s$, and consider $g(s) = (1-\tanh^2 s)^\alpha f(\tanh s)$. Then we have $\|(1-x^2)^\alpha f\|_{C^0([-1,1])} = \|g\|_{L^\infty(\mathbb{R})}$. Along with the standard Sobolev embedding $\|g\|_{L^\infty(\mathbb{R})} \le \|g\|_{H^1(\mathbb{R})}$ and $\frac{\mathrm{d}x}{\mathrm{d}s} = 1-x^2$, this yields

$$\begin{aligned}\|(1-x^2)^\alpha f\|^2_{C^0([-1,1])} &\le \int_{\mathbb{R}} (g^2 + (\partial_s g)^2)\,\mathrm{d}s \\ &= \int_{-1}^{1} \big((1-x^2)^{2\alpha-1} f^2 + (1-x^2)(\partial_x(1-x^2)^\alpha f)^2\big)\,\mathrm{d}x.\end{aligned}$$

Using the elementary estimate $(a-b)^2 \le 2a^2 + 2b^2$, we conclude

$$\|(1-x^2)^\alpha f\|^2_{C^0([-1,1])} \le (1+8\alpha^2)\int_{-1}^{1}(1-x^2)^{2\alpha-1} f^2 + 2\int_{-1}^{1}(1-x^2)^{2\alpha+1}(\partial_x f)^2.$$

□

We recall the following Hardy inequality, and refer to [17, Lemma A.1] for a proof (see [20] for the original).

Lemma B.1 (Hardy)**.** *Let $\alpha \ne \frac{1}{2}$, and let $\|x^{\alpha+1}\partial_x f\|_{L^2(\mathbb{R}_+)} < \infty$. Suppose that $f(x_k) \to 0$ for some sequence $x_k \to c$ as $k \to \infty$, where $c = 0$ if $\alpha < -\frac{1}{2}$ and $c = \infty$ if $\alpha > -\frac{1}{2}$. Then*

$$\|x^\alpha f\|_{L^2(\mathbb{R}_+)} \le \frac{2}{|2\alpha+1|}\|x^{\alpha+1}\partial_x f\|_{L^2(\mathbb{R}_+)}. \tag{B.1}$$

One may choose f such that $f(x) = \log(\log \frac{1}{x})$ near $x = 0$ to show that the assumption $\alpha \ne -\frac{1}{2}$ is necessary.

Proof of Lemma 2.2. The proof follows similar arguments to those for Lemma 2.1. We begin by writing

$$\int_{-1}^{1}(1-x^2)^\alpha f^2\,\mathrm{d}x = \int_{-1}^{0}(1-x^2)^\alpha f^2\,\mathrm{d}x + \int_{0}^{1}(1-x^2)^\alpha f^2\,\mathrm{d}x. \tag{B.2}$$

As both terms on the right-hand side of the identity (B.2) are symmetric, we will only look at the first one. Let $\eta \in C^\infty(\mathbb{R})$ be a cut-off function with $\eta(x) \equiv 1$ for $x \le 0$ and $\eta(x) \equiv 0$ for $x \ge \frac{1}{2}$. Also, set $g(s) = f(x)\eta(x)$ for $s = 1+x$. As $1 \le (1-x) \le 2$ in $(-1,0)$, we have

$$\int_{-1}^{0}(1-x^2)^\alpha f^2\,\mathrm{d}x \le C_\alpha \int_{-1}^{0}(1+x)^\alpha f^2\,\mathrm{d}x = C_\alpha\int_0^1 s^\alpha g^2\,\mathrm{d}s \le C_\alpha \int_0^\infty s^\alpha g^2\,\mathrm{d}s, \tag{B.3}$$

where $C_\alpha = 2^\alpha$ if $\alpha > 0$ and 1 otherwise. We make use of the Hardy inequality (B.1) on the right-most term in (B.3), which yields

$$\int_{-1}^{0}(1-x^2)^\alpha f^2\,\mathrm{d}x \le \frac{C_\alpha}{(\alpha+1)^2}\int_0^\infty s^{\alpha+2}(\partial_s g)^2\,\mathrm{d}s \quad \text{for } \alpha > -1. \tag{B.4}$$

Now, a straightforward computation gives

$$(\partial_s g)^2 = (f\partial_x\eta + \eta\partial_x f)^2 \le 2((f\partial_x\eta)^2 + (\eta\partial_x f)^2)$$

for $s < \frac{3}{2}$ i.e. $x < \frac{1}{2}$, and so from (B.4) we can deduce

$$\begin{aligned}\int_{-1}^{0}(1-x^2)^\alpha f^2\,\mathrm{d}x &\le \frac{C_\alpha}{(\alpha+1)^2}\int_{-1}^{1}(1-x^2)^{\alpha+2}((f\partial_x\eta)^2+(\eta\partial_x f)^2)\,\mathrm{d}x\\ &\le \frac{C_\alpha}{(\alpha+1)^2}\left(\int_0^{\frac{1}{2}} f^2\,\mathrm{d}x+\int_{-1}^{1}(1-x^2)^{\alpha+2}(\partial_x f)^2\,\mathrm{d}x\right)\\ &\lesssim_\beta \frac{C_\alpha}{(\alpha+1)^2}\int_{-1}^{1}\left((1-x^2)^\beta f^2+(1-x^2)^{\alpha+2}(\partial_x f)^2\right)\mathrm{d}x,\end{aligned}$$

where on the last line we used $1-x^2 \ge \frac{3}{4}$ in $(0,\frac{1}{2})$. As per (B.2), this implies the desired result. □

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