FREE BOUNDARY MONGE-AMPÈRE EQUATIONS

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ABSTRACT. In this paper, we consider a class of Monge-Ampère equations in a free boundary domain of $\mathbb{R}^2$ where the value of the unknown function is prescribed on the free boundary. From a variational point of view, these equations describe an optimal transport problem from an a priori undetermined source domain to a prescribed target domain. We prove the existence and uniqueness of a variational solution to these Monge-Ampère equations under a singularity condition on the density function on the source domain. Furthermore, we provide regularity results under some conditions on the prescribed domain.

1. INTRODUCTION

Let Ω be a bounded subset of $\mathbb{R}^2$ and $\Lambda = [0,1] \times [0,1)$. Let $P : \mathbb{R}^2 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\mathbf{b} : [0,1] \longrightarrow [0,1)$. We consider the following Monge-Ampère equations where the unknown $(P, \mathbf{b})$ is such that P is convex and $(P, \mathbf{b})$ satisfies

$$\begin{cases} \sigma(\nabla P)\det(\nabla^2 P) = E \\ \nabla P(\Lambda_{\mathbf{b}}) = \Omega \\ P(\mathbf{b}(z), z) = f(\mathbf{b}(z), z) \quad \text{on} \quad \{\mathbf{b} > 0\}. \end{cases} \tag{1.1}$$

Here, σ, E and f are prescribed. The function $E : \Lambda \longrightarrow \mathbb{R}$ is positive and $\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}$ non negative such that σ vanishes outside Ω. In addition, σ and E satisfy the compatibility condition

$$\int_\Omega \sigma(y)dy = \int_{\Lambda_{\mathbf{b}}} E(x)dx \tag{1.2}$$

with $\Lambda_{\mathbf{b}}$, a subset of Λ, defined by

$$\Lambda_{\mathbf{b}} = \left\{x = (x_1, x_2) : 0 \le x_1 \le \mathbf{b}(x_2), x_2 \in [0,1]\right\}.$$

Monge-Ampère equations of the type (1.1) include equations that arise in the study of axisymmetric flows [1, 2]. These flows appear in Meteorology and model the dynamics of tropical cyclones [3, 4, 5]. In the context of axisymmetric flows, P represents the pressure of the flow in a set of transformed variables while $\mathbf{b}$ determines the free boundary domain in which the system is considered. Typically, $f(x_1, x_2) \approx 1/(1-x_1)$ describes the density of the fluid in a set of transformed variables and $E \approx f^2$.

The compatibility condition (1.2) allows for a simple interpretation of (1.1) in the context of the theory of optimal transport. Indeed, Brenier has developed a framework where weak solutions for such Monge-Ampère equations with a second boundary condition are studied. In that framework, the system of equations (1.1) expresses the fact that the map ∇P transports the density function

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$E\chi_{\Lambda_{\mathbf{b}}}$ onto σ, and P is prescribed on the a priori undetermined boundary given by the graph of $\mathbf{b}$.

A simple case of the problem in (1.1) is considered in [6]. There, f and $E = f^2$, while defined on $\mathbb{R}^2$, depends only on their second variable with a singularity point. Under the assumption that f is smooth and one-sided Lipschitz continuous, the existence of a weak solution to (1.1) was obtained.

The system (1.1) shares some similarity with the Monge-Ampère double obstacle problems considered by L. Caffarelli and R. McCann in [7] in that the free boundary domains are in fact determined by the potential P. However, the two problems appear not to be of the same class. In [7], the obstacles are set on both the source and target domains. Moreover, the densities functions both on the source domain and target domain are assumed to be in $L^1(\mathbb{R}^2)$. The Monge-Ampère double obstacle problem has been shown to be intimately linked to the partial optimal transfer problem. Here, the problem considered is different. While the target domain is fixed with a density function in $L^1(\mathbb{R}^2)$, the density function on the a priori undetermined source domain is not a $L^1(\mathbb{R}^2)$ function.

To obtain a solution to (1.1), we consider the following variational problem:

$$\inf_{\mathbf{b}\in\mathscr{B}_0} \frac{1}{2} W^2\left(\sigma, E\chi_{\Lambda_{\mathbf{b}}}\right)^2 + \int_{\mathbb{R}^2}\left[f(x) - \frac{|x|^2}{2}\right] E(x)dx, \tag{1.3}$$

where $\mathscr{B}_0$ is the set of Borel functions $\mathbf{b} : [0,1] \longrightarrow [0,1)$ such that (1.2) holds. We impose, among other things, the following key singularity condition on E:

$$\inf_{x_2\in[0,1]} \int_0^1 E(x_1, x_2)\, dx_1 = +\infty. \tag{1.4}$$

In the context of tropical cyclones, (1.4), written in an appropriate coordinates system, indicates that at each height level, the total mass of air is infinite. Moreover, the smoothness of E as later assumed in the paper, arises from the smoothness of the coordinates system (see [1]).
In (1.3), we assume that the function f satisfies the following estimate:

$$E^\alpha(x_1, x_2) \le f(x_1, x_2) \le aE^\alpha(x_1, x_2) + \mathbf{B}(x_1) \qquad 0 < \alpha < 1, \quad a > 1 \tag{1.5}$$

for all $(x_1, x_2) \in \Lambda$ with $\mathbf{B} \in C^1([0,1])$. Under conditions (1.4) and (1.5), we show that (1.3) has a dual formulation given by

$$\sup_{(P,\Psi)\in\mathcal{V}} -\int_{\mathbb{R}^2} \Psi(y)\sigma(y)dy + \inf_{\mathbf{b}\in\mathcal{B}} \int_0^1 \int_0^{\mathbf{b}(x_2)} \big(f(x_1,x_2) - P(x_1,x_2)\big)\, E(x_1,x_2)dx_1dx_2 \tag{1.6}$$

where $\mathcal{V}$ consists of $(P,\Psi) \in C(\mathbb{R}^2)\times C(\bar\Omega)$ such that $P(x) + \Psi(y) \ge \langle x, y\rangle$ for all $x \in \Lambda$ and $y \in \Omega$.
In fact, the set of admissible functions in (1.6) can be reduced to $\mathcal{V}_0$ consisting of (P,Ψ) such that

$$P(x) = \sup_{y\in\bar\Omega}\langle x,y\rangle - \Psi(y) \text{ for all } x\in\mathbb{R}^2 \qquad\text{and}\qquad \Psi(y) = \sup_{x\in\bar\Lambda}\langle x,y\rangle - P(x) \text{ for all } y\in\Omega \tag{1.7}$$

In this work, we show that if $\mathbf{b}_0$ solves the variational problem (1.3), and (P_0,Ψ_0) satisfies (1.7) and solves the variational problem (1.6), then $(P_0,\mathbf{b}_0)$ solves (1.1) in a weak sense, that is, ∇P_0 pushes-forward $E\chi_{\Lambda_{\mathbf{b}_0}}$ onto σ and $P(\mathbf{b}_0(x_2), x_2) = f(\mathbf{b}_0(x_2), x_2)$ whenever $\mathbf{b}_0(x_2) > 0$.
We point out that the condition (1.4) is used to guarantee the existence of the minimizer $\mathbf{b}_0$. However, the uniqueness of the minimizer $\mathbf{b}_0$ is crucial to determine the free boundary of $\Lambda_{\mathbf{b}_0}$. For

this purpose, we impose a Lipschitz bound on f on a subset of Λ and require that

$$\mathrm{spt}(\sigma) \subset [0, R_1] \times [R_0, R_1] \qquad \text{with} \quad R_0 \quad \text{large enough} \tag{1.8}$$

and that E and f satisfy the following estimates

$$0 \leq \partial_{x_2} E(x_1, x_2) \leq k_0 E^{1-\alpha}(x_1, x_2), \qquad 0 < \alpha < 1 \tag{1.9}$$

and

$$0 \leq \partial_{x_2} f(x_1, x_2) \leq k_1 \tag{1.10}$$

for some $0 < k_0 \leq 1/R_1$. We point out that in the theory of optimal transport, in general, regularity is not assumed on the transported density in order to ensure existence of an optimal map. Here, due to the presence of a free boundary, stronger conditions appear to be in order. The conditions (1.4) (1.8), (1.9) and (1.10) ensure existence and uniqueness of the weak solution in (1.1). Furthermore, we investigate the regularity of the weak solution to (1.1) in the sense of Brenier. Exploiting bounds on σ and its support, we show that the boundary of the domain $\Lambda_{\mathbf{b}}$ is piecewise Lipschitz continuous and obtain better regularity for P_0 thanks to standard results established by Caffarelli [10] and more recently by Figalli [8, 9].

This work is organized in the following way: In section 2, we introduce notation, definitions and some standard results. In section 3, we study the variational problem (1.6) by focusing on the second term in the functional involved. In section 4, we establish and exploit the duality between (1.3) and (1.6) to obtain the existence of a unique minimizer in (1.3) and a unique maximizer in (1.6). Lastly, in section 5, we show that the Monge-Ampère equation has a solution and that the regularity theories developed by Caffarelli and Figalli apply.

2. Notation and Preliminaries

Throughout the paper, we use the following notation for $x,\ y \in \mathbb{R}^2$:

$$x = (x_1, x_2) \qquad \text{and} \qquad y = (y_1, y_2).$$

For $A \subset \mathbb{R}^d$, we denote respectively by $\bar{A}$ and $\mathring{A}$ the closure and the interior of A in $\mathbb{R}^d$.

If $h : A \subset \mathbb{R}^d \longmapsto \mathbb{R}$ is a Lipschitz continuous, then $Lip(h) \equiv \inf_{\substack{\mathbf{x},\mathbf{y} \in A \\ \mathbf{x} \neq \mathbf{y}}} \dfrac{|h(\mathbf{x}) - h(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|}$ denotes the Lipschitz constant of h on A.

If A is a convex subset of $\mathbb{R}^d$ and $h : A \longmapsto \mathbb{R}$ is convex, then the subdifferential of h at $x_0 \in A$ is given by

$$\partial h(x_0) := \Big\{ y \in \mathbb{R}^d : h(x) - h(x_0) \geq \langle y, x - x_0 \rangle \quad \text{for all } x \in A \Big\}.$$

Given two Borel measures μ_0 and μ_1 of the same finite mass and a Borel function $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, we say that T pushes μ_0 onto μ_1 if

$$\mu_1(A) = \mu_0(T^{-1}(A))$$

for any Borel set $A \subset \mathbb{R}^d$ and we write $T\#\mu_0 = \mu_1$.

Given two measures μ_0 and μ_1 in the space of Borel measures with fixed total mass, we denote by $\mathscr{T}(\mu_0, \mu_1)$ the set of all Borel measures γ on the product space $\mathbb{R}^d \times \mathbb{R}^d$ such that $\mu_0 = \pi^0\#\gamma$ and $\mu_1 = \pi^1\#\gamma$ where π^0 and π^1 denote respectively the first and second projection on $\mathbb{R}^d \times \mathbb{R}^d$.

Assume, in addition, that μ_0 and μ_1 have finite second moments. Then, the (2-th) Wasserstein distance between measures μ_0 and μ_1 is defined by

$$W_2^2(\mu_0,\mu_1) = \inf\left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma : \gamma \in \mathscr{T}(\mu_0,\mu_1) \right\}. \tag{2.1}$$

See [11, 14]. The infimum in (2.1) is attained and the set of minimizers in (2.1) is denoted by $\mathscr{T}_0(\mu_0,\mu_1)$.

Throughout this article, we assume that Ω is an open bounded subset of $\mathbb{R}_+\times\mathbb{R}_+$ and we set

$$\Lambda = [0,1]\times[0,1).$$

We denote by $\mathscr{B}$ the set of Borel functions $\mathbf{b} : [0,1] \longrightarrow [0,1)$ such that the graph of $\mathbf{b}$ is contained in Λ. To any $\mathbf{b}\in\mathscr{B}$, we associate the set $\Lambda_{\mathbf{b}}$ defined by

$$\Lambda_{\mathbf{b}} = \left\{(x_1,x_2)\in\Lambda : 0 \le x_1 \le \mathbf{b}(x_2),\ x_2\in[0,1]\right\}. \tag{2.2}$$

If $\mathbf{b}\in\mathscr{B}$ is a constant function with value b_0, then we write for simplicity Λ_{b_0} for $\Lambda_{\mathbf{b}}$.
We consider the following Monge-Ampère equations

$$\begin{cases} \sigma(\nabla P)\det(\nabla^2 P) = E & \text{on} \quad \mathring{\Lambda}_{\mathbf{b}} \\ \nabla P(\Lambda_{\mathbf{b}}) = \Omega & \\ P(\mathbf{b}(x_2),x_2) = f(\mathbf{b}(x_2),x_2) & \text{on} \quad \{\mathbf{b}>0\} \end{cases} \tag{2.3}$$

where σ and E satisfy the compatibility condition

$$\int_\Omega \sigma(y)dy = \int_{\Lambda_{\mathbf{b}}} E(x)dx. \tag{2.4}$$

We denote by $\mathscr{B}_0$ the set of $\mathbf{b}\in\mathscr{B}$ for which (2.4) holds. We say that $(P,\mathbf{b})$ is a weak solution to (2.3) in the sense of Brenier if P is convex and

$$\begin{cases} \nabla P \# E\chi_{\Lambda_{\mathbf{b}}} = \sigma \\ P(\mathbf{b}(z),z) = f(\mathbf{b}(z),z) \quad \text{on} \quad \{\mathbf{b}>0\}. \end{cases} \tag{2.5}$$

We consider the functional

$$\mathcal{F}(\mathbf{b}) = \begin{cases} \frac{1}{2}W^2(\sigma,E\chi_{\Lambda_{\mathbf{b}}})^2 + \displaystyle\int_{\Lambda_{\mathbf{b}}} \left(f(x_1,x_2) - \frac{x_1^2+x_2^2}{2}\right) E(x_1,x_2)dx_1dx_2 & \mathbf{b}\in\mathscr{B}_0 \\ +\infty & \mathbf{b}\notin\mathscr{B}_0 \end{cases} \tag{2.6}$$

where $\mathscr{B}_0$ is the set of $\mathbf{b}\in\mathscr{B}$ such that $E\chi_{\Lambda_{\mathbf{b}}}$ is a density of probability. To formulate a dual problem as described in (1.6), we consider the functional

$$\mathscr{G}(P,\Psi) = -\int_{\mathbb{R}^2} \Psi(y)\sigma(y)dy + \inf_{\mathbf{b}\in\mathscr{B}} \int_0^1 \mathscr{K}_P(\mathbf{b}(x_2),x_2)dx_2 \tag{2.7}$$

with $\mathscr{K}_P$ is defined by

$$\mathscr{K}_P(b,x_2) = \int_0^b \left[f(x_1,x_2) - P(x_1,x_2)\right] E(x_1,x_2)dx_1$$

for $0\le b<1$ and $0\le x_2\le 1$. The functional $\mathscr{G}$ is defined on the set $\mathcal{V}$ consisting of (P,Ψ) such that $P(x)+\Psi(y)\ge\langle x,y\rangle$ for all $x\in\Lambda$ and $y\in\Omega$.

Assumptions:

Throughout this work, we assume that

$$\Omega \subset [0, R_1] \times [R_0, R_1] \tag{2.8}$$

for some $0 < R_0 < R_1$. We make the following assumptions on E and f:

(α**1**) $E \geq 1$ and satisfies the following singularity condition:

$$\inf_{x_2 \in [0,1]} \int_0^1 E(x_1, x_2)\, dx_1 = +\infty. \tag{2.9}$$

(α**2**) α, a are fixed constants such that $0 < \alpha < 1$ and $a > 1$. We require that $f \in C^1(\Lambda)$ Lipschitz on sets Λ_b for $b \in (0,1)$ and satisfies the estimate

$$E^\alpha(x_1, x_2) \leq f(x_1, x_2) \leq aE^\alpha(x_1, x_2) + \mathbf{B}(x_1) \tag{2.10}$$

for all $(x_1, x_2) \in \Lambda$ and $\mathbf{B} \in C^1([0,1])$.

(α**3**) E and f are $C^1(\mathring{\Lambda}) \cap C(\Lambda_b)$ for any $b \in (0,1)$ and satisfies the following estimates

$$0 \leq \partial_{x_2} E(x_1, x_2) \leq k_0 E^{1-\alpha}(x_1, x_2) \tag{2.11}$$

and

$$0 \leq \partial_{x_2} f(x_1, x_2) \leq k_1 \tag{2.12}$$

for some $0 < k_0 \leq 1/R_1$, $k_1 > 1$ and $0 < \alpha < 1$.

3. The study of the dual problem

In this section, we consider the functional defined in (2.7) and show the existence of a maximizer. We first focus our attention on the second term of (2.7) and exploit (2.9) to obtain the existence of minimizer.

For any $P : \mathbb{R}^2 \longrightarrow \mathbb{R}$, we define

$$P^*(y) = \sup_{x \in \bar{\Lambda}} \langle x, y \rangle - P(x) \quad y \in \Omega. \tag{3.1}$$

In a similar way, for any $\Psi : \Omega \longrightarrow \mathbb{R}$, we define

$$\Psi^*(x) = \sup_{y \in \bar{\Omega}} \langle x, y \rangle - \Psi(y) \quad x \in \mathbb{R}^2. \tag{3.2}$$

We recall that $\mathcal{V}$ consists of (P, Ψ) such that

$$P(x) + \Psi(y) \geq \langle x, y \rangle \qquad \text{for all } x \in \Lambda \text{ and } y \in \Omega.$$

We denote by $\mathcal{V}_0$ the set of functions (P, Ψ) such that

$$P = \Psi^* \text{on } \Lambda \qquad \Psi = P^* \text{on } \Omega.$$

3.1. Compactness of super level sets of $\mathscr{G}$.

Here, we begin with establishing some useful estimates on the elements of $\mathcal{V}_0$. The first lemma is straightforward.

Lemma 3.1. *Let $(P, \Psi) \in \mathcal{V}$ such that $P = \Psi^*$. Then, the following holds:*

(i) *P is convex and Lipschitz continuous such that $\partial P(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$. Moreover,*

$$-2R_1 \leq P(x) - P(0,0) \leq 2R_1 \qquad x \in \bar{\Lambda}. \tag{3.3}$$

(ii) *Assume that* $\Psi = P^*$. *Then,* Ψ *is convex and Lipschitz continuous such that* $\partial\Psi(y) \subset \overline{conv}(\Lambda)$ *for* $y \in \bar{\Omega}$. *Moreover,*

$$-2R_1 - P(0,0) \leq \Psi(y) \leq 4R_1 - P(0,0) \qquad y \in \overline{\Omega}. \tag{3.4}$$

Proof: We note that $P = \Psi^*$ implies that P is convex as supremum of a collection of linear functions with gradients in $\bar{\Omega}$ and that $\partial P(x) \subset \overline{conv}(\Omega)$ for all $x \in \overline{\Lambda}$. Let $x, \bar{x} \in \overline{\Lambda}$ and $y_0 \in \partial P(x)$. Then, $y_0 \in \overline{conv}(\Omega)$ such that $P(x) + \Psi(y_0) = \langle x, y_0 \rangle$. We note that $P(\bar{x}) \geq \langle \bar{x}, y_0 \rangle - \Psi(y_0)$. It follows that

$$P(x) - P(\bar{x}) \leq \langle y_0, x - \bar{x} \rangle \leq |y_0||x - \bar{x}| \leq \sqrt{2}R_1|x - \bar{x}|. \tag{3.5}$$

By exchanging x and $\bar{x}$ in (3.5), we obtain that

$$|P(x) - P(\bar{x})| \leq \sqrt{2}R_1|x - \bar{x}|. \tag{3.6}$$

As x and $\bar{x}$ are arbitrary in Λ, (3.6) implies that P in Lipschitz continuous. Setting $\bar{x} = 0$ in (3.6) and taking into account that $|x| \leq \sqrt{2}$ for all $x \in \bar{\Lambda}$, we obtain (3.3).
A similar reasoning as above leads to the fact that Ψ is convex Lipschitz continuous and that $\partial\Psi(y) \subset \bar{\Lambda}$ for $y \in \overline{\Omega}$. Let $y \in \overline{\Omega}$. There exists $x_0 \in \bar{\Lambda}$ such that $\Psi(y) = \langle x_0, y \rangle - P(x_0)$. We note that

$$0 \leq \langle x_0, y \rangle \leq |x_0||y| \leq \sqrt{2}(\sqrt{2}R_1) = 2R_1. \tag{3.7}$$

We combine (3.3) and (3.7) to obtain (3.4). □

The following lemma establishes the compactness of certain super level sets of $\mathscr{G}$ on $\mathcal{V}_0$. We first observe that

$$\int_0^b E(x_1, x_2)dx_1 \geq \inf_{\bar{x}_2 \in [0,1]} \int_0^b E(x_1, \bar{x}_2)dx_1, \tag{3.8}$$

for all $b \in [0, 1)$ and $x_2 \in [0, 1]$. It follows that

$$\int_0^1 \int_0^b E(x_1, x_2)dx_1dx_2 \geq \int_0^1 \inf_{\bar{x}_2 \in [0,1]} \int_0^b E(x_1, \bar{x}_2)dx_1dx_2 = \inf_{\bar{x}_2 \in [0,1]} \int_0^b E(x_1, \bar{x}_2)dx_1 \tag{3.9}$$

for all $b \in [0, 1)$. In light of the condition (2.9), we choose in the sequel a constant $\mathfrak{b}_0$ such that $0 \leq \mathfrak{b}_0 < 1$ and

$$\int_0^1 \int_0^{\mathfrak{b}_0} E(x_1, x_2)dx_1dx_2 > 2. \tag{3.10}$$

Lemma 3.2. *Let* $C_0 > 0$. *There exist* $A_0, B_0 > 0$ *such that for* $(P, \Psi) \in \mathcal{V}_0$ *and* $-C_0 \leq \mathscr{G}(P, \Psi)$ *we have that* P *and* Ψ *are convex Lipschitz continuous with*

$$Lip(P) \leq B_0, \quad Lip(\Psi) \leq B_0 \tag{3.11}$$

$$P(0,0) \leq A_0, \quad -C_0 - 4R_1 \leq P(x) \leq A_0 + 2R_1 \qquad \text{for all } x \in \Lambda \tag{3.12}$$

and

$$-A_0 - 2R_1 \leq \Psi(y) \leq C_0 + 6R_1 \qquad \text{for all } y \in \Omega. \tag{3.13}$$

Consequently, the set of $(P, \Psi) \in \mathcal{V}_0$ *with* $-C_0 \leq \mathscr{G}(P, \Psi)$ *is compact with respect to the uniform topology on* $C(\bar{\Lambda}) \times C(\bar{\Omega})$.

Proof : Let $(P,\Psi)\in\mathcal{V}_0$. Set $B_0=\max(R_1,\sqrt{2})$. By lemma 3.1, P and Ψ are convex Lipschitz continuous with $\partial P(x)\subset\overline{conv}(\Omega)$ for all $x\in\Lambda$ and $\partial\Psi(y)\subset\overline{conv}(\Lambda)$ for all $y\in\Omega$. It follows that (3.11) holds. We note that

$$\inf_{\bar{\mathbf{b}}\in\mathcal{B}}\int_{\Lambda_{\bar{\mathbf{b}}}}\big(f(x)-P(x)\big)\,E(x)dx\le\int_0^1\int_0^{\mathbf{b}(x_2)}\big(f(x_1,x_2)-P(x_1,x_2)\big)\,E(x_1,x_2)dx_1dx_2 \tag{3.14}$$

for all $\mathbf{b}\in\mathcal{B}$. For $(P,\Psi)\in\mathcal{V}_0$ such that $-C_0\le\mathcal{G}(P,\Psi)$, we have

$$-C_0\le-\int_{\mathbb{R}^2}\Psi(y)\sigma(y)dy+\inf_{\mathbf{b}\in\mathcal{B}}\int_0^1\int_0^{\mathbf{b}(x_2)}\big(f(x_1,\ x_2)-P(x_1,x_2)\big)\,E(x_1,x_2)dx_1dx_2. \tag{3.15}$$

It follows that

$$-C_0\le-\int_{\mathbb{R}^2}\Psi(y)\sigma(y)dy+\int_0^1\int_0^{b}\big(f(x_1,\ x_2)-P(x_1,x_2)\big)\,E(x_1,x_2)dx_1dx_2 \tag{3.16}$$

for all constant functions $\mathbf{b}=b$ with $0\le b<1$. Setting $b=0$ and using (3.4) we obtain that

$$-C_0\le 2R_1+P(0,0). \tag{3.17}$$

Using (3.4) again, together with (3.3), the equation (3.16) implies

$$-C_0\le 2R_1+P(0,0)+\int_0^1\int_0^{b}\big(f(x_1,\ x_2)+2R_1-P(0,0)\big)\,E(x_1,x_2)dx_1dx_2. \tag{3.18}$$

For any $u\in L^\infty_{loc}(\Lambda)$ and $0\le b<1$ we set

$$\mathcal{R}(u,b)=\int_0^1\int_0^b u(x_1,x_2)E(x_1,x_2)dx_1dx_2. \tag{3.19}$$

Clearly, if $u\ge 0$, then $\mathcal{R}(u,b)\ge 0$. We rewrite (3.18) as

$$-C_0\le 2R_1+P(0,0)+\mathcal{R}(f,b)+\big(2R_1-P(0,0)\big)\,\mathcal{R}(1,b) \tag{3.20}$$

By (3.10),

$$\mathcal{R}(1,\mathfrak{b}_0)>2. \tag{3.21}$$

Setting

$$A_0=\frac{2R_1+\mathcal{R}(f,\mathfrak{b}_0)+2R_1\mathcal{R}(1,\mathfrak{b}_0)+C_0}{\mathcal{R}(1,\mathfrak{b}_0)-1}, \tag{3.22}$$

it follows that $A_0>0$ and

$$P(0,0)\le A_0. \tag{3.23}$$

The equations (3.17) and (3.23) combined with (3.3) yields (3.12) while (3.17) and (3.23) together with (3.4) yields (3.13).

In light of (3.11) - (3.13), we use the Arzela Ascoli theorem to conclude that the set of $(P,\Psi)\in\mathcal{V}_0$ with $-C_0\le\mathscr{G}(P,\Psi)$ is compact with respect to the uniform topology on $C(\bar{\Lambda})\times C(\bar{\Omega})$. □

3.2. Existence and uniqueness of a minimizer for the $\mathscr{K}_P$ problem.

Lemma 3.3. *Assume* (α**1**) *and* (α**2**) *hold. Let* $A > 0$. *Assume* P *is continuous such that* $P(0,0) + 2R_1 \geq P(x)$ *for* $x \in \Lambda$. *Then, there exists a positive* $\bar{\varepsilon}_0 = \bar{\varepsilon}_0(P) < 1$ *such that*

$$\left\{\mathscr{K}_P(\cdot, x_2) \leq 0\right\} \subset [0, \bar{\varepsilon}_0] \tag{3.24}$$

for any $x_2 \in [0,1]$. *If, in addition,* $P(0,0) < A$, *then there exists* $\varepsilon_0 = \varepsilon_0(A) < 1$ *such that*

$$\left\{\mathscr{K}_P(\cdot, x_2) \leq 0\right\} \subset [0, \varepsilon_0] \tag{3.25}$$

for any $x_2 \in [0,1]$.

Remark 3.4. We recall that a sufficient condition for a convex Lipschitz function P to satisfy the inequality $P(0,0) + 2R_1 \geq P(x)$ is given by $\partial P(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$. If $(P, \Psi) \in \mathcal{V}_0$, then, by lemma 3.1, $P(0,0) + 2R_1 \geq P(x)$ for all $x \in \Lambda$.

Proof : Let $x_2 \in [0,1]$. We note that

$$\begin{aligned} \mathscr{K}_P(b, x_2) &= \int_0^b f(x_1,x_2)E(x_1,x_2)dx_1 - \int_0^b P(x_1,x_2)E(x_1,x_2)dx_1 \\ &\geq \int_0^b E^{\alpha+1}(x_1,x_2)\, dx_1 - (P(0,0)+2R_1)\int_0^b E(x_1,x_2)\, dx_1 \end{aligned} \tag{3.26}$$

for all $0 \leq b < 1$. We have used (2.10) to obtain the second equation in (3.26). Using the Jensen inequality, we have that

$$\begin{aligned} \mathscr{K}_P(b, x_2) &\geq \frac{1}{b^\alpha}\left(\int_0^b E(x_1,x_2)dx_1\right)^{\alpha+1} - (P(0,0)+2R_1)\int_0^b E(x_1,x_2)\, dx_1 \\ &\geq \left(\int_0^b E(x_1,x_2)\, dx_1\right)\left[\left(\int_0^b E(x_1,x_2)dx_1\right)^\alpha - (P(0,0)+2R_1)\right] \\ &\geq \left(\inf_{\bar{x}_2\in[0,1]}\int_0^b E(x_1,\bar{x}_2)\, dx_1\right)\left[\left(\inf_{\bar{x}_2\in[0,1]}\int_0^b E(x_1,\bar{x}_2)dx_1\right)^\alpha - (P(0,0)+2R_1)\right] \\ &:= F\big(b, P(0,0)\big) \end{aligned} \tag{3.27}$$

where

$$F(b,q) = \left(\inf_{\bar{x}_2\in[0,1]}\int_0^b E(x_1,\bar{x}_2)\, dx_1\right)\left[\left(\inf_{\bar{x}_2\in[0,1]}\int_0^b E(x_1,\bar{x}_2)dx_1\right)^\alpha - (q+2R_1)\right].$$

It follows that

$$\left\{0 \leq b < 1 : \mathscr{K}_P(b,x_2) \leq 0\right\} \subset \left\{0 \leq b < 1 : F(b, P(0,0)) \leq 0\right\}. \tag{3.28}$$

In light of (2.9), we choose b_0 such that $0 \leq b_0 < 1$ and

$$\inf_{x_2\in[0,1]}\int_0^{b_0} E(x_1,x_2)\, dx_1 > B \tag{3.29}$$

where

$$B = \begin{cases} (P(0,0)+2R_1)^{1/\alpha} & \text{if} \quad P(0,0)+2R_1 > 0 \\ 0 & \text{if} \quad P(0,0)+2R_1 < 0. \end{cases}$$

It follows that

$$F\big(b_0, P(0,0)\big) > 0.$$

Thus,

$$\big\{0 \le b < 1 : F(b, P(0,0)) \le 0\big\} \subset \Big\{0 \le b < 1 : F(b, P(0,0)) < F\big(b_0, P(0,0)\big)\Big\}. \tag{3.30}$$

As $F(\cdot, P(0,0))$ is monotone increasing, (3.28) and (3.30) implies that (3.24) holds for the choice

$$\bar{\varepsilon}_0 := b_0 < 1. \tag{3.31}$$

Assume that $P(0,0) \le A$. As $F(b,\cdot)$ is monotone decreasing, for $0 \le b < 1$ such that $0 \ge \mathscr{K}_P(b,x_2)$, we use (3.27) to get

$$0 \ge \mathscr{K}_P(b,x_2) \ge F\big(b, P(0,0)\big) \ge F(b,A). \tag{3.32}$$

We further choose b_0 such that

$$\inf_{x_2\in[0,1]} \int_0^{b_0} E(x_1,x_2)\,dx_1 > (A+2R_1)^{1/\alpha}. \tag{3.33}$$

It follows that $F(b_0, A) > 0$ and that

$$\big\{0 \le b < 1 : \mathscr{K}_P(b,x_2) \le 0\big\} \subset \big\{0 \le b < 1 : F(b,A) \le F(b_0,A)\big\}. \tag{3.34}$$

Again, as $F(\cdot, A)$ is monotone increasing, (3.34) implies that (3.25) holds for the choice

$$\bar{\varepsilon}_0 := b_0 < 1.$$

□

Proposition 3.5. *Assume (α**1**) and (α**2**) hold. Let $A > 0$. There exists a positive $\varepsilon_0 = \varepsilon_0(A) < 1$ such that for any P convex Lipschitz such that $\partial P(x) \subset \overline{conv}(\Omega)$ with $P(0,0) \le A$ and any $x_2 \in [0,1]$, there exists $b^* \in [0,\varepsilon_0]$ satisfying*

$$\mathscr{K}_P(b^*,x_2) \le \mathscr{K}_P(b,x_2) \qquad \text{for all } b \in [0,1). \tag{3.35}$$

If $b^* > 0$, then

$$P(b^*,x_2) = f(b^*,x_2). \tag{3.36}$$

Let $\{P_n\}_{n=0}^{\infty}$ be convex and uniformly Lipschitz such that $\partial P_n(x) \subset \overline{conv}(\Omega)$ with $P_n(0,0) \le A$ for all $n \ge 1$ such that $\{P_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of Λ to P_0. Then there exists $h_0 = h_0(\varepsilon_0) > 0$ such that

$$|\mathscr{K}_{P_n}(b,x_2)| \le h_0 \tag{3.37}$$

for all $b \in [0,\varepsilon_0]$ and $x_2 \in [0,1]$ and $n \ge 0$. Moreover, if $\{b_n^*\}_{n=1}^{\infty}$ and $\{x_2^n\}_{n=1}^{\infty}$ are non negative sequences respectively of $[0,\varepsilon_0]$ and $[0,1]$ such that b_n^* satisfies (3.35) when P is replaced by P_n and x_2 is replaced by x_2^n, and $\{b_n^*\}_{n=1}^{\infty}$ converges to b_0^{**} and $\{x_n^n\}_{n=1}^{\infty}$ converges to x_2^0, then

$$\lim_{n\to\infty} \mathscr{K}_{P_n}(b_n^*,x_2^n) = \mathscr{K}_{P_0}(b_0^{**},x_2^0) \tag{3.38}$$

and b_0^{**} satisfies (3.35) when P is replaced by P_0 and x_2 is replaced by x_2^0. Furthermore, fix $a^* \in [0,\varepsilon]$. The set of $x_2 \in [0,1]$ such that (3.35) holds, when b^* is replaced by a^*, is compact.

Proof: (i) Let $x_2 \in [0,1]$. Note that $\mathscr{K}_P(0,x_2)=0$. As P is Lipschitz $\mathscr{K}_P$ is continuous. It follows that the set $K_{x_2}=\left\{\mathscr{K}_P(\cdot,x_2)\le \mathscr{K}_P(0,x_2)\right\}$ is closed. We next choose $\varepsilon_0>0$ as in lemma 3.3. Then, $K_{x_2}\subset[0,\varepsilon_0]$ so that K_{x_2} is compact. As $\mathscr{K}_P(\cdot,x_2)$ is continuous on K_{x_2}, $\mathscr{K}_P(\cdot,x_2)$ has a minimizer b^* satisfying (3.35). If $b^*>0$, then $\mathscr{K}_P(\cdot,x_2)$ is differentiable at b^* and

$$\partial_b\mathscr{K}_P(b^*,x_2)=\big(f(b^*,x_2)-P(b^*,x_2)\big)\,E(b^*,x_2)=0.$$

As $E>0$ on Λ_{ε_0}, the last displayed equation yields (3.36). We consider now the sequence $\{P_n\}_{n=0}^{\infty}$. One may further choose ε_0 independent of n in light of remark 3.4. We note that as $\{P_n\}_{n=1}^{\infty}$ converges uniformly on Λ_{ε_0} we have $\|P_n\|_{L^\infty(\Lambda_{\varepsilon_0})}\le e_0$, for some constant $e_0>0$. Furthermore, one may assume that

$$\begin{aligned}|\big(f(x_1,x_2)-P_n(x_1,x_2)\big)\,E(x_1,x_2)| &\le |f(x_1,x_2)E(x_1,x_2)|+|P_n(x_1,x_2)E(x_1,x_2)|\\ &\le a\|E\|^{\alpha+1}_{L^\infty(\Lambda_{\varepsilon_0})}+\|\mathbf{B}\|_\infty\|E\|_{L^\infty(\Lambda_{\varepsilon_0})}+e_0\|E\|_{L^\infty(\Lambda_{\varepsilon_0})}\\ &:= m_0\end{aligned} \tag{3.39}$$

for all $x\in\Lambda_{\varepsilon_0}$. Thus,

$$\big|\mathscr{K}_{P_n}(b,x_2)\big|\le \int_0^{\varepsilon_0}|\big(f(x_1,x_2)-P_n(x_1,x_2)\big)\,E(x_1,x_2)|dx_1\le \varepsilon_0 m_0 \tag{3.40}$$

for any $b\in[0,\varepsilon_0]$ and $x_2\in[0,1]$, which proves (3.37). Next, we observe that

$$\big|\mathscr{K}_{P_n}(b_n^*,x_2^n)-\mathscr{K}_{P_n}(b^{**},x_2^n)\big|\le\left|\int_{b_n^*}^{b_0^{**}}|f(x_1,x_2^n)-P_n(x_1,x_2^n)|E(x_1,x_2^n)dx_1\right|\le m_0\big|b_n^*-b_0^{**}\big|. \tag{3.41}$$

As $\{b_n^*\}_{n=1}^{\infty}$ converges to b_0^{**}, we use (3.41) have

$$\limsup_{n\to\infty}\big|\mathscr{K}_{P_n}(b_n^*,x_2^n)-\mathscr{K}_{P_n}(b_0^{**},x_2^n)\big|=0. \tag{3.42}$$

Using the continuity of E,f and P, and the Lebesgue dominated convergence theorem we also observe that

$$\begin{aligned}\lim_{n\to\infty}\mathscr{K}_{P_n}(b_0^{**},x_2^n)&=\lim_{n\to\infty}\int_0^{b_0^{**}}\big(f(x_1,x_2^n)-P_n(x_1,x_2^n)\big)\,E(x_1,x_2^n)dx_1\\ &=\lim_{n\to\infty}\int_0^{b_0^{**}}\Big(f(x_1,x_2^0)-P_n(x_1,x_2^0)\Big)\,E(x_1,x_2^0)dx_1\\ &=\lim_{n\to\infty}\mathscr{K}_{P_n}(b_0^{**},x_2^0)<\infty.\end{aligned} \tag{3.43}$$

In view of (3.43),

$$\limsup_{n\to\infty}\Big|\mathscr{K}_{P_n}(b_0^{**},x_2^n)-\mathscr{K}_{P_n}(b_0^{**},x_2^0)\Big|=0. \tag{3.44}$$

In light of (3.42) and (3.44), to obtain (3.38), it suffices to show that

$$\limsup_{n\to\infty}\Big|\mathscr{K}_{P_n}(b_0^{**},x_2^0)-\mathscr{K}_{P_0}(b_0^{**},x_2^0)\Big|=0. \tag{3.45}$$

We obtain the following estimate

$$
\begin{aligned}
\left|\mathscr{K}_{P_n}(b_0^{**},x_2^0)-\mathscr{K}_{P_0}(b_0^{**},x_2)\right| &= \left|\int_0^{b_0^{**}}\left(P_n(x_1,x_2^0)-P_0(x_1,x_2^0)\right)E(x_1,x_2^0)dx_1\right| \\
&\leq \int_0^{\varepsilon_0}\left|P_n(x_1,x_2^0)-P_0(x_1,x_2^0)\right|E(x_1,x_2^0)dx_1 \\
&\leq \varepsilon_0\|E\|_{L^\infty(\Lambda_{\epsilon_0})}\sup_{x\in\Lambda_{\varepsilon_0}}\left|P_n(x)-P_0(x)\right|.
\end{aligned}
\tag{3.46}
$$

The equation (3.46) yields the desired result. Let $\{x_2^n\}_{n=0}^\infty \subset [0,1]$ such that a^* satisfies

$$\mathscr{K}_P(a^*,x_2^n)\leq \mathscr{K}_P(b,x_2^n) \qquad \text{for all } b\in[0,1).$$

Assume that $\{x_2^n\}_{n=1}^\infty$ converges x_2^0. As $\mathscr{K}_P$ is continuous, we obtain in the limit

$$\mathscr{K}_P(a^*,x_2^0)\leq \mathscr{K}_P(b,x_2^0) \qquad \text{for all } b\in[0,1).$$

It follows that the set of $x_2\in[0,1]$ such that (3.35) holds, when b^* is replaced by a^*, is a closed subset of [0, 1] and thus compact.

Proposition 3.6. *Assume that (α**1**), (α**2**) and (α**3**) hold. Set $d=a+\|\mathbf{B}\|_\infty$. Let $C_0>0$, $A>0$ and $\delta\geq 0$. Let $\varepsilon_0:=\varepsilon_0(A)<1$ as provided in lemma 3.3. Assume in addition that*

$$-R_0+k_1+\frac{d+C_0}{R_1}+4<0. \tag{3.47}$$

Let P convex Lipschitz such that $P(0,0)\leq A$ and $P(x)\geq -C_0-4R_1-\delta$. Assume $\partial P(x)\subset \overline{conv}(\Omega)$ for all $x\in\Lambda$. Let $x_2,\bar{x}_2\in[0,1]$, $b^*,\bar{b}^*\in[0,\varepsilon_0]$ such that

$$\mathscr{K}_P(b^*,x_2)\leq \mathscr{K}_P(b,x_2) \qquad \textit{for all } b\in[0,1) \tag{3.48}$$

and

$$\mathscr{K}_P(\bar{b}^*,\bar{x}_2)\leq \mathscr{K}_P(b,\bar{x}_2) \qquad \textit{for all } b\in[0,1). \tag{3.49}$$

For $\delta=0$ or sufficiently small, the following holds: if $x_2<\bar{x}_2$, then $b^*\leq \bar{b}^*$.

Proof: We note that $\mathscr{K}_P(\cdot,x_2)$ differentiable on $(0,1)$ and

$$\frac{\partial\mathscr{K}_P}{\partial b}(b,x_2)=(f(b,x_2)-P(b,x_2))E(b,x_2) \tag{3.50}$$

for $b\in(0,1)$ and $x_2\in[0,1]$. The Lipschitz condition on P ensures that $\mathscr{K}_P$ has a mixed partial derivative almost everywhere and

$$\frac{\partial^2\mathscr{K}_P}{\partial b\partial x_2}(b,x_2)=(-\partial_{x_2}P)E(b,x_2)+(\partial_{x_2}f(b,x_2))E(b,x_2)+(f(b,x_2)-P(b,x_2))\partial_{x_2}E \tag{3.51}$$

for a.e. $(b,x_2)\in\Lambda_{\varepsilon_0}$. As $\partial P(x)\subset\overline{conv}(\Omega)$ for all $x\in\Lambda$, we have that $\partial_{x_2}P\geq R_0$ a.e. This, together with (2.11) and $P(x)>-C_0-4R_1-\delta$, implies that

$$\frac{\partial^2\mathscr{K}_P}{\partial b\partial x_2}(b,x_2)\leq -R_0E(b,x_2)+k_1E(b,x_2)+k_0(aE^\alpha(b,x_2)+\mathbf{B}(x_1)+C_0+4R_1+\delta)E^{1-\alpha}(b,x_2) \tag{3.52}$$

a.e. on Λ_{ε_0}. In view of the conditions $E\geq 1$ and $0<\alpha<1$, (3.52) implies that

$$\frac{\partial^2\mathscr{K}_P}{\partial b\partial x_2}(b,x_2)\leq -R_0E(b,x_2)+k_1E(b,x_2)+k_0aE(b,x_2)+k_0(\|\mathbf{B}\|_\infty+C_0+4R_1+\delta)E(b,x_2). \tag{3.53}$$

As $0 < k_0 \le 1/R_1$, we obtain that

$$\frac{\partial^2 \mathscr{K}_P}{\partial x_1 \partial x_2}(x_1, x_2) \le \left(-R_0 + k_1 + \frac{1}{R_1}(d + C_0 + 4R_1 + \delta)\right) E(x_1, x_2). \tag{3.54}$$

For $\delta = 0$ or sufficiently small, we note that the condition (3.47) guarantees that

$$\frac{\partial^2 \mathscr{K}_P}{\partial x_1 \partial x_2}(x_1, x_2) \le 0. \tag{3.55}$$

Let $x_2, \bar{x}_2 \in [0,1]$, $b^*, \bar{b}^* \in [0, \varepsilon_0]$ such that (3.48) and (3.49) hold. As P is Lipschitz, we have

$$0 \le \Big(\mathscr{K}_P(b^*, \bar{x}_2) - \mathscr{K}_P(\bar{b}^*, \bar{x}_2)\Big) + \Big(\mathscr{K}_P(\bar{b}^*, x_2) - \mathscr{K}_P(b^*, x_2)\Big) = -\int_{x_2}^{\bar{x}_2} dz \int_{b^*}^{\bar{b}^*} \frac{\partial^2 \mathscr{K}_P}{\partial x_1 \partial z}(x_1, z) dx_1. \tag{3.56}$$

In light of (3.55) and (3.56), we conclude that if $x_2 < \bar{x}_2$, then $b^* \le \bar{b}^*$. □

3.3. **Monotonicity and estimate for minimizer in the $\mathscr{K}_P$ functional.** In this section, we establish the monotonicity of the minimizer in the second term of $\mathscr{G}$. We further derive an estimate that is useful in the study of the regularity of the domain source domain.

Proposition 3.7. *Assume that (α**1**), (α**2**) and (α**3**) hold. Let $C_0 > 0$ and $A > 0$. Let $\varepsilon_0 := \varepsilon_0(A) < 1$ as provided in lemma 3.3. Assume in addition that (3.47) holds. Let P convex Lipschitz such that $P(0,0) \le A$ and $P(x) \ge -C_0 - 4R_1 - \delta$. Assume $\partial P(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$. Let $\mathbf{b} : [0,1] \longrightarrow [0,1)$ such that for any $x_2 \in [0,1]$, $\mathbf{b}(x_2) \in [0, \varepsilon_0]$ and*

$$\mathscr{K}_P(\mathbf{b}(x_2), x_2) \le \mathscr{K}_P(b, x_2) \qquad \text{for all } b \in [0,1) \tag{3.57}$$

as provided by proposition 3.5. Then, for $\delta = 0$ or sufficiently small the following hold:

(i) *$\mathbf{b}$ is monotone increasing.*
(ii) *If $\bar{\mathbf{b}} : [0,1] \longrightarrow [0,1)$ is such that for any $x_2 \in [0,1]$, $\bar{\mathbf{b}}(x_2) \in [0, \varepsilon_0]$ satisfies (3.57) and $z \in (0,1)$ is a point of continuity of $\mathbf{b}$, then $\mathbf{b}(z) = \bar{\mathbf{b}}(z)$. Consequently, $\mathbf{b} = \bar{\mathbf{b}}$ outside a countable subset of $[0,1]$.*

Moreover, assume that $\{P_n\}_{n=0}^{\infty}$ is a sequence of convex Lipschitz functions such that for each $n \ge 0$, $P_n(0,0) \le A$ and $P_n(x) \ge -C_0 - 4R_1 - \delta$. Assume $\partial P_n(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$ and that $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P_0 on compact subsets of Λ. Let $\{\mathbf{b}_n\}_{n=0}^{\infty}$ such that for any $x_2 \in [0,1]$, $\mathbf{b}_n(x_2) \in [0,\ \varepsilon_0]$ and

$$\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) \le \mathscr{K}_{P_n}(b, x_2) \qquad \text{for all } b \in [0,1). \tag{3.58}$$

Then, for $\delta = 0$ or sufficiently small, $\{\mathbf{b}_n\}_{n=1}^{\infty}$ converges pointwise to $\mathbf{b}_0$ at point of continuity of $\mathbf{b}_0$ and

$$\lim_{n\to\infty} \mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) = \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2) \text{ for almost every } x_2 \in [0,1]. \tag{3.59}$$

Proof: By proposition 3.6, it is straightforward that $\mathbf{b}$ is monotone increasing. It follows that $\mathbf{b}$ is continuous everywhere on $[0,1]$ except for a countable number of points.
Let $\bar{\mathbf{b}}$ be such that for any $x_2 \in [0,1]$, $\bar{\mathbf{b}}(x_2) \in [0, \varepsilon_0]$ and satisfies (3.57). Let $z \in (0,1)$ be a point of continuity for $\mathbf{b}$ and let $\kappa > 0$. By proposition 3.6, $\bar{\mathbf{b}}(z) \le \mathbf{b}(z+\kappa)$ and $\mathbf{b}(z-\kappa) \le \bar{\mathbf{b}}(z)$ for κ small enough. In light of the continuity of $\mathbf{b}$ at z, as κ goes to 0, we have that $\bar{\mathbf{b}}(z) = \mathbf{b}(z)$, which proves (ii).

Fix $x_2 \in [0,1]$. As $\mathbf{b}_n(x_2) \in [0, \varepsilon_0]$, there exists a subsequence of $\left\{\mathbf{b}_n(x_2)\right\}_{n=1}^{\infty}$ that we denote again by $\left\{\mathbf{b}_n(x_2)\right\}_{n=1}^{\infty}$ such that $\left\{\mathbf{b}_n(x_2)\right\}_{n=1}^{\infty}$ converges to $\mathbf{p}_0(x_2) \in [0,\ \varepsilon_0]$.

We observe that

$$\begin{aligned}\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2) = &\left(\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_n(x_2), x_2)\right) \\ &+ \left(\mathscr{K}_{P_0}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2)\right).\end{aligned} \tag{3.60}$$

Thus,

$$\begin{aligned}\left|\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2)\right| \le &\left|\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_n(x_2), x_2)\right| \\ &+\left|\mathscr{K}_{P_0}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2)\right| \\ \le &\int_0^{\mathbf{b}_n(x_2)} \left|P_n(x_1,x_2) - P_0(x_1,x_2)\right| E(x_1,x_2)dx_1 \\ &+\left|\int_{\mathbf{b}_n(x_2)}^{\mathbf{p}_0(x_2)} \left|f(x_1,x_2) - P_0(x_1,x_2)\right| E(x_1,x_2)dx_1\right|.\end{aligned} \tag{3.61}$$

It follows that

$$\left|\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2)\right| \le 2\varepsilon_0 \|E\|_{L^\infty(\Lambda_{\varepsilon_0})} \|P_n - P_0\|_{L^\infty(\Lambda_{\varepsilon_0})} + m_0|\mathbf{b}_n(x_2) - \mathbf{p}_0(x_2)|. \tag{3.62}$$

$m_0 := a\|E\|^{\alpha+1}_{L^\infty(\Lambda_{\varepsilon_0})} + \|\mathbf{B}\|_\infty \|E\|_{L^\infty(\Lambda_{\varepsilon_0})} + e_0\|E\|_{L^\infty(\Lambda_{\varepsilon_0})}$. As $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P_0 on Λ_{ε_0}, and $\left\{\mathbf{b}_n(x_2)\right\}_{n=1}^{\infty}$ converges to $\mathbf{p}_0(x_2)$ we have that $\left\{\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2)\right\}_{n=1}^{\infty}$ converges to $\mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2)$ on Λ_{ε_0}. Thus, in the limit (3.58) yields

$$\mathscr{K}_{P_0}(\mathbf{p}_0(x_2), x_2) \le \mathscr{K}_{P_0}(b, x_2) \qquad \text{for all } b \in [0,1). \tag{3.63}$$

By (i), $\mathbf{p}_0$ is monotone increasing. If x_2 is a point of continuity for $\mathbf{b}_0$, then we have that $\mathbf{p}_0(x_2) = \mathbf{b}_0(x_2)$ by (ii). And so, as the limit $\mathbf{p}_0(x_2)$ is independent of the subsequence of $\left\{\mathbf{b}_n(x_2)\right\}_{n=1}^{\infty}$, we have that $\{\mathbf{b}_n\}_{n=1}^{\infty}$ converges pointwise to $\mathbf{b}_0$ at points of continuity of $\mathbf{b}_0$. □

Proposition 3.8. *Assume that* (α**1**), (α**2**) *and* (α**3**) *hold. Let* $C_0 > 0$, $A_0 > 0$ *and* $\delta > 0$. *Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume in addition that* (3.47) *holds. Let* P_0 *convex Lipschitz such that* $P_0(0,0) \le A_0$ *and* $P_0(x) \ge -C_0 - 4R_1$ *for all* $x \in \Lambda$. *Assume* $\partial P_0(x) \subset \overline{conv}(\Omega)$ *for all* $x \in \Lambda$. *Let* $\mathbf{b}_0 : [0,1] \longrightarrow [0,1)$ *such that for any* $x_2 \in [0,1]$, $\mathbf{b}_0(x_2) \in [0,\ \varepsilon_0(A_0)]$ *and*

$$\mathscr{K}_P(\mathbf{b}_0(x_2), x_2) \le \mathscr{K}_P(b, x_2) \qquad \textit{for all } b \in [0,1) \tag{3.64}$$

as provided by proposition 3.5. Let $\{P_n\}_{n=1}^{\infty}$ *be a sequence of convex Lipschitz functions such that* $\partial P_n(x) \subset \overline{conv}(\Omega)$ *for all* $x \in \Lambda$ *and that* $\{P_n\}_{n=1}^{\infty}$ *converges uniformly to* P_0 *on compact subsets of* Λ. *Then, there exists* $n_0 \in \mathbb{N}$ *such that*

$$P_n(0,0) \le A_0 + \delta \quad \textit{and} \quad P_n(x) \ge -C_0 - 4R_1 - \delta \tag{3.65}$$

for $n \ge n_0$ *and* $x \in \Lambda$. *Let* $\{\mathbf{b}_n\}_{n=n_0}^{\infty}$ *such that for any* $x_2 \in [0,1]$, $\mathbf{b}_n(x_2) \in [0,\ \varepsilon_0(A_0+\delta)]$ *and*

$$\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) \le \mathscr{K}_{P_n}(b, x_2) \qquad \textit{for all } b \in [0,1) \tag{3.66}$$

as provided by proposition 3.5. Then, $\{\mathbf{b}_n\}_{n=1}^{\infty}$ *converges pointwise to* $\mathbf{b}_0$ *at point of continuity of* $\mathbf{b}_0$.

Proof: Note that as $\{P_n\}_{n=1}^{\infty}$ converges uniformly to P_0, there exists n_0 such that for $n \geq n_0$ we have $-\delta \leq P_n(x) - P_0(0,0) \leq \delta$ for all $x \in \Lambda$. It follows that $P_n(0,0) \leq A_0 + \delta$ and $P_n(x) \geq -C_0 - 4R_1 - \delta$ for $n \geq n_0$. Since $P_0(0,0) \leq A_0$, $P_0(x) \geq -C_0 - 4R_1$ and $\delta > 0$ we have also $P_0(0,0) \leq A_0 + \delta$ and $P_0(x) \geq -C_0 - 4R_1 - \delta$. In addition, we note that we have for any $x_2 \in [0,1]$, $\mathbf{b}_0(x_2) \in [0,\ \varepsilon_0(A_0+\delta)]$ and $\mathbf{b}_0$ satisfies (3.64) by proposition 3.5. Using proposition 3.7, we obtain that $\{\mathbf{b}_n\}_{n=1}^{\infty}$ converges pointwise to $\mathbf{b}_0$ at point of continuity of $\mathbf{b}_0$.

□

The following lemmas establish estimates useful for the Lipschitz regularity of the boundary of the domain $\Lambda_{\mathbf{b}}$, solution to problem (2.3).

Lemma 3.9. *Assume that* $(\alpha\mathbf{1})$, $(\alpha\mathbf{2})$ *and* $(\alpha\mathbf{3})$ *hold. Let* $C_0 > 0$ *and* $A_0 > 0$ *as provided by lemma 3.2. Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume in addition that* (3.47) *holds. Let* P *be Lipschitz such that* $P(0,0) < A_0$ *and* $P(x) \geq -C_0 - 4R_1$. *Assume* $\partial P(x) \subset \overline{conv}(\Omega)$ *for all* $x \in \Lambda$. *Let* $\mathcal{A}$ *be the set of all* $x_2 \in [0,1]$ *such that*

$$\mathscr{K}_P(0, x_2) \leq \mathscr{K}_P(b, x_2) \qquad \textit{for all } b \in [0,1). \tag{3.67}$$

If $\mathcal{A}$ *is non-empty, then* $\mathcal{A}$ *is a closed interval.*

Proof: Assume that $\mathcal{A}$ is non-empty. By proposition 3.5, $\mathcal{A}$ is a compact subset of $[0,1]$. Let us denote by m the maximal element of $\mathcal{A}$ so that $\mathcal{A} \subset [0,m]$. Let $a \in [0,m]$ and $b^* > 0$ such that

$$\mathscr{K}_P(b^*, a) \leq \mathscr{K}_P(b, a) \qquad \text{for all } b \in [0,1) \tag{3.68}$$

In light of the monotonicity result in proposition 3.6, as $a \leq m$ we have that $b^* = 0$. It follows that $a \in \mathcal{A}$. We thus conclude that $[0,m] \subset \mathcal{A}$ so that $\mathcal{A} = [0,m]$. □

Proposition 3.10. *Assume that* $(\alpha\mathbf{1})$, $(\alpha\mathbf{2})$ *and* $(\alpha\mathbf{3})$ *hold. Let* $C_0 > 0$ *and* $A_0 > 0$ *as provided by lemma 3.2. Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume that* (3.47) *holds. Assme additionally that* f *is* k_1-*Lipschitz on* Λ_{ε_0}. *Let* P *be a convex Lipschitz continuous function such that* $P(0,0) < A_0$ *and* $P(x) \geq -C_0 - 4R_1$. *Assume* $\partial P(x) \subset \overline{conv}(\Omega)$ *for all* $x \in \Lambda$. *Let* $\mathcal{A}$ *be as defined in lemma 3.9. Let* m *denotes the maximal element of* $\mathcal{A}$ *when* $\mathcal{A}$ *is non-empty and set* $m = 0$ *when* $\mathcal{A}$ *is empty. Let* $x_2, \bar{x}_2 \in [m,1]$ *such that* $x_2 < \bar{x}_2$ *and* $b^*, \bar{b}^* \in [0,1)$ *such that*

$$\mathscr{K}_P(b^*, x_2) \leq \mathscr{K}_P(b, x_2) \qquad \textit{for all } b \in [0,1) \tag{3.69}$$

and

$$\mathscr{K}_P(\bar{b}^*, \bar{x}_2) \leq \mathscr{K}_P(b, \bar{x}_2) \qquad \textit{for all } b \in [0,1). \tag{3.70}$$

Then, there exists a constant $c_0 = (R_0 - k_1)/k_1$ *such that if* $x_2, \bar{x}_2 \in [m,1]$, *then*

$$b^* - \bar{b}^* \geq c_0(x_2 - \bar{x}_2). \tag{3.71}$$

Proof: Assume $x_2, \bar{x}_2 \in (m,1]$ such that $x_2 < \bar{x}_2$. Then, lemma 3.9 ensures that $b^*, \bar{b}^* > 0$. As $b^*, \bar{b}^*$ satisfy respectively (3.69) and (3.70), we have that

$$f(b^*, x_2) - P(b^*, x_2) = f(\bar{b}^*, \bar{x}_2) - P(\bar{b}^*, \bar{x}_2) = 0, \tag{3.72}$$

in light of (3.36). We note that P is convex Lipschitz with $\partial P(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$. In view of (2.8)

$$0 \leq \partial_{x_1} P \leq R_1 \qquad \text{and} \qquad R_0 \leq \partial_{x_2} P \leq R_1 \quad a.e. \tag{3.73}$$

We use (3.72), the first equation of (3.73) and the bound of the derivatives of f to obtain that

$$\begin{aligned} f(\bar{b}^*, x_2) - P(\bar{b}^*, x_2) &= f(\bar{b}^*, x_2) - P(\bar{b}^*, x_2) - \big[f(b^*, x_2) - P(b^*, x_2)\big] \\ &= \int_{b^*}^{\bar{b}^*} \partial_u \big[f(u, x_2) - P(u, x_2)\big]\, du \\ &\le k_1(\bar{b}^* - b^*). \end{aligned} \tag{3.74}$$

In a similar way, we use (3.72), the second equation of (3.73) and the bound of the derivatives of f to obtain

$$\begin{aligned} f(\bar{b}^*, x_2) - P(\bar{b}^*, x_2) &= f(\bar{b}^*, x_2) - P(\bar{b}^*, x_2) - \big[f(\bar{b}^*, \bar{x}_2) - P(\bar{b}^*, \bar{x}_2)\big] \\ &= \int_{\bar{x}_2}^{x_2} \partial_v \big(f(\bar{b}^*, v) - P(\bar{b}^*, v)\big)\, dv \\ &\ge (R_0 - k_1)(\bar{x}_2 - x_2). \end{aligned} \tag{3.75}$$

We combine (3.74) and (3.75) to prove that

$$(\bar{b}^* - b^*) \ge \frac{R_0 - k_1}{k_1}(\bar{x}_2 - x_2) \tag{3.76}$$

which proves (3.71). We note that the inequality (3.71) can be extended by continuity to $[m, 1]$ in light of the proposition 3.5. □

Remark 3.11. If $\mathbf{b}$ denotes a function defined in such a way that for each $x_2 \in [m, 1]$ we have that $\mathbf{b}(x_2)$ satisfies (3.69) and c_0 is positive, then the inequality (3.71) implies that $\mathbf{b}$ is strictly monotone increasing on $[0, m]$. Furthermore, (3.71) implies that the generalized inverse of $\mathbf{b}$ is Lipschitz continuous so that the boundary of domain $\Lambda_{\mathbf{b}}$ is piecewise Lipschitz continuous up to a change of coordinates. This result can be found in [1].

3.4. Existence of a maximizer for the functional $\mathscr{G}$.

Proposition 3.12. *Assume that* $(\alpha\mathbf{1})$, $(\alpha\mathbf{2})$ *and* $(\alpha\mathbf{3})$ *hold. Set* $d = a + \|\mathbf{B}\|_\infty$ *and* $C_0 = 5R_1$. *Let* $A_0 > 0$ *as provided by lemma 3.2. Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume, in addition, that* R_0 *satisfies*

$$2R_0 > k_1 + 9 + \sqrt{(k_1 + 9)^2 + 4d}. \tag{3.77}$$

The functional $\mathscr{G}$ *defined by*

$$\mathscr{G}(P, \Psi) = -\int_{\mathbb{R}^2} \Psi\sigma(y)dy + \inf_{\mathbf{b}\in\mathscr{B}} \int_0^1 \int_0^{\mathbf{b}(x_2)} \big(f(x_1, x_2) - P(x_1, x_2)\big) E(x_1, x_2) dx_1 dx_2 \tag{3.78}$$

has a maximizer over $\mathcal{V}$ *in the set* $(P, \Psi) \in \mathcal{V}_0$ *such that* $-C_0 \le \mathscr{G}(P, \Psi)$.

Proof: Let $\big\{(P_n, \Psi_n)\big\}_{n=1}^{\infty} \subset \mathcal{V}$ be a maximizing sequence for $\mathscr{G}$. By the standard double convexification argument (see [14]), we assume that $\big\{(P_n, \Psi_n)\big\}_{n=1}^{\infty} \subset \mathcal{V}_0$. It follows that, by virtue of lemma 3.2, $\{P_n\}_{n=1}^{\infty}$ are (uniformly) Lipschitz such that $\partial P_n(x) \subset \overline{conv}(\Omega)$ and $P_n(0, 0) < A_0$. We observe that

$$\sup_{x\in\Lambda, y\in\Omega} \langle x, y\rangle \le |x||y| < 4R_1 = C_0 - R_1.$$

Set

$$P_{**}(x) = 0 \qquad x \in \Lambda \quad \text{and} \quad \Psi_{**}(y) = C_0 - R_1 \quad y \in \Omega.$$

Then, a simple computation shows that $\mathscr{G}(P_{**}, \Psi_{**}) = R_1 - C_0$. As $\{(P_n, \Psi_n)\}_{n=1}^\infty$ is a maximizing sequence, we assume without loss of generality that $-C_0 \le \mathscr{G}(P_n, \Psi_n)$ for $n \ge 1$. In light of lemma 3.2 again, $P_n(x) \ge -C_0 - 4R_1$ for $n \ge 1$ and $\{(P_n, \Psi_n)\}_{n=1}^\infty$ is compact with respect to the uniform topology on $C(\bar{\Lambda}) \times C(\bar{\Omega})$. It follows that there exists a subsequence of $\{(P_n, \Psi_n)\}_{n=1}^\infty$ still denoted by $\{(P_n, \Psi_n)\}_{n=1}^\infty$ such that $\{(P_n, \Psi_n)\}_{n=1}^\infty$ converges uniformly to some $(P_0, \Psi_0) \in \mathcal{V}_0$. As σ is a density of a probability measure,

$$\lim_{n\to\infty} \int_\Omega \Psi_n(y)\sigma(y)dy = \int_\Omega \Psi_0(y)\sigma(y)dy. \tag{3.79}$$

Note that as $R_0 < R_1$,

$$-R_0 + k_1 + \frac{d + C_0}{R_1} + 4 \le -R_0 + k_1 + \frac{d}{R_0} + 9.$$

Since R_0 satisfies (3.77), we have

$$-R_0 + k_1 + \frac{d + C_0}{R_1} + 4 < 0.$$

In light of proposition 3.7, let $\{\mathbf{b}_n\}_{n=0}^\infty$ a sequence of functions defined on $[0,1]$ such that $\mathbf{b}_n(x_2) \in [0, \varepsilon_0]$ for all $x_2 \in [0,1]$ and

$$\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) \le \mathscr{K}_{P_n}(b, x_2) \qquad \text{for all } b \in [0,1),\ n \ge 0. \tag{3.80}$$

Then,

$$\inf_{\mathbf{b}\in\mathscr{B}} \int_0^1 \int_0^{\mathbf{b}(x_2)} \big(f(x_1,x_2) - P_n(x_1,x_2)\big) E(x_1,x_2)dx_1dx_2 = \mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2).$$

As $\{P_n\}_{n=1}^\infty$ converges uniformly to P_0, in light of proposition 3.7, $\{\mathbf{b}_n\}_{n=1}^\infty$ converges almost everywhere to $\mathbf{b}_0$ and

$$\lim_{n\to\infty} \mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2) = \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2) \text{ for almost every } x_2 \in [0,1]. \tag{3.81}$$

Moreover, we use (3.37) to obtain

$$\|\mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2)\| \le h_0 \tag{3.82}$$

for some $h_0 > 0$ and all $n \ge 1$. Using the Lebesgue dominated convergence theorem, it follows from (3.81) and (3.82) that

$$\lim_{n\to\infty} \int_0^1 \mathscr{K}_{P_n}(\mathbf{b}_n(x_2), x_2)dx_2 = \int_0^1 \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)dx_2. \tag{3.83}$$

We combine (3.79) and (3.83) to obtain that

$$\lim_{n\to\infty} \mathcal{G}(P_n, \Psi_n) = \mathcal{G}(P_0, \Psi_0). \tag{3.84}$$

As $\{(P_n, \Psi_n)\}_{n=1}^\infty$ is a maximizing sequence, (3.84) implies that (P_0, Ψ_0) is a maximizer for the function $\mathcal{G}$ over $\mathcal{V}$. □

4. Study of primal problem

In this section, we study first the duality problem between the functional $\mathscr{F}$ and $\mathscr{G}$ and derive a condition of optimality. This condition is further exploited to obtain existence and uniquess both in the primal and dual problems. To begin, we set

$$\mathcal{M}(\sigma) = \frac{1}{2}\int_{\Omega} |y|^2 \sigma(y) dy.$$

Proposition 4.1. *We have the following:*

$$\mathcal{G}(P, \Psi) + \mathcal{M}(\sigma) \leq \mathcal{F}(\mathbf{b})$$

for all $(P, \Psi) \in \mathcal{V}$ *and* $\mathbf{b} \in \mathscr{B}_0$. *Moreover,* $\mathcal{F}(\mathbf{b}) = \mathcal{G}(P, \Psi) + \mathcal{M}(\sigma)$ *if and only if* $\nabla P \# E\chi_{\Lambda_{\mathbf{b}}} = \sigma$ *and*

$$\mathscr{K}_P(\mathbf{b}(x_2), x_2) \leq \mathscr{K}_P(b, x_2) \quad \textit{for a.e } x_2 \in [0,1] \textit{ and for all } b \in [0,1). \tag{4.1}$$

Proof: 1) Let $(P, \Psi) \in \mathcal{V}$. We recall that

$$\mathcal{G}(P, \Psi) = -\int_{\mathbb{R}^2} \Psi(y)\sigma(y)dy + \inf_{\mathbf{b}\in\mathscr{B}} \int_0^1 \int_0^{\mathbf{b}(x_2)} \big(f(x_1, x_2) - P(x_1, x_2)\big) E(x_1, x_2) dx_1 dx_2. \tag{4.2}$$

Then, for any $\mathbf{b} \in \mathscr{B}_0$ and $\gamma \in \Gamma\left(\sigma, E\chi_{\Lambda_{\mathbf{b}}}\right)$, we have

$$\begin{aligned}
\mathcal{G}(P, \Psi) &\leq -\int_{\mathbb{R}^2} \Psi(y)\sigma(y)\, dy + \int_0^1 \int_0^{\mathbf{b}(x_2)} \big(f(x_1, x_2) - P(x_1, x_2)\big) E(x_1, x_2) dx_1 dx_2 \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} -(P(x) + \Psi(y))\, d\gamma(x, y) + \int_{\Lambda_{\mathbf{b}}} f(x_1, x_2) E(x_1, x_2) dx_1 dx_2 \\
&\leq \int_{\mathbb{R}^2\times\mathbb{R}^2} -\langle x, y\rangle\, d\gamma(x, y) + \int_{\Lambda_{\mathbf{b}}} f(x_1, x_2) E(x_1, x_2) dx_1 dx_2.
\end{aligned} \tag{4.3}$$

It follows that

$$\mathcal{G}(P, \Psi) \leq \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |x - y|^2 d\gamma + \int_{\Lambda_{\mathbf{b}}} \left(f(x_1, x_2) - \frac{x_1^2 + x_2^2}{2}\right) E(x_1, x_2) dx_1 dx_2 - \int_{\mathbb{R}^2} \frac{|y|^2}{2}\sigma(y)dy. \tag{4.4}$$

By taking the infinitum over such γ's, we thus obtain

$$\mathcal{G}(P, \Psi) + \mathcal{M}(\sigma) \leq \mathcal{F}(\mathbf{b}). \tag{4.5}$$

In light of (4.3), the equality in (4.5) holds for some $(P_0, \Psi_0) \in \mathcal{V}_0$ and $\mathbf{b}_0 \in \mathscr{B}_0$ if and only if $P_0(x) + \Psi_0(y) = \langle x, y\rangle$ *a.e.* $-\gamma_0$ for some $\gamma_0 \in \Gamma\left(\sigma, E\chi_{\Lambda_{\mathbf{b}_0}}\right)$ and $\mathbf{b}_0(x_2)$ minimizes $\mathscr{K}_P(\cdot, x_2)$ for almost every x_2. The first condition means that $\nabla P_0 \# E\chi_{\Lambda_{\mathbf{b}_0}} = \sigma$, and the second condition is equivalent to (4.1). □

In the next proposition we show the existence and uniqueness of the minimizer of $\mathscr{F}$ and provide a characterization of the minimizer.

Proposition 4.2. *Assume that* (α**1**), (α**2**) *and* (α**3**) *hold. Let* $C_0 = 5R_1$ *and* $A_0 > 0$ *as provided by lemma 3.2. Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume that* R_0 *satisfies* (3.77).

The functional $\mathcal{F}$ has a unique minimizer $\mathbf{b}_0$ (up to a set of zero Lebesgue measure) over $\mathscr{B}_0$ that is monotone. If $(P_0, \Psi_0) \in \mathcal{V}_0$ is a maximizer of $\mathcal{G}$ over $\mathcal{V}$ as obtained in proposition 3.12, then

$$\nabla P_0 \# E\chi_{\Lambda_{\mathbf{b}_0}} = \sigma \quad \text{and} \quad P_0(\mathbf{b}_0(x_2), x_2) = f(\mathbf{b}_0(x_2), x_2) \tag{4.6}$$

for all $x_2 \in [0,1]$ such that $\mathbf{b}_0(x_2) > 0$. Moreover, for any other maximizer $(\bar{P}_0, \bar{\Psi}_0)$ of $\mathcal{G}$ over $\mathcal{V}$, we have that $P_0 = \bar{P}_0$ on $\Lambda_{\mathbf{b}_0}$.

Proof: We divide the proof into several parts.

1) Assume $(P_0, \Psi_0) \in \mathcal{V}_0$ is a maximizer of $\mathscr{G}$ as obtained in proposition 3.12. Then, P_0 convex Lipschitz such that $P_0(0,0) \le A_0$ and $P_0(x) \ge -C_0 - 4R_1$ for all $x \in \Lambda$. Moreover, $\partial P_0(x) \subset \overline{conv}(\Omega)$ for all $x \in \Lambda$. Let $\mathbf{b}_0 : [0,1] \longrightarrow [0,1)$ such that for any $x_2 \in [0,1]$, $\mathbf{b}_0(x_2) \in [0,\ \varepsilon_0(A_0)]$ and

$$\mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2) \le \mathscr{K}_{P_0}(b, x_2) \qquad \text{for all } b \in [0,1) \tag{4.7}$$

as provided by proposition 3.5. Let $\xi \in C_c(\mathbb{R}^2)$ and η a real number such that $|\eta| < 1$. Set

$$\Psi_\eta(y) = \Psi_0(y) + \eta\xi(y) \qquad \text{and} \qquad P_\eta(x) = \sup_{y \in \bar{\Omega}} \langle x, y\rangle - \Psi_0(y) - \eta\xi(y). \tag{4.8}$$

We note that P_η is convex and Lipschitz continuous with $\partial P_\eta(x) \subset \overline{conv}(\Omega)$. We then obtain that (cfr [12], [13])

$$\|P_\eta - P_0\|_{L^\infty(\bar{\Lambda})} \le |\eta|\ \|\xi\|_\infty \tag{4.9}$$

and

$$\lim_{\eta \to 0} \frac{P_\eta(x) - P_0(x)}{\eta} = -\xi\left(\nabla P_0(x)\right) \tag{4.10}$$

for all point of differentiability of x of P_0. In light of the first equation in (4.9), $\{P_\eta\}_{0<|\eta|<-1}$ converges uniformly to P_0. Let $\delta > 0$. Then, for $|\eta|$ small enough, $P_\eta(0,0) \le A_0 + \delta$ and $P_\eta(x) \ge -C_0 - 4R_1 - \delta$ for all $x \in \Lambda$. In light of proposition 3.7, let $\mathbf{b}_\eta$ with $0 < |\eta| < 1$ such that $\mathbf{b}_\eta \in [0,\ \varepsilon_0(A + \delta)]$ and

$$\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) \le \mathscr{K}_{P_\eta}(b, x_2) \qquad \text{for all } b \in [0,1). \tag{4.11}$$

Let $\{\eta_n\}_{n=1}^\infty$ be such that $0 < |\eta_n| < 1$ and $\{\eta_n\}_{n=1}^\infty$ converges to 0. Then, for δ small enough, it follows from proposition 3.8 that $\{\mathbf{b}_{\eta_n}\}_{n=1}^\infty$ converges pointwise to $\mathbf{b}_0$ at points of continuity of $\mathbf{b}_0$. As $\{\eta_n\}_{n=1}^\infty$ is arbitrary, $\{\mathbf{b}_\eta\}_\eta$ converges pointwise to $\mathbf{b}_0$ at points of continuity of $\mathbf{b}_0$ as η goes to 0.

Note that

$$\int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_\eta(x_2), x_2)}{\eta} dx_2 = -\int_0^1 \int_0^{\mathbf{b}_\eta} \frac{P_\eta - P_0}{\eta} E(x_1, x_2) dx_1 x_2 \tag{4.12}$$

for $0 < |\eta| < 1$. We observe that E is continuous and thus bounded on Λ_{ε_0}. As $\{\mathbf{b}_\eta\}_\eta$ converges almost everywhere to $\mathbf{b}_0$ and $\mathbf{b}_\eta \in [0, \varepsilon_0]$, we use (4.9), (4.10), (4.12) and invoke the Lebesgue dominated convergence theorem to obtain that

$$\begin{aligned}\lim_{\eta \to 0} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_\eta(x_2), x_2)}{\eta} dx_2 &= -\lim_{\eta \to 0} \int_0^1 \int_0^{\mathbf{b}_\eta} \frac{P_\eta - P_0}{\eta} E(x_1, x_2) dx_1 dx_2 \\ &= \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2.\end{aligned} \tag{4.13}$$

Using the minimizing property of $\mathbf{b}_0$ we obtain

$$\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_\eta(x_2), x_2) \leq \mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2). \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$\begin{aligned}\int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2 &= \liminf_{\eta \to 0} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_\eta(x_2), x_2)}{\eta} dx_2 \\ &\leq \liminf_{\substack{\eta \to 0 \\ \eta > 0}} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2.\end{aligned} \tag{4.15}$$

Similarly, we note that

$$\int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_0(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 = -\int_0^1 \int_0^{\mathbf{b}_0} \frac{P_\eta - P_0}{\eta} E(x_1, x_2) dx_1 dx_2 \tag{4.16}$$

for $0 < |\eta| < 1$. We use again (4.9), (4.10), (4.16) and invoke the Lebesgue dominated convergence theorem to obtain that

$$\begin{aligned}\lim_{\eta \to 0} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_0(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 &= -\lim_{\eta \to 0} \int_0^1 \int_0^{\mathbf{b}_0} \frac{P_\eta - P_0}{\eta} E(x_1, x_2) dx_1 dx_2 \\ &= \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2.\end{aligned} \tag{4.17}$$

Using the minimizing property of $\mathbf{b}_\eta$, for $0 < |\eta| < 1$, we obtain

$$\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2) \leq \mathscr{K}_{P_\eta}(\mathbf{b}_0(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2). \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$\begin{aligned}&\limsup_{\substack{\eta \to 0 \\ \eta > 0}} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 \\ &\qquad \leq \limsup_{\eta \to 0} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_0(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 \\ &\qquad = \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2.\end{aligned} \tag{4.19}$$

We combine (4.15) and (4.19) to get that

$$\lim_{\substack{\eta \to 0 \\ \eta > 0}} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 = \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2. \tag{4.20}$$

Exploiting (4.14) and (4.18) again, one can establish in a similar manner as above that

$$\lim_{\substack{\eta \to 0 \\ \eta < 0}} \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2 = \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2. \tag{4.21}$$

We note that

$$\frac{\mathscr{G}(P_\eta, \Psi_\eta) - \mathscr{G}(P_0, \Psi_0)}{\eta} = -\int_{\mathbb{R}^2} \xi(y) \sigma(y) dy + \int_0^1 \frac{\mathscr{K}_{P_\eta}(\mathbf{b}_\eta(x_2), x_2) - \mathscr{K}_{P_0}(\mathbf{b}_0(x_2), x_2)}{\eta} dx_2. \tag{4.22}$$

As (P_0, Ψ_0) is a maximizer for $\mathscr{G}$, we have that

$$0 = \lim_{\eta \to 0} \frac{\mathscr{G}(P_\eta, \Psi_\eta) - \mathscr{G}(P_0, \Psi_0)}{\eta} = -\int_{\mathbb{R}^2} \xi(y)\sigma(y)dy + \int_0^1 \int_0^{\mathbf{b}_0} \xi(\nabla P_0) E(x_1, x_2) dx_1 dx_2. \quad (4.23)$$

Since (4.23) holds for all $\xi \in C_c(\mathbb{R}^2)$, we conclude that

$$\nabla P_0 \text{ pushes-forward } E\chi_{\Lambda_{\mathbf{b}_0}} \text{onto } \sigma. \quad (4.24)$$

2) It follows from (4.24) that, as σ is a probability measure, $E\chi_{\Lambda_{\mathbf{b}_0}}$ is a probability measure which implies that $\mathbf{b}_0 \in \mathscr{B}_0$. Also, we note that $\nabla P_0 \# E\chi_{\Lambda_{\mathbf{b}_0}} = \sigma$ combined with (4.11), when $\eta = 0$, yields that $\mathcal{G}(P_0, \Psi_0) = \mathcal{M}(\sigma) + \mathcal{F}(\mathbf{b}_0)$ in light of the equivalence result established in proposition 4.1. It follows that $\mathbf{b}_0$ is a minimizer of $\mathcal{F}$. Assume that $\bar{\mathbf{b}}_0$ is another minimizer of $\mathcal{F}$ over $\mathscr{B}_0$. Then,

$$\mathcal{G}(P_0, \Psi_0) + \mathcal{M}(\sigma) = \mathcal{F}(\mathbf{b}_0) = \mathcal{F}(\bar{\mathbf{b}}_0).$$

Using again the equivalence result obtained in proposition 4.1, the last equation displayed implies that ∇P_0 pushes forward $E\chi_{\Lambda_{\bar{\mathbf{b}}_0}}$ onto σ and

$$\mathscr{K}_{P_0}(\bar{\mathbf{b}}_0(x_2), x_2) \leq \mathscr{K}_{P_0}(b, x_2) \qquad \text{for all } b \in [0, b), \quad (4.25)$$

and for a.e. $x_2 \in [0,1]$ fixed. In light of proposition 3.7, We combine (4.25) and the definition of $\mathbf{b}_0$ in (4.11) to obtain that $\bar{\mathbf{b}}_0 = \mathbf{b}_0$ a.e. As a result, $\mathcal{F}$ has a unique minimizer over $\mathscr{B}_0$. The monotonicity of the unique minimizer is guaranteed again by proposition 3.7. Furthermore, by (3.36), we have

$$P_0(\mathbf{b}_0(x_2), x_2) = f(\mathbf{b}_0(x_2), x_2) \qquad \text{for all } x_2 \text{ such that } \mathbf{b}_0(x_2) > 0. \quad (4.26)$$

3) Assume that $\mathcal{G}$ admits another maximizer $(\bar{P}_0, \bar{\Psi}_0)$. Then, using the result obtained in part 1) we have that

$$\nabla \bar{P}_0 \# E\chi_{\Lambda_{\mathbf{b}_0}} = \nabla P_0 \# E\chi_{\Lambda_{\mathbf{b}_0}} = \sigma.$$

As P_0 and $\bar{P}_0$ are convex, it follows that $\nabla P_0 = \nabla \bar{P}_0$ on $\Lambda_{\mathbf{b}_0}$ a.e. by the Brenier theorem. Since $\Lambda_{\mathbf{b}_0}$ is connected, (4.26) implies that $P_0 = \bar{P}_0$ on $\Lambda_{\mathbf{b}_0}$. □

5. Monge-Ampère Problem and Regularity results

In this section, we reconsider the Monge-Ampère equation (2.3). We collect previous results leading to the existence and uniqueness of a solution $(P_0, \mathbf{b}_0)$ into the next theorem. We investigate the regularity of the solution P_0 and $\mathbf{b}_0$.

The key ingredient toward regularity is the assumption obtained by Caffarelli that the target domain Ω is convex. We also explore a weaker result by Figalli in the absence of convexity of the domains.

Theorem 5.1. *Assume that* $(\alpha\mathbf{1})$, $(\alpha\mathbf{2})$ *and* $(\alpha\mathbf{3})$ *hold. Let* $0 < R_0 < R_1$ *and* $C_0 = 5R_1$. *Assume that* R_0 *satisfies* (3.77). *Let* Ω *be an open set such that* (2.8) *holds. Let* $A_0 > 0$ *as provided by lemma 3.2. Let* $\varepsilon_0 := \varepsilon_0(A_0) < 1$ *as provided in lemma 3.3. Assume that* $\lambda_0 \leq \sigma \leq \frac{1}{\lambda_0}$ *on* Ω *for some* $\lambda_0 > 0$. *Then, there exists* $P_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *Lipschitz convex and* $\mathbf{b}_0 : [0,1] \longrightarrow [0, \varepsilon_0]$ *monotone increasing such that* $(P_0, \mathbf{b}_0)$ *uniquely solves* (2.3) *in the sense of Brenier. Furthermore the following hold:*

(i) *The boundary of domain* $\Lambda_{\mathbf{b}_0}$ *is piecewise Lipschitz continuous.*

(ii) *Assume that* Ω *is convex. Then, there exists* $\alpha \in (0,1)$ *such that* P_0 *is* $C^{1,\alpha}_{loc}(\mathring{\Lambda}_{\mathbf{b}_0})$.

(iii) *There exists two open sets* $\Omega' \subset \Omega$ *and* $\Lambda' \subset \Lambda_{\mathbf{b}_0}$ *with* $\mathcal{L}^2(\Omega \setminus \Omega') = \mathcal{L}^2(\Lambda \setminus \Lambda_{\mathbf{b}_0}) = 0$ *such that* $P_0 \in C^{1,\alpha}(\Lambda')$, P_0 *is strictly convex inside* Λ'. *Moreover,* ∇P_0 *is a bi-Holder homeomorphism between* Λ' *and* Ω'.

Proof: We note that existence and uniqueness of the solution $(P_0, \mathbf{b}_0)$ to (2.3) is guaranteed by proposition 4.2. (i) follows from (3.77) and remark 3.11. We note that by assumption E is continuous and positive on Λ_b for $b \in (0,1)$. It follows that E is bounded away from zero and infinity on $\Lambda_{\mathbf{b}_0}$. As a result, by theorem 4.23 in [8] we obtain (ii) and by theorem 3.1 in [9] we obtain (iii). □

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