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INVERSE POTENTIAL PROBLEMS IN DIVERGENCE FORM FOR MEASURES IN THE PLANE*, **

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Abstract. We study inverse potential problems with source term the divergence of some unknown ($\mathbb{R}^3$-valued) measure supported in a plane; e.g., inverse magnetization problems for thin plates. We investigate methods for recovering a magnetization $\boldsymbol{\mu}$ by penalizing the measure-theoretic total variation norm $\|\boldsymbol{\mu}\|_{TV}$, and appealing to the decomposition of divergence-free measures in the plane as superpositions of unit tangent vector fields on rectifiable Jordan curves. In particular, we prove for magnetizations supported in a plane that TV-regularization schemes always have a unique minimizer, even in the presence of noise. It is further shown that TV-norm minimization (among magnetizations generating the same field) uniquely recovers planar magnetizations in the following two cases: (i) when the magnetization is carried by a collection of sufficiently separated line segments and a set that is purely 1-unrectifiable; (ii) when a superset of the support is tree-like. We note that such magnetizations can be recovered via TV-regularization schemes in the zero noise limit by taking the regularization parameter to zero. This suggests definitions of sparsity in the present infinite dimensional context, that generate results akin to compressed sensing.

Résumé. Nous considérons les problèmes inverses de potentiel dont le terme source est la divergence d'une mesure, à valeurs dans $\mathbb{R}^3$ et à support plan ; par exemple, les problèmes inverses d'aimantation pour des plaques minces. On étudie la reconstruction d'une telle mesure $\boldsymbol{\mu}$ par minimisation d'un critère quadratique d'attache aux données régularisé qui pénalise la norme de la variation totale $\|\boldsymbol{\mu}\|_{TV}$, dont on analyse le comportement grâce à la décomposition des mesures 2-D à divergence nulle dans le plan comme superposition d'intégrales de contour le long de courbes de Jordan. On montre en particulier l'unicité de l'argument du minimum, même en présence de bruit, ainsi que la consistance de la reconstruction dans les deux cas suivants : (i) quand l'aimantation a un support contenu dans l'union d'un ensemble purement non rectifiable et d'une collection de segments suffisamment séparés dans un plan ; (ii) quand son support est contenu dans une structure arborescente. La consistance a lieu lorsque le bruit et le paramètre de régularisation tendent vers zéro, de façon conjointe, en suivant le principe de Morozov. Ceci suggère, dans ce contexte de dimension infinie, des notions de parcimonie fondées sur la dimension du support qui sont réminiscentes de l'identification parcimonieuse en dimension finie.

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1. Introduction

This paper considers inverse potential problems with source term in divergence form, in connection with the structure of 2-D divergence-free measures in the plane. A typical application, and the main motivation of the authors to carry out this research, is to inverse magnetization problems on thin plates. When magnetizations are modelled by $\mathbb{R}^3$-valued measures supported on a set S (in the thin plate case $S \subset \mathbb{R}^2$), the inverse magnetization problem consists in recovering such a measure, say $\boldsymbol{\mu}$, from knowledge of the magnetic field $\mathbf{b}(\boldsymbol{\mu})$ that it generates, see Section 1.1 for details. Magnetizations supported in a plane generate the zero magnetic field if and only if they are tangent to that plane and divergence-free there (see Lemmas **??** and **??**). Thus, the kernel of the forward operator mapping μ to $\mathbf{b}(\boldsymbol{\mu})$ consists precisely of planar divergence-free measures in this case. Now, a 2-D divergence-free measure $\boldsymbol{\nu}$ in the plane can be decomposed as a superposition of elementary "loops"; *i.e.*, contour integrations along rectifiable Jordan curves, in such a way that the Radon-Nykodim derivative $d\boldsymbol{\nu}/d|\boldsymbol{\nu}|(x)$ is essentially the unit tangent to any of these curves through x, (*cf.* Theorem **??**). This result, which is more precise (though limited to the planar case) than the general decomposition theorem for solenoids given in [19], gives us insight on the structure of the kernel of the forward operator, enabling us to give sufficient conditions for a magnetization to be *TV-minimal on S*; *i.e.*, the magnetization has minimum total variation among those magnetizations supported on S that generate the same field. When a TV-minimal magnetization on S is unique among magnetizations generating the same field, we call it *strictly TV-minimal on S*. By standard regularization theory, strictly TV-minimal magnetizations can in principle be recovered by solving a sequence of minimization problems for the so-called regularizing functional, which is the sum of the quadratic residuals and a penalty term consisting of the product of a regularization parameter $\lambda > 0$ and the total variation of the unknown measure, see (4). Specifically, any sequence of minimizers of the regularizing functional converges weak-$*$ to the strictly TV-minimal measure generating the data (when it exists), as the regularizing parameter and the noise tend jointly to zero in a suitable manner, see *e.g.* [7]. In short: regularizing schemes that penalize the total variation are consistent to recover strictly TV-minimal magnetizations, and thus, any assumption ensuring strict TV-minimality gives rise to a consistency result. For the larger class of magnetizations supported on a slender set S (see Section 1.1 for a definition), such a consistency result is obtained in [3, Theorem 2.6] by showing, using results from [19], that magnetizations supported on a purely 1-unrectifiable set are strictly TV-minimal. Specializing to the case of planar S and appealing to the above-mentioned loop decomposition, we shall obtain more general conditions: for instance magnetizations carried by the union of a purely 1-unrectifiable set and a collection of sufficiently separated line segments are strictly TV-minimal, see Corollary **??** and Theorem **??**.

The results just mentioned are reminiscent of compressed sensing, where underdetermined systems of linear equations in $\mathbb{R}^n$ are approximately solved by minimizing the residuals while penalizing the l^1-norm. This favors the recovery of sparse solutions (*i.e.*, solutions having a large number of zero components) when they exist, see *e.g.* [10] and [5]. In this connection, the gist of [3, Theorem 2.6] and Corollary **??** is to define notions of "sparsity" in the present, infinite-dimensional context. Our results warrant the use of regularizing schemes that penalize the total variation (a natural analog of the l^1-norm), in order to recover magnetizations which are sparse according to such definitions.

Our second application of the loop decomposition to inverse magnetization problems on thin plates is to prove that, for each value of the regularization parameter, the minimizer of the regularizing functional is unique (Theorem **??**). This result is important for algorithmic approaches, because it tells us that for every choice of the regularization parameter there is a unique estimate of the unknown magnetization based on the regularization scheme (5). It is also surprising, for in the case that a magnetization is TV-minimal but not strictly TV minimal, one would rather expect the regularizing functional to have several minimizers, at least for small values of the regularizing parameter.

Let us stress that magnetizations supported in a plane are commonly considered in paleomagnetic studies, where thin slabs of rock are modeled by planar regions [1, 14, 15, 21]. In particular, our analysis applies to this setting. It would be quite interesting to carry over the contents of the present paper to more general slender surfaces in $\mathbb{R}^3$ than planes, as the results could be of interest in other situations from geosciences or medical imaging. In practice, the development of numerically effective algorithms for such inverse problem raises delicate issues of discretization. These are not addressed in this paper, but will be taken up in future work.

The loop decomposition of planar divergence-free measures, which plays a fundamental role in our proofs, is hinted at on page 843 of [19]. To the best of our knowledge, it was first shown in [20] along the lines suggested by [19]. This method furnishes in addition a specific parametrization of the loops representing a given divergence-free measure. Two other proofs of the general loop decomposition of divergence-free planar measures are given in the manuscript [4], but without the above-mentioned parametrization. This parametrization is further refined in the manuscript [2], whose version of the loop decomposition appears in the present paper as Theorem **??**, see the discussion in Section **??**. This refinement is useful to establish Theorem **??**, showing that measures carried by a collection of sufficiently separated line segments are strictly TV-minimal.

In the rest of this introductory section, we describe the inverse magnetization problem, explain our main results and set up notation.

1.1. Background and Overview of Results

Let us first describe the inverse magnetization problem. For a closed subset $S \subset \mathbb{R}^3$, let $\mathcal{M}(S)$ denote the space of finite signed Borel measures supported on S. We use the space $\mathcal{M}(S)^3$ of $\mathbb{R}^3$-valued measures supported on S to model physical magnetizations distributed on S, and shall often speak of "magnetization on S" to mean "element of $\mathcal{M}(S)^3$". For $\boldsymbol{\mu} \in \mathcal{M}(S)^3$, we let $|\boldsymbol{\mu}|$ denote the *total variation measure* of $\boldsymbol{\mu}$. The latter is a positive measure, and we put $\|\boldsymbol{\mu}\|_{TV} := |\boldsymbol{\mu}|(\mathbb{R}^3)$ for the *total variation* of $\boldsymbol{\mu}$, see Section 1.2.

The *magnetic field* $\mathbf{b}(\boldsymbol{\mu})$ generated by a magnetization $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is defined, at a point x not in the support of $\boldsymbol{\mu}$, in terms of the *scalar magnetic potential* $\Phi(\boldsymbol{\mu})$ by (see [13]):

$$\mathbf{b}(\boldsymbol{\mu})(x) = -\mu_0 \nabla \Phi(\boldsymbol{\mu})(x), \qquad x \notin \operatorname{supp} \boldsymbol{\mu}, \tag{1}$$

where μ_0 is the *magnetic constant* and ∇ indicates the gradient. Here, $\Phi(\boldsymbol{\mu})(x)$ is given by

$$\Phi(\boldsymbol{\mu})(x) := \frac{1}{4\pi} \int \nabla_y \frac{1}{|x-y|} \cdot d\boldsymbol{\mu}(y) = \frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \cdot d\boldsymbol{\mu}(y), \tag{2}$$

where, for $x, y \in \mathbb{R}^3$, $x \cdot y$ and $|x|$ denote the Euclidean scalar product and norm and ∇_y the gradient with respect to y. Clearly, $\Phi(\boldsymbol{\mu})$ and the components of $\mathbf{b}(\boldsymbol{\mu})$ are harmonic functions on $\mathbb{R}^3 \setminus S$. Moreover, formula (2) defines $\Phi(\boldsymbol{\mu})$ on the whole of $\mathbb{R}^3$ as a member of $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$ (see [3, Proposition 2.1]) so that $\mathbf{b}(\boldsymbol{\mu})$, initially defined on $\mathbb{R}^3 \setminus S$, extends to a $\mathbb{R}^3$-valued divergence-free distribution on $\mathbb{R}^3$. Indeed, we may write

$$\Delta \Phi = \nabla \cdot \boldsymbol{\mu} \quad \text{and} \quad \mathbf{b}(\boldsymbol{\mu}) = \mu_0 \left(\boldsymbol{\mu} - \nabla \Phi(\boldsymbol{\mu}) \right), \tag{3}$$

where $\nabla \cdot \boldsymbol{\mu}$ indicates the divergence of $\boldsymbol{\mu}$. Note that (3) yields a Helmholtz-Hodge decomposition of $\boldsymbol{\mu}$, as the sum of a gradient and a divergence-free distribution. However, neither term is a measure in general but rather a distribution of order -1.

The inverse magnetization problem is to recover $\boldsymbol{\mu}$ from measurements of $\mathbf{b}(\boldsymbol{\mu})$ taken on a set $Q \subset \mathbb{R}^3 \setminus S$ which, due to the oriented nature of sensors (coils), are typically observed in one direction only, say along some unit vector $v \in \mathbb{R}^3$. We assume for simplicity that v is the same at each measurement point. For instance, it is usually so in Scanning Magnetic Microscopy experiments (SMM) where data consist of point-wise values of the normal component of the magnetic field on a planar region not intersecting S, see [14, 15, 21]. Geometric conditions on Q, S and v, ensuring that such measurements suffice to determine $\mathbf{b}(\boldsymbol{\mu})$ in the entire region $\mathbb{R}^3 \setminus S$, are given in [3, Lemma 2.3] and recalled when S is planar in Section **??**, for the convenience of the reader. In the remainder of this introduction, we assume that these assumptions are satisfied.

Still, the mapping $\boldsymbol{\mu} \to \mathbf{b}(\boldsymbol{\mu})$ is generally not injective, which is a major difficulty with this inverse problem. In this connection, we say that $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{M}(S)^3$ are *S-equivalent* if $\mathbf{b}(\boldsymbol{\mu})$ and $\mathbf{b}(\boldsymbol{\nu})$ agree on $\mathbb{R}^3 \setminus S$. A magnetization $\boldsymbol{\mu}$ is said to be *S-silent* if $\boldsymbol{\mu}$ is S-equivalent to the zero magnetization; *i.e.*, if $\mathbf{b}(\boldsymbol{\mu})$ vanishes on $\mathbb{R}^3 \setminus S$.

Since no nonzero harmonic function lies in $L^2(\mathbb{R}^3) + L^1(\mathbb{R}^3)$, it follows from (3) that a divergence-free magnetization is S-silent; here and below, the divergence is understood in the distributional sense. A partial converse is given in [3, Theorem 2.2], namely a S-silent magnetization is divergence-free provided that S is *slender*, meaning it has Lebesgue measure zero and each connected component of $\mathbb{R}^3 \setminus S$ has infinite Lebesgue measure. The slenderness assumption is a strong one: for instance it rules out the case where S is a volumic sample or a closed surface. However, it is satisfied in important special cases, for example in paleomagnetic studies, as mentioned already, or in Geomagnetism where some regions of the Earth's crust are assumed to be non-magnetic (or much less magnetic) than the others [11], or even in Electro-Encephalography where sources of primary current are often considered to lie on the surface of the encephalon (which is closed and therefore not slender) but their support should arguably leave out the brain stem connecting to the spinal cord (therefore the support is contained in a slender set).

In [19], Smirnov describes $\mathbb{R}^n$-valued divergence-free measures in $\mathbb{R}^n$, also known as *solenoids*, in terms of integrals of elementary components that are absolutely continuous with respect to 1-dimensional Hausdorff measure $\mathcal{H}^1$. Consequently, if S is slender and $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ is carried by a *purely 1-unrectifiable* set (*i.e.*, whose intersection with any 1-rectifiable set has $\mathcal{H}^1$- measure zero, see [16]), then $\boldsymbol{\mu}$ is mutually singular to every S-silent magnetization and so has minimum total variation amongst all magnetizations that are S-equivalent to $\boldsymbol{\mu}$. This observation led the authors in [3] to consider the following extremal problem involving the quantity $M_S(\boldsymbol{\mu})$, defined for $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ by

$$M_S(\boldsymbol{\mu}) := \inf\{\|\boldsymbol{\nu}\|_{TV} : \boldsymbol{\nu} \text{ is } S\text{-equivalent to } \boldsymbol{\mu}\}.$$

Extremal Problem 1. Given $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$, find $\boldsymbol{\mu}$ that is S-equivalent to $\boldsymbol{\mu}_0$ satisfying

$$\|\boldsymbol{\mu}\|_{TV} = M_S(\boldsymbol{\mu}_0).$$

A solution to Extremal Problem 1 is, by definition, TV-minimal on S, and it is strictly TV-minimal on S if this solution is unique. When $S \subset \mathbb{R}^3$ is slender and $\boldsymbol{\mu}_0 \in \mathcal{M}(S)^3$, we find that $\boldsymbol{\mu}_0$ is strictly TV-minimal on S for the three cases listed below. Here case (a) is essentially [3, Theorem 2.6] and a special case of Theorem **??** to come, while (b) is contained in [3, Theorem 2.11] and (c) follows from Corollary **??** further below.

(a) there is a purely 1-unrectifiable set of full $|\boldsymbol{\mu}_0|$ measure;
(b) the set S is a finite disjoint union of compact sets $S_1, \dots S_k$ and

$$\boldsymbol{\mu}_0\lfloor_{S_i} = \mathbf{u}_i|\boldsymbol{\mu}_0|\lfloor_{S_i},$$

for some collection of unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{R}^3$, in which case we say $\boldsymbol{\mu}_0$ is *piecewise unidirectional*;
(c) there is a set of full $|\boldsymbol{\mu}_0|$-measure contained in a countable union of coplanar disjoint line segments L_k such that the distance from any L_k to any L_j, $j \neq k$, is greater than or equal to $\mathcal{H}^1(L_k)$.

Corollary **??** also implies that (a) can be combined with (c), namely if a measure satisfies (c) and we add to it a measure on S carried by a purely 1-unrectifiable set, then we get a measure which is strictly TV-minimal again.

Now, for ρ a positive measure on Q, let $A : \mathcal{M}(S)^3 \to L^2(Q, \rho)$ be the *forward operator* mapping $\boldsymbol{\mu}$ to the restriction of $\mathbf{b}(\boldsymbol{\mu}) \cdot v$ on Q (see (**??**)). The measure ρ does not play a significant role in what follows (e.g., it could be chosen to be Lebesgue measure on Q), but it is important for practical applications. To recover solutions of Extremal Problem 1 knowing the restriction f of $\mathbf{b}(\boldsymbol{\mu}_0) \cdot v$ to Q, the theory of regularization for convex problems [7] suggests to minimize with respect to $\boldsymbol{\mu} \in \mathcal{M}(S)^3$ the functional

$$\mathcal{F}_{f,\lambda}(\boldsymbol{\mu}) := \|f - A\boldsymbol{\mu}\|^2_{L^2(Q,\rho)} + \lambda\|\boldsymbol{\mu}\|_{TV} \tag{4}$$

for some suitable value of the *regularization parameter* $\lambda > 0$. That is, we consider:

Extremal Problem 2. Given $f \in L^2(Q)$ and $\lambda > 0$, find $\boldsymbol{\mu}_\lambda \in \mathcal{M}(S)^3$ such that

$$\mathcal{F}_{f,\lambda}(\boldsymbol{\mu}_{\boldsymbol{\lambda}}) = \inf_{\boldsymbol{\mu} \in \mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\boldsymbol{\mu}). \tag{5}$$

When Q and S are positively separated, the existence of at least one minimizer in (5) is a consequence of the weak-$*$ compactness of the unit ball in $\mathcal{M}(S)^3$ see *e.g.* [6, Proposition 3.6]. Solving Extremal Problem 2 is a particular *regularization scheme* for the Inverse Magnetization Problem, namely one that penalizes the total variation of the unknown.

It is standard that if $f = A\boldsymbol{\mu}_0$ and $\lambda_n \to 0$, then any subsequence of $\boldsymbol{\mu}_{\lambda_n}$ has a subsequence converging weak-$*$ to a solution of Extremal Problem 1. To account for measurement noise, one usually replaces f by $f_n = A\boldsymbol{\mu}_0 + e_n$, and then the same result holds for a sequence $\boldsymbol{\mu}_n$ minimizing (4) with $f = f_n$ and $\lambda = \lambda_n$, provided that both λ_n and $\|e_n\lambda_n^{-1/2}\|_{L^2(Q,\rho)}$ tend to 0, see [7, Theorems 2&5] or [12, Theorems 3.5&4.4]. In particular, if there is a unique solution $\boldsymbol{\mu}_0$ of Extremal Problem 1, then we get weak-$*$ convergence of $\boldsymbol{\mu}_n$ to $\boldsymbol{\mu}_0$. A stronger result, involving narrow convergence of the total variation measure $|\boldsymbol{\mu}_n|$, can be found in [3, Theorem 4.3]. This strengthening is important, for it implies that "no mass is lost" as could be the case with weak-$*$ convergence, namely $\|\boldsymbol{\mu}_n\|_{TV}$ tends to $\|\boldsymbol{\mu}_0\|_{TV}$. To recap, we have a consistency property asserting that a magnetization meeting a certain assumptions (*e.g.* either (a), (b) or (c) above) can be approximately recovered *via* the regularization scheme (5), when the noise is small and the regularization parameter λ is chosen small but still larger than the square of the noise (the so-called Morozov discrepancy principle). Note that (5) might *a priori* have several minimizers, for the objective function (4) is not strictly convex, as is easy to see.

In Section **??**, we sharpen the analysis of [3] regarding Extremal Problems 1 and 2 in the case where S is contained in a plane. We prove that $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ is the unique solution to Extremal Problem 1 in case (c) listed above (Theorem **??**), and also that Extremal Problem 2 has a unique solution for any data (Theorem **??**). Both results depend on Theorem **??**, asserting that a two-dimensional divergence-free measure $\boldsymbol{\nu}$ can be decomposed into loops; *i.e.*, contour integrations along rectifiable Jordan curves such that $d\boldsymbol{\nu}/d|\boldsymbol{\nu}|(x)$ is, for $|\boldsymbol{\nu}|$-a.e. x, the unit tangent to any of these curves through x. This theorem is stated in Section **??**, after some measure-theoretic preparation in Sections **??** and **??**.

1.2. **Notation**

We conclude this section with some notation and definitions regarding measures and distributions. For a vector x in the Euclidean space $\mathbb{R}^n$ (we mainly deal with $n = 2$ or 3), we denote the j-th component of x by x_j and the partial derivative with respect to x_j by ∂_{x_j}. By default, we consider vectors as column vectors; e.g., for $x \in \mathbb{R}^3$ we write $x = (x_1, x_2, x_3)^T$ where "T" denotes "transpose". We write $\mathbb{N}$ for the nonnegative integers, $\mathbb{N}^*$ for the positive integers, and $\mathbb{R}^+$ for the nonnegative real numbers. We use bold symbols to represent vector-valued functions and measures, and the corresponding nonbold symbols with subscripts to denote the respective components; e.g., $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ or $\mathbf{b}(\boldsymbol{\mu}) = (b_1(\boldsymbol{\mu}), b_2(\boldsymbol{\mu}), b_3(\boldsymbol{\mu}))^T$. For $x \in \mathbb{R}^n$ and $R > 0$, we let $\mathbb{B}(x, R)$ indicate the open ball centered at x with radius R, and $\mathbb{S}(x, R)$ the boundary sphere. This notation does not show dependence on n, but no confusion will arise. We denote by $\mathcal{M}(E)$ the space of finite signed Borel measures on $E \subset \mathbb{R}^n$.

We write χ_E for the characteristic function of a set E and δ_x for the Dirac delta measure at x. Given a $\mathbb{R}^m$-valued measure in $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$ and a Borel set $E \subset \mathbb{R}^n$, we denote by $\boldsymbol{\mu}\lfloor E$ the measure obtained by restricting $\boldsymbol{\mu}$ to E (*i.e.*, for every Borel set $B \subset \mathbb{R}^n$, $\boldsymbol{\mu}\lfloor E(B) := \boldsymbol{\mu}(E \cap B)$).

For $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$, the *total variation measure* $|\boldsymbol{\mu}|$ is defined on Borel sets $B \subset \mathbb{R}^n$ by

$$|\boldsymbol{\mu}|(B) := \sup_{\mathcal{P}} \sum_{P \in \mathcal{P}} |\boldsymbol{\mu}(P)|, \tag{6}$$

where the supremum is taken over all finite Borel partitions $\mathcal{P}$ of B. The *total variation norm of* $\boldsymbol{\mu}$ is then defined as

$$\|\boldsymbol{\mu}\|_{TV} := |\boldsymbol{\mu}|(\mathbb{R}^n). \tag{7}$$

The *support* of $\boldsymbol{\mu}$ is the complement of the largest open set U such that $|\boldsymbol{\mu}|(U) = 0$; it is denoted by $\operatorname{supp}\boldsymbol{\mu}$. A *carrier* for $\boldsymbol{\mu}$ is any $|\boldsymbol{\mu}|$-measurable set E with full $|\boldsymbol{\mu}|$-measure; we also say that $\boldsymbol{\mu}$ is *carried by* E. Since $|\boldsymbol{\mu}|$ is a Radon measure, the Radon-Nikodym derivative $\mathbf{u}_{\boldsymbol{\mu}} := d\boldsymbol{\mu}/d|\boldsymbol{\mu}|$ exists as a $\mathbb{R}^m$-valued $|\boldsymbol{\mu}|$-integrable function and it satisfies $|\mathbf{u}_{\boldsymbol{\mu}}| = 1$ a.e. with respect to $|\boldsymbol{\mu}|$.

For $\Omega \subset \mathbb{R}^n$ an open set, we denote by $C_c(\Omega, \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued continuous functions with compact support on Ω, equiped with the sup-norm. When $m = 1$, we drop the dependence on m and simply write $C_c(\Omega)$. A similar notational simplification is used for other functional spaces introduced below.

We shall identify $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^n)^m$ with the linear form on $C_c(\mathbb{R}^n, \mathbb{R}^m)$ given by

$$\langle \boldsymbol{\mu}, \mathbf{f} \rangle := \int \mathbf{f} \cdot d\boldsymbol{\mu}, \qquad \mathbf{f} \in C_c(\mathbb{R}^n, \mathbb{R}^m). \tag{8}$$

The norm of the functional (8), is $\|\boldsymbol{\mu}\|_{TV}$. More generally, for $\Omega \subset \mathbb{R}^n$ an open set, it follows from Lusin's theorem [17, Cor. to Theorem 2.23], applied to the restriction of $\mathbf{u}_{\boldsymbol{\mu}}$ to "large" compact sets in Ω, and from the dominated convergence theorem that

$$|\boldsymbol{\mu}|(\Omega) = \sup\{\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle,\ \boldsymbol{\varphi} \in C_c(\Omega, \mathbb{R}^m),\ |\boldsymbol{\varphi}| \le 1\}. \tag{9}$$

The functional (8) extends naturally with the same norm to the Banach space $C_0(\mathbb{R}^n, \mathbb{R}^m)$ of $\mathbb{R}^m$-valued continuous functions on $\mathbb{R}^n$ vanishing at infinity.

At places, we also identify $\boldsymbol{\mu}$ with the restriction of (8) to $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, the space of C^∞-smooth functions with compact support, equiped with the usual topology of test functions [18]. We refer to a continuous linear functional on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ as being a distribution, and put ∂_{x_i} to mean distributional derivative with respect to the variable x_i.

We denote Lebesgue measure on $\mathbb{R}^n$ by $\mathcal{L}_n$ and d-dimensional Hausdorff measure by $\mathcal{H}^d$, see [9] for the definitions. We normalize $\mathcal{H}^d$ for $d = 1$ and 2 so that it coincides with arclength and surface area for smooth curves and surfaces, and more generally that it agrees with d-dimensional volume for nice d-dimensional subsets of $\mathbb{R}^n$. We denote the Hausdorff dimension of a set E by $\dim_{\mathcal{H}}(E)$. Recall that $E \subset R^n$ is *m-rectifiable* if it is the countable union of images of Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, up to a set of $\mathcal{H}^m$-measure zero, see [16, Def. 15.3].

For $E \subset \mathbb{R}^n$ a measurable set and $1 \le p \le \infty$, we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of $\mathcal{L}_n$-a.e. coinciding) real-valued measurable functions on E whose p-th power is integrable, with norm $\|g\|_{L^p(E)} = (\int_E |g|^p d\mathcal{L}_n)^{1/p}$ (ess. $\sup_E |g|$ if $p = \infty$). If E is open, we set $L^1_{loc}(E)$ to consist of functions f whose restriction $f_{|K}$ to K lies in $L^1(K)$, for every compact $K \subset E$. For $\nu \in \mathcal{M}(\mathbb{R}^n)$ a positive measure different from $\mathcal{L}_n$, we put $L^1[d\nu]$ for the space of real-valued integrable functions against ν.

We are particularly concerned with magnetizations supported on $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and hence, with a slight abuse of notation, given $S \subset \mathbb{R}^2$ and $\boldsymbol{\mu} \in \mathcal{M}(S \times \{0\})^3$, we shall identify S with $S \times \{0\} \subset \mathbb{R}^3$ and $\boldsymbol{\mu}$ with $\boldsymbol{\mu}\lfloor(\mathbb{R}^2 \times \{0\})$. In addition, we let $\mathfrak{R}$ denote the rotation by $\pi/2$ in $\mathbb{R}^2$; i.e., $\mathfrak{R}((x_1, x_2)^T) = (-x_2, x_1)^T$.

For an open set $\Omega \subset \mathbb{R}^n$, recall the space $BV(\Omega)$ of functions of *bounded variation* comprised of functions in $L^1(\Omega)$ whose distributional derivatives are signed measures on Ω (see, [22]). We let $BV_{loc}(\Omega)$ denote the space of functions whose restriction to any relatively compact open subset Ω_1 of Ω lies in $BV(\Omega_1)$. We define the space $\dot{BV}(\Omega)$ of "homogeneous" BV-functions to consist of locally integrable functions whose distributional derivatives are finite signed measures on Ω. Note that $\phi \in \dot{BV}(\Omega)$ if and only if it is a distribution on Ω such that $\nabla\phi \in \mathcal{M}(\Omega)^n$, by [8, Theorem 6.7.7].

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