Dynamic programming principle and Hamilton-Jacobi-Bellman equation under nonlinear expectation

Mingshang Hu *　Shaolin Ji†　Xiaojuan Li‡

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Abstract. In this paper, we study a stochastic recursive optimal control problem in which the value functional is defined by the solution of a backward stochastic differential equation (BSDE) under $\tilde{G}$-expectation. Under standard assumptions, we establish the comparison theorem for this kind of BSDE and give a novel and simple method to obtain the dynamic programming principle. Finally, we prove that the value function is the unique viscosity solution to a type of fully nonlinear HJB equation.

Key words. Dynamic programming principle, Hamilton-Jacobi-Bellman equation, Stochastic recursive optimal control, Backward stochastic differential equation

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

Motivated by the model uncertainty in finance, Peng [21–23] established the theory of G-expectation which is a consistent sublinear expectation and does not require a probability space. The representation of G-expectation as the supremum of expectations over a set of nondominated probability measures was obtained in [4, 15]. Due to this set of nondominated probability measures, the backward stochastic differential equation (BSDE for short) is completely different from the classical one. Hu et al. [12] obtained an existence and uniqueness theorem for a new kind of BSDE driven by G-Brownian motion. In addition, there are other advances in this direction. Denis and Martini [5] developed quasi-sure stochastic analysis. Soner et al. [28] obtained an existence and uniqueness theorem for a new type of BSDE (2BSDE) under a family of nondominated probability measures.

Recently, Hu and Ji [11] studied the following stochastic recursive optimal control problem under G-

*Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan, Shandong 250100, PR China. humingshang@sdu.edu.cn. Research supported by National Key R&D Program of China (No. 2018YFA0703900) and NSF (No. 11671231).

†Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan, Shandong 250100, PR China. jsl@sdu.edu.cn. Research supported by NSF (No. 11971263 and 11871458).

‡Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan 250100, China. Email: lixiaojuan@mail.sdu.edu.cn.

expectation:

$$\begin{cases} dX_s^{t,x,u} = & b(s, X_s^{t,x,u}, u_s)ds + h_{ij}(s, X_s^{t,x,u}, u_s)d\langle B^i, B^j\rangle_s + \sigma(s, X_s^{t,x,u}, u_s)dB_s, \\ X_t^{t,x,u} = & x, \end{cases} \tag{1.1}$$

$$\begin{aligned} Y_s^{t,x,u} = \quad & \Phi(X_T^{t,x,u}) + \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, u_r)dr + \int_s^T g_{ij}(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, u_r)d\langle B^i, B^j\rangle_r \\ & - \int_s^T Z_r^{t,x,u}dB_r - (K_T^{t,x,u} - K_s^{t,xu}), \ s \in [t, T]. \end{aligned} \tag{1.2}$$

The value function is defined as

$$V(t, x) := \underset{u\in\mathcal{U}[t,T]}{ess \inf} Y_t^{t,x,u}. \tag{1.3}$$

As pointed out in [11], the value function defined in (1.3) is a inf sup problem, which is known as the robust optimal control problem. For recent development of robust control problem under a set of nondominated probability measures, we refer the readers to [6, 8, 9, 18, 27] and the references therein. When G is linear, the above optimal control problem is classical stochastic recursive optimal control problem, which was first studied by Peng in [24]. For the development of classical stochastic recursive optimal control problem, we refer the readers to [1, 2, 7, 14, 17, 19, 20, 29–31] and the references therein.

The nonlinear part with respect to $\partial^2_{xx}V$ in the HJB equation related to the optimal control problem (1.1) and (1.2) is the inf sup of a family of linear part with respect to $\partial^2_{xx}V$. Up to our knowledge, this inf sup representation is the only result that has been made so far in the optimal control problem. In order to obtain the fully nonlinear representation, we want to study the stochastic recursive optimal control problem under $\tilde{G}$-expectation. Here $\tilde{G}$ is any function dominated by G in the meaning of (2.1). More precisely, we consider the following BSDE under $\tilde{G}$-expectation:

$$Y_s^{t,x,u} = \tilde{\mathbb{E}}_s\left[\Phi(X_T^{t,x,u}) + \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, u_r)dr + \int_s^T g_{ij}(r, X_r^{t,x,u}, Y_r^{t,x,u}, u_r)d\langle B^i, B^j\rangle_r\right]. \tag{1.4}$$

The new optimal control problem is (1.1) and (1.4), and the value function is still defined as (1.3). It is worth pointing out that the BSDE (1.4) under $\tilde{G}$-expectation does not contain Z, which is an important open problem.

In this paper, we study the dynamic programming principle (DPP) and HJB equation for optimal control problem (1.1) and (1.4). Firstly, we establish the comparison theorem for BSDE (1.4), which is new in the literature. Secondly, for each $\xi \in L^2_G(\Omega)$, we prove that there exists a sequence of simple random variables $\xi_k \in L^2_G(\Omega)$ such that ξ_k converges to ξ in the sense of L^2_G. Based on this approximation, we give a novel method to prove the DPP, which still holds for the optimal control problem (1.1) and (1.2) and is easier than the implied partition method in [11]. At last, we prove that V is the unique viscosity solution to a type of fully nonlinear HJB equation, which is not the inf sup representation with respect to $\partial^2_{xx}V$.

This paper is organized as follows. We recall some basic results on G-expectation and $\tilde{G}$-expectation in Section 2. In Section 3, we formulate our stochastic recursive optimal control problem under $\tilde{G}$-expectation. In Section 4, we prove the properties of the value function and obtain the DPP. We prove that the value function is the unique viscosity solution to a type of fully nonlinear HJB equation in Section 5.

2 Preliminaries

Let $T > 0$ be given and let $\Omega_T = C_0([0,T];\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ with $\omega_0 = 0$. The canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0,T]$. Set

$$Lip(\Omega_T) := \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : N \geq 1, t_1 < \cdots < t_N \leq T, \varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})\},$$

where $C_{b.Lip}(\mathbb{R}^{d\times N})$ denotes the space of bounded Lipschitz functions on $\mathbb{R}^{d\times N}$.

Let $G : \mathbb{S}_d \to \mathbb{R}$ be a given monotonic and sublinear function, where $\mathbb{S}_d$ denotes the set of $d \times d$ symmetric matrices. Peng [22, 23] constructed a G-expectation space $(\Omega_T, Lip(\Omega_T), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t\in[0,T]})$, which is a consistent sublinear expectation space. The canonical process $(B_t)_{t\in[0,T]}$ is called G-Brownian motion under $\hat{\mathbb{E}}$. Throughout this paper, we suppose that G is non-degenerate, i.e., there exists a $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \frac{1}{2}\underline{\sigma}^2 \mathrm{tr}[A - B]$ for any $A \geq B$. Furthermore, let $\tilde{G} : \mathbb{S}_d \to \mathbb{R}$ be any given monotonic function dominated by G, i.e., for A_1, $A_2 \in \mathbb{S}_d$,

$$\begin{cases} \tilde{G}(0) = 0, \\ \tilde{G}(A_1) \geq \tilde{G}(A_2) \text{ if } A_1 \geq A_2, \\ \tilde{G}(A_1) - \tilde{G}(A_2) \leq G(A_1 - A_2). \end{cases} \tag{2.1}$$

Peng also constructed a $\tilde{G}$-expectation space $(\Omega_T, Lip(\Omega_T), \tilde{\mathbb{E}}, (\tilde{\mathbb{E}}_t)_{t\in[0,T]})$ in [21, 26], which is a consistent nonlinear expectation space satisfying:

(i) for each fixed $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, $u(t,x) := \tilde{\mathbb{E}}[\varphi(x + B_T - B_t)]$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is the viscosity solution to the following partial differential equation

$$\partial_t u + \tilde{G}(\partial^2_{xx} u) = 0, \ u(T,x) = \varphi(x);$$

(ii) for each $X, Y \in Lip(\Omega_T)$, $t \in [0,T]$,

$$\tilde{\mathbb{E}}_t[X] - \tilde{\mathbb{E}}_t[Y] \leq \hat{\mathbb{E}}_t[X - Y]. \tag{2.2}$$

Denote by $L^p_G(\Omega_T)$ the completion of $Lip(\Omega_T)$ under the norm $||X||_{L^p_G} := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$. For each $t \in [0,T]$, the conditional G-expectation and $\tilde{G}$-expectation can be continuously extended to $L^1_G(\Omega_T)$ under the norm $||\cdot||_{L^1_G}$, and still satisfy the relation (2.2) for $X, Y \in L^1_G(\Omega_T)$.

Definition 2.1 *Let $M^0_G(0,T)$ be the space of simple processes in the following form: for each $N \in \mathbb{N}$ and $0 = t_0 < \cdots < t_N = T$,*

$$\eta_t = \sum_{k=0}^{N-1} \xi_k I_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in Lip(\Omega_{t_k})$ for $k = 0, 1, \ldots, N-1$.

Denote by $M^p_G(0,T)$ the completion of $M^0_G(0,T)$ under the norm $||\eta||_{M^p_G} := (\hat{\mathbb{E}}[\int_0^T |\eta_t|^p dt])^{1/p}$ for $p \geq 1$. For each $\eta^k \in M^2_G(0,T)$, $k = 1, \ldots, d$, denote $\eta = (\eta^1, \ldots, \eta^d)^T \in M^2_G(0,T;\mathbb{R}^d)$, the G-Itô integral $\int_0^T \eta^T_t dB_t$ is well defined, see Peng [22, 23, 26].

Theorem 2.2 *([4, 15]) There exists a weakly compact set of probability measures $\mathcal{P}$ on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that*

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X] \text{ for all } X \in L_G^1(\Omega_T).$$

$\mathcal{P}$ is called a set that represents $\hat{\mathbb{E}}$.

For this $\mathcal{P}$, we define capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A) \text{ for } A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s.

3 Stochastic optimal control problem

Let U be a given compact set of $\mathbb{R}^m$. For each $t \in [0,T]$, we denote by

$$\mathcal{U}[t,T] := \{u : u \in M_G^2(t,T;\mathbb{R}^m) \text{ with values in } U\}$$

the set of admissible controls on $[t,T]$.

In the following, we use Einstein summation convention. For each given $t \in [0,T]$, $\xi \in L_G^p(\Omega_t;\mathbb{R}^n)$ with $p \geq 2$ and $u \in \mathcal{U}[t,T]$, we consider the following forward and backward SDEs:

$$\begin{cases} dX_s^{t,\xi,u} = & b(s,X_s^{t,\xi,u},u_s)ds + h_{ij}(s,X_s^{t,\xi,u},u_s)d\langle B^i,B^j\rangle_s + \sigma(s,X_s^{t,\xi,u},u_s)dB_s, \\ X_t^{t,\xi,u} = & \xi, \end{cases} \tag{3.1}$$

and

$$Y_s^{t,\xi,u} = \tilde{\mathbb{E}}_s\left[\Phi(X_T^{t,\xi,u}) + \int_s^T f(r,X_r^{t,\xi,u},Y_r^{t,\xi,u},u_r)dr + \int_s^T g_{ij}(r,X_r^{t,\xi,u},Y_r^{t,\xi,u},u_r)d\langle B^i,B^j\rangle_r\right], \tag{3.2}$$

where $s \in [t,T]$, $\langle B\rangle = (\langle B^i,B^j\rangle)_{i,j=1}^d$ is the quadratic variation of B.

Suppose that b, $h_{ij} : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times d}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$, f, g_{ij} : $[0,T]\times\mathbb{R}^n\times\mathbb{R}\times U \to \mathbb{R}$ are deterministic functions and satisfy the following conditions:

(H1) There exists a constant $L > 0$ such that for any (s,x,y,u), $(s,x',y',v) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times U$,

$$\begin{aligned} &|b(s,x,u) - b(s,x',v)| + |h_{ij}(s,x,u) - h_{ij}(s,x',v)| + |\sigma(s,x,u) - \sigma(s,x',v)| \\ &\quad \leq L(|x-x'| + |u-v|), \\ &|\Phi(x) - \Phi(x')| \leq L|x-x'|, \\ &|f(s,x,y,u) - f(s,x',y',v)| + |g_{ij}(s,x,y,u) - g_{ij}(s,x',y',v)| \\ &\quad \leq L(|x-x'| + |y-y'| + |u-v|); \end{aligned}$$

(H2) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$; b, h_{ij}, σ, f, g_{ij} are continuous in s.

We have the following theorems.

Theorem 3.1 *([26]) Let Assumptions (H1) and (H2) hold. Then, for each $\xi \in L_G^2(\Omega_t; \mathbb{R}^n)$ and $u \in \mathcal{U}[t,T]$, there exists a unique solution $(X, Y) \in M_G^2(t, T; \mathbb{R}^{n+1})$ for the forward-backward SDE (3.1) and (3.2).*

Theorem 3.2 *([11, 26]) Let Assumptions (H1) and (H2) hold, and let ξ, $\xi' \in L_G^p(\Omega_t; \mathbb{R}^n)$ with $p \geq 2$ and u, $v \in \mathcal{U}[t,T]$. Then, for each $\delta \in [0, T-t]$, we have*

$$\begin{aligned}
&\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi,u} - X_{t+\delta}^{t,\xi',v}|^2] \leq C(|\xi - \xi'|^2 + \hat{\mathbb{E}}_t[\int_t^{t+\delta} |u_s - v_s|^2 ds]), \\
&\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,\xi,u}|^p] \leq C(1 + |\xi|^p), \\
&\hat{\mathbb{E}}_t\left[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi,u} - \xi|^p\right] \leq C(1 + |\xi|^p)\delta^{p/2},
\end{aligned}$$

where C depends on T, G, p and L.

Our stochastic optimal control problem is to find $u \in \mathcal{U}[t,T]$ which minimizes the objective function $Y_t^{t,x,u}$ for each given $x \in \mathbb{R}^n$. For this purpose, we need the following definition of essential infimum of $\{Y_t^{t,x,u} : u \in \mathcal{U}[t,T]\}$.

Definition 3.3 *([11]) The essential infimum of $\{Y_t^{t,x,u} : u \in \mathcal{U}[t,T]\}$, denoted by $\underset{u\in\mathcal{U}[t,T]}{ess\inf} Y_t^{t,x,u}$, is a random variable $\zeta \in L_G^2(\Omega_t)$ satisfying:*

(i) *for any $u \in \mathcal{U}[t,T]$, $\zeta \leq Y_t^{t,x,u}$ q.s.;*

(ii) *if η is a random variable satisfying $\eta \leq Y_t^{t,x,u}$ q.s. for any $u \in \mathcal{U}[t,T]$, then $\zeta \geq \eta$ q.s.*

For each $(t, x) \in [0, T] \times \mathbb{R}^n$, we define the value function

$$V(t, x) := \underset{u\in\mathcal{U}[t,T]}{ess\inf} Y_t^{t,x,u}. \tag{3.3}$$

In the following we will prove that $V(\cdot, \cdot)$ is deterministic and $V(t, \xi) = \underset{u\in\mathcal{U}[t,T]}{ess\inf} Y_t^{t,\xi,u}$ for each $\xi \in L_G^2(\Omega_t; \mathbb{R}^n)$. Furthermore, we will obtain the dynamic programming principle and the related fully nonlinear HJB equation.

4 Dynamic programming principle

In the following, the constant C will change from line to line in our proof. We use the following notations: for each given $0 \leq t \leq s \leq T$,

$Lip(\Omega_s^t) := \{\varphi(B_{t_1} - B_t, \dots, B_{t_N} - B_t) : N \geq 1, t_1, \dots, t_N \in [t, s], \varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})\}$;

$L_G^2(\Omega_s^t) := \{$the completion of $Lip(\Omega_s^t)$ under the norm $||\cdot||_{L_G^2}\}$;

$M_G^{0,t}(t,T) := \{\eta_s = \sum_{k=0}^{N-1} \xi_k I_{[t_k, t_{k+1})}(s) : t = t_0 < \cdots < t_N = T, \xi_k \in Lip(\Omega_{t_k}^t)\}$;

$M_G^{2,t}(t,T) := \{$the completion of $M_G^{0,t}(t,T)$ under the norm $||\cdot||_{M_G^2}\}$;

$\mathcal{U}^t[t,T] := \{u : u \in M_G^{2,t}(t,T;\mathbb{R}^m)$ with values in $U\}$;

$\mathbb{U}[t,T] := \{u = \sum_{k=1}^N I_{A_k} u^k : N \geq 1, u^k \in \mathcal{U}^t[t,T], I_{A_k} \in L_G^2(\Omega_t), (A_k)_{k=1}^N$ is a partition of $\Omega\}$.

In order to prove that $V(\cdot,\cdot)$ is deterministic, we need the following two lemmas. The first lemma can be found in [11].

Lemma 4.1 *([11]) Let $u \in \mathcal{U}[t,T]$ be given. Then there exists a sequence $(u^k)_{k\geq 1}$ in $\mathbb{U}[t,T]$ such that*

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_t^T |u_s - u_s^k|^2 ds\right] = 0.$$

Lemma 4.2 *Let Assumptions (H1) and (H2) hold, and let $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$, $u \in \mathcal{U}[t,T]$ and $v = \sum_{k=1}^N I_{A_k} v^k \in \mathbb{U}[t,T]$. Then there exists a constant C depending on T, G and L such that*

$$\hat{\mathbb{E}}\left[\left|Y_t^{t,\xi,u} - \sum_{k=1}^N I_{A_k} Y_t^{t,\xi,v^k}\right|^2\right] \leq C\hat{\mathbb{E}}\left[\int_t^T |u_s - v_s|^2 ds\right].$$

Proof. Similar to the proof of Lemma 15 in [11], we can get

$$X_s^{t,\xi,v} = \sum_{k=1}^N I_{A_k} X_s^{t,\xi,v^k} \text{ and } Y_s^{t,\xi,v} = \sum_{k=1}^N I_{A_k} Y_s^{t,\xi,v^k} \text{ for } s \in [t,T].$$

Since $\tilde{G}$-expectation $\tilde{\mathbb{E}}$ is dominated by G-expectation $\hat{\mathbb{E}}$, by (3.2), we obtain

$$|Y_s^{t,\xi,u} - Y_s^{t,\xi,v}| \leq C\hat{\mathbb{E}}_s\left[|X_T^{t,\xi,u} - X_T^{t,\xi,v}| + \int_s^T (|Y_r^{t,\xi,u} - Y_r^{t,\xi,v}| + |X_r^{t,\xi,u} - X_r^{t,\xi,v}| + |u_r - v_r|)dr\right],$$

where $s \in [t,T]$ and C depends on G and L. By the Hölder inequality, we get

$$|Y_s^{t,\xi,u} - Y_s^{t,\xi,v}|^2 \leq C\hat{\mathbb{E}}_s\left[|X_T^{t,\xi,u} - X_T^{t,\xi,v}|^2 + \int_s^T (|Y_r^{t,\xi,u} - Y_r^{t,\xi,v}|^2 + |X_r^{t,\xi,u} - X_r^{t,\xi,v}|^2 + |u_r - v_r|^2)dr\right],$$

where $s \in [t,T]$ and C depends on T, G and L. By the Gronwall inequality under $\hat{\mathbb{E}}$ (see Theorem 3.10 in [13]), we deduce

$$|Y_t^{t,\xi,u} - Y_t^{t,\xi,v}|^2 \leq C\hat{\mathbb{E}}_t\left[|X_T^{t,\xi,u} - X_T^{t,\xi,v}|^2 + \int_t^T (|X_r^{t,\xi,u} - X_r^{t,\xi,v}|^2 + |u_r - v_r|^2)dr\right], \tag{4.1}$$

where C depends on T, G and L. By Theorem 3.2, we have

$$\hat{\mathbb{E}}_t\left[|X_s^{t,\xi,u}-X_s^{t,\xi,v}|^2\right]\leq C\hat{\mathbb{E}}_t\left[\int_t^T|u_r-v_r|^2dr\right], \tag{4.2}$$

where $s\in[t,T]$ and C depends on T, G and L. Thus we obtain the desired result by (4.1) and (4.2). $\Box$

Theorem 4.3 *Let Assumptions (H1) and (H2) hold. Then the value function $V(t,x)$ exists and*

$$V(t,x)=\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u}.$$

Proof. Since $(B_{t+s}-B_t)_{s\geq 0}$ is a G-Brownian motion, we know that $(Y_s^{t,x,u})_{s\in[t,T]}\in M_G^{2,t}(t,T)$ for each $u\in\mathcal{U}^t[t,T]$. Thus $Y_t^{t,x,u}\in\mathbb{R}$ for each $u\in\mathcal{U}^t[t,T]$. In order to prove that $\underset{u\in\mathcal{U}[t,T]}{ess\inf}\, Y_t^{t,x,u}=\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u}$ q.s., by Definition 3.3 and $\mathcal{U}^t[t,T]\subset\mathcal{U}[t,T]$, we only need to show that $Y_t^{t,x,v}\geq\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u}$ q.s. for each $v\in\mathcal{U}[t,T]$.

For each given $v\in\mathcal{U}[t,T]$, by Lemma 4.1, there exists a sequence $u^k=\sum_{i=1}^{N_k}I_{A_i^k}u^{i,k}\in\mathbb{U}[t,T]$, $k\geq 1$, such that $\hat{\mathbb{E}}\left[\int_t^T|v_s-u_s^k|^2ds\right]\to 0$ as $k\to\infty$. It follows from Lemma 4.2 that

$$\hat{\mathbb{E}}\left[\left|Y_t^{t,x,v}-\sum_{i=1}^{N_k}I_{A_i^k}Y_t^{t,x,u^{i,k}}\right|^2\right]\leq C\hat{\mathbb{E}}\left[\int_t^T|v_s-u_s^k|^2ds\right]\to 0\text{ as }k\to\infty.$$

Thus there exists a subsequence of $\{\sum_{i=1}^{N_k}I_{A_i^k}Y_t^{t,x,u^{i,k}},\ k\geq 1\}$ which converges to $Y_t^{t,x,v}$ q.s. Since

$$\sum_{i=1}^{N_k}I_{A_i^k}Y_t^{t,x,u^{i,k}}\geq\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u},$$

we get $Y_t^{t,x,v}\geq\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u}$ q.s. Thus, we obtain the desired result. $\Box$

Now we study the properties of $V(\cdot,\cdot)$.

Proposition 4.4 *Let Assumptions (H1) and (H2) hold. Then there exists a constant C depending on T, G and L such that, for any $t\in[0,T]$, x, $y\in\mathbb{R}^n$,*

$$|V(t,x)-V(t,y)|\leq C|x-y|\ \text{and}\ |V(t,x)|\leq C(1+|x|).$$

Proof. Similar to the proof of inequality (4.1), we can obtain that, for any $u\in\mathcal{U}^t[t,T]$,

$$|Y_t^{t,x,u}-Y_t^{t,y,u}|^2\leq C\hat{\mathbb{E}}_t\left[|X_T^{t,x,u}-X_T^{t,y,u}|^2+\int_t^T|X_r^{t,x,u}-X_r^{t,y,u}|^2dr\right], \tag{4.3}$$

where C depends on T, G and L. By Theorem 3.2, we have

$$\hat{\mathbb{E}}_t\left[|X_s^{t,x,u}-X_s^{t,y,u}|^2\right]\leq C|x-y|^2, \tag{4.4}$$

where $s\in[t,T]$ and C depends on T, G and L. Thus we get $|V(t,x)-V(t,y)|\leq C|x-y|$ by (4.3) and (4.4). Similarly, we can obtain $|V(t,x)|\leq C(1+|x|)$. $\Box$

Theorem 4.5 *Let Assumptions (H1) and (H2) hold. Then, for any* $\xi \in L_G^2(\Omega_t;\mathbb{R}^n)$*, we have*

$$V(t,\xi) = \underset{u\in\mathcal{U}[t,T]}{ess\inf} Y_t^{t,\xi,u}.$$

Proof. For each given $u \in \mathcal{U}[t,T]$, we first prove that $V(t,\xi) \le Y_t^{t,\xi,u}$ q.s.

For each $\varepsilon > 0$, we can find a $\xi_\varepsilon = \sum_{k=1}^\infty x_k I_{A_k}$ such that $|\xi - \xi_\varepsilon| \le \varepsilon$, where $x_k \in \mathbb{R}^n$ and $\{A_k\}_{k=1}^\infty$ is a $\mathcal{B}(\Omega_t)$-partition of Ω. By Proposition 4.4, we get

$$\left|V(t,\xi) - \sum_{k=1}^\infty V(t,x_k) I_{A_k}\right| = |V(t,\xi) - V(t,\xi_\varepsilon)| \le C\varepsilon. \tag{4.5}$$

Similar to the proof of inequalities (4.3) and (4.4), we can get

$$|Y_t^{t,\xi,u} - Y_t^{t,x_k,u}| \le C|\xi - x_k|, \ k \ge 1,$$

where C depends on T, G and L. Then, we obtain

$$\left|Y_t^{t,\xi,u} - \sum_{k=1}^\infty Y_t^{t,x_k,u} I_{A_k}\right| = \sum_{k=1}^\infty |Y_t^{t,\xi,u} - Y_t^{t,x_k,u}| I_{A_k} \le C|\xi - \xi_\varepsilon| \le C\varepsilon. \tag{4.6}$$

By (3.3), we have

$$\sum_{k=1}^\infty V(t,x_k) I_{A_k} \le \sum_{k=1}^\infty Y_t^{t,x_k,u} I_{A_k}, \text{ q.s.} \tag{4.7}$$

It follows from (4.5), (4.6) and (4.7) that

$$V(t,\xi) \le Y_t^{t,\xi,u} + C\varepsilon, \text{ q.s.},$$

where C is independent of ε. Thus we obtain $V(t,\xi) \le Y_t^{t,\xi,u}$ q.s.

Second, if η is a random variable satisfying $\eta \le Y_t^{t,\xi,u}$ q.s. for any $u \in \mathcal{U}[t,T]$, then we prove that $V(t,\xi) \ge \eta$ q.s.

It follows from (4.6) that

$$\eta \le \sum_{k=1}^\infty Y_t^{t,x_k,u} I_{A_k} + C\varepsilon, \text{ q.s., for any } u \in \mathcal{U}[t,T],$$

where C depends on T, G and L. By Theorem 4.3 and the above inequality, we can get

$$\eta \le \sum_{k=1}^\infty V(t,x_k) I_{A_k} + C\varepsilon, \text{ q.s.} \tag{4.8}$$

Thus we obtain $V(t,\xi) \ge \eta$ q.s. by (4.5) and (4.8), which implies the desired result. □

Finally, we study the dynamic programming principle. The following lemmas are useful in deriving the dynamic programming principle. The first lemma is a special case of Theorem 3.20 in [16].

Lemma 4.6 *([16])Let* $0 < t < t' \le T$ *and* $c,\ c' \in \mathbb{R}^d$ *with* $c \le c'$*. Then* $I_{\{B_{t'} - B_t \in [c,c')\}} \in L_G^2(\Omega_T)$*.*

Remark 4.7 *For each* $0 = t_0 < t_1 < \cdots < t_N \le T$ *and* $c_i,\ c_i' \in \mathbb{R}^d$ *with* $c_i \le c_i'$*,* $i = 1,\ldots,N$*, we have*

$$\prod_{i=1}^N I_{\{B_{t_i} - B_{t_{i-1}} \in [c_i,c_i')\}} \in L_G^2(\Omega_T).$$

Lemma 4.8 *Let $\xi \in L_G^2(\Omega_s)$ with fixed $s \in [0,T]$. Then there exists a sequence $\xi_k = \sum_{i=1}^{N_k} x_i^k I_{A_i^k}$, $k \geq 1$, such that*

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[|\xi - \xi_k|^2\right] = 0,$$

where $x_i^k \in \mathbb{R}$, $I_{A_i^k} \in L_G^2(\Omega_s)$, $i \leq N_k$, $k \geq 1$ and $(A_i^k)_{i=1}^{N_k}$ is a $\mathcal{B}(\Omega_s)$-partition of Ω.

Proof. Since $L_G^2(\Omega_s)$ is the completion of $Lip(\Omega_s)$ under the norm $||\cdot||_2$, we only need to prove the case

$$\xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}),$$

where $N \geq 1$, $0 < t_1 < \cdots < t_N \leq s$, $\varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})$.

By Remark 4.7, we know that

$$I_{\{(B_{t_1}, B_{t_2}-B_{t_1},\ldots,B_{t_N}-B_{t_{N-1}})\in[c,c')\}} \in L_G^2(\Omega_s)$$

for each c, $c' \in \mathbb{R}^{d\times N}$ with $c \leq c'$. For each $k \geq 1$, we can find

$$A_i^k = \{(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) \in [c_{i,k}, c'_{i,k})\}, i = 1, \ldots, N_k - 1,$$

such that $[-ke, ke) = \cup_{i\leq N_k-1}[c_{i,k}, c'_{i,k})$ with $e = [1, \ldots, 1]^T \in \mathbb{R}^{d\times N}$, $|c'_{i,k} - c_{i,k}| \leq k^{-1}$ and $A_i^k \cap A_j^k = \emptyset$ for $i \neq j$. Set $A_{N_k}^k = \Omega \backslash \cup_{i\leq N_k-1} A_i^k$ and

$$\xi_k = \sum_{i=1}^{N_k-1} \varphi(c_{i,k}) I_{A_i^k} + 0 I_{A_{N_k}^k}.$$

Then we obtain

$$|\xi - \xi_k| \leq \frac{L_\varphi}{k} + \frac{M_\varphi}{k}(|B_{t_1}| + |B_{t_2} - B_{t_1}| + \cdots + |B_{t_N} - B_{t_{N-1}}|),$$

where L_φ is the Lipschitz constant of φ and M_φ is the bound of φ. Thus

$$\hat{\mathbb{E}}\left[|\xi - \xi_k|^2\right] \leq \frac{C}{k^2},$$

which yields the desired result. $\square$

In order to give the dynamic programming principle, we define the following backward semigroup $\mathbb{G}_{t,t+\delta}^{t,x,u}[\cdot]$ which was first introduced by Peng in [25].

For each given $(t,x) \in [0,T) \times \mathbb{R}^n$, $\delta \in [0, T-t]$, $u \in \mathcal{U}[t, t+\delta]$ and $\eta \in L_G^2(\Omega_{t+\delta})$, define

$$\mathbb{G}_{s,t+\delta}^{t,x,u}[\eta] = \tilde{Y}_s^{t,x,u} \text{ for } s \in [t, t+\delta],$$

where $(X_s^{t,x,u}, \tilde{Y}_s^{t,x,u})_{s\in[t,t+\delta]}$ is the solution of the following forward and backward SDEs:

$$\begin{cases} dX_s^{t,x,u} = & b(s, X_s^{t,x,u}, u_s)ds + h_{ij}(s, X_s^{t,x,u}, u_s)d\langle B^i, B^j\rangle_s + \sigma(s, X_s^{t,x,u}, u_s)dB_s, \\ X_t^{t,x,u} = & x, \end{cases} \tag{4.9}$$

and

$$\tilde{Y}_s^{t,x,u} = \tilde{\mathbb{E}}_s\left[\eta + \int_s^{t+\delta} f(r, X_r^{t,x,u}, \tilde{Y}_r^{t,x,u}, u_r)dr + \int_s^{t+\delta} g_{ij}(r, X_r^{t,x,u}, \tilde{Y}_r^{t,x,u}, u_r)d\langle B^i, B^j\rangle_r\right]. \tag{4.10}$$

The following lemma is the comparison theorem of backward SDE under $\tilde{\mathbb{E}}$.

Lemma 4.9 *Let Assumptions (H1) and (H2) hold, and let* $(t,x)\in[0,T)\times\mathbb{R}^n$, $\delta\in[0,T-t]$, $u\in\mathcal{U}[t,t+\delta]$ *and* η_1, $\eta_2\in L_G^2(\Omega_{t+\delta})$ *be given. If* $\eta_1\geq\eta_2$ *q.s., then* $\mathbb{G}_{t,t+\delta}^{t,x,u}[\eta_1]\geq\mathbb{G}_{t,t+\delta}^{t,x,u}[\eta_2]$ *q.s.*

Proof. Denote $Y_s^1=\mathbb{G}_{s,t+\delta}^{t,x,u}[\eta_1]$, $Y_s^2=\mathbb{G}_{s,t+\delta}^{t,x,u}[\eta_2]$, $\hat{Y}_s=Y_s^1-Y_s^2$ for $s\in[t,t+\delta]$, and $\hat{\eta}=\eta_1-\eta_2$. For each given $\varepsilon>0$, just like the proof of Theorem 3.6 in [13], we can find $(a_s^\varepsilon)_{s\in[t,t+\delta]}$, $(m_s^\varepsilon)_{s\in[t,t+\delta]}$, $(c_s^{ij,\varepsilon})_{s\in[t,t+\delta]}$, $(n_s^{ij,\varepsilon})_{s\in[t,t+\delta]}\in M_G^2(t,t+\delta)$ such that $|a_s^\varepsilon|\leq L$, $|c_s^{ij,\varepsilon}|\leq L$, $|m_s^\varepsilon|\leq 2L\varepsilon$, $|n_s^{ij,\varepsilon}|\leq 2L\varepsilon$,

$$f(r,X_r^{t,x,u},Y_r^1,u_r)-f(r,X_r^{t,x,u},Y_r^2,u_r)=a_r^\varepsilon\hat{Y}_r+m_r^\varepsilon$$

and

$$g_{ij}(r,X_r^{t,x,u},Y_r^1,u_r)-g_{ij}(r,X_r^{t,x,u},Y_r^2,u_r)=c_r^{ij,\varepsilon}\hat{Y}_r+n_r^{ij,\varepsilon}.$$

Then

$$\hat{Y}_s=\tilde{\mathbb{E}}_s\left[\hat{\eta}+\tilde{\eta}+\int_s^{t+\delta}(a_r^\varepsilon\hat{Y}_r+m_r^\varepsilon)dr+\int_s^{t+\delta}(c_r^{ij,\varepsilon}\hat{Y}_r+n_r^{ij,\varepsilon})d\langle B^i,B^j\rangle_r\right]-\tilde{\mathbb{E}}_s[\tilde{\eta}],\tag{4.11}$$

where $s\in[t,t+\delta]$ and $\tilde{\eta}=\eta_2+\int_t^{t+\delta}f(r,X_r^{t,x,u},Y_r^2,u_r)dr+\int_t^{t+\delta}g_{ij}(r,X_r^{t,x,u},Y_r^2,u_r)d\langle B^i,B^j\rangle_r$.

For each given $k\geq 1$, set $t_l^k=t+l\delta k^{-1}$, $l=0,1,\ldots,k$. By (4.11), one can check that, for $s\in[t_l^k,t_{l+1}^k]$, $l=k-1,\ldots,0$,

$$\hat{Y}_s=\tilde{\mathbb{E}}_s\left[\hat{Y}_{t_{l+1}^k}+\tilde{\eta}+\int_s^{t_{l+1}^k}(a_r^\varepsilon\hat{Y}_r+m_r^\varepsilon)dr+\int_s^{t_{l+1}^k}(c_r^{ij,\varepsilon}\hat{Y}_r+n_r^{ij,\varepsilon})d\langle B^i,B^j\rangle_r\right]-\tilde{\mathbb{E}}_s[\tilde{\eta}].\tag{4.12}$$

Define $(\hat{Y}_l^k)_{l=0}^n$ backwardly as follows: set $\hat{Y}_k^k=\hat{\eta}$, for $l=k-1,\ldots,0$,

$$\hat{Y}_l^k=\tilde{\mathbb{E}}_{t_l^k}\left[\hat{Y}_{l+1}^k+\tilde{\eta}+\int_{t_l^k}^{t_{l+1}^k}(a_r^\varepsilon\hat{Y}_{l+1}^k+m_r^\varepsilon)dr+\int_{t_l^k}^{t_{l+1}^k}(c_r^{ij,\varepsilon}\hat{Y}_{l+1}^k+n_r^{ij,\varepsilon})d\langle B^i,B^j\rangle_r\right]-\tilde{\mathbb{E}}_{t_l^k}[\tilde{\eta}].\tag{4.13}$$

Note that $|\int_{s_1}^{s_2}\zeta_r d\langle B^i,B^j\rangle_r|\leq(\hat{\mathbb{E}}\left[|B^i|^2\right]\hat{\mathbb{E}}\left[|B^j|^2\right])^{1/2}\int_{s_1}^{s_2}|\zeta_r|dr$ for each s_1, $s_2\in[t,t+\delta]$ and $\zeta\in M_G^1(t,t+\delta)$, then one can verify that

$$\left|\int_{t_l^k}^{t_{l+1}^k}a_r^\varepsilon dr+\int_{t_l^k}^{t_{l+1}^k}c_r^{ij,\varepsilon}d\langle B^i,B^j\rangle_r\right|\leq C\int_{t_l^k}^{t_{l+1}^k}(|a_r^\varepsilon|+|c_r^{ij,\varepsilon}|)dr\leq Ck^{-1}$$

and

$$\left|\int_{t_l^k}^{t_{l+1}^k}m_r^\varepsilon dr+\int_{t_l^k}^{t_{l+1}^k}n_r^{ij,\varepsilon}d\langle B^i,B^j\rangle_r\right|\leq C\int_{t_l^k}^{t_{l+1}^k}(|m_r^\varepsilon|+|n_r^{ij,\varepsilon}|)dr\leq C\varepsilon k^{-1},$$

where C is dependent of L and δ and independent of l. For each $k\geq k_0$ with $Ck_0^{-1}\leq 2^{-1}$, we have

$$\hat{Y}_{k-1}^k\geq\tilde{\mathbb{E}}_{t_{k-1}^k}[\tilde{\eta}-C\varepsilon k^{-1}]-\tilde{\mathbb{E}}_{t_{k-1}^k}[\tilde{\eta}]=-C\varepsilon k^{-1}$$

and

$$\hat{Y}_{k-2}^k\geq\tilde{\mathbb{E}}_{t_{k-2}^k}[-(1+Ck^{-1})C\varepsilon k^{-1}+\tilde{\eta}-C\varepsilon k^{-1}]-\tilde{\mathbb{E}}_{t_{k-2}^k}[\tilde{\eta}]=-[(1+Ck^{-1})+1]C\varepsilon k^{-1}.$$

Continuing this process, we obtain

$$\hat{Y}_0^k\geq -C\varepsilon k^{-1}\sum_{l=0}^{k-1}(1+Ck^{-1})^l\geq-(e^C-1)\varepsilon.\tag{4.14}$$

For each given $\eta \in Lip(\Omega_{t+\delta})$, define $\phi(s_1, s_2) = \hat{\mathbb{E}}[|\tilde{\mathbb{E}}_{s_1}[\eta] - \tilde{\mathbb{E}}_{s_2}[\eta]|]$ for $s_1,\ s_2 \in [t, t+\delta]$. By the definition of $\tilde{\mathbb{E}}_s[\eta]$, one can verify that ϕ is a continuous function. Then we get

$$\sup_{|s_1 - s_2| \leq \delta k^{-1}} \hat{\mathbb{E}}[|\tilde{\mathbb{E}}_{s_1}[\eta] - \tilde{\mathbb{E}}_{s_2}[\eta]|] \to 0 \text{ as } k \to \infty. \tag{4.15}$$

Note that

$$Y_s^2 = \tilde{\mathbb{E}}_s[\tilde{\eta}] - \int_t^s f(r, X_r^{t,x,u}, Y_r^2, u_r)dr - \int_t^s g_{ij}(r, X_r^{t,x,u}, Y_r^2, u_r)d\langle B^i, B^j\rangle_r,$$

then, by (4.15) and $\tilde{\eta} \in L_G^2(\Omega_{t+\delta})$, one can check that

$$\sup_{|s_1 - s_2| \leq \delta k^{-1}} \hat{\mathbb{E}}[|Y_{s_1}^2 - Y_{s_2}^2|] \to 0 \text{ as } k \to \infty. \tag{4.16}$$

Similarly, the relation (4.16) still holds for Y^1. Thus we obtain

$$\gamma_k := \sup_{|s_1 - s_2| \leq \delta k^{-1}} \hat{\mathbb{E}}[|\hat{Y}_{s_1} - \hat{Y}_{s_2}|] \to 0 \text{ as } k \to \infty. \tag{4.17}$$

Define $\Delta_l^k = \hat{Y}_{t_l^k} - \hat{Y}_l^k$ for $l = 0,1,\ldots,k$. By (4.12), (4.13) and (4.17), we get

$$\hat{\mathbb{E}}[|\Delta_l^k|] \leq (1 + Ck^{-1})\hat{\mathbb{E}}[|\Delta_{l+1}^k|] + Ck^{-1}\gamma_k, \tag{4.18}$$

where $l = k-1,\ldots,0$, $\Delta_k^k = 0$, C depends on L and δ. Similar to (4.14), we deduce

$$\hat{\mathbb{E}}[|\Delta_0^k|] = \hat{\mathbb{E}}[|\hat{Y}_t - \hat{Y}_0^k|] \leq (e^C - 1)\gamma_k. \tag{4.19}$$

It follows from (4.14), (4.17) and (4.19) that $\hat{Y}_t \geq -(e^C - 1)\varepsilon$ q.s. Since ε is arbitrary, we obtain the desired result. □

The following theorem is the dynamic programming principle.

Theorem 4.10 *Let Assumptions (H1) and (H2) hold. Then, for each $(t, x) \in [0, T) \times \mathbb{R}^n$, $\delta \in [0, T-t]$, we have*

$$V(t, x) = \operatorname*{\mathit{ess}\,inf}_{u \in \mathcal{U}[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] = \inf_{u \in \mathcal{U}^t[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})]. \tag{4.20}$$

Proof. By Theorem 4.3, we have

$$\operatorname*{\mathit{ess}\,inf}_{u \in \mathcal{U}[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] = \inf_{u \in \mathcal{U}^t[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})].$$

For any $u \in \mathcal{U}^t[t, T]$, by Theorem 4.5, we get

$$Y_{t+\delta}^{t,x,u} = Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x,u}, u} \geq V(t+\delta, X_{t+\delta}^{t,x,u}) \text{ q.s.}$$

Then, by Lemma 4.9, we obtain

$$Y_t^{t,x,u} = \mathbb{G}_{t,t+\delta}^{t,x,u}[Y_{t+\delta}^{t,x,u}] \geq \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})],$$

which implies

$$V(t, x) \geq \inf_{u \in \mathcal{U}^t[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})].$$

Now we prove the converse inequality. For each given $\varepsilon > 0$, there exists a $v \in \mathcal{U}^t[t, t+\delta]$ such that

$$\mathbb{G}^{t,x,v}_{t,t+\delta}[V(t+\delta, X^{t,x,v}_{t+\delta})] \leq \varepsilon + \inf_{u \in \mathcal{U}^t[t,t+\delta]} \mathbb{G}^{t,x,u}_{t,t+\delta}[V(t+\delta, X^{t,x,u}_{t+\delta})]. \tag{4.21}$$

Since $X^{t,x,v}_{t+\delta} \in L^2_G(\Omega^t_{t+\delta}; \mathbb{R}^n)$, by Lemma 4.8, we can find a sequence $\xi_k = \sum_{l=1}^{N_k} x^k_l I_{A^k_l}$, $k \geq 1$, such that

$$\hat{\mathbb{E}}\left[|X^{t,x,v}_{t+\delta} - \xi_k|^2\right] \leq k^{-1}, \tag{4.22}$$

where $x^k_l \in \mathbb{R}^n$, $I_{A^k_l} \in L^2_G(\Omega^t_{t+\delta})$, $l \leq N_k$, $k \geq 1$ and $(A^k_l)^{N_k}_{l=1}$ is a $\mathcal{B}(\Omega^t_{t+\delta})$-partition of Ω. For each x^k_l, we can find $v^k_l \in \mathcal{U}^{t+\delta}[t+\delta, T]$ such that

$$V(t+\delta, x^k_l) \leq Y^{t+\delta, x^k_l, v^k_l}_{t+\delta} \leq V(t+\delta, x^k_l) + \varepsilon. \tag{4.23}$$

Set

$$v^k(s) = \sum_{l=1}^{N_k} v^k_l(s) I_{A^k_l} \text{ for } s \in [t+\delta, T],$$

and

$$u^k(s) = v(s) I_{[t,t+\delta)}(s) + v^k(s) I_{[t+\delta,T]}(s) \text{ for } s \in [t, T],$$

it is easy to verify that $v^k \in \mathcal{U}[t+\delta, T]$ and $u^k \in \mathcal{U}^t[t, T]$. Thus we get

$$V(t,x) \leq Y^{t,x,u^k}_t = \mathbb{G}^{t,x,v}_{t,t+\delta}[Y^{t,x,u^k}_{t+\delta}]. \tag{4.24}$$

Similarly to the proof of inequality (4.1), we obtain that

$$\left|\mathbb{G}^{t,x,v}_{t,t+\delta}[Y^{t,x,u^k}_{t+\delta}] - \mathbb{G}^{t,x,v}_{t,t+\delta}[V(t+\delta, X^{t,x,v}_{t+\delta})]\right|^2 \leq C\hat{\mathbb{E}}\left[\left|Y^{t,x,u^k}_{t+\delta} - V(t+\delta, X^{t,x,v}_{t+\delta})\right|^2\right] \tag{4.25}$$

and

$$\hat{\mathbb{E}}\left[\left|Y^{t+\delta, X^{t,x,v}_{t+\delta}, v^k}_{t+\delta} - Y^{t+\delta,\xi_k,v^k}_{t+\delta}\right|^2\right] \leq C \sup_{s \in [t+\delta, T]} \hat{\mathbb{E}}\left[\left|X^{t+\delta, X^{t,x,v}_{t+\delta}, v^k}_s - X^{t+\delta,\xi_k,v^k}_s\right|^2\right], \tag{4.26}$$

where C depends on T, G and L. By Theorem 3.2, (4.22) and (4.26), we have

$$\hat{\mathbb{E}}\left[\left|Y^{t+\delta, X^{t,x,v}_{t+\delta}, v^k}_{t+\delta} - Y^{t+\delta,\xi_k,v^k}_{t+\delta}\right|^2\right] \leq C\hat{\mathbb{E}}\left[\left|X^{t,x,v}_{t+\delta} - \xi_k\right|^2\right] \leq Ck^{-1}, \tag{4.27}$$

where C depends on T, G and L. Noting that $Y^{t,x,u^k}_{t+\delta} = Y^{t+\delta, X^{t,x,v}_{t+\delta}, v^k}_{t+\delta}$, we deduce from (4.24), (4.25) and (4.27) that

$$V(t,x) \leq \mathbb{G}^{t,x,v}_{t,t+\delta}[V(t+\delta, X^{t,x,v}_{t+\delta})] + C\left(\sqrt{k^{-1}} + \sqrt{\hat{\mathbb{E}}\left[\left|Y^{t+\delta,\xi_k,v^k}_{t+\delta} - V(t+\delta, X^{t,x,v}_{t+\delta})\right|^2\right]}\right), \tag{4.28}$$

where C depends on T, G and L. It is easy to check that

$$Y^{t+\delta,\xi_k,v^k}_{t+\delta} = \sum_{l=1}^{N_k} Y^{t+\delta, x^k_l, v^k_l}_{t+\delta} I_{A^k_l}. \tag{4.29}$$

Then we obtain from (4.23) and (4.29) that

$$V(t+\delta,\xi_k)=\sum_{l=1}^{N_k}V(t+\delta,x_l^k)I_{A_l^k}\leq Y_{t+\delta}^{t+\delta,\xi_k,v^k}\leq V(t+\delta,\xi_k)+\varepsilon. \tag{4.30}$$

By Proposition 4.4 and (4.30), we get

$$\hat{\mathbb{E}}\left[\left|Y_{t+\delta}^{t+\delta,\xi_k,v^k}-V(t+\delta,X_{t+\delta}^{t,x,v})\right|^2\right]\leq C\left(\varepsilon^2+\hat{\mathbb{E}}\left[\left|X_{t+\delta}^{t,x,v}-\xi_k\right|^2\right]\right)\leq C(\varepsilon^2+k^{-1}), \tag{4.31}$$

where C depends on T, G and L. By (4.21), (4.28) and (4.31), we deduce that

$$V(t,x)\leq C(\varepsilon+\sqrt{k^{-1}})+\inf_{u\in\mathcal{U}^t[t,t+\delta]}\mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta,X_{t+\delta}^{t,x,u})],$$

which implies the desired result by letting $k\to\infty$ and then $\varepsilon\downarrow 0$. □

Remark 4.11 *In the above proof, we use Lemma 4.8 to find v^k, which can be used to simplify the proof of the dynamic programming principle and is easier than the implied partition method in [11].*

Now we use the dynamic programming principle to prove the continuity of $V(\cdot,\cdot)$ in t.

Lemma 4.12 *Let Assumptions (H1) and (H2) hold. Then the value function $V(\cdot,\cdot)$ is $\frac{1}{2}$ Hölder continuous in t.*

Proof. For each $(t,x)\in[0,T)\times\mathbb{R}^n$, $\delta\in[0,T-t]$, by Theorem 4.10, we get

$$|V(t,x)-V(t+\delta,x)|\leq\sup_{u\in\mathcal{U}^t[t,t+\delta]}|\mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta,X_{t+\delta}^{t,x,u})]-V(t+\delta,x)|. \tag{4.32}$$

For each given $u\in\mathcal{U}^t[t,t+\delta]$, by the definition of the backward semigroup, we know $\mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta,X_{t+\delta}^{t,x,u})]=Y_t$, where $(Y_s)_{s\in[t,t+\delta]}$ is the solution of the following backward SDE:

$$Y_s=\tilde{\mathbb{E}}_s\left[V(t+\delta,X_{t+\delta}^{t,x,u})+\int_s^{t+\delta}f(r,X_r^{t,x,u},Y_r,u_r)dr+\int_s^{t+\delta}g_{ij}(r,X_r^{t,x,u},Y_r,u_r)d\langle B^i,B^j\rangle_r\right].$$

By Assumptions (H1), (H2) and Proposition 4.4, one can verify that

$$|Y_s-V(t+\delta,x)|\leq C\hat{\mathbb{E}}_s\left[|X_{t+\delta}^{t,x,u}-x|+\int_s^{t+\delta}(1+|x|+|X_r^{t,x,u}|+|Y_r-V(t+\delta,x)|)dr\right],$$

where C depends on T, G and L. It follows from the Gronwall inequality under $\hat{\mathbb{E}}$ that

$$|Y_t-V(t+\delta,x)|\leq C\hat{\mathbb{E}}_t\left[|X_{t+\delta}^{t,x,u}-x|+\int_t^{t+\delta}(1+|x|+|X_r^{t,x,u}|)dr\right],$$

where C depends on T, G and L. Since $\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x,u}-x|]\leq(\hat{\mathbb{E}}_t[|X_{t+\delta}^{t,x,u}-x|^2])^{1/2}$ and $\hat{\mathbb{E}}_t[|X_r^{t,x,u}|]\leq(\hat{\mathbb{E}}_t[|X_r^{t,x,u}|^2])^{1/2}$, we obtain

$$|\mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta,X_{t+\delta}^{t,x,u})]-V(t+\delta,x)|\leq C(1+|x|)\sqrt{\delta}$$

by Theorem 3.2, where C depends on T, G and L. Thus we obtain $|V(t,x)-V(t+\delta,x)|\leq C(1+|x|)\sqrt{\delta}$ by inequality (4.32). □

5 The viscosity solution to the HJB equation

In this section, we prove that the value function $V(\cdot,\cdot)$ is the unique viscosity solution to the following HJB equation:

$$\begin{cases} \partial_t V(t,x)+\inf_{u\in U} H(t,x,V(t,x),\partial_x V(t,x),\partial_{xx}^2 V(t,x),u)=0, \\ V(T,x)=\Phi(x),\ x\in\mathbb{R}^n, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} H(t,x,v,p,A,u)&=\tilde{G}(F(t,x,v,p,A,u))+\langle p,b(t,x,u)\rangle+f(t,x,v,u), \\ F_{ij}(t,x,v,p,A,u)&=(\sigma^T(t,x,u)A\sigma(t,x,u))_{ij}+2\langle p,h_{ij}(t,x,u)\rangle+2g_{ij}(t,x,v,u), \end{aligned} \tag{5.2}$$

$(t,x,v,p,A,u)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}_n\times U$, $\tilde{G}$ is defined in (2.1).

The following is the definition of the viscosity solution to (5.1) (see [3]).

Definition 5.1 *A function $V(\cdot,\cdot)\in C([0,T]\times\mathbb{R}^n)$ is called a viscosity subsolution (resp. supersolution) to (5.1) if $V(T,x)\le\Phi(x)$ (resp. $V(T,x)\ge\Phi(x)$) for each $x\in\mathbb{R}^n$, and for each given $(t,x)\in[0,T)\times\mathbb{R}^n$, $\varphi\in C_b^{2,3}([0,T]\times\mathbb{R}^n)$ such that $\varphi(t,x)=V(t,x)$ and $\varphi\ge V$ (resp. $\varphi\le V$) on $[0,T]\times\mathbb{R}^n$, we have*

$$\partial_t\varphi(t,x)+\inf_{u\in U}H(t,x,\varphi(t,x),\partial_x\varphi(t,x),\partial_{xx}^2\varphi(t,x),u)\ge 0 \ \textit{(resp. } \le 0\textit{)}.$$

A function $V(\cdot,\cdot)\in C([0,T]\times\mathbb{R}^n)$ is called a viscosity solution to (5.1) if it is both a viscosity subsolution and a viscosity supersolution to (5.1).

Remark 5.2 *$C_b^{2,3}([0,T]\times\mathbb{R}^n)$ denotes the set of real-valued functions that are continuously differentiable up to the second order (resp. third order) in t-variable (resp. x-variable) and whose derivatives are bounded.*

Remark 5.3 *According to Theorem C.3.5 in [26], for the case that*

$$\Phi\in C_0(\mathbb{R}^n)=\{\phi\in C(\mathbb{R}^n):\lim_{|x|\to\infty}\phi(x)=0\},$$

the viscosity solution to (5.1) is unique; for the case that $\Phi\in C(\mathbb{R}^n)$ satisfying $|\Phi(x)|\le C(1+|x|^p)$ for some positive constants C and p, the meaning of the uniqueness is that, for each $\Phi_k\in C_0(\mathbb{R}^n)$ such that Φ_k converges uniformly to Φ on each compact set and $|\Phi_k|\le C(1+|x|^p)$, we have $V^{\Phi_k}(t,x)\to V^{\Phi}(t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^n$.

Theorem 5.4 *Let Assumptions (H1) and (H2) hold. Then the value function $V(\cdot,\cdot)$ defined in (3.3) is the unique viscosity solution to the HJB equation (5.1).*

In order to prove this theorem, we need the following lemmas. Let $\varphi\in C_b^{2,3}([0,T]\times\mathbb{R}^n)$ be given. For each given $(t,x)\in[0,T)\times\mathbb{R}^n$, $\delta\in[0,T-t]$ and $u\in\mathcal{U}[t,t+\delta]$, we consider the following BSDEs

$$Y_s^u=\tilde{\mathbb{E}}_s\left[\varphi(t+\delta,X_{t+\delta}^{t,x,u})+\int_s^{t+\delta}f(r,X_r^{t,x,u},Y_r^u,u_r)dr+\int_s^{t+\delta}g_{ij}(r,X_r^{t,x,u},Y_r^u,u_r)d\langle B^i,B^j\rangle_r\right], \tag{5.3}$$

$$Y_s^{1,u}=\tilde{\mathbb{E}}_s\left[\int_s^{t+\delta}F_1(r,X_r^{t,x,u},Y_r^{1,u},u_r)dr+\int_s^{t+\delta}F_2^{ij}(r,X_r^{t,x,u},Y_r^{1,u},u_r)d\langle B^i,B^j\rangle_r\right] \tag{5.4}$$

and

$$Y_s^{2,u}=\tilde{\mathbb{E}}_s\left[\int_s^{t+\delta}F_1(r,x,0,u_r)dr+\int_s^{t+\delta}F_2^{ij}(r,x,0,u_r)d\langle B^i,B^j\rangle_r\right],\tag{5.5}$$

where $s\in[t,t+\delta]$, $(X_s^{t,x,u})_{s\in[t,t+\delta]}$ is the solution of the SDE (4.9),

$$F_1(s,x,y,u)=\partial_t\varphi(s,x)+\langle b(s,x,u),\partial_x\varphi(s,x)\rangle+f(s,x,y+\varphi(s,x),u),$$

$$F_2^{ij}(s,x,y,u)=\frac{1}{2}F_{ij}(s,x,y+\varphi(s,x),\partial_x\varphi(s,x),\partial_{xx}^2\varphi(s,x),u).$$

Lemma 5.5 *For each $u\in\mathcal{U}[t,t+\delta]$, we have*

$$Y_s^{1,u}=Y_s^u-\varphi(s,X_s^{t,x,u})\ \textit{for}\ s\in[t,t+\delta].$$

Proof. Applying Itô's formula to $\varphi(r,X_r^{t,x,u})$ on $[s,t+\delta]$, we obtain that $(Y_s^u-\varphi(s,X_s^{t,x,u}))_{s\in[t,t+\delta]}$ satisfies the backward SDE (5.4), which implies the desired result by the uniqueness of the solution. □

Lemma 5.6 *For each $u\in\mathcal{U}^t[t,t+\delta]$, we have*

$$|Y_t^{1,u}-Y_t^{2,u}|\leq C(1+|x|^3)\delta^{3/2},$$

where the constant C is dependent on T, G, L and independent of u.

Proof. Noting that $\varphi\in C_b^{2,3}([0,T]\times\mathbb{R}^n)$ and U is compact, one can verify that

$$|F_1(r,x,0,u_r)|\leq C(1+|x|)\text{ and }|F_2^{ij}(r,x,0,u_r)|\leq C(1+|x|^2),$$

where C is dependent on L and independent of u. Thus

$$|Y_s^{2,u}|\leq C(1+|x|^2)\delta\text{ for }s\in[t,t+\delta],\tag{5.6}$$

where C is dependent on G, L and independent of u. Set $\hat{Y}_s=Y_s^{1,u}-Y_s^{2,u}$ for $s\in[t,t+\delta]$, by (5.4) and (5.5), we get

$$|\hat{Y}_s|\leq C\hat{\mathbb{E}}_s\left[\int_s^{t+\delta}(\hat{F}_r+|\hat{Y}_r|)dr\right],$$

where $C>0$ is dependent on G, L and independent of u,

$$\hat{F}_r=|F_1(r,X_r^{t,x,u},Y_r^{2,u},u_r)-F_1(r,x,0,u_r)|+|F_2^{ij}(r,X_r^{t,x,u},Y_r^{2,u},u_r)-F_2^{ij}(r,x,0,u_r)|.$$

Note that $Y_t^{1,u}\in\mathbb{R}$ and $Y_t^{2,u}\in\mathbb{R}$ for each $u\in\mathcal{U}^t[t,t+\delta]$, then, by the Gronwall inequality under $\hat{\mathbb{E}}$, we obtain

$$|Y_t^{1,u}-Y_t^{2,u}|\leq C\hat{\mathbb{E}}\left[\int_t^{t+\delta}\hat{F}_rdr\right],\tag{5.7}$$

where $C>0$ is dependent on T, G, L and independent of u. One can check that

$$\hat{F}_r\leq C\left[(1+|x|^2)|X_r^{t,x,u}-x|+|X_r^{t,x,u}-x|^2+|Y_r^{2,u}|\right],\tag{5.8}$$

where C is dependent on L and independent of u. It follows from (5.6), (5.7), (5.8) and Theorem 3.2 that

$$\begin{aligned}|Y_t^{1,u}-Y_t^{2,u}| &\leq C\left\{(1+|x|^2)\delta\left(\hat{\mathbb{E}}\left[\sup_{r\in[t,t+\delta]}|X_r^{t,x,u}-x|^2\right]\right)^{1/2}+\delta\hat{\mathbb{E}}\left[\sup_{r\in[t,t+\delta]}|X_r^{t,x,u}-x|^2\right]+(1+|x|^2)\delta^2\right\}\\ &\leq C(1+|x|^3)\delta^{3/2},\end{aligned}$$

where C is dependent on T, G, L and independent of u. □

Lemma 5.7 *Let $\eta=(\eta^{ij})_{i,j=1}^d\in M_G^1(0,T;\mathbb{S}_d)$. Then, for each $s\leq T$, we have*

$$\tilde{\mathbb{E}}_s\left[\int_s^T\eta_r^{ij}d\langle B^i,B^j\rangle_r-\int_s^T\tilde{G}(2\eta_r)dr\right]=0.$$

Proof. For each η, $\tilde{\eta}\in M_G^1(0,T;\mathbb{S}_d)$, one can verify that

$$\begin{aligned}&\hat{\mathbb{E}}\left[\left|\tilde{\mathbb{E}}_s\left[\int_s^T\eta_r^{ij}d\langle B^i,B^j\rangle_r-\int_s^T\tilde{G}(2\eta_r)dr\right]-\tilde{\mathbb{E}}_s\left[\int_s^T\tilde{\eta}_r^{ij}d\langle B^i,B^j\rangle_r-\int_s^T\tilde{G}(2\tilde{\eta}_r)dr\right]\right|\right]\\ &\leq C\hat{\mathbb{E}}\left[\int_s^T|\eta_r-\tilde{\eta}_r|dr\right],\end{aligned}$$

where C only depends on G. Thus we only need to prove the case $\eta\in M_G^0(0,T;\mathbb{S}_d)$, i.e.,

$$\eta_r=\sum_{k=0}^{N-1}\eta_{t_k}I_{[t_k,t_{k+1})}(r),$$

where $s=t_0<\cdots<t_N=T$, $\eta_{t_k}\in Lip(\Omega_{t_k};\mathbb{S}_d)$. Since $\tilde{\mathbb{E}}_s[\cdot]=\tilde{\mathbb{E}}_s[\tilde{\mathbb{E}}_{t_k}[\cdot]]$, we only need to prove

$$\tilde{\mathbb{E}}_{t_k}\left[\eta_{t_k}^{ij}(\langle B^i,B^j\rangle_{t_{k+1}}-\langle B^i,B^j\rangle_{t_k})-\tilde{G}(2\eta_{t_k})(t_{k+1}-t_k)\right]=0. \tag{5.9}$$

Applying Itô's formular to $\langle\eta_{t_k}(B_r-B_{t_k}),B_r-B_{t_k}\rangle$ on $[t_k,t_{k+1}]$, we get

$$\tilde{\mathbb{E}}_{t_k}\left[\eta_{t_k}^{ij}(\langle B^i,B^j\rangle_{t_{k+1}}-\langle B^i,B^j\rangle_{t_k})\right]=\tilde{\mathbb{E}}_{t_k}\left[\langle\eta_{t_k}(B_{t_{k+1}}-B_{t_k}),B_{t_{k+1}}-B_{t_k}\rangle\right].$$

For each given $A\in\mathbb{S}_d$, define

$$u(t,x)=\tilde{\mathbb{E}}\left[\langle A(x+B_t),x+B_t\rangle\right]\text{ for }(t,x)\in[0,\infty)\times\mathbb{R}^d.$$

By Theorem C.3.5 in [26], we know that u is a viscosity solution to the following PDE

$$\partial_t u-\tilde{G}(\partial_{xx}^2u)=0,\ u(0,x)=\langle Ax,x\rangle. \tag{5.10}$$

On the other hand, by the proof of Theorem 3.8.2 in [26], we have

$$u(t,x)=\langle Ax,x\rangle+\tilde{\mathbb{E}}\left[\langle AB_t,B_t\rangle\right]=\langle Ax,x\rangle+\tilde{\mathbb{E}}\left[\langle AB_1,B_1\rangle\right]t. \tag{5.11}$$

By (5.10) and (5.11), we obtain $\tilde{\mathbb{E}}\left[\langle AB_1,B_1\rangle\right]=\tilde{G}(2A)$, which implies $\tilde{\mathbb{E}}\left[\langle AB_t,B_t\rangle\right]=\tilde{G}(2A)t$. Thus we have

$$\tilde{\mathbb{E}}_{t_k}\left[\eta_{t_k}^{ij}(\langle B^i,B^j\rangle_{t_{k+1}}-\langle B^i,B^j\rangle_{t_k})\right]=\tilde{G}(2\eta_{t_k})(t_{k+1}-t_k),$$

which implies (5.9). □

Remark 5.8 *It is important to note that we can not derive* $\tilde{\mathbb{E}}[\langle AB_1, B_1\rangle] = \tilde{G}(2A)$ *by* $u(t,x) = \langle Ax, x\rangle + \tilde{G}(2A)t$ *satisfying (5.10). Because, in this case of* $u(0,x) = \langle Ax, x\rangle \notin C_0(\mathbb{R}^n)$*, the meaning of uniqueness of viscosity solution is stated as in Remark 5.3.*

Lemma 5.9 *We have*

$$\inf_{u\in\mathcal{U}^t[t,t+\delta]} Y_t^{2,u} = \int_t^{t+\delta} F_0(r,x)dr,$$

where

$$F_0(r,x) = \inf_{v\in U}\{F_1(r,x,0,v) + \tilde{G}(2(F_2^{ij}(r,x,0,v))_{ij=1}^d)\}.$$

Proof. For each $u \in \mathcal{U}^t[t,t+\delta]$, by Lemma 5.7, we get

$$\begin{aligned}
Y_t^{2,u} &= \tilde{\mathbb{E}}_t\left[\int_t^{t+\delta} F_1(r,x,0,u_r)dr + \int_t^{t+\delta} F_2^{ij}(r,x,0,u_r)d\langle B^i, B^j\rangle_r\right] \\
&\geq \tilde{\mathbb{E}}_t\left[\int_t^{t+\delta} F_0(r,x)dr + \int_t^{t+\delta} F_2^{ij}(r,x,0,u_r)d\langle B^i, B^j\rangle_r - \int_t^{t+\delta} \tilde{G}(2(F_2^{ij}(r,x,0,u_r))_{ij=1}^d)dr\right] \\
&= \int_t^{t+\delta} F_0(r,x)dr.
\end{aligned}$$

Hence, $\inf_{u\in\mathcal{U}^t[t,t+\delta]} Y_t^{2,u} \geq \int_t^{t+\delta} F_0(r,x)dr$. On the other hand, we can choose a deterministic control $u^* \in \mathcal{U}^t[t,t+\delta]$ such that

$$\int_t^{t+\delta} [F_1(r,x,0,u_r^*) + \tilde{G}(2(F_2^{ij}(r,x,0,u_r^*))_{ij=1}^d)]dr = \int_t^{t+\delta} F_0(r,x)dr.$$

Then we obtain $Y_t^{2,u^*} = \int_t^{t+\delta} F_0(r,x)dr$ by Lemma 5.7, which implies $\inf_{u\in\mathcal{U}^t[t,t+\delta]} Y_t^{2,u} \leq \int_t^{t+\delta} F_0(r,x)dr$. Thus we obtain the desired result. □

Proof of Theorem 5.4. By Proposition 4.4 and Lemma 4.12, we know that $V(\cdot,\cdot)$ is continuous on $[0,T]\times\mathbb{R}^n$. Now, we first prove that $V(\cdot,\cdot)$ is the viscosity subsolution to (5.1).

For each given $(t,x) \in [0,T)\times\mathbb{R}^n$, suppose $\varphi \in C_b^{2,3}([0,T]\times\mathbb{R}^n)$ such that $\varphi(t,x) = V(t,x)$ and $\varphi \geq V$ on $[0,T]\times\mathbb{R}^n$. For each $\delta \in [0,T-t]$, by Theorem 4.10, we get

$$V(t,x) = \inf_{u\in\mathcal{U}^t[t,t+\delta]} \mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})].$$

Since $\varphi(t+\delta, X_{t+\delta}^{t,x,u}) \geq V(t+\delta, X_{t+\delta}^{t,x,u})$, by Lemma 4.9, we obtain $\mathbb{G}_{t,t+\delta}^{t,x,u}[V(t+\delta, X_{t+\delta}^{t,x,u})] \leq Y_t^u$. It follows from $\varphi(t,x) = V(t,x)$, Lemmas 5.5 and 5.6 that

$$\begin{aligned}
\inf_{u\in\mathcal{U}^t[t,t+\delta]} Y_t^{2,u} &\geq \inf_{u\in\mathcal{U}^t[t,t+\delta]} Y_t^{1,u} - C(1+|x|^3)\delta^{3/2} \\
&= \inf_{u\in\mathcal{U}^t[t,t+\delta]} (Y_t^u - \varphi(t,x)) - C(1+|x|^3)\delta^{3/2} \\
&\geq -C(1+|x|^3)\delta^{3/2},
\end{aligned}$$

where C is dependent on T, G, L. By Lemma 5.9, we get

$$\delta^{-1}\int_t^{t+\delta} F_0(r,x)dr \geq -C(1+|x|^3)\delta^{1/2}.$$

One can verify that $F_0(\cdot,x)$ is continuous in r. Hence we obtain $F_0(t,x)\geq 0$ by letting $\delta\downarrow 0$, which implies that $V(\cdot,\cdot)$ is the viscosity subsolution to (5.1). By the same method, we can prove that $V(\cdot,\cdot)$ is the viscosity supersolution to (5.1). Thus $V(\cdot,\cdot)$ is the viscosity solution to (5.1).

For the uniqueness of the viscosity solution, we only need to prove the case $\Phi\in C_0(\mathbb{R}^n)$ according to Remark 5.3. Hoverever, by the proof of Theorem C.2.9 with $l=0$ in [26], we see that in order to get the uniqueness we just need to know that $\inf_{u\in U}H(t,x,v,p,A,u)$ satisfies assumption (G′). For each $t\in[0,T)$, x, $y\in\mathbb{R}^n$, $v\in\mathbb{R}$, $\alpha>0$, A, $B\in\mathbb{S}_n$ such that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix},$$

we have

$$\begin{aligned}
&\inf_{u\in U}H(t,x,v,\alpha(x-y),A,u)-\inf_{u\in U}H(t,y,v,\alpha(x-y),-B,u)\\
&\leq \sup_{u\in U}\left[H(t,x,v,\alpha(x-y),A,u)-H(t,y,v,\alpha(x-y),-B,u)\right]\\
&\leq \sup_{u\in U}G(F(t,x,v,\alpha(x-y),A,u)-F(t,y,v,\alpha(x-y),-B,u))+L(|x-y|+\alpha|x-y|^2)\\
&\leq \sup_{u\in U}G(\sigma^T(t,x,u)A\sigma(t,x,u)+\sigma^T(t,y,u)B\sigma(t,y,u))+C(|x-y|+\alpha|x-y|^2)\\
&\leq \sup_{u\in U}G(3\alpha(\sigma(t,x,u)-\sigma(t,y,u))^T(\sigma(t,x,u)-\sigma(t,y,u)))+C(|x-y|+\alpha|x-y|^2)\\
&\leq C(|x-y|+\alpha|x-y|^2),
\end{aligned}$$

where C depends on L and G. Thus $\inf_{u\in U}H(t,x,v,p,A,u)$ satisfies assumption (G′), which implies that $V(\cdot,\cdot)$ is the unique viscosity solution to (5.1). □

Finally, we give the following stochastic verification theorem.

Theorem 5.10 *Let Assumptions (H1) and (H2) hold. Suppose that $\tilde{V}\in C^{1,2}([0,T]\times\mathbb{R}^n)$ is a solution to the HJB equation (5.1), and $\partial_t\tilde{V}$, $\partial^2_{xx}\tilde{V}$ are functions of polynomial growth. For each given $(t,x)\in[0,T)\times\mathbb{R}^n$, if $\bar{u}\in\mathcal{U}^t[t,T]$ satisfies*

$$\partial_s\tilde{V}(s,X_s^{t,x,\bar{u}})+H(s,X_s^{t,x,\bar{u}},\tilde{V}(s,X_s^{t,x,\bar{u}}),\partial_x\tilde{V}(s,X_s^{t,x,\bar{u}}),\partial^2_{xx}\tilde{V}(s,X_s^{t,x,\bar{u}}),\bar{u}_s)=0,\ \ s\in[t,T],$$

then

$$Y_t^{t,x,\bar{u}}=\inf_{u\in\mathcal{U}^t[t,T]}Y_t^{t,x,u}.$$

Proof. For each $u\in\mathcal{U}^t[t,T]$, applying Itô's formula to $\tilde{V}(r,X_r^{t,x,u})$ on $[s,T]$ for $s\in[t,T]$, we obtain

$$\begin{aligned}
&\tilde{V}(s,X_s^{t,x,u})+\int_s^T\frac{1}{2}F_{ij}(\Theta_r)d\langle B^i,B^j\rangle_r-\int_s^T\tilde{G}(F(\Theta_r))dr+\int_s^T(\partial_x\tilde{V}(r,X_r^{t,x,u}))^T\sigma(r,X_r^{t,x,u},u_r)dB_r \qquad (5.12)\\
&=\Phi(X_T^{t,x,u})-\int_s^T l_r dr+\int_s^T f(\Theta_r')dr+\int_s^T g_{ij}(\Theta_r')d\langle B^i,B^j\rangle_r,
\end{aligned}$$

where $F_{ij}(\cdot)$ is defined in (5.2),

$$\Theta_r = (r, X_r^{t,x,u}, \tilde{V}(r, X_r^{t,x,u}), \partial_x \tilde{V}(r, X_r^{t,x,u}), \partial_{xx}^2 \tilde{V}(r, X_r^{t,x,u}), u_r),$$

$$l_r = \partial_s \tilde{V}(r, X_r^{t,x,u}) + H(\Theta_r),\ \Theta'_r = (r, X_r^{t,x,u}, \tilde{V}(r, X_r^{t,x,u}), u_r).$$

Noting that $\partial_t \tilde{V}$ and $\partial_{xx}^2 \tilde{V}$ are functions of polynomial growth, we obtain $(F_{ij}(\Theta_r))_{r\in[t,T]}$, $(l_r)_{r\in[t,T]} \in M_G^2(t,T)$ by Theorem 3.2. By Lemma 5.7 and taking $\tilde{\mathbb{E}}_s[\cdot]$ on both sides of (5.12), we get that $\tilde{Y}_s = \tilde{V}(s, X_s^{t,x,u})$ is the solution of the following BSDE

$$\tilde{Y}_s = \tilde{\mathbb{E}}_s\left[\Phi(X_T^{t,x,u}) - \int_s^T l_r dr + \int_s^T f(r, X_r^{t,x,u}, \tilde{Y}_r, u_r)dr + \int_s^T g_{ij}(r, X_r^{t,x,u}, \tilde{Y}_r, u_r)d\langle B^i, B^j\rangle_r\right]. \quad (5.13)$$

Since $\tilde{V}$ is a solution to the HJB equation (5.1), we know that $l_r \geq 0$ for $r \in [t,T]$. The same proof of Lemma 4.9 for BSDEs (1.4) and (5.13), we obtain

$$Y_t^{t,x,u} \geq \tilde{Y}_t = \tilde{V}(t,x).$$

If $u = \bar{u}$, then $l_r = 0$ for $r \in [t,T]$ by the assumption. Thus $Y_t^{t,x,\bar{u}} = \tilde{V}(t,x)$, which implies the desired result. □

Example 5.11 *For $n = m = d = 1$, consider the following simple stochastic linear model:*

$$\begin{cases} dX_s^{t,x,u} = X_s^{t,x,u} ds + u_s dB_s,\ X_t^{t,x,u} = x, \\ Y_s^{t,x,u} = \tilde{\mathbb{E}}_s\left[|X_T^{t,x,u}|^2 + \int_s^T (-2Y_r^{t,x,u} - u_r)dr\right], \end{cases} \quad (5.14)$$

where $U = [1,2]$. The related HJB equation is

$$\begin{cases} \partial_t V + \inf_{u\in[1,2]} \left[\tilde{G}(u^2 \partial_{xx}^2 V) + x\partial_x V + (-2V - u)\right] = 0, \\ V(T,x) = x^2. \end{cases}$$

It is easy to check that $V(t,x) = \frac{\lambda}{2}(1 - e^{2(t-T)}) + x^2$ is a solution to the above HJB equation, where

$$\lambda = \inf_{u\in[1,2]} \left[\tilde{G}(2u^2) - u\right].$$

Note that there exists a $c \in [1,2]$ satisfying $\left[\tilde{G}(2c^2) - c\right] = \lambda$. Then, by Theorem 5.10, we obtain that $\bar{u} = c$ is an optimal control for (5.14).

Remark 5.12 *Under the weak framework, [10] and [31] studied the existence of optimal Markov control policy, i.e., $\bar{u}_s = \bar{u}(s, X_s)$. However, in our strong framework, we can not get this type of optimal control policy in general.*

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