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DYNAMIC OPTIMIZATION PROBLEMS FOR MEAN-FIELD STOCHASTIC LARGE-POPULATION SYSTEMS*

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Abstract. This paper considers dynamic optimization problems for a class of control average mean-field stochastic large-population systems. For each agent, the state system is governed by a linear mean-field stochastic differential equation with individual noise and common noise, and the weight coefficients in the corresponding cost functional can be indefinite. The decentralized optimal strategies are characterized by stochastic Hamiltonian system, which turns out to be an algebra equation and a mean-field forward-backward stochastic differential equation. Applying the decoupling method, the feedback representation of decentralized optimal strategies is further obtained through two Riccati equations. The solvability of stochastic Hamiltonian system and Riccati equations under indefinite condition is also derived. The explicit structure of the control average limit and the related mean-field Nash certainty equivalence equation systems are discussed by some separation techniques. Moreover, the decentralized optimal strategies are proved to satisfy the approximate Nash equilibrium property. The good performance of the proposed theoretical results is illustrated by a practical example from the engineering field.

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1. Introduction

In recent years, the modeling and analysis for dynamics systems with a large number of agents, also known as large-population systems or large-scale systems, has been gained intensive and consistent attention from various fields, including social science, operational research, mathematical finance, economics and engineering, etc. The agents in large-population systems are individually negligible at the microscopic level but their collective behaviors are significant at the macroscopic level, which cannot be ignored. The most prominent feature of large-population systems is that there exists a coupling structure in the state system and cost functional. Since the highly complex interactions among considerable individuals, it is difficult to obtain the so-called centralized

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strategies which are built on the whole information of all agents as the agent number becomes large. By contrast, it is efficient and feasible to design the decentralized strategies based on local information only. The local information mentioned here is involved with own state of the given agent and some certain quantities. In such scenarios, one powerful tool is the mean-field game (MFG). Generally speaking, the main procedure of MFG can be boiled down to the following four steps: (i) To avoid the complex coupling information structure in large-population systems, the corresponding asymptotic behavior is analyzed when the agent number tends to infinity and the state average or control average limit process is introduced. At this time, the limit term can be regarded as a frozen term, which needs to be determined later; (ii) By replacing the state average or control average term with the frozen limit, homologous limiting control problems are introduced. Consequently, the given agent only depends on its own state and the frozen limit, which essentially formulates a class of stochastic LQ optimal control problems. The decentralized optimal strategies can be obtained by standard methods like stochastic maximum principle and dynamic programming principle; (iii) The frozen limit term needs to be determined by the Nash certainty equivalence (NCE) equation systems, which play a vital role in the MFG approach; (iv) The decentralized optimal strategies derived in the second step satisfy the ϵ-Nash equilibrium property, which completes the whole procedure.

The MFG is currently one of the most active and fast developing research directions and there exist a lot of literature on this topic. In the early years, Lasry and Lions [1], Huang, Malhamé, and Caines [2] independently studied large-population problems and related MFGs from the perspective of mathematical and engineering. For further analysis of MFGs, the interested readers may refer to Guéant, Lasry, and Lions [3] for more details about their motivations and applications. Recently, there are some associated works following up [1] and [2]. For example, Huang, Malhamé, and Caines [4] proposed the state aggregation method and the NCE principle to construct decentralized optimal strategies. Huang and Wang [5] discussed stochastic large-population problems under the framework of partial information. Hu, Huang, and Li [6] considered a class of control constrained MFG problems. Huang [7], Hu, Huang, and Nie [8] studied mixed large-population problems where the major-minor structure was taken into consideration. For other aspects of MFGs, the readers are referred to Wang and Zhang [9], Huang and Li [10], Li, Li, and Wu [11], Li, Nie, and Wu [12] with diverse system dynamics, cost functionals and interaction mechanisms. Huang, Wang, and Wu [13] investigated a LQ MFG problem when the state system was governed by a backward stochastic differential equation (BSDE). Huang and Zhou [14] and Huang, Wang, and Yong [15] used the direct approach to study corresponding MFGs in the noncooperative game and cooperative game, respectively. Some comparisons between the fixed point approach and the direct approach were also given.

Another related but distinct concept is the mean-field type control. Different from the MFG, mean-field type control is associated with a special class of stochastic differential equations (SDEs), which are the so-called McKean-Vlasov or mean-field SDEs (MF-SDEs). Historically, such equations were used to describe the behavior of large particle systems in statistical physics. As its name suggests, MF-SDE means that the coefficients of equation do not only depend on the system state and control, but also on their expected values. The research on MF-SDEs can be traced back to 1950s, when Kac [16] first introduced it as a mathematical model for the Vlasov kinetic equation of plasma and then McKean [17] studied it continuously. Since then, many theoretical and application-oriented results have emerged which greatly promote the development of MF-SDEs theory, especially in the combination with optimal control and differential game, see for example, Buckdahn, Djehiche, and Li [18], Li [19], Yong [20], Buckdahn, Li, and Ma [21], Graber [22], Li and Wu [23]. In addition, the readers may refer to Buckdahn, Djehiche, Li, and Peng [24], Buckdahn, Li, and Peng [25] and Li, Sun, and Xiong [26] for the relevant progress in MF-BSDEs.

To clarify the relationship between MFG and mean-field type control, it is necessary to make some comparisons. Just as literature mentioned before, these two problems are quite different in methods used and equilibriums obtained. The MFG is facilitated to address with large-population problems. In order to derive decentralized optimal strategies, the state average or control average should be treated as a frozen term. However, in the mean-field type control, the state and control expectation are underlying terms in the state system and cost functional, which can vary with respect to admissible controls. Furthermore, it will be proved later that the decentralized optimal strategies satisfy the ϵ-Nash equilibrium property while the counterpart in mean-field type control results in a franchise equilibrium, which can refer to Carmona and Delarue [28] for more details.

To guarantee the well-posedness of problems, all mentioned LQ MFGs and mean-field type control problems assume standard positive definite condition. Different from the deterministic LQ problem, the state and control weight coefficients in the cost functional of stochastic LQ setting are allowed to be indefinite. The interesting and surprising phenomenon quickly attracted widespread attention. Chen, Li, and Zhou [29] used the Riccati equation to study a kind of indefinite stochastic LQ problems earlier. Since then, the solvability of Riccati equation has always been the core difficulty in the study of indefinite problems, see for example Qian and Zhou [30] and Huang and Yu [31]. A new method called equivalent cost functional was proposed by Yu [32] to study an indefinite stochastic LQ problem with random coefficients. Li, Li, and Yu [33] advanced the relaxed compensator to solve indefinite mean-field LQ stochastic optimal control problems directly. The above studies have enriched the development of indefinite stochastic LQ control problems. However, to our best acknowledge, there exists little work related to indefinite MFG problems.

This paper first studies indefinite large-population problems for a class of mean-filed stochastic systems, where considerable agents are coupled via the control average. Here, each agent is not only affected by its own noise but also disturbed by the common noise, which describes some exogenous factors that are identical to all agents. Moreover, the weight coefficients for state and control in the cost functional can be indefinite. In fact, the co-existence of mean-field term and control average structure in our large-population problems results in a highly complex coupled one, which is difficult to decouple. To add insult to injury, the weight matrices of the cost functional in our setting are allowed to be indefinite. For these reasons, we need to develop new methods to solve our problems.

With the help of the MFG approach, the decentralized strategies for the control average mean-field stochastic large-population problem are derived. Firstly, we introduce a limiting problem, which is an indefinite LQ optimal control problem for a stochastic mean-field system with nonhomogeneous terms. Secondly, the decentralized optimal strategies can be characterized by corresponding stochastic Hamiltonian system, which consists of an algebra equation and a mean-field forward-backward stochastic differential equation (MF-FBSDE). By the decoupling method, the feedback representation of decentralized optimal strategies is further obtained through two Riccati equations. The solvability of stochastic Hamiltonian system and Riccati equations under indefinite condition is also derived. Thirdly, the explicit structure of the control average limit and the corresponding mean-field NCE equation systems are obtained by some separation techniques. Moreover, the decentralized optimal strategies are verified to satisfy the ϵ-Nash equilibrium property. For illustration, one example stemming from the engineering field is further discussed.

In most existing literature, the dynamic optimization of a large-population system is always formulated by the state average dynamics system and cost functional without mean-field terms. Besides, the positive definite assumption is compulsory. However, considerable realistic situations (e.g., performance evaluation and decision-making problem of some financial models, cooperative control problem of the unmanned aerial vehicles) suggest that the problem to be studied should be formulated in a general control average mean-field setup, where the weight coefficients in the cost functional can be indefinite. Motivated by these observations, a new class of *control average* large-population problems for *mean-field* stochastic systems without positive definite condition are considered in this paper. By introducing a relaxed compensator, the existence and uniqueness of solutions to MF-FBSDE and Riccati equations under indefinite condition are discussed, thus the decentralized optimal strategies are designed in the feedback representation form. Distinguished from the conventional state average large-population setup, some *separation techniques* are proposed to determine the explicit structure of the frozen control average limit and the corresponding mean-field NCE equation systems.

The rest of this paper is organized as follows. Section 2 gives some basic notations and preliminaries, the large-population problem of mean-field type is also formulated. Section 3 studies the corresponding limiting control problem and establishes the characterization of decentralized optimal strategies by stochastic Hamiltonian system and Riccati equations under indefinite condition. The structure of the control average limit and the corresponding mean-field NCE equation systems are also discussed. Section 4 proves the ϵ-Nash equilibrium property. In Section 5, one case is discussed for the illustrating example. The decentralized optimal strategies are solved explicitly. Section 6 concludes this work.

2. Preliminaries and Problem Formulation

We consider a large-population system with N individual agents $\{\mathcal{A}_i\}_{1\le i\le N}$. For a fixed time $T>0$, let $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$ be a complete filtered probability space on which a standard $(d+N)$-dimensional Brownian motion $\left\{W_t^0,W_t^i,1\le i\le N\right\}$ is defined and the filtration $\mathbb{F}=\{\mathcal{F}_t,0\le t\le T\}$ is assumed to be a nature one of $W^0,W^i,1\le i\le N$ augmented by all $\mathbb{P}$-null sets $\mathcal{N}$. Here, W^0 signifies the common noise which is identical to all agents. For $1\le i\le N$, W^i denotes the noise only for the ith agent which varies from person to person. For brief statement in the following sections, we define $\mathcal{F}_t^0=\sigma\{W_s^0,0\le s\le t\}\vee\mathcal{N}$, $\mathcal{F}_t^{W^i}=\sigma\left\{W_s^i,0\le s\le t\right\}\vee\mathcal{N}$, $\mathcal{F}_t^i=\sigma\left\{W_s^0,W_s^i,0\le s\le t\right\}\vee\mathcal{N}$, $\mathbb{F}^0=\{\mathcal{F}_t^0,0\le t\le T\}$ and $\mathbb{F}^i=\{\mathcal{F}_t^i,0\le t\le T\}$.

For n-dimensional Euclidean space $\mathbb{R}^n$ with usual norm $|\cdot|$, let $\langle\,\cdot,\cdot\,\rangle$ be the inner product. $\top$ appearing in superscript represents the transpose of a matrix or vector. $\mathcal{S}^d$ stands for the set of all $d\times d$ symmetric matrices and $\mathcal{S}^d_+$ denotes the semi-positive matrices of $\mathcal{S}^d$. I denotes the identity matrix with appropriate dimension. The following spaces will be used throughout the paper.

- $\mathcal{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^n)=\{f:[0,T]\times\Omega\to\mathbb{R}^n\mid f(\cdot)$ is an $\mathbb{F}$-adapted process satisfying $\mathbb{E}[\sup_{0\le t\le T}|f(t)|^2]<\infty\}$.
- $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)=\{f:[0,T]\times\Omega\to\mathbb{R}^n\mid f(\cdot)$ is an $\mathbb{F}$-adapted process satisfying $\mathbb{E}\int_0^T|f(t)|^2dt<\infty\}$.
- $L^2(0,T;\mathbb{R}^n)=\{f:[0,T]\to\mathbb{R}^n\mid f(\cdot)$ is a deterministic Lebesgue measurable function satisfying $\int_0^T|f(t)|^2dt<\infty\}$.
- $L^\infty(0,T;\mathbb{R}^n)=\{f:[0,T]\to\mathbb{R}^n\mid f(\cdot)$ is an uniformly bounded deterministic function$\}$.
- $L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)=\{\xi:\Omega\to\mathbb{R}^n\mid\xi$ is an $\mathcal{F}_T$-adapted, uniformly bounded random variable$\}$.
- $C([0,T];\mathbb{R}^n)=\{f:[0,T]\to\mathbb{R}^n\mid f(\cdot)$ is a continuous function$\}$.

We consider a mean-field stochastic large-population system with N individual agents $\{\mathcal{A}_i\}_{1\le i\le N}$. The state dynamics of $\mathcal{A}_i$ is governed by the following MF-SDE

$$\begin{cases} dx_t^i=\Big\{A_tx_t^i+\widetilde{A}_t\mathbb{E}[x_t^i]+B_tu_t^i+\widetilde{B}_t\mathbb{E}[u_t^i]+\dfrac{1}{N-1}\displaystyle\sum_{j=1,j\ne i}^N\bar{B}_tu_t^j\Big\}dt\\[2mm] \qquad\quad+\Big\{\widetilde{C}_t\mathbb{E}[x_t^i]+\widetilde{D}_t\mathbb{E}[u_t^i]+\sigma_t\Big\}dW_t^i+\sigma_t^0dW_t^0,\\ x_0^i=a, \end{cases}\tag{2.1}$$

where $a\in\mathbb{R}^n$ is the initial state, $x^i_\cdot$ and $u^i_\cdot$ are the state process and control process, respectively. For any $t\in[0,T]$, $A_t,\widetilde{A}_t,\widetilde{C}_t\in L^\infty(0,T;\mathbb{R}^{n\times n}),B_t,\widetilde{B}_t,\bar{B}_t,\widetilde{D}_t\in L^\infty(0,T;\mathbb{R}^{n\times k}),\sigma_t\in L^\infty(0,T;\mathbb{R}^n),\sigma_t^0\in L^\infty(0,T;\mathbb{R}^{n\times d})$. With such coefficients, MF-SDE (2.1) admits a unique solution $x^i_\cdot\in\mathcal{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ for any control process $u^i_\cdot$. Moreover, $\frac{1}{N-1}\sum_{j=1,j\ne i}^N\bar{B}_\cdot u^j_\cdot$ denotes the control average term.

Now, we specify the strategy set of the large-population system. $u^i_\cdot$ is called the centralized strategy if $u^i_\cdot\in\mathcal{U}_i^c$, where $\mathcal{U}_i^c=\{u^i_\cdot\mid u^i_\cdot\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)\}$. Moreover, we define the decentralized strategy $u^i_\cdot$ as $u^i_\cdot\in\mathcal{U}_i^d$, where $\mathcal{U}_i^d=\{u^i_\cdot\mid u^i_\cdot\in L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^k)\}$. Let $u=(u^1,\ldots,u^N)$ be the strategy set of all N agents, $u^{-i}=(u^1,\ldots,u^{i-1},u^{i+1},\ldots,u^N)$ be the strategy set except the ith agent.

Then the cost functional of $\mathcal{A}_i$ is subject to

$$\begin{aligned}&\mathcal{J}^i\left(u^i_\cdot,u^{-i}_\cdot\right)\\ =\;&\mathbb{E}\Big\{\int_0^T\Big[\langle Q_tx_t^i,x_t^i\rangle+\langle\widetilde{Q}_t\mathbb{E}[x_t^i],\mathbb{E}[x_t^i]\rangle+2\langle S_tu_t^i,x_t^i\rangle+2\langle\widetilde{S}_t\mathbb{E}[u_t^i],\mathbb{E}[x_t^i]\rangle+\langle R_tu_t^i,u_t^i\rangle+\langle\widetilde{R}_t\mathbb{E}[u_t^i],\mathbb{E}[u_t^i]\rangle\\ &+2\langle q_t,x_t^i\rangle+2\langle\widetilde{q}_t,\mathbb{E}[x_t^i]\rangle+2\langle\rho_t,u_t^i\rangle+2\langle\widetilde{\rho}_t,\mathbb{E}[u_t^i]\rangle\Big]dt+\langle Gx_T^i,x_T^i\rangle+\langle\widetilde{G}\mathbb{E}[x_T^i],\mathbb{E}[x_T^i]\rangle+2\langle g,x_T^i\rangle+2\langle\widetilde{g},\mathbb{E}[x_T^i]\rangle\Big\},\end{aligned}\tag{2.2}$$

where $Q_t, \widetilde{Q}_t \in L^\infty(0,T;\mathcal{S}^n), S_t, \widetilde{S}_t \in L^\infty(0,T;\mathbb{R}^{k\times n}), R_t, \widetilde{R}_t \in L^\infty(0,T;\mathcal{S}^k), q_t, \widetilde{q}_t \in L^\infty(0,T;\mathbb{R}^n), \rho_t, \widetilde{\rho}_t \in L^\infty(0,T;\mathbb{R}^k), G, \widetilde{G} \in \mathcal{S}^n, g, \widetilde{g} \in \mathbb{R}^n$, for any $t \in [0,T]$. It is obvious that for given initial $a \in \mathbb{R}^n$ and any control process $u^i_\cdot$, cost functional (2.2) is well-defined.

Remark 2.1. The *motivations* for studying control average mean-field large-population problem come from the following. In various financial and engineering problems, the mean-field large-population problem has been extensively studied for two reasons. On one hand, MF-SDE can be used to describe the particle systems at the microscopic level, which has important value in some applications. On the other hand, the given agent hopes that the optimal state process and/or control process could be not too sensitive with the possible variation of the random events. To achieve this, one may keep the variances $\mathrm{var}[x^i_\cdot]$ and $\mathrm{var}[u^i_\cdot]$ small. Therefore, it is natural to include $\mathrm{var}[x^i_\cdot]$ and $\mathrm{var}[u^i_\cdot]$ in the cost functional. Note that $\mathrm{var}[x^i_\cdot] = \mathbb{E}[|x^i_\cdot|^2] - (\mathbb{E}[x^i_\cdot])^2$ and $\mathrm{var}[u^i_\cdot] = \mathbb{E}[|u^i_\cdot|^2] - (\mathbb{E}[u^i_\cdot])^2$, then the problem consists of (2.1) and (2.2) is actually a mean-field large-population problem. Another motivation is that in some decision-making problems, the input or control of given agent will have immediate and transient impact on the state of oneself and others, thus the control average term arises.

Now, we formulate dynamic optimization problems for mean-field large-population (MFL) systems.

Problem (MFL). For $1 \le i \le N$, find a strategy set $\bar{u} = (\bar{u}^1, \dots, \bar{u}^N)$ such that

$$\mathcal{J}^i\left(\bar{u}^i_\cdot, \bar{u}^{-i}_\cdot\right) = \inf_{u^i_\cdot \in \mathcal{U}^c_i} \mathcal{J}^i\left(u^i_\cdot, \bar{u}^{-i}_\cdot\right),$$

where $\bar{u}^{-i} = (\bar{u}^1, \dots, \bar{u}^{i-1}, \bar{u}^{i+1}, \dots, \bar{u}^N)$.

In particular, the control strategy $\bar{u}$ is the so-called Nash equilibrium for Problem (MFL). For the sake of comparison, we also present the definition of the ϵ-Nash equilibrium, which will be applied in the later section, for more details one can refer to Carmona and Delarue [28], Brezis [34].

Definition 2.1. For $1 \le i \le N$, $\bar{u}^i_\cdot \in \mathcal{U}^c_i$, the strategy set $\bar{u} = (\bar{u}^1, \dots, \bar{u}^N)$ is called an ϵ-Nash equilibrium with respect to costs $\mathcal{J}^i$, if there exists an $\epsilon = \epsilon(N)$, $\lim\limits_{N\to\infty} \epsilon(N) = 0$, such that

$$\mathcal{J}^i\left(\bar{u}^i_\cdot, \bar{u}^{-i}_\cdot\right) \le \mathcal{J}^i\left(u^i_\cdot, \bar{u}^{-i}_\cdot\right) + \epsilon,$$

where any alternative strategy $u^i_\cdot \in \mathcal{U}^c_i$ is applied by $\mathcal{A}_i$.

Obviously, if $\epsilon = 0$ in the above definition, we see that it reduces to the exact Nash equilibrium.

3. Mean-Field Nash Certainty Equivalence Equation Systems

Due to the highly complex interactions among peers, it is not implementable and efficient for the given agent to collect global information of all other ones in the framework of noncooperative games. Consequently, the centralized strategies based on global information are intractable to realize. Another alternative choice is to determine an approximate equilibrium depending on local information, which is known as the decentralized strategies. In this section, we will study the limiting problem, which is an indefinite LQ optimal control problem for a stochastic mean-field system with nonhomogeneous terms. By introducing a relaxed compensator, we first prove the solvability of stochastic Hamiltonian system and Riccati equations under indefinite condition. Then the decentralized optimal strategies are designed in the feedback form. The explicit structure of the control average limit and the corresponding mean-field NCE equation systems are also obtained by some separation techniques.

As the agent number N tends to infinity, we denote that $\frac{1}{N-1}\sum_{j=1,j\neq i}^N \bar{B}_\cdot u^j_\cdot$ is approximated by $m_\cdot$. Some subtle analysis for $m_\cdot$ will be given latter. Then the limiting state of $\mathcal{A}_i$ is controlled by

$$\begin{cases} dx^i_t = \Big\{A_t x^i_t + \widetilde{A}_t \mathbb{E}[x^i_t] + B_t u^i_t + \widetilde{B}_t \mathbb{E}[u^i_t] + m_t\Big\}dt \\ \qquad\quad + \Big\{\widetilde{C}_t \mathbb{E}[x^i_t] + \widetilde{D}_t \mathbb{E}[u^i_t] + \sigma_t\Big\}dW^i_t + \sigma^0_t dW^0_t, \\ x^i_0 = a. \end{cases} \tag{3.1}$$

The limiting cost functional becomes

$$
\begin{aligned}
&J^i\left(u^i_\cdot\right)\\
&=\mathbb{E}\Big\{\int_0^T\Big[\langle Q_t x^i_t,x^i_t\rangle+\langle\widetilde{Q}_t\mathbb{E}[x^i_t],\mathbb{E}[x^i_t]\rangle+2\langle S_t u^i_t,x^i_t\rangle+2\langle\widetilde{S}_t\mathbb{E}[u^i_t],\mathbb{E}[x^i_t]\rangle+\langle R_t u^i_t,u^i_t\rangle+\langle\widetilde{R}_t\mathbb{E}[u^i_t],\mathbb{E}[u^i_t]\rangle\\
&\quad+2\langle q_t,x^i_t\rangle+2\langle\widetilde{q}_t,\mathbb{E}[x^i_t]\rangle+2\langle\rho_t,u^i_t\rangle+2\langle\widetilde{\rho}_t,\mathbb{E}[u^i_t]\rangle\Big]dt+\langle Gx^i_T,x^i_T\rangle+\langle\widetilde{G}\mathbb{E}[x^i_T],\mathbb{E}[x^i_T]\rangle+2\langle g,x^i_T\rangle+2\langle\widetilde{g},\mathbb{E}[x^i_T]\rangle\Big\}.
\end{aligned}
\tag{3.2}
$$

Remark 3.1. One should pay attention to distinguishing these two symbols: $\mathcal{J}^i\left(u^i_\cdot,u^{-i}_\cdot\right)$ and $J^i\left(u^i_\cdot\right)$. Here, we write $\mathcal{J}^i\left(u^i_\cdot,u^{-i}_\cdot\right)$ to emphasize the dependence of all agents due to the coupling structure in state equation. By contrast, $J^i\left(u^i_\cdot\right)$ is only involved with the ith agent and $m_\cdot$.

Now, the limiting mean-field large-population problem (LMFL) can be introduced as follows.

Problem (LMFL). For $1\le i\le N$, find $\bar{u}^i_\cdot\in\mathcal{U}^d_i$ such that

$$J^i\left(\bar{u}^i_\cdot\right)=\inf_{u^i_\cdot\in\mathcal{U}^d_i}J^i\left(u^i_\cdot\right).$$

For $1\le i\le N$, if the infimum of $J^i\left(u^i_\cdot\right)$ over admissible controls is finite, Problem (LMFL) is called *well-posed*. Then $\bar{u}^i_\cdot$ (if exists) is called the decentralized strategy and $\bar{x}^i_\cdot$ is called the corresponding decentralized optimal state trajectory with respect to $\bar{u}^i_\cdot$. Moreover, $(\bar{x}^i_\cdot,\bar{u}^i_\cdot)$ is called the decentralized optimal pair for Problem (LMFL).

For more in-depth reveal the essence of problem, we give another version of (3.1) and (3.2) similar to Yong [20]. To ease the presentation, we introduce the following notations:

$$\widehat{\varphi}=\varphi+\widetilde{\varphi},\ \text{ with }\varphi=A_\cdot,B_\cdot,Q_\cdot,S_\cdot,R_\cdot,G,q_\cdot,\rho_\cdot,g,$$

and

$$
\boldsymbol{Q}_t=\begin{pmatrix}Q_t&\mathbf{0}\\\mathbf{0}&\widehat{Q}_t\end{pmatrix},\ \boldsymbol{S}_t=\begin{pmatrix}S_t&\mathbf{0}\\\mathbf{0}&\widehat{S}_t\end{pmatrix},\ \boldsymbol{R}_t=\begin{pmatrix}R_t&\mathbf{0}\\\mathbf{0}&\widehat{R}_t\end{pmatrix},\ \boldsymbol{G}=\begin{pmatrix}G&\mathbf{0}\\\mathbf{0}&\widehat{G}\end{pmatrix},
$$
$$
\boldsymbol{q}_t=\begin{pmatrix}q_t\\\widehat{q}_t\end{pmatrix},\ \boldsymbol{\rho}_t=\begin{pmatrix}\rho_t\\\widehat{\rho}_t\end{pmatrix},\ \boldsymbol{g}=\begin{pmatrix}g\\\widehat{g}\end{pmatrix},\ \boldsymbol{x}^i_t=\begin{pmatrix}x^i_t-\mathbb{E}[x^i_t]\\\mathbb{E}[x^i_t]\end{pmatrix},\ \boldsymbol{u}^i_t=\begin{pmatrix}u^i_t-\mathbb{E}[u^i_t]\\\mathbb{E}[u^i_t]\end{pmatrix},
$$

where $\mathbf{0}$ denotes zero matrices with appropriate dimensions.

After taking expectation on the both side of (3.1), we can obtain

$$
\begin{cases}
d\mathbb{E}[x^i_t]=\Big\{\widehat{A}_t\mathbb{E}[x^i_t]+\widehat{B}_t\mathbb{E}[u^i_t]+\mathbb{E}[m_t]\Big\}dt,\\
\mathbb{E}[x^i_0]=a,
\end{cases}
\tag{3.3}
$$

then the difference between (3.1) and (3.3) reads

$$
\begin{cases}
d(x^i_t-\mathbb{E}[x^i_t])=\Big\{A_t(x^i_t-\mathbb{E}[x^i_t])+B_t(u^i_t-\mathbb{E}[u^i_t])+m_t-\mathbb{E}[m_t]\Big\}dt\\
\qquad\qquad\qquad\qquad+\Big\{\widetilde{C}_t\mathbb{E}[x^i_t]+\widetilde{D}_t\mathbb{E}[u^i_t]+\sigma_t\Big\}dW^i_t+\sigma^0_t dW^0_t,\\
x^i_0-\mathbb{E}[x^i_0]=0.
\end{cases}
\tag{3.4}
$$

With these notations, the cost functional (3.2) can be rewritten as

$$
J^i\left(u^i_\cdot\right)=\mathbb{E}\Big\{\int_0^T\Big[\langle\boldsymbol{Q}_t\boldsymbol{x}^i_t,\boldsymbol{x}^i_t\rangle+2\langle\boldsymbol{S}_t\boldsymbol{u}^i_t,\boldsymbol{x}^i_t\rangle+\langle\boldsymbol{R}_t\boldsymbol{u}^i_t,\boldsymbol{u}^i_t\rangle+2\langle\boldsymbol{q}_t,\boldsymbol{x}^i_t\rangle+2\langle\boldsymbol{\rho}_t,\boldsymbol{u}^i_t\rangle\Big]dt+\langle\boldsymbol{G}\boldsymbol{x}^i_T,\boldsymbol{x}^i_T\rangle+2\langle\boldsymbol{g},\boldsymbol{x}^i_T\rangle\Big\}.
\tag{3.5}
$$

Now, we introduce the following positive definite (PD) assumption:

Assumption (PD).

$$\begin{pmatrix} \boldsymbol{Q}_t & \boldsymbol{S}_t \\ \boldsymbol{S}_t^\top & \boldsymbol{R}_t \end{pmatrix}, \quad \boldsymbol{R}_t \geq \delta I, \quad \boldsymbol{G} \geq 0, \quad \text{for some } \delta > 0 \text{ and any } \ t \in [0,T].$$

If Assumption (PD) holds, it is easy to verify that Problem (LMFL) is well-posed. Inspired by the results in Li, Li, and Yu [33], we are interested in studying Problem (LMFL) under indefinite condition in this paper. It should be emphasized that the relaxed compensator plays a key role in this process.

To start with, we define space

$$\Upsilon[0,T] = \{F_\cdot \mid F_t = F_0 + \int_0^t f_s ds, \ t \in [0,T], \ f_\cdot \in L^\infty(0,T;\mathcal{S}^n), \text{ where } F_\cdot \text{ does not depend on } x_\cdot^i \text{ and } u_\cdot^i.\}.$$

For any given $(H_\cdot, K_\cdot) \in \Upsilon[0,T] \times \Upsilon[0,T]$, we let

$$\boldsymbol{Q}_t^{H,K} = \begin{pmatrix} Q_t^{H,K} & \mathbf{0} \\ \mathbf{0} & \widehat{Q}_t^{H,K} \end{pmatrix}, \ \boldsymbol{S}_t^{H,K} = \begin{pmatrix} S_t^{H,K} & \mathbf{0} \\ \mathbf{0} & \widehat{S}_t^{H,K} \end{pmatrix}, \ \boldsymbol{R}_t^{H,K} = \begin{pmatrix} R_t^{H,K} & \mathbf{0} \\ \mathbf{0} & \widehat{R}_t^{H,K} \end{pmatrix},$$

$$\boldsymbol{G}^{H,K} = \begin{pmatrix} G^{H,K} & \mathbf{0} \\ \mathbf{0} & \widehat{G}^{H,K} \end{pmatrix}, \ \boldsymbol{q}_t^{H,K} = \begin{pmatrix} q_t^{H,K} \\ \widehat{q}_t^{H,K} \end{pmatrix}, \ \boldsymbol{\rho}_t^{H,K} = \begin{pmatrix} \rho_t^{H,K} \\ \widehat{\rho}_t^{H,K} \end{pmatrix}, \ \boldsymbol{g}^{H,K} = \begin{pmatrix} g^{H,K} \\ \widehat{g}^{H,K} \end{pmatrix},$$

where

$$\begin{aligned}
&Q_t^{H,K} = \dot{H}_t + H_t A_t + A_t^\top H_t + Q_t, \quad \widehat{Q}_t^{H,K} = \dot{K}_t + K_t \widehat{A}_t + \widehat{A}_t^\top K_t + \widetilde{C}_t^\top H_t \widetilde{C}_t + \widehat{Q}_t, \\
&S_t^{H,K} = S_t + H_t B_t, \quad \widehat{S}_t^{H,K} = \widehat{S}_t + K_t \widehat{B}_t + \widetilde{C}_t^\top H_t \widetilde{D}_t, \quad R_t^{H,K} = R_t, \quad \widehat{R}_t^{H,K} = \widehat{R}_t + \widetilde{D}_t^\top H_t \widetilde{D}_t, \\
&q_t^{H,K} = q_t + H_t(m_t - \mathbb{E}[m_t]), \quad \widehat{q}_t^{H,K} = \widehat{q}_t + \widetilde{C}_t^\top H_t \sigma_t + K_t \mathbb{E}[m_t], \quad \rho_t^{H,K} = \rho_t, \\
&\widehat{\rho}_t^{H,K} = \widehat{\rho}_t + \widetilde{D}_t^\top H_t \sigma_t, \quad G^{H,K} = G - H_T, \quad \widehat{G}^{H,K} = \widehat{G} - K_T, \quad g^{H,K} = g, \quad \widehat{g}^{H,K} = \widehat{g}.
\end{aligned}$$

Then we introduce

$$\begin{aligned}
J^{i,H,K}\left(u_\cdot^i\right) = \mathbb{E}\Big\{ \int_0^T & \Big[\langle \boldsymbol{Q}_t^{H,K} \boldsymbol{x}_t^i, \boldsymbol{x}_t^i \rangle + 2\langle \boldsymbol{S}_t^{H,K} \boldsymbol{u}_t^i, \boldsymbol{x}_t^i \rangle + \langle \boldsymbol{R}_t^{H,K} \boldsymbol{u}_t^i, \boldsymbol{u}_t^i \rangle + 2\langle \boldsymbol{q}_t^{H,K}, \boldsymbol{x}_t^i \rangle + 2\langle \boldsymbol{\rho}_t^{H,K}, \boldsymbol{u}_t^i \rangle \Big] dt \\
&+ \langle \boldsymbol{G}^{H,K} \boldsymbol{x}_T^i, \boldsymbol{x}_T^i \rangle + 2\langle \boldsymbol{g}^{H,K}, \boldsymbol{x}_T^i \rangle \Big\}.
\end{aligned} \tag{3.6}$$

For further study, we introduce an auxiliary problem with H and K, denoted by Problem (LMFL)^{H,K}.

Problem (LMFL)^{H,K}. For $1 \leq i \leq N$, find $\bar{u}_\cdot^i \in \mathcal{U}_i^d$ such that

$$J^{i,H,K}\left(\bar{u}_\cdot^i\right) = \inf_{u_\cdot^i \in \mathcal{U}_i^d} J^{i,H,K}\left(u_\cdot^i\right).$$

We notice that $J^i\left(u_\cdot^i\right)$ and $J^{i,H,K}\left(u_\cdot^i\right)$ are equivalent in the following lemma.

Lemma 3.1. For any given quantity $m_\cdot \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$ and any $u_\cdot^i \in \mathcal{U}_i^d$, let $(H_\cdot, K_\cdot) \in \Upsilon[0,T] \times \Upsilon[0,T]$, then

$$J^i\left(u_\cdot^i\right) = J^{i,H,K}\left(u_\cdot^i\right) + \langle K_0 a, a \rangle + \int_0^T [\langle H_t \sigma_t, \sigma_t \rangle + \langle H_t \sigma_t^0, \sigma_t^0 \rangle] dt.$$

Proof. Applying Itô's formula to $\langle H_\cdot(x_\cdot^i - \mathbb{E}[x_\cdot^i]), x_\cdot^i - \mathbb{E}[x_\cdot^i] \rangle$ and $\langle K_\cdot \mathbb{E}[x_\cdot^i], \mathbb{E}[x_\cdot^i] \rangle$, and then taking integral and expectation on the both side, we have

$$
\begin{aligned}
&\mathbb{E}[\langle H_T(x_T^i-\mathbb{E}[x_T^i]),x_T^i-\mathbb{E}[x_T^i]\rangle]+\mathbb{E}[\langle K_T\mathbb{E}[x_T^i],\mathbb{E}[x_T^i]\rangle]-\langle K_0a,a\rangle-\int_0^T[\langle H_t\sigma_t,\sigma_t\rangle+\langle H_t\sigma_t^0,\sigma_t^0\rangle]dt\\
=&\mathbb{E}\int_0^T[\langle(\dot H_t+H_tA_t+A_t^\top H_t)(x_t^i-\mathbb{E}[x_t^i]),x_t^i-\mathbb{E}[x_t^i]\rangle+\langle(\dot K_t+K_t\widehat A_t+\widehat A_t^\top K_t+\widetilde C_t^\top H_t\widetilde C_t)\mathbb{E}[x_t^i],\mathbb{E}[x_t^i]\rangle\\
&+2\langle H_tB_t(u_t^i-\mathbb{E}[u_t^i]),x_t^i-\mathbb{E}[x_t^i]\rangle+2\langle(K_t\widehat B_t+\widetilde C_t^\top H_t\widetilde D_t)\mathbb{E}[u_t^i],\mathbb{E}[x_t^i]\rangle+\langle\widetilde D_t^\top H_t\widetilde D_t\mathbb{E}[u_t^i],\mathbb{E}[u_t^i]\rangle\\
&+2\langle H_t(m_t-\mathbb{E}[m_t]),x_t^i-\mathbb{E}[x_t^i]\rangle+2\langle(\widetilde C_t^\top H_t\sigma_t+K_t\mathbb{E}[m_t]),\mathbb{E}[x_t^i]\rangle+2\langle\widetilde D_t^\top H_t\sigma_t,\mathbb{E}[u_t^i]\rangle]dt,
\end{aligned}
$$

where $H_\cdot$, $\sigma_\cdot$ and $\sigma_\cdot^0$ are deterministic functions.

Combining the above relationship with (3.5) and (3.6), we can obtain the desired result. □

Definition 3.1. If there exists a pair of functions $(H_\cdot,K_\cdot)\in\Upsilon[0,T]\times\Upsilon[0,T]$ such that $(\boldsymbol{Q}_\cdot^{H,K},\boldsymbol{S}_\cdot^{H,K},\boldsymbol{R}_\cdot^{H,K},\boldsymbol{G}^{H,K})$ satisfies Assumption (PD), then $(H_\cdot,K_\cdot)$ is called a relaxed compensator of Problem (LMFL).

It is clear that if there exists a corresponding relaxed compensator, Problem (LMFL) is well-posed. Next, we provide the open-loop decentralized optimal strategies of Problem (LMFL) under indefinite condition.

Theorem 3.1. For any given quantity $m_\cdot\in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$, if there exists a relaxed compensator $(H_\cdot,K_\cdot)\in\Upsilon[0,T]\times\Upsilon[0,T]$, then the following stochastic Hamiltonian system, which consists of an algebra equation and a MF-FBSDE

$$
\begin{cases}
0=R_t\bar u_t^i+\widetilde R_t\mathbb{E}[\bar u_t^i]+B_t^\top\bar y_t^i+\widetilde B_t^\top\mathbb{E}[\bar y_t^i]+\widetilde D_t^\top\mathbb{E}[\bar z_t^i]\\
\qquad+S_t^\top\bar x_t^i+\widetilde S_t^\top\mathbb{E}[\bar x_t^i]+\widehat\rho_t,\\
d\bar x_t^i=\Big\{A_t\bar x_t^i+\widetilde A_t\mathbb{E}[\bar x_t^i]+B_t\bar u_t^i+\widetilde B_t\mathbb{E}[\bar u_t^i]+m_t\Big\}dt\\
\qquad+\Big\{\widetilde C_t\mathbb{E}[\bar x_t^i]+\widetilde D_t\mathbb{E}[\bar u_t^i]+\sigma_t\Big\}dW_t^i+\sigma_t^0dW_t^0,\\
d\bar y_t^i=-\Big\{A_t^\top\bar y_t^i+\widetilde A_t^\top\mathbb{E}[\bar y_t^i]+\widetilde C_t^\top\mathbb{E}[\bar z_t^i]+Q_t\bar x_t^i+\widetilde Q_t\mathbb{E}[\bar x_t^i]\\
\qquad+S_t\bar u_t^i+\widetilde S_t\mathbb{E}[\bar u_t^i]+\widehat q_t\Big\}dt+\bar z_t^idW_t^i+\bar z_t^0dW_t^0,\\
\bar x_0^i=a,\ \bar y_T^i=G\bar x_T^i+\widetilde G\mathbb{E}[\bar x_T^i]+\widehat g,
\end{cases}
\tag{3.7}
$$

admits a unique solution $(\bar x_\cdot^i,\bar u_\cdot^i,\bar y_\cdot^i,\bar z_\cdot^i,\bar z_\cdot^0)\in\mathcal{L}^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times\mathcal{U}_i^d\times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)$. Moreover, $(\bar x_\cdot^i,\bar u_\cdot^i)$ is the unique decentralized optimal pair of Problem (LMFL).

Proof. If $(H_\cdot,K_\cdot)\in\Upsilon[0,T]\times\Upsilon[0,T]$ is a relaxed compensator, then $(\boldsymbol{Q}_\cdot^{H,K},\boldsymbol{S}_\cdot^{H,K},\boldsymbol{R}_\cdot^{H,K},\boldsymbol{G}^{H,K})$ satisfies Assumption (PD). Then by Theorem 3.4 in [27], we can prove the following stochastic Hamiltonian system

$$
\begin{cases}
0=\ R_t^{H,K}\bar u_t^{i,H,K}+(\widehat R_t^{H,K}-R_t^{H,K})\mathbb{E}[\bar u_t^{i,H,K}]+B_t^\top\bar y_t^{i,H,K}+\widetilde B_t^\top\mathbb{E}[\bar y_t^{i,H,K}]\\
\qquad+\widetilde D_t^\top\mathbb{E}[\bar z_t^{i,H,K}]+(S_t^{H,K})^\top\bar x_t^{i,H,K}+(\widehat S_t^{H,K}-S_t^{H,K})^\top\mathbb{E}[\bar x_t^{i,H,K}]+\widehat\rho_t^{H,K},\\
d\bar x_t^{i,H,K}=\ \Big\{A_t\bar x_t^{i,H,K}+\widetilde A_t\mathbb{E}[\bar x_t^{i,H,K}]+B_t\bar u_t^{i,H,K}+\widetilde B_t\mathbb{E}[\bar u_t^{i,H,K}]+m_t\Big\}dt\\
\qquad+\Big\{\widetilde C_t\mathbb{E}[\bar x_t^{i,H,K}]+\widetilde D_t\mathbb{E}[\bar u_t^{i,H,K}]+\sigma_t\Big\}dW_t^i+\sigma_t^0dW_t^0,\\
d\bar y_t^{i,H,K}=-\Big\{A_t^\top\bar y_t^{i,H,K}+\widetilde A_t^\top\mathbb{E}[\bar y_t^{i,H,K}]+\widetilde C_t^\top\mathbb{E}[\bar z_t^{i,H,K}]+Q_t^{H,K}\bar x_t^{i,H,K}\\
\qquad+(\widehat Q_t^{H,K}-Q_t^{H,K})\mathbb{E}[\bar x_t^{i,H,K}]+S_t^{H,K}\bar u_t^{i,H,K}+(\widehat S_t^{H,K}-S_t^{H,K})\mathbb{E}[\bar u_t^{i,H,K}]\\
\qquad+q_t^{H,K}+\mathbb{E}[\widehat q_t^{H,K}-q_t^{H,K}]\Big\}dt+\bar z_t^{i,H,K}dW_t^i+\bar z_t^{0,H,K}dW_t^0,\\
\bar x_0^{i,H,K}=\ a,\ \bar y_T^{i,H,K}=\ G^{H,K}\bar x_T^{i,H,K}+(\widehat G^{H,K}-G^{H,K})\mathbb{E}[\bar x_T^{i,H,K}]+\widehat g^{H,K},
\end{cases}
\tag{3.8}
$$

linked to Problem (LMFL)^{H,K} admits a unique solution $(\bar{x}^{i,H,K}_\cdot, \bar{u}^{i,H,K}_\cdot, \bar{y}^{i,H,K}_\cdot, \bar{z}^{i,H,K}_\cdot, \bar{z}^{0,H,K}_\cdot) \in \mathcal{L}^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times \mathcal{U}^d_i \times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)$. Moreover, $(\bar{x}^{i,H,K}_\cdot, \bar{u}^{i,H,K}_\cdot)$ is the unique decentralized optimal pair of Problem (LMFL)^{H,K}.

We next prove the solvability equivalence between (3.7) and (3.8). In fact, if $(\bar{x}^{i,H,K}_\cdot, \bar{u}^{i,H,K}_\cdot, \bar{y}^{i,H,K}_\cdot, \bar{z}^{i,H,K}_\cdot, \bar{z}^{0,H,K}_\cdot)$ is a solution to (3.8), the following relationships

$$\begin{aligned}
&\bar{x}^i_\cdot = \bar{x}^{i,H,K}_\cdot, \quad \bar{u}^i_\cdot = \bar{u}^{i,H,K}_\cdot, \quad \bar{y}^i_\cdot = \bar{y}^{i,H,K}_\cdot + H_\cdot(\bar{x}^i_\cdot - \mathbb{E}[\bar{x}^i_\cdot]) + K_\cdot\mathbb{E}[\bar{x}^i_\cdot],\\
&\bar{z}^i_\cdot = \bar{z}^{i,H,K}_\cdot + H_\cdot(\widetilde{C}_\cdot\mathbb{E}[\bar{x}^i_\cdot] + \widetilde{D}_\cdot\mathbb{E}[\bar{u}^i_\cdot] + \sigma_\cdot), \quad \bar{z}^0_\cdot = \bar{z}^{0,H,K}_\cdot + H_\cdot\sigma^0_\cdot,
\end{aligned}$$

provide a solution $(\bar{x}^i_\cdot, \bar{u}^i_\cdot, \bar{y}^i_\cdot, \bar{z}^i_\cdot, \bar{z}^0_\cdot)$ to (3.7). On the other hand, since the above relationships are invertible, we have that the transformation yields a solution to (3.8).

Therefore, due to the existence and uniqueness of solution to (3.8) and the solvability equivalence between (3.7) and (3.8), we conclude that (3.7) admits a unique solution $(\bar{x}^i_\cdot, \bar{u}^i_\cdot, \bar{y}^i_\cdot, \bar{z}^i_\cdot, \bar{z}^0_\cdot)$. Moreover, by Lemma 3.1, the unique decentralized optimal pair $(\bar{x}^i_\cdot, \bar{u}^i_\cdot) = (\bar{x}^{i,H,K}_\cdot, \bar{u}^{i,H,K}_\cdot)$ of Problem (LMFL)^{H,K} is also the unique decentralized optimal pair of Problem (LMFL), which completes the proof. □

Noting that the decentralized optimal strategies are characterized by $(\bar{x}^i_\cdot, \bar{u}^i_\cdot, \bar{y}^i_\cdot, \bar{z}^i_\cdot, \bar{z}^0_\cdot)$, which is the unique solution of (3.7). In the below, we will focus on the feedback representation of decentralized strategies.

Theorem 3.2. For any given quantity $m_\cdot \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$, if there exists a relaxed compensator $(H_\cdot, K_\cdot) \in \Upsilon[0,T]\times\Upsilon[0,T]$, then the following Riccati equations system

$$\begin{cases}
\dot{P}_t + P_tA_t + A_t^\top P_t + Q_t - (S_t + P_tB_t)R_t^{-1}(S_t+P_tB_t)^\top = 0,\\
P_T = G,
\end{cases} \tag{3.9}$$

$$\begin{cases}
\dot{\Sigma}_t + \Sigma_t\widehat{A}_t + \widehat{A}_t^\top\Sigma_t + \widetilde{C}_t^\top P_t\widetilde{C}_t + \widehat{Q}_t - \Gamma_t\Lambda_t^{-1}\Gamma_t^\top = 0,\\
\Sigma_T = \widehat{G},
\end{cases} \tag{3.10}$$

admits a unique solution $(P_\cdot, \Sigma_\cdot) \in C([0,T];\mathcal{S}^n)\times C([0,T];\mathcal{S}^n)$ and the following MF-BSDE

$$\begin{cases}
d\Phi_t = -\Big\{[A_t^\top - (S_t+P_tB_t)R_t^{-1}B_t^\top](\Phi_t - \mathbb{E}[\Phi_t]) + \widehat{A}_t^\top\mathbb{E}[\Phi_t] - \Gamma_t\Lambda_t^{-1}\Pi_t\\
\qquad\quad + \widetilde{C}_t^\top P_t\sigma_t + P_t(m_t - \mathbb{E}[m_t]) + \Sigma_t\mathbb{E}[m_t] + \widehat{q}_t\Big\}dt + \Theta_t dW^0_t,\\
\Phi_T = \widehat{g},
\end{cases} \tag{3.11}$$

admits a unique pair of solutions $(\Phi_\cdot, \Theta_\cdot) \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^{n\times d})$ with

$$\Lambda_t = \widehat{R}_t + \widetilde{D}_t^\top P_t\widetilde{D}_t, \quad \Gamma_t = \Sigma_t\widehat{B}_t + \widetilde{C}_t^\top P_t\widetilde{D}_t + \widehat{S}_t, \quad \Pi_t = \widehat{B}_t^\top\mathbb{E}[\Phi_t] + \widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t.$$

Moreover, Problem (LMFL) admits a unique feedback representation of decentralized optimal strategies

$$\bar{u}^i_t = -R_t^{-1}(S_t+P_tB_t)^\top(\bar{x}^i_t - \mathbb{E}[\bar{x}^i_t]) - \Lambda_t^{-1}\Gamma_t^\top\mathbb{E}[\bar{x}^i_t] - R_t^{-1}B_t^\top(\Phi_t - \mathbb{E}[\Phi_t]) - \Lambda_t^{-1}\Pi_t, \tag{3.12}$$

where the decentralized optimal state trajectory $\bar{x}^i_\cdot$ is determined by the following MF-SDE

$$\begin{cases}
d\bar{x}^i_t = \Big\{[A_t - B_tR_t^{-1}(S_t+P_tB_t)^\top](\bar{x}^i_t - \mathbb{E}[\bar{x}^i_t]) + (\widehat{A}_t - \widehat{B}_t\Lambda_t^{-1}\Gamma_t^\top)\mathbb{E}[\bar{x}^i_t]\\
\qquad\quad - B_tR_t^{-1}B_t^\top(\Phi_t - \mathbb{E}[\Phi_t]) - \widehat{B}_t\Lambda_t^{-1}\Pi_t + m_t\Big\}dt\\
\qquad\quad + \Big\{(\widetilde{C}_t - \widetilde{D}_t\Lambda_t^{-1}\Gamma_t^\top)\mathbb{E}[\bar{x}^i_t] - \widetilde{D}_t\Lambda_t^{-1}\Pi_t + \sigma_t\Big\}dW^i_t + \sigma^0_t dW^0_t,\\
\bar{x}^i_0 = a.
\end{cases} \tag{3.13}$$

Proof. If $(H_\cdot, K_\cdot) \in \Upsilon[0,T] \times \Upsilon[0,T]$ is a relaxed compensator, then $(\boldsymbol{Q}_\cdot^{H,K}, \boldsymbol{S}_\cdot^{H,K}, \boldsymbol{R}_\cdot^{H,K}, \boldsymbol{G}^{H,K})$ satisfies Assumption (PD). Inspired by Yong [20] and from (3.8), we let

$$\bar{y}_t^{i,H,K} = P_t^{H,K}(\bar{x}_t^{i,H,K} - \mathbb{E}[\bar{x}_t^{i,H,K}]) + \Sigma_t^{H,K}\mathbb{E}[\bar{x}_t^{i,H,K}] + \Phi_t^{H,K},$$

where $(P_\cdot^{H,K}, \Sigma_\cdot^{H,K})$ is the unique solution to the following Riccati equations system

$$\begin{cases} \dot{P}_t^{H,K} + P_t^{H,K}A_t + A_t^\top P_t^{H,K} + Q_t^{H,K} - (S_t^{H,K} + P_t^{H,K}B_t)(R_t^{H,K})^{-1}(S_t^{H,K} + P_t^{H,K}B_t)^\top = 0, \\ P_T^{H,K} = G^{H,K}, \end{cases} \tag{3.14}$$

$$\begin{cases} \dot{\Sigma}_t^{H,K} + \Sigma_t^{H,K}\widehat{A}_t + \widehat{A}_t^\top \Sigma_t^{H,K} + \widetilde{C}_t^\top P_t^{H,K}\widetilde{C}_t + \widehat{Q}_t^{H,K} - \Gamma_t^{H,K}(\Lambda_t^{H,K})^{-1}(\Gamma_t^{H,K})^\top = 0, \\ \Sigma_T^{H,K} = \widehat{G}^{H,K}, \end{cases} \tag{3.15}$$

and $(\Phi_\cdot^{H,K}, \Theta_\cdot^{H,K})$ is the unique pair of solutions to the following MF-BSDE

$$\begin{cases} d\Phi_t^{H,K} = -\Big\{[A_t^\top - (S_t^{H,K} + P_t^{H,K}B_t)(R_t^{H,K})^{-1}B_t^\top](\Phi_t^{H,K} - \mathbb{E}[\Phi_t^{H,K}]) + \widehat{A}_t^\top \mathbb{E}[\Phi_t^{H,K}] \\ \qquad\qquad - \Gamma_t^{H,K}(\Lambda_t^{H,K})^{-1}\Pi_t^{H,K} + \widetilde{C}_t^\top P_t^{H,K}\sigma_t + P_t^{H,K}(m_t - \mathbb{E}[m_t]) \\ \qquad\qquad + \Sigma_t^{H,K}\mathbb{E}[m_t] + q_t^{H,K} + \mathbb{E}[\widehat{q}_t^{H,K} - q_t^{H,K}]\Big\}dt + \Theta_t^{H,K}dW_t^0, \\ \Phi_T^{H,K} = \widehat{g}^{H,K}, \end{cases} \tag{3.16}$$

with

$$\Lambda_t^{H,K} = \widehat{R}_t^{H,K} + \widetilde{D}_t^\top P_t^{H,K}\widetilde{D}_t, \quad \Gamma_t^{H,K} = \Sigma_t^{H,K}\widehat{B}_t + \widetilde{C}_t^\top P_t^{H,K}\widetilde{D}_t + \widehat{S}_t^{H,K}, \quad \Pi_t^{H,K} = \widehat{B}_t^\top \mathbb{E}[\Phi_t^{H,K}] + \widetilde{D}_t^\top P_t^{H,K}\sigma_t + \widehat{\rho}_t^{H,K}.$$

Noting that Riccati equations (3.14) and (3.15) are the special cases of (12) and (13) in Li, Li, and Yu [35]. By Theorem 3.1 in [35], (3.14) and (3.15) admit the the unique solutions $(P_\cdot^{H,K}, \Sigma_\cdot^{H,K}) \in C([0,T];\mathcal{S}_+^n) \times C([0,T];\mathcal{S}_+^n)$. Moreover, we can verify that MF-BSDE (3.16) admits a unique pair of solutions $(\Phi_\cdot^{H,K}, \Theta_\cdot^{H,K}) \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n) \times L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^{n\times d})$ by Proposition 2.6 in [27].

Therefore, Problem (LMFL)^{H,K} admits a unique feedback representation of decentralized optimal strategies

$$\begin{aligned} \bar{u}_t^i = & -(R_t^{H,K})^{-1}(S_t^{H,K} + P_t^{H,K}B_t)^\top(\bar{x}_t^{i,H,K} - \mathbb{E}[\bar{x}_t^{i,H,K}]) - (\Lambda_t^{H,K})^{-1}(\Gamma_t^{H,K})^\top \mathbb{E}[\bar{x}_t^{i,H,K}] \\ & - (R_t^{H,K})^{-1}B_t^\top(\Phi_t^{H,K} - \mathbb{E}[\Phi_t^{H,K}]) - (\Lambda_t^{H,K})^{-1}\Pi_t^{H,K}, \end{aligned} \tag{3.17}$$

where the decentralized optimal state trajectory $\bar{x}_\cdot^{i,H,K}$ is determined by the following MF-SDE

$$\begin{cases} d\bar{x}_t^{i,H,K} = \Big\{[A_t - B_t(R_t^{H,K})^{-1}(S_t^{H,K} + P_t^{H,K}B_t)^\top](\bar{x}_t^{i,H,K} - \mathbb{E}[\bar{x}_t^{i,H,K}]) + [\widehat{A}_t - \widehat{B}_t(\Lambda_t^{H,K})^{-1}(\Gamma_t^{H,K})^\top]\mathbb{E}[\bar{x}_t^{i,H,K}] \\ \qquad\qquad - B_t(R_t^{H,K})^{-1}B_t^\top(\Phi_t^{H,K} - \mathbb{E}[\Phi_t^{H,K}]) - \widehat{B}_t(\Lambda_t^{H,K})^{-1}\Pi_t^{H,K} + m_t\Big\}dt \\ \qquad\qquad + \Big\{[\widetilde{C}_t - \widetilde{D}_t(\Lambda_t^{H,K})^{-1}(\Gamma_t^{H,K})^\top]\mathbb{E}[\bar{x}_t^{i,H,K}] - \widetilde{D}_t(\Lambda_t^{H,K})^{-1}\Pi_t^{H,K} + \sigma_t\Big\}dW_t^i + \sigma_t^0 dW_t^0, \\ \bar{x}_0^{i,H,K} = a. \end{cases} \tag{3.18}$$

If $P_\cdot^{H,K}, \Sigma_\cdot^{H,K}$ and $(\Phi_\cdot^{H,K}, \Theta_\cdot^{H,K})$ are the solutions to (3.14)-(3.16), then

$$P_\cdot = P_\cdot^{H,K} + H_\cdot, \quad \Sigma_\cdot = \Sigma_\cdot^{H,K} + K_\cdot, \quad \Phi_\cdot = \Phi_\cdot^{H,K}, \tag{3.19}$$

provide solutions to (3.9)-(3.11). By contrast, if $P_\cdot, \Sigma_\cdot$ and $(\Phi_\cdot, \Theta_\cdot)$ are solutions to (3.9)-(3.11), the invertible transformations present solutions to (3.14)-(3.16). Consequently, we obtain the solvability equivalence between (3.9)-(3.11) and (3.14)-(3.16). Due to the existence and uniqueness of solutions to (3.14)-(3.16), we prove that (3.9)-(3.11) admit unique solutions. Based on (3.19), we can easily verify that $\bar{x}^i_\cdot = \bar{x}^{i,H,K}_\cdot$ and $\bar{u}^i_\cdot = \bar{u}^{i,H,K}_\cdot$. Since $(\bar{x}^{i,H,K}_\cdot, \bar{u}^{i,H,K}_\cdot)$ is the unique decentralized optimal pair of Problem (LMFL)^{H,K} and then by cost functional equivalence results in Lemma 3.1, we also conclude that $(\bar{x}^i_\cdot, \bar{u}^i_\cdot)$ is the unique decentralized optimal pair of Problem (LMFL). □

In the following, we aim to investigate the frozen control average limit and the related mean-field NCE equation systems. Firstly, the decentralized optimal state trajectory $\bar{x}^i_\cdot$ can be rewritten as

$$\begin{cases} d\bar{x}^i_t = \Big\{[A_t - B_t R_t^{-1}(S_t + P_t B_t)^\top]\bar{x}^i_t + [\widetilde{A}_t + B_t R_t^{-1}(S_t + P_t B_t)^\top - \widehat{B}_t \Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[\bar{x}^i_t] \\ \qquad - B_t R_t^{-1} B_t^\top \Phi_t + (B_t R_t^{-1} B_t^\top - \widehat{B}_t \Lambda_t^{-1} \widehat{B}_t^\top)\mathbb{E}[\Phi_t] - \widehat{B}_t \Lambda_t^{-1}(\widetilde{D}_t^\top P_t \sigma_t + \widehat{\rho}_t) + m_t\Big\}dt \\ \qquad + \Big\{(\widetilde{C}_t - \widetilde{D}_t \Lambda_t^{-1}\Gamma_t^\top)\mathbb{E}[\bar{x}^i_t] - \widetilde{D}_t \Lambda_t^{-1}\widehat{B}_t^\top \mathbb{E}[\Phi_t] - \widetilde{D}_t \Lambda_t^{-1}(\widetilde{D}_t^\top P_t \sigma_t + \widehat{\rho}_t) + \sigma_t\Big\}dW^i_t + \sigma^0_t dW^0_t, \\ \bar{x}^i_0 = a. \end{cases} \tag{3.20}$$

For further study, we introduce two equations as follows.

$$\begin{cases} dx^{i,1}_t = \Big\{[A_t - B_t R_t^{-1}(S_t + P_t B_t)^\top]x^{i,1}_t + [\widetilde{A}_t + B_t R_t^{-1}(S_t + P_t B_t)^\top - \widehat{B}_t \Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x^{i,1}_t]\Big\}dt \\ \qquad + \Big\{(\widetilde{C}_t - \widetilde{D}_t \Lambda_t^{-1}\Gamma_t^\top)(\mathbb{E}[x^{i,1}_t] + \mathbb{E}[x^2_t]) - \widetilde{D}_t \Lambda_t^{-1}\widehat{B}_t^\top \mathbb{E}[\Phi_t] - \widetilde{D}_t \Lambda_t^{-1}(\widetilde{D}_t^\top P_t \sigma_t + \widehat{\rho}_t) + \sigma_t\Big\}dW^i_t, \\ x^{i,1}_0 = a^1, \end{cases} \tag{3.21}$$

and

$$\begin{cases} dx^2_t = \Big\{[A_t - B_t R_t^{-1}(S_t + P_t B_t)^\top]x^2_t + [\widetilde{A}_t + B_t R_t^{-1}(S_t + P_t B_t)^\top - \widehat{B}_t \Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x^2_t] \\ \qquad - B_t R_t^{-1} B_t^\top \Phi_t + (B_t R_t^{-1} B_t^\top - \widehat{B}_t \Lambda_t^{-1} \widehat{B}_t^\top)\mathbb{E}[\Phi_t] - \widehat{B}_t \Lambda_t^{-1}(\widetilde{D}_t^\top P_t \sigma_t + \widehat{\rho}_t) + m_t\Big\}dt + \sigma^0_t dW^0_t, \\ x^2_0 = a^2, \end{cases} \tag{3.22}$$

where $a = a^1 + a^2$.

Noting that (3.11) can be rewritten as

$$\begin{cases} d\Phi_t = -\Big\{[A_t^\top - (S_t + P_t B_t)R_t^{-1} B_t^\top]\Phi_t + [\widetilde{A}_t^\top + (S_t + P_t B_t)R_t^{-1} B_t^\top - \Gamma_t \Lambda_t^{-1}\widehat{B}^\top]\mathbb{E}[\Phi_t] \\ \qquad - \Gamma_t \Lambda_t^{-1}(\widetilde{D}_t^\top P_t \sigma_t + \widehat{\rho}_t) + \widetilde{C}_t^\top P_t \sigma_t + P_t(m_t - \mathbb{E}[m_t]) + \Sigma_t \mathbb{E}[m_t] + \widehat{q}_t\Big\}dt + \Theta_t dW^0_t, \\ \Phi_T = \widehat{g}. \end{cases} \tag{3.23}$$

We emphasize that (3.21) is driven by individual noise W^i while (3.22) is driven by common noise W^0. Equation (3.22) includes the frozen limit term while (3.21) not. Moreover, the diffusion term of (3.21) only depends on $\mathbb{E}[x^2_\cdot]$ and $\mathbb{E}[\Phi_\cdot]$, which are all the deterministic functions. Based on these facts, we conclude that the individual noise W^i and the common noise W^0 are completely seperated.

Anything else, since state $\bar{x}^i_\cdot$ satisfies (3.20), by comparing the three equations (3.20)-(3.22), it is easy to verify that $\bar{x}^i_\cdot = x^{i,1}_\cdot + x^2_\cdot$, for $1 \le i \le N$.

To start with, we introduce the following ordinary differential equation (ODE)

$$\begin{cases} dx_t^1 = (\widehat{A}_t - \widehat{B}_t\Lambda_t^{-1}\Gamma_t^\top)x_t^1 dt, \\ x_0^1 = a^1, \end{cases} \tag{3.24}$$

a MF-SDE

$$\begin{cases} dx_t^2 = \Big\{[A_t - (B_t+\bar{B}_t)R_t^{-1}(S_t+P_tB_t)^\top]x_t^2 + [\widetilde{A}_t + (B_t+\bar{B}_t)R_t^{-1}(S_t+P_tB_t)^\top \\ \qquad - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x_t^2] - \bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x_t^1 - (B_t+\bar{B}_t)R_t^{-1}B_t^\top\Phi_t + [(B_t+\bar{B}_t)R_t^{-1}B_t^\top \\ \qquad - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}\widehat{B}_t^\top]\mathbb{E}[\Phi_t] - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t)\Big\}dt + \sigma_t^0 dW_t^0, \\ x_0^2 = a^2, \end{cases} \tag{3.25}$$

and a MF-BSDE

$$\begin{cases} d\Phi_t = -\Big\{[A_t^\top - (S_t+P_tB_t+P_t\bar{B}_t)R_t^{-1}B_t^\top]\Phi_t + [\widetilde{A}_t^\top + (S_t+P_tB_t+P_t\bar{B}_t)R_t^{-1}B_t^\top \\ \qquad - (\Gamma_t+\Sigma_t\bar{B}_t)\Lambda_t^{-1}\widehat{B}^\top]\mathbb{E}[\Phi_t] - \Sigma_t\bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x_t^1 - P_t\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top x_t^2 \\ \qquad + [P_t\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top - \Sigma_t\bar{B}_t\Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x_t^2] - (\Gamma_t+\Sigma_t\bar{B}_t)\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) \\ \qquad + \widetilde{C}_t^\top P_t\sigma_t + \widehat{q}_t\Big\}dt + \Theta_t dW_t^0, \\ \Phi_T = \widehat{g}. \end{cases} \tag{3.26}$$

We declare that the above three equations (3.24)-(3.26) are called the mean-field NCE equation systems, which enable the agents to determine the frozen control average limit $m_\cdot$ in current setup. Some subtle analysis can be found in the proposition below.

Proposition 3.1. If there exists a relaxed compensator $(H_\cdot, K_\cdot) \in \Upsilon[0,T]\times\Upsilon[0,T]$, then the control average limit $m_\cdot \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$ has the following form

$$\begin{aligned} m_t =& -\bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x_t^1 - \bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top x_t^2 - \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top]\mathbb{E}[x_t^2] \\ & - \bar{B}_tR_t^{-1}B_t^\top\Phi_t - \bar{B}_t(\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top)\mathbb{E}[\Phi_t] - \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t), \end{aligned} \tag{3.27}$$

where $P_\cdot$ and $\Sigma_\cdot$ are the solutions of Riccati equations (3.9), (3.10) and $x_\cdot^1$, $x_\cdot^2$, $\Phi_\cdot$ are determined by the equations (3.24)-(3.26), respectively.

Proof. Noting that if there exists a relaxed compensator $(H_\cdot, K_\cdot) \in \Upsilon[0,T]\times\Upsilon[0,T]$, the decentralized optimal strategies has feedback representation form. Moreover, based on the above analysis, we have $\bar{x}_\cdot^j = x_\cdot^{j,1} + x_\cdot^2$, for $1 \le j \le N$, $j \ne i$. By (3.12), we deduce

$$\begin{aligned} m_t =& -\lim_{N\to\infty}\frac{1}{N-1}\sum_{j=1,j\neq i}^N \bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top\bar{x}_t^j - \lim_{N\to\infty}\frac{1}{N-1}\sum_{j=1,j\neq i}^N \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top]\mathbb{E}[\bar{x}_t^j] \\ & - \bar{B}_tR_t^{-1}B_t^\top\Phi_t - \bar{B}_t[\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top]\mathbb{E}[\Phi_t] - \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) \\ =& -\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top(\mathbb{E}[x_t^{i,1}] + x_t^2) - \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top](\mathbb{E}[x_t^{i,1}] + \mathbb{E}[x_t^2]) \\ & - \bar{B}_tR_t^{-1}B_t^\top\Phi_t - \bar{B}_t[\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top]\mathbb{E}[\Phi_t] - \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) \\ =& -\bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x_t^1 - \bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top x_t^2 - \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top]\mathbb{E}[x_t^2] \\ & - \bar{B}_tR_t^{-1}B_t^\top\Phi_t - \bar{B}_t(\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top)\mathbb{E}[\Phi_t] - \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t), \end{aligned} \tag{3.28}$$

where $x^1_\cdot = \mathbb{E}[x^{i,1}_\cdot]$ satisfies (3.24), $x^2_\cdot$, $\Phi_\cdot$ satisfy (3.22) and (3.23). Substituting (3.28) into (3.22) and (3.23), one can get the equivalent forms (3.25) and (3.26).

If introduce the following notations

$$\begin{aligned}\Xi^1_t :=& -\bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x^1_t - \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top]\mathbb{E}[x^2_t]\\ &- \bar{B}_t[\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top]\mathbb{E}[\Phi_t] - \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t),\\ \Xi^2_t :=& -\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top x^2_t - \bar{B}_tR_t^{-1}B_t^\top\Phi_t,\end{aligned}$$

we can obtain $m_\cdot = \Xi^1_\cdot + \Xi^2_\cdot$. Due to $\Xi^1_\cdot$ is a deterministic function and $\Xi^2_\cdot$ belongs to $L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$, we can conclude that $m_\cdot \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$. Then the desired result is clear. $\square$

Remark 3.2. In our setting, some *separation techniques* can be applied to determine the control average limit $m_\cdot$ and the mean-field NCE equation systems. By introducing (3.21) and (3.22), the original complicated system (3.20) can be separated into two mean-field SDEs, which are relatively simple to handle (W^i and W^0 are in completely separated manner). Moreover,

(i) If the control average appears in the diffusion term in W^0 of state equation (2.1), the separation technique can work. In fact, we can still use (3.21) and introduce the following equation

$$\begin{cases} dx^2_t = \Big\{[A_t - B_tR_t^{-1}(S_t+P_tB_t)^\top]x^2_t + [\widetilde{A}_t + B_tR_t^{-1}(S_t+P_tB_t)^\top - \widehat{B}_t\Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x^2_t]\\ \qquad - B_tR_t^{-1}B_t^\top\Phi_t + (B_tR_t^{-1}B_t^\top - \widehat{B}_t\Lambda_t^{-1}\widehat{B}_t^\top)\mathbb{E}[\Phi_t] - \widehat{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) + m_t\Big\}dt\\ \qquad + (m_t + \sigma^0_t)dW^0_t,\\ x^2_0 = a^2,\end{cases}$$

to separate the original system

$$\begin{cases} d\bar{x}^i_t = \Big\{[A_t - B_tR_t^{-1}(S_t+P_tB_t)^\top]\bar{x}^i_t + [\widetilde{A}_t + B_tR_t^{-1}(S_t+P_tB_t)^\top - \widehat{B}_t\Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[\bar{x}^i_t]\\ \qquad - B_tR_t^{-1}B_t^\top\Phi_t + (B_tR_t^{-1}B_t^\top - \widehat{B}_t\Lambda_t^{-1}\widehat{B}_t^\top)\mathbb{E}[\Phi_t] - \widehat{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) + m_t\Big\}dt\\ \qquad + \Big\{(\widetilde{C}_t - \widetilde{D}_t\Lambda_t^{-1}\Gamma_t^\top)\mathbb{E}[\bar{x}^i_t] - \widetilde{D}_t\Lambda_t^{-1}\widehat{B}_t^\top\mathbb{E}[\Phi_t] - \widetilde{D}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) + \sigma_t\Big\}dW^i_t\\ \qquad + (m_t + \sigma^0_t)dW^0_t,\\ \bar{x}^i_0 = a.\end{cases}$$

By similar arguments, we can obtain that $m_\cdot$ has the same form as (3.27), but the corresponding mean-filed NCE equation systems will be composed of (3.24), (3.26) and

$$\begin{cases} dx^2_t = \Big\{[A_t - (B_t+\bar{B}_t)R_t^{-1}(S_t+P_tB_t)^\top]x^2_t + [\widetilde{A}_t + (B_t+\bar{B}_t)R_t^{-1}(S_t+P_tB_t)^\top\\ \qquad - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}\Gamma_t^\top]\mathbb{E}[x^2_t] - \bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x^1_t - (B_t+\bar{B}_t)R_t^{-1}B_t^\top\Phi_t + [(B_t+\bar{B}_t)R_t^{-1}B_t^\top\\ \qquad - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}\widehat{B}_t^\top]\mathbb{E}[\Phi_t] - (\widehat{B}_t+\bar{B}_t)\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t)\Big\}dt\\ \qquad - \Big\{\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top x^2_t + \bar{B}_t[\Lambda_t^{-1}\Gamma_t^\top - R_t^{-1}(S_t+P_tB_t)^\top]\mathbb{E}[x^2_t] + \bar{B}_t\Lambda_t^{-1}\Gamma_t^\top x^1_t\\ \qquad + \bar{B}_tR_t^{-1}B_t^\top\Phi_t + \bar{B}_t(\Lambda_t^{-1}\widehat{B}_t^\top - R_t^{-1}B_t^\top)\mathbb{E}[\Phi_t] + \bar{B}_t\Lambda_t^{-1}(\widetilde{D}_t^\top P_t\sigma_t + \widehat{\rho}_t) - \sigma^0_t\Big\}dW^0_t,\\ x^2_0 = a^2.\end{cases}$$

Other results in the following part of this paper can also be obtained similarly in this case. Considering the length and heavy notation, we do not repeat here.

(ii) If the control average appears in the diffusion term in W^i, or $x^i_\cdot$ and $u^i_\cdot$ appear in the diffusion term in W^0 of state equation (2.1), the separation technique does not work. In particular, there will be $m_t dW^i_t$ or $x^i_t dW^0_t$ in (3.20) correspondingly, which yields the technical obstacle. Since $m_\cdot \in L^2_{\mathbb{F}^0}(0,T;\mathbb{R}^n)$ and $x^i_\cdot \in \mathcal{L}^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)$, we cannot get the structure of control average limit and the mean-field NCE equation systems due to W^i and W^0 are coupled together in such two cases. Thus, this paper only considers the control average in the drift term and $x^i_\cdot$, $u^i_\cdot$ in the diffusion term in W^i of state equation. It should be emphasized that the two cases are important and meaningful, but still challenging problems for technical reasons, and we will explore them in our future work.

So far, we find that the existence of a relaxed compensator is crucial in solving the indefinite problem. Next, we will provide a necessary and sufficient condition for a relaxed compensator. Since the proof is similar to the Proposition 4.9 in [35], we do not repeat here. Moreover, for the relationship between the relaxed compensator and the solutions to Riccati equations system, the interested reader can refer to [33] and [35] for more details.

Proposition 3.2. A pair of functions $(H_\cdot, K_\cdot) \in \Upsilon[0,T] \times \Upsilon[0,T]$ is a relaxed compensator of Problem (LMFL) if and only if the following conditions hold

$$\begin{cases} \dot{H}_t + H_t A_t + A_t^\top H_t + Q_t - (S_t + H_t B_t) R_t^{-1} (S_t + H_t B_t)^\top \geq 0, \\ H_T \leq G, \quad R_t \geq \delta I, \quad t \in [0,T], \end{cases} \tag{3.29}$$

and

$$\begin{cases} \dot{K}_t + K_t \widehat{A}_t + \widehat{A}_t^\top K_t + \widetilde{C}_t^\top H_t \widetilde{C}_t + \widehat{Q}_t \\ \qquad - (K_t \widehat{B}_t + \widetilde{C}_t^\top H_t \widetilde{D}_t + \widehat{S}_t)(\widehat{R}_t + \widetilde{D}_t^\top H_t \widetilde{D}_t)^{-1} (K_t \widehat{B}_t + \widetilde{C}_t^\top H_t \widetilde{D}_t + \widehat{S}_t)^\top \geq 0, \\ K_T \leq \widehat{G}, \quad \widehat{R}_t + \widetilde{D}_t^\top H_t \widetilde{D}_t \geq \delta I, \quad t \in [0,T]. \end{cases} \tag{3.30}$$

Remark 3.3. It is clear that, if Assumption (PD) satisfies, all the results obtained above still hold. For saving space, we omit them here. In section 5, we will use an example to illustrate this case.

4. ϵ-Nash Equilibrium Analysis

Up to now, we have characterized the decentralized optimal strategies $\bar{u}^i_\cdot$ by introducing the limiting control problem and the mean-field NCE equation systems. In this section, we aim to verify the ϵ-Nash equilibrium property for Problem (MFL). Before giving the main results, we need serval lemmas, which cover the relevant estimates of the state process and cost functional.

Denote $\breve{x}^i_\cdot$ as the centralized state trajectory with respect to $\bar{u}^i_\cdot$ in (2.1), which satisfies

$$\begin{cases} d\breve{x}^i_t = \Big\{ A_t \breve{x}^i_t + \widetilde{A}_t \mathbb{E}[\breve{x}^i_t] + B_t \bar{u}^i_t + \widetilde{B}_t \mathbb{E}[\bar{u}^i_t] + \dfrac{1}{N-1} \displaystyle\sum_{j=1, j\neq i}^{N} \bar{B}_t \bar{u}^j_t \Big\} dt \\ \qquad + \Big\{ \widetilde{C}_t \mathbb{E}[\breve{x}^i_t] + \widetilde{D}_t \mathbb{E}[\bar{u}^i_t] + \sigma_t \Big\} dW^i_t + \sigma^0_t dW^0_t, \\ \breve{x}^i_0 = a, \end{cases} \tag{4.1}$$

and $\bar{x}^i_\cdot$ as the decentralized one with respect to $\bar{u}^i_\cdot$ in (3.1), which follows

$$\begin{cases} d\bar{x}^i_t = \Big\{ A_t \bar{x}^i_t + \widetilde{A}_t \mathbb{E}[\bar{x}^i_t] + B_t \bar{u}^i_t + \widetilde{B}_t \mathbb{E}[\bar{u}^i_t] + m_t \Big\} dt \\ \qquad + \Big\{ \widetilde{C}_t \mathbb{E}[\bar{x}^i_t] + \widetilde{D}_t \mathbb{E}[\bar{u}^i_t] + \sigma_t \Big\} dW^i_t + \sigma^0_t dW^0_t, \\ \bar{x}^i_0 = a. \end{cases} \tag{4.2}$$

For simplicity, we suppose that C^0 is a positive constant which may often vary from line to line. Firstly, we have the following results about the estimates of two different states.

Lemma 4.1. It holds that

$$\sup_{1\leq i\leq N}\Big[\sup_{0\leq t\leq T}\mathbb{E}|\breve{x}_t^i-\bar{x}_t^i|^2\Big]=O\left(\frac{1}{N}\right), \tag{4.3}$$

$$\sup_{1\leq i\leq N}\Big[\sup_{0\leq t\leq T}\mathbb{E}\Big||\breve{x}_t^i|^2-|\bar{x}_t^i|^2\Big|\Big]=O\left(\frac{1}{\sqrt{N}}\right). \tag{4.4}$$

Proof. By (4.1) and (4.2), we have

$$\begin{cases} d(\breve{x}_t^i-\bar{x}_t^i)=\Big\{A_t(\breve{x}_t^i-\bar{x}_t^i)+\widetilde{A}_t\mathbb{E}[\breve{x}_t^i-\bar{x}_t^i]+\dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\bar{B}_t\bar{u}_t^j \\ \qquad\qquad -m_t\Big\}dt+\widetilde{C}_t\mathbb{E}[\breve{x}_t^i-\bar{x}_t^i]dW_t^i, \\ \breve{x}_0^i-\bar{x}_0^i=0. \end{cases} \tag{4.5}$$

According to (3.12) and Proposition 3.1, we obtain

$$\begin{aligned}\mathbb{E}\Big|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{B}_t\bar{u}_t^j-m_t\Big|^2=&\mathbb{E}\Big|\bar{B}_tR_t^{-1}(S_t+P_tB_t)^\top\big(\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big)\Big|^2\\ \leq& C^0\mathbb{E}\Big|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\Big|^2.\end{aligned}$$

Moreover, we can verify

$$\begin{aligned}\mathbb{E}\Big|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\Big|^2=&\frac{1}{(N-1)^2}\sum_{j=1,j\neq i}^{N}\mathbb{E}\big|\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big|^2\\ &+\frac{1}{(N-1)^2}\sum_{j=1,j\neq i,k}^{N}\mathbb{E}\big(\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big)^\top\big(\bar{x}_t^k-\mathbb{E}[\bar{x}_t^i]\big).\end{aligned} \tag{4.6}$$

Noting that for $1\leq j,k\leq N, j\neq k$, $\bar{x}_t^j$ and $\bar{x}_t^k$ are independent identically distributed under $\mathbb{E}[\cdot]$, which leads to $\mathbb{E}[\bar{x}_t^j]=\mathbb{E}[\bar{x}_t^k]$ and $\mathbb{E}[\bar{x}_t^j\bar{x}_t^k]=\mathbb{E}[\bar{x}_t^j]\mathbb{E}[\bar{x}_t^k]=(\mathbb{E}[\bar{x}_t^i])^2$.

Based on these properties, we can check

$$\begin{aligned}\mathbb{E}\big(\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big)^\top\big(\bar{x}_t^k-\mathbb{E}[\bar{x}_t^i]\big)=&\mathbb{E}\big[\mathbb{E}\big(\bar{x}_t^j\bar{x}_t^k-\bar{x}_t^j\mathbb{E}[\bar{x}_t^i]-\bar{x}_t^k\mathbb{E}[\bar{x}_t^i]+(\mathbb{E}[\bar{x}_t^i])^2\big)\big]\\ =&\mathbb{E}\big(\mathbb{E}[\bar{x}_t^j\bar{x}_t^k]-(\mathbb{E}[\bar{x}_t^i])^2\big)=0.\end{aligned} \tag{4.7}$$

Since $\mathbb{E}\big|\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big|^2<\infty$, from (4.6) and (4.7), we have

$$\frac{1}{(N-1)^2}\sum_{j=1,j\neq i}^{N}\mathbb{E}\big|\bar{x}_t^j-\mathbb{E}[\bar{x}_t^i]\big|^2=O\left(\frac{1}{N}\right),$$

thus

$$\sup_{1\le i\le N}\Big[\sup_{0\le t\le T}\mathbb{E}\Big|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{B}_t\bar{u}_t^j-m_t\Big|^2\Big]=O\left(\frac{1}{N}\right). \tag{4.8}$$

Combining (4.5) with (4.8), we get

$$\begin{aligned}\mathbb{E}|\breve{x}_t^i-\bar{x}_t^i|^2 \le& C^0\Big\{\int_0^t\mathbb{E}|\breve{x}_s^i-\bar{x}_s^i|^2ds+\int_0^t\mathbb{E}\Big|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\bar{B}_s\bar{u}_s^j-m_s\Big|^2ds\Big\}\\ =&C^0\int_0^t\mathbb{E}|\breve{x}_s^i-\bar{x}_s^i|^2ds+O\left(\frac{1}{N}\right),\end{aligned}$$

then (4.3) holds from the Gronwall's inequality.

Moreover, applying the Cauchy-Schwarz inequality, we have

$$\sup_{0\le t\le T}\mathbb{E}||\breve{x}_t^i|^2-|\bar{x}_t^i|^2|\le\sup_{0\le t\le T}\mathbb{E}|\breve{x}_t^i-\bar{x}_t^i|^2+2\Big(\sup_{0\le t\le T}\mathbb{E}|\bar{x}_t^i|^2\Big)^{\frac12}\Big(\sup_{0\le t\le T}\mathbb{E}|\breve{x}_t^i-\bar{x}_t^i|^2\Big)^{\frac12}.$$

Due to coefficients setting and (4.3), we show that $\sup_{0\le t\le T}\mathbb{E}|\bar{x}_t^i|^2<+\infty$, thus (4.4) holds. □

Lemma 4.2. It holds that

$$|\mathcal{J}^i\left(\bar{u}_t^i,\bar{u}_t^{-i}\right)-J^i\left(\bar{u}_t^i\right)|=O\left(\frac{1}{\sqrt{N}}\right). \tag{4.9}$$

Proof. Recalling (2.2) and (3.2), we have

$$\begin{aligned}&\mathcal{J}^i\left(\bar{u}_t^i,\bar{u}_t^{-i}\right)\\ =&\ \mathbb{E}\Big\{\int_0^T\Big[\langle Q_t\breve{x}_t^i,\breve{x}_t^i\rangle+\langle\widetilde{Q}_t\mathbb{E}[\breve{x}_t^i],\mathbb{E}[\breve{x}_t^i]\rangle+2\langle S_t\bar{u}_t^i,\breve{x}_t^i\rangle+2\langle\widetilde{S}_t\mathbb{E}[\bar{u}_t^i],\mathbb{E}[\breve{x}_t^i]\rangle+\langle R_t\bar{u}_t^i,\bar{u}_t^i\rangle+\langle\widetilde{R}_t\mathbb{E}[\bar{u}_t^i],\mathbb{E}[\bar{u}_t^i]\rangle\\ &+2\langle q_t,\breve{x}_t^i\rangle+2\langle\widetilde{q}_t,\mathbb{E}[\breve{x}_t^i]\rangle+2\langle\rho_t,\bar{u}_t^i\rangle+2\langle\widetilde{\rho}_t,\mathbb{E}[\bar{u}_t^i]\rangle\Big]dt+\langle G\breve{x}_T^i,\breve{x}_T^i\rangle+\langle\widetilde{G}\mathbb{E}[\breve{x}_T^i],\mathbb{E}[\breve{x}_T^i]\rangle+2\langle g,\breve{x}_T^i\rangle+2\langle\widetilde{g},\mathbb{E}[\breve{x}_T^i]\rangle\Big\},\end{aligned}$$

$$\begin{aligned}&J^i\left(\bar{u}_t^i\right)\\ =&\ \mathbb{E}\Big\{\int_0^T\Big[\langle Q_t\bar{x}_t^i,\bar{x}_t^i\rangle+\langle\widetilde{Q}_t\mathbb{E}[\bar{x}_t^i],\mathbb{E}[\bar{x}_t^i]\rangle+2\langle S_t\bar{u}_t^i,\bar{x}_t^i\rangle+2\langle\widetilde{S}_t\mathbb{E}[\bar{u}_t^i],\mathbb{E}[\bar{x}_t^i]\rangle+\langle R_t\bar{u}_t^i,\bar{u}_t^i\rangle+\langle\widetilde{R}_t\mathbb{E}[\bar{u}_t^i],\mathbb{E}[\bar{u}_t^i]\rangle\\ &+2\langle q_t,\bar{x}_t^i\rangle+2\langle\widetilde{q}_t,\mathbb{E}[\bar{x}_t^i]\rangle+2\langle\rho_t,\bar{u}_t^i\rangle+2\langle\widetilde{\rho}_t,\mathbb{E}[\bar{u}_t^i]\rangle\Big]dt+\langle G\bar{x}_T^i,\bar{x}_T^i\rangle+\langle\widetilde{G}\mathbb{E}[\bar{x}_T^i],\mathbb{E}[\bar{x}_T^i]\rangle+2\langle g,\bar{x}_T^i\rangle+2\langle\widetilde{g},\mathbb{E}[\bar{x}_T^i]\rangle\Big\}.\end{aligned}$$

Since $\mathbb{E}[|\bar{u}_t^i|^2]\le\infty$ and by (4.3), (4.4), we can estimate

$$\begin{aligned}&|\mathcal{J}^i\left(\bar{u}_t^i,\bar{u}_t^{-i}\right)-J^i\left(\bar{u}_t^i\right)|\\ =&\mathbb{E}\Big\{\int_0^T\Big[\langle Q_t\breve{x}_t^i,\breve{x}_t^i\rangle-\langle Q_t\bar{x}_t^i,\bar{x}_t^i\rangle+\langle\widetilde{Q}_t\mathbb{E}[\breve{x}_t^i],\mathbb{E}[\breve{x}_t^i]\rangle-\langle\widetilde{Q}_t\mathbb{E}[\bar{x}_t^i],\mathbb{E}[\bar{x}_t^i]\rangle+2\langle S_t\bar{u}_t^i,\breve{x}_t^i-\bar{x}_t^i\rangle\\ &+2\langle\widetilde{S}_t\mathbb{E}[\bar{u}_t^i],\mathbb{E}[\breve{x}_t^i-\bar{x}_t^i]\rangle+2\langle q_t,\breve{x}_t^i-\bar{x}_t^i\rangle+2\langle\widetilde{q}_t,\mathbb{E}[\breve{x}_t^i-\bar{x}_t^i]\rangle\Big]dt+\langle G\breve{x}_T^i,\breve{x}_T^i\rangle-\langle G\bar{x}_T^i,\bar{x}_T^i\rangle\\ &+\langle\widetilde{G}\mathbb{E}[\breve{x}_T^i],\mathbb{E}[\breve{x}_T^i]\rangle-\langle\widetilde{G}\mathbb{E}[\bar{x}_T^i],\mathbb{E}[\bar{x}_T^i]\rangle+2\langle g,\breve{x}_T^i-\bar{x}_t^i\rangle+2\langle\widetilde{g},\mathbb{E}[\breve{x}_T^i-\bar{x}_t^i]\rangle\Big\}\\ \le&\ C^0\Big\{\int_0^T\Big[\mathbb{E}||\breve{x}_t^i|^2-|\bar{x}_t^i|^2|+(\mathbb{E}|\breve{x}_t^i-\bar{x}_t^i|^2)^{\frac12}\Big]dt+\mathbb{E}||\breve{x}_T^i|^2-|\bar{x}_T^i|^2|+(\mathbb{E}|\breve{x}_T^i-\bar{x}_T^i|^2)^{\frac12}\Big\}=O\left(\frac{1}{\sqrt{N}}\right),\end{aligned} \tag{4.10}$$

which implies the desired result. □

Inspired by the definition of ϵ-Nash equilibrium, we will discuss the estimate for $\mathcal{J}^i\left(u^i_\cdot, \bar{u}^{-i}_\cdot\right)$, which means that the given ith agent take any $u^i_\cdot$ as its own strategy while other peers still keep the control $\bar{u}^j_\cdot$, for $1 \leq i, j \leq N$, $j \neq i$. For this reason, we consider the following perturbed state system.

$$
\begin{cases}
dl^i_t = \Big\{A_t l^i_t + \widetilde{A}_t \mathbb{E}[l^i_t] + B_t u^i_t + \widetilde{B}_t \mathbb{E}[u^i_t] + \dfrac{1}{N-1} \displaystyle\sum_{\varrho=1,\varrho\neq i}^{N} \bar{B}_t \bar{u}^\varrho_t \Big\} dt \\
\qquad + \Big\{\widetilde{C}_t \mathbb{E}[l^i_t] + \widetilde{D}_t \mathbb{E}[u^i_t] + \sigma_t \Big\} dW^i_t + \sigma^0_t dW^0_t, \\
dl^j_t = \Big\{A_t l^j_t + \widetilde{A}_t \mathbb{E}[l^j_t] + B_t \bar{u}^j_t + \widetilde{B}_t \mathbb{E}[\bar{u}^j_t] + \dfrac{1}{N-1} \displaystyle\sum_{\kappa=1,\kappa\neq i,j}^{N} \bar{B}_t (\bar{u}^\kappa_t + u^i_t) \Big\} dt \\
\qquad + \Big\{\widetilde{C}_t \mathbb{E}[l^j_t] + \widetilde{D}_t \mathbb{E}[\bar{u}^j_t] + \sigma_t \Big\} dW^j_t + \sigma^0_t dW^0_t, \\
l^i_0 = l^j_0 = a,
\end{cases}
\tag{4.11}
$$

for any $1 \leq i, j \leq N$, $j \neq i$.

Accordingly, the limiting state is subject to

$$
\begin{cases}
dh^i_t = \Big\{A_t h^i_t + \widetilde{A}_t \mathbb{E}[h^i_t] + B_t u^i_t + \widetilde{B}_t \mathbb{E}[u^i_t] + m_t \Big\} dt \\
\qquad + \Big\{\widetilde{C}_t \mathbb{E}[h^i_t] + \widetilde{D}_t \mathbb{E}[u^i_t] + \sigma_t \Big\} dW^i_t + \sigma^0_t dW^0_t, \\
h^i_0 = a.
\end{cases}
\tag{4.12}
$$

Next, we address with the new estimates of the state and cost functional, which are characterized by the following lemma. Its proof can be given similarly as Lemmas 4.1 and 4.2, we omit it here for saving space.

Lemma 4.3. It holds that

$$
\sup_{1\leq i\leq N} \Big[\sup_{0\leq t\leq T} \mathbb{E}|l^i_t - h^i_t|^2\Big] = O\left(\frac{1}{N}\right),
\tag{4.13}
$$

$$
\sup_{1\leq i\leq N} \Big[\sup_{0\leq t\leq T} \mathbb{E}\Big||l^i_t|^2 - |h^i_t|^2\Big|\Big] = O\left(\frac{1}{\sqrt{N}}\right),
\tag{4.14}
$$

$$
|\mathcal{J}^i\left(u^i_t, \bar{u}^{-i}_t\right) - J^i\left(u^i_t\right)| = O\left(\frac{1}{\sqrt{N}}\right).
\tag{4.15}
$$

Now, we provide the main results in this section.

Theorem 4.1. If there exists a relaxed compensator $(H_\cdot, K_\cdot) \in \Upsilon[0,T] \times \Upsilon[0,T]$, then the strategy set $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^N)$, where $\bar{u}^i_\cdot$ is given by (3.12), is an ϵ-Nash equilibrium of Problem (MFL).

Proof. Since the optimality of $\bar{u}^i_\cdot$, it follows that $J^i(\bar{u}^i_t) \leq J^i(u^i_t)$, for any alternative $u^i_\cdot$.

Based on (4.9) and (4.15), we have

$$
\begin{aligned}
\mathcal{J}^i\left(\bar{u}^i_t, \bar{u}^{-i}_t\right) =& J^i\left(\bar{u}^i_t\right) + O\left(\frac{1}{\sqrt{N}}\right) \leq J^i\left(u^i_t\right) + O\left(\frac{1}{\sqrt{N}}\right) \\
=& \mathcal{J}^i\left(u^i_t, \bar{u}^{-i}_t\right) + O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}
$$

which completes the proof with $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$. □

5. Example and Simulation

In recent years, unmanned aerial vehicles (UAVs) have received extensive attention with their high mobility and low cost in military and civilian domains including typical examples like weather monitoring, forest fire detection, traffic evacuation, cargo transport, emergency search and rescue, communication relaying, etc. In particular, UAVs are considered a promising solution for handling complex communication scenarios, which can be used as airborne mobile base stations to enhance terrestrial wireless communication systems. Based on the fact that the serviced ground users are scattered across a wide range, ultra-dense UAVs need to be deployed for better quality communication, which means the number of UAV is large. However, it should not be neglected that the communication performance has been extremely limited due to the poor battery life of UAVs. Therefore, how to control the energy consumption of transit power is a challenging but appealing topic when facing sophisticated communication scenarios. In this section, the energy control problem for a large number of UAV air-to-ground wireless communication systems is studied.

We consider an ultra-dense UAVs wireless communication system which is composed of N autonomous drones. Among the system, each UAV acts as a base station for conducting data transmission from air to ground. We also assume all the UAVs are randomly distributed and share an identical channel at the same time. Despite the fact that one single drone may be requested by multiple ground users, we suppose that it can only serve one user at a fixed time. As shown in Figure 1, the user i, which is served by the considered UAV i, can be influenced by other UAVs. Moreover, the UAV i may also disturb the signal transmission for other users. For this reason, there have been complicated interactions among all the dynamics of N drones.

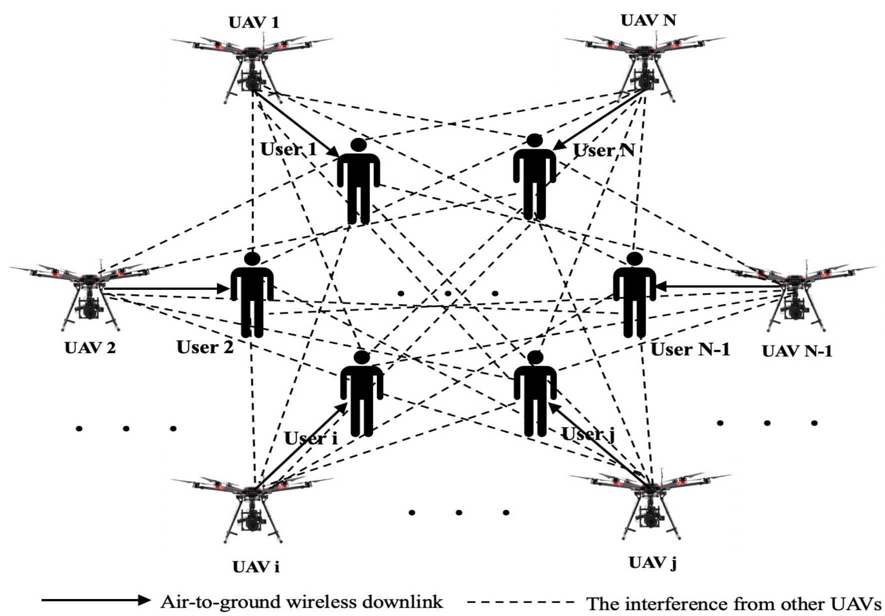

Figure 1. The Multi-UAV Wireless Communication System.

For a given time T, since the controls will be affected by other UAVs, we consider a large-population system made up of N UAVs and there exists a coupling structure among peers. The state system of the ith UAV is governed by the following MF-SDE

$$\begin{cases} dx_t^i = \Big\{ a_t x_t^i + \tilde{a}_t \mathbb{E}\left[x_t^i\right] + b_t u_t^i + \dfrac{1}{N-1} \displaystyle\sum_{j=1, j\neq i}^{N} \bar{b}_t u_t^j \Big\} dt + \sigma_t^i dW_t^i + \sigma_t^0 dW_t^0, \\ x_0^i = a^i, \end{cases} \tag{5.1}$$

where x_t^i is the emission energy state of the ith UAV at time t. u_t^i is the transmit power level varying from person to person, which is the input affecting the energy state. $\mathbb{E}[x_\cdot^i]$ characterizes the average energy state of ith UAV. The control average term indicates that the given ith UAV will be interfered by the other peers. Moreover, a, $\tilde{a}$, b, $\bar{b}$ are the weight coefficients, which are the uniformly bounded deterministic functions. Accept the various individual random noise W^i, the dynamics of emission energy is also influenced by the common noise W^0, like the temperature of the region, cloud cover and so on. Thus, we take these external factors into account in our controlled system.

In fact, the energy storage of each UAV is certain and limited. For better communication performance, we assume that each UAV needs to choose a transmit power level to minimize the following cost functional

$$\mathcal{J}^i\left(u_t^i, u_t^{-i}\right) = \mathbb{E}\int_0^T \Big[\eta_t(x_t^i - \mathbb{E}[x_t^i])^2 + r_t(u_t^i)^2\Big]dt, \tag{5.2}$$

where $\eta_t \geq 0$, $r_t > 0$ are the uniformly bounded deterministic functions. We notice that the integral item consists of two parts: the first is the minimum square criterion on energy state, which indicates that the current energy of the UAV cannot deviate from the average level to maintain the basic performance of the UAV; the second part is the running cost for the signal transmission process.

Now we formulate the energy control (EC) problem of the multi-UAV wireless communication system in the following:

Problem (EC). For $1 \leq i \leq N$, find the transmit power set $(\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^N)$ such that

$$\mathcal{J}^i\left(\bar{u}_\cdot^i, \bar{u}_\cdot^{-i}\right) = \inf_{u_\cdot^i \in \mathcal{U}_i^c} \mathcal{J}^i\left(u_\cdot^i, u_\cdot^{-i}\right),$$

where the system of emission energy satisfying (5.1).

It is obvious that the coefficients setting satisfies Assumption (PD), we will illustrate the results derived in Section 3 still hold in this case. We consider the corresponding limiting control problem by replacing the term $\frac{1}{N-1}\sum_{j=1,j\neq i}^N \bar{b}_\cdot u_\cdot^j$ with certain quantity $m_\cdot$. Firstly, we introduce the following Riccati equation

$$\begin{cases} \dot{P}_t + 2a_tP_t + \eta_t - \dfrac{P_t^2b_t^2}{r_t} = 0, \\ P_T = 0. \end{cases} \tag{5.3}$$

Applying the separation techniques introduced in Proposition 3.1, we have

$$m_t = -\frac{b_t\bar{b}_t}{r_t}\Big\{P_t(x_t^2 - \mathbb{E}[x_t^2]) + \Phi_t\Big\}, \tag{5.4}$$

where $(x_\cdot^2, \Phi_\cdot, \Theta_\cdot)$ satisfy the following equations and $a^i = a_1^i + a_2$

$$\begin{cases} dx_t^2 = \Big\{\Big[a_t - \dfrac{P_t(b_t^2 + b_t\bar{b}_t)}{r_t}\Big]x_t^2 + \Big[\tilde{a}_t + \dfrac{P_t(b_t^2 + b_t\bar{b}_t)}{r_t}\Big]\mathbb{E}[x_t^2] \\ \qquad - \dfrac{b_t^2 + b_t\bar{b}_t}{r_t}\Phi_t\Big\}dt + \sigma_t^0 dW_t^0, \\ d\Phi_t = \Big\{\Big[\dfrac{P_t(b_t^2 + b_t\bar{b}_t)}{r_t} - a_t\Big]\Phi_t - \Big[\dfrac{P_t(b_t^2 + b_t\bar{b}_t)}{r_t} + \tilde{a}_t\Big]\mathbb{E}[\Phi_t] \\ \qquad + \dfrac{P_t^2 b_t\bar{b}_t}{r_t}(x_t^2 - \mathbb{E}[x_t^2])\Big\}dt + \Theta_t dW_t^0, \\ x_0^2 = a_2, \ \Phi_T = 0. \end{cases} \tag{5.5}$$

Noting that (5.5) is a coupled MF-FBSDE. To decoupling (5.5), we introduce an extra Riccati equation in the following.

$$\begin{cases}\dot{\Psi}_t+2\Big[a_t-\dfrac{P_t(b_t^2+b_t\bar{b}_t)}{r_t}\Big]\Psi_t-\dfrac{b_t^2+b_t\bar{b}_t}{r_t}\Psi_t^2-\dfrac{P_t^2b_t\bar{b}_t}{r_t}=0,\\ \Psi_T=\ 0.\end{cases}\tag{5.6}$$

Consequently, we have $\Phi_t=\Psi_t(x_t^2-\mathbb{E}[x_t^2])$, where $x_\cdot^2$ is determined by

$$\begin{cases}dx_t^2=\bigg\{\Big[a_t-\dfrac{(P_t+\Psi_t)(b_t^2+b_t\bar{b}_t)}{r_t}\Big]x_t^2+\Big[\tilde{a}_t+\dfrac{(P_t+\Psi_t)(b_t^2+b_t\bar{b}_t)}{r_t}\Big]\mathbb{E}[x_t^2]\bigg\}dt+\sigma_t^0dW_t^0,\\ x_0^2=a_2.\end{cases}$$

After taking expectation on the both side of above equation, we have the difference $x_t^2-\mathbb{E}[x_t^2]$ satifies

$$\begin{cases}d(x_t^2-\mathbb{E}[x_t^2])=\bigg\{\Big[a_t-\dfrac{(P_t+\Psi_t)(b_t^2+b_t\bar{b}_t)}{r_t}\Big](x_t^2-\mathbb{E}[x_t^2])\bigg\}dt,\\ \quad x_0^2-\mathbb{E}[x_0^2]=0,\end{cases}$$

so we deduce

$$x_t^2-\mathbb{E}[x_t^2]=e^{\int_0^t\Big[a_s-\frac{(P_s+\Psi_s)(b_s^2+b_s\bar{b}_s)}{r_s}\Big]ds}\int_0^t e^{\int_0^s\Big[\frac{(P_\tau+\Psi_\tau)(b_\tau^2+b_\tau\bar{b}_\tau)}{r_\tau}-a_\tau\Big]d\tau}\sigma_s^0dW_s^0.$$

Then the feedback representation of decentralized optimal transmit power levels has form with

$$\bar{u}_t^i=-\frac{b_t}{r_t}\Big\{P_t(\bar{x}_t^i-\mathbb{E}[\bar{x}_t^i])+\Phi_t\Big\},\tag{5.7}$$

where the optimal state trajectory $\bar{x}_\cdot^i$ is determined by the following equation

$$\begin{cases}d\bar{x}_t^i=\bigg\{\Big[a_t-\dfrac{P_tb_t^2}{r_t}\Big]\bar{x}_t^i+\Big[\tilde{a}_t+\dfrac{P_tb_t^2}{r_t}\Big]\mathbb{E}[\bar{x}_t^i]-\dfrac{P_tb_t\bar{b}_t}{r_t}\Big(x_t^2-\mathbb{E}[x_t^2]\Big)-\dfrac{b_t^2+b_t\bar{b}_t}{r_t}\Phi_t\bigg\}dt\\ \qquad\quad+\sigma_t^idW_t^i+\sigma_t^0dW_t^0,\\ \bar{x}_0^i=a^i.\end{cases}$$

By similar calculations, we shall obtain

$$\bar{x}_t^i=e^{\int_0^t\Big(a_s-\frac{P_sb_s^2}{r_s}\Big)ds}\bigg\{a^i+\int_0^t e^{\int_0^s\Big(\frac{P_\tau b_\tau^2}{r_\tau}-a_\tau\Big)d\tau}\Delta_sds+\int_0^t e^{\int_0^s\Big(\frac{P_\tau b_\tau^2}{r_\tau}-a_\tau\Big)d\tau}(\sigma_s^idW_s^i+\sigma_s^0dW_s^0)\bigg\},\tag{5.8}$$

where

$$\Delta_s:=\Big(\tilde{a}_s+\frac{P_sb_s^2}{r_s}\Big)e^{\int_0^s(a_\tau+\tilde{a}_\tau)d\tau}a^i-\frac{P_sb_s\bar{b}_s}{r_s}\Big(x_s^2-\mathbb{E}[x_s^2]\Big)-\frac{b_s^2+b_s\bar{b}_s}{r_s}\Phi_s.$$

Therefore, the decentralized optimal cost of Problem (EC) can be estimated as follows.

$$J^i\left(\bar{u}_\cdot^i\right)=\mathbb{E}\int_0^T\Big[(\eta_t+\frac{b_t^2P_t^2}{r_t})(\bar{x}_t^i-\mathbb{E}[\bar{x}_t^i])^2+\frac{2b_t^2P_t\Phi_t}{r_t}(\bar{x}_t^i-\mathbb{E}[\bar{x}_t^i])+\frac{b_t^2\Phi_t^2}{r_t}\Big]dt,\tag{5.9}$$

where $\bar{x}_\cdot^i$ is determined by (5.8).

Based on the above analysis, we have the following results, which proof is similar to Section 3.

Proposition 5.1. Problem (EC) admits the decentralized optimal transmit powers with the form of (5.7), where the decentralized optimal energy state satisfies (5.8). The decentralized optimal cost is given by (5.9). Moreover, (5.7) is the ϵ-Nash equilibrium of Problem (EC).

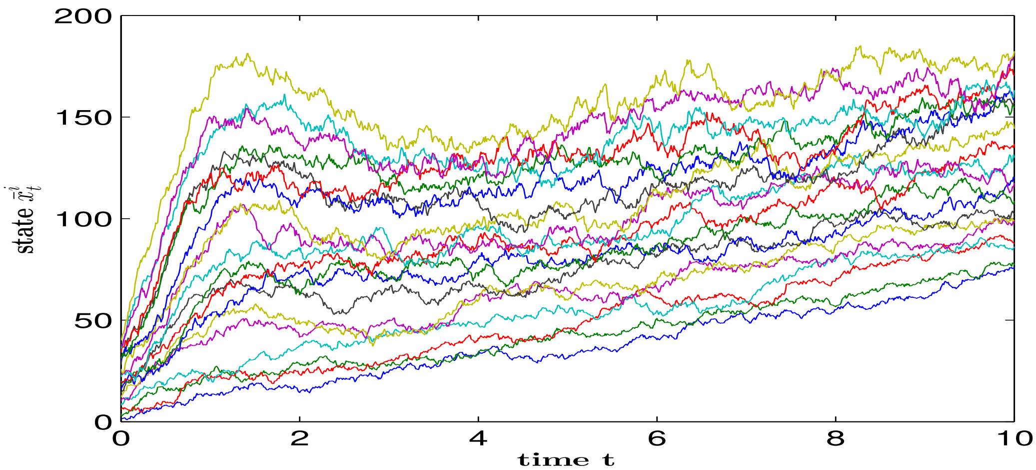

FIGURE 2. The different decentralized optimal states.

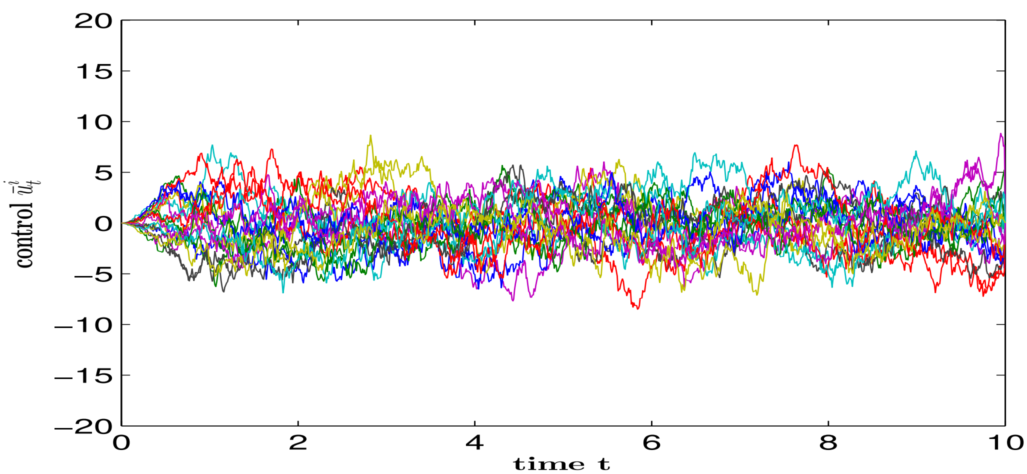

FIGURE 3. The different decentralized control strategies.

To be more intuitive, we present some numerical simulation results for energy control problem of multi-UAV wireless communication system with the decentralized optimal transmit power laws as proposed. In our case, we consider that there are $N = 20$ UAVs randomly distributed in the specified area, which share the identical channel at the same time. Other parameters are set as $T = 10$ hours, $a = 1$, $\tilde{a} = 1.5$, $b = 1$, $\bar{b} = 2$, $\sigma^0 = 5$, $\eta = 0.8$, $r = 1$, $a^i = 2, 4, 6, \ldots, 40$, $\sigma^i = 0.5, 1, 1.5, \ldots, 10$. Figure 2 presents the decentralized optimal states of different UAVs. We can observe that the overall trend of the different UAVs' states is consistent, the difference between them mainly comes from the different initial values a^i and volatility σ^i. Figure 3 shows the corresponding decentralized optimal control strategies of 20 UAVs. It is obvious that 20 UAVs' control strategies fluctuate around zero, which are consistent with our theoretical results. In fact, by (5.7), we have $\mathbb{E}[\bar{u}^i_\cdot] = 0$, for $1 \leq i \leq N$.

6. Conclusions

In this paper, a class of indefinite control average mean-field large-population problems have been studied. The solvability of stochastic Hamiltonian system and Riccati equations under indefinite condition has been proved. The decentralized strategies in open-loop form and feedback form have been obtained. The explicit structure of the control average limit and the mean-field NCE equation systems have been determined by some separation techniques. Moreover, the decentralized optimal strategies have been verified to satisfy the ϵ-Nash equilibrium property. For illustration, a practical example from the engineering field has been solved by the theoretical results.

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