

Null controllability and inverse source problem for stochastic Grushin equation with boundary degeneracy and singularity

Lin Yan^a, Bin Wu^{a*}, Shiping Lu^a, Yuchan Wang^a

^aSchool of Mathematics and Statistics

Nanjing University of Information Science and Technology,
Nanjing 210044, P. R. China

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Abstract

In this paper, we consider a null controllability and an inverse source problem for stochastic Grushin equation with boundary degeneracy and singularity. We construct two special weight functions to establish two Carleman estimates for the whole stochastic Grushin operator with singular potential by a weighted identity method. One is for the backward stochastic Grushin equation with singular weight function. We then apply it to prove the null controllability for stochastic Grushin equation for any T and any degeneracy $\gamma > 0$, when our control domain touches the degeneracy line $\{x = 0\}$. In order to study the inverse source problem of determining two kinds of sources simultaneously, we prove the other Carleman estimate, which is for the forward stochastic Grushin equation with regular weight function. Based on this Carleman estimate, we obtain the uniqueness of the inverse source problem.

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined. Let $I = I_x \times I_y$ with $I_x = (0, 1)$, $I_y = (0, 1)$, $Q_T = I \times (0, T)$, $Q_t = I \times (0, t)$, $\Sigma_T = \partial I \times (0, T)$. Then we consider the following stochastic

*Corresponding author. email: binwu@nuist.edu.cn

by the boundary observation $u_y|_{\Sigma_T}$, $u_x|_{\Gamma_T}$ and final time observation $u|_{t=T}$ in I .

Physically, controlling a system can be understood simply as driving its solution to the point as desired by giving an action on some domain. In practical situations, some physical parameters are very hard to be measured directly in advance. The goal of inverse problem is to determine these unknown physical parameters by additional observations. These problems are also very important for real applications.

When no singular term was involved, the null controllability of deterministic Grushin equation with $I = (-1, 1) \times (0, 1)$ was studied in [2,3]. The null controllability for Grushin-type equations was obtained for any time $T > 0$ and for any degeneracy $\gamma > 0$, with a control that acts on one strip, touching the degeneracy line $\{x = 0\}$ in [5]. When restricting the domain to one side only of the singular set, i.e. $I = (0, 1) \times (0, 1)$, [8] proved that there exists T^* such that for every $T > T^*$ the Grushin-type equation is null controllable for $\gamma = 1$, $\sigma < \frac{1}{4}$. Next, [1] showed a similar null controllability in large time T when the degeneracy of the diffusion coefficient and singularity of the potential occur at the interior of the domain. The key ingredient in these papers is applying a Fourier decomposition to reduce the problem to the validity of a uniform observability inequality with respect to the Fourier frequency. As for the inverse source problem for deterministic Grushin equation, [4] proved a Lipschitz stability result of determining a source function h depending on x and y , by the observation data $\partial_t u|_{\omega \times (T_1, T_2)}$ with a suitable subdomain ω .

It is well known that Carleman estimate is one of the key tool to study null controllability and inverse problems, which is a class of weighted energy estimates in connection with deterministic/stochastic differential operators. As its applications to deterministic differential equations, we refer to [11,14,20,21,22,36,40] for inverse problems, [7,32,33,37] for unique continuation problems, [12,16,19,28] for control theory. For Carleman estimates related to deterministic Grushin equation, we refer to [2,3,23,31]. In recent years, many efforts have been devoted to studying the Carleman estimate for stochastic partial differential equations, for example [6,24,34,39] for stochastic heat equation, [42] for stochastic wave equation, [15] for stochastic Korteweg-de Vries equation, [17] for stochastic Kuramoto-Sivashinsky equation, [27] for stochastic Schrödinger equation, [18] for the stochastic Kawahara equation, and so on. To the best of our knowledge, there are only two papers about Carleman estimates for one dimensional stochastic degenerate operator $du - (x^\alpha u_x)_x dt$ [25,38], which is very different from the degenerate Grushin operator $du - u_{xx} dt - x^{2\gamma} u_{yy} dt$. In these works, Carleman estimates were mainly applied to deal with stochastic control problems. Since the solution of a stochastic differential equation is not differentiable with respect to time variable, which leads to that some traditional methods for deterministic inverse problems cannot be applied to the corresponding ones in the stochastic case. Therefore, [29] proposed a regular weight function in Carleman estimates to study an stochastic inverse problem related to the stochastic hyperbolic equation. We also refer to [26,41] for stochastic inverse problems.

Although there are numerous results for Carleman estimates for deterministic Grushin equation, little has been known for Carleman estimates related to the stochastic Grushin

equation. In this paper, we first construct a special weight function ψ to obtain a Carleman estimate for backward stochastic Grushin operator with singular potential and then apply this Carleman estimate to prove the null controllability for system (1.2). We do not apply the method based on Fourier decomposition as [1,8]. A weakness of Fourier decomposition is that in proving the observation inequality the authors have to deal with the eigenvalues in Fourier decomposition $\mu_n \rightarrow +\infty$ as $n \rightarrow \infty$, which is the reason that the condition $T > T^*$ is introduced in [8]. In order to obtain the null controllability result for any time T and any degeneracy γ , we consider the Grushin operator with singular potential, i.e. $u_{xx} + x^{2\gamma}u_{yy} + \frac{\sigma}{x^2}u$, as a whole to establish our Carleman estimate, not as [8] only for its Fourier components with respect to u , i.e. $(u_n)_{xx} - [(n\pi)^2x^{2\gamma} - \frac{\sigma}{x^2}]u_n$. It seems impossible to adopt the cut-off function method proposed in [9] to establish our Carleman estimate directly. The main purpose of the cut-off function introduced in [9] is to eliminate the boundary term on the non-degenerate boundary $x = 1$, which then was bounded by the local observation of ω not touching the generate boundary in Carleman estimate. However, in our paper, in order to establish Carleman estimate for the whole Grushin operator, we construct a special weight function to overcome the difficulties arising from degeneracy and singularity. Due to the choice of the weight function, the boundary term on degenerate boundary is left in Carleman estimate, rather than the term on non-degenerate boundary.

Secondly, we introduce a regular weight function in the Carleman estimate for forward stochastic Grushin equations to study our inverse problem of determining two source functions simultaneously. Based on such a regular weight function, we can put the random source function H on the left-hand side of this Carleman estimate, which allows us to determine H . However the derivatives of H with respect to spatial variables still lie on the right-hand side of Carleman estimate. For this reason, the random source function H to be determined could not depend on x and y . Moreover, similar to [26] or [41], we can only determine h in partial domain $I_x \times (0, T)$, since in the proof of the uniqueness result we have to differentiate the equation (1.3) with respect to y , rather than t as the deterministic case. This is also the result arising from the random effect of the equation. Such form of separation is also meaningful in some situations, for example, a heat source generated by the decay of radioactive isotope, which depends on variables x, y, t . Here, our main goal is to find the spatial distribution x and the time distribution t of the radioactive isotope.

Throughout this paper, we denote by $L^2_{\mathcal{F}}(0, T)$ the space of all progressively measurable stochastic process X such that $\mathbb{E}(\int_0^T |X|^2 dt) < \infty$. For a Banach space H , we denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{L^2(0, T; H)}^2) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted continuous processes X such that $\mathbb{E}(\|X\|_{C([0, T]; H)}^2) < \infty$, with the canonical norm.

Now we state the main results in this paper. The first one is the following null controllability for any T and any degeneracy $\gamma > 0$.

Theorem 1.1. *Let $\gamma > 0$, $0 \leq \sigma < \frac{1}{4}$ and $\alpha, \beta \in L^\infty(0, T; L^\infty(I))$. Then for any $u_0 \in$*

$L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I))$, there exists a pair $(g, G) \in L^2_{\mathcal{F}}(0, T; L^2(\omega)) \times L^2_{\mathcal{F}}(0, T; L^2(I))$ such that the corresponding solution u of (1.2) satisfies $u(T) = 0$ in I , \mathbb{P} -a.s. for any $T > 0$.

Remark 1.1. As explained in [8] for the deterministic case, the lower bound $\sigma < \frac{1}{4}$ is used to guarantee well-posedness issues linked to the use of the following Hardy inequality [10]

$$\int_0^1 \frac{z^2(x)}{x^2} dx \leq 4 \int_0^1 z_x^2(x) dx, \quad \forall z \in H_0^1(0, 1). \quad (1.4)$$

Moreover, it is noted that our control domain touches the line $\{x = 0\}$, which allows us to prove our controllability result for any $\gamma > 0$ and any time $T > 0$. However, a coming flaw with such a control domain is that the null controllability could not hold for $\sigma = \frac{1}{4}$. This is because that we need $\frac{1}{4} - \sigma > 0$ to prove the Cacciopoli inequality (3.47), when our control domain ω touches the line $\{x = 0\}$. Also $\sigma < \frac{1}{4}$ is necessary for the null controllability, even for the deterministic setting, see Theorem 1 and Theorem 2 in [8]. Since we choose a special weight function ψ to prove our Carleman estimate for the whole Grushin operator with singular potential, we need additional condition $\sigma \geq 0$ to deal with the singular term.

The other one is the following uniqueness result for our inverse source problem.

Theorem 1.2. Let $\gamma > 0$, $0 \leq \sigma < \frac{1}{4}$, $h \in L^2_{\mathcal{F}}(0, T; H^1(I_x))$, $H \in L^2_{\mathcal{F}}(0, T)$ and $R_1, R_2 \in C^3(\overline{Q_T})$ such that

$$|R_i| \neq 0 \quad \text{in } Q_T, \quad i = 1, 2, \quad (1.5)$$

$$\left| \nabla \left(\frac{R_2}{R_1} \right)_y \right| \leq C \left| \left(\frac{R_2}{R_1} \right)_y \right| \quad \text{in } Q_T. \quad (1.6)$$

If

$$u_y|_{\Sigma_T} = u_x|_{\Gamma_T} = 0, \quad \mathbb{P} - a.s., \quad (1.7)$$

$$u(T) = 0 \quad \text{in } I, \quad \mathbb{P} - a.s., \quad (1.8)$$

then

$$h(x, t) = 0, \quad (x, t) \in I_x \times [0, T], \quad \mathbb{P} - a.s. \quad (1.9)$$

and

$$H(t) = 0, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad (1.10)$$

where u is the solution of (1.3) corresponding to h and H .

Remark 1.2. Obviously, condition (1.6) is correct for $\frac{R_2}{R_1}$ not depending on y . Or when $\left| \nabla \ln \left(\frac{R_2}{R_1} \right)_y \right| \leq C$ in $\overline{Q_T}$, i.e. $\frac{R_2}{R_1}$ sufficiently smooth in $\overline{Q_T}$, (1.6) is also correct.

The rest of this paper is organized as follows. In next section, we prove the well-posedness of the system (1.1). In section 3, we establish two Carleman estimates for stochastic forward/backward Grushin equation with singular potential, respectively. In section 4, we prove the null controllability for system (1.2), i.e. Theorem 1.1. In last section, we show the uniqueness for our inverse source problem, i.e. Theorem 1.2.

2 Well-posedness

In this section, we show the well-posedness of the following stochastic Grushin equation with singular potential:

$$\begin{cases} du - u_{xx}dt - x^{2\gamma}u_{yy}dt - \frac{\sigma}{x^2}udt = fdt + FdB(t), & (x, y, t) \in Q_T, \\ u(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ u(x, y, 0) = u_0(x, y), & (x, y) \in I. \end{cases} \quad (2.1)$$

In order to deal with the degeneracy and the singularity, we introduce some suitable spaces. For $\gamma > 0$, we define $H_\gamma^1(I)$ and $H_\gamma^2(I)$ as the completion of $C_0^\infty(I)$ in the following norms

$$\begin{aligned} \|u\|_{H_\gamma^1(I)} &= \left[\int_I \left(|u_x|^2 + x^{2\gamma}|u_y|^2 - \frac{\sigma}{x^2}|u|^2 \right) dx dy \right]^{\frac{1}{2}}, \\ \|u\|_{H_\gamma^2(I)} &= \left[\int_I \left| u_{xx} + x^{2\gamma}u_{yy} + \frac{\sigma}{x^2}u \right|^2 dx dy \right]^{\frac{1}{2}}. \end{aligned}$$

The Hardy inequality (1.4) implies that $H_\gamma^1(I)$ is a Banach space endowed with the above norm for all $\sigma < \frac{1}{4}$. Further we introduce

$$\begin{aligned} \mathcal{G}_T &= L_{\mathcal{F}}^2(\Omega; C([0, T]; L^2(I))) \cap L_{\mathcal{F}}^2(0, T; H_\gamma^1(I)), \\ \mathcal{H}_T &= L_{\mathcal{F}}^2(\Omega; C([0, T]; L^2(I))) \cap L_{\mathcal{F}}^2(0, T; H_0^1(I)), \\ \mathcal{S}_T &= L_{\mathcal{F}}^2(\Omega; C([0, T]; H_\gamma^1(I))) \cap L_{\mathcal{F}}^2(0, T; H_\gamma^2(I)). \end{aligned}$$

Definition 2.1. A stochastic process $u \in \mathcal{G}_T$ is called a weak solution of (2.1) if for any $t \in [0, T]$, $\vartheta \in C_0^1(\bar{I})$, it holds that

$$\begin{aligned} & \int_I [u(t) - u_0] \vartheta dx dy + \int_{Q_t} \left(u_x \vartheta_x + x^{2\gamma} u_y \vartheta_y - \frac{\sigma}{x^2} u \vartheta \right) dx dy dt \\ &= \int_{Q_t} f \vartheta dx dy dt + \int_{Q_t} F \vartheta dx dy dB(t), \quad \mathbb{P} - a.s. \end{aligned} \quad (2.2)$$

Theorem 2.2. Let $\gamma > 0$ and $\sigma < \frac{1}{4}$. Then for any $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I))$, system (2.1) admits a unique weak solution $u \in \mathcal{G}_T$ such that

$$\|u\|_{\mathcal{G}_T} \leq C \left(\|u_0\|_{L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I))} + \|f\|_{L_{\mathcal{F}}^2(0, T; L^2(I))} + \|F\|_{L_{\mathcal{F}}^2(0, T; L^2(I))} \right), \quad (2.3)$$

where C is depending on I, T, γ and σ .

Proof. Letting $0 < \varepsilon < 1$, we consider the following approximate problem:

$$\begin{cases} du^\varepsilon - u_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} u_{yy}^\varepsilon dt - \frac{\sigma}{(x+\varepsilon)^2} u^\varepsilon dt = fdt + FdB(t), & (x, y, t) \in Q_T, \\ u^\varepsilon(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ u^\varepsilon(x, y, 0) = u_0^\varepsilon(x, y), & (x, y) \in I, \end{cases} \quad (2.4)$$

where

$$u_0^\varepsilon \rightarrow u_0 \quad \text{in } L^2(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(I)).$$

Then by [30], we know that (2.4) admits a unique solution $u^\varepsilon \in \mathcal{H}_T$ for any $0 < \varepsilon < 1$.

By Itô's formula $d(|u^\varepsilon|^2) = 2u^\varepsilon du^\varepsilon + (du^\varepsilon)^2$ and the equation of u^ε , we have

$$\begin{aligned}
& \int_I |u^\varepsilon(x, y, t)|^2 dx dy \\
&= \int_I |u_0^\varepsilon|^2 dx dy + \int_{Q_t} 2u^\varepsilon \left(u_{xx}^\varepsilon + (x + \varepsilon)^{2\gamma} u_{yy}^\varepsilon + \frac{\sigma}{(x + \varepsilon)^2} u^\varepsilon + f \right) dx dy dt \\
&\quad + 2 \int_{Q_t} u^\varepsilon F dx dy dB(t) + \int_{Q_t} |F|^2 dx dy dt \\
&\leq \int_I |u_0^\varepsilon|^2 dx dy - 2 \int_{Q_t} \left[|u_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |u_y^\varepsilon|^2 - \frac{\sigma}{(x + \varepsilon)^2} |u^\varepsilon|^2 \right] dx dy dt \\
&\quad + \int_{Q_t} |u^\varepsilon|^2 dx dy dt + \int_{Q_t} (|f|^2 + |F|^2) dx dy dt + 2 \int_{Q_t} u^\varepsilon F dx dy dB(t). \tag{2.5}
\end{aligned}$$

Then, taking mathematical expectation on both sides of (2.5) and applying Grönwall's inequality yields that

$$\begin{aligned}
& \sup_{t \in [0, T]} \mathbb{E} \|u^\varepsilon(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T \|u^\varepsilon(t)\|_{H_\gamma^1(I)}^2 dt \\
&\leq C e^{CT} \mathbb{E} \int_I |u_0^\varepsilon|^2 dx dy + C(T) \mathbb{E} \int_{Q_T} (|f|^2 + |F|^2) dx dy dt. \tag{2.6}
\end{aligned}$$

Applying Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, T]} \left| \int_{Q_t} u^\varepsilon F dx dy dB(t) \right| \leq \mathbb{E} \left[\int_0^T \left(\int_I |u^\varepsilon F| dx dy \right)^2 dt \right]^{\frac{1}{2}} \\
&\leq C \mathbb{E} \left[\int_0^T \left(\int_I |u^\varepsilon|^2 dx dy \right) \left(\int_I |F|^2 dx dy \right) dt \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4} \mathbb{E} \sup_{t \in [0, T]} \int_I |u^\varepsilon(x, y, t)|^2 dx dy + C \mathbb{E} \int_{Q_T} |F|^2 dx dy dt,
\end{aligned}$$

which together with (2.5) implies

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, T]} \int_I |u^\varepsilon(t)|^2 dx dy + \mathbb{E} \int_0^T \|u^\varepsilon(t)\|_{H_\gamma^1(I)}^2 dt \\
&\leq C(T) \left[\mathbb{E} \int_I |u_0^\varepsilon|^2 dx dy + \mathbb{E} \int_{Q_T} |u^\varepsilon|^2 dx dy dt + \mathbb{E} \int_{Q_T} (|f|^2 + |F|^2) dx dy dt \right]. \tag{2.7}
\end{aligned}$$

By (2.6) and (2.7), we have

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T \|u^\varepsilon(t)\|_{H_\gamma^1(I)}^2 dt \\
&\leq C(T) \mathbb{E} \int_I |u_0^\varepsilon|^2 dx dy + C(T) \mathbb{E} \int_{Q_T} (|f|^2 + |F|^2) dx dy dt. \tag{2.8}
\end{aligned}$$

Similarly, we could prove for any $\varepsilon_1, \varepsilon_2 \in (0, 1)$ that

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \|(u^{\varepsilon_1} - u^{\varepsilon_2})(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T \|(u^{\varepsilon_1} - u^{\varepsilon_2})(t)\|_{H_\gamma^1(I)}^2 dt \\ & \leq C \mathbb{E} \int_I |u_0^{\varepsilon_1} - u_0^{\varepsilon_2}|^2 dx dy. \end{aligned} \quad (2.9)$$

Hence, we observe that $\{u^\varepsilon\}$ is a Cauchy sequence in \mathcal{G}_T . By a standard limiting process we find that (2.1) admits a weak solution $u \in \mathcal{G}_T$ (the limit of u^ε in \mathcal{G}_T) such that

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T]} \|u(t)\|_{L^2(I)}^2 + \mathbb{E} \int_0^T \|u\|_{H_\gamma^1(I)}^2 dt \\ & \leq C(T) \mathbb{E} \int_I |u_0|^2 dx dy + C(T) \mathbb{E} \int_{Q_T} (|f|^2 + |F|^2) dx dy dt, \end{aligned} \quad (2.10)$$

which implies (2.3). The uniqueness of the weak solution of (2.1) could be directly deduced from (2.10). \square

To study our inverse problem, we also need the following existence and uniqueness of strong solution.

Theorem 2.3. *Let $\gamma > 0$ and $\sigma < \frac{1}{4}$. Then for any $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\gamma^1(I))$, system (2.1) admits a unique strong solution $u \in \mathcal{S}_T$ such that*

$$\|u\|_{\mathcal{S}_T} \leq C \left(\|u_0\|_{L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H_\gamma^1(I))} + \|f\|_{L_{\mathcal{F}}^2(0, T; L^2(I))} + \|F\|_{L_{\mathcal{F}}^2(0, T; H_\gamma^1(I))} \right), \quad (2.11)$$

where C is depending on I, T, γ and σ .

Proof. For the proof of Theorem 2.3, we borrow some ideas from [18]. We consider the operator

$$\mathcal{A} : D(\mathcal{A}) \rightarrow L^2(I), \quad v \mapsto -v_{xx} - x^{2\gamma} v_{yy} - \frac{\sigma}{x^2} v,$$

where

$$D(\mathcal{A}) = \{v \in H_\gamma^2(I) \mid v(x, y, t) = 0, \quad (x, y, t) \in \Sigma_T\}.$$

Let $\{\phi_k\}_{k=1}^\infty$ be the corresponding eigenfunctions of \mathcal{A} , such that $\|\phi_k\|_{L^2(I)} = 1$ ($k = 1, 2, 3, \dots$), which serves as an orthonormal basis of $L^2(I)$, $\{\lambda_k\}_{k=1}^\infty$ be the corresponding eigenvalues of \mathcal{A} such that $\mathcal{A}\phi_k = \lambda_k \phi_k$. We construct approximate solutions to (2.1) in the form

$$u^n(x, t) = \sum_{k=1}^n c_k^n(t) \phi_k(x),$$

where the unknown function c_k^n are solutions to the Cauchy problem for stochastic differential equations:

$$\begin{cases} dc_k^n - \lambda_k c_k^n dt = f_k dt + F_k dB(t), k = 1, 2, 3, \dots n \\ c_k(0) = (u_0, \phi_k)_{L^2(I)}, \end{cases} \quad (2.12)$$

where $f_k = (f, \phi_k)_{L^2(I)}$, $F_k = (F, \phi_k)_{L^2(I)}$. From the classical theory of stochastic differential equations, we can obtain there is a pathwise unique solution c_k^n adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, such that $c_k^n \in C([0, T])$ for almost all $\omega \in \Omega$.

By Itô's formula, we have

$$d(c_k^n)^2 - 2\lambda_k(c_k^n)^2 dt = 2c_k^n f_k dt + 2c_k^n F_k dB(t) + F_k^2 dt. \quad (2.13)$$

Multiplying both sides of (2.13) by λ_k , integration on $(0, t)$ and taking sums from 1 to n about k , we have

$$\begin{aligned} & (u^n, \mathcal{A}u^n)_{L^2(I)}(t) + 2 \int_0^t (\mathcal{A}u^n, \mathcal{A}u^n)_{L^2(I)} dt \\ &= (u^n, \mathcal{A}u^n)_{L^2(I)}(0) + 2 \int_0^t (f^n, \mathcal{A}u^n)_{L^2(I)} dt + 2 \int_0^t (F^n, \mathcal{A}u^n)_{L^2(I)} dB(t) \\ & \quad + \int_0^t (F^n, \mathcal{A}F^n)_{L^2(I)} dt, \end{aligned} \quad (2.14)$$

where

$$f^n(x, t) = \sum_{k=1}^n f_k(t) \phi_k(x), \quad F^n(x, t) = \sum_{k=1}^n F_k(t) \phi_k(x).$$

Obviously, we have $(v, \mathcal{A}v)_{L^2(I)} = \|v\|_{H_\gamma^1(I)}^2$. Then taking mathematical expectation on both sides of (2.14) and applying Grönwall's inequality yields that

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \|u^n(t)\|_{H_\gamma^1(I)}^2 + \mathbb{E} \int_0^T \|u^n(t)\|_{H_\gamma^2(I)}^2 dt \\ & \leq C \left[\mathbb{E} \|u_0^n\|_{H_\gamma^1(I)}^2 + \mathbb{E} \int_0^T \|f^n\|_{L^2(I)}^2 dt + \mathbb{E} \int_0^T \|F^n\|_{H_\gamma^1(I)}^2 dt \right]. \end{aligned} \quad (2.15)$$

Similarly, we could prove for any $n_1, n_2 \geq 1$ that

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \|(u^{n_1} - u^{n_2})(t)\|_{H_\gamma^1(I)}^2 + \mathbb{E} \int_0^T \|(u^{n_1} - u^{n_2})(t)\|_{H_\gamma^2(I)}^2 dt \\ & \leq C \left(\mathbb{E} \|u_0^{n_1} - u_0^{n_2}\|_{H_\gamma^1(I)}^2 + \|f^{n_1} - f^{n_2}\|_{L_\mathbb{F}^2(0, T; L^2(I))}^2 + \|F^{n_1} - F^{n_2}\|_{L_\mathbb{F}^2(0, T; H_\gamma^1(I))}^2 \right). \end{aligned}$$

Hence, $\{u^n\}$ is a Cauchy sequence in \mathcal{S}_T . By a standard limiting process we obtain that (2.1) admits a strong solution $u \in \mathcal{S}_T$ (the limit of u^n in \mathcal{S}_T). Additionally, from (2.15) we deduce (2.11) directly. The uniqueness of the solution of (2.1) could be directly deduced from (2.11). \square

3 Carleman estimates for stochastic Grushin equation

In this section, we will show two Carleman estimates for stochastic Grushin equation with singular potential, which will be used to study the null controllability and the inverse source problem, respectively. One is for the backward stochastic Grushin equation with singular weight function. The other one is for the forward stochastic Grushin equation with regular weight function.

3.1 Carleman estimate for backward stochastic Grushin equation with singular weight function

In this subsection, we will use a singular weight function to prove a Carleman estimate for the backward stochastic Grushin equation with singular potential

$$\begin{cases} dv + v_{xx}dt + x^{2\gamma}v_{yy}dt + \frac{\sigma}{x^2}vdt = f_1dt + F_1dB(t), & (x, y, t) \in Q_T, \\ v(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ v(x, y, T) = v_T(x, y), & (x, y) \in I, \end{cases} \quad (3.1)$$

where $v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))$, $f_1 \in L^2_{\mathcal{F}}(0, T; L^2(I))$. This Carleman estimate will be used to prove the null controllability result for (1.2).

To formulate our Carleman estimate, we introduce some weight functions. For $\omega = (0, a) \times I_y$, we choose $\omega^{(i)} = (0, a_i) \times I_y$ for $i = 1, 2$ with $0 < a_1 < a_2 < a$. Then we know that $\omega^{(1)} \subset \omega^{(2)} \subset \omega$. We define

$$\phi(x, y) = e^{\lambda\psi(x, y)}, \quad \varphi(x, y, t) = (e^{\lambda\psi(x, y)} - e^{2\lambda\|\psi\|_{C(\bar{I})}})\xi(t), \quad \theta(x, y, t) = e^{s\varphi(x, y, t)},$$

with

$$\psi(x, y) = x^{2+2\gamma}y(1-y) - \mu x + M, \quad \xi(t) = \frac{1}{t^4(T-t)^4}. \quad (3.2)$$

Here μ is a positive constant such that

$$\mu > \sup_{(x, y) \in \bar{I}} (2 + 2\gamma)(x + 1)^{1+2\gamma}y(1-y) + \delta_0 \quad (3.3)$$

with some $\delta_0 > 1$, which will be specified below. M is chosen sufficiently large to satisfy $\psi(x, y) > 0$ for all $(x, y) \in \bar{I}$.

Obviously, the function ξ satisfies the following essential properties

$$\xi(t) \rightarrow +\infty \quad \text{as } t \rightarrow 0^+ \text{ or } T^- \quad \text{and} \quad \xi > 0, \quad |\xi_t| \leq cT\xi^{\frac{5}{4}}. \quad (3.4)$$

Here and henceforth we use c denote a positive constant independent of any parameters such that $c > 1$. Additionally, in order to give the explicit expression of λ , s and C in Carleman estimate, we introduce notation C_1 defined by

$$C_1 = 2^{2\gamma}(2 + 2\gamma)(1 + 2\gamma)2\gamma|2\gamma - 1| + c(2 + 2\gamma)(1 + 2\gamma) + \sigma.$$

Remark 3.1. Different from the weight function used to establish Carleman estimate for Grushin operators with singular potential by Fourier decomposition method [8], we choose a special weight function ψ to prove a new Carleman estimate for the whole Grushin operator with singular potential. Such a weight function is introduced first in the Carleman estimate for degenerate Grushin operator. Traditional weight function including separate power functions of x and y could not be applied to the whole Grushin operator. To deal with $x^{2\gamma}v_{yy}$, we construct $\psi(x, y)$ to guarantee the positive lower bound of the integral terms related to the decomposition of $x^{2\gamma}v_{yy}$.

Our main result in the subsection is the following Carleman estimate for (3.1).

Theorem 3.1. *Let $\gamma > 0$, $0 \leq \sigma < \frac{1}{4}$, $v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))$, $f_1 \in L^2_{\mathcal{F}}(0, T; L^2(I))$. Then there exist constants λ_1 , $s_1(\lambda)$ and $C(\lambda)$ such that*

$$\begin{aligned} & \mathbb{E} \int_{Q_T} s \xi \theta^2 |v_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \xi \theta^2 x^{2\gamma} |v_y|^2 dx dy dt + \mathbb{E} \int_{Q_T} s^3 \xi^3 \theta^2 |v|^2 dx dy dt \\ & \leq C(\lambda) \left[\mathbb{E} \int_{Q_T} \theta^2 |f_1|^2 dx dy dt + \mathbb{E} \int_{Q_T} s^2 \xi^2 \theta^2 |F_1|^2 dx dy dt + \mathbb{E} \int_{\omega_T} s^3 \xi^3 \theta^2 |v|^2 dx dy dt \right] \end{aligned} \quad (3.5)$$

for all sufficiently large $\lambda \geq \lambda_1$ and $s \geq s_1(\lambda)$ and all solution $(v, F_1) \in \mathcal{G}_T \times L^2_{\mathcal{F}}(0, T; L^2(I))$ satisfies (3.1), where

$$\begin{aligned} \lambda_1 &= c(T^{23} + 1)2^{6+10\gamma}C_1^4, \quad s_1(\lambda) = c(T^8 + 1)\delta_0^4 2^{8+12\gamma}C_1^5 e^{c\|\psi\|_{C(\bar{T})}\lambda}, \\ C(\lambda) &= c(T^{40} + 1)\delta_0^4 2^{8+12\gamma}\lambda^2 e^{c\|\psi\|_{C(\bar{T})}\lambda}C_1^5. \end{aligned}$$

Since the system (3.1) is not only degenerate, but also singular on $\{x = 0\} \times I_y$, we first transfer to study an approximate version of (3.1). To do this, letting $0 < \varepsilon < 1$ and $v_T^\varepsilon \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H_0^1(I))$ such that

$$v_T^\varepsilon \rightarrow v_T \quad \text{in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I)),$$

we then consider

$$\begin{cases} dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt = f_1 dt + F_1^\varepsilon dB(t), & (x, y, t) \in Q_T, \\ v^\varepsilon(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ v^\varepsilon(x, y, T) = v_T^\varepsilon(x, y), & (x, y) \in I. \end{cases} \quad (3.6)$$

According to [30], we know that the system (3.6) admits a unique solution $(v^\varepsilon, F_1^\varepsilon) \in \mathcal{H}_T \times L^2_{\mathcal{F}}(0, T; L^2(I))$. Set

$$\widehat{\varphi}(x, y, t) = \varphi(x + \varepsilon, y, t), \quad \widehat{\theta}(x, y, t) = \theta(x + \varepsilon, y, t).$$

In the sequel, $\widehat{\phi}$ and $\widehat{\psi}$ are defined analogously. Then we have the following weighted identity for (3.6).

Lemma 3.2. *Let τ be a constant such that $2 < \tau < 3$. Assume that v^ε is an $H^2(\mathbb{R}^2)$ -valued continuous semimartingale. Set $l = s\widehat{\varphi}$, $z = \widehat{\theta}v^\varepsilon$ and*

$$\begin{aligned} P_1 &= dz - 2l_x z_x dt - 2(x + \varepsilon)^{2\gamma} l_y z_y dt - \tau l_{xx} z dt, \\ P_2 &= z_{xx} + (x + \varepsilon)^{2\gamma} z_{yy} + l_x^2 z + (x + \varepsilon)^{2\gamma} l_y^2 z + \frac{\sigma}{(x + \varepsilon)^2} z, \\ P &= (\tau - 1)l_{xx} z - l_t z - (x + \varepsilon)^{2\gamma} l_{yy} z. \end{aligned}$$

Then for a.e. $(x, y) \in \mathbb{R}^2$, it holds that

$$\begin{aligned} & P_2 \widehat{\theta} \left[dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt \right] \\ & = |P_2|^2 dt + P_2 P dt + \sum_{i=1}^5 X_i dt + dY + \{\cdot\}_x + \{\cdot\}_y + J, \quad \mathbb{P} - a.s., \end{aligned} \quad (3.7)$$

where

$$\begin{aligned}
X_1 &= [(\tau + 1)l_{xx} - (x + \varepsilon)^{2\gamma}l_{yy}] z_x^2, \\
X_2 &= [-2\gamma(x + \varepsilon)^{2\gamma-1}l_x + (\tau - 1)(x + \varepsilon)^{2\gamma}l_{xx} + (x + \varepsilon)^{4\gamma}l_{yy}] z_y^2, \\
X_3 &= 4[\gamma(x + \varepsilon)^{2\gamma-1}l_y + (x + \varepsilon)^{2\gamma}l_{xy}] z_x z_y, \\
X_4 &= [(3 - \tau)l_x^2 l_{xx} + 2\gamma(x + \varepsilon)^{2\gamma-1}l_x l_y^2 + 3(x + \varepsilon)^{4\gamma}l_y^2 l_{yy}] z^2 \\
&\quad + (x + \varepsilon)^{2\gamma} [4l_x l_y l_{xy} + l_x^2 l_{yy} + (1 - \tau)l_{xx} l_y^2] z^2 \\
&\quad + \left[(1 - \tau) \frac{\sigma}{(x + \varepsilon)^2} l_{xx} - \frac{2\sigma}{(x + \varepsilon)^3} l_x + \frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} l_{yy} \right] z^2, \\
X_5 &= \left[-l_x l_{xt} - (x + \varepsilon)^{2\gamma} l_y l_{yt} - \frac{\tau}{2} l_{xxxx} - \frac{\tau}{2} (x + \varepsilon)^{2\gamma} l_{xxyy} \right] z^2, \\
Y &= -\frac{1}{2} z_x^2 - \frac{1}{2} (x + \varepsilon)^{2\gamma} z_y^2 + \frac{1}{2} \left[l_x^2 + (x + \varepsilon)^{2\gamma} l_y^2 + \frac{\sigma}{(x + \varepsilon)^2} \right] z^2, \\
\{\cdot\} &= z_x dz + \left[-l_x z_x^2 + (x + \varepsilon)^{2\gamma} l_x z_y^2 - l_x^3 z^2 - (x + \varepsilon)^{2\gamma} l_x l_y^2 z^2 \right. \\
&\quad \left. - \frac{\sigma}{(x + \varepsilon)^2} l_x z^2 - 2(x + \varepsilon)^{2\gamma} l_y z_x z_y - \tau l_{xx} z z_x + \frac{\tau}{2} l_{xxx} z^2 \right] dt, \\
\{\cdot\cdot\} &= (x + \varepsilon)^{2\gamma} z_y dz + \left[-2(x + \varepsilon)^{2\gamma} l_x z_x z_y + (x + \varepsilon)^{2\gamma} l_y z_x^2 \right. \\
&\quad \left. - (x + \varepsilon)^{4\gamma} l_y z_y^2 - (x + \varepsilon)^{2\gamma} l_x^2 l_y z^2 - (x + \varepsilon)^{4\gamma} l_y^3 z^2 \right. \\
&\quad \left. - \frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} l_y z^2 - \tau(x + \varepsilon)^{2\gamma} l_{xx} z z_y + \frac{\tau}{2} (x + \varepsilon)^{2\gamma} l_{xxy} z^2 \right] dt, \\
J &= \frac{1}{2} (dz_x)^2 + \frac{1}{2} (x + \varepsilon)^{2\gamma} (dz_y)^2 - \frac{1}{2} \left[l_x^2 + (x + \varepsilon)^{2\gamma} l_y^2 + \frac{\sigma}{(x + \varepsilon)^2} \right] (dz)^2.
\end{aligned}$$

Proof. Notice that $\hat{\theta} = e^l$, $l = s\hat{\varphi}$ and $z = \hat{\theta}v^\varepsilon$. Then we have

$$\hat{\theta} \left[dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt \right] = P_1 + (P_2 + P) dt.$$

Hence

$$P_2 \hat{\theta} \left[dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt \right] = P_1 P_2 + |P_2|^2 dt + P_2 P dt. \quad (3.8)$$

We easily see that

$$P_1 P_2 = P_2 dz - 2l_x z_x P_2 dt - 2(x + \varepsilon)^{2\gamma} l_y z_y P_2 dt - \tau l_{xx} z P_2 dt. \quad (3.9)$$

Now we calculate the terms on the right-hand side of (3.9) one by one. For the first one, by Itô's formula, we have

$$\begin{aligned}
P_2 dz &= \left[z_{xx} + (x + \varepsilon)^{2\gamma} z_{yy} + l_x^2 z + (x + \varepsilon)^{2\gamma} l_y^2 z + \frac{\sigma}{(x + \varepsilon)^2} z \right] dz \\
&= (z_x dz)_x - \frac{1}{2} d(z_x^2) + \frac{1}{2} (dz_x)^2 + [(x + \varepsilon)^{2\gamma} z_y dz]_y - \frac{1}{2} d[(x + \varepsilon)^{2\gamma} z_y^2]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(x + \varepsilon)^{2\gamma} (dz_y)^2 + \frac{1}{2} d(l_x^2 z^2) - l_x l_{xt} z^2 dt - \frac{1}{2} l_x^2 (dz)^2 \\
& + \frac{1}{2} d[(x + \varepsilon)^{2\gamma} l_y^2 z^2] - (x + \varepsilon)^{2\gamma} l_y l_{yt} z^2 dt - \frac{1}{2} (x + \varepsilon)^{2\gamma} l_y^2 (dz)^2 \\
& + \frac{1}{2} d\left[\frac{\sigma}{(x + \varepsilon)^2} z^2\right] - \frac{1}{2} \frac{\sigma}{(x + \varepsilon)^2} (dz)^2.
\end{aligned} \tag{3.10}$$

By a direct calculation, we have

$$\begin{aligned}
& - 2l_x z_x P_2 dt \\
& = - 2l_x z_x \left[z_{xx} + (x + \varepsilon)^{2\gamma} z_{yy} + l_x^2 z + (x + \varepsilon)^{2\gamma} l_y^2 z + \frac{\sigma}{(x + \varepsilon)^2} z \right] dt \\
& = - (l_x z_x^2)_x dt + l_{xx} z_x^2 dt - 2[(x + \varepsilon)^{2\gamma} l_x z_x z_y]_y dt + [(x + \varepsilon)^{2\gamma} l_x z_y^2]_x dt \\
& \quad + 2(x + \varepsilon)^{2\gamma} l_{xy} z_x z_y dt - [2\gamma(x + \varepsilon)^{2\gamma-1} l_x + (x + \varepsilon)^{2\gamma} l_{xx}] z_y^2 dt - (l_x^3 z^2)_x dt \\
& \quad + 3l_x^2 l_{xx} z^2 dt - [(x + \varepsilon)^{2\gamma} l_x l_y^2 z^2]_x dt + 2(x + \varepsilon)^{2\gamma} l_x l_y l_{xy} z^2 dt \\
& \quad + [2\gamma(x + \varepsilon)^{2\gamma-1} l_x l_y^2 + (x + \varepsilon)^{2\gamma} l_{xx} l_y^2] z^2 dt - \left[\frac{\sigma}{(x + \varepsilon)^2} l_x z^2 \right]_x dt \\
& \quad + \left[\frac{\sigma}{(x + \varepsilon)^2} l_{xx} - \frac{2\sigma}{(x + \varepsilon)^3} l_x \right] z^2 dt
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
& - 2(x + \varepsilon)^{2\gamma} l_y z_y P_2 dt \\
& = - 2(x + \varepsilon)^{2\gamma} l_y z_y \left[z_{xx} + (x + \varepsilon)^{2\gamma} z_{yy} + l_x^2 z + (x + \varepsilon)^{2\gamma} l_y^2 z + \frac{\sigma}{(x + \varepsilon)^2} z \right] dt \\
& = - 2[(x + \varepsilon)^{2\gamma} l_y z_x z_y]_x dt + [(x + \varepsilon)^{2\gamma} l_y z_x^2]_y dt \\
& \quad + [4\gamma(x + \varepsilon)^{2\gamma-1} l_y + 2(x + \varepsilon)^{2\gamma} l_{xy}] z_x z_y dt - (x + \varepsilon)^{2\gamma} l_{yy} z_x^2 dt \\
& \quad - [(x + \varepsilon)^{4\gamma} l_y z_y^2]_y dt + (x + \varepsilon)^{4\gamma} l_{yy} z_y^2 dt - [(x + \varepsilon)^{2\gamma} l_x^2 l_y z^2]_y dt \\
& \quad + (x + \varepsilon)^{2\gamma} (l_x^2 l_y)_y z^2 dt - [(x + \varepsilon)^{4\gamma} l_y^3 z^2]_y dt + 3(x + \varepsilon)^{4\gamma} l_y^2 l_{yy} z^2 dt \\
& \quad - \left[\frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} l_y z^2 \right]_y dt + \frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} l_{yy} z^2 dt.
\end{aligned} \tag{3.12}$$

The last term can be rewritten as

$$\begin{aligned}
& - \tau l_{xx} z P_2 dt \\
& = - \tau l_{xx} z \left[z_{xx} + (x + \varepsilon)^{2\gamma} z_{yy} + l_x^2 z + (x + \varepsilon)^{2\gamma} l_y^2 z + \frac{\sigma}{(x + \varepsilon)^2} z \right] dt \\
& = - \tau (l_{xx} z z_x)_x dt + \frac{\tau}{2} (l_{xxx} z^2)_x dt - \frac{\tau}{2} l_{xxx} z^2 dt + \tau l_{xx} z_x^2 dt - \tau [(x + \varepsilon)^{2\gamma} l_{xx} z z_y]_y dt \\
& \quad + \frac{\tau}{2} [(x + \varepsilon)^{2\gamma} l_{xxy} z^2]_y dt - \frac{\tau}{2} (x + \varepsilon)^{2\gamma} l_{xxyy} z^2 dt + \tau (x + \varepsilon)^{2\gamma} l_{xx} z_y^2 dt \\
& \quad - \tau l_x^2 l_{xx} z^2 - \tau (x + \varepsilon)^{2\gamma} l_{xx} l_y^2 z^2 - \tau \frac{\sigma}{(x + \varepsilon)^2} l_{xx} z^2 dt.
\end{aligned} \tag{3.13}$$

Combining (3.8)-(3.13), we can obtain (3.7) and then complete the proof of Lemma 3.2. \square

Now, integrating both sides of (3.7) in Q_T , taking mathematical expectation in Ω and using $\widehat{\theta}(x, y, 0) = \widehat{\theta}(x, y, T) = 0$ in I , we obtain that

$$\begin{aligned} & \mathbb{E} \int_{Q_T} P_2 \widehat{\theta} \left[dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt \right] dx dy \\ & \geq \frac{1}{2} \mathbb{E} \int_{Q_T} |P_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} |P|^2 dx dy dt + \sum_{i=1}^5 \mathbb{E} \int_{Q_T} X_i dx dy dt \\ & \quad + \mathbb{E} \int_{Q_T} (\{\cdot\}_x + \{\cdot\}_y) dx dy + \mathbb{E} \int_{Q_T} J dx dy. \end{aligned} \quad (3.14)$$

In the following we estimate the last three terms in (3.14) one by one.

Lemma 3.3. *There exist positive constant*

$$M_1 = \frac{3 - \tau}{2} \delta_0^4 > 1$$

such that

$$\begin{aligned} \sum_{i=1}^5 \mathbb{E} \int_{Q_T} X_i dx dy dt & \geq M_1 \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |z|^2 dx dy dt + M_1 \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\phi} \xi |z_x|^2 dx dy dt \\ & \quad + M_1 \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |z_y|^2 dx dy dt \end{aligned} \quad (3.15)$$

for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma} C_1^4, \quad s \geq 1.$$

Proof. Notice that $\widehat{\psi}(x, y) = (x + \varepsilon)^{2+2\gamma} y(1 - y) - \mu(x + \varepsilon) + M$. Together with (3.3), we obtain the following properties of $\widehat{\psi}$:

$$\begin{cases} \widehat{\psi}_x < -\delta_0, & \widehat{\psi}_x \widehat{\psi}_{xx} \leq 0, \\ |\widehat{\psi}_{xxxx}| \leq C_1 (x + \varepsilon)^{-2}, & |\widehat{\psi}_y| + |\widehat{\psi}_{yy}| \leq C_1 (x + \varepsilon)^{2+2\gamma}, \\ |\widehat{\psi}_{xy}| + |\widehat{\psi}_{xx}| + |\widehat{\psi}_{xxy}| + |\widehat{\psi}_{xyy}| + |\widehat{\psi}_{xxyy}| \leq C_1 (x + \varepsilon)^{2\gamma}. \end{cases} \quad (3.16)$$

Recalling $l = s\widehat{\phi}$, we have

$$\begin{cases} X_1 = (\tau + 1) s \lambda^2 \widehat{\phi} \widehat{\psi}_x^2 \xi |z_x|^2 + K_1 |z_x|^2, \\ X_2 = \left[s \lambda^2 (x + \varepsilon)^{4\gamma} \widehat{\psi}_y^2 + (\tau - 1) s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\psi}_x^2 - 2\gamma s \lambda (x + \varepsilon)^{2\gamma-1} \widehat{\psi}_x \right] \widehat{\phi} \xi |z_y|^2 \\ \quad + K_2 |z_y|^2, \\ X_3 = 4s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \widehat{\psi}_x \widehat{\psi}_y \xi z_x z_y + K_3 z_x z_y, \end{cases} \quad (3.17)$$

where

$$\begin{aligned} K_1 &= \left[-s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\psi}_y^2 + s \lambda \left((\tau + 1) \widehat{\psi}_{xx} - (x + \varepsilon)^{2\gamma} \widehat{\psi}_{yy} \right) \right] \widehat{\phi} \xi, \\ K_2 &= s \lambda \left[(\tau - 1) (x + \varepsilon)^{2\gamma} \widehat{\psi}_{xx} + (x + \varepsilon)^{4\gamma} \widehat{\psi}_{yy} \right] \widehat{\phi} \xi, \\ K_3 &= 4s \lambda \left[(x + \varepsilon)^{2\gamma} \widehat{\psi}_{xy} + \gamma (x + \varepsilon)^{2\gamma-1} \widehat{\psi}_y \right] \widehat{\phi} \xi, \end{aligned}$$

satisfy

$$\begin{cases} |K_1| \leq C_2 s \lambda^2 \widehat{\phi} \xi, \\ |K_2| \leq C_2 s \lambda (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi, \\ |K_3| \leq C_2 s \lambda (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi, \end{cases} \quad (3.18)$$

due to (3.16), where

$$C_2 = c(\gamma + 1)2^{4+6\gamma}C_1^2 < c2^{4+6\gamma}C_1^3, \quad \lambda > 1.$$

By Young's inequality, we obtain for all $\varepsilon > 0$ that

$$\left| 4s\lambda^2(x + \varepsilon)^{2\gamma} \widehat{\phi} \widehat{\psi}_x \widehat{\psi}_y \xi z_x z_y \right| \leq \varepsilon s \lambda^2 \widehat{\psi}_x^2 \widehat{\phi} \xi |z_x|^2 + c(\varepsilon) s \lambda^2 (x + \varepsilon)^{4\gamma} \widehat{\psi}_y^2 \widehat{\phi} \xi |z_y|^2. \quad (3.19)$$

Therefore, by (3.17)-(3.19) we have the following estimate

$$\begin{aligned} & \sum_{i=1}^3 \mathbb{E} \int_{Q_T} X_i dx dy dt \\ & \geq \mathbb{E} \int_{Q_T} \left[(\tau + 1 - \varepsilon) s \lambda^2 \widehat{\psi}_x^2 - C_2 s \lambda^2 - 2^{2\gamma} C_2 s \lambda \right] \widehat{\phi} \xi |z_x|^2 dx dy dt \\ & \quad + \mathbb{E} \int_{Q_T} \left[(\tau - 1) s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\psi}_x^2 - 2\gamma s \lambda (x + \varepsilon)^{2\gamma-1} \widehat{\psi}_x - c(\varepsilon) s \lambda^2 (x + \varepsilon)^{4\gamma} \widehat{\psi}_y^2 \right. \\ & \quad \left. - C_2 s \lambda (x + \varepsilon)^{2\gamma} \right] \widehat{\phi} \xi |z_y|^2 dx dy dt. \end{aligned} \quad (3.20)$$

Fixing $0 < \varepsilon < \frac{1}{2}$ and choosing $\delta_0 > 1$ sufficiently large to satisfy

$$\begin{cases} \left(\frac{1}{2} - \varepsilon \right) \delta_0^2 s \lambda^2 - C_2 s \lambda^2 - 2^{2\gamma} C_2 s \lambda > 0, \\ (\tau - 2) \delta_0^2 s \lambda^2 - c(\varepsilon) C_1^2 2^{4+6\gamma} s \lambda^2 - C_2 s \lambda > 0, \end{cases} \quad (3.21)$$

and noticing that $\widehat{\psi}_x < 0$, we further find that

$$\begin{aligned} & \sum_{i=1}^3 \mathbb{E} \int_{Q_T} X_i dx dy dt \\ & \geq \left(\tau + \frac{1}{2} \right) \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\psi}_x^2 \widehat{\phi} \xi |z_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\psi}_x^2 \widehat{\phi} \xi |z_y|^2 dx dy dt. \end{aligned} \quad (3.22)$$

By definitions of l , $\widehat{\varphi}$, we have the following estimate for X_4 :

$$\begin{aligned} X_4 = & s^3 \lambda^4 \left[(3 - \tau) \widehat{\psi}_x^4 + 3(x + \varepsilon)^{4\gamma} \widehat{\psi}_y^4 + (6 - \tau)(x + \varepsilon)^{2\gamma} \widehat{\psi}_x^2 \widehat{\psi}_y^2 \right] \widehat{\phi}^3 \xi^3 |z|^2 \\ & + \frac{s\sigma}{(x + \varepsilon)^2} \left[(1 - \tau) \left(\lambda^2 \widehat{\psi}_x^2 + \lambda \widehat{\psi}_{xx} \right) - \frac{2}{(x + \varepsilon)} \lambda \widehat{\psi}_x \right] \widehat{\phi} \xi |z|^2 + K_4 |z|^2, \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} K_4 = & s^3 \lambda^3 \left[(3 - \tau) \widehat{\psi}_x^2 \widehat{\psi}_{xx} + 2\gamma(x + \varepsilon)^{2\gamma-1} \widehat{\psi}_x \widehat{\psi}_y^2 + 3(x + \varepsilon)^{4\gamma} \widehat{\psi}_y^2 \widehat{\psi}_{yy} \right. \\ & \left. + (x + \varepsilon)^{2\gamma} \left(4\widehat{\psi}_x \widehat{\psi}_y \widehat{\psi}_{xy} + \widehat{\psi}_x^2 \widehat{\psi}_{yy} + (1 - \tau) \widehat{\psi}_y^2 \widehat{\psi}_{xx} \right) \right] \widehat{\phi}^3 \xi^3 \\ & + s\sigma(x + \varepsilon)^{-2+2\gamma} \left(\lambda^2 \widehat{\psi}_y^2 + \lambda \widehat{\psi}_{yy} \right) \widehat{\phi} \xi \end{aligned}$$

satisfies

$$|K_4| \leq C_3 s^3 \lambda^3 \widehat{\phi}^3 \xi^3, \quad (3.24)$$

where

$$C_3 = c(T^{16} + 1)\delta_0^2 2^{6+10\gamma} C_1^4, \quad \lambda > 1, \quad s > 1.$$

Then we obtain that

$$\begin{aligned} & \mathbb{E} \int_{Q_T} X_4 dx dy dt \\ & \geq \mathbb{E} \int_{Q_T} \left[(3 - \tau) s^3 \lambda^4 \widehat{\psi}_x^4 \widehat{\phi}^3 \xi^3 - \tau \frac{\sigma}{(x + \varepsilon)^2} s \lambda^2 \widehat{\psi}_x^2 \widehat{\phi} \xi - C_3 s^3 \lambda^3 \widehat{\phi}^3 \xi^3 \right] |z|^2 dx dy dt. \end{aligned} \quad (3.25)$$

Moreover, by (3.4) and (3.16) we have

$$\begin{aligned} & \mathbb{E} \int_{Q_T} X_5 dx dy dt \\ & = - \mathbb{E} \int_{Q_T} s^2 \lambda^2 \left[\widehat{\psi}_x^2 + (x + \varepsilon)^{2\gamma} \widehat{\psi}_y^2 \right] \widehat{\phi}^2 \xi \xi_t |z|^2 dx dy dt \\ & \quad - \frac{\tau}{2} \mathbb{E} \int_{Q_T} s \left[\lambda^4 \widehat{\psi}_x^4 + 6\lambda^3 \widehat{\psi}_x^2 \widehat{\psi}_{xx} + \lambda^2 (3\widehat{\psi}_{xx}^2 + 4\widehat{\psi}_x \widehat{\psi}_{xxx}) + \lambda \widehat{\psi}_{xxxx} \right] \widehat{\phi} \xi |z|^2 dx dy dt \\ & \quad - \frac{\tau}{2} \mathbb{E} \int_{Q_T} (x + \varepsilon)^{2\gamma} s \left[\lambda^4 \widehat{\psi}_x^2 \widehat{\psi}_y^2 + \lambda^3 (4\widehat{\psi}_x \widehat{\psi}_y \widehat{\psi}_{xy} + \widehat{\psi}_{xx} \widehat{\psi}_y^2 + \widehat{\psi}_x^2 \widehat{\psi}_{yy}) \right. \\ & \quad \quad \left. + \lambda^2 (2\widehat{\psi}_{xy}^2 + 2\widehat{\psi}_y \widehat{\psi}_{xxy} + 2\widehat{\psi}_x \widehat{\psi}_{xyy} + \widehat{\psi}_{xx} \widehat{\psi}_{yy}) + \lambda \widehat{\psi}_{xxyy} \right] \widehat{\phi} \xi |z|^2 dx dy dt \\ & \geq - C_4 \mathbb{E} \int_{Q_T} \left(s^2 \lambda^2 \widehat{\phi}^2 \xi^3 + s \lambda^4 \widehat{\phi} \xi \right) |z|^2 dx dy dt \\ & \quad - C_1 \mathbb{E} \int_{Q_T} s \lambda \frac{1}{(x + \varepsilon)^2} \widehat{\phi} \xi |z|^2 dx dy dt, \end{aligned} \quad (3.26)$$

where

$$C_4 = c\delta_0^2 2^{4+6\gamma} C_1^2 T^7 + c\delta_0^4 2^{4+6\gamma} C_1^3 < c(T^7 + 1)\delta_0^4 2^{4+6\gamma} C_1^3, \quad \lambda > 1.$$

By Hardy inequality (1.4), we have

$$\begin{aligned} & - \mathbb{E} \int_{Q_T} s \lambda \frac{1}{(x + \varepsilon)^2} \widehat{\phi} \xi |z|^2 dx dy dt \geq -4 \mathbb{E} \int_{Q_T} s \lambda \xi \left| \left(\widehat{\phi}^{\frac{1}{2}} z \right)_x \right|^2 dx dy dt \\ & \geq -c \mathbb{E} \int_{Q_T} s \lambda \widehat{\phi} \xi |z_x|^2 dx dy dt - c \mathbb{E} \int_{Q_T} s \lambda^3 \widehat{\psi}_x^2 \widehat{\phi} \xi |z|^2 dx dy dt. \end{aligned} \quad (3.27)$$

Then, it follows from (3.22), (3.25)-(3.27) that

$$\begin{aligned}
& \sum_{i=1}^5 \mathbb{E} \int_{Q_T} X_i dx dy dt \\
\geq & \mathbb{E} \int_{Q_T} [(3-\tau)\delta_0^4 s^3 \lambda^4 - C_3 s^3 \lambda^3 - C_4 s^2 \lambda^2 - C_4 T^{16} s \lambda^4 - c\delta_0^2 C_1 T^{16} s \lambda^3] \widehat{\phi}^3 \xi^3 |z|^2 dx dy dt \\
& + \tau \mathbb{E} \int_{Q_T} s \lambda^2 \left[|z_x|^2 - \frac{\sigma}{(x+\varepsilon)^2} |z|^2 \right] \widehat{\psi}_x^2 \widehat{\phi} \xi dx dy dt + \frac{1}{2} \mathbb{E} \int_{Q_T} (s \lambda^2 \widehat{\psi}_x^2 - c C_1 s \lambda) \widehat{\phi} \xi |z_x|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s \lambda^2 (x+\varepsilon)^{2\gamma} \widehat{\psi}_x^2 \widehat{\phi} \xi |z_y|^2 dx dy dt. \tag{3.28}
\end{aligned}$$

Then noticing that $\tau < 3$ and choosing sufficiently large δ_0 such that

$$M_1 = \frac{3-\tau}{2} \delta_0^4 > 1,$$

and taking

$$\lambda > c(T^{23} + 1)2^{6+10\gamma} C_1^4 > \frac{2}{3-\tau} \delta_0^{-4} (C_3 + C_4 + C_4 T^{16} + c\delta_0^2 C_1 T^{16}), \quad s > 1,$$

we could obtain the desired estimate (3.15). \square

Lemma 3.4. *There exists positive constant $M_2 = \mu$ such that*

$$\mathbb{E} \int_{Q_T} (\{\cdot\}_x + \{\cdot\}_y) dx dy \geq -M_2 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\phi} \xi |z_x|^2 dy dt. \tag{3.29}$$

Proof. From the homogeneous Dirichlet boundary condition in (3.6), it follows that

$$\begin{cases} z_x(x, 0, t) = z_x(x, 1, t) = 0, & (x, t) \in I_x \times (0, T), \\ z_y(0, y, t) = z_y(1, y, t) = 0, & (y, t) \in I_y \times (0, T), \\ z_t(x, y, t) = 0, & (x, y, t) \in \Sigma_T. \end{cases} \tag{3.30}$$

Moreover, we easily see that

$$\begin{cases} \widehat{\varphi}_x(0, y, t) \leq 0, & \widehat{\varphi}_x(1, y, t) \leq 0, & (y, t) \in I_y \times (0, T), \\ \widehat{\varphi}_y(x, 0, t) \geq 0, & \widehat{\varphi}_y(x, 1, t) \leq 0, & (x, t) \in I_x \times (0, T). \end{cases} \tag{3.31}$$

Therefore, by (3.30) we have

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} (\{\cdot\}_x + \{\cdot\}_y) dx dy \\
= & -\mathbb{E} \int_0^T \int_{I_y} [s \widehat{\varphi}_x z_x^2]_{x=0}^{x=1} dy dt - \mathbb{E} \int_0^T \int_{I_x} [s(x+\varepsilon)^{4\gamma} \widehat{\varphi}_y z_y^2]_{y=0}^{y=1} dx dt. \tag{3.32}
\end{aligned}$$

Finally, by using (3.31) and (3.32), we obtain (3.29). \square

Lemma 3.5. *There exists positive constant*

$$M_3 = \frac{1}{2}(1-4\sigma)\delta_0^2 + c2^{4+6\gamma} C_1^2 + 2\sigma\delta_0^2$$

such that

$$\mathbb{E} \int_{Q_T} J dx dy \geq -M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt. \tag{3.33}$$

Proof. By using $(dz)^2 = \widehat{\theta}^2 |F_1^\varepsilon|^2 dt$ and Hardy inequality, we find that

$$\begin{aligned} & \mathbb{E} \int_{Q_T} \frac{\sigma}{(x+\varepsilon)^2} (dz)^2 dx dy \leq 4\sigma \mathbb{E} \int_{Q_T} \left| \left(\widehat{\theta} F_1^\varepsilon \right)_x \right|^2 dx dy dt \\ & \leq 4\sigma \mathbb{E} \int_{Q_T} \left(l_x^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 + 2l_x \widehat{\theta}^2 F_1^\varepsilon F_{1,x}^\varepsilon + \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 \right) dx dy dt. \end{aligned} \quad (3.34)$$

By (3.34) and

$$(dz_x)^2 = l_x^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dt + 2l_x \widehat{\theta}^2 F_1^\varepsilon F_{1,x}^\varepsilon dt + \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dt, \quad (3.35)$$

we further obtain

$$\begin{aligned} & \mathbb{E} \int_{Q_T} J dx dy \\ & \geq \mathbb{E} \int_{Q_T} l_x \widehat{\theta}^2 F_1^\varepsilon F_{1,x}^\varepsilon dx dy dt + \frac{1}{2} \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} (x+\varepsilon)^{2\gamma} l_y^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\ & \quad - 2\sigma \mathbb{E} \int_{Q_T} \left(l_x^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 + 2l_x \widehat{\theta}^2 F_1^\varepsilon F_{1,x}^\varepsilon + \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 \right) dx dy dt \\ & \geq -\frac{\varepsilon}{2} \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dx dy dt - \frac{1}{2\varepsilon} (1-4\sigma)^2 \mathbb{E} \int_{Q_T} l_x^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\ & \quad + \frac{1}{2} \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} (x+\varepsilon)^{2\gamma} l_y^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\ & \quad - 2\sigma \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dx dy dt - 2\sigma \mathbb{E} \int_{Q_T} l_x^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\ & \geq \left(\frac{1}{2} - \frac{\varepsilon}{2} - 2\sigma \right) \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |F_{1,x}^\varepsilon|^2 dx dy dt - C_5 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt, \end{aligned} \quad (3.36)$$

with

$$C_5 = \frac{1}{2\varepsilon} (1-4\sigma)^2 \delta_0^2 + c 2^{4+6\gamma} C_1^2 + 2\sigma \delta_0^2.$$

Taking $\varepsilon = 1 - 4\sigma > 0$ due to $0 \leq \sigma < \frac{1}{4}$, from (3.36) we deduce (3.33). \square

Combining Lemma 3.3-Lemma 3.5, we have the following result.

Lemma 3.6. *There exist positive constant*

$$M_4(\lambda) = c(T^{24} + 1) \delta_0^2 2^{4+6\gamma} \lambda^4 e^{c\|\psi\|_{C(\bar{T})}} \lambda C_1^2$$

such that

$$\begin{aligned} & \mathbb{E} \int_{Q_T} s \lambda^2 \xi \widehat{\theta}^2 |v_x^\varepsilon|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x+\varepsilon)^{2\gamma} \xi \widehat{\theta}^2 |v_y^\varepsilon|^2 dx dy dt \\ & \quad + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt \\ & \leq M_4(\lambda) \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt + M_4(\lambda) \mathbb{E} \int_{Q_T} s^2 \lambda^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\ & \quad + M_4(\lambda) \mathbb{E} \int_{\omega_T^{(2)}} \widehat{\theta}^2 [s \xi |v_x^\varepsilon|^2 + s(x+\varepsilon)^{2\gamma} \xi |v_y^\varepsilon|^2 + s^3 \xi^3 |v^\varepsilon|^2] dx dy dt \end{aligned} \quad (3.37)$$

for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma}C_1^4, \quad s \geq c(T^8 + 1)\delta_0^4 2^{8+12\gamma}C_1^4 e^{c\|\psi\|_{C(\bar{T})}}.$$

Proof. By substituting (3.15), (3.29) and (3.33) into (3.14), we find that

$$\begin{aligned} & \mathbb{E} \int_{Q_T} P_2 \widehat{\theta} \left[dv^\varepsilon + v_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon dt \right] dx dy \\ & \geq \frac{1}{2} \mathbb{E} \int_{Q_T} |P_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} |P|^2 dx dy dt + M_1 \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |z|^2 dx dy dt \\ & \quad + M_1 \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\phi} \xi |z_x|^2 dx dy dt + M_1 \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |z_y|^2 dx dy dt \\ & \quad - M_2 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\phi} \xi |z_x|^2 dy dt - M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \end{aligned} \quad (3.38)$$

with $z = \widehat{\theta} v^\varepsilon$.

In order to eliminate the boundary term, we introduce a cut-function $\chi \in C^2(\bar{I})$ such that

$$\begin{cases} \chi(x, y) = 0, & (x, y) \in \overline{\omega^{(1)}}, \\ 0 < \chi(x, y) < 1, & (x, y) \in \omega^{(2)} \setminus \overline{\omega^{(1)}}, \\ \chi(x, y) = 1, & (x, y) \in \overline{I \setminus \omega^{(2)}}. \end{cases} \quad (3.39)$$

Then $\tilde{v}^\varepsilon := \chi v^\varepsilon$, $\tilde{F}_1^\varepsilon := \chi F_1^\varepsilon$ satisfy

$$\begin{cases} d\tilde{v}^\varepsilon + \tilde{v}_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} \tilde{v}_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} \tilde{v}^\varepsilon dt = \tilde{f}_1 dt + \tilde{F}_1^\varepsilon dB(t), & (x, y, t) \in Q_T, \\ \tilde{v}^\varepsilon(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ \tilde{v}^\varepsilon(x, y, T) = \chi(x, y) v_T^\varepsilon(x, y), & (x, y) \in I, \end{cases} \quad (3.40)$$

where

$$\tilde{f}_1 = 2\chi_x v_x^\varepsilon + \chi_{xx} v^\varepsilon + (x + \varepsilon)^{2\gamma} (2\chi_y v_y^\varepsilon + \chi_{yy} v^\varepsilon) + \chi f_1.$$

Let $\tilde{z} = \widehat{\theta} \tilde{v}^\varepsilon$ and \tilde{P}_2, \tilde{P} denote the same expressions as P_2, P by replacing z with \tilde{z} . By the definition of χ , we obtain $\tilde{z}_x = 0$ on Γ_T . Then noting that $M_1 > 1$ and applying (3.38) to \tilde{v}^ε yields

$$\begin{aligned} & \mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \left[d\tilde{v}^\varepsilon + \tilde{v}_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} \tilde{v}_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} \tilde{v}^\varepsilon dt \right] dx dy \\ & \geq \frac{1}{2} \mathbb{E} \int_{Q_T} |\tilde{P}_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} |\tilde{P}|^2 dx dy dt \\ & \quad + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |\tilde{z}|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\phi} \xi |\tilde{z}_x|^2 dx dy dt \\ & \quad + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |\tilde{z}_y|^2 dx dy dt - M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |\tilde{F}_1^\varepsilon|^2 dx dy dt, \end{aligned} \quad (3.41)$$

which implies

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\phi} \xi |\tilde{z}_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |\tilde{z}_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |\tilde{z}|^2 dx dy dt + \mathbb{E} \int_{Q_T} |\tilde{P}_2|^2 dx dy dt \\
& \leq 2 \mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \left[d\tilde{v}^\varepsilon + \tilde{v}_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} \tilde{v}_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} \tilde{v}^\varepsilon dt \right] dx dy \\
& + 2 \mathbb{E} \int_{Q_T} |\tilde{P}|^2 dx dy dt + 2M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt. \tag{3.42}
\end{aligned}$$

Using the equation of \tilde{v}^ε , $\text{Supp}(\chi_x), \text{Supp}(\chi_y) \subset \omega^{(2)}$ and noticing that

$$\mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \tilde{F}_1^\varepsilon dx dy dB(t) = 0,$$

we see that

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \left[d\tilde{v}^\varepsilon + \tilde{v}_{xx}^\varepsilon dt + (x + \varepsilon)^{2\gamma} \tilde{v}_{yy}^\varepsilon dt + \frac{\sigma}{(x + \varepsilon)^2} \tilde{v}^\varepsilon dt \right] dx dy \\
& = \mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \tilde{f}_1 dx dy dt + \mathbb{E} \int_{Q_T} \tilde{P}_2 \widehat{\theta} \tilde{F}_1^\varepsilon dx dy dB(t) \\
& \leq \frac{1}{2} \mathbb{E} \int_{Q_T} |\tilde{P}_2|^2 dx dy dt + c \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt \\
& + c2^{4\gamma} \mathbb{E} \int_{\omega_T^{(2)}} \widehat{\theta}^2 [|v_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2 + |v^\varepsilon|^2] dx dy dt. \tag{3.43}
\end{aligned}$$

From (3.2), (3.4) and (3.16), we obtain

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} |\tilde{P}|^2 dx dy dt = \mathbb{E} \int_{Q_T} |(\tau - 1)l_{xx}\tilde{z} - l_t\tilde{z} - (x + \varepsilon)^{2\gamma}l_{yy}\tilde{z}|^2 dx dy dt \\
& \leq C_6 \mathbb{E} \int_{Q_T} s^2 \lambda^4 \widehat{\phi}^2 \xi^3 |\tilde{z}|^2 dx dy dt + ce^{c\|\psi\|_{C(\bar{T})}} \lambda T^2 \mathbb{E} \int_{Q_T} s^2 \xi^{\frac{5}{2}} |\tilde{z}|^2 dx dy dt, \tag{3.44}
\end{aligned}$$

where

$$C_6 = c2^{4+6\gamma} \delta_0^4 C_1^2 T^8 + c2^{8+12\gamma} C_1^4 T^8.$$

Substituting (3.43) and (3.44) into (3.42) and choosing

$$s \geq c(T^8 + 1) \delta_0^4 2^{8+12\gamma} C_1^4 e^{c\|\psi\|_{C(\bar{T})}} \lambda > 4C_6 + ce^{c\|\psi\|_{C(\bar{T})}} \lambda T^6,$$

we obtain that

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \xi |\tilde{z}_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |\tilde{z}_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |\tilde{z}|^2 dx dy dt \\
& \leq c \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt + 2M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\
& + c2^{4\gamma} \mathbb{E} \int_{\omega_T^{(2)}} \widehat{\theta}^2 [|v_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2 + |v^\varepsilon|^2] dx dy dt. \tag{3.45}
\end{aligned}$$

Using $\tilde{z} = z$ on $I \setminus \overline{\omega^{(2)}}$, we further have

$$\begin{aligned}
& \mathbb{E} \int_{Q_T \setminus \omega_T^{(2)}} s \lambda^2 \widehat{\phi} \xi |z_x|^2 dx dy dt + \mathbb{E} \int_{Q_T \setminus \omega_T^{(2)}} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\phi} \xi |z_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T \setminus \omega_T^{(2)}} s^3 \lambda^4 \widehat{\phi}^3 \xi^3 |z|^2 dx dy dt \\
& \leq c \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt + 2M_3 \mathbb{E} \int_{Q_T} s^2 \lambda^2 \widehat{\phi}^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\
& + c2^{4\gamma} \mathbb{E} \int_{\omega_T^{(2)}} \widehat{\theta}^2 [|v_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2 + |v^\varepsilon|^2] dx dy dt. \tag{3.46}
\end{aligned}$$

Finally using $z = \widehat{\theta} v^\varepsilon$ and going back to v^ε , we obtain the desired inequality (3.37) with

$$\begin{aligned}
M_4(\lambda) &= c(T^{24} + 1) \delta_0^2 2^{4+6\gamma} \lambda^4 e^{c\|\psi\|_{C(\bar{T})}} \lambda C_1^2 \\
&> c2^{4\gamma} T^{24} + c2^{4\gamma} T^8 + c(\delta_0^2 + 2^{4+6\gamma} C_1^2 + 1) \lambda^4 e^{c\|\psi\|_{C(\bar{T})}} \lambda.
\end{aligned}$$

This completes the proof of this lemma. \square

In order to prove Theorem 3.1, we also need the following the Cacciopoli inequality.

Lemma 3.7. *Let $\gamma > 0$, $0 \leq \sigma < \frac{1}{4}$, $v_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))$, $f_1 \in L^2_{\mathcal{F}}(0, T; L^2(I))$. Then there exists positive constant*

$$M_5(\lambda) = c(T^{16} + 1) \delta_0^2 2^{4+6\gamma} \lambda^2 e^{c\|\psi\|_{C(\bar{T})}} \lambda C_1^3$$

such that the solution $(v^\varepsilon, F_1^\varepsilon) \in \mathcal{H}_T \times L^2_{\mathcal{F}}(0, T; L^2(I))$ of the backward stochastic Grushin equation (3.6) satisfies

$$\begin{aligned}
& \mathbb{E} \int_{\omega_T^{(2)}} \widehat{\theta}^2 [s \xi |v_x^\varepsilon|^2 + s(x + \varepsilon)^{2\gamma} \xi |v_y^\varepsilon|^2 + s \xi |F_1^\varepsilon|^2] dx dy dt \\
& \leq M_5(\lambda) \mathbb{E} \int_{\omega_T} s^3 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt + M_5(\lambda) \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt. \tag{3.47}
\end{aligned}$$

Proof. We choose a cut-function $\zeta \in C^2(\bar{I})$ such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ in $\omega^{(2)}$, $\zeta = 0$ in $I \setminus \omega$. By Itô formula, we have

$$d(\xi \widehat{\theta}^2 |v^\varepsilon|^2) = (\xi_t \widehat{\theta}^2 + 2\xi \widehat{\theta} \widehat{\theta}_t) |v^\varepsilon|^2 dt + 2\xi \widehat{\theta}^2 v^\varepsilon dv^\varepsilon + \xi \widehat{\theta}^2 (dv^\varepsilon)^2. \tag{3.48}$$

Together with the equation of v^ε in (3.6), we find that

$$\begin{aligned}
0 &= \mathbb{E} \int_{Q_T} \zeta^2 d(\xi \widehat{\theta}^2 |v^\varepsilon|^2) dx dy \\
&= \mathbb{E} \int_{Q_T} \zeta^2 \left[(\xi_t \widehat{\theta}^2 + 2\xi \widehat{\theta} \widehat{\theta}_t) |v^\varepsilon|^2 dt + 2\xi \widehat{\theta}^2 v^\varepsilon dv^\varepsilon + \xi \widehat{\theta}^2 (dv^\varepsilon)^2 \right] dx dy \\
&= \mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 \left[(\xi^{-1} \xi_t + 2s \widehat{\varphi}_t) |v^\varepsilon|^2 + 2v^\varepsilon \left(-v_{xx}^\varepsilon - (x + \varepsilon)^{2\gamma} v_{yy}^\varepsilon \right. \right. \\
&\quad \left. \left. - \frac{\sigma}{(x + \varepsilon)^2} v^\varepsilon + f_1 \right) + |F_1^\varepsilon|^2 \right] dx dy dt
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 \left[(\xi^{-1} \xi_t + 2s \widehat{\varphi}_t) |v^\varepsilon|^2 + 2|v_x^\varepsilon|^2 + 2(x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2 - \frac{2\sigma}{(x + \varepsilon)^2} |v^\varepsilon|^2 \right] dx dy dt \\
&\quad - \mathbb{E} \int_{Q_T} \xi \left(\zeta^2 \widehat{\theta}^2 \right)_{xx} |v^\varepsilon|^2 dx dy dt - \mathbb{E} \int_{Q_T} (x + \varepsilon)^{2\gamma} \xi \left(\zeta^2 \widehat{\theta}^2 \right)_{yy} |v^\varepsilon|^2 dx dy dt \\
&\quad + \mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 (2f_1 v^\varepsilon + |F_1^\varepsilon|^2) dx dy dt, \tag{3.49}
\end{aligned}$$

which implies

$$\begin{aligned}
&2\mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 [|v_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2] dx dy dt + \mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\
&\leq \mathbb{E} \int_{Q_T} \xi \left[-\zeta^2 \xi^{-1} \xi_t \widehat{\theta}^2 - 2s \zeta^2 \widehat{\varphi}_t \widehat{\theta}^2 + \left(\zeta^2 \widehat{\theta}^2 \right)_{xx} + (x + \varepsilon)^{2\gamma} \left(\zeta^2 \widehat{\theta}^2 \right)_{yy} + s \zeta^2 \xi \widehat{\theta}^2 \right] |v^\varepsilon|^2 dx dy dt \\
&\quad + 2\sigma \mathbb{E} \int_{Q_T} \frac{1}{(x + \varepsilon)^2} \zeta^2 \xi \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt + \mathbb{E} \int_{Q_T} s^{-1} \zeta^2 \widehat{\theta}^2 |f_1|^2 dx dy dt. \tag{3.50}
\end{aligned}$$

On the other hand, by Hardy inequality (1.4), we have

$$\begin{aligned}
&\sigma \mathbb{E} \int_{Q_T} \frac{1}{(x + \varepsilon)^2} \zeta^2 \xi \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt \leq 4\sigma \mathbb{E} \int_{Q_T} \xi \left| \left(\zeta \widehat{\theta} v^\varepsilon \right)_x \right|^2 dx dy dt \\
&\leq 4\sigma \mathbb{E} \int_{Q_T} \zeta^2 \xi \widehat{\theta}^2 |v_x^\varepsilon|^2 dx dy dt + C_7(\lambda) \mathbb{E} \int_{Q_T} (\zeta^2 + \zeta_x^2) s^2 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt, \tag{3.51}
\end{aligned}$$

where

$$C_7(\lambda) = c\sigma\lambda^2\delta_0^2 e^{c\|\psi\|_{C(\bar{T})}\lambda}.$$

Therefore, by the definition of ζ and

$$\begin{aligned}
&\xi \left| -\zeta^2 \xi^{-1} \xi_t \widehat{\theta}^2 - 2s \zeta^2 \widehat{\varphi}_t \widehat{\theta}^2 + \left(\zeta^2 \widehat{\theta}^2 \right)_{xx} + (x + \varepsilon)^{2\gamma} \left(\zeta^2 \widehat{\theta}^2 \right)_{yy} + s \zeta^2 \xi \widehat{\theta}^2 \right| \\
&\leq C_8(\lambda) s^2 (\zeta^2 + \zeta_x^2 + \zeta_y^2 + \zeta_{xx}^2 + \zeta_{yy}^2) \xi^3 \widehat{\theta}^2,
\end{aligned}$$

where

$$\begin{aligned}
C_8(\lambda) &= c(T^{15} + T^{16} + 2^{2\gamma} T^{16} + e^{c\|\psi\|_{C(\bar{T})}\lambda} T^7) + c(T^8 + 1)(\delta_0^2 + 2^{4+6\gamma} C_1^2) \lambda^2 e^{c\|\psi\|_{C(\bar{T})}\lambda} \\
&< c(T^{16} + 1) \delta_0^2 2^{4+6\gamma} \lambda^2 e^{c\|\psi\|_{C(\bar{T})}\lambda} C_1^2,
\end{aligned}$$

we deduce from (3.50) and (3.51) that

$$\begin{aligned}
&\mathbb{E} \int_{\omega_T^{(2)}} \xi \widehat{\theta}^2 [(1 - 4\sigma) |v_x^\varepsilon|^2 + (x + \varepsilon)^{2\gamma} |v_y^\varepsilon|^2] dx dy dt \\
&\leq \left(C_7(\lambda) + \frac{1}{2} C_8(\lambda) \right) \mathbb{E} \int_{\omega_T} s^2 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt + \mathbb{E} \int_{Q_T} s^{-1} \widehat{\theta}^2 |f_1|^2 dx dy dt. \tag{3.52}
\end{aligned}$$

Noticing that $0 \leq \sigma < \frac{1}{4}$ and multiplying (3.52) by s , choosing

$$M_5(\lambda) = c(T^{16} + 1) \delta_0^2 2^{4+6\gamma} \lambda^2 e^{c\|\psi\|_{C(\bar{T})}\lambda} C_1^3 > \frac{1}{1 - 4\sigma} \left(C_7(\lambda) + \frac{1}{2} C_8(\lambda) \right),$$

we immediately obtain (3.47). \square

Now we prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \xi \widehat{\theta}^2 |v_x^\varepsilon|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \xi \widehat{\theta}^2 |v_y^\varepsilon|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt \\
& \leq M_4(\lambda)(M_5(\lambda) + 1) \mathbb{E} \int_{Q_T} \widehat{\theta}^2 |f_1|^2 dx dy dt + M_4(\lambda) \mathbb{E} \int_{Q_T} s^2 \lambda^2 \xi^2 \widehat{\theta}^2 |F_1^\varepsilon|^2 dx dy dt \\
& + M_4(\lambda)(M_5(\lambda) + 1) \mathbb{E} \int_{\omega_T} s^3 \xi^3 \widehat{\theta}^2 |v^\varepsilon|^2 dx dy dt
\end{aligned} \tag{3.53}$$

for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma}C_1^4, \quad s \geq c(T^8 + 1)\delta_0^4 2^{8+12\gamma}C_1^4 e^{c\|\psi\|_{C(\bar{T})\lambda}}.$$

Since $v_T^\varepsilon \rightarrow v_T$ in $L(\Omega, \mathcal{F}_T, \mathbb{P}; L^2(I))$, by a similar argument as the end proof of Theorem 2.2, together with the standard energy estimate for the backward stochastic parabolic equation, we could prove $(v^\varepsilon, F_1^\varepsilon) \rightarrow (v, F_1)$ in $\mathcal{G}_T \times L^2_{\mathcal{F}}(0, T; L^2(I))$ as $\varepsilon \rightarrow 0$. Therefore, taking

$$C(\lambda) = c(T^{40} + 1)\delta_0^4 2^{8+12\gamma} \lambda^2 e^{c\|\psi\|_{C(\bar{T})\lambda}} C_1^5 > M_4(\lambda)(M_5(\lambda) + 1)\lambda^{-2}, \tag{3.54}$$

we can obtain (3.5). This completes the proof of Theorem 3.1. \square

3.2 Carleman estimate for forward stochastic Grushin equation with regular weight function

In this subsection, we introduce a new regular weight function to establish the other Carleman estimate for the forward stochastic Grushin equation with singular potential

$$\begin{cases} dw - w_{xx} dt - x^{2\gamma} w_{yy} dt - \frac{\sigma}{x^2} w dt = f_2 dt + F_2 dB(t), & (x, y, t) \in Q_T, \\ w(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ w(x, y, 0) = 0, & (x, y) \in I, \end{cases} \tag{3.55}$$

with $f_2 \in L^2_{\mathcal{F}}(0, T; L^2(I))$, $F_2 \in L^2_{\mathcal{F}}(0, T; H^1(I))$. The regular weight function allows us to put the random source on the left-hand side of this Carleman estimate. Based on such a Carleman estimate we can obtain the uniqueness for our inverse problem.

We set

$$\Phi(x, y, t) = e^{\lambda \varrho(x, y, t)}, \quad \Theta(x, y, t) = e^{s \Phi(x, y, t)} \tag{3.56}$$

with

$$\varrho(x, y, t) = x^{2+2\gamma} y(1-y) - \mu x - (\lambda - t)^2 + 2\lambda^2. \tag{3.57}$$

Here μ is the same as the one in Section 3.1. We easily see that $\varrho > 0$ in Q_T if we choose λ suitable large.

Our main result in this subsection is the following Carleman estimate for (3.55) with regular weight function.

Theorem 3.8. *Let $\gamma > 0$, $0 \leq \sigma < \frac{1}{4}$, $f_2 \in L^2_{\mathcal{F}}(0, T; L^2(I))$, $F_2 \in L^2_{\mathcal{F}}(0, T; H^1(I))$. Then there exist constants λ_2 , s_2 and C such that*

$$\begin{aligned} & \mathbb{E} \int_{Q_T} s\lambda^2 \Phi \Theta^2 |w_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s\lambda^2 \Phi \Theta^2 x^{2\gamma} |w_y|^2 dx dy dt \\ & + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \Phi^3 \Theta^2 |w|^2 dx dy dt + \mathbb{E} \int_{Q_T} s\lambda \Phi \Theta^2 |F_2|^2 dx dy dt \\ \leq & C \mathbb{E} \int_{Q_T} \Theta^2 |f_2|^2 dx dy dt + C \mathbb{E} \int_{Q_T} s\Phi \Theta^2 |\nabla F_2|^2 dx dy dt \\ & + C \mathbb{E} \int_I s^2 \lambda^2 \Phi^2(T) \Theta^2(T) w^2(T) dx dy + C \mathbb{E} \int_{\Gamma_T} s\lambda \Phi \Theta^2 |w_x|^2 dy dt \end{aligned} \quad (3.58)$$

for all sufficiently large $\lambda \geq \lambda_2$ and $s \geq s_2(\lambda)$ and all $w \in \mathcal{G}_T$ satisfies (3.55), where

$$\begin{aligned} \lambda_2 &= c(T^{23} + 1)2^{6+10\gamma} C_1^4, & s_2 &= c\delta_0^4 2^{8+12\gamma} C_1^4, \\ C &= c(1 + \mu)\delta_0^2 2^{4+8\gamma} C_1^2. \end{aligned}$$

Remark 3.2. We could not eliminate the term of ∇F_2 on the right-hand side of (3.58). Based on this reason, the random source H to be determined in (1.3) does not depend on spatial variables.

Remark 3.3. In form we do not use λ in the proof of the null controllability. So we hide the second large parameter λ in Theorem 3.1. In fact, λ lies in s and C in (3.5). However, λ plays a very important role in the proof of the stability of determining the random source H . Therefore we have to separate λ from constant C .

We still transfer to consider an approximate version of (3.55):

$$\begin{cases} dw^\varepsilon - w^\varepsilon_{xx} dt - (x + \varepsilon)^{2\gamma} w^\varepsilon_{yy} dt - \frac{\sigma}{(x+\varepsilon)^2} w^\varepsilon dt = f_2 dt + F_2 dB(t), & (x, y, t) \in Q_T, \\ w^\varepsilon(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ w^\varepsilon(x, y, 0) = 0, & (x, y) \in I, \end{cases} \quad (3.59)$$

where $0 < \varepsilon < 1$. Set

$$\widehat{\Phi}(x, y, t) = \Phi(x + \varepsilon, y, t), \quad \widehat{\varrho}(x, y, t) = \varrho(x + \varepsilon, y, t), \quad \widehat{\Theta}(x, y, t) = \Theta(x + \varepsilon, y, t).$$

We first give a weighted identity for the approximate problem (3.59).

Lemma 3.9. *Let τ be a constant such that $2 < \tau < 3$. Assume that w^ε is an $H^2(\mathbb{R}^2)$ -valued continuous semimartingale. Set $L = s\widehat{\Phi}$, $Z = \widehat{\Theta}w^\varepsilon$ and*

$$\begin{aligned} Q_1 &= dZ + 2L_x Z_x dt + 2(x + \varepsilon)^{2\gamma} L_y Z_y dt + \tau L_{xx} Z dt, \\ Q_2 &= -L_t Z - Z_{xx} - (x + \varepsilon)^{2\gamma} Z_{yy} - L_x^2 Z - (x + \varepsilon)^{2\gamma} L_y^2 Z - \frac{\sigma}{(x + \varepsilon)^2} Z, \\ Q &= -(\tau - 1)L_{xx} Z + (x + \varepsilon)^{2\gamma} L_{yy} Z. \end{aligned}$$

Then for a.e. $(x, y) \in \mathbb{R}^2$, it holds that

$$\begin{aligned} & Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] \\ & = |Q_2|^2 dt + Q_2 Q dt + \sum_{i=1}^5 \overline{X}_i dt + d\overline{Y} + \overline{\{\cdot\}}_x + \overline{\{\cdot\}}_y + \overline{J}, \quad \mathbb{P} - a.s., \end{aligned} \quad (3.60)$$

where

$$\begin{aligned} \overline{X}_1 &= [(\tau + 1)L_{xx} - (x + \varepsilon)^{2\gamma} L_{yy}] Z_x^2, \\ \overline{X}_2 &= [-2\gamma(x + \varepsilon)^{2\gamma-1} L_x + (\tau - 1)(x + \varepsilon)^{2\gamma} L_{xx} + (x + \varepsilon)^{4\gamma} L_{yy}] Z_y^2, \\ \overline{X}_3 &= 4[\gamma(x + \varepsilon)^{2\gamma-1} L_y + (x + \varepsilon)^{2\gamma} L_{xy}] Z_x Z_y, \\ \overline{X}_4 &= [(3 - \tau)L_x^2 L_{xx} + 2\gamma(x + \varepsilon)^{2\gamma-1} L_x L_y^2 + 3(x + \varepsilon)^{4\gamma} L_y^2 L_{yy}] Z^2 \\ & \quad + (x + \varepsilon)^{2\gamma} [4L_x L_y L_{xy} + L_x^2 L_{yy} + (1 - \tau)L_{xx} L_y^2] Z^2 \\ & \quad + \left[(1 - \tau) \frac{\sigma}{(x + \varepsilon)^2} L_{xx} - \frac{2\sigma}{(x + \varepsilon)^3} L_x + \frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} L_{yy} \right] Z^2, \\ \overline{X}_5 &= \left[\frac{1}{2} L_{tt} + (1 - \tau)L_{xx} L_t + 2L_x L_{xt} + 2(x + \varepsilon)^{2\gamma} L_y L_{yt} + (x + \varepsilon)^{2\gamma} L_{yy} L_t \right. \\ & \quad \left. - \frac{\tau}{2} L_{xxxx} - \frac{\tau}{2} (x + \varepsilon)^{2\gamma} L_{xxyy} \right] Z^2, \\ \overline{Y} &= \frac{1}{2} Z_x^2 + \frac{1}{2} (x + \varepsilon)^{2\gamma} Z_y^2 - \frac{1}{2} \left[L_t + L_x^2 + (x + \varepsilon)^{2\gamma} L_y^2 + \frac{\sigma}{(x + \varepsilon)^2} \right] Z^2, \\ \overline{\{\cdot\}} &= -Z_x dZ + \left[-L_x L_t Z^2 - L_x Z_x^2 + (x + \varepsilon)^{2\gamma} L_x Z_y^2 - L_x^3 Z^2 - (x + \varepsilon)^{2\gamma} L_x L_y^2 Z^2 \right. \\ & \quad \left. - \frac{\sigma}{(x + \varepsilon)^2} L_x Z^2 - 2(x + \varepsilon)^{2\gamma} L_y Z_x Z_y - \tau L_{xx} Z Z_x + \frac{\tau}{2} L_{xxx} Z^2 \right] dt, \\ \overline{\{\cdot\}} &= -(x + \varepsilon)^{2\gamma} Z_y dZ + [-(x + \varepsilon)^{2\gamma} L_y L_t Z^2 - 2(x + \varepsilon)^{2\gamma} L_x Z_x Z_y + (x + \varepsilon)^{2\gamma} L_y Z_x^2 \\ & \quad - (x + \varepsilon)^{4\gamma} L_y Z_y^2 - (x + \varepsilon)^{2\gamma} L_x^2 L_y Z^2 - (x + \varepsilon)^{4\gamma} L_y^3 Z^2 - \frac{\sigma}{(x + \varepsilon)^{2-2\gamma}} L_y Z^2 \\ & \quad - \tau(x + \varepsilon)^{2\gamma} L_{xx} Z Z_y + \frac{\tau}{2} (x + \varepsilon)^{2\gamma} L_{xxy} Z^2] dt, \\ \overline{J} &= -\frac{1}{2} (dZ_x)^2 - \frac{1}{2} (x + \varepsilon)^{2\gamma} (dZ_y)^2 + \frac{1}{2} \left[L_t + L_x^2 + (x + \varepsilon)^{2\gamma} L_y^2 + \frac{\sigma}{(x + \varepsilon)^2} \right] (dZ)^2. \end{aligned}$$

Proof. Notice that $\widehat{\Theta} = e^L$, $L = s\widehat{\Phi}$ and $Z = \widehat{\Theta}w^\varepsilon$. Then we have

$$\widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] = Q_1 + (Q_2 + Q) dt.$$

Hence

$$Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] = Q_1 Q_2 + |Q_2|^2 dt + Q_2 Q dt. \quad (3.61)$$

We only need to deal with $-L_t Z Q_1$ in $Q_1 Q_2$. The calculations of the other terms are similar to the ones in $P_1 P_2$. Therefore, by using a similar argument similar to Lemma 3.2,

together with

$$\begin{aligned}
-L_t Z Q_1 &= -L_t Z \left[dZ + 2L_x Z_x dt + 2(x + \varepsilon)^{2\gamma} L_y Z_y dt + \tau L_{xx} Z dt \right] \\
&= -\frac{1}{2} d(L_t Z^2) + \frac{1}{2} L_{tt} Z^2 dt + \frac{1}{2} L_t (dZ)^2 - (L_x L_t Z^2)_x dt + (L_{xx} L_t + L_x L_{xt}) Z^2 dt \\
&\quad - \left[(x + \varepsilon)^{2\gamma} L_y L_t Z^2 \right]_y dt + (x + \varepsilon)^{2\gamma} (L_{yy} L_t + L_y L_{yt}) Z^2 dt - \tau L_{xx} L_t Z^2 dt,
\end{aligned}$$

we obtain (3.60). \square

Now, integrating both sides of (3.60) in Q_T , taking mathematical expectation in Ω , we obtain

$$\begin{aligned}
&\mathbb{E} \int_{Q_T} Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] dx dy \\
&\geq \frac{1}{2} \mathbb{E} \int_{Q_T} |Q_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} |Q|^2 dx dy dt + \sum_{i=1}^5 \mathbb{E} \int_{Q_T} \overline{X}_i dx dy dt \\
&\quad + \mathbb{E} \int_{Q_T} d\overline{Y} dx dy + \mathbb{E} \int_{Q_T} (\overline{\{\cdot\}}_x + \overline{\{\cdot\}}_y) dx dy + \mathbb{E} \int_{Q_T} \overline{J} dx dy. \tag{3.62}
\end{aligned}$$

In the following we estimate the last four terms on the right-hand side of (3.62).

Lemma 3.10. *There exist positive constant*

$$\overline{M}_1 = \frac{3 - \tau}{2} \delta_0^4 > 1$$

such that

$$\begin{aligned}
\sum_{i=1}^5 \mathbb{E} \int_{Q_T} \overline{X}_i dx dy dt &\geq \overline{M}_1 \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\Phi}^3 |Z|^2 dx dy dt + \overline{M}_1 \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\Phi} |Z_x|^2 dx dy dt \\
&\quad + \overline{M}_1 \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\Phi} |Z_y|^2 dx dy dt \tag{3.63}
\end{aligned}$$

for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma} C_1^4, \quad s \geq 1.$$

Proof. For regular weight function $\widehat{\varrho}(x, y, t) = (x + \varepsilon)^{2+2\gamma} y(1 - y) - \mu(x + \varepsilon) - (\lambda - t)^2 + 2\lambda^2$, we have the following properties of $\widehat{\varrho}$:

$$\begin{cases} \widehat{\varrho}_t = 2(\lambda - t), & \widehat{\varrho}_{tt} = -2, & \widehat{\varrho}_{xt} = \widehat{\varrho}_{yt} = 0, \\ \widehat{\varrho}_x < -\delta_0, & \widehat{\varrho}_x \widehat{\varrho}_{xxx} \leq 0, & |\widehat{\varrho}_{xxx}| \leq C_1(x + \varepsilon)^{-2}, & |\widehat{\varrho}_y| + |\widehat{\varrho}_{yy}| \leq C_1(x + \varepsilon)^{2+2\gamma}, \\ |\widehat{\varrho}_{xy}| + |\widehat{\varrho}_{xx}| + |\widehat{\varrho}_{yy}| + |\widehat{\varrho}_{xyy}| + |\widehat{\varrho}_{xyx}| + |\widehat{\varrho}_{xyy}| \leq C_1(x + \varepsilon)^{2\gamma}. \end{cases} \tag{3.64}$$

Then, by a similar process to Lemma 3.3, we could obtain (3.63) for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma} C_1^4, \quad s \geq 1.$$

\square

Lemma 3.11. *There exists constant*

$$\overline{M}_2 = c(1 + \delta_0^2)2^{4+6\gamma}C_1^2$$

such that

$$\mathbb{E} \int_{Q_T} d\overline{Y} dx dy \geq -\overline{M}_2 \mathbb{E} \int_I s^2 \lambda^2 \widehat{\Phi}^2(T) Z^2(T) dx dy. \quad (3.65)$$

Proof. By $Z|_{t=0} = 0$, \mathbb{P} -a.s. in I , we have

$$\begin{aligned} & \mathbb{E} \int_{Q_T} d\overline{Y} dx dy \\ &= \frac{1}{2} \mathbb{E} \int_I [|Z_x(T)|^2 + (x + \varepsilon)^{2\gamma} |Z_y(T)|^2] dx dy - \frac{1}{2} \mathbb{E} \int_I \left[L_t(T) + L_x^2(T) + (x + \varepsilon)^{2\gamma} L_y^2(T) \right. \\ & \quad \left. + \frac{\sigma}{(x + \varepsilon)^2} |Z(T)|^2 \right] dx dy \\ & \geq \frac{1}{2} \mathbb{E} \int_I \left[|Z_x(T)|^2 - \frac{\sigma}{(x + \varepsilon)^2} |Z(T)|^2 \right] dx dy - \overline{M}_2 \mathbb{E} \int_I s^2 \lambda^2 \widehat{\Phi}^2(T) |Z(T)|^2 dx dy, \end{aligned} \quad (3.66)$$

where

$$\overline{M}_2 = c(1 + \delta_0^2)2^{4+6\gamma}C_1^2.$$

Together with $0 \leq \sigma < \frac{1}{4}$, (3.66) implies (3.65). \square

Lemma 3.12. *There exists constant $\overline{M}_3 = \mu$ such that*

$$\mathbb{E} \int_{Q_T} (\overline{\{\cdot\}}_x + \overline{\{\cdot\}}_y) dx dy \geq -\overline{M}_3 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\Phi} |Z_x|^2 dy dt. \quad (3.67)$$

Proof. Since $\widehat{\Phi}$ has the same property (3.31) as $\widehat{\varphi}$ on the boundary of I . Therefore we immediately obtain the estimate (3.67) for boundary term on the right-hand side of (3.62). \square

Lemma 3.13. *There exists positive constant*

$$\overline{M}_4 = \delta_0^2 + 2^{4+8\gamma}C_1^2$$

such that

$$\mathbb{E} \int_{Q_T} \overline{J} dx dy \geq \mathbb{E} \int_{Q_T} s \lambda \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt - \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 |\nabla F_2|^2 dx dy dt \quad (3.68)$$

for all

$$\lambda > 2(T + 1).$$

Proof. It is easily to see that

$$\begin{cases} (dZ)^2 = \widehat{\Theta}^2 |F_2|^2 dt, \\ (dZ_x)^2 = L_x^2 \widehat{\Theta}^2 |F_2|^2 dt + 2L_x \widehat{\Theta}^2 F_2 F_{2,x} dt + \widehat{\Theta}^2 |F_{2,x}|^2 dt, \\ (dZ_y)^2 = L_y^2 \widehat{\Theta}^2 |F_2|^2 dt + 2L_y \widehat{\Theta}^2 F_2 F_{2,y} dt + \widehat{\Theta}^2 |F_{2,y}|^2 dt. \end{cases}$$

Therefore, we have

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} \bar{J} dx dy \\
&= \frac{1}{2} \mathbb{E} \int_{Q_T} \left[L_t + \frac{\sigma}{(x+\varepsilon)^2} \right] \widehat{\Theta}^2 |F_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} \widehat{\Theta}^2 [|F_{2,x}|^2 + (x+\varepsilon)^{2\gamma} |F_{2,y}|^2] dx dy dt \\
&\quad - \mathbb{E} \int_{Q_T} [L_x \widehat{\Theta}^2 F_2 F_{2,x} + (x+\varepsilon)^{2\gamma} L_y \widehat{\Theta}^2 F_2 F_{2,y}] dx dy dt. \tag{3.69}
\end{aligned}$$

By $\widehat{\varrho}_t = 2(\lambda - t)$ and $0 \leq \sigma < \frac{1}{4}$, we have

$$\frac{1}{2} \mathbb{E} \int_{Q_T} \left[L_t + \frac{\sigma}{(x+\varepsilon)^2} \right] \widehat{\Theta}^2 |F_2|^2 dx dy dt \geq \mathbb{E} \int_{Q_T} s\lambda(\lambda - T) \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt. \tag{3.70}$$

On the other hand, there exists a positive constant

$$\overline{M}_4 = \delta_0^2 + 2^{4+8\gamma} C_1^2$$

such that

$$\begin{aligned}
& - \mathbb{E} \int_{Q_T} [L_x \widehat{\Theta}^2 F_2 F_{2,x} + (x+\varepsilon)^{2\gamma} L_y \widehat{\Theta}^2 F_2 F_{2,y}] dx dy dt \\
&= - \mathbb{E} \int_{Q_T} s\lambda [\widehat{\varrho}_x F_2 F_{2,x} + (x+\varepsilon)^{2\gamma} \widehat{\varrho}_y F_2 F_{2,y}] \widehat{\Phi} \widehat{\Theta}^2 dx dy dt \\
&\geq - \frac{1}{2} \mathbb{E} \int_{Q_T} s\lambda^2 \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt - \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 (|F_{2,x}|^2 + |F_{2,y}|^2) dx dy dt. \tag{3.71}
\end{aligned}$$

Therefore, from (3.70) and (3.71) we deduce that

$$\begin{aligned}
\mathbb{E} \int_{Q_T} \bar{J} dx dy &\geq \mathbb{E} \int_{Q_T} s\lambda \left[\frac{1}{2} \lambda - T \right] \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt \\
&\quad - \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 (|F_{2,x}|^2 + |F_{2,y}|^2) dx dy dt. \tag{3.72}
\end{aligned}$$

Taking

$$\lambda > 2(T + 1),$$

then we obtain (3.68). \square

Now we prove Theorem 3.8.

Proof of Theorem 3.8. Substituting (3.63), (3.65), (3.67) and (3.68) into (3.62) and using $\overline{M}_1 > 1$, we find that

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x+\varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x+\varepsilon)^2} w^\varepsilon dt \right] dx dy \\
&\geq \frac{1}{2} \mathbb{E} \int_{Q_T} |Q_2|^2 dx dy dt - \frac{1}{2} \mathbb{E} \int_{Q_T} |Q|^2 dx dy dt \\
&\quad + \overline{M}_1 \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\Phi}^3 |Z|^2 dx dy dt + \overline{M}_1 \mathbb{E} \int_{Q_T} s\lambda^2 \widehat{\Phi} |Z_x|^2 dx dy dt \\
&\quad + \overline{M}_1 \mathbb{E} \int_{Q_T} s\lambda^2 (x+\varepsilon)^{2\gamma} \widehat{\Phi} |Z_y|^2 dx dy dt + \mathbb{E} \int_{Q_T} s\lambda \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt
\end{aligned}$$

$$\begin{aligned}
& - \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 |\nabla F_2|^2 dx dy dt - \overline{M}_2 \mathbb{E} \int_I s^2 \lambda^2 \widehat{\Phi}^2(T) Z^2(T) dx dy \\
& - \overline{M}_3 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\Phi} |Z_x|^2 dy dt,
\end{aligned} \tag{3.73}$$

which implies

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\Phi} |Z_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\Phi} |Z_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\Phi}^3 |Z|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt + \frac{1}{2} \mathbb{E} \int_{Q_T} |Q_2|^2 dx dy dt \\
\leq & \mathbb{E} \int_{Q_T} Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] dx dy \\
& + \frac{1}{2} \mathbb{E} \int_{Q_T} |Q|^2 dx dy dt + \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 |\nabla F_2|^2 dx dy dt + \overline{M}_2 \mathbb{E} \int_I s^2 \lambda^2 \widehat{\Phi}^2(T) Z^2(T) dx dy \\
& + \overline{M}_3 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\Phi} |Z_x|^2 dy dt.
\end{aligned} \tag{3.74}$$

Using the equation of w^ε and noting that $\mathbb{E} \int_{Q_T} Q_2 \widehat{\Theta} F_2 dx dy dB(t) = 0$, we have

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} Q_2 \widehat{\Theta} \left[dw^\varepsilon - w_{xx}^\varepsilon dt - (x + \varepsilon)^{2\gamma} w_{yy}^\varepsilon dt - \frac{\sigma}{(x + \varepsilon)^2} w^\varepsilon dt \right] dx dy \\
\leq & \frac{1}{2} \mathbb{E} \int_{Q_T} |Q_2|^2 dx dy dt + \frac{1}{2} \mathbb{E} \int_{Q_T} \widehat{\Theta}^2 |f_2|^2 dx dy dt.
\end{aligned} \tag{3.75}$$

Obviously, there exists a positive constant

$$C_9 = c \delta_0^4 2^{8+12\gamma} C_1^4$$

such that

$$|Q|^2 \leq C_9 s^2 \lambda^4 \widehat{\Phi}^2 |Z|^2. \tag{3.76}$$

From (3.74)-(3.76), it follows that

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \widehat{\Phi} |Z_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 (x + \varepsilon)^{2\gamma} \widehat{\Phi} |Z_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \widehat{\Phi}^3 |Z|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda \widehat{\Phi} \widehat{\Theta}^2 |F_2|^2 dx dy dt \\
\leq & \mathbb{E} \int_{Q_T} \widehat{\Theta}^2 |f_2|^2 dx dy dt + 2 \overline{M}_4 \mathbb{E} \int_{Q_T} s \widehat{\Phi} \widehat{\Theta}^2 |\nabla F_2|^2 dx dy dt \\
& + 2 \overline{M}_2 \mathbb{E} \int_I s^2 \lambda^2 \widehat{\Phi}^2(T) Z^2(T) dx dy + 2 \overline{M}_3 \mathbb{E} \int_{\Gamma_T} s \lambda \widehat{\Phi} |Z_x|^2 dy dt
\end{aligned} \tag{3.77}$$

for all sufficiently large λ and s such that

$$\lambda \geq c(T^{23} + 1)2^{6+10\gamma} C_1^4, \quad s \geq c \delta_0^4 2^{8+12\gamma} C_1^4.$$

Finally, going back to the original variable w^ε and letting $\varepsilon \rightarrow 0$ in (3.77), the desired estimate (3.58) holds with

$$C = c(1 + \mu)\delta_0^2 2^{4+8\gamma} C_1^2 > 2(\overline{M}_2 + \overline{M}_3 + \overline{M}_4).$$

This completes the proof of Theorem 3.8. \square

4 Proof of Theorem 1.1

This section is devoted to proving the null controllability result for the forward stochastic Grushin equation (1.2), i.e. Theorem 1.1.

Proof of Theorem 1.1. It is well known that the key ingredient for proving Theorem 1.1 is to obtain the observability inequality for the corresponding adjoint system

$$\begin{cases} dv + v_{xx}dt + x^{2\gamma}v_{yy}dt + \frac{\sigma}{x^2}vdt = (-\alpha v - \beta V)dt + VdB(t), & (x, y, t) \in Q_T, \\ v(x, y, t) = 0, & (x, y, t) \in \Sigma_T, \\ v(x, y, T) = v_T(x, y), & (x, y) \in I. \end{cases} \quad (4.1)$$

More precisely, we will prove the following observability inequality for (4.1):

$$\mathbb{E} \int_I |v(0)|^2 dx dy \leq C \mathbb{E} \int_{\omega_T} |v|^2 dx dy dt + C \mathbb{E} \int_{Q_T} |V|^2 dx dy dt, \quad (4.2)$$

where C is depending on $I, T, \gamma, \omega, \sigma, \alpha$ and β .

We apply Theorem 3.1 to (4.1) to obtain

$$\mathbb{E} \int_{Q_T} s^3 \xi^3 \theta^2 |v|^2 dx dy dt \leq C \mathbb{E} \int_{\omega_T} s^3 \xi^3 \theta^2 |v|^2 dx dy dt + C \mathbb{E} \int_{Q_T} s^2 \xi^2 \theta^2 |V|^2 dx dy dt \quad (4.3)$$

for all large $\lambda > \lambda_1$ and $s > s_1$. We fix $\lambda = \lambda_1$ and $s = s_1$. By

$$m_1 := \max_{Q_T} (\xi^2 + \xi^3) \theta^2 < +\infty,$$

we further obtain

$$\begin{aligned} \mathbb{E} \int_{Q_T} \xi^3 \theta^2 |v|^2 dx dy dt &\leq C \mathbb{E} \int_{Q_T} \xi^2 \theta^2 |V|^2 dx dy dt + \mathbb{E} \int_{\omega_T} \xi^3 \theta^2 |v|^2 dx dy dt \\ &\leq C(\lambda_1, s_1, m_1) \left(\mathbb{E} \int_{Q_T} |V|^2 dx dy dt + \mathbb{E} \int_{\omega_T} |v|^2 dx dy dt \right). \end{aligned} \quad (4.4)$$

On the other hand, by

$$m_2 := \min_{I \times (\frac{T}{4}, \frac{3T}{4})} \xi^3 \theta^2 > 0,$$

we obtain

$$\mathbb{E} \int_{Q_T} \xi^3 \theta^2 |v|^2 dx dy dt \geq \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_I \xi^3 \theta^2 |v|^2 dx dy dt \geq m_2 \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_I |v|^2 dx dy dt. \quad (4.5)$$

From (4.4) and (4.5), we deduce

$$\mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_I |v|^2 dx dy dt \leq C(\lambda_1, s_1, m_1, m_2) \left(\mathbb{E} \int_{\omega_T} |v|^2 dx dy dt + \mathbb{E} \int_{Q_T} |V|^2 dx dy dt \right). \quad (4.6)$$

By the standard estimate for the backward stochastic equation (4.1), we obtain for any $0 \leq \tau < \tilde{\tau} \leq T$ that

$$\mathbb{E} \int_I |v(\tau)|^2 dx dy \leq \mathbb{E} \int_I |v(\tilde{\tau})|^2 dx dy + C \mathbb{E} \int_{\tau}^{\tilde{\tau}} \int_I |v|^2 dx dy dt. \quad (4.7)$$

Then from Grönwall's inequality, it follows that

$$\mathbb{E} \int_I |v(\tau)|^2 dx dy \leq e^{CT} \mathbb{E} \int_I |v(\tilde{\tau})|^2 dx dy, \quad 0 \leq \tau < \tilde{\tau} \leq T. \quad (4.8)$$

Further, letting $\tau = 0$ and integrating (4.8) over $(\frac{T}{4}, \frac{3T}{4})$ with respect to $\tilde{\tau}$, we obtain

$$\mathbb{E} \int_I |v(0)|^2 dx dy \leq C \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_I |v|^2 dx dy dt. \quad (4.9)$$

Combining (4.6) and (4.9), we obtain (4.2). Then by a standard dual argument, e.g. as [34] or [39], we could obtain a pair $(g, G) \in L^2_{\mathcal{F}}(0, T; L^2(\omega)) \times L^2_{\mathcal{F}}(0, T; L^2(I))$ that drives the corresponding solution u of (1.2) to zero at time T . This completes the proof of Theorem 1.1. \square

5 Proof of Theorem 1.2

In this section, we prove the uniqueness for our inverse source problem, i.e. Theorem 1.2.

Proof of Theorem 1.2. For convenience, we use C to denote generic constant depends on $I, T, \gamma, \sigma, \mu, M$. Let $u = R_1 p$. By virtue of u as a solution of equation (1.3), we know that p solves

$$\begin{cases} dp - p_{xx} dt - x^{2\gamma} p_{yy} dt - \frac{\sigma}{x^2} p dt = \frac{2R_{1,x}}{R_1} p_x dt + \frac{2x^{2\gamma} R_{1,y}}{R_1} p_y dt \\ \quad + \left(-\frac{R_{1,t}}{R_1} + \frac{R_{1,xx}}{R_1} + \frac{x^{2\gamma} R_{1,yy}}{R_1} \right) p dt \\ \quad + h(x, t) dt + \frac{R_2}{R_1} H(t) dB(t), \quad (x, y, t) \in Q_T, \\ p(x, y, t) = 0, \quad (x, y, t) \in \Sigma_T, \\ p(x, y, 0) = 0, \quad (x, y) \in I. \end{cases} \quad (5.1)$$

Letting $w = p_y$, together with $u_y|_{\Sigma_T} = 0$, $\mathbb{P} - a.s.$, we obtain

$$\left\{ \begin{aligned} dw - w_{xx}dt - x^{2\gamma}w_{yy}dt - \frac{\sigma}{x^2}wdt &= \frac{2R_{1,x}}{R_1}w_xdt + \frac{2x^{2\gamma}R_{1,y}}{R_1}w_ydt \\ &+ \left(-\frac{R_{1,t}}{R_1} + \frac{R_{1,xx}}{R_1} + \frac{x^{2\gamma}R_{1,yy}}{R_1} \right) wdt \\ &+ \left(\frac{2R_{1,x}}{R_1} \right)_y p_x dt + \left(\frac{2x^{2\gamma}R_{1,y}}{R_1} \right)_y p_y dt \\ &+ \left(-\frac{R_{1,t}}{R_1} + \frac{R_{1,xx}}{R_1} + \frac{x^{2\gamma}R_{1,yy}}{R_1} \right)_y p dt \\ &+ \left(\frac{R_2}{R_1} \right)_y H(t)dB(t), \quad (x, y, t) \in Q_T, \\ w(x, y, t) &= 0, \quad (x, y, t) \in \Sigma_T, \\ w(x, y, 0) &= 0, \quad (x, y) \in I. \end{aligned} \right. \quad (5.2)$$

Applying Theorem 3.8 to w , we find that

$$\begin{aligned} &\mathbb{E} \int_{Q_T} s\lambda^2\Phi\Theta^2|w_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s\lambda^2\Phi\Theta^2x^{2\gamma}|w_y|^2 dx dy dt \\ &+ \mathbb{E} \int_{Q_T} s^3\lambda^4\Phi^3\Theta^2|w|^2 dx dy dt + \mathbb{E} \int_{Q_T} s\lambda\Phi\Theta^2 \left| \left(\frac{R_2}{R_1} \right)_y \right|^2 |H|^2 dx dy dt \\ &\leq C\mathbb{E} \int_{Q_T} \Theta^2 (|w_x|^2 + x^{2\gamma}|w_y|^2 + |w|^2 + |p_x|^2 + |p_y|^2 + |p|^2) dx dy dt \\ &+ C\mathbb{E} \int_{Q_T} s\Phi\Theta^2 \left| \nabla \left(\frac{R_2}{R_1} \right)_y \right|^2 |H|^2 dx dy dt \\ &+ C\mathbb{E} \int_I s^2\lambda^2\Phi^2(T)\Theta^2(T)w^2(T) dx dy + C\mathbb{E} \int_{\Gamma_T} s\lambda\Phi\Theta^2|w_x|^2 dy dt \end{aligned} \quad (5.3)$$

for all $\lambda \geq \lambda_2$, $s \geq s_2$. By means of $w = p_y$ and $p(x, 0, t) = 0$ for $(x, t) \in I_x \times (0, T)$, we see that

$$p(x, y, t) = \int_0^y w(x, \eta, t) d\eta. \quad (5.4)$$

Therefore, we obtain

$$\mathbb{E} \int_{Q_T} \Theta^2 (|p|^2 + |p_x|^2 + |p_y|^2) dx dy dt \leq C\mathbb{E} \int_{Q_T} \Theta^2 (|w|^2 + |w_x|^2) dx dy dt. \quad (5.5)$$

By (1.6), we have

$$\mathbb{E} \int_{Q_T} s\Phi\Theta^2 \left| \nabla \left(\frac{R_2}{R_1} \right)_y \right|^2 |H|^2 dx dy dt \leq C\mathbb{E} \int_{Q_T} s\Phi\Theta^2 \left| \left(\frac{R_2}{R_1} \right)_y \right|^2 |H|^2 dx dy dt. \quad (5.6)$$

Thus, substituting (5.5) and (5.6) into (5.3) and choosing λ sufficiently large to absorb the first two terms on the right-hand side of (5.3) by the terms on the left-hand side of (5.3),

we find that

$$\begin{aligned}
& \mathbb{E} \int_{Q_T} s \lambda^2 \Phi \Theta^2 |w_x|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda^2 \Phi \Theta^2 x^{2\gamma} |w_y|^2 dx dy dt \\
& + \mathbb{E} \int_{Q_T} s^3 \lambda^4 \Phi^3 \Theta^2 |w|^2 dx dy dt + \mathbb{E} \int_{Q_T} s \lambda \Phi \Theta^2 \left| \left(\frac{R_2}{R_1} \right)_y \right|^2 |H|^2 dx dy dt \\
& \leq C \mathbb{E} \int_I s^2 \lambda^2 \Phi^2(T) \Theta^2(T) w^2(T) dx dy + C \mathbb{E} \int_{\Gamma_T} s \lambda \Phi \Theta^2 |w_x|^2 dy dt. \tag{5.7}
\end{aligned}$$

Since $u|_{\Gamma_T} = u_x|_{\Gamma_T} = 0$, \mathbb{P} -a.s., we have $u_y|_{\Gamma_T} = u_{xy}|_{\Gamma_T} = 0$ and further $w_x|_{\Gamma_T} = 0$, \mathbb{P} -a.s. Moreover $w(T) = 0$ in I , due to (1.8). Then from (5.7) we deduce

$$w = 0 \quad \text{in } Q_T, \quad \mathbb{P} - a.s. \tag{5.8}$$

which implies

$$u = 0 \quad \text{in } Q_T, \quad \mathbb{P} - a.s. \tag{5.9}$$

By (5.9) and the equation (1.3) of u , we have

$$\int_0^t h(x, \tau) R_1(x, y, \tau) d\tau + \int_0^t H(\tau) R_2(x, y, \tau) dB(\tau) = 0, \quad t \in (0, T), \tag{5.10}$$

Together with (1.5), we finally obtain (1.9) and (1.10). The proof of Theorem 1.2 is completed. \square

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