# Optimal energy decay rates for abstract second order evolution equations with non-autonomous damping *

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**Abstract**

We consider an abstract second order non-autonomous evolution equation in a Hilbert space $H: u'' + Au + \gamma(t)u' + f(u) = 0$, where $A$ is a self-adjoint and nonnegative operator on $H$, $f$ is a conservative $H$-valued function with polynomial growth (not necessarily to be monotone), and $\gamma(t)u'$ is a time-dependent damping term. How exactly the decay of the energy is affected by the damping coefficient $\gamma(t)$ and the exponent associated with the nonlinear term $f$? There seems to be little development on the study of such problems, with regard to *non-autonomous* equations, even for strongly positive operator $A$. By an idea of asymptotic rate-sharpening (among others), we obtain the optimal decay rate of the energy of the non-autonomous evolution equation in terms of $\gamma(t)$ and $f$. As a byproduct, we show the optimality of the energy decay rates obtained previously in the literature when $f$ is a monotone operator.

*Keywords:* Non-autonomous; abstract second order evolution equation; time dependent damping; energy estimates; slow solutions; nonlinear source; Hilbert space.

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## 1 Introduction

In view of the importance in mathematical theory and applications in physics, engineering, mechanics, biology and others, various nonautonomous equations including the non-autonomous evolution equations in abstract (infinite-dimensional) spaces, have been studied by many researchers and a lot of good results on this issue have been established (cf., e.g. [1, 3, 8–17, 24, 25, 27, 28, 30, 32, 34, 35] and references therein).

In this paper, we are concerned with an abstract second order non-autonomous evolution equation

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in a real Hilbert space $H$, with time dependent damping, as follows

$$\begin{cases} u''(t) + Au(t) + \gamma(t)u'(t) + f(u(t)) = 0, \quad \forall t \geq 0, \\ \\ u(0) = u_0, \ u'(0) = u_1, \end{cases} \tag{1.1}$$

where $A$ is a nonnegative self-adjoint linear operator on $H$, the nonlinear term $f : D(A^{1/2}) \to H$ is assumed to be conservative with polynomial growth, and $\gamma u'$ is the time dependent damping with

$$R_1(1+t)^{-\alpha_1} \leq \gamma(t) \leq R_2(1+t)^{-\alpha_2}, \quad \forall t \geq 0, \tag{1.2}$$

where $R_1$ and $R_2$ are two positive constants, and $0 \leq \alpha_2 \leq \alpha_1 < 1$.

For the study of the asymptotic behaviors and decay rates for the damped autonomous evolution equations, there have been many developments. We refer the reader to, e.g., [2, 4, 6, 12, 13, 15, 17–21, 26, 31, 35] and references therein. Especially, when $A$ is strongly positive, the asymptotic behaviors for the autonomous case of (1.1) have been extensively studied too, cf. [10, 22, 23, 29, 30, 32, 33]. In this paper, we assume $A$ to just have a nontrivial kernel; for the case of $\gamma(t)$ being constant, optimal decay rates have been obtained, and fast and slow solutions are classified subtly (see [18–20]).

It was shown in [7] that any bounded solution of (1.1) converges weakly in $H_1$ to a stationary point of the potential energy, when $f$ is a general monotone operator. Following the work, it was proved in [31] that any solution energy $E(t)$ satisfies that for every $\bar{\alpha} < \alpha_1$,

$$E(t) = \mathrm{o}\left(\frac{1}{t^{1+\bar{\alpha}}}\right) \quad \text{as } t \to +\infty, \tag{1.3}$$

under the condition that $\gamma(t)$ is decreasing,

$$\gamma(t) \geq c(1+t)^{-\alpha_1}, \quad t \geq 0 \tag{1.4}$$

(with a constant $c > 0$), and

$$\gamma'(t) \leq -\alpha_1 \frac{\gamma(t)}{1+t}, \quad t \geq t_0 \geq 0 \ \text{ a.e.} \tag{1.5}$$

We note that the two inequalities (1.4) and (1.5) together imply that

$$\gamma(t) \leq C(1+t)^{-\alpha_1}, \quad t \geq 0$$

(with a constant $C > 0$). For

$$\gamma(t) = (1+t)^{-\alpha} \quad (\alpha \in [0, 1)), \tag{1.6}$$

the estimate (1.3) was later improved in [4] as

$$E(t) = \mathrm{o}\left(\frac{1}{t^{1+\alpha}}\right) \quad \text{as } t \to +\infty. \tag{1.7}$$

However, what is the optimal decay rate of the energy of the non-autonomous equation (1.1), i.e., how exactly the decay of the energy of (1.1) is affected by the damping coefficient $\gamma(t)$ and the exponent associated with the nonlinear term $f$? This problem is still unsolved. To the best of our knowledge, there has been little development so far on the study of such problems, with regard to non-autonomous equations, even for strongly positive operator $A$; see [30, Section 1.3] about an optimality result for one dimensional wave equation damped by a time-dependent boundary feedback, whose proof relies on d'Alembert's formula.

In this paper, we devote ourselves to studying the problem.

Let $\gamma(t)$ be as in (1.6), and $f(u) = |u|^p u$ $(p > 0)$. By [28, Theorem 2.4], the energy has the upper estimate

$$E(t) \le C(1+t)^{-(1-\alpha)(p+2)/p}. \tag{1.8}$$

On the other hand, to get a lower bound we may exploit the hyperbolic version of the Dirichlet quotient

$$G(t) := \frac{\|u'(t)\|^2 + \|A^{1/2}u(t)\|^2}{\|u(t)\|^{2p+2}}, \tag{1.9}$$

introduced in [18]. Estimating the time-derivative of $\hat{G}(t)$, a small perturbation of $G$ with $\gamma$ as a product factor (see (4.9) in Section 4), with the aid of either (1.7) or (1.8) we might probably obtain a lower estimate

$$E(t) \ge c(1+t)^{-(1+\alpha)(p+2)/p}$$

for a nonempty open set of initial data, but under the restriction

$$\alpha < p/(p+4) \quad \text{or} \quad \alpha < 1/3.$$

Obviously, there is a gap between the exponents $-(1-\alpha)(p+2)/p$ and $-(1+\alpha)(p+2)/p$, when $\alpha > 0$. Actually, it is still unknown what is the best possible upper estimate.

By an idea of asymptotic rate-sharpening, we obtain the optimal decay rate of the energy of this abstract second order non-autonomous evolution equation in terms of $\gamma(t)$ and $f$. Moreover, as a byproduct, we show that, when f is a monotone operator, some energy decay rates given previously in the literature are optimal.

More explicitly, we obtain upper estimates for the solutions and their energies of (1.1) in a general setting, and single out slow solutions (decaying exactly at the rates corresponding to the upper estimates) with a nonempty open set of initial data. When $\gamma(t)$ is a constant, the estimates recover those given in [18, Theorems 2.2 and 2.3]. Moreover, specialized to the case of (1.6), the exponents of the upper and lower bounds coincide and read $-(1+\alpha)(p+2)/p$ for all $\alpha \in [0,1)$. In addition, since

$$-(1+\alpha)(p+2)/p \to -(1+\alpha) \quad \text{as } p \to +\infty, \tag{1.10}$$

we see that the decay rate in (1.7) is the best possible for a general monotone $f$ (as considered in [4, 7, 31]).

The outline of this paper is the following. In Section 2 we state assumptions and present our main theorems. Section 3 is devoted to formulating several preliminary results. We employ these results in Section 4 to prove the main theorems. Finally, in Section 5 we give two examples of our theorems being applied to Neumann or Dirichlet problems for nonautonomous, semilinear wave equations.

## 2 Main Theorems

We denote by $\langle v, w\rangle$ the inner product of two vectors $v, w$ in $H$, and by $\|v\|$ the $H$-norm of $v$, and we define

$$H_1 = D(A^{1/2}) \quad \text{with norm} \ \ \|v\|_{H_1} := \left(\|v\|^2 + \|A^{1/2}v\|^2\right)^{1/2}.$$

The following is the basic assumptions on $\gamma$ and $f$.

**Assumption (H1)**

(i) $\gamma \in W^{1,\infty}_{\text{loc}}(R^+)$ satisfies (1.2);

(ii) $f = \nabla F : \ H_1 \to H$ is a locally Lipschitz continuous gradient operator of some nonnegative functional $F$ on $H_1$, with $F(0) = 0$.

One knows

$$F(v) = \int_0^1 \langle f(tv), v\rangle dt, \quad \forall v \in H_1.$$

A function $u \in C([0, +\infty); H_1) \cap C^1([0, +\infty); H)$ is called a *mild solution* of problem (1.1), if it satisfies the integral equation

$$u(t) = S'(t)u_0 + S(t)u_1 - \int_0^t S(t-\tau)[\gamma(\tau)u'(\tau) + f(u)]d\tau, \quad t \geq 0.$$

Here $S(\cdot) : \ [0, +\infty) \to \mathcal{L}(H)$ (the space of bounded linear operators on $H$) is a solution operator for the linear equation

$$u''(t) + Au(t) = 0, \quad t \geq 0,$$

with $S(0) = 0$ and $S'(0) = I$ (the identity; the derivative being in the sense of strong topology).

In particular, $u$ is called a *strong solution* if $u \in C^1([0, +\infty); H_1) \cap C^2([0, +\infty); H)$ and (1.1) holds.

The following result concerning the wellposedness of (1.1) is from [28, Proposition 2.3].

**Proposition 2.1.** *Assume (H1). Then for every $(u_0, u_1) \in H_1 \times H$, problem (1.1) admits a unique mild solution $u$, which depends continuously on the initial data. In particular, $u$ is a strong solution if $(u_0, u_1) \in D(A) \times H_1$. Moreover, defining the energy*

$$E(t) := \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 + F(u(t)), \quad t \geq 0, \tag{2.1}$$

*one has*

$$\frac{dE(t)}{dt} = -\gamma(t)\|u'(t)\|^2, \quad t \geq 0 \tag{2.2}$$

*(for strong solutions).*

For energy decay, more conditions are required.

**Assumption** (H2)

(1) For $v \in H_1$,

$$\langle f(v), v\rangle \geq cF(v) \geq 0,$$

with some constant $c > 0$;

(2) for $v \in H_1$,

$$\|v\|^{p+2} \le M_0(\|A^{1/2}v\|)\left(\|A^{1/2}v\|^2 + F(v)\right),$$

with constant $p > 0$ and some positive function $M_0$ on $R_+$ that is bounded on bounded sets;

(3) for $t \ge 0$,

$|\gamma'(t)| \le C\gamma^2(t)$ with some constant $C > 0$, or $\alpha_1 = \alpha_2$;

(4) for $t \ge 0$,

$|\gamma'(t)| \le C\gamma(t)(1+t)^{-1}$ with some constant $C > 0$ or $\gamma'(t) \le 0$, in the case $\alpha_1 > 1/2$;

(5) $2\alpha_1 < 1 + \alpha_2$, in the case $\alpha_1 > 1/2$.

Now, we present our first main result.

**Theorem 2.2.** *Let Assumptions* (H1) *and* (H2) *hold. Let* $(u_0, u_1) \in H_1 \times H$*, and let* $u(t)$ *be the unique global mild solution of problem (1.1). Then*

$$E(t) \le M_1(E(0))\left(1 + \int_0^t \gamma(\tau)^{-1}d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \ge 0, \tag{2.3}$$

$$\|u(t)\| \le M_2(E(0))\left(1 + \int_0^t \gamma(\tau)^{-1}d\tau\right)^{-\frac{1}{p}}, \quad \forall t \ge 0,$$

*for some positive functions* $M_1$ *and* $M_2$ *on* $R_+$ *that are bounded on bounded sets.*

For the existence of slow solutions, we need additional conditions.

**Assumption** (H3)

(1) $\ker A \ne \{0\}$, and there exists $\zeta > 0$ such that

$$\|A^{1/2}u\|^2 \ge \zeta\|u\|^2, \quad \forall u \in H_1 \cap (\ker A)^{\perp};$$

(2) there exist positive number $\xi$ and $M$ such that

$$\|f(u)\| \le M\left(\|u\|^{1+p} + \|A^{1/2}u\|^{1+p}\right), \quad \forall u \in H_1 \text{ with } \|u\|_{H_1} \le \xi;$$

(3) $\gamma(t)\gamma(s)^{-1} \le C_0, \quad \forall t \ge s \ge 0$, with some constant $C_0 > 0$.

The following is our second main result.

**Theorem 2.3.** *Let Assumptions* (H1)-(H3) *hold. Then there exists a nonempty open set* $S \subset H_1 \times H$ *such that for every* $(u_0, u_1) \in S$*, the unique global solution of problem (1.1) satisfies*

$$\|u(t)\| \ge c_0\left(1 + \int_0^t \gamma(\tau)^{-1}d\tau\right)^{-\frac{1}{p}}, \quad \forall t \ge 0,$$

$$E(t) \ge c_0\left(1 + \int_0^t \gamma(\tau)^{-1}d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \ge 0, \tag{2.4}$$

*for some positive function* $c_0$ *depending on* $\|u_0\|$, $\|A^{1/2}u_0\|$ *and* $\|u_1\|$.

**Remark 2.4.** (1) When $\gamma$ is decreasing and $\alpha_1 = \alpha_2 = \alpha$, we know that (H2)(3)-(5) and (H3)(3) are satisfied, and the upper estimate of the energy in (2.3) reads

$$E(t) \leq M_1(E(0))\,(1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0,$$

and the lower estimate in (2.4) reads

$$E(t) \geq c_0\,(1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \geq 0.$$

(2) The inequality $|\gamma'(t)| \leq C\gamma^2(t)$ in (H2)(3) is weaker than $|\gamma'(t)| \leq C\gamma(t)(1+t)^{-1}$ in (H2)(4).

(3) The inequality $2\alpha_1 < 1 + \alpha_2$ (in (H2)(5)) holds automatically, when $\alpha_1 \in [0, 1/2)$.

(4) Condition (H3)(2) is stronger than the corresponding one in [18, (2.14)]:

$$\|f(u)\| \leq M\left(\|u\|^{1+p} + \|A^{1/2}u\|^{1+\beta}\right), \quad \forall u \in H_1 \text{ with } \|u\|_{H_1} \leq \xi, \tag{2.5}$$

where $M, \xi, \beta$ are positive constants. It remains open whether condition (H3)(2) in Theorem 2.3 can be improved as (2.5).

# 3 Preliminary results and proofs

Throughout this section, $\bar{C}, \tilde{C}, C_1, C_2, ..., C_{17}$ denote positive constants depending on the values of $E(0)$ and being bounded on bounded sets of the values, and they may be different at different positions.

From (2.2),

$$\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 \leq E(t) \leq E(0),$$

for strong solutions (and so for mild solution as well, by a density argument). Thus, from (H2)(2) we have

$$\|u\|^{p+2} \leq \bar{C}(E(0))E(t), \quad \forall u \in H_1,\ t \geq 0, \tag{3.1}$$

for some positive function $\bar{C}$ on $R_+$ that is bounded on bounded sets.

Now, we establish an important lemma with respect to a perturbed energy functional $E_{\varepsilon,\eta}(t)$ (see below), which is useful in the proof of Theorem 2.2. The functional with $\eta(t) = 1$ was employed in [18] to establish the upper estimate of the energy when $\gamma(t) = 1$.

**Lemma 3.1.** *Assume* (H1) *and* (H2)(1). *Let $u(t)$ be a strong solution of* (1.1), *and let $\eta \in W^{1,\infty}_{\mathrm{loc}}(R^+)$ be a positive function, $\varepsilon \in (0,1)$, and*

$$r := \frac{p}{p+2}.$$

*Define*

$$E_{\varepsilon,\eta}(t) := E(t) + \varepsilon\eta(t)[E(t)]^r\langle u'(t), u(t)\rangle.$$

*Then*

$$E'_{\varepsilon,\eta}(t) \leq -\frac{1}{2}\gamma(t)\|u'(t)\|^2 + \varepsilon\tilde{C}(E(0))V_1(t)\eta(t)\|u'(t)\|^2 + \varepsilon[\varepsilon\tilde{C}(E(0))V_2(t) - \tilde{c}]\eta(t)[E(t)]^{r+1}, \quad t \geq 0,$$

*with some $\tilde{c} \in (0,1)$ and some positive function $\tilde{C}$ that are bounded on bounded sets. Here,*

$$
\begin{aligned}
&V_1(t) := [E(t)]^{\frac{p}{2p+4}}\gamma(t) + [E(t)]^r,\\
&V_2(t) := \gamma(t)\eta(t)\left[1 + \left(\frac{\eta'(t)}{\eta(t)\gamma(t)}\right)^2\right].
\end{aligned}
\tag{3.2}
$$

*Proof.* The time-derivative of $E_{\varepsilon,\eta}(t)$ is

$$
\begin{aligned}
E'_{\varepsilon,\eta}(t) =& -\gamma(t)\|u'\|^2 - \varepsilon r\eta(t)[E(t)]^{r-1}\gamma(t)\langle u', u\rangle\|u'\|^2 + \varepsilon\eta(t)[E(t)]^r\|u'\|^2\\
&- \varepsilon\eta(t)[E(t)]^r\gamma(t)\langle u', u\rangle - \varepsilon\eta(t)[E(t)]^r\left(\|A^{1/2}u(t)\|^2 + \langle f(u), u\rangle\right)\\
&+ \varepsilon\eta'(t)[E(t)]^r\langle u', u\rangle\\
=:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

We first estimate $I_2$. By (3.1) and the Cauchy-Schwarz inequality we obtain

$$
r[E(t)]^{r-1}\langle u', u\rangle \le r[E(t)]^{r-1}\|u'\|\|u\| \le C_1[E(t)]^{r-1+\frac{1}{2}+\frac{1}{p+2}} = C_1[E(t)]^{\frac{p}{2p+4}},
$$

with a constant $C_1 > 0$. Hence

$$
I_2 \le C_1\varepsilon\eta(t)[E(t)]^{\frac{p}{2p+4}}\gamma(t)\|u'\|^2. \tag{3.3}
$$

For $I_4$, we use Young's inequality, as well as the Cauchy-Schwarz inequality and (3.1) to infer that

$$
\begin{aligned}
\varepsilon\eta(t)[E(t)]^r\gamma(t)\langle u', u\rangle &\le \frac{1}{4}\gamma(t)\|u'\|^2 + \gamma(t)\varepsilon^2\eta^2(t)[E(t)]^{2r}\|u\|^2\\
&\le \frac{1}{4}\gamma(t)\|u'\|^2 + C_2\varepsilon^2\gamma(t)\eta^2(t)[E(t)]^{2r+\frac{2}{p+2}}.
\end{aligned}
$$

Since $2r + \frac{2}{p+2} = r + 1$, this means that

$$
I_4 \le \frac{1}{4}\gamma(t)\|u'\|^2 + C_2\varepsilon^2\gamma(t)\eta^2(t)[E(t)]^{r+1}. \tag{3.4}
$$

Similarly, we have

$$
I_6 \le \frac{1}{4}\gamma(t)\|u'\|^2 + C_3\varepsilon^2\gamma(t)^{-1}[\eta'(t)]^2[E(t)]^{r+1}. \tag{3.5}
$$

Moreover, in view of (H2)(1) and (2.1) we deduce that

$$
\begin{aligned}
[E(t)]^r\left(\|A^{1/2}u(t)\|^2 + \langle f(u), u\rangle\right) &\ge c_1[E(t)]^r\left(\frac{1}{2}\|A^{1/2}u(t)\|^2 + F(u)\right)\\
&= c_1[E(t)]^r\left(E(t) - \frac{1}{2}\|u'\|^2\right)\\
&= c_1[E(t)]^{r+1} - \frac{c_1}{2}[E(t)]^r\|u'\|^2,
\end{aligned}
$$

with some constant $c_1 > 0$, so that

$$
I_5 \le -c_1\varepsilon\eta(t)[E(t)]^{r+1} + \frac{c_1}{2}\varepsilon[E(t)]^r\eta(t)\|u'\|^2. \tag{3.6}
$$

Finally, combining (3.3)-(3.6) together we have

$$
\begin{aligned}
E'_{\varepsilon,\eta}(t) \le& -\frac{1}{2}\gamma(t)\|u'\|^2 + C_1\varepsilon\eta(t)[E(t)]^{\frac{p}{2p+4}}\gamma(t)\|u'\|^2 + \frac{c_1}{2}\varepsilon\eta(t)[E(t)]^r\|u'\|^2\\
&+ C_2\varepsilon^2\gamma(t)\eta^2(t)[E(t)]^{r+1} - c_1\varepsilon\eta(t)[E(t)]^{r+1} + C_3\varepsilon^2\gamma(t)^{-1}[\eta'(t)]^2[E(t)]^{r+1}.
\end{aligned}
$$

This gives the estimate of $E'_{\varepsilon,\eta}(t)$ as required. □

Next, we derive a preliminary upper estimate of energy, by an application of Lemma 3.1.

**Proposition 3.2.** *Let Assumptions* (H1), (H2)(1), *and* (H2)(3) *hold. Then*

$$E(t) \leq M_1(E(0)) \left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{p+2}{p}}, \quad \forall t \geq 0,$$

*for some positive function $M_1$ on $R_+$ that is bounded on bounded sets.*

*Proof.* It suffices to show the estimate for strong solutions, by a density argument.

First we consider the case $\alpha_1 = \alpha_2 = \alpha$ and take $\eta(t) = R_1(1+t)^{-\alpha}$. By (1.2) we see that

$$\left|\frac{\eta'(t)}{\eta(t)\gamma(t)}\right| \leq \frac{\alpha}{R_1}(1+t)^{\alpha-1}.$$

Thus, for $V_1$ and $V_2$ in (3.2) we have

$$V_1(t) \leq C_4, \quad V_2(t) \leq C_4, \quad t \geq 0,$$

with a constant $C_4 > 0$. Accordingly,

$$E'_{\varepsilon,\gamma}(t) \leq -\gamma(t)\left(\frac{1}{2} - \varepsilon\tilde{C}C_4\right)\|u'\|^2 + \varepsilon R_1(\varepsilon\tilde{C}C_4 - \tilde{c})(1+t)^{-\alpha}[E(t)]^{1+r} \tag{3.7}$$

by virtue of Lemma 3.1. On the other hand, using the Cauchy-Schwarz inequality, (1.2) and (3.1), we obtain

$$\begin{aligned}\varepsilon R_1(1+t)^{-\alpha}[E(t)]^r|\langle u', u\rangle| &\leq \varepsilon R_1[E(t)]^r\|u'\|\|u\| \\ &\leq C_5\varepsilon[E(t)]^{r+\frac{1}{2}+\frac{1}{p+2}} = C_5\varepsilon[E(t)]^{1+\frac{p}{2(p+2)}} \\ &\leq C_6\varepsilon E(t). \end{aligned} \tag{3.8}$$

Take $\varepsilon = \tilde{c}(2\tilde{C}C_4 + 2C_6 + \tilde{c})^{-1}$. Then $\varepsilon \in (0,1)$, and

$$\varepsilon\tilde{C}C_4 \leq c/2 < 1/2, \quad C_6\varepsilon \leq 1/2.$$

Thus, it follows from (3.8) and (3.7) that

$$\frac{1}{2}E(t) \leq E_{\varepsilon,\gamma}(t) \leq 2E(t), \quad \forall t \geq 0,$$

and

$$E'_{\varepsilon,\gamma}(t) \leq -\frac{1}{2}\tilde{c}R_1\varepsilon(1+t)^{-\alpha}[E(t)]^{1+r} \leq -\left(\frac{1}{2}\right)^{2+r}\tilde{c}R_1\varepsilon(1+t)^{-\alpha}[E_{\varepsilon,\gamma}(t)]^{1+r}.$$

Therefore,

$$E(t) \leq 2E_{\varepsilon,\gamma}(t) \leq C_7\left(1 + \int_0^t (1+\tau)^{-\alpha} d\tau\right)^{-\frac{1}{r}} \leq C_7\left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{1}{r}}. \tag{3.9}$$

When $\alpha_1 \neq \alpha_2$, by (H2)(3) we derive

$$|\gamma'(t)| \leq C_0\gamma^2(t), \quad t \geq 0.$$

Therefore, taking $\eta(t)=\gamma(t)$, we see that

$$\left|\frac{\eta'(t)}{\eta(t)\gamma(t)}\right| = \left|\frac{\gamma'(t)}{\gamma(t)^2}\right|$$

is also bounded. Then, arguing similarly as in the paragraph above, and using (1.2) whenever in need, we still obtain the energy estimate as in (3.9).

The proof is complete. □

Finally, we make an improvement over the upper estimate of energy in Proposition 3.2, which will play a key role for the case $\alpha_1 > 1/2$, by following the way as in [31, Section 2] (see also [7, Section 3.1]) with effective adaptations for the present situation.

**Proposition 3.3.** *Suppose* $1/2 < \alpha_1 < 1$. *Let Assumptions* (H1) *and* (H2)(1)-(4) *hold. Then*

$$E(t) \le M_2(E(0)) \left(\frac{1}{1+t}\right)^{(\alpha_1+\alpha_2)\cdot\frac{p+2}{p}}, \quad \forall t \ge 0,$$

*for some positive function* $M_2$ *that are bounded on bounded sets.*

*Proof.* First we write

$$k_0 := (1-\alpha_1)\cdot\frac{p+2}{p}, \quad k_* := (\alpha_1+\alpha_2)\frac{p+2}{p}.$$

It is clear that $k_0 < k_*$. Applying Proposition 3.2 enables us to obtain

$$E(t) \le C_8 \left(\frac{1}{1+\int_0^t \gamma(\tau)d\tau}\right)^{\frac{p+2}{p}} \le C_8(1+t)^{-k_0}, \quad \forall t \ge 0, \tag{3.10}$$

by (1.2).

Consider the function $q(t) := \frac{1}{2}\|u(t)\|^2$. Using (H2)(1) and (1.1), we have

$$\begin{aligned} q''(t) + \gamma(t)q'(t) &= \|u'\|^2 - \|A^{1/2}u\|^2 - \langle f(u), u\rangle \\ &\le \|u'\|^2 - \min\{2,c\}\left(\frac{1}{2}\|A^{1/2}u\|^2 + F(u)\right) \\ &\le 2\|u'\|^2 - \min\{2,c\}E(t). \end{aligned}$$

Therefore,

$$\min\{2,c\}\int_0^T \lambda_k(t)E(t)dt \le 2\int_0^T \lambda_k(t)\|u'(t)\|^2dt - \int_0^T \lambda_k(t)q''(t)dt - \int_0^T \lambda_k(t)\gamma(t)q'(t)dt,$$

where

$$T>0, \quad \lambda_k(t) := (1+t)^k \text{ with } k \in [k_0-1, k_*].$$

Integrating by parts yields that

$$\begin{aligned} \min\{2,c\}\int_0^T \lambda_k(t)E(t)dt \le& 2\int_0^T \lambda_k\|u'(t)\|^2dt - \lambda_k(T)q'(T) + [\lambda_k' - (\gamma\lambda_k)](T)q(T) \\ &+ \int_0^T [(\lambda_k\gamma)' - \lambda_k''](t)q(t)dt + q'(0) + (\gamma(0)-k)q(0). \end{aligned}$$

Notice

$$\lambda_k'(t) = k(1+t)^{k-1} \quad \text{and} \quad k \le k_* < \frac{2p+4}{p}.$$

We see

$$\lambda_k'(T) - (\gamma\lambda_k)(T) \le -\frac{1}{2}(\gamma\lambda_k)(T), \quad \forall T \ge T_0,$$

where

$$T_0 := \left(\frac{4p+8}{pR_1}\right)^{1/(1-\alpha_1)}.$$

Also,

$$|q'(t)| \le \|u'(t)\|\|u(t)\| \le 2\sqrt{E(t)}\sqrt{q(t)}.$$

Therefore,

$$\begin{aligned}
&-\lambda_k(T)q'(T) + [\lambda_k' - (\gamma\lambda_k)](T)q(T) \\
\le& \left[2(\lambda_k\sqrt{E})(T)\right]\sqrt{q(T)} - \left[\frac{1}{2}(\lambda_k\gamma)(T)\right]\left(\sqrt{q(T)}\right)^2 \\
\le& \frac{\left[2(\lambda_k\sqrt{E})(T)\right]^2}{2(\lambda_k\gamma)(T)} \quad \left(\text{the maximum of the above formal function of } \sqrt{q(T)}\right) \\
\le& \frac{2}{R_1}\lambda_{k+\alpha_1}(T)E(T) \quad \text{(by (1.2))}, \quad \forall T \ge T_0.
\end{aligned}$$

Accordingly, we infer, for $T \ge T_0$,

$$\begin{aligned}
\int_0^T \lambda_k(t)E(t)dt \le& \bar{C}_0\int_0^T \lambda_k(t)\|u'(t)\|^2dt + \bar{C}_0\lambda_{k+\alpha_1}(T)E(T) + \bar{C}_0\int_0^T [(\lambda_k\gamma)' - \lambda_k''](t)q(t)dt + C_9 \\
:=& \bar{C}_0(J_1 + J_2 + J_3) + C_9, \quad \forall T \ge T_0,
\end{aligned} \tag{3.11}$$

with constants $\bar{C}_0, C_9 > 0$.

From (1.2) and (3.10) it follows that

$$\begin{aligned}
J_1 =& \int_0^T \frac{\lambda_k}{\gamma(t)}\gamma(t)\|u'(t)\|^2dt \le \frac{1}{R_1}\int_0^T \lambda_{k+\alpha_1}(-E')dt \\
\le& -\frac{1}{R_1}\lambda_{k+\alpha_1}(T)E(T) + \frac{E(0)}{R_1} + \frac{k+\alpha_1}{R_1}\int_0^T \lambda_{k+\alpha_1-1}(t)E(t)dt \\
\le& \frac{E(0)}{R_1} + \frac{C_8(k+\alpha_1)}{R_1}\int_0^T \lambda_{k+\alpha_1-1-k_0}(t)dt.
\end{aligned} \tag{3.12}$$

Also using (3.10) gives

$$J_2 \le \frac{2C_8}{R_1}\lambda_{k+\alpha_1-k_0}(T). \tag{3.13}$$

As for $J_3$, we observe

$$(\lambda_k\gamma)'(t) = \lambda_k'(t)\gamma(t) + \lambda_k(t)\gamma'(t) \le (k+C)R_2(1+t)^{k-1}(1+t)^{-\alpha_2},$$

by (1.2) and (H2)(4),

$$\lambda_k''(t) = k(k-1)(1+t)^{k-2},$$

and

$$q(t) = \frac{1}{2}\|u(t)\|^2 \le C_{10}[E(t)]^{\frac{2}{p+2}} \le C_{11}(1+t)^{-\frac{2k_0}{p+2}},$$

by (3.10) again. Accordingly,

$$J_3 \le C_{12} \int_0^T (1+t)^{k-\alpha_2-1-\frac{2k_0}{p+2}} dt. \tag{3.14}$$

Take $k = k_0 - \alpha_1 - \epsilon$, where $\epsilon$ is a fixed positive number satisfying $1 - \alpha_1 - \epsilon > 0$. Noticing

$$\tilde{k} - \alpha_1 \le \frac{2\tilde{k}}{p+2} + \alpha_2, \quad \text{whenever } \tilde{k} \in [k_0, k_*],$$

we see that

$$\begin{cases} k < k_0 - \alpha_1, \\ k < \frac{2k_0}{p+2} + \alpha_2. \end{cases}$$

Accordingly, combining (3.11) with (3.12)-(3.14) yields that

$$\int_0^{+\infty} \lambda_k E(t) dt \le C_{13}.$$

Since $E(t)$ is decreasing, we see

$$E(t) \int_{t/2}^{t} \lambda_k(s) ds \le \int_{t/2}^{t} \lambda_k E(s) ds.$$

Hence,

$$E(t) \le C_{14} \left(\frac{1}{1+t}\right)^{k+1} = C_{14} \left(\frac{1}{1+t}\right)^{k_1}, \quad t \ge 0, \tag{3.15}$$

where

$$k_1 := k_0 + 1 - \alpha_1 - \epsilon.$$

If $k_1 \ge k_*$, then we have obtained the required energy estimate. If not, then

$$E(t) \le C_{15} \left(\frac{1}{1+t}\right)^{k_2}, \quad t \ge 0,$$

where

$$k_2 := k_1 + 1 - \alpha_1 - \epsilon = k_0 + 2(1 - \alpha_1 - \epsilon);$$

this is because (3.12)-(3.14), with $k_0$ replaced by $k_1$, are satisfied, by using (3.15) (instead of (3.10)). Thus, proceeding like this and denoting by $n$ the positive integer satisfying

$$\begin{cases} k_0 + n(1 - \alpha_1 - \epsilon) < k_*, \\ k_n := k_0 + (n+1)(1 - \alpha_1 - \epsilon) \ge k_*, \end{cases}$$

we obtain

$$E(t) \le C_{16} \left(\frac{1}{1+t}\right)^{k_n} \le C_{17} \left(\frac{1}{1+t}\right)^{k_*}, \quad t \ge 0.$$

This ends the proof.

□

## 4 Proofs of the Theorems

Throughout the section, $\bar{C}_1, \bar{C}_2, ..., \bar{C}_{15}$ denote positive constants depending on the values of $E(0)$ and being bounded on bounded sets of the values, and they may be different at different positions.

### 4.1 Proof of Theorem 2.2

It suffices to deal with the strong solution case.

We divide the proof into three steps.

**Step 1.** Obtain a better estimate of energy for case $\alpha_1 \leq 1/2$.

Take $\eta(t) = 1$ in Lemma 3.1. Then $V_2(t) = \gamma(t)$ is bounded. For $V_1(t)$ we exploit Proposition 3.2 to deduce that

$$E(t)^r \leq \bar{C}_1 \left(1 + \int_0^t \gamma(\tau) d\tau\right)^{-\frac{p+2}{p}\cdot r} \leq \bar{C}_1 \left(\frac{1}{1+t}\right)^{1-\alpha_1} \leq \bar{C}_1 \left(\frac{1}{1+t}\right)^{\alpha_1} \leq \bar{C}_1 \gamma(t),$$

since $\alpha_1 \leq 1/2$. Accordingly,

$$V_1(t) \leq \bar{C}_2 \gamma(t).$$

Thus, similarly as in the proof of Proposition 3.2, we can let $\varepsilon$ small enough such that

$$\frac{1}{2} E(t) \leq E_{\varepsilon,1}(t) \leq 2E(t), \quad \forall t \geq 0,$$

and

$$E'_{\varepsilon,1}(t) \leq -\frac{1}{2} \tilde{c} \varepsilon [E(t)]^{1+r}.$$

This gives that

$$E(t) \leq \bar{C}_3 (1+t)^{-\frac{1}{r}} = \bar{C}_3 (1+t)^{-\frac{p+2}{p}}. \tag{4.1}$$

**Step 2.** Obtain a better estimate of energy for case $\alpha_1 > 1/2$.

Take

$$\eta(t) = (1+t)^{2\alpha_1 - 1}$$

in Lemma 3.1. Making use of Proposition 3.3 we infer that

$$\eta(t)[E(t)]^{\frac{p}{2p+4}} \leq \bar{C}_4 (1+t)^{2\alpha_1 - 1} (1+t)^{-\frac{\alpha_1+\alpha_2}{2}} \leq \bar{C}_4 (1+t)^{2\alpha_1 - 1} (1+t)^{-\alpha_2} \leq \bar{C}_4, \tag{4.2}$$

by (H2)(5). This yields that

$$\varepsilon \eta(t) [E(t)]^r |\langle u', u \rangle| \leq \varepsilon \eta(t) [E(t)]^r \|u'\| \|u\| \leq \bar{C}_5 \varepsilon \eta(t) [E(t)]^{1+\frac{p}{2p+4}} \leq \bar{C}_5 \varepsilon E(t),$$

which implies that

$$\frac{1}{2} E(t) \leq E_{\varepsilon,\eta}(t) \leq 2E(t), \quad \forall t \geq 0, \tag{4.3}$$

whenever $\varepsilon$ is small enough.

Moreover, using Proposition 3.3 again, we obtain

$$\eta(t)[E(t)]^r \le \bar{C}_6(1+t)^{2\alpha_1-1}(1+t)^{-\alpha_1-\alpha_2} = \bar{C}_6(1+t)^{2\alpha_1-1-\alpha_2}(1+t)^{-\alpha_1} \le \bar{C}_6\gamma(t),$$

due to (H2)(5) and (1.2). Furthermore, we have

$$\gamma(t)\eta(t) \le R_2(1+t)^{-\alpha_2}(1+t)^{2\alpha_1-1} \le R_2,$$

and

$$\eta'(t) = (2\alpha_1-1)(1+t)^{2\alpha_1-2} \le \bar{C}_7\gamma(t)\eta(t),$$

by (1.2). These estimates combined with (4.2) indicate that $V_2(t)$ is bounded, and

$$V_1(t)\eta(t) \le \bar{C}_8\gamma(t).$$

Therefore, there exists a sufficiently small $\varepsilon$ such that (4.3) is satisfied, and

$$E'_{\varepsilon,\eta}(t) \le -\frac{1}{2}\tilde{c}\varepsilon\eta(t)[E(t)]^{1+r}.$$

Consequently,

$$E_{\varepsilon,\eta}(t) \le \bar{C}_9\left(1+\int_0^t \eta(\tau)d\tau\right)^{-1/r}.$$

So

$$E(t) \le \bar{C}_{10}(1+t)^{-2\alpha_1/r} = \bar{C}_{10}(1+t)^{-2\alpha_1\cdot\frac{p+2}{p}}. \tag{4.4}$$

**Step 3.** Achieve the desired estimate of energy.

We take

$$\eta(t) = \eta_0(t) := \begin{cases} (1+t)^{\alpha}, & \text{if} \quad \alpha_1=\alpha_2=\alpha, \\ \gamma(t)^{-1}, & \text{if} \quad \alpha_1\ne\alpha_2 \end{cases}$$

in Lemma 3.1. The boundedness of $V_2(t)$ is easy to see by (H2)(3).

Employing (4.1) and (4.4), we obtain

$$[E(t)]^r \le \bar{C}_{11}(1+t)^{-2\alpha_1} \le \bar{C}_{11}\gamma(t)^2$$

by (1.2). This yields that

$$[E(t)]^{\frac{p}{2p+4}} = [E(t)]^{r/2} \le \bar{C}_{12}\gamma(t).$$

Hence

$$V_1(t)\eta_0(t) = \left([E(t)]^{\frac{p}{2p+4}}\gamma(t) + [E(t)]^r\right)\eta_0(t) \le \bar{C}_{13}\gamma(t).$$

Moreover,

$$\varepsilon\eta_0(t)[E(t)]^r|\langle u',u\rangle| \le \bar{C}_{14}\varepsilon\eta_0(t)[E(t)]^{1+\frac{p}{2(p+2)}} \le \bar{C}_{14}\varepsilon E(t).$$

Accordingly, choosing $\varepsilon$ small enough we deduce that

$$\frac{1}{2}E(t) \le E_{\varepsilon,\eta_0}(t) \le 2E(t), \quad \forall t\ge 0,$$

$$E'_{\varepsilon,\eta_0}(t) \le -\frac{1}{2}\tilde{c}\varepsilon\eta_0(t)[E(t)]^{1+r}.$$

Therefore,

$$E(t) \le \bar{C}_{15}\left(1+\int_0^t \eta_0(\tau)d\tau\right)^{-1/r} \le \bar{C}_{15}\left(1+\int_0^t \gamma(\tau)^{-1}d\tau\right)^{-1/r}.$$

## 4.2 Proof of Theorem 2.3

We follow the strategy in [18, Section 3.4] (for the case of $\gamma(t)=1$) to prove the theorem. It is worth noting that exploitations of the (already obtained) fine upper bound estimates will play an important role.

Let $u(t)$ be a mild solution of (1.1) with $u_0 \neq 0$. Set

$$\gamma_1 = \frac{R_2^2(C_0^2+1)}{\|u_0\|^{2p+2}}\left(\|u_1\|^2+\|A^{1/2}u_0\|^2\right)+\frac{129C_0^2M^2}{\delta^2}, \tag{4.5}$$

where $R_2$ is as in (1.2), $C_0$ as in (H3)(3), $M$ as in (H3)(2), and $\delta \in (0,1/2)$ is a constant that will be determined later; set

$$G(t) := \frac{\|u'(t)\|^2+\|A^{1/2}u(t)\|^2}{2\|u(t)\|^{2p+2}}. \tag{4.6}$$

We will show the existence of a nonempty open set $S \subset (H_1\setminus\{0\})\times H$ such that for $(u_0,u_1)\in S$, the solution $u(t)$ of problem (1.1) satisfies the property: given $T>0$, one has

$$u(T)\neq 0 \quad \text{and} \quad G(T) < \frac{2\gamma_1}{\gamma^2(T)}, \tag{4.7}$$

whenever

$$u(t)\neq 0 \quad \text{and} \quad G(t) \le \frac{2\gamma_1}{\gamma^2(t)} \quad \text{for all } t\in[0,T). \tag{4.8}$$

Next, we assume (4.8). We will find such a set $S$, and with it prove (4.7) (for strong solutions), by seven steps.

**Step 1.** Construct a small perturbation of $G(t)$.

Denoting by $Q$ the orthogonal projection from $H$ to $(\ker A)^\perp$, we set

$$\hat{G}(t) = G(t) + \delta\gamma(t)\frac{\langle u'(t), Qu(t)\rangle}{\|u(t)\|^{2p+2}}, \quad t\in[0,T), \tag{4.9}$$

for the case of $\alpha_1 \neq \alpha_2$; replace $\gamma(t)$ by $(1+t)^{-\alpha}$ in (4.9) for the case of $\alpha_1=\alpha_2=\alpha$. Below we only address the former case (the latter case can be dealt with similarly). We have

$$\begin{aligned}\hat{G}'(t) =& -\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t)\frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} + \delta\gamma(t)\frac{\|Qu'\|^2-\gamma(t)\langle u', Qu\rangle}{\|u(t)\|^{2p+2}} \\ &- \frac{\langle f(u), u'+\delta\gamma(t)Qu\rangle}{\|u\|^{2p+2}} - (2p+2)\frac{\langle u',u\rangle}{\|u\|^2}\cdot\hat{G}(t) + \delta\gamma'(t)\frac{\langle u', Qu\rangle}{\|u\|^{2p+2}} \\ =&: K_1+K_2+K_3+K_4+K_5+K_6, \quad t\in[0,T).\end{aligned} \tag{4.10}$$

**Step 2.** Estimate $K_3$ and $K_6$.

It follows from (1.2) and (H3)(1) that

$$\gamma(t)\langle u', Qu\rangle \le \frac{R_2}{\sqrt{\zeta}}\|u'\|\sqrt{\zeta}\|Qu\| \le \frac{R_2}{\sqrt{\zeta}}\|u'\|\|A^{1/2}u\| \le \frac{2R_2^2}{\zeta}\|u'\|^2+\frac{1}{8}\|A^{1/2}u\|^2,$$

noting that $\|A^{1/2}Qu\| = \|A^{1/2}u\|$. Hence, we have

$$K_3 \le \tilde{D}_1 \delta\gamma(t) \frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{1}{8}\delta\gamma(t)\frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \le \tilde{D}_1 \delta\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{1}{4}\delta\gamma(t)G(t), \quad t \in [0,T), \tag{4.11}$$

where $\tilde{D}_1 := (2R_2^2 + \zeta)/\zeta$.

As for $K_6$, we use (H2)(3) to get

$$|\gamma'(t)| \le C\gamma^2(t) \le CR_2\gamma(t), \quad \forall t \ge 0.$$

Also,

$$|\langle u', Qu\rangle| \le \frac{1}{\sqrt{\zeta}}\|u'\|\sqrt{\zeta}\|Qu\| \le \frac{1}{\sqrt{\zeta}}\|u'\|\|A^{1/2}u\| \le \frac{2CR_2}{\zeta}\|u'\|^2 + \frac{1}{8CR_2}\|A^{1/2}u\|^2.$$

Accordingly,

$$K_6 \le \tilde{D}_2\delta\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\delta\gamma(t)}{4}G(t), \quad t \in [0,T), \tag{4.12}$$

where $\tilde{D}_2 := 2C^2R_2^2/\zeta$.

**Step 3.** Estimate $K_4$ and $K_5$ in the case $\alpha < 1/2$.

From now on, we assume

$$E(0) \le 1, \text{ and } \left(\sup_{s\in[0,1]} \bar{C}(s) + 2\right)[E(0)]^{\frac{2}{p+2}} \le \xi^2, \tag{4.13}$$

where $\bar{C}$ and $\xi$ are, respectively, as in (3.1) and (H3)(2). Then

$$\begin{aligned}\|u(t)\|_{H_1}^2 = \|u(t)\|^2 + \|A^{1/2}u(t)\|^2 &\le \left(\sup_{s\in[0,1]}\bar{C}(s)\right)[E(t)]^{\frac{2}{p+2}} + 2E(t)\\ &\le \left(\sup_{s\in[0,1]}\bar{C}(s) + 2\right)[E(0)]^{\frac{2}{p+2}} \le \xi^2.\end{aligned}$$

Thus, in view of (2.1), (4.1), (3.1), (1.2) and (H3)(2), we deduce that

$$\begin{aligned}\frac{\|f(u)\|}{\|u\|^{p+1}} &\le M\left(\frac{\|u\|^{p+1} + \|A^{1/2}u\|^{p+1}}{\|u\|^{p+1}}\right)\\ &\le M\left(1 + \frac{\|A^{1/2}u\|}{\|u\|^{p+1}}\cdot[2E(t)]^{p/2}\right)\\ &\le M\left(1 + \sqrt{2G(t)}\cdot[2E(t)]^{p/4}[2E(0)]^{p/4}\right)\\ &\le M\left(1 + D_1\sqrt{G(t)}\cdot(1+t)^{-\frac{p+2}{4}}[E(0)]^{p/4}\right)\\ &\le M\left(1 + D_2[E(0)]^{p/4}\gamma(t)\sqrt{G(t)}\right), \quad t \in [0,T),\end{aligned}$$

noting

$$\frac{p+2}{4} > \frac{1}{2} > \alpha_1.$$

Here, $D_1, D_2$ are positive constants independent of initial data (because of $E(0) \le 1$). Also, we have

$$\delta\gamma(t)\|Qu\| \le \frac{\delta R_2}{\sqrt{\zeta}}\|A^{1/2}u\|,$$

by (H3)(1). Hence,

$$\frac{\|u_t\| + \delta\gamma(t)\|Qu\|}{\|u\|^{p+1}} \le \frac{\|u_t\| + R_2/\sqrt{\zeta}\|A^{1/2}u\|}{\|u\|^{p+1}} \le D_3\sqrt{G(t)}, \quad t \in [0,T),$$

where $D_3 := 2(1 + R_2/\sqrt{\zeta})$. Thus,

$$K_4 \le \frac{f(u)}{\|u\|^{p+1}} \cdot \frac{\|u_t\| + \delta\gamma(t)\|Qu\|}{\|u\|^{p+1}} \le D_3 M\sqrt{G(t)} + MD_2D_3[E(0)]^{p/4}\gamma(t)G(t).$$

So

$$K_4 \le \frac{4M^2}{\delta\gamma(t)} + \frac{\delta\gamma(t)}{4}G(t) + \tilde{D}_3[E(0)]^{p/4}\gamma(t)G(t), \quad t \in [0,T). \tag{4.14}$$

where $\tilde{D}_3$ is a positive constant independent of initial data.

For $K_5$, we note that

$$\exists\, \theta > 0 \text{ such that } \alpha_1 + \theta = 1/2,$$

since $\alpha_1 < 1/2$. From the definition of $G(t)$, (3.1), (4.8) and (4.1), it follows that for $t \in [0,T)$,

$$\begin{aligned}\left|\frac{\langle u', u\rangle}{\|u\|^2}\right| &\le \frac{\|u'\|}{\|u\|^{1+p}} \cdot \|u\|^p \\ &\le \sqrt{2G(t)} \cdot \|u\|^{2p\alpha_1}\|u\|^{2p\theta} \\ &\le \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_4(1+t)^{-2\alpha_1}[E(0)]^{\frac{2p\theta}{p+2}} \\ &\le D_5\sqrt{\gamma_1}\gamma(t)[E(0)]^{\frac{2p\theta}{p+2}},\end{aligned}$$

with positive constants $D_4, D_5$ independent of initial data. Therefore

$$K_5 \le \tilde{D}_4[E(0)]^{\frac{p\theta'}{p+2}}\sqrt{\gamma_1}\gamma(t) \cdot \hat{G}(t), \quad t \in [0,T), \tag{4.15}$$

for some $\theta' > 0$, and some positive constant $\tilde{D}_4$ independent of initial data.

**Step 4.** Estimate $K_4$ and $K_5$ in the case $\alpha \ge 1/2$.

In view of (2.1), (4.4) and (H3)(2), we obtain

$$\begin{aligned}\frac{\|f(u)\|}{\|u\|^{p+1}} &\le M\left(1 + \frac{\|A^{1/2}u\|}{\|u\|^{p+1}} \cdot [2E(t)]^{p/4}[2E(0)]^{p/4}\right) \\ &\le M\left(1 + D_6\sqrt{G(t)} \cdot (1+t)^{-\alpha_1\frac{p+2}{2}}[E(0)]^{p/4}\right) \\ &\le M\left(1 + D_7E(0)]^{p/4}\gamma(t)\sqrt{G(t)}\right), \quad t \in [0,T),\end{aligned}$$

with positive constants $D_6, D_7$ independent of initial data. This means that the estimate (4.14) for $K_4$ holds too in this case.

By virtue of Theorem 2.2, we get

$$\|u(t)\| \le D_8\left(1 + \int_0^t \gamma(\tau)^{-1}d\tau\right)^{-\frac{1}{p}} \le D_9(1+t)^{-\frac{1+\alpha_2}{p}},$$

where $D_8, D_9$ are positive constants independent of initial data. In addition,

$$\exists\, \theta_1 \in (0,1) \text{ such that } 2\alpha_1 = \theta_1(1+\alpha_2)$$

by (H2)(5). Hence

$$\begin{aligned}\left|\frac{\langle u', u\rangle}{\|u\|^2}\right| &\le \sqrt{2G(t)} \cdot \|u\|^{p\theta_1}\|u\|^{(1-\theta_1)p} \\ &\le \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_{10}(1+t)^{-\theta_1(1+\alpha_2)} \cdot [E(0)]^{(1-\theta_1)\frac{p}{p+2}} \\ &= \frac{\sqrt{\gamma_1}}{\gamma(t)} \cdot D_{10}(1+t)^{-2\alpha_1}[E(0)]^{(1-\theta_1)\frac{p}{p+2}} \\ &\le D_{11}\sqrt{\gamma_1}\gamma(t)[E(0)]^{\frac{(1-\theta_1)p}{p+2}}, \quad t \in [0,T),\end{aligned}$$

where $D_{10}, D_{11}$ are positive constants independent of initial data. So the estimate (4.15) for $K_5$ holds too in this case.

**Step 5.** Estimate $\hat{G}'$.

Plugging (4.11)-(4.15) into (4.10), we obtain

$$\begin{aligned}\hat{G}'(t) \le &-\left(1-\tilde{D}_1\delta-\tilde{D}_2\delta\right)\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t)\frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} + \frac{4M^2}{\delta\gamma(t)} \\ &+ \left(3\delta/4 + \tilde{D}_3[E(0)]^{p/4}\right)\gamma(t)G(t) + \tilde{D}_4\sqrt{\gamma_1}[E(0)]^{\frac{p\theta'}{p+2}} \cdot \gamma(t)\hat{G}(t), \quad t \in [0,T).\end{aligned}$$

From (H3)(1) and (1.2), we know that

$$\gamma(t)\frac{\langle u', Qu\rangle}{\|u\|^{2p+2}} \le 2\left(R_2^2\zeta^{-1} + 1/4\right)G(t), \quad t \in [0,T).$$

Now, we choose $\delta \in (0, 1/2)$ such that

$$\begin{cases} \left(R_2^2\zeta^{-1} + 1/4\right)\delta \le 1/10, \\ (\tilde{D}_1 + \tilde{D}_2)\delta \le 1/2; \end{cases}$$

then, choose a set $S$ of initial data satisfying (4.13) and

$$\begin{cases} \tilde{D}_3[E(0)]^{p/4} \le \delta/32, \\ \tilde{D}_4\sqrt{\gamma_1}[E(0)]^{\frac{p\theta'}{p+2}} \le \delta/32. \end{cases} \tag{4.16}$$

We here emphasize that $\tilde{D}_3, \tilde{D}_4$ are independent of all the initial data satisfying $E(0) \le 1$. Thus,

$$\frac{5}{6}G(t) \le \hat{G}(t) \le 2G(t), \quad t \in [0,T),$$

and

$$\begin{aligned}\hat{G}'(t) \le &-\delta\gamma(t)\left(\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}}\right) + \frac{4M^2}{\delta\gamma(t)} + \frac{25}{32}\delta\gamma(t)G(t) + \frac{1}{32}\delta\gamma(t)\hat{G}(t) \\ \le &-\delta\gamma(t)\hat{G}(t) + \frac{4M^2}{\delta\gamma(t)} + \frac{15}{16}\delta\gamma(t)\hat{G}(t) + \frac{1}{32}\delta\gamma(t)\hat{G}(t), \quad t \in [0,T),\end{aligned}$$

by noting

$$\begin{aligned}&-\left(1-\tilde{D}_1\delta-\tilde{D}_2\delta\right)\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t)\frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \\ \le &-(1-1/2)\gamma(t)\frac{\|u'\|^2}{\|u\|^{2p+2}} - \delta\gamma(t)\frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}} \\ \le &-\min\{1/2, \delta\}\gamma(t)\left(\frac{\|u'\|^2}{\|u\|^{2p+2}} + \frac{\|A^{1/2}u\|^2}{\|u\|^{2p+2}}\right),\end{aligned}$$

and $\delta \in (0, 1/2)$. Hence,

$$\hat{G}'(t) \le -\frac{1}{32}\delta\gamma(t)\hat{G}(t) + \frac{4M^2}{\delta\gamma(t)}, \quad t \in [0, T). \tag{4.17}$$

**Step 6.** Estimate $\hat{G}$ .

Integrating (4.17) and setting

$$H(t) := \exp\left(\frac{\delta}{32}\int_0^t \gamma(\tau)d\tau\right), \quad t \ge 0,$$

we deduce that

$$\hat{G}(t) \le \left(\hat{G}(0) + \frac{4M^2}{\delta}\int_0^t \gamma(s)^{-1}H(s)ds\right)H(t)^{-1}, \quad t \in [0, T).$$

By (H3)(3), we have

$$\gamma^{-1}(s) \le C_0^2\frac{\gamma(s)}{\gamma^2(t)}, \quad \forall t \ge s > 0.$$

Then

$$\hat{G}(t) \le \hat{G}(0)H(t)^{-1} + \frac{4C_0^2M^2}{\delta\gamma^2(t)}\frac{\int_0^t \gamma(s)H(s)ds}{H(t)}, \quad t \in [0, T).$$

From

$$H'(t) = \frac{\delta}{32}\gamma(t)\exp\left(\frac{\delta}{32}\int_0^t \gamma(\tau)d\tau\right) = \frac{\delta}{32}\gamma(t)H(t),$$

it follows that

$$\int_0^t \gamma(s)H(s)ds = \frac{32}{\delta}(H(t) - H(0)) \le \frac{32}{\delta}H(t).$$

Therefore, noting $H(t) \ge 1$ for $t \in [0, T)$, we derive

$$\hat{G}(t) \le \hat{G}(0) + \frac{128C_0^2M^2}{\delta^2\gamma^2(t)}, \quad t \in [0, T). \tag{4.18}$$

**Step 7.** Obtain (4.7).

Using (4.8), we have

$$\left|\frac{d}{dt}\|u(t)\|^2\right| = 2|\langle u'(t), u(t)\rangle| \le 2\sqrt{2G(t)}\cdot\|u\|^{2+p} \le \frac{4\sqrt{\gamma_1}}{\gamma(t)}\left(\|u(t)\|^2\right)^{1+\frac{p}{2}}, \quad t \in [0, T).$$

So we obtain

$$\|u(t)\|^2 \ge \left(2p\sqrt{\gamma_1}\int_0^t \gamma(s)^{-1}ds + \|u_0\|^{-p}\right)^{-\frac{2}{p}}, \quad t \in [0, T).$$

Thus, we see that $u(T) \ne 0$.

Letting $t \to T^-$ in (4.18) and using (H3)(3), we get

$$\hat{G}(T) \le \left(\frac{1}{C_0^2}\cdot C_0^2\hat{G}(0) + \frac{128C_0^2M^2}{\delta^2\gamma^2(T)}\right) \le \left(\frac{\gamma^2(0)}{\gamma^2(T)}\cdot C_0^2\hat{G}(0) + \frac{128C_0^2M^2}{\delta^2\gamma^2(T)}\right).$$

So

$$G(T) \le 2\hat{G}(T) \le \frac{2}{\gamma^2(T)}\left(2R_2^2C_0^2\cdot G(0) + \frac{128C_0^2M^2}{\delta^2}\right) < \frac{2\gamma_1}{\gamma^2(T)}.$$

Therefore, (4.7) is satisfied.

Consequently, for any $(u_0, u_1) \in S$ (given by (4.13) and (4.16)) and any $T > 0$, we have

$$u(T) \neq 0 \quad \text{and} \quad G(T) < \frac{2\gamma_1}{\gamma^2(T)},$$

provided $u(t)$ is a strong solution, and

$$u(t) \neq 0 \quad \text{and} \quad G(t) \leq \frac{2\gamma_1}{\gamma^2(t)}, \quad \text{for all } t \in [0, T). \tag{4.19}$$

This implies that

$$\sup\{T > 0 : \text{ (4.19) holds}\} = +\infty,$$

due to the continuity of $G$ and the fact that (4.19) is indeed satisfied for some $T > 0$, as can be seen from the estimate

$$\gamma^2(0)G(0) \leq R_2^2 G(0) < \gamma_1/2.$$

Therefore, we obtain

$$u(t) \neq 0 \quad \text{and} \quad G(t) \leq \frac{2\gamma_1}{\gamma^2(t)}, \quad \text{for all } t \geq 0.$$

Accordingly,

$$\left| \frac{d}{dt} \|u(t)\|^2 \right| \leq \frac{4\sqrt{\gamma_1}}{\gamma(t)} \left( \|u(t)\|^2 \right)^{1+\frac{p}{2}}, \quad t \geq 0,$$

for strong solutions (and so for mild solution as well). Hence,

$$\|u(t)\| \geq \left( 2p\sqrt{\gamma_1} \int_0^t \gamma^{-1}(s) ds + \|u_0\|^{-p} \right)^{-\frac{1}{p}}, \quad \text{for all } t \geq 0.$$

This combined with (3.1) gives (2.4). Thus, we complete the proof.

# 5 Applications

In this section, $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\nu$ is the unit outward normal on $\partial\Omega$, and $\gamma(t)$ is a decreasing function on $R^+$ satisfying (1.2) with $\alpha_1 = \alpha_2 = \alpha$.

**Example 5.1.** We consider the following nonautonomous wave equation with the Neumann boundary condition:

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \gamma(t) u_t(t,x) + \left( \int_\Omega |u(t,x)|^2 dx \right)^{p/2} u(t,x) = 0, \quad \text{in } [0, +\infty) \times \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \quad x \in \Omega, \\ \dfrac{\partial u(t,x)}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{5.1}$$

where $p \geq 1$.

Take $H = L^2(\Omega)$, and define operator $A$ by

$$Av(x) = -\Delta v(x), \quad x \in \Omega \ \text{ a.e.}$$

with

$$v \in D(A) := \Big\{ u \in H^2(\Omega);\ \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \Big\}.$$

It is known that $A$ is a self-adjoint nonnegative operator on $H$, $H_1 := D(A^{1/2}) = H^1(\Omega)$, and (H3)(1) holds.

Moreover, we set

$$f(v)(x) = \left( \int_\Omega |v(x)|^2 dx \right)^{p/2} v(x) = \|v\|^p v(x), \quad v \in H_1.$$

Then

$$\|f(v)\| = \|v\|^{1+p}, \quad \forall v \in H_1,$$

and so (H3)(2) is satisfied. Besides (H1)(ii) and (H2)(1)-(2) are also satisfied (cf. [28, Example 4.2] for details). Therefore, applying Theorems 2.2 and 2.3 (seeing also Remark 2.4 (1)), we obtain the following conclusions regarding the mild solution $u(t)$ and the energy

$$E_u(t) := \frac{1}{2} \int_\Omega (|u_t|^2 + |\nabla u|^2) dx + \frac{1}{p+2} \left( \int_\Omega |u(t,x)|^2 dx \right)^{\frac{p+2}{2}}$$

of the problem (5.1).

(i) For some positive function $M_1$ on $R_+$ that are bounded on bounded sets,

$$E(t),\ \|u(t)\|^{p+2} \le M_1(E(0))\,(1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \ge 0;$$

(ii) there exists a nonempty open set $S \subset H_1 \times H$ such that for some positive function $c_0$ depending on $\|u_0\|$, $\|A^{1/2}u_0\|$ and $\|u_1\|$,

$$E(t),\ \|u(t)\|^{p+2} \ge c_0\,(1+t)^{-(1+\alpha)(p+2)/p}, \quad \forall t \ge 0,$$

whenever $(u_0, u_1) \in S$,

**Example 5.2.** Consider the Dirichlet problem for a nonautonomous wave equation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) - \lambda_1 u(t,x) + \gamma(t) u_t + |u(t,x)|^p u(t,x) = 0, \\ \qquad\qquad \text{in } [0,+\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\ u(t,x) = 0, \quad \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{5.2}$$

where $\lambda_1$ is the first eigenvalue of the negative Dirichlet-Laplacian on $\Omega$, $p > 0$ and

$$p \le 2/(n-2) \quad \text{if } n > 2. \tag{5.3}$$

Take $H = L^2(\Omega)$, and define operator $A$ by

$$Av(x) = -\Delta v(x) - \lambda_1 v(x), \quad x \in \Omega \ \text{ a.e.}$$

with $v \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then we know that $A$ is self-adjoint and nonnegative, (H3)(1) is satisfied, and $H_1 := D(A^{1/2}) = H_0^1(\Omega)$. Set

$$f(v)(x) = |v(x)|^p v(x), \quad v \in H_1.$$

Using (5.3) gives

$$\|f(v)\| = \|v\|_{L^{2p+2}}^{1+p} \leq c_2 \|v\|_{H^1}^{1+p}, \quad \forall v \in H_1,$$

with a constant $c_2 > 0$. So (H3)(2) holds. Moreover, (H1)(ii) and (H2)(1)-(2) hold too (cf. [18, the proof of Theorem 4.1] for details). Accordingly, Theorems 2.2 and 2.3 are applicable to problem (5.2), and so for the mild solution $u(t)$ and the energy

$$E_u(t) := \frac{1}{2}\int_\Omega (|u_t|^2 + |\nabla u|^2 - \lambda_1 |u|^2)dx + \frac{1}{p+2}\left(\int_\Omega |u(t,x)|^{p+2}dx\right)$$

of the problem (5.2), we have the same conclusions as in Example 5.1.

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