

ERGODIC RISK-SENSITIVE CONTROL OF MARKOV PROCESSES ON COUNTABLE STATE SPACE REVISITED

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ABSTRACT. We consider a large family of discrete and continuous time controlled Markov processes and study an ergodic risk-sensitive minimization problem. Under a blanket stability assumption, we provide a complete analysis to this problem. In particular, we establish uniqueness of the value function and verification result for optimal stationary Markov controls, in addition to the existence results. We also revisit this problem under a near-monotonicity condition but without any stability hypothesis. Our results also include policy improvement algorithms both in discrete and continuous time frameworks.

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1. INTRODUCTION

Let \mathbf{X} be a controlled Markov process (CMP), either discrete or continuous time, taking values in a discrete state space S . Let \mathfrak{U} be the class of admissible controls which also includes history dependent controls. We study the minimization problem of ergodic exponential cost given by

$$\mathcal{E}_i(c, \zeta) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right] \quad (\text{for discrete time}),$$

$$\mathcal{E}_i(c, \zeta) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\int_0^T c(X_t, \zeta_t) dt} \right] \quad (\text{for continuous time}),$$

where c is the running cost and $\zeta \in \mathfrak{U}$. More precisely, we are interested in the optimal value

$$\lambda^* = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta),$$

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and the characterization of all optimal stationary Markov controls. Our main results, [Theorems 2.1](#) and [3.1](#), establish existence of an eigenpair (λ^*, ψ^*) , $\psi^* > 0$, satisfying

$$\left. \begin{aligned} e^{\lambda^*} \psi^*(i) &= \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_j \psi^*(j) P(j|i, u) \right] \quad (\text{for discrete time}), \\ \lambda^* \psi^*(i) &= \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad (\text{for continuous time}), \end{aligned} \right\} \quad (1.1)$$

and also show that any minimizing selector of (1.1) is an optimal control. In addition, We show that ψ^* is unique upto a normalization and any optimal stationary Markov control is a measurable selector of (1.1). We also propose a policy improvement algorithm (PIA) and establish its convergence.

To the best of our knowledge, risk-sensitive optimization problems have been first considered in the seminal paper by Howard and Matheson [31], but it was only lately that this topic gained renewed interest due to applications in finance and large deviation theory. In this respect we mention the interesting work of Kontoyiannis and Meyn [35] studying multiplicative ergodic theorem and large deviation principle for geometrically stable Markov processes. Some of the early works on finite horizon risk-sensitive control includes Jacobson [32], Speyer, Deyst and Jacobson [44], Speyer [43], Gheorghie [20], Whittle [48], James, Baras and Elliott [33], etc. Bielecki, Hernández-Hernández and Pliska [15] consider the ergodic risk-sensitive problem for CMP with a finite state space and establish the existence of a unique solution to (1.1). Ergodic risk-sensitive control for discrete time CMP with countable state space is studied by Borkar and Meyn [17] (see also, Hernández-Hernández and Marcus [28]). Di Masi and Stettner [18, 19] consider the problem in a general state space. Later Shen, Stannat and Obermayer [42], Bäuerle and Rieder [10] extend these results for a wider class of utility functions. Let us also mention the work of Basu and Ghosh [9], Bäuerle and Rieder [11] which study zero-sum game with ergodic risk-sensitive cost criterion. Most of the above mentioned works, with the exception of [11, 18], establish existence of a solution to (1.1) and show that every minimizing selector is an optimal control. So a natural question is whether all the optimal stationary Markov controls are obtained in this fashion from (1.1). Also, given any eigenpair (λ^*, V) , $V > 0$, satisfying (1.1) whether we can find an optimal Markov control through a measurable selection. This is also related with the uniqueness of ψ^* . In [11, 18], the authors establish uniqueness of ψ^* under a more restrictive setting (see [18, (A1)][11, (E2)]). The uniqueness of ψ^* in these papers is a consequence of the contraction property of certain map associated to (1.1). Also, this uniqueness result is established among a class of functions belonging to a certain weighted Banach space and arguments of these articles do not give uniqueness in the class of all positive continuous functions. Furthermore, the results of [11, 18] can not be used to obtain a verification result for optimal stationary Markov controls. Let us also mention our result [Theorem 2.4](#) which establishes existence of an optimal stationary Markov control under a near-monotonicity assumption on the cost but does not impose any stability hypothesis.

On the other hand, the literature on risk-sensitive control problems for continuous time CMP are very few, especially for ergodic risk-sensitive control problems. Wei [46], Guo, Liu and Zhang [27] investigate finite horizon risk-sensitive criterion for continuous time CMP taking values in a countable state space. An infinite horizon discounted cost criterion is considered by Guo and Liao [23]. Wei and Chen [47] consider the ergodic risk-sensitive criterion for a finite-state continuous time CMP and establish the existence of an optimal control using the nonlinear eigen-equation (1.1). The articles that are close to the problem we are considering in this paper are by Ghosh and Saha [21], Suresh Kumar and Pal [45], Guo and Huang [26]. [21] studies the problem under a stability hypothesis whereas [45] imposes a near-monotonicity assumption on the cost. Both the articles obtain the existence of a principal eigenfunction satisfying (1.1) and show that any minimizing selector is an optimal Markov control. Recently, Guo and Huang [26] study a similar control problem for continuous time CMP satisfying a blanket geometric-stability condition, and the existence of solutions to (1.1) and the existence of an optimal stationary Markov control are

established. It should be noted that the stability hypothesis in [26, Assumption 3.1] is stronger than our [Assumption 3.4](#). Uniqueness of the value function ψ^* is also established in [26] by imposing a further set of conditions (see Assumption 5.1 and 6.1 there). In this article we do not impose any such conditions to obtain the uniqueness of ψ^* . We take a different approach to attack this problem and establish the existence of a unique eigenfunction and verification result for optimal stationary Markov controls (see [Theorem 3.1](#)), under a blanket stability hypothesis. In [Theorems 3.2](#) and [3.3](#) we consider the problem under a condition of near-monotonicity on the cost function and establish existence of optimal stationary Markov controls. Our approach in this article is inspired from the work of Biswas [16], Arapostathis et. al. [7] (see also [5]) where ergodic risk-sensitive control is studied for non-degenerate controlled diffusion. It should be observed that the ideas of [5, 7, 16] can not be adapted in a straightforward manner to the present setting. These papers use several analytic tools such as Harnack's inequality, Sobolev estimate, monotonicity of Dirichlet principal eigenvalues for their analysis. We do not have similar estimates in hand. On the other hand, our state space being discrete, we get an advantage in the passage of several limits using a standard diagonalization argument.

As mentioned above we also provide a PIA for both discrete and continuous time setup. In their work [17], Borkar and Meyn propose a PIA for norm-like cost function and establish its convergence, provided some additional hypotheses hold [17, Theorem 5.4]. Some of these conditions are not easily verifiable. Ghosh and Saha [21] (see also [26]) prove convergence of PIA for a finite state CMP. Both the papers [17, 21] assume their action space to be finite. In a recent work, Arapostathis, Biswas and Pradhan [6] establish convergence of their proposed PIA for non-degenerate controlled diffusions, provided the diffusion satisfies certain stability hypothesis. In [Section 4](#) we propose a PIA and show that the algorithm converges to the optimal value λ^* . For our result we make use of a characterization of the Perron-Frobenius eigenvalue (see (4.1)).

The rest of the article is organized as follows: In [Section 2](#) we consider the discrete time CMP and our main results of this section are [Theorems 2.1](#) and [2.4](#). [Section 3](#) studies a similar problem for continuous time CMP. Finally, in [Section 4](#) we consider the policy improvement algorithms.

2. RISK-SENSITIVE CONTROL FOR DISCRETE TIME CMP

2.1. Description of the problem. We consider a controlled Markov process $\mathbf{X} := \{X_0, X_1, \dots\}$ on a denumerable state space $S := \{1, 2, \dots\}$ controlled by a control process $\zeta := \{\zeta_0, \zeta_1, \dots\}$ taking values in \mathbb{U} . Here \mathbb{U} is a Borel space endowed with the Borel σ algebra $\mathfrak{B}(\mathbb{U})$. For every $i \in S$, $\mathbb{U}(i) \in \mathfrak{B}(\mathbb{U})$ stands for the nonempty compact set of all admissible actions when the system is at the state i . The space of all admissible state action pairs is given by $\mathcal{K} := \{(i, u) : i \in S, u \in \mathbb{U}(i)\}$. For each $A \in \mathfrak{B}(S)$ the controlled stochastic kernel $P(A|\cdot) : \mathcal{K} \rightarrow [0, 1]$ is Borel measurable. We denote by $c : \mathcal{K} \rightarrow \mathbb{R}_+$ the one-stage cost function. For each $t \in \mathbb{N}$, the space \mathcal{H}_t denotes the *admissible histories* upto time t , where $\mathcal{H}_0 := S$, $\mathcal{H}_t = \mathcal{K} \times \mathcal{H}_{t-1}$. A generic element h_t of \mathcal{H}_t is a vector of the form

$$h_t = (x_0, u_0, x_1, u_1, \dots, x_{t-1}, u_{t-1}, x_t), \quad \text{with } (x_s, u_s) \in \mathcal{K}, \quad 0 \leq s \leq t-1,$$

and $x_t \in S$, denotes the observable history of the process upto time t . We also denote by $\mathfrak{F}_n = \mathfrak{B}(\mathcal{H}_n)$. An *admissible control* is a sequence $\zeta = \{\zeta_0, \zeta_1, \dots\}$ where for each $t \in \mathbb{N}$, $\zeta_t : \mathcal{H}_t \rightarrow \mathbb{U}$ is a measurable map satisfying $\zeta_t(h_t) \in \mathbb{U}(x_t)$, for all $h_t \in \mathcal{H}_t$. The set of all admissible policies is denoted by \mathfrak{U} . It is well known that for a given initial state $i \in S$ and policy $\zeta \in \mathfrak{U}$ there exist unique probability measure \mathbb{P}_i^ζ on $(\Omega, \mathfrak{B}(\Omega))$, where $\Omega = (S \times \mathbb{U})^\infty$, (see [29, p.4], [3]) satisfying the following

$$\mathbb{P}_i^\zeta(X_0 = i) = 1, \quad \text{and} \quad \mathbb{P}_i^\zeta(X_{t+1} \in A | \mathcal{H}_t, \zeta_t) = P(A | X_t, \zeta_t) \quad \forall A \in \mathfrak{B}(S). \quad (2.1)$$

The corresponding expectation operator is denoted by \mathbb{E}_i^ζ . A policy $\zeta \in \mathfrak{U}$ is said to be a Markov policy if $\zeta_t(h_t) = v_t(x_t)$ for all $h_t \in \mathcal{H}_t$, for some measurable map $v_t : S \rightarrow \mathbb{U}$ such that $v_t(i) \in \mathbb{U}(i)$

for all $i \in S$. The set of all Markov policies is denoted by \mathfrak{U}_m . If the map v_t does not have any explicit time dependence, that is, $\zeta_t(h_t) = v(x_t)$ for all $h_t \in \mathcal{H}_t$, then ζ is called a stationary Markov strategy and we denote the set of all stationary Markov strategies by \mathfrak{U}_{sm} . From [29, p.6] (also see [3]), it is easy to see that under any Markov policy $\zeta \in \mathfrak{U}_m$ the corresponding stochastic process \mathbf{X} is strong Markov. For each $\zeta \in \mathfrak{U}$ the ergodic risk-sensitive cost is given by

$$\mathcal{E}_i(c, \zeta) := \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right], \quad (2.2)$$

where \mathbf{X} is the discrete time CMP (DTCMP) corresponding to the control $\zeta \in \mathfrak{U}$, with initial state i . Our aim is to minimize (2.2) over all admissible policies \mathfrak{U} . In other words, we are interested in the quantity

$$\lambda^* = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta). \quad (2.3)$$

A policy $\zeta^* \in \mathfrak{U}$ is said to be optimal if for all $i \in S$

$$\mathcal{E}_i(c, \zeta^*) = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta).$$

One of our chief goals in this article is to characterize all the optimal stationary Markov controls.

Assumption 2.1. We impose the following conditions on the DTCMP

- (a) For each $i \in S$ and any bounded measurable function $f : S \rightarrow \mathbb{R}$ the maps $u \mapsto c(i, u)$ and $u \mapsto \sum_{j \in S} f(j)P(j|i, u)$ are continuous on $\mathbb{U}(i)$.
- (b) There exists a state $i_0 \in S$ such that

$$P(j|i_0, u) > 0 \quad \text{for all } j \in S \setminus \{i_0\}, u \in \mathbb{U}(i_0).$$

Assumption 2.1(a) is a quite routine assumption for discrete time CMP. **Assumption 2.1(b)** will be used to show that the sequence of Dirichlet eigenfunctions does not vanish in the limit (see **Lemma 2.5** below). It is also possible to consider other type of condition instead **Assumption 2.1(b)**. We refer to **Remark 2.3** for further discussion.

A function $g : S \rightarrow \mathbb{R}$ is said to be *norm-like* if for every $\kappa \in \mathbb{R}$ the set $\{i \in S : g(i) \leq \kappa\}$ is either empty or finite. We also impose the following Foster–Lyapunov condition on the dynamics.

Assumption 2.2. We assume that the DTCMP \mathbf{X} is irreducible under every stationary Markov control in \mathfrak{U}_{sm} . In (a) and (b) below the function \mathcal{V} on S takes values in $[1, \infty)$ and \widehat{C} is a positive constant. We assume that one of the following holds.

- (a) For some positive constant $\beta \in (0, 1)$ and a finite set \mathcal{K} it holds that

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j)P(j|i, u) \leq (1 - \beta)\mathcal{V}(i) + \widehat{C} \mathbf{1}_{\mathcal{K}}(i) \quad \forall i \in S. \quad (2.4)$$

Also, assume that $\|c\|_\infty := \sup_{i \in S} \sup_{u \in \mathbb{U}(i)} c(i, u) < \gamma$ where $\beta = (1 - e^{-\gamma})$ (i.e., $\gamma = \log(\frac{1}{1-\beta})$).

- (b) For a finite set \mathcal{K} and a norm-like function $\ell : S \rightarrow \mathbb{R}_+$ it holds that

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j)P(j|i, u) \leq \widehat{C} \mathbf{1}_{\mathcal{K}}(i) + (1 - \beta_i)\mathcal{V}(i) \quad \forall i \in S. \quad (2.5)$$

where $1 - e^{-\ell(i)} = \beta_i$. Moreover, the function $\ell(\cdot) - \max_{u \in \mathbb{U}(\cdot)} c(\cdot, u)$ is norm-like.

(2.5) will be useful to treat problems with unbounded running cost. Among others, (2.5) implies that (2.3) is finite. Similar condition is also used by Balaji and Meyn [8, Theorem 1.2] in the study of multiplicative ergodicity. (2.5) also used in the work of Arapostathis et. al. [7] to study the ergodic risk-sensitive control of diffusions. It is easily seen that $u \mapsto \sum_{j \in S} f(j)P(j|i, u)$ is lower-semicontinuous in $\mathbb{U}(i)$ for all positive $f \in \mathcal{O}(\mathcal{V})$ and $i \in S$, where $\mathcal{O}(\mathcal{V})$ denotes the space

of all functions f satisfying $\sup_{k \in S} \frac{|f(k)|}{\mathcal{V}(k)} < \infty$. By $\mathfrak{o}(\mathcal{V})$ we denote the subset of $\mathcal{O}(\mathcal{V})$ consists of function f satisfying $\lim_{k \rightarrow \infty} \frac{|f(k)|}{\mathcal{V}(k)} = 0$.

Example 2.1. Assumptions 2.1 and 2.2 are satisfied by a large family of controlled Markov chains. To illustrate, we consider the following elementary queueing model

$$Q_{k+1} = [(1 - \theta)Q_k - \zeta_k + A_{k+1}]_+, \quad k \geq 0,$$

where $\theta > 0$ denotes the reneging rate. The control ζ_k takes integer values in some bounded set and $\{A_k, k \geq 1\}$ is an i.i.d. sequence and the support of the common marginal distribution is equal to \mathbb{Z}_+ . Also, assume that $\mathbb{E}[A_1] = a < \infty$. It is easy to see that $i_0 = 0$ satisfies Assumption 2.1(b). Take $\mathcal{V}(i) = i + 1$. Then it is easy to check that for any $v \in \mathfrak{U}_{\text{sm}}$, we have

$$\mathbb{E}_i^v[\mathcal{V}(Q_1)] \leq 1 + (1 - \theta)\mathcal{V}(i) + a.$$

Thus, we can choose $\beta < \theta$ in (2.4).

Furthermore, if we assume that $a_1 := \log \mathbb{E}[e^{\gamma A_1}] < \infty$ for some $\gamma > 0$, then letting $\mathcal{V}(i) = e^{\gamma i}$ we see that

$$\mathbb{E}_i^v[\mathcal{V}(Q_1)] \leq e^{\gamma(1-\theta)i} \mathbb{E}[e^{\gamma A_1}] = e^{-\theta i + a_1} \mathcal{V}(i) \leq \widehat{C} \mathbf{1}_{\mathcal{K}} + (1 - (1 - e^{-[\theta i - a_1]_+})) \mathcal{V}(i),$$

where $\mathcal{K} = \{j \in S : \theta j - a_1 \leq 0\}$ and $\widehat{C} = \max_{j \in \mathcal{K}} e^{-\theta j + a_1} \mathcal{V}(j)$. Thus (2.5) holds for the choice of $\beta_i = (1 - e^{-[\theta i - a_1]_+})$, $i \in S$.

It can be easily shown that \mathcal{E}_i is finite for any $\zeta \in \mathfrak{U}$ under Assumption 2.2.

Lemma 2.1. Grant Assumption 2.2. Then there exists a constant κ such that

$$\mathcal{E}_i(c, \zeta) \leq \kappa \quad \text{for all } i \in S, \zeta \in \mathfrak{U}. \quad (2.6)$$

Proof. We only provide a proof under Assumption 2.2(b) and the proof under Assumption 2.2(a) is obvious since c is bounded. Since \mathcal{K} is finite, for some constant κ_1 we can write (2.5) as

$$\sup_{u \in \mathfrak{U}(i)} \sum_{j \in S} \mathcal{V}(j) P(j|i, u) \leq e^{\kappa_1 - \ell(i)} \mathcal{V}(i) \quad \forall i \in S. \quad (2.7)$$

Thus, by successive conditioning and using (2.1), we deduce from (2.7) that

$$\mathbb{E}_i^\zeta \left[e^{\sum_{t=0}^{T-1} (\ell(X_t) - \kappa_1)} \mathcal{V}(X_T) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in S. \quad (2.8)$$

Since $\mathcal{V} \geq 1$, taking logarithm on both side of (2.8), dividing both sides by T and letting $T \rightarrow \infty$ we obtain

$$\mathcal{E}_i(\ell, \zeta) \leq \kappa_1 \quad \text{for all } i \in S.$$

On the other hand, $\ell - \max_{u \in \mathfrak{U}(\cdot)} c(\cdot, u)$ is norm-like. Thus, for some constant κ_2 , we have $\max_{u \in \mathfrak{U}(i)} c(i, u) \leq \ell(i) + \kappa_2$ for all $i \in S$. Hence we obtain

$$\mathcal{E}_i(c, \zeta) \leq \kappa_1 + \kappa_2 \quad \text{for all } i \in S, \zeta \in \mathfrak{U}.$$

This completes the proof. \square

Now we are ready to state our first main result of this section.

Theorem 2.1. Grant Assumptions 2.1 and 2.2. Then the following hold.

- (i) There exists a unique positive function ψ^* , $\psi^*(i_0) = 1$, (where i_0 is a reference state as in Assumption 2.1) satisfying

$$e^{\lambda^*} \psi^*(i) = \min_{u \in \mathfrak{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \text{for } i \in S. \quad (2.9)$$

(ii) A stationary Markov control $v \in \mathfrak{U}_{\text{sm}}$ is optimal if and only if it satisfies

$$\min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] = \left[e^{c(i,v(i))} \sum_{j \in S} \psi^*(j) P(j|i, v(i)) \right] \quad \text{for } i \in S. \quad (2.10)$$

As mentioned in the introduction the existence part of [Theorem 2.1](#) is not new and existence of ψ^* and stationary optimal Markov control has been obtained for a more general class of DTCMP; for instance, CMP taking values in a general state space. Our contribution in [Theorem 2.1](#) comes from the uniqueness of ψ^* and the verification result which shows that all the optimal stationary Markov controls are nothing but measurable selectors of (2.10). The rest of this section is devoted to the proof of [Theorem 2.1](#). Our approach is quite different from the one considered in [17–19]. We take a more direct approach by considering the *Dirichlet eigenvalue* problems on finite sets and then pass to the limit by increasing the finite sets to S . This approach was first considered by Biswas in [16] to study the risk-sensitive control problem for controlled diffusions with a near-monotone cost function. The key steps in proving [Theorem 2.1](#) can be summarized as follows: we consider a collection of increasing finite sets \mathcal{D}_n , increasing to S , and find a Dirichlet eigenpair (ρ_n, ψ_n) in \mathcal{D}_n for every n (see [Lemma 2.2](#) below). Then in [Lemma 2.5](#) we show that there is a subsequence of these eigenpairs converging to a positive eigenpair (ψ^*, ρ) of (2.9) and $\rho = \lambda^*$ (see [Lemma 2.7](#)). In [Lemma 2.7](#) we then show that ψ^* is unique upto a normalizing constant.

Let \mathcal{D} be a finite set in S such that $i_0 \in \mathcal{D}$. Define a space

$$\mathcal{B}_{\mathcal{D}} = \{f : S \rightarrow \mathbb{R} \mid f \text{ is Borel measurable and } f(i) = 0 \forall i \in \mathcal{D}^c\}.$$

We begin with the following standard result which is required to apply Kreĭn-Rutman theorem in [Lemma 2.2](#).

Proposition 2.1. *Suppose $c < 0$ in \mathcal{D} . Then for each $f \in \mathcal{B}_{\mathcal{D}}$, there exists a unique solution $\phi \in \mathcal{B}_{\mathcal{D}}$ satisfying*

$$\min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \phi(j) P(j|i, u) + f(i) \right] = \phi(i), \quad \forall i \in \mathcal{D}, \quad \text{and } \phi(i) = 0 \quad \forall i \in \mathcal{D}^c. \quad (2.11)$$

Moreover, the unique solution ϕ is given by

$$\phi(i) = \inf_{\zeta \in \mathfrak{U}_{\text{m}}} \mathbb{E}_i^{\zeta} \left[\sum_{k=0}^{\tau-1} e^{\sum_{t=0}^{k-1} c(X_t, \zeta_t)} f(X_k) \right] \quad \forall i \in \mathcal{D}, \quad (2.12)$$

where $\tau = \tau(\mathcal{D}) = \inf\{t > 0 : X_t \notin \mathcal{D}\}$ denotes the first exit time from \mathcal{D} .

Proof. Fix $f \in \mathcal{B}_{\mathcal{D}}$. Define a map $\mathcal{T} : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ as follows: for $g \in \mathcal{B}_{\mathcal{D}}$, $\mathcal{T}g$ is given by

$$\mathcal{T}g(i) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} g(j) P(j|i, u) + f(i) \right] \quad i \in \mathcal{D}, \quad \mathcal{T}g(i) = 0 \quad \text{for } i \in \mathcal{D}^c.$$

Letting $\|g\|_{\mathcal{D}} = \max\{|g(i)| : i \in \mathcal{D}\}$ we get from above that

$$\|\mathcal{T}g_1 - \mathcal{T}g_2\|_{\mathcal{D}} \leq \vartheta \|g_1 - g_2\|_{\mathcal{D}} \quad \text{where } \vartheta = \max_{i \in \mathcal{D}} \max_{u \in \mathbb{U}(i)} e^{c(i,u)} < 1.$$

Thus \mathcal{T} is a contraction. From Banach fixed point theorem we find a unique ϕ satisfying (2.11). (2.12) is standard and follows from Dynkin's formula. \square

Next we recall a version of the nonlinear Kreĭn-Rutman theorem from [4, Section 3.1] (cf. [36]).

Theorem 2.2. *Let \mathcal{X} be an ordered Banach space and $\mathcal{C} \subset \mathcal{X}$ be a nonempty closed cone satisfying $\mathcal{X} = \mathcal{C} - \mathcal{C} := \{x - y : x, y \in \mathcal{C}\}$. Let $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be an order-preserving, 1-homogeneous (that is, $\mathcal{T}(\alpha x) = \alpha \mathcal{T}(x)$ for all $x \in \mathcal{X}$ and $\alpha \in \mathbb{R}_+$), completely continuous map such that for some nonzero $\xi \in \mathcal{C}$ and $M > 0$, we have $M\mathcal{T}(\xi) \succeq \xi$. Then there exist nontrivial $x_0 \in \mathcal{C}$ and $\lambda_0 > 0$ satisfying $\mathcal{T}x_0 = \lambda_0 x_0$.*

Here \succeq denotes the partial ordering in \mathcal{X} with respect to the cone \mathcal{C} , that is, $x \succeq y$ if and only if $x - y \in \mathcal{C}$. Also, we recall that a map $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is called completely continuous if it is continuous and compact. We let $\mathcal{C} = \mathcal{B}_{\mathcal{D}}^+ \subset \mathcal{B}_{\mathcal{D}}$, the cone of nonnegative functions vanishing outside \mathcal{D} . Applying [Theorem 2.2](#) we then establish the existence of an eigenpair to the Dirichlet problem in \mathcal{D} .

Lemma 2.2. *There exists a pair $(\lambda_{\mathcal{D}}, \psi_{\mathcal{D}}) \in \mathbb{R}_+ \times \mathcal{B}_{\mathcal{D}}^+$, $\psi_{\mathcal{D}} \neq 0$, satisfying*

$$\lambda_{\mathcal{D}} \psi_{\mathcal{D}}(i) = \min_{u \in \mathcal{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_{\mathcal{D}}(j) P(j|i, u) \right]. \quad (2.13)$$

Moreover, we have

$$\rho_{\mathcal{D}} := \log \lambda_{\mathcal{D}} \leq \inf_{\zeta \in \mathcal{U}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right], \quad (2.14)$$

for all i satisfying $\psi_{\mathcal{D}}(i) \neq 0$.

Proof. Suppose $c < -\delta$ in \mathcal{D} , for some positive constant δ . Define an operator $\mathcal{T} : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ by

$$\mathcal{T}(f)(i) := \phi(i) = \inf_{\zeta \in \mathcal{U}_m} \mathbb{E}_i^{\zeta} \left[\sum_{k=0}^{\tau-1} e^{\sum_{t=0}^{k-1} c(X_t, \zeta_t)} f(X_k) \right] \quad \text{for } i \in \mathcal{D}, \quad \text{and } \phi(i) = 0 \quad \text{for } i \in \mathcal{D}^c. \quad (2.15)$$

In view of [Proposition 2.1](#), it is clear that the map \mathcal{T} is well defined. From (2.15), it is also easily seen that $\mathcal{T}(\lambda f) = \lambda \mathcal{T}(f)$ for all $\lambda \geq 0$. Again, since $c < -\delta$, from the definition of \mathcal{T} it is straightforward to check that

$$\|\mathcal{T}(f) - \mathcal{T}(g)\|_{\mathcal{D}} \leq \kappa_1 \|f - g\|_{\mathcal{D}},$$

for some constant $\kappa_1 > 0$. Therefore $\mathcal{T} : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ is continuous.

Let $f, g \in \mathcal{B}_{\mathcal{D}}$ such that $f \succeq g$. Then, we have

$$(\mathcal{T}(f) - \mathcal{T}(g))(i) \geq \inf_{\zeta \in \mathcal{U}_m} \mathbb{E}_i^{\zeta} \left[\sum_{k=0}^{\tau-1} e^{\sum_{t=0}^{k-1} c(X_t, \zeta_t)} (f - g)(X_k) \right] \geq 0.$$

Thus, we obtain $\mathcal{T}(f) \succeq \mathcal{T}(g)$. Let $\{f_m\}$ be a bounded sequence in $\mathcal{B}_{\mathcal{D}}$. From (2.15) we then have $\|\phi_m\|_{\mathcal{D}} \leq \kappa_2$ for some constant $\kappa_2 > 0$, where $\phi_m = \mathcal{T}f_m$. Then by a standard diagonalization argument we deduce that there exists a $\phi \in \mathcal{B}_{\mathcal{D}}$ satisfying $\|\phi_{m_k} - \phi\|_{\mathcal{D}} \rightarrow 0$, as $m_k \rightarrow \infty$, for some subsequence $\{\phi_{m_k}\}$. This implies that $\mathcal{T} : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ is compact and hence \mathcal{T} is completely continuous.

Let $f \in \mathcal{B}_{\mathcal{D}}$ be such that $f(i_0) = 1$ and $f(j) = 0$ for all $j \neq i_0$. By (2.11) we then have $\phi(i_0) \geq f(i_0) > 0$. Thus $\mathcal{T}f \succeq f$.

Thus we can apply [Theorem 2.2](#) to find a nonzero $\psi_{\mathcal{D}} \in \mathcal{C}$ and $\lambda > 0$ such that $\mathcal{T}(\psi_{\mathcal{D}}) = \lambda \psi_{\mathcal{D}}$. Applying [Proposition 2.1](#) we obtain

$$\min_{u \in \mathcal{U}(i)} \left[\lambda e^{c(i,u)} \sum_{j \in S} \psi_{\mathcal{D}}(j) P(j|i, u) + \psi_{\mathcal{D}}(i) \right] = \lambda \psi_{\mathcal{D}}(i), \quad \forall i \in \mathcal{D}, \quad \text{and } \psi_{\mathcal{D}}(i) = 0 \quad \forall i \in \mathcal{D}^c.$$

Defining $\lambda_{\mathcal{D}} = \frac{\lambda-1}{\lambda}$, we get from above that

$$\min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_{\mathcal{D}}(j) P(j|i, u) \right] = \lambda_{\mathcal{D}} \psi_{\mathcal{D}}(i) \quad \forall i \in \mathcal{D}. \quad (2.16)$$

Since $\psi_{\mathcal{D}} \geq 0$ and $\psi_{\mathcal{D}}(i) > 0$ for some $i \in \mathcal{D}$, it follows from (2.16) that $\lambda_{\mathcal{D}} \geq 0$. For general c , we can start with $c - \|c\|_{\mathcal{D}} - \delta$ and then multiply both sides of (2.16) with $e^{\|c\|_{\mathcal{D}} + \delta}$. This gives (2.13).

To prove (2.14) consider a state i satisfying $\psi_{\mathcal{D}}(i) \neq 0$. Note that there is nothing to prove if $\lambda_{\mathcal{D}} = 0$. So we assume $\lambda_{\mathcal{D}} > 0$. Now, due to (2.1), for any bounded function F we have

$$Y_n := \sum_{k=1}^{n-1} \left(e^{\sum_{t=0}^{k-1} c(X_t, \zeta_t)} F(X_k) - e^{\sum_{t=0}^{k-1} c(X_t, \zeta_t)} \sum_{j \in S} F(j) P(j|X_{k-1}, \zeta_{k-1}) \right) \quad (2.17)$$

is a $\mathfrak{F}_n = \mathfrak{B}(\mathcal{H}_n)$ martingale. Thus by optional sampling theorem, $\{Y_{n \wedge \tau}, \mathfrak{F}_{n \wedge \tau}\}$ is also a Martingale. Since, by (2.16),

$$e^{\sum_{t=0}^k (c(X_t, \zeta_t) - \rho_{\mathcal{D}})} \sum_{j \in S} \psi_{\mathcal{D}}(j) P(j|X_k, \zeta_k) - e^{\sum_{t=0}^{k-1} (c(X_t, \zeta_t) - \rho_{\mathcal{D}})} \psi_{\mathcal{D}}(X_k) \geq 0 \quad \text{for } k = 0, \dots, \tau - 1,$$

($\rho_{\mathcal{D}}$ is given by (2.14)) it follows that

$$\psi_{\mathcal{D}}(i) \leq \mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{T-1} (c(X_t, \zeta_t) - \rho_{\mathcal{D}})} \psi_{\mathcal{D}}(X_T) \mathbb{1}_{\{T < \tau\}} \right].$$

This in turn, gives

$$e^{T\rho_{\mathcal{D}}} \psi_{\mathcal{D}}(i) \leq \max_{\mathcal{D}} \psi_{\mathcal{D}} \mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right].$$

Now taking logarithm both sides, dividing by T and letting $T \rightarrow \infty$, we obtain

$$\rho_{\mathcal{D}} \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right].$$

Since $\zeta \in \mathfrak{U}$ is arbitrary we have (2.14). This completes the proof. \square

Now we prove certain estimate which will play important role in our subsequent analysis. Let \mathcal{D}_n be an increasing sequence of finite subsets of S such that $\cup_{n=1}^{\infty} \mathcal{D}_n = S$. With no loss of generality, we assume that $i_0 \in \mathcal{D}_n$ for all n .

Lemma 2.3. *Suppose that Assumption 2.2 holds and consider a finite set $\mathcal{B} \subset S$ containing K . Then for any $\zeta \in \mathfrak{U}$ we have the following.*

(i) Under Assumption 2.2(a) we have

$$\mathbb{E}_i^{\zeta} \left[e^{\gamma \check{\tau}} \mathcal{V}(X_{\check{\tau}}) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in \mathcal{B}^c, \quad (2.18)$$

where $\check{\tau} = \check{\tau}(\mathcal{B}) = \inf\{t > 0 : X_t \in \mathcal{B}\}$.

(ii) Under Assumption 2.2(b) we have

$$\mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{\check{\tau}-1} \ell(X_t)} \mathcal{V}(X_{\check{\tau}}) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in \mathcal{B}^c. \quad (2.19)$$

Proof. We only show (ii) and (i) follows by setting $\ell = \gamma$. Fix \mathcal{D}_n so that $\mathcal{B} \subset \mathcal{D}_n$ and $i \in \mathcal{D}_n \setminus \mathcal{B}$. Choose m large enough so that

$$\sup_{u \in \mathbb{U}(i)} e^{\ell(i)} \sum_{j \in S} \mathcal{V}_m(j) P(j|i, u) \leq \mathcal{V}_m(i) \quad \text{for all } i \in \mathcal{D}_n \setminus \mathcal{B}, \quad (2.20)$$

where $\mathcal{V}_m = \min\{\mathcal{V}, m\}$. Taking $F = \mathcal{V}_m$ in (2.17), it follows that $\{Y_{t \wedge \tau_o}, \mathfrak{F}_{t \wedge \tau_o}\}$ is a martingale where $\tau_o = \check{\tau} \wedge \tau_n$ and τ_n is the first exit time of \mathbf{X} from \mathcal{D}_n . Using (2.20) we thus obtain

$$\mathbb{E}_i^{\zeta} \left[e^{\sum_{t=0}^{T \wedge \check{\tau} \wedge \tau_n - 1} \ell(X_t)} \mathcal{V}_m(X_{T \wedge \check{\tau} \wedge \tau_n}) \right] \leq \mathcal{V}_m(i).$$

First we let $m \rightarrow \infty$ and then $T \rightarrow \infty$ to obtain

$$\mathbb{E}_i^\zeta \left[e^{\sum_{t=0}^{\tilde{\tau} \wedge \tau_n - 1} \ell(X_t)} \mathcal{V}(X_{\tilde{\tau}}) \right] \leq \mathcal{V}(i).$$

Now let $n \rightarrow \infty$ and use Fatou's lemma to obtain (2.19). \square

Denote by (ρ_n, ψ_n) the eigenpair in the domain \mathcal{D}_n , obtained by Lemma 2.2. Next we are interested to find a limit of these eigenpairs as $n \rightarrow \infty$. Recall that in case of nondegenerate controlled diffusion such limits are easily obtained by applying Harnack's inequality and monotonicity property ρ_n (cf. [16, Lemma 2.1],[5, Theorem 3.4]). Such tools are not available in the current situation. Below we produce a different argument to pass this limit.

Lemma 2.4. *Grant Assumptions 2.1 and 2.2. Let (ρ_n, ψ_n) be the Dirichlet eigenpair in \mathcal{D}_n satisfying*

$$e^{\rho_n} \psi_n(i) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] \quad \forall i \in \mathcal{D}_n. \quad (2.21)$$

Then the following hold.

(i)

$$\limsup_{n \rightarrow \infty} \rho_n \leq \inf_{\zeta \in \mathfrak{U}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_{i_0}^\zeta \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right]. \quad (2.22)$$

Furthermore, ρ_n is bounded for all large n .

(ii) $\liminf_{n \rightarrow \infty} \rho_n \geq 0$.

Proof. First we consider (i). (2.22) follows from (2.14) since $\psi_n(i_0) > 0$ for all n , by Assumption 2.1(b). In view of Lemma 2.2 and (2.6), we see that ρ_n is bounded from above by κ . Next we show that ρ_n is bounded below. Suppose, on the contrary, that along a subsequence $\rho_n \rightarrow -\infty$ as $n \rightarrow \infty$. This implies that $\rho_n < 0$ for all large enough n . Using Assumption 2.1(b) and (2.21), we have $\psi_n(i_0) > 0$. Dividing ψ_n by $\psi_n(i_0)$ we can ensure that $\psi_n(i_0) = 1$ for all n . Rewriting (2.21), we obtain

$$1 = \psi_n(i_0) = e^{-\rho_n} \min_{u \in \mathbb{U}(i_0)} \left[e^{c(i_0,u)} \sum_{j \in S} \psi_n(j) P(j|i_0, u) \right]. \quad (2.23)$$

Since $(c(i_0, u) - \rho_n) > 0$ for all n large, we get

$$1 \geq \sum_{j \in S} \psi_n(j) P(j|i_0, \hat{v}_n(i_0)), \quad (2.24)$$

for any minimizing selector \hat{v}_n of (2.23). This in turn, implies that

$$\psi_n(j) \leq \sup_{u \in \mathbb{U}(i_0)} \frac{1}{P(j|i_0, u)} \quad \text{for } j \neq i_0,$$

for all n large. Applying a standard diagonalization argument we find a non-negative function ψ with $\psi(i_0) = 1$ such that along a further subsequence $\psi_n \rightarrow \psi$ componentwise. Also, since $\mathbb{U}(i)$ is compact for each $i \in S$, we have $\hat{v}_n \rightarrow \hat{v}$ along a further subsequence. Hence letting $n \rightarrow \infty$ in (2.24) we obtain

$$1 \geq \sum_{j \in S} \psi(j) P(j|i_0, \hat{v}(i_0)). \quad (2.25)$$

Writing (2.21) as

$$e^{\rho_n} \psi_n(i) \geq \left[\sum_{j \in S} \psi_n(j) P(j|i, v_n(i)) \right]$$

and letting $n \rightarrow \infty$ we obtain

$$0 \geq \sum_{j \in S} \psi(j) P(j|i, \hat{v}(i)).$$

Choosing $i = i_0$ and applying [Assumption 2.1\(b\)](#) it follows that $\psi(i_0) = 1$ and $\psi(j) = 0$ for all $j \neq i_0$. It is also easily seen that $\{\psi(X_n), \mathfrak{F}_n\}$ is a super-martingale where \mathbf{X} is the Markov process under the stationary Markov control \hat{v} . Hence by Doob's martingale convergence theorem $\psi(X_n) \rightarrow Y$ almost surely, as $n \rightarrow \infty$. On the other hand, \mathbf{X} is recurrent, which follows from [Assumption 2.2](#), \mathbf{X} visits every state (in particular, i_0) of S infinitely often. Thus, $\psi(X_n)$ can not converge. This is a contradiction. Hence ρ_n must be bounded from below. This completes the proof of (i).

Next we consider (ii). Suppose that $\hat{\rho} = \liminf_{n \rightarrow \infty} \rho_n < 0$. Then along a suitable subsequence ρ_n converges to $\hat{\rho}$. Hence, for all n large enough we have $(c(i, u) - \rho_n) > 0$. Thus, repeating the above argument we find a nonnegative function $\phi : S \rightarrow \mathbb{R}$ with $\phi(i_0) = 1$ satisfying (see [\(2.25\)](#))

$$\phi(i) \geq \mathbb{E}_i^{\hat{v}} [\phi(X_1)] \quad \forall i \in S,$$

for some stationary Markov control \hat{v} . This in turn implies that $\{\phi(X_t)\}$ is a supermartingale and therefore, the above argument gives us $\phi \equiv 1$. Now passing limit in [\(2.23\)](#) and using Fatou's lemma we have

$$1 = \phi(i) \geq e^{c(i, \hat{v}(i)) - \hat{\rho}} > 1.$$

This is a contradiction. This gives us (ii). \square

Remark 2.1. It should be observed from the proof of [Lemma 2.4](#) that if we assume $\inf_{\zeta \in \mathcal{U}} \mathcal{E}_i(c, \zeta)$ to be finite for every i then the conclusion of [Lemma 2.4](#) holds, provided the CMP is recurrent under every stationary Markov control and [Assumption 2.1](#) holds.

Now with the help of [Lemma 2.4](#) we can pass the limit in (ρ_n, ψ_n) to obtain the following result.

Lemma 2.5. *Grant [Assumptions 2.1](#) and [2.2](#). Then there exists $(\rho, \psi^*) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi^* > 0$, satisfying*

$$e^\rho \psi^*(i) = \min_{u \in \mathcal{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.26)$$

Moreover, we have the following.

(i)

$$\rho \leq \inf_{\zeta \in \mathcal{U}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\sum_{t=0}^{T-1} c(X_t, \zeta_t)} \right] \quad \text{for all } i \in S. \quad (2.27)$$

In particular, $\rho \leq \inf_{i \in S} \inf_{\zeta \in \mathcal{U}} \mathcal{E}_i(c, \zeta) = \lambda^*$.

(ii) *There exists a finite set \mathcal{B} containing \mathcal{K} such that for any minimizing selector v^* of [\(2.26\)](#), it holds that*

$$\psi^*(i) = \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \rho)} \psi^*(X_{\check{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.28)$$

Proof. Since $\{\rho_n\}$ is a bounded sequence and $\liminf_{n \rightarrow \infty} \rho_n \geq 0$, by [Lemma 2.4](#), there exist a subsequence, denoted by $\{\rho_n\}$ itself, such that $\rho_n \rightarrow \rho$, as $n \rightarrow \infty$, and $\rho \geq 0$. Again, since $c \geq 0$ we can choose a finite set \mathcal{B} containing \mathcal{K} such that

- under [Assumption 2.2\(a\)](#), since $\|c\|_\infty < \gamma$, we have

$$\left(\max_{u \in \mathcal{U}(i)} c(i, u) - \rho_n \right) < \gamma \quad \forall i \in \mathcal{B}^c, \quad \text{for all large } n; \quad (2.29)$$

- under [Assumption 2.2\(b\)](#), since $\ell(\cdot) - \max_{u \in \mathcal{U}(\cdot)} c(\cdot, u)$ is norm-like, we have

$$\left(\max_{u \in \mathcal{U}(i)} c(i, u) - \rho_n \right) < \ell(i) \quad \forall i \in \mathcal{B}^c, \quad \text{for all large } n. \quad (2.30)$$

Now we scale ψ_n by multiplying a suitable scalar so that it touches \mathcal{V} from below. To do so, define

$$\theta_n = \sup\{\kappa > 0 : (\mathcal{V} - \kappa\psi_n) > 0 \text{ in } S\}.$$

Since ψ_n vanishes in \mathcal{D}_n^c and $\psi_n \geq 0$, it is easily seen that θ_n is finite. We replace ψ_n by $\theta_n\psi_n$ and claim that ψ_n touches \mathcal{V} inside \mathcal{B} . Suppose, on the contrary, that the claim is not true. Then there exists a state $i_1 \in \mathcal{B}^c$ so that $(\mathcal{V} - \psi_n)(i_1) = 0$ and $\mathcal{V} - \psi_n > 0$ in $\mathcal{B} \cup \mathcal{D}_n^c$. For all n large enough, from (2.21) and (2.29), for any $\zeta \in \mathfrak{U}_{\text{sm}}$ we have under Assumption 2.2(a)

$$\begin{aligned} \psi_n(i_1) &\leq \mathbb{E}_{i_1}^\zeta \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge N-1} (c(X_t, \zeta_t) - \rho_n)} \psi_n(X_{(\check{\tau}(\mathcal{B}) \wedge N)}) \mathbf{1}_{\{\check{\tau}(\mathcal{B}) \wedge N < \tau_n\}} \right] \\ &\leq \mathbb{E}_{i_1}^\zeta \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge N-1} \gamma} \psi_n(X_{(\check{\tau}(\mathcal{B}) \wedge N)}) \mathbf{1}_{\{\check{\tau}(\mathcal{B}) \wedge N < \tau_n\}} \right], \end{aligned}$$

where $\tau_n = \tau(\mathcal{D}_n)$. Since $\psi_n \leq \mathcal{V}$, using Lemma 2.3 and dominated convergence, we let $N \rightarrow \infty$ to obtain

$$\psi_n(i_1) \leq \mathbb{E}_{i_1}^\zeta \left[e^{\gamma \check{\tau}(\mathcal{B})} \psi_n(X_{\check{\tau}(\mathcal{B})}) \right]. \quad (2.31)$$

Combining (2.18) and (2.31), we deduce that

$$0 = (\mathcal{V} - \psi_n)(i_1) \geq \mathbb{E}_{i_1}^\zeta \left[e^{\gamma \check{\tau}(\mathcal{B})} (\mathcal{V} - \psi_n)(X_{\check{\tau}(\mathcal{B})}) \right] > 0.$$

This is a contradiction. Thus ψ_n touches \mathcal{V} inside \mathcal{B} . In view of (2.30), it is easily seen that a similar conclusion holds under Assumption 2.2(b).

Thus, by a standard diagonalization argument, there exist $\psi^* \leq \mathcal{V}$ such that $\psi_n(i) \rightarrow \psi^*(i)$, as $n \rightarrow \infty$, for all $i \in S$. Next we show that

$$\lim_{n \rightarrow \infty} \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] \rightarrow \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.32)$$

Since $\psi_n \leq \mathcal{V}$, by dominated convergence theorem, for any $\zeta_0 \in \mathbb{U}(i)$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] &\leq \limsup_{n \rightarrow \infty} \left[e^{c(i, \zeta_0)} \sum_{j \in S} \psi_n(j) P(j|i, \zeta_0) \right] \\ &= \left[e^{c(i, \zeta_0)} \sum_{j \in S} \psi^*(j) P(j|i, \zeta_0) \right]. \end{aligned}$$

From the arbitrariness of ζ_0 it then follows that

$$\limsup_{n \rightarrow \infty} \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] \leq \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right]. \quad (2.33)$$

Again, let v_n be a minimizing selector of (2.21). Since $\mathbb{U}(i)$ is compact, by a standard diagonalization argument we have along some subsequence $v_n(i)$ converges to $v(i)$ in $\mathbb{U}(i)$ for all $i \in S$. Thus by generalized Fatou's lemma [30, Lemma 8.3.7], we deduce that

$$\lim_{n \rightarrow \infty} \left[e^{c(i, v_n(i))} \sum_{j \in S} \psi_n(j) P(j|i, v_n(i)) \right] = \left[e^{c(i, v(i))} \sum_{j \in S} \psi^*(j) P(j|i, v(i)) \right]. \quad (2.34)$$

Combining (2.33) and (2.34) we thus obtain (2.32). Therefore, letting $n \rightarrow \infty$ in (2.21), we see that the pair $(\rho, \psi^*) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$ satisfies

$$e^\rho \psi^*(i) = \inf_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.35)$$

Moreover, since $(\mathcal{V} - \psi_n) = 0$ at some point in \mathcal{B} for all n large enough, we have $(\mathcal{V} - \psi^*) = 0$ at some point in \mathcal{B} . Since $\mathcal{V} \geq 1$, we have ψ is nontrivial. In fact, we have $\psi^* > 0$. If not, let $\psi^*(j) = 0$ for some $j \in S$. Then for any minimizing selector v of (2.35), ψ^* satisfies

$$e^{c(i,v(i))-\rho} \mathbb{E}_i^v [\psi^*(X_1)] = \psi^*(i) \quad \forall i \in S. \quad (2.36)$$

Let $\psi^*(i) > 0$ for some $i \in \mathcal{B}$. Since \mathbf{X} is irreducible under v , there exists a $n \in \mathbb{N}$ and distinct $i_1, i_2, \dots, i_n \in S$ satisfying

$$P(i|i_n, v(i_n))P(i_n|i_{n-1}, v(i_{n-1})) \cdots P(i_1|j, v(j)) > 0.$$

From (2.36) this implies that $0 = \psi^*(j) = \psi^*(i_1) = \dots = \psi^*(i_n) = \psi^*(i)$ which is a contradiction. Thus we must have $\psi^* > 0$ in S . This gives us (2.26).

Next we consider (i). Since $\psi_n \rightarrow \psi^* > 0$ as $n \rightarrow \infty$, for any given state $i \in S$ we have $\psi_n(i) > 0$ for all large n . Now (2.27) follows from (2.14) and the fact $\rho_n \rightarrow \rho$, as $n \rightarrow \infty$.

Now we consider (ii). Suppose that Assumption 2.2(a) holds. Let $v^* \in \mathfrak{U}_{\text{sm}}$ be a minimizing selector of (2.35). Using (2.21), for all n large enough, we have (see (2.17))

$$\begin{aligned} \psi_n(i) &\leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge N-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_{\check{\tau}(\mathcal{B}) \wedge N}) \mathbb{1}_{\{\check{\tau}(\mathcal{B}) \wedge N < \tau_n\}} \right] \\ &\leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_{\check{\tau}(\mathcal{B})}) \mathbb{1}_{\{\check{\tau}(\mathcal{B}) < \tau_n \wedge N\}} \right] \\ &\quad + \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{N-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_N) \mathbb{1}_{\{N < \tau_n \wedge \check{\tau}(\mathcal{B})\}} \right] \end{aligned} \quad (2.37)$$

for all $i \in \mathcal{B}^c \cap \mathcal{D}_n$. Since $\psi_n \leq \mathcal{V}$, we obtain

$$\begin{aligned} \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{N-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_N) \mathbb{1}_{\{N < \tau_n \wedge \check{\tau}(\mathcal{B})\}} \right] &\leq e^{N(\|c\|_\infty - \rho_n - \gamma)} \mathbb{E}_i^{v^*} \left[e^{N\gamma} \mathcal{V}(X_N) \mathbb{1}_{\{N < \tau_n \wedge \check{\tau}(\mathcal{B})\}} \right] \\ &\leq e^{N(\|c\|_\infty - \rho_n - \gamma)} \mathcal{V}(i) \end{aligned} \quad (2.38)$$

by (2.18). Letting $N \rightarrow \infty$ in (2.38), it follows that

$$\lim_{N \rightarrow \infty} \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{N-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_N) \mathbb{1}_{\{N < \tau_n \wedge \check{\tau}(\mathcal{B})\}} \right] = 0.$$

Thus taking limit $N \rightarrow \infty$ in (2.37), we deduce

$$\psi_n(i) \leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \rho_n)} \psi_n(X_{\check{\tau}(\mathcal{B})}) \mathbb{1}_{\{\check{\tau}(\mathcal{B}) < \tau_n\}} \right]. \quad (2.39)$$

Again, since $\psi_n \leq \mathcal{V}$, using Lemma 2.3 and dominated convergence theorem, we let $n \rightarrow \infty$ in (2.39) to obtain

$$\psi^*(i) \leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \rho)} \psi^*(X_{\check{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.40)$$

On the other hand, from (2.35), we have

$$\psi^*(i) = \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge N-1} (c(X_t, v^*(X_t)) - \rho)} \psi^*(X_{\check{\tau}(\mathcal{B}) \wedge N}) \right] \quad \forall i \in \mathcal{B}^c.$$

Thus letting $N \rightarrow \infty$ and using Fatou's lemma, we obtain

$$\psi^*(i) \geq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \rho)} \psi^*(X_{\check{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.41)$$

Combining (2.40) and (2.41) we get (2.28). A similar argument works under Assumption 2.2(b). Hence the proof. \square

Remark 2.2. It is easy to check that one can also apply the argument of Lemma 2.5 for every stationary Markov control. More precisely, if we impose Assumptions 2.1 and 2.2, then for every Markov control $v \in \mathfrak{U}_{\text{sm}}$ there exists $(\rho_v, \psi_v) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi_v > 0$, satisfying

$$e^{\rho_v} \psi_v(i) = e^{c(i,v(i))} \sum_{j \in S} \psi_v(j) P(j|i, v(i)) \quad \forall i \in S. \quad (2.42)$$

Furthermore, $\rho_v \leq \inf_i \mathcal{E}_i(c, v)$ and for some finite set $\mathcal{B} \supset \mathcal{K}$

$$\psi_v(i) = \mathbb{E}_i^v \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v(X_t)) - \rho_v)} \psi_v(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.43)$$

Next we show that $\rho = \lambda^*$. The proof in the controlled diffusion setting uses Girsanov transformation and ergodicity property of the twisted process (cf. Arapostathis et. al. [7, Theorem 3.2]). It seems difficult to apply a similar approach in the present setting. To overcome this difficulty we follow an approach used in [2, Lemma 4.4]. It should also be kept in mind that the Dirichlet eigenvalues obtained in [2] has certain monotonicity property which is not very clear in the present setting.

Lemma 2.6. *Assume Assumption 2.1 and Assumption 2.2(a). Let (ρ, ψ^*) be the eigenpair in (2.26). Then $\rho < \|c\|_\infty$, provided c is not a constant.*

Proof. Note that $\rho \leq \|c\|_\infty$, by (2.27). We suppose, on the contrary, that $\rho = \|c\|_\infty$. Let v^* be a minimizing selector of (2.26). It then follows from (2.26) that

$$\psi^*(i) = e^{c(i, v^*(i)) - \rho} \mathbb{E}_i^{v^*} [\psi^*(X_1)] \leq \mathbb{E}_i^{v^*} [\psi^*(X_1)] \quad \forall i \in S.$$

Thus, $\{\psi^*(X_t), \mathfrak{F}_t\}$ is a sub-martingale. On the other hand, (2.28) implies that ψ^* is bounded. Hence by Doob's theorem, $\psi^*(X_t)$ must converge as $t \rightarrow \infty$. Since \mathbf{X} is recurrent under v^* , this is possible only if ψ^* is a constant. From (2.26) we then get $\|c\|_\infty = \rho = \min_{u \in \mathbb{U}(i)} c(i, u)$ which is possible if c is a constant. This is a contradiction. Hence we must have $\rho < \|c\|_\infty$. \square

Now we are ready to show that $\rho = \lambda^*$. To this aim we perturb the cost function as follows:

- Under Assumption 2.2(a): let $\alpha > 0$ be a small number such that $\|c\|_\infty + \alpha < \gamma$. We define

$$\tilde{c}_n(u, i) = c(u, i) \mathbf{1}_{\mathcal{D}_n}(i) + (\|c\|_\infty + \alpha) \mathbf{1}_{\mathcal{D}_n^c}(i) \quad \forall u \in \mathbb{U}(i), i \in S.$$

It is evident that $\|\tilde{c}_n\|_\infty < \gamma$.

- Under Assumption 2.2(b): We define

$$\tilde{c}_n(u, i) = c(u, i) + \frac{1}{n} h(i) \quad \forall u \in \mathbb{U}(i), i \in S,$$

where $h(i) = [\ell(i) - \max_{u \in \mathbb{U}(i)} c(i, u)]_+$. Recall that h is a norm-like function. For large enough n it is evident that $\ell(\cdot) - \max_{u \in \mathbb{U}(\cdot)} \tilde{c}_n(\cdot, u)$ is also norm-like.

Thus, the conclusion of Lemma 2.5 hold if we replace c by \tilde{c}_n .

Lemma 2.7. *Assume Assumptions 2.1 and 2.2. Then any minimizing selector of (2.26), that is, any $v^* \in \mathfrak{U}_{\text{sm}}$ satisfying*

$$\min_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] = \left[e^{c(i, v^*(i))} \sum_{j \in S} \psi^*(j) P(j|i, v^*(i)) \right] \quad \forall i \in S, \quad (2.44)$$

is an optimal control and $\rho = \lambda^*$. Moreover, ψ^* is the unique solution of (2.26) with $\psi^*(i_0) = 1$.

Proof. Let $v^* \in \mathfrak{U}_{\text{sm}}$ be a minimizing selector as in (2.44). From Remark 2.2, there exists an eigenpair $(\rho_{v^*, n}, \psi_{v^*, n}) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi_{v^*, n} > 0$, satisfying

$$e^{\rho_{v^*, n}} \psi_{v^*, n}(i) = \left[e^{\tilde{c}_n(i, v^*(i))} \sum_{j \in S} \psi_{v^*, n}(j) P(j|i, v^*(i)) \right] \quad \forall i \in S, \quad (2.45)$$

and

$$0 \leq \rho_{v^*, n} \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{T-1} \tilde{c}_n(X_t, v^*(X_t))} \right] = \mathcal{E}_i(\tilde{c}_n, v^*) \quad \forall i \in S. \quad (2.46)$$

From the proof of Lemma 2.5, there exist a finite set $\tilde{\mathcal{B}}$, dependent on n , containing \mathcal{K} such that $(\tilde{c}_n(i, u) - \rho_{v^*, n}) \geq 0$ in $\tilde{\mathcal{B}}^c$ (under Assumption 2.2(a) we may take $\tilde{\mathcal{B}} = \mathcal{D}_n$, by Lemma 2.6, and

since under [Assumption 2.2\(b\)](#), \tilde{c}_n is norm-like we can choose suitable finite set $\tilde{\mathcal{B}}$ satisfying the required condition). Rewrite [\(2.45\)](#) as

$$\psi_{v^*,n}(i) = \left[e^{(\tilde{c}_n(i,v^*(i))-\rho_{v^*,n})} \sum_{j \in S} \psi_{v^*,n}(j) P(j|i, v^*(i)) \right] \quad \forall i \in S. \quad (2.47)$$

Then by the Markov property of \mathbf{X} , it follows from [\(2.47\)](#) that

$$\psi_{v^*,n}(i) = \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\tilde{\mathcal{B}}) \wedge N-1} (\tilde{c}_n(X_t, v^*(X_t)) - \rho_{v^*,n})} \psi_{v^*,n}(X_{\tilde{\tau}(\tilde{\mathcal{B}}) \wedge N}) \right].$$

Letting $N \rightarrow \infty$ and using Fatou's lemma, for all $i \in \tilde{\mathcal{B}}^c$, we deduce that

$$\begin{aligned} \psi_{v^*,n}(i) &\geq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\tilde{\mathcal{B}})-1} (\tilde{c}_n(X_t, v^*(X_t)) - \rho_{v^*,n})} \psi_{v^*,n}(X_{\tilde{\tau}(\tilde{\mathcal{B}})}) \right] \\ &\geq \left(\min_{\tilde{\mathcal{B}}} \psi_{v^*,n} \right) \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\tilde{\mathcal{B}})-1} (\tilde{c}_n(X_t, v^*(X_t)) - \rho_{v^*,n})} \right] \\ &\geq \left(\min_{\tilde{\mathcal{B}}} \psi_{v^*,n} \right). \end{aligned}$$

Thus $\psi_{v^*,n}$ is bounded below by a positive constant. Again using the Markov property of \mathbf{X} and applying Fatou's lemma in [\(2.47\)](#), we obtain that

$$\begin{aligned} \psi_{v^*,n}(i) &\geq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{T-1} (\tilde{c}_n(X_t, v^*(X_t)) - \rho_{v^*,n})} \psi_{v^*,n}(X_T) \right] \\ &\geq \left(\min_{\tilde{\mathcal{B}}} \psi_{v^*,n} \right) \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{T-1} (\tilde{c}_n(X_t, v^*(X_t)) - \rho_{v^*,n})} \right]. \end{aligned}$$

Taking logarithm on both sides, dividing by T and letting $T \rightarrow \infty$, we get

$$\begin{aligned} \rho_{v^*,n} &\geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{T-1} \tilde{c}_n(X_t, v^*(X_t))} \right] \\ &\geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{T-1} c(X_t, v^*(X_t))} \right]. \end{aligned}$$

Using [\(2.46\)](#), this of course, implies $\mathcal{E}_i(c, v^*) \leq \mathcal{E}_i(\tilde{c}_n, v^*) = \rho_{v^*,n}$ for all n . From the definition of \tilde{c}_n , it is evident that $\{\rho_{v^*,n}\}$ is a decreasing sequence which is bounded from below. Since $\|\tilde{c}_n\|_\infty < \gamma$ (for [Assumption 2.2\(a\)](#)), it is easily seen that the stochastic representation [\(2.43\)](#) holds for $\psi_{v^*,n}$ with the same choice of \mathcal{B} , independent of n . In view of [Lemma 2.5](#), we can even have $\psi_{v^*,n} \leq \mathcal{V}$ and it touches \mathcal{V} inside \mathcal{B} . Thus, using a diagonalization argument, there exists a pair $(\tilde{\rho}, \psi_{v^*}) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi_{v^*} > 0$ satisfying

$$e^{\tilde{\rho}} \psi_{v^*}(i) = \left[e^{c(i,v^*(i))} \sum_{j \in S} \psi_{v^*}(j) P(j|i, v^*(i)) \right] \quad \forall i \in S, \quad (2.48)$$

and $\lim_{n \rightarrow \infty} \rho_{v^*,n} = \tilde{\rho} \geq \mathcal{E}_i(c, v^*) \geq \rho$ for all $i \in S$. To complete the first part of the proof we only need to show that $\tilde{\rho} = \rho$. From [Lemma 2.3](#) and dominated convergence theorem (on [\(2.43\)](#) for each n), we obtain that

$$\psi_{v^*}(i) = \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \tilde{\rho})} \psi_{v^*}(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.49)$$

Since $\tilde{\rho} \geq \rho$, using [\(2.28\)](#) we have

$$\psi^*(i) \geq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \tilde{\rho})} \psi^*(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.50)$$

Then (2.49) and (2.50) implies that

$$(\psi^* - \psi_{v^*})(i) \geq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \tilde{\rho})} (\psi^* - \psi_{v^*})(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.51)$$

Rescale ψ_{v^*} , by multiplying with a suitable positive constant, so that $(\psi^* - \psi_{v^*}) \geq 0$ in \mathcal{B} and $(\psi^* - \psi_{v^*})(\hat{i}) = 0$ for some $\hat{i} \in \mathcal{B}$. Thus from (2.51), we deduce that $(\psi^* - \psi_{v^*}) \geq 0$ in S . From (2.44) and (2.48), we get

$$0 = (\psi^* - \psi_{v^*})(\hat{i}) \geq \mathbb{E}_i^{v^*} \left[e^{(c(\hat{i}, v^*(\hat{i})) - \tilde{\rho})} (\psi^* - \psi_{v^*})(X_1) \right].$$

This is similar to (2.36) and thus a similar argument gives $\psi^* = \psi_{v^*}$ in S . (2.44) and (2.48) then give us $\tilde{\rho} = \rho$. Hence $\rho = \lambda^* = \mathcal{E}_i(c, v^*)$ for all i . This completes the first part of the proof.

Next we show that ψ^* is unique upto a normalization. Let V be a positive solution to

$$e^{\lambda^*} V(i) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} V(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.52)$$

Choose a minimizing selector v of (2.52), that is,

$$e^{\lambda^*} V(i) = e^{c(i, v(i))} \sum_{j \in S} V(j) P(j|i, v(i)) \quad \forall i \in S. \quad (2.53)$$

From the proof of (2.40) we then get

$$\psi^*(i) \leq \mathbb{E}_i^v \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v(X_t)) - \lambda^*)} \psi^*(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c, \quad (2.54)$$

for some suitable finite set \mathcal{B} . On the other hand using (2.53), we have (see (2.41))

$$V(i) \geq \mathbb{E}_i^v \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v(X_t)) - \lambda^*)} \psi^*(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.55)$$

Combining (2.52), (2.53), and (2.55) and using the arguments above (see (2.51)) we can conclude that $\psi^* = V$ (upto a multiplicative constant). Hence the proof. \square

Now we complete the proof of [Theorem 2.1](#)

Proof of Theorem 2.1. (i) follows from [Lemmas 2.5](#) and [2.7](#). Furthermore, [Lemma 2.7](#) also gives us that any measurable selector of (2.44) is an optimal stationary Markov control. Thus to complete the proof of (ii) we only need to show that if for any $\hat{v} \in \mathfrak{U}_{sm}$ we have $\mathcal{E}_i(c, \hat{v}) = \lambda^*$ then \hat{v} satisfies (2.44).

From [Remark 2.2](#) there exist $(\rho_{\hat{v}}, \psi_{\hat{v}}) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi_{\hat{v}} > 0$, satisfying

$$e^{\rho_{\hat{v}}} \psi_{\hat{v}}(i) = e^{c(i, \hat{v}(i))} \sum_{j \in S} \psi_{\hat{v}}(j) P(j|i, \hat{v}(i)) \quad \forall i \in S. \quad (2.56)$$

Moreover, for some finite set \mathcal{B} containing \mathcal{K}

$$\psi_{\hat{v}}(i) = \mathbb{E}_i^{\hat{v}} \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, \hat{v}(X_t)) - \rho_{\hat{v}})} \psi_{\hat{v}}(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (2.57)$$

Proof of [Lemma 2.7](#) also gives us $\rho_{\hat{v}} = \lambda^*$. Then using the arguments of [Lemma 2.7](#) (see (2.54) and (2.55)) it can be easily shown that $\psi_{\hat{v}} = \kappa \psi^*$ for some positive κ . Hence the result follows from (2.56) and (2.26). This completes the proof. \square

2.2. Near-monotone cost. In this section we replace the [Assumption 2.2](#) with a near-monotone assumption stated below.

Assumption 2.3. Define $\lambda_m = \inf_{i \in S} \inf_{v \in \mathfrak{U}_{sm}} \mathcal{E}_i(c, v)$. We assume that

$$\inf_{v \in \mathfrak{U}_{sm}} \mathcal{E}_i(c, v) < \infty \quad \forall i \in S,$$

and the cost function c satisfies the *near-monotone* condition with respect to λ_m , that is,

$$\liminf_{n \rightarrow \infty} \inf_{k \geq n} \inf_{u \in \mathbb{U}(k)} c(k, u) > \lambda_m. \quad (2.58)$$

We recall $\lambda^* = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta)$ from [\(2.3\)](#). Note that by [Assumption 2.3](#), $\lambda^* \leq \lambda_m < \infty$. Near-monotone cost penalizes transient behaviour of the CMP. The following result gives an existence of an optimal stationary Markov control. In the following we are not able to get a nonlinear Poisson equation, instead we obtain an inequality.

Theorem 2.3. *Grant [Assumptions 2.1](#) and [2.3](#). We also assume that \mathbf{X} is recurrent under any control $v \in \mathfrak{U}_{sm}$. Then there exists a positive function ψ^* satisfying*

$$e^{\lambda^*} \psi^*(i) \geq \min_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \text{for } i \in S. \quad (2.59)$$

Futhermore, we have $\lambda^ = \lambda_m$ and any measurable selector of [\(2.59\)](#) is an optimal stationary Markov control.*

Proof. From [Lemma 2.2](#), let (ρ_n, ψ_n) be the Dirichlet eigenpair in \mathcal{D}_n satisfying

$$e^{\rho_n} \psi_n(i) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] \quad \forall i \in \mathcal{D}_n. \quad (2.60)$$

As mentioned in [Remark 2.1](#), the conclusion of [Lemma 2.4](#) holds under the hypothesis of [Theorem 2.3](#). Since $\psi_n(i_0) > 0$, we normalize ψ_n to satisfy $\psi_n(i_0) = 1$. Since $c \geq 0$, using [\(2.60\)](#) and [Lemma 2.4\(i\)](#) we obtain

$$\psi_n(j) \leq \sup_{u \in \mathbb{U}(i_0)} \frac{1}{P(j|i_0, u)} \exp\left(\inf_{v \in \mathfrak{U}_{sm}} \mathcal{E}_{i_0}(c, v)\right) \quad \text{for } j \neq i_0,$$

for all n large. Thus, using a diagonalization argument, we can extract a subsequence, denoting it by (ρ_n, ψ_n) , such that $\rho_n \rightarrow \rho$ and $\psi_n \rightarrow \psi^*$ componentwise, as $n \rightarrow \infty$. Moreover, $\psi^*(i_0) = 1$. Using the compactness of the measurable selectors of [\(2.60\)](#) and applying Fatou's lemma it is also easy to see that

$$e^{\rho} \psi^*(i) \geq \min_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.61)$$

From the arguments of [Lemma 2.5](#) (see [\(2.36\)](#)) it is also evident that $\psi^* > 0$ in S . Therefore, given a state $i \in S$, we can find large n so that $\psi_n(i) > 0$ and thus, applying [\(2.14\)](#), we obtain $\rho \leq \lambda^*$. Since $\rho \leq \lambda_m$, applying the near-monotonicity condition [\(2.58\)](#), we find a finite set \mathcal{B} such that

$$\inf_{u \in \mathbb{U}(i)} c(i, u) - \rho > 0 \quad \text{for } i \in \mathcal{B}^c. \quad (2.62)$$

Now consider a measurable selector v of [\(2.61\)](#) and by $\tau = \tau(\mathcal{B})$ we denote the first hitting time to \mathcal{B} . From the proof of [Lemma 2.3](#) we then see that

$$\psi^*(i) \geq \mathbb{E}_i^v \left[e^{\sum_{t=0}^{\tau-1} (c(X_t, v(X_t)) - \rho)} \psi^*(X_\tau) \right] \geq \min_{j \in \mathcal{B}} \psi^*(j) \quad i \in \mathcal{B}^c.$$

Thus, $\inf_{i \in S} \psi^*(i) > 0$. Using the Markov property of \mathbf{X} , we obtain from (2.61) that

$$\begin{aligned} \psi^*(i) &\geq \mathbb{E}_i^v \left[e^{\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \rho)} \psi^*(X_T) \right] \\ &\geq \left(\min_{\mathcal{B}} \psi^* \right) \mathbb{E}_i^v \left[e^{\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \rho)} \right]. \end{aligned}$$

Taking logarithm on both sides, dividing by T and letting $T \rightarrow \infty$, we get $\rho \geq \mathcal{E}_i(c, v) \geq \lambda_m$ for all $i \in S$. Thus $\rho = \lambda^* = \lambda_m$ and v is an optimal stationary Markov control. \square

Theorem 2.3 should be compared with [17, Theorem 3.6]. Though our condition on the controlled Markov chains is little stronger than those of [17], our method neither need the cost c to be norm-like nor we assume the action set to be finite.

Remark 2.3. **Assumption 2.1**(b) can be replaced by other similar assumption. For instance, if the killed process communicates with every state in \mathcal{D}_n from i_0 before leaving the domain \mathcal{D}_n , for large n , then our method applies. More precisely, we can replace **Assumption 2.1**(b) with the following: for all large n we have

$$\inf_{v \in \mathfrak{U}_{\text{sm}}} \mathbb{P}_{i_0}^v(\check{\tau}_j < \tau_{\mathcal{D}_n}) > 0 \quad \text{for all } j \in \mathcal{D}_n \setminus \{i_0\},$$

where $\check{\tau}_{i_0}$ denotes the hitting time to i_0 . In other words, for every $j \in \mathcal{D}_n \setminus \{i_0\}$ and $v \in \mathfrak{U}_{\text{sm}}$, there exists distinct $i_1, i_2, \dots, i_m \in \mathcal{D}_n \setminus \{i_0\}$ satisfying

$$P(i_1|i_0, v(i_0))P(i_2|i_1, v(i_1)) \cdots P(j|i_m, v(i_m)) > 0.$$

The Birth-Death Markov chain is a typical example of a CMP satisfying the above condition. Note that under this new assumption we can show that $\psi_n(i_0) > 0$ for large n .

It is also possible to relax the recurrence hypothesis under stationary Markov control. To this aim we introduce the following assumption.

Assumption 2.4. There exists a function $W : S \rightarrow [1, \infty)$ satisfying $W(i) \geq i$ for all large i and

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} (W(j) - W(i))P(j|i, u) \leq g(i) \quad \text{for } i \in S, \quad (2.63)$$

for some function $g : S \rightarrow \mathbb{R}$ satisfying $\lim_{i \rightarrow \infty} g(i) = 0$. Furthermore, for some $\eta > 0$ we have

$$\min_{u \in \mathbb{U}(i)} P(i-1|i, u) \geq \eta \quad \text{for all } i \geq 2, \quad (2.64)$$

and $P(\cdot|1, u)$ supported in a finite set C , independent of u . In addition, also assume that for $\mathcal{D}_n := \{1, \dots, n\}$, $v \in \mathfrak{U}_{\text{sm}}$ and any $j \in \mathcal{D}_n \setminus \{1\}$ there exists distinct $i_1, i_2, \dots, i_k \in \mathcal{D}_n$ we have

$$P(i_1|1, v(1))P(i_2|i_1, v(i_1)) \cdots P(j|i_k, v(i_k)) > 0. \quad (2.65)$$

Note that (2.63) does not guarantee that DTCMP \mathbf{X} is recurrent under every stationary Markov control. To illustrate, let us consider the following classical example of Birth-Death process.

Example 2.2. Suppose that \mathbb{U} is a compact metric space and $\lambda, \mu : S \times \mathbb{U} \rightarrow [0, 1]$ be such that $\lambda(i, u) + \mu(i, u) = 1$ and $\mu(i, u) \geq \eta > 0$ for all $i \geq 1$. Let $\{p_{ij}\}$ be a collection of nonnegative numbers satisfying

$$\sum_{j \geq 1} p_{ij} = 1, \quad \sum_{j \geq 1} j p_{ij} < \infty \quad \text{for all } i \geq 1.$$

We define

$$P(j|i, u) = \begin{cases} \lambda(i, u)p_{ik} & \text{if } j = i + k, i, k \geq 1, \\ \mu(i, u) & \text{if } j = i - 1, i \geq 1, \\ 0 & \text{if } j \leq i - 2, i \geq 1, \\ 1 & \text{if } j = 1, i = 0. \end{cases}$$

Then for $W(i) = i + 1$ we have

$$\sup_{u \in \mathbb{U}} \sum_{j \in S} (W(j) - W(i))P(j|i, u) \leq \sup_{u \in \mathbb{U}} |\lambda(i, u) \sum_{k \geq 1} k p_{ik} - \mu(i, u)| := g(i).$$

Thus if we assume, $g(i) \rightarrow 0$ as $i \rightarrow \infty$, we get (2.63) and (2.64). Furthermore, if we assume $\lambda(i, u) > 0, p_{i,1} > 0$, then we also have (2.65).

\mathbf{X} need not be recurrent under (2.63). For instance, if we take $p_{i,1} = 1$ for all i and

$$\lambda(i, u) = \lambda(i) = \frac{(i+1)^2}{i^2 + (i+1)^2}, \quad \mu(i, u) = \mu(i) = \frac{i^2}{(i+1)^2} \lambda(i).$$

Then $\lambda(i) + \mu(i) = 1$ and

$$|\lambda(i) - \mu(i)| \leq \frac{2i+1}{i^2 + (i+1)^2} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

But

$$\sum_{n \geq 1} \prod_{i=1}^n \frac{\mu(i)}{\lambda(i)} = \sum_{n \geq 1} \frac{1}{(n+1)^2} < \infty.$$

This implies that \mathbf{X} is transient.

We establish the following result.

Theorem 2.4. *Grant Assumptions 2.3 and 2.4. Also assume that \mathbf{X} is irreducible under any stationary Markov control. Then there exists a positive function ψ^* satisfying*

$$e^{\lambda^*} \psi^*(i) \geq \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \text{for } i \in S. \quad (2.66)$$

Furthermore, we have $\lambda^* = \lambda_m$ and any measurable selector of (2.66) is an optimal stationary Markov control.

Proof. Since \mathbf{X} may not be recurrent under a stationary Markov control, the proof of Theorem 2.3 does not work. We have to modify the proof. We begin with the Dirichlet eigenpair (ρ_n, ψ_n) in \mathcal{D}_n satisfying

$$e^{\rho_n} \psi_n(i) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)} \sum_{j \in S} \psi_n(j) P(j|i, u) \right] \quad \forall i \in \mathcal{D}_n. \quad (2.67)$$

Due to (2.65) we must have $\psi_n(1) > 0$. This in turn, implies from (2.64) that $\psi_n > 0$ in \mathcal{D}_n . As a consequence we have $\rho_n \leq \lambda^*$, by Lemma 2.2, and $\rho_n > -\infty$ for all n . Note that we can not apply (2.1)(ii) anymore to find an upper bound for ψ_n . Instead we use (2.65). Denote by $\check{\tau}_j$ the first hitting time to j . We claim that if $j \in \mathcal{D}_n$, then

$$\inf_{v \in \mathfrak{U}_{sm}} \mathbb{P}_1^v(\check{\tau}_j \leq n \wedge \tau_n) > \kappa(n, j) \quad (2.68)$$

for some positive constant $\kappa(n, j)$. Suppose, on the contrary, that the claim is not true. Then there exists $v_m \in \mathfrak{U}_{sm}$ such that $\mathbb{P}_1^{v_m}(\check{\tau}_j \leq n \wedge \tau_n) \rightarrow 0$ as $m \rightarrow \infty$. Using the compactness of $\mathbb{U}(i)$, we can extract a subsequence of v_m , denoted by the original one, so that $v_m(i) \rightarrow v(i)$ for all i as $m \rightarrow \infty$. By Assumption 2.1(i) we then see that the law of \mathbf{X}_m converges to \mathbf{X} , where \mathbf{X}_m (\mathbf{X}) is the DTCMP governed by v_m (v , respectively). Therefore, for every $k \leq n$,

$$\begin{aligned} & \mathbb{P}_1^v(X_i \in \mathcal{D}_n \setminus \{1, j\}, X_k = j \quad \text{for all } 1 \leq i \leq k-1) \\ &= \lim_{m \rightarrow \infty} \mathbb{P}_1^{v_m}(X_{m,i} \in \mathcal{D}_n \setminus \{1, j\}, X_{m,k} = j \quad \text{for all } 1 \leq i \leq k-1) \\ &\leq \lim_{m \rightarrow \infty} \mathbb{P}_1^{v_m}(\check{\tau}_j \leq n \wedge \tau_n) = 0. \end{aligned}$$

This clearly, contradicts (2.65). Hence we must have (2.68). From the monotonicity of τ_n it then follows that for $\mathcal{D}_n \supset \mathcal{D}_m \ni j$, we have

$$\inf_{v \in \mathcal{U}_{sm}} \mathbb{P}_1^v(\check{\tau}_j \leq m \wedge \tau_n) \geq \inf_{v \in \mathcal{U}_{sm}} \mathbb{P}_1^v(\check{\tau}_j \leq m \wedge \tau_m) \geq \kappa(m, j). \quad (2.69)$$

Now we normalize ψ_n to satisfy $\psi_n(1) = 1$. Let v_n be a minimizing selector of (2.67). Thus, using optional sampling theorem (see (2.17)), it follows from (2.67) that

$$\begin{aligned} 1 = \psi_n(1) &= \mathbb{E}_1^{v_n} \left[e^{\sum_{t=0}^{\check{\tau}_j \wedge m \wedge \tau_n - 1} (c(X_t, v_n(X_t)) - \rho_n)} \psi_n(X_{\check{\tau}_j \wedge m \wedge \tau_n}) \right] \\ &\geq e^{-\lambda^* m} \psi_n(j) \inf_{v \in \mathcal{U}_{sm}} \mathbb{P}_1^v(\check{\tau}_j \leq m \wedge \tau_n) \\ &\geq e^{-\lambda^* m} \psi_n(j) \kappa(m, j) \end{aligned}$$

using (2.69). Choosing $m = j + 1$, this indeed gives us

$$\psi_n(j) \leq \frac{1}{\kappa(j+1, j)} e^{(j+1)\lambda^*} \quad \text{for all } n > j. \quad (2.70)$$

This gives the upper bound on $\{\psi_n\}$. Therefore, from the proof of Lemma 2.4 it can be easily seen that $\{\rho_n\}$ is bounded from below. (Otherwise, we must have a $\psi^* \geq 0$ satisfying $\psi^*(1) = 1$ and

$$0 \geq \sum_{j \in S} \psi^*(j) P(j|i, \hat{v}(i)) \quad \text{for all } i \in S.$$

Then $\psi^*(i) = 0$ on the support of $P(\cdot|1, \hat{v}(1))$. Repeated use of (2.64) thus gives us $\psi^*(1) = 0$ which is not possible). Thus we can find a subsequence of (ρ_n, ψ_n) converging to (ρ, ψ^*) and

$$e^\rho \psi^*(i) \geq \min_{u \in \mathcal{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} \psi^*(j) P(j|i, u) \right] \quad \forall i \in S. \quad (2.71)$$

Since \mathbf{X} is irreducible under any stationary Markov policy, it follows that $\psi^* > 0$ (see (2.36)).

Next we show that $\rho = \lambda^* = \mathcal{E}_i(c, v)$ for all i , where v is a minimizing selector of (2.36). Actually, the proof would follow from the arguments of Theorem 2.3 if we could show that \mathbf{X} is recurrent under the policy v . Observe that for any $i \in S$, we have from (2.71) that

$$\psi^*(i+1) \geq e^{-\rho} \psi^*(i) P(i|i+1, v(i+1)) \geq \eta e^{-\lambda^*} \psi^*(i).$$

Thus

$$\psi^*(j) \geq (\eta e^{-\lambda^*})^{(j-1)} \psi^*(1) = e^{-\tilde{\kappa} j} \quad j \geq 2, \quad (2.72)$$

for some constant $\tilde{\kappa}$. Also, applying Dynkin's formula in (2.63) we obtain

$$\mathbb{E}_i^v[W(X_T)] - W(i) \leq \mathbb{E}_i^v \left[\sum_{t=0}^{T-1} g(X_t) \right] \quad T \geq 0. \quad (2.73)$$

In view of (2.72) and Assumption 2.4 we have $\log \psi^*(i) \geq -\tilde{\kappa} W(i) - \hat{\kappa}$ for some constant $\hat{\kappa}$. Using (2.71) and the Markov property of \mathbf{X} it follows that

$$\psi^*(1) \geq \mathbb{E}_1^v \left[e^{\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \rho)} \psi^*(X_T) \right].$$

Taking logarithm on both sides and dividing by T , we get

$$\begin{aligned} 0 &\geq \frac{1}{T} \mathbb{E}_0^v \left[\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \rho) \right] - \frac{1}{T} (\tilde{\kappa} \mathbb{E}_1^v[W(X_T)] + \hat{\kappa}) \\ &\geq \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \tilde{\kappa} g(X_t) - \rho) \right] - \frac{1}{T} (\tilde{\kappa} W(1) + \hat{\kappa}), \end{aligned} \quad (2.74)$$

by (2.73). Now suppose that \mathbf{X} is not recurrent under v . Since \mathbf{X} is irreducible, it must be transient and therefore, it can not have any invariant probability measure by [37, Proposition 10.1.1]. Thus, by [37, Theorem 12.1.2], for any finite set $B \subset S$ we have

$$\frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} \mathbb{1}_B(X_t) \right] \rightarrow 0 \quad \text{as } T \rightarrow \infty. \quad (2.75)$$

Let B_\circ be a finite set such that $\min_{u \in \mathbb{U}(i)} (c(i, u) - \tilde{\kappa}g(i) - \rho) > \delta > 0$, for some $\delta > 0$, and $i \in B_\circ^c$. This is possible due to (2.58) and the fact $\lim_{i \rightarrow \infty} g(i) = 0$. Applying (2.75) we obtain

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} (c(X_t, v(X_t)) - \tilde{\kappa}g(X_t) - \rho) \right] \\ & \geq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} \mathbb{1}_{B_\circ^c}(X_t) (c(X_t, v(X_t)) - \tilde{\kappa}g(X_t) - \rho) \right] - \max_{B_\circ} |g - \rho| \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} \mathbb{1}_{B_\circ}(X_t) \right] \\ & \geq \delta \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} \mathbb{1}_{B_\circ^c}(X_t) \right] \\ & = \delta \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_1^v \left[\sum_{t=0}^{T-1} \mathbb{1}_S(X_t) \right] \\ & = \delta. \end{aligned}$$

But this leads to a contradiction to (2.74) when we let $T \rightarrow \infty$ in (2.74). Therefore \mathbf{X} must be recurrent under v . Now rest of the argument follows from Theorem 2.3. \square

3. RISK-SENSITIVE CONTROL OF CONTINUOUS TIME CMP

In this section we consider continuous time CMP $\mathbf{X} = \{X_t, t \geq 0\}$, on a countable state space S , controlled by the control process ζ_t , $t \geq 0$, taking values in \mathbb{U} . As before, \mathbb{U} is the action space of the controller, which is assumed to be a Borel space with Borel σ algebra $\mathfrak{B}(\mathbb{U})$. For each $i \in S$, let $\mathbb{U}(i)$ be the space of all admissible actions of the controller when the system is at state i . Let $\mathcal{K} := \{(i, u) : i \in S, u \in \mathbb{U}(i)\}$ be set of all feasible state action pair. As before, we denote by $c : \mathcal{K} \rightarrow \mathbb{R}_+$ the running cost function. The transition rates $q(j|i, u)$, $u \in \mathbb{U}(i)$, $i, j \in S$, satisfy the condition $q(j|i, u) \geq 0$ for all $u \in \mathbb{U}(i)$, $i, j \in S$ and $j \neq i$. In addition, we also impose that

Assumption 3.1. (a) For each $i \in S$, the admissible action space $\mathbb{U}(i)$ is a nonempty compact subset of \mathbb{U} .

(b) The model is conservative:

$$\sum_{j \in S} q(j|i, u) = 0 \quad \forall u \in \mathbb{U}(i), i \in S.$$

(c) The model is stable:

$$q(i) := \sup_{u \in \mathbb{U}(i)} (-q(i|i, u)) = \sup_{u \in \mathbb{U}(i)} \sum_{j \neq i} q(j|i, u) < \infty \quad \forall i \in S.$$

For each $i, j \in S$, $q(j|i, u)$ is a measurable map on $\mathbb{U}(i)$.

Following [34] (see also [23, 25, 40]) we briefly describe the evolution of the continuous time CMP (CTCMP). Let $S_\infty := S \cup \{i_\infty\}$ for an isolated point $i_\infty \notin S$. Define the canonical sample space $\Omega := (S \times (0, \infty))^\infty \cup \{(i_0, \theta_1, i_1, \dots, \theta_m, i_m, \infty, i_\infty, \infty, i_\infty, \dots) \mid \theta_k \neq \infty, i_k \neq i_\infty \text{ for all } 0 \leq k \leq m, m \geq 1\}$, with Borel σ -algebra $\mathfrak{B}(\Omega)$. For each sample point $\omega = (i_0, \theta_1, i_1, \dots, \theta_m, i_m, \dots) \in \Omega$,

we set $T_0(\omega) = 0$, $T_k(\omega) = \theta_1 + \theta_2 + \dots + \theta_k$, and define $T_\infty(\omega) = \lim_{k \rightarrow \infty} T_k(\omega)$. Now we define a controlled process $\{X_t\}_{t \geq 0}$ on $(\Omega, \mathfrak{B}(\Omega))$ by

$$X_t = \sum_{k \geq 0} \mathbb{1}_{\{T_k \leq t < T_{k+1}\}} i_k + \mathbb{1}_{\{t \geq T_\infty\}} i_\infty \quad \text{for } t \geq 0. \quad (3.1)$$

From (3.1), it is clear that for any $m \geq 1$ and $\omega \in \Omega$, $T_m(\omega)$ denotes the m -th jump moment of the process X_t , i_m is the state of the controlled process on $[T_m, T_{m+1})$ and $\theta_m = T_m - T_{m-1}$ denotes the waiting time between jumps (or, sojourn time) at state i_{m-1} . Also, we add an isolated point $u_\infty \notin \mathbb{U}$ to \mathbb{U} and let $\mathbb{U}_\infty = \mathbb{U} \cup \{u_\infty\}$ and $\mathbb{U}(i_\infty) = \{u_\infty\}$. We do not want to consider our process after the time T_∞ . Thus we assume that i_∞ is an absorbing state, that is, $q(j|i_\infty, u_\infty) = 0$ for all $j \in S$. Also, assume that $c(i_\infty, u) = 0$ for all $u \in \mathbb{U}_\infty$. Consider a filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ where $\mathfrak{F}_t := \sigma((T_m \leq s, X_{T_m} \in A) : 0 \leq s \leq t, m \geq 0, A \subset S)$, and let $\tilde{\mathfrak{F}} := \sigma(\mathcal{A} \times \{0\}, \mathcal{B} \times (s, \infty) : \mathcal{A} \in \mathfrak{F}_0, \mathcal{B} \in \mathfrak{F}_{s-})$ be the σ -algebra of predictable sets in $\Omega \times (0, \infty)$ with respect to \mathfrak{F}_t , where $\mathfrak{F}_{s-} := \vee_{t < s} \mathfrak{F}_t$.

An admissible policy $\zeta = \{\zeta_t\}_{t \geq 0}$ is a measurable map from $(\Omega \times (0, \infty), \tilde{\mathfrak{F}})$ to $(\mathbb{U}_\infty, \mathfrak{B}(\mathbb{U}_\infty))$ satisfying $\zeta_t(\omega) \in \mathbb{U}(X_{t-}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$. Let \mathfrak{U} be the space of all admissible policies. An admissible policy ζ is said to be a Markov policy if $\zeta_t(\omega) = \zeta_t(X_{t-}(\omega))$ for all $\omega \in \Omega$ and $t \geq 0$. The space of all Markov policies is denoted by \mathfrak{U}_m . If the Markov policy ζ does not have any explicit time dependency then it is called a stationary Markov policy and \mathfrak{U}_{sm} denotes the space of all stationary Markov strategies. For each $i \in S$ and $\zeta \in \mathfrak{U}$, it is well known that (cf. [14, 23, 25, 34]) there exist unique probability measure \mathcal{P}_i^ζ on $(\Omega, \mathfrak{B}(\Omega))$ such that $\mathcal{P}_i^\zeta(X_0 = i) = 1$. Let \mathbb{E}_i^ζ be the corresponding expectation operator. Also, from [22, pp.13-15], we know that $\{X_t\}_{t \geq 0}$ is a Markov process under any $\zeta \in \mathfrak{U}_m$ (in fact, strong Markov).

Under some policies the process $\{X_t\}_{t \geq 0}$ may be explosive, in order to avoid explosion of the CTCMP, we impose the following (see [23, 25], [22, Assumption 2.2]).

Assumption 3.2. There exist a function $\tilde{\mathcal{V}} : S \rightarrow [1, \infty)$ and constants $C_0 \neq 0, C_1 > 0$ and $C_2 \geq 0$ such that

- (a) $\sum_{j \in S} \tilde{\mathcal{V}}(j) q(j|i, u) \leq C_0 \tilde{\mathcal{V}}(i) + C_2$ for all $(i, u) \in \mathcal{K}$;
- (b) $q(i) \leq C_1 \tilde{\mathcal{V}}(i)$ for all $i \in S$.

For the rest of this section we are going to assume that [Assumption 3.2](#) holds. Note that [Assumption 3.2](#) holds if $\sup_{i \in S} q(i) < \infty$. In this case we can choose $\tilde{\mathcal{V}}$ to be a suitable constant. From [24, Theorem 3.1] (see also, [23, Proposition 2.2]) it also follows that, under [Assumption 3.2](#), $\mathbb{P}_i^\zeta(T_\infty = \infty) = 1$ for all $i \in S$ and $\zeta \in \mathfrak{U}$.

We also assume the following for our CTCMP (compare with [Assumption 2.1](#))

- Assumption 3.3.**
- (a) For each $i \in S$, the map $u \mapsto c(i, u)$ is continuous on $\mathbb{U}(i)$.
 - (b) For each $i \in S$ and bounded measurable function $f : S \rightarrow \mathbb{R}$, the map $u \mapsto \sum_{j \in S} f(j) q(j|i, u)$ is continuous on $\mathbb{U}(i)$.
 - (c) There exists $i_0 \in S$ such that $q(j|i_0, u) > 0$ for all $j \neq i_0$ and $u \in \mathbb{U}(i_0)$.

For each admissible control ζ the ergodic risk-sensitive cost is given by

$$\mathcal{E}_i(c, \zeta) := \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\int_0^T c(X_t, \zeta_t) dt} \right], \quad (3.2)$$

where \mathbf{X} is the CTCMP corresponding to ζ with initial state i . As before, our aim is to minimize (3.2) over all admissible policies in \mathfrak{U} . A policy $\zeta^* \in \mathfrak{U}$ is said to be optimal if for all $i \in S$

$$\mathcal{E}_i(c, \zeta^*) = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta) = \lambda^* \quad \text{for all } i. \quad (3.3)$$

We also introduce the following Lyapunov condition. Recall that a stationary Markov process \mathbf{X} with rate matrix $Q = [q(j|i)]$ is irreducible if for any $i, j \in S, i \neq j$, there exists distinct $i_1, i_2, \dots, i_k \in S$ satisfying $q(i_1|i) \cdots q(j|i_k) > 0$ (cf. [22, p. 107]).

Assumption 3.4. We assume that the CTCMP \mathbf{X} is irreducible under every stationary Markov control in \mathfrak{U}_{sm} . In (a) and (b) below the function \mathcal{V} on S takes values in $[1, \infty)$ and \widehat{C} is a positive constant. We assume that one of the following hold.

(a) For some positive constant γ and a finite set \mathcal{K} it holds that

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j)q(j|i, u) \leq \widehat{C}\mathbf{1}_{\mathcal{K}}(i) - \gamma\mathcal{V}(i) \quad \forall \quad i \in S. \quad (3.4)$$

Also, assume that $\|c\|_{\infty} := \sup_{i \in S} \sup_{u \in \mathbb{U}(i)} c(i, u) < \gamma$.

(b) For a finite set \mathcal{K} and a norm-like function $\ell : S \rightarrow \mathbb{R}_+$ it holds that

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j)q(j|i, u) \leq \widehat{C}\mathbf{1}_{\mathcal{K}}(i) - \ell(i)\mathcal{V}(i) \quad \forall \quad i \in S. \quad (3.5)$$

Moreover, the function $\ell(\cdot) - \max_{u \in \mathbb{U}(\cdot)} c(\cdot, u)$ is norm-like.

Applying [1, Proposition 6.3.3], we see that the CTCMP is strongly ergodic under every stationary Markov control, provided **Assumption 3.4** holds. Moreover, letting $\check{\tau}_j$ to be the first hitting time to j , it also follows that $\mathbb{E}_i^{\zeta}[\check{\tau}_j] < \infty$ for all $i \neq j$ and $\zeta \in \mathfrak{U}_{\text{sm}}$.

Before we proceed further, let us present a modified example from [22, Example 1.3].

Example 3.1. For $i \geq 2$, we suppose that

$$q(j|i, u) = \begin{cases} \lambda i + u & \text{for } j = i + 1, \\ \mu i + u & \text{for } j = i - 1, \\ -(\lambda i + \mu i + 2u) & \text{for } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let the control parameter u take values in some bounded set. Also, assume that $q(j|1, u) = q(j|1)$ is positive for every $j \geq 2$ and decays exponentially fast with j . Therefore, **Assumption 3.3(c)** holds. Suppose that $\mu > \lambda > 0$ and define $\mathcal{V}(i) = e^{\theta i}$ for some $\theta > 0$ to be chosen later. Then note that

$$\begin{aligned} \sum_{j \in S} \mathcal{V}(j)q(j|i, u) &= e^{\theta i} \left((\lambda i + u)e^{\theta} + (\mu i + u)e^{-\theta} - (\lambda i + \mu i + 2u) \right) \\ &= i\mathcal{V}(i) \left(\lambda(e^{\theta} - 1) + \mu(e^{-\theta} - 1) + \frac{u}{i}(e^{\theta} + e^{-\theta} - 2) \right). \end{aligned} \quad (3.6)$$

Since for every small $\theta > 0$ we have

$$\mu(e^{-\theta} - 1) + \lambda(e^{\theta} - 1) < 0 \Leftrightarrow (e^{\theta} - 1)(\lambda - \mu e^{-\theta}) < 0 \Leftrightarrow \lambda < \mu e^{-\theta},$$

letting $\ell(i) = \alpha i$, for $2\alpha = -\mu(e^{-\theta} - 1) - \lambda(e^{\theta} - 1) > 0$, we get from (3.6) that

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j)q(j|i, u) \leq -\ell(i)\mathcal{V}(i),$$

for $i \in \mathcal{K}^c$ where \mathcal{K} is some finite set satisfying $\alpha > \frac{u}{i}(e^{\theta} + e^{-\theta} - 2)$ for all $i \in \mathcal{K}^c$ and all control parameter u . Now we let θ small enough so that $\sum_{j \in S} \mathcal{V}(j)q(j|1) < \infty$. Hence **Assumption 3.4** holds.

Let us now state our first main result of this section (compare it with **Theorem 2.1**)

Theorem 3.1. *Grant **Assumptions 3.1 to 3.4**. Then the following hold.*

(i) There exists a unique positive function ψ^* , $\psi^*(i_0) = 1$, satisfying

$$\lambda^* \psi^*(i) = \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad \text{for } i \in S. \quad (3.7)$$

(ii) A stationary Markov control $v \in \mathfrak{U}_{\text{sm}}$ is optimal if and only if it satisfies

$$\min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] = \left[\sum_{j \in S} \psi^*(j) q(j|i, v(i)) + c(i, v(i)) \psi^*(i) \right] \quad (3.8)$$

for all $i \in S$.

The rest of this section is dedicated to the proof of [Theorem 3.1](#). The main strategy of the proof is same as the proof of [Theorem 2.1](#). We begin with our next result which is a counterpart of [Proposition 2.1](#) for CTCMP.

Proposition 3.1. *Grant [Assumption 3.1](#) and [Assumption 3.3\(a\)-\(b\)](#). Suppose $c < -\delta$ in \mathcal{D} for some positive constant δ and a finite set \mathcal{D} . Then for any $f \in \mathcal{B}_{\mathcal{D}}$ there exist unique $\phi \in \mathcal{B}_{\mathcal{D}}$ satisfying*

$$\min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \phi(j) q(j|i, u) + c(i, u) \phi(i) \right] = -f(i), \quad \forall i \in \mathcal{D}, \quad \text{and } \phi(i) = 0 \quad \forall i \in \mathcal{D}^c. \quad (3.9)$$

Let $\tau = \tau(\mathcal{D}) := \inf\{t > 0 : X_t \notin \mathcal{D}\}$. Then the unique solution satisfies

$$\phi(i) = \inf_{\zeta \in \mathfrak{U}_m} \mathbb{E}_i^\zeta \left[\int_0^\tau e^{\int_0^t c(X_s, \zeta_s) ds} f(X_t) dt \right] \quad \forall i \in S. \quad (3.10)$$

Proof. Given a tuple $(y_i)_{i \in \mathcal{D}}$ and a fixed $i \in \mathcal{D}$, let us consider the map

$$\mathbb{R} \ni x \mapsto G(x) = \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in \mathcal{D}} y_j q(j|i, u) + (q(i|i, u) + c(i, u))x \right].$$

We note that G is strictly decreasing. For, $x_1 > x_2$ we have

$$G(x_2) - G(x_1) \geq \min_{u \in \mathbb{U}(i)} [(q(i|i, u) + c(i, u))(x_2 - x_1)] \geq \delta(x_1 - x_2) > 0,$$

since $q(i|i, u)(x_2 - x_1) \geq 0$ for all $u \in \mathbb{U}(i)$ and $i \in S$. Furthermore, $\lim_{x \rightarrow \pm\infty} G(x) = \mp\infty$. Therefore, for every $y \in \mathbb{R}$ there exists a unique x satisfying $G(x) = y$. Using G we can now define a map $\mathcal{T}_1 : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ that satisfies

$$\min_{u \in \mathbb{U}(i)} \left[\sum_{j \in \mathcal{D}} \phi(j) q(j|i, u) + (q(i|i, u) + c(i, u))(\mathcal{T}_1 \phi(i)) \right] = -f(i) \quad i \in \mathcal{D}. \quad (3.11)$$

We now show that \mathcal{T}_1 is a contraction. Recall the norm $\|\cdot\|_{\mathcal{D}}$ from [Proposition 2.1](#). Let $\psi_m = \mathcal{T}_1 \phi_m$ for $m = 1, 2$. For each $i \in \mathcal{D}$ we then have from (3.11) that

$$\begin{aligned} 0 &\geq \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in \mathcal{D}} \phi_1(j) q(j|i, u) + (q(i|i, u) + c(i, u)) \psi_1(i) \right] \\ &\quad - \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in \mathcal{D}} \phi_2(j) q(j|i, u) + (q(i|i, u) + c(i, u)) \psi_2(i) \right] \end{aligned}$$

$$\geq \min_{u \in \mathbb{U}(i)} \left[\sum_{j \neq i \in \mathcal{D}} (\phi_1(j) - \phi_2(j))q(j|i, u) + (q(i|i, u) + c(i, u))(\psi_1(i) - \psi_2(i)) \right].$$

Let $\tilde{u} \in \mathbb{U}(i)$ be point where the minimum on RHS is attained. Then we get from above

$$(q(i|i, \tilde{u}) + c(i, \tilde{u}))(\psi_1(i) - \psi_2(i)) + q(i|i, \tilde{u})\|\phi_1 - \phi_2\|_{\mathcal{D}} \leq 0,$$

which in turn, gives

$$(\psi_2(i) - \psi_1(i)) \leq \sup_{u \in \mathbb{U}(i)} \frac{-q(i|i, u)}{-q(i|i, u) - c(i, u)} \|\phi_1 - \phi_2\|_{\mathcal{D}} \leq \vartheta \|\phi_1 - \phi_2\|_{\mathcal{D}}$$

for some $\vartheta < 1$. Interchanging ψ_1 and ψ_2 in the above calculation and using the arbitrariness of i we have

$$\|\mathcal{T}_1\phi_1 - \mathcal{T}_1\phi_2\|_{\mathcal{D}} \leq \vartheta \|\phi_1 - \phi_2\|_{\mathcal{D}}.$$

Therefore, \mathcal{T}_1 is a contraction and for Banach fixed point theorem, we get a unique solution to (3.9). (3.10) follows from Dynkin's formula. \square

As before, applying [Theorem 2.2](#) and [Proposition 3.1](#), we obtain the existence of an eigenpair.

Lemma 3.1. *Grant [Assumption 3.1](#), [Assumption 3.2](#) and [Assumption 3.3\(a\)-\(b\)](#). Then there exists $(\rho_{\mathcal{D}}, \psi_{\mathcal{D}}) \in \mathbb{R} \times \mathcal{B}_{\mathcal{D}}^+$, $\psi_{\mathcal{D}} \gneq 0$, satisfying*

$$\rho_{\mathcal{D}}\psi_{\mathcal{D}}(i) = \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi_{\mathcal{D}}(j)q(j|i, u) + c(i, u)\psi_{\mathcal{D}}(i) \right]. \quad (3.12)$$

Moreover, we have

$$\rho_{\mathcal{D}} \leq \inf_{\zeta \in \mathcal{U}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^{\zeta} \left[e^{\int_0^T c(X_t, \zeta_t) dt} \right], \quad (3.13)$$

for all $i \in S$ such that $\psi_{\mathcal{D}}(i) > 0$.

Proof. It is evident from (3.12) that we may assume $c < -\delta$ in \mathcal{D} for some positive constant δ . Otherwise, subtract $-\sup_{\mathcal{D}} c - \delta$ from both sides of (3.12). Let $\mathcal{T} : \mathcal{B}_{\mathcal{D}} \rightarrow \mathcal{B}_{\mathcal{D}}$ be an operator defined as

$$\mathcal{T}(f)(i) := \phi(i) = \inf_{\zeta \in \mathcal{U}_m} \mathbb{E}_i^{\zeta} \left[\int_0^{\tau} e^{\int_0^t c(X_s, \zeta_s) ds} f(X_t) dt \right] \quad \forall i \in S. \quad (3.14)$$

Then ϕ is the solution to (3.9). It is fairly straightforward to show that \mathcal{T} is completely continuous, order-preserving and 1-homogeneous. Now choose $f \in \mathcal{B}_{\mathcal{D}}$ such that $f(i) = 1$ for some $i \in \mathcal{D}$ and zero elsewhere. Then, from (3.14), it follows that

$$\begin{aligned} \phi(i) &\geq \inf_{\zeta \in \mathcal{U}_m} \mathbb{E}_i^{\zeta} \left[\int_0^{T_1} e^{\int_0^t c(X_s, \zeta_s) ds} f(X_t) dt \right] \\ &\geq f(i) \mathbb{E}_i^{\zeta} \left[\int_0^{T_1} e^{-t\|c\|_{\mathcal{D}}} dt \right] \\ &\geq \frac{f(i)}{\|c\|_{\mathcal{D}}} \inf_{\zeta \in \mathcal{U}_m} \mathbb{E}_i^{\zeta} \left[1 - e^{-\|c\|_{\mathcal{D}} T_1} \right], \end{aligned}$$

where T_1 denotes the first jump time. It is well-known (cf. [40]) that

$$\mathbb{P}_i^{\zeta}(T_1 > t) = e^{\int_0^t q(i|i, \zeta_s(i)) ds}. \quad (3.15)$$

Therefore,

$$\mathbb{E}_i^{\zeta} \left[1 - e^{-\|c\|_{\mathcal{D}} T_1} \right] = 1 - \mathbb{E}_i^{\zeta} \left[e^{-\|c\|_{\mathcal{D}} T_1} \right]$$

$$\begin{aligned}
&= 1 - \int_0^\infty \|c\|_{\mathcal{D}} e^{-\|c\|_{\mathcal{D}} s} \mathbb{P}_i^\zeta(T_1 \leq t) dt \\
&= \|c\|_{\mathcal{D}} \int_0^\infty e^{-\|c\|_{\mathcal{D}} s} \mathbb{P}_i^\zeta(T_1 > t) dt \\
&= \|c\|_{\mathcal{D}} \int_0^\infty e^{-\|c\|_{\mathcal{D}} s} e^{\int_0^t q(i|i, \zeta_s(i)) ds} dt \\
&\geq \|c\|_{\mathcal{D}} \int_0^\infty e^{-\|c\|_{\mathcal{D}} s} e^{-tq(i)} dt = \frac{\|c\|_{\mathcal{D}}}{\|c\|_{\mathcal{D}} + q(i)},
\end{aligned}$$

where in the fourth line we use (3.15). Hence

$$\phi(i) \geq \frac{f(i)}{\|c\|_{\mathcal{D}} + q(i)}.$$

Thus for some $M > 0$ we have $M\mathcal{T}(f) \succeq f$.

By Theorem 2.2 there exist a nontrivial $\psi_{\mathcal{D}} \in \mathcal{B}_{\mathcal{D}}^+$, $\psi_{\mathcal{D}} \neq 0$, and $\lambda_{\mathcal{D}} > 0$ such that $\mathcal{T}(\psi_{\mathcal{D}}) = \lambda_{\mathcal{D}}\psi_{\mathcal{D}}$. Applying Proposition 3.1 we then obtain

$$\min_{u \in \mathbb{U}(i)} \left[\sum_{j \in \mathcal{S}} \psi_{\mathcal{D}}(j) q(j|i, u) + \psi_{\mathcal{D}}(i) c(i, u) \right] = \rho_{\mathcal{D}} \psi_{\mathcal{D}}(i) \quad \forall i \in \mathcal{D}, \quad (3.16)$$

where $\rho_{\mathcal{D}} = -[\lambda_{\mathcal{D}}]^{-1}$. This gives us (3.12).

Next we show (3.13). Consider $i \in \mathcal{D}$ satisfying $\psi_{\mathcal{D}}(i) > 0$. Choose an admissible control $\zeta \in \mathcal{U}$. We plan to apply Dynkin's formula upto the stopping time $t \wedge \tau$ where $\tau = \tau(\mathcal{D})$ is the first exit time from \mathcal{D} . To apply the results from [25] (see also, [46, Theorem 3.1]), we define $\tilde{q}(j|i, u) = q(j|i, u)$ for $j, i \in \mathcal{D}$, $\tilde{q}(\Delta|i, u) = \sum_{j \in \mathcal{D}^c} q(j|i, u)$, and $q(j|\Delta, u) = 0$ for all $j \in \mathcal{S}, u \in \mathbb{U}$. Here Δ is an absorbing state. Also, define $\mathbb{U}(\Delta) = u_\infty$. Recall the definition (3.1) and given a history dependent control $\zeta \in \mathcal{U}$ we can redefine another admissible control $\tilde{\zeta}$ to satisfy $\tilde{\zeta}_t \in \mathbb{U}(\Delta)$ if $X_{t-} \in \mathcal{D}^c$. Let $\tilde{\mathbf{X}}$ be a process, corresponding to the control $\tilde{\zeta}$, taking values in $\mathcal{D} \cup \{\Delta\}$. Note that the law of $\{(X_t, \zeta_t) : t < \tau\}$ is same as $\{(\tilde{X}_t, \tilde{\zeta}_t) : t < \tau_*\}$, where τ_* denotes the first hitting time to Δ by $\tilde{\mathbf{X}}$. Now we can apply Dynkin's formula [25, Lemma 3.2] to $\tilde{\mathbf{X}}$. We apply it on $\psi_{\mathcal{D}}$. Set $\psi_{\mathcal{D}}(\Delta) = 0$. Then

$$\begin{aligned}
\psi_{\mathcal{D}}(i) &= \tilde{\mathbb{E}}_i^{\tilde{\zeta}} \left[e^{\int_0^T (c(\tilde{X}_s, \tilde{\zeta}_s) - \rho_{\mathcal{D}}) ds} \psi_{\mathcal{D}}(\tilde{X}_T) \mathbf{1}_{\{T < \tau_*\}} \right] \\
&\quad - \tilde{\mathbb{E}}_i^{\tilde{\zeta}} \left[\int_0^{T \wedge \tau_*} e^{\int_0^t (c(\tilde{X}_s, \tilde{\zeta}_s) - \rho_{\mathcal{D}}) ds} \left(\sum_{j \in \mathcal{D}} \psi_{\mathcal{D}}(j) q(j|\tilde{X}_t, \tilde{\zeta}_t) + (c(\tilde{X}_t, \tilde{\zeta}_t) - \rho_{\mathcal{D}}) \psi_{\mathcal{D}}(\tilde{X}_t) \right) dt \right] \\
&= \mathbb{E}_i^\zeta \left[e^{\int_0^T (c(X_s, \zeta_s) - \rho_{\mathcal{D}}) ds} \psi_{\mathcal{D}}(X_T) \mathbf{1}_{\{T < \tau\}} \right] \\
&\quad - \mathbb{E}_i^\zeta \left[\int_0^{T \wedge \tau} e^{\int_0^t (c(X_s, \zeta_s) - \rho_{\mathcal{D}}) ds} \left(\sum_{j \in \mathcal{D}} \psi_{\mathcal{D}}(j) q(j|X_t, \zeta_t) + (c(X_t, \zeta_t) - \rho_{\mathcal{D}}) \psi_{\mathcal{D}}(X_t) \right) dt \right] \\
&\leq \mathbb{E}_i^\zeta \left[e^{\int_0^T (c(X_s, \zeta_s) - \rho_{\mathcal{D}}) ds} \psi_{\mathcal{D}}(X_T) \mathbf{1}_{\{T < \tau\}} \right] \\
&\leq (\sup_{\mathcal{D}} \psi_{\mathcal{D}}) \mathbb{E}_i^\zeta \left[e^{\int_0^T (c(X_s, \zeta_s) - \rho_{\mathcal{D}}) ds} \right],
\end{aligned}$$

where in the first inequality we use (3.16). Now taking logarithm on both sides, dividing by T and letting $T \rightarrow \infty$, we obtain

$$\rho_{\mathcal{D}} \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_i^\zeta \left[e^{\int_0^T c(X_t, \zeta_t)} \right].$$

Since $\zeta \in \mathfrak{U}$ is arbitrary, we obtain (3.13). This completes the proof. \square

We begin with the following hitting time estimate which follows from [Assumption 3.4](#) (compare with [Lemma 2.3](#)).

Lemma 3.2. *Suppose that [Assumption 3.4](#) holds. Let \mathcal{B} be a finite set containing \mathcal{K} . Then for any $\zeta \in \mathfrak{U}_m$ we get the following.*

(i) Under [Assumption 3.4\(a\)](#), we have

$$\mathbb{E}_i^\zeta \left[e^{\gamma \check{\tau}(\mathcal{B})} \mathcal{V}(X_{\check{\tau}(\mathcal{B})}) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in \mathcal{B}^c, \quad (3.17)$$

where $\check{\tau}(\mathcal{B}) = \inf\{t > 0: X_t \in \mathcal{B}\}$.

(ii) Under [Assumption 3.4\(b\)](#), we have

$$\mathbb{E}_i^\zeta \left[e^{\int_0^{\check{\tau}(\mathcal{B})} \ell(X_s) ds} \mathcal{V}(X_{\check{\tau}(\mathcal{B})}) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in \mathcal{B}^c. \quad (3.18)$$

Proof. We only provide a proof for (i) and the proof for (ii) would be analogous. Suppose [Assumption 3.4\(a\)](#) holds. Let \mathcal{D}_n be a collection of finite, increasing sets converging to S . By τ_n we denote the first exit time from \mathcal{D}_n . Choose n large enough so that $\mathcal{B} \Subset \mathcal{D}_n$. Applying Dynkin's formula [[22](#), Appendix C.3] and using (3.4) it follows that

$$\mathbb{E}_i^\zeta \left[e^{\gamma(\check{\tau}(\mathcal{B}) \wedge T \wedge \tau_n)} \mathcal{V}(X_{\check{\tau}(\mathcal{B}) \wedge T \wedge \tau_n}) \right] \leq \mathcal{V}(i) \quad \text{for all } i \in \mathcal{B}^c \cap \mathcal{D}_n,$$

for $T > 0$. Letting $T \rightarrow \infty$ first and, then $n \rightarrow \infty$ and applying Fatou's lemma we obtain (3.17). This completes the proof. \square

Let $\{\mathcal{D}_n\}$ be a collection of finite, increasing sets converging to S . Denote by (ρ_n, ψ_n) the eigenpair in the domain \mathcal{D}_n obtained by [Lemma 3.1](#). Next we study limit of ρ_n as $n \rightarrow \infty$.

Lemma 3.3. *Grant [Assumptions 3.1](#) to [3.4](#). Then the following holds.*

(i)

$$\rho_n \leq \inf_{\zeta \in \mathfrak{U}} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_{i_0}^\zeta \left[e^{\int_0^T c(X_t, \zeta_t) dt} \right] = \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_{i_0}(c, \zeta), \quad (3.19)$$

and $\{\rho_n\}$ is bounded above.

(ii) The sequence $\{\rho_n\}$ is bounded and we have $\liminf_{n \rightarrow \infty} \rho_n \geq 0$.

Proof. Since

$$\rho_n \psi_n(i) = \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi_n(j) q(j|i, u) + c(i, u) \psi_n(i) \right] \quad i \in \mathcal{D}_n, \quad (3.20)$$

and $\psi_n \geq 0$, it follows from [Assumption 3.3\(c\)](#) that $\psi_n(i_0) > 0$. Then (3.19) follows from (3.13). To complete the proof of (i) we only need to show that

$$\mathcal{E}_i(c, \zeta) \leq \kappa \quad \forall \zeta \in \mathfrak{U}, \quad i \in S, \quad (3.21)$$

for some constant κ . Since c is bounded under [Assumption 3.4\(a\)](#), the above is immediate. Under [Assumption 3.4\(b\)](#), we write (3.5) as

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} \mathcal{V}(j) q(j|i, u) \leq (\kappa_1 - \ell(i)) \mathcal{V}(i) \quad \forall \quad i \in S,$$

where $\kappa_1 = \widehat{C} \max_{\mathcal{K}} [\mathcal{V}]^{-1}$. Applying the arguments of [Lemma 3.2](#) we then get

$$\mathbb{E}_i^\zeta \left[e^{\int_0^T (\ell(X_t) - \kappa_1) dt} \mathcal{V}(X_T) \right] \leq \mathcal{V}(i).$$

Since $\mathcal{V} \geq 1$, taking logarithm in the above, dividing by T and letting $T \rightarrow \infty$ we obtain that

$$\mathcal{E}_i(\ell, \zeta) \leq \kappa_1 \quad \forall \zeta \in \mathfrak{U}, \quad i \in S.$$

Again, since $\max_{u \in \mathbb{U}(\cdot)} c(\cdot, u) \leq \ell(\cdot) + \kappa_2$ for some $\kappa_2 > 0$ by [Assumption 3.4\(b\)](#), we have [\(3.21\)](#) from the above estimate.

Next we consider (ii). First we show that ρ_n is bounded below. Since $\psi_n(i_0) > 0$, normalizing ψ_n we can assume that $\psi_n(i_0) = 1$. Since $c \geq 0$, from [\(3.20\)](#), it follows that

$$\rho_n \geq \min_{u \in \mathbb{U}(i_0)} \left[\sum_{j \in S} \psi_n(j) q(j|i_0, u) \right] \geq \min_{u \in \mathbb{U}(i_0)} q(i_0|i_0, u).$$

Thus ρ_n is bounded from below.

Thus we remain to show that $\hat{\rho} = \liminf_{n \rightarrow \infty} \rho_n \geq 0$. Suppose, on the contrary, that $\hat{\rho} < 0$. We therefore have, along some subsequence, $\rho_n \rightarrow \hat{\rho}$, as $n \rightarrow \infty$. Thus, using [Assumption 3.1\(c\)](#) and [\(3.20\)](#), we have

$$\psi_n(j) \leq \sup_{u \in \mathbb{U}(i_0)} \frac{-q(i_0|i_0, u)}{q(j|i_0, u)} := \kappa_1 \quad \text{for all } j \in S \setminus \{i_0\}, \quad (3.22)$$

for all large n . Hence, by a standard diagonalization argument, there exists a function ψ with $\psi(i_0) = 1$ such that along some subsequence $\psi_n(i) \rightarrow \psi(i)$, as $n \rightarrow \infty$, for all $i \in S$. Let \tilde{v}_n be a minimizing selector of [\(3.20\)](#). Since $\mathbb{U}(i)$ is compact for each $i \in S$, along a further subsequence, $\tilde{v}_n(i) \rightarrow \tilde{v}(i)$, as $n \rightarrow \infty$, for all $i \in S$. Therefore, letting $n \rightarrow \infty$ in

$$\rho_n \psi_n(i) = \left[\sum_{j \in S} \psi_n(j) q(j|i, \tilde{v}_n(i)) + c(i, \tilde{v}_n(i)) \psi_n(i) \right],$$

using [Assumption 3.3\(a\)-\(b\)](#) and Fatou's lemma, we obtain

$$\hat{\rho} \psi(i) \geq \left[\sum_{j \in S} \psi(j) q(j|i, \tilde{v}(i)) + c(i, \tilde{v}(i)) \psi(i) \right] \quad i \in S. \quad (3.23)$$

Since $\hat{\rho} < 0$ and $c \geq 0$, from [\(3.23\)](#) we deduce that

$$\left[\sum_{j \in S} \psi(j) q(j|i, \tilde{v}(i)) \right] \leq 0 \quad \forall i \in S. \quad (3.24)$$

Applying Dynkin's formula to [\(3.24\)](#), for any $t > 0$ and $i \in S$, it follows that

$$\mathbb{E}_i^{\tilde{v}}[\psi(X_t)] \leq \psi(i).$$

Therefore, $\{\psi(X_t)\}$ is a supermartingale with respect to the canonical filtration of \mathbf{X} , and thus, by Doob's martingale convergence theorem $\psi(X_t)$ converges as $t \rightarrow \infty$. By [Assumption 3.4](#), \mathbf{X} is recurrent implying the skeleton process $\{X_n : n \in \mathbb{N}\}$ is also recurrent (cf. [\[1, Proposition 5.1.1\]](#)). Therefore, $\{X_n : n \in \mathbb{N}\}$ visits every state of S infinitely often and this is possible only if $\psi \equiv 1$. This contradicts [\(3.23\)](#). Thus we must have $\hat{\rho} \geq 0$. This completes the proof. \square

Using [Lemma 3.3](#) and ideas from [Lemma 2.5](#) we can now establish the existence of an eigenfunction on S .

Lemma 3.4. *Consider [Assumptions 3.1 to 3.4](#). Then there exists $(\rho, \psi^*) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, with $\psi^* > 0$, satisfying*

$$\rho \psi^*(i) = \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad i \in S. \quad (3.25)$$

Moreover, we have

$$(i) \quad \rho \leq \lambda^*. \quad (3.26)$$

(ii) There exists a finite set $\mathcal{B} \supset \mathcal{K}$ such that for any minimizing selector v^* of (3.25) we have

$$\psi^*(i) = \mathbb{E}_i^{v^*} \left[e^{\int_0^{\tilde{\tau}(\mathcal{B})} (c(X_t, v^*(X_t)) - \rho) ds} \psi^*(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (3.27)$$

Proof. From Lemma 3.3 we know that the sequence $\{\rho_n\}$ is a bounded and $\liminf_{n \rightarrow \infty} \rho_n \geq 0$. Thus one can find a subsequence such that along this subsequence, ρ_n converges to some $\rho \geq 0$, as $n \rightarrow \infty$. Now we repeat the method of Lemma 2.5. Using Assumption 3.4 and the fact $c \geq 0$, we can find a finite set \mathcal{B} containing \mathcal{K} such that

(i) Under Assumption 3.4(a):

$$\left(\max_{u \in \mathbb{U}(i)} c(i, u) - \rho_n \right) < \gamma \quad \forall i \in \mathcal{B}^c, \text{ for all } n \text{ large.} \quad (3.28)$$

(ii) Under Assumption 3.4(b):

$$\left(\max_{u \in \mathbb{U}(i)} c(i, u) - \rho_n \right) < \ell(i) \quad \forall i \in \mathcal{B}^c, \text{ for all } n \text{ large.} \quad (3.29)$$

Now we scale ψ_n by multiplying a suitable scalar so that it touches \mathcal{V} from below. In particular, define

$$\theta_n = \sup\{\kappa > 0 : (\mathcal{V} - \kappa\psi_n) > 0 \text{ in } S\}.$$

Replacing ψ_n by $\theta_n\psi_n$ and from the arguments of Lemma 2.5 we see that ψ_n touches \mathcal{V} inside \mathcal{B} . Since $\psi_n \leq \mathcal{V}$ for all large n , by a standard diagonalization argument, one can extract a subsequence so that along this subsequence, $\psi_n(i) \rightarrow \psi^*(i)$ for all $i \in S$, as $n \rightarrow \infty$, and $\psi^* \leq \mathcal{V}$. Using (3.20) and Fatou's lemma we get that (see (3.23))

$$\rho\psi^*(i) \geq \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j)q(j|i, u) + c(i, u)\psi^*(i) \right] \quad i \in S. \quad (3.30)$$

On the other hand, for every $i \in S$ and $u \in \mathbb{U}(i)$, we obtain from (3.20) that

$$\begin{aligned} \rho\psi^*(i) &= \lim_{n \rightarrow \infty} \rho_n\psi_n = \lim_{n \rightarrow \infty} \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi_n(j)q(j|i, u) + c(i, u)\psi_n(i) \right] \\ &\leq \lim_{n \rightarrow \infty} \left[\sum_{j \in S} \psi_n(j)q(j|i, u) + c(i, u)\psi_n(i) \right] \\ &= \sum_{j \in S} \psi^*(j)q(j|i, u) + c(i, u)\psi^*(i), \end{aligned}$$

by dominated convergence theorem, where we use the fact that $\psi_n \leq \mathcal{V}$ for all large n . Since u is arbitrary, combining with (3.30) we get (3.25). Next we show that $\psi^* > 0$. By our construction we have $(\mathcal{V} - \psi_n) = 0$ at some point in \mathcal{B} for all n large. Hence $(\mathcal{V} - \psi^*) = 0$ at some point in \mathcal{B} . Since $\mathcal{V} \geq 1$, we deduce that ψ^* is nonzero. We claim that $\psi^* > 0$. If not, then we must have $\psi^*(i) = 0$ for some $i \in S$. Then for any minimizing selector v^* of (3.25), we have

$$\sum_{j \neq i} \psi^*(j)q(j|i, v^*(i)) = 0. \quad (3.31)$$

Since the Markov chain \mathbf{X} is irreducible under v^* , from (3.31) it follows that $\psi^* \equiv 0$. This is a contradiction to fact that ψ^* is nontrivial. This proves the claim.

Now we prove (i). In view of (3.13), it is enough to show that given $i \in S$, $\psi_n(i) > 0$ for all n large. Since $\psi^* > 0$ and $\psi_n(i) \rightarrow \psi^*(i)$ as $n \rightarrow \infty$, we have $\psi_n(i) > 0$ for all large enough n . Hence $\lim_{n \rightarrow \infty} \rho_n \leq \inf_{\zeta \in \mathcal{U}} \mathcal{E}_i(c, \zeta)$ for all i . This gives us (3.26).

(ii) follows from an argument similar to Lemma 2.5. \square

Remark 3.1. It is easy to check that we can also apply the argument of Lemma 3.4 for every stationary Markov control. More precisely, if we impose Assumptions 3.1 to 3.4, then for every Markov control $v \in \mathfrak{U}_{\text{sm}}$ there exists $(\psi_v, \rho_v) \in \mathbb{R}_+ \times \mathcal{O}(\mathcal{V})$, $\psi_v > 0$, satisfying

$$\rho_v \psi_v(i) = \sum_{j \in S} \psi_v(j) q(j|i, v(i)) + c(i, v(i)) \psi_v(i) \quad \forall i \in S. \quad (3.32)$$

Furthermore, $\rho_v \leq \inf_i \mathcal{E}_i(c, v)$ and for some finite set $\mathcal{B} \supset \mathcal{K}$

$$\psi_v(i) = \mathbb{E}_i^v \left[e^{\int_0^{\tilde{\tau}(\mathcal{B})} (c(X_t, v(X_t)) - \rho) dt} \psi_v(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (3.33)$$

Next we show that $\rho = \lambda^*$. To do so we can use the perturbed cost \tilde{c}_n introduced in Section 2. In fact, following an argument similar to Lemma 2.7 we can prove the following.

Lemma 3.5. *Assume Assumptions 3.1 to 3.4. Then any minimizing selector of (3.25), that is, any $v^* \in \mathfrak{U}_{\text{sm}}$ satisfying*

$$\min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] = \left[\sum_{j \in S} \psi(j) q(j|i, v^*(i)) + c(i, v^*(i)) \psi^*(i) \right] \quad \forall i \in S, \quad (3.34)$$

is an optimal control and $\rho = \lambda^*$. Moreover, ψ^* is the unique solution of (3.25) with $\psi^*(i_0) = 1$.

Now we are ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. (i) follows from Lemmas 3.4 and 3.5. By Lemma 3.5 we also get that any minimizing selector of (3.8) is an optimal Markov control. Using Remark 3.1 and the arguments of Theorem 2.1 we can also show the converse direction, that is, if for some $v \in \mathfrak{U}_{\text{sm}}$ we have $\mathcal{E}_i(c, v) = \lambda^*$ then v satisfies (3.34). This gives us (ii). \square

We conclude this section with the following remark.

Remark 3.2. Assumption 3.3(c) can be replaced by other similar assumption. For instance, if the killed process communicates with every state from i_0 before leaving the domain \mathcal{D}_n , for large n , then our method applies. More precisely, for every \mathcal{D}_n , $v \in \mathfrak{U}_{\text{sm}}$ and for every $j \in \mathcal{D}_n \setminus \{i_0\}$, if there exists distinct $i_1, i_2, \dots, i_m \in \mathcal{D}_n \setminus \{i_0\}$ satisfying

$$q(i_1|i_0, v(i_0)) q(i_2|i_1, v(i_1)) \cdots q(j|i_m, v(i_m)) > 0,$$

then the conclusion of Theorem 3.1 holds. Note that in this case we also get $\psi_n(i_0) > 0$ in \mathcal{D}_n .

3.1. Near-monotone cost. In this section we replace Assumption 3.4 with a near-monotone assumption stated below.

Assumption 3.5. Define $\lambda_m = \inf_{i \in S} \inf_{v \in \mathfrak{U}_{\text{sm}}} \mathcal{E}_i(c, v)$, and

$$\inf_{v \in \mathfrak{U}_{\text{sm}}} \mathcal{E}_i(c, v) < \infty \quad \forall i \in S,$$

and the cost function c satisfies the *near-monotone* condition with respect to λ_m , that is,

$$\liminf_{n \rightarrow \infty} \inf_{k \geq n} \inf_{u \in \mathbb{U}(k)} c(k, u) > \lambda_m. \quad (3.35)$$

Note that by [Assumption 3.5](#), $\lambda^* \leq \lambda_m < \infty$ where $\lambda^* = \inf_{i \in S} \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_i(c, \zeta)$ is given by [\(3.3\)](#). The following result gives the existence of an optimal stationary Markov control. This result should be compared with [\[45\]](#) where existence of optimal stationary Markov control is obtained under [\(3.35\)](#), but our hypotheses are weaker and we also allow history dependent controls (see [\(A1\)](#), [\(A2\)\(ii\)](#) in [\[45\]](#)).

Theorem 3.2. *Grant [Assumptions 3.1 to 3.3](#) and [3.5](#). We also assume that \mathbf{X} is recurrent under any control $v \in \mathfrak{U}_{\text{sm}}$. Then there exists a positive function ψ^* satisfying*

$$\lambda^* \psi^*(i) \geq \min_{u \in \mathfrak{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad \text{for } i \in S. \quad (3.36)$$

Furthermore, we have $\lambda^* = \lambda_m$ and any measurable selector of [\(3.36\)](#) is an optimal stationary Markov control.

Proof. The proof is similar to [Theorem 2.3](#). Recall the eigenpair (ρ_n, ψ_n) satisfying

$$\rho_n \psi_n(i) = \min_{u \in \mathfrak{U}(i)} \left[\sum_{j \in S} \psi_n(j) q(j|i, u) + c(i, u) \psi_n(i) \right] \quad i \in \mathcal{D}_n. \quad (3.37)$$

Since the CTCMP \mathbf{X} is recurrent under any stationary Markov control, the argument of [Lemma 3.3](#) works, and therefore, we have

$$0 \leq \liminf_{n \rightarrow \infty} \rho_n \leq \limsup_{n \rightarrow \infty} \rho_n \leq \inf_{\zeta \in \mathfrak{U}} \mathcal{E}_{i_0}(c, \zeta). \quad (3.38)$$

Normalize ψ_n to satisfying $\psi_n(i_0) = 1$. Let \tilde{v}_n be a minimizing selector of [\(3.37\)](#)–[\(3.38\)](#) it then follows that $\psi_n(i) \leq \kappa_i$ for all n (see [\(3.22\)](#)), for some constant κ_i . Using a standard diagonalization argument, we can find a subsequence along which we have

$$\rho_n \rightarrow \rho, \quad \psi_n(i) \rightarrow \psi^*(i) \quad \text{and} \quad \tilde{v}_n(i) \rightarrow v(i),$$

for all $i \in S$. From Fatou's lemma, we then have

$$\begin{aligned} \rho \psi^*(i) &\geq \sum_{j \in S} \psi^*(j) q(j|i, v(i)) + c(i, v(i)) \psi^*(i) \\ &\geq \min_{u \in \mathfrak{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad i \in S. \end{aligned} \quad (3.39)$$

Note that $\psi^*(i_0) = 1$. Using irreducibility it is then easy to see that $\psi^* > 0$. This in particular, implies that for any $i \in S$ we have $\psi_n(i) > 0$ for all large n . Using [\(3.13\)](#) and [\(3.37\)](#) we obtain $\rho \leq \lambda^* \leq \lambda_m$. Since $\rho \leq \lambda_m$, applying the near-monotonicity condition [\(3.35\)](#), we find a finite set \mathcal{B} such that

$$\inf_{u \in \mathfrak{U}(i)} c(i, u) - \rho > 0 \quad \text{for } i \in \mathcal{B}^c. \quad (3.40)$$

Now consider a measurable selector v of [\(2.61\)](#) and by $\check{\tau} = \check{\tau}(\mathcal{B})$ we denote the first hitting time to \mathcal{B} . Since \mathbf{X} is recurrent under v , we have $\mathbb{P}_i(\check{\tau} < \infty) = 1$ for all $i \in \mathcal{B}^c$ (cf. [\[1, Proposition 5.1.1\]](#)). Thus applying Dynkin's formula on [\(3.39\)](#), followed by Fatou's lemma, we get

$$\psi^*(i) \geq \mathbb{E}_i^v \left[e^{\int_{t=0}^{\check{\tau}-1} (c(X_t, v(X_t)) - \rho) dt} \psi^*(X_{\check{\tau}}) \right] \geq \min_{j \in \mathcal{B}} \psi^*(j) \quad i \in \mathcal{B}^c,$$

using [\(3.40\)](#). Thus, $\inf_{i \in S} \psi^*(i) > 0$. Now we can repeat the argument of [Theorem 3.2](#) to show that $\rho = \lambda^* = \lambda_m = \mathcal{E}_i(c, v)$ for all i . This completes the proof. \square

In a similar fashion we can extend [Theorem 2.4](#) to a continuous time set-up.

Theorem 3.3. *Grant Assumption 3.5. Also, assume that there exists a function $W : S \rightarrow [1, \infty)$ satisfying $W(i) \geq i$ for all large i and*

$$\sup_{u \in \mathbb{U}(i)} \sum_{j \in S} W(i) q(j|i, u) \leq g(i) \quad \text{for } i \in S,$$

for some function $g : S \rightarrow \mathbb{R}$ satisfying $\lim_{i \rightarrow \infty} g(i) = 0$. Furthermore, for some $\eta > 0$ we have

$$\min_{u \in \mathbb{U}(i)} q(i-1|i, u) \geq \eta \quad \text{for all } i \geq 1,$$

and $q(\cdot|1, u)$ supported in a finite set C , independent of u . We also assume that for $\mathcal{D}_n := \{1, \dots, n\}$, $v \in \mathfrak{U}_{\text{sm}}$ and any $j \in \mathcal{D}_n \setminus \{1\}$ there exists distinct $i_1, i_2, \dots, i_k \in \mathcal{D}_n$ we have

$$q(i_1|1, v(1))q(i_2|i_1, v(i_1)) \cdots q(j|i_k, v(i_k)) > 0.$$

Furthermore, \mathbf{X} is irreducible under any stationary Markov control. Then there exists a positive function ψ^* satisfying

$$\lambda^* \psi^*(i) \geq \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \psi^*(j) q(j|i, u) + c(i, u) \psi^*(i) \right] \quad \text{for } i \in S. \quad (3.41)$$

Furthermore, we have $\lambda^* = \lambda_m$ and any measurable selector of (3.41) is an optimal stationary Markov control.

The proof is analogous to Theorem 2.4 and thus omitted.

4. POLICY ITERATION

In this section we propose policy improvement algorithms (PIA) and establish its convergence to the optimal value. To do so, we introduce the notion of generalized Perron-Frobenius eigenvalue. Our definition below can be seen as the counterpart of the elliptic generalized eigenvalue in the case of discrete Markov chain (cf. [7, 12, 13, 39]). For discrete time CMP we define the generalized Perron-Frobenius eigenvalue as follows

$$\lambda_1^d = \inf \{ \lambda \in \mathbb{R} : \exists \Psi > 0 \text{ satisfying } \min_{u \in \mathbb{U}(i)} e^{c(i, u)} \sum_{j \in S} \Psi(j) P(j|i, u) \leq e^\lambda \Psi(i) \forall i \in S \}. \quad (4.1)$$

Similarly, we can also define a generalized Perron-Frobenius eigenvalue for every stationary Markov control v as follows.

$$\lambda_1^d(v) = \inf \{ \lambda \in \mathbb{R} : \exists \Psi > 0 \text{ satisfying } e^{c(i, v(i))} \sum_{j \in S} \Psi(j) P(j|i, v(i)) \leq e^\lambda \Psi(i) \forall i \in S \}. \quad (4.2)$$

In relation with the risk-sensitive control we would be interested to know whether $\lambda_1^d = \lambda^*$. Note that for nondegenerate elliptic operator this is not true in general. See for instance, Example 3.1 in [7]. Our next result would be helpful in answering this question.

Lemma 4.1. *Assume that \mathbf{X} is irreducible under every stationary Markov control and λ_1^d is finite. Let \mathcal{D} be a finite domain and $(\rho, \psi) \in \mathbb{R} \times \mathcal{B}_{\mathcal{D}}^+$, $\psi \neq 0$, be such that*

$$\min_{u \in \mathbb{U}(i)} e^{c(i, u)} \sum_{j \in S} \psi(j) P(j|i, u) = e^\rho \psi(i), \quad \forall i \in \mathcal{D}, \quad \text{and } \psi(i) = 0 \quad \forall i \in \mathcal{D}^c. \quad (4.3)$$

Then we must have $\rho \leq \lambda_1^d$. Similar result also holds under every stationary Markov control.

Proof. Suppose, on the contrary, that $\rho > \lambda_1^d$. From the definition of λ_1^d , we find a pair (λ, Ψ) with $\Psi > 0$ satisfying

$$\min_{u \in \mathbb{U}(i)} e^{c(i, u)} \sum_{j \in S} \Psi(j) P(j|i, u) \leq e^\lambda \Psi(i) \quad \text{for all } i \in S. \quad (4.4)$$

Consider a minimizing selector $v \in \mathfrak{U}_{\text{sm}}$ of the left-hand side of (4.4). From (4.3) we then have

$$e^{c(i,v(i))} \sum_{j \in S} \psi(j) P(j|i, v(i)) \geq e^\rho \psi(i), \quad \forall i \in \mathcal{D}, \quad \text{and } \psi(i) = 0 \quad \forall i \in \mathcal{D}^c. \quad (4.5)$$

Define

$$\theta = \sup\{\kappa > 0 : \Psi - \kappa\psi > 0 \text{ in } \mathcal{D}\}.$$

Since $\psi \geq 0$, we have $\theta \in (0, \infty)$ and $V := \Psi - \theta\psi$ is non-negative in S . Furthermore, V must vanish at some point, say j_0 , in \mathcal{D} and $V > 0$ in \mathcal{D}^c .

Since $\rho \geq \lambda$, using (4.4) and (4.5) we also get

$$V(i) \geq e^{c(i,v(i))-\lambda} \sum_{j \in S} V(j) P(j|i, v(i)) \quad \forall i \in \mathcal{D}. \quad (4.6)$$

Denote by $\tau = \tau(\mathcal{D})$ the first exit time from \mathcal{D} . From optional sampling theorem we then obtain from (4.6) that (see (2.16))

$$0 = V(j_0) \geq \mathbb{E}_{j_0}^v \left[e^{\sum_{t=0}^{\tau \wedge T-1} (c(X_t, v(X_t)) - \lambda)} V(X_{\tau \wedge T}) \right].$$

Letting $T \rightarrow \infty$ and applying Fatou's lemma we get from above that

$$0 \geq \mathbb{E}_{j_0}^v \left[e^{\sum_{t=0}^{\tau-1} (c(X_t, v(X_t)) - \lambda)} V(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right].$$

Since \mathbf{X} is irreducible, we have $\mathbb{P}_{j_0}^v(\tau < \infty) > 0$ and also $V > 0$ in \mathcal{D}^c . This is clearly a contradiction to the above. Thus we must have $\rho \leq \lambda_1^d$. \square

The following remark is immediate from Remark 2.1, Lemma 4.1 and the proof of Theorem 2.3.

Remark 4.1. Suppose that Assumption 2.1 holds and \mathbf{X} is irreducible under every stationary Markov control. Then $\lim_{n \rightarrow \infty} \rho_n = \lambda_1^d$. Moreover, if $\lambda_1^d < \infty$, there exists a positive eigenvector V satisfying

$$\min_{u \in \mathbb{U}(i)} e^{c(i,u)} \sum_{j \in S} V(j) P(j|i, u) \leq e^{\lambda_1^d} V(i) \quad i \in S.$$

4.1. Discrete time stable case. In this section we propose a policy improvement algorithm (PIA) and show that it converges to the optimal value λ^* . To this aim we use a stronger hypothesis compared to Assumption 2.2.

Assumption 4.1. We suppose that Assumption 2.2 holds for a norm-like function \mathcal{V} . Furthermore, in case of Assumption 2.2(b), we have $\max_{u \in \mathbb{U}(i)} c(i, u) \leq \eta \ell(i)$ for $i \in S$ and some $\eta \in (0, 1)$. Also, there exists a state z_o in S such that

$$\inf_{u \in \mathbb{U}(i)} P(z_o|i, u) > 0 \quad \text{for all } i \in S. \quad (4.7)$$

(4.7) will be used to construct a suitable small set and to apply certain convergence result from [38]. Assumptions 2.1 and 4.1 are imposed throughout this section. Suppose that v is a stationary Markov control and (ρ_v, ψ_v) be the corresponding eigenpair obtained in Remark 2.2. Let $\mathcal{B} \supset \mathcal{K}$ be such that for $i \in \mathcal{B}^c$ we have

$$\left. \begin{aligned} \max_{u \in \mathbb{U}(i)} c(i, u) - \rho_v &< \alpha\gamma, \quad \text{under Assumption 2.2(a),} \\ \max_{u \in \mathbb{U}(i)} c(i, u) - \rho_v &< \alpha\ell(i), \quad \text{under Assumption 2.2(b),} \end{aligned} \right\} \quad (4.8)$$

for some $\alpha \in (0, 1)$. Then the arguments of Lemma 2.5 gives us

$$\psi_v(i) = \mathbb{E}_i^v \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c(X_t, v(X_t)) - \rho_v)} \psi_v(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c. \quad (4.9)$$

We are going to use this observation in the later part of this section. We now describe our PIA.

Algorithm 4.1. Policy iteration.

1. Initialization. Set $k = 0$ and select any $v_0 \in \mathfrak{U}_{\text{sm}}$.
2. Value determination. Let V_k be the unique principal eigenfunction satisfying $V_k(i_0) = 1$ and

$$e^{\rho_k} V_k(i) = e^{c(i, v_k(i))} \sum_{j \in S} V_k(j) P(j|i, v_k(i)) \quad i \in S. \quad (4.10)$$

Existence of a unique principal eigenfunction in (4.10) follows from [Remark 2.2](#) and [Lemma 2.7](#). We let $\lambda_k := \lambda_1^d(v_k) = \mathcal{E}_i(c, v_k) = \rho_k$.

3. Policy improvement. Choose any $v_{k+1} \in \mathfrak{U}_{\text{sm}}$ satisfying

$$v_{k+1}(i) \in \text{Arg min}_{u \in \mathbb{U}(i)} \left[e^{c(i, u)} \sum_{j \in S} V_k(j) P(j|i, u) \right], \quad i \in S.$$

Then our main result of this section is

Theorem 4.1. *Under [Assumptions 2.1](#) and [4.1](#) the following hold.*

- (i) *For all $k \in \mathbb{N}$, we have $\lambda_{k+1} \leq \lambda_k$ and $\lim_{k \rightarrow \infty} \lambda_k = \lambda^*$.*
- (ii) *V_k converges pointwise, as $k \rightarrow \infty$, to ψ^* where ψ^* is the unique solution to (2.9).*

Our proof of [Theorem 4.1](#) is inspired from [6] which also establishes convergence of PIA for controlled diffusions. The proof technique of [6] uses several estimates from elliptic partial differential equations which are not obvious in the present situation. So our proofs requires a more careful analysis. We denote by $c_k(i) = c(i, v_k(i))$. It is also obvious from (4.2) that $\lambda_{k+1} \leq \lambda_k$ for all $k \geq 0$. Fix a set \mathcal{B} containing \mathcal{K} . Since $\rho_v \geq 0$, from [Assumption 4.1](#) (4.8) holds for $\rho_{v_0} = \lambda_0$. Let $\kappa_k = \min_{\mathcal{B}} \frac{\mathcal{V}}{V_k}$ and replace V_k by $\kappa_k V_k$. Using (4.9) and [Lemma 2.3](#), it then follows that $V_k \leq \mathcal{V}$ in S . Define

$$\theta_{k+1}(i) = 1 - \frac{1}{V_k(i)} e^{c_{k+1}(i) - \lambda_k} \sum_{j \in S} V_k(j) P(j|i, v_{k+1}(i)). \quad (4.11)$$

Since

$$V_k(i) \geq \min_{u \in \mathbb{U}(i)} \left[e^{c(i, u) - \lambda_k} \sum_{j \in S} V_k(j) P(j|i, u) \right] = e^{c_{k+1}(i) - \lambda_k} \sum_{j \in S} V_k(j) P(j|i, v_{k+1}(i)),$$

we have $0 \leq \theta_k \leq 1$ for all $k \in \mathbb{N}$. We begin with the following estimate which will be useful to establish convergence of V_k .

Lemma 4.2. *Grant [Assumptions 2.1](#) and [4.1](#). Then the following hold.*

- (i) *There exists κ , independent of k , such that*

$$V_k(i) \leq \kappa (\mathcal{V}(i))^\alpha \quad \text{for all } i \in S, \quad (4.12)$$

where α is given by (4.8).

- (ii) *For every $i \in S$ we have $\inf_{k \in \mathbb{N}} V_k(i) > 0$.*

Proof. (i) actually follows from (4.8) and [Lemma 2.3](#). Since $\lambda_k \leq \lambda_0$, we see from (4.8) that

$$\begin{aligned} \max_{u \in \mathbb{U}(i)} c(i, u) - \lambda_k &< \alpha \gamma, \quad \text{under [Assumption 2.2\(a\)](#),} \\ \max_{u \in \mathbb{U}(i)} c(i, u) - \lambda_k &< \alpha \ell(i), \quad \text{under [Assumption 2.2\(b\)](#),} \end{aligned}$$

for all $i \in \mathcal{B}^c$. Therefore, the stochastic representation of V_k is possible with respect to \mathcal{B} , that is,

$$V_k(i) = \mathbb{E}_i^{v_k} \left[e^{\sum_{t=0}^{\tilde{\tau}(\mathcal{B})-1} (c_k(X_t) - \lambda_k)} V_k(X_{\tilde{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c.$$

Let us consider [Assumption 2.2\(a\)](#) first. Since $\alpha \in (0, 1)$, from [Lemma 2.3](#) it follows that

$$\begin{aligned} V_k(i) &\leq \mathbb{E}_i^{v_k} \left[e^{\alpha \gamma \check{\tau}(\mathcal{B})} V_k(X_{\check{\tau}(\mathcal{B})}) \right] \\ &\leq \mathbb{E}_i^{v_k} \left[e^{\gamma \check{\tau}(\mathcal{B})} (V_k)^{1/\alpha} (X_{\check{\tau}(\mathcal{B})}) \right]^\alpha \\ &\leq (\max_{\mathcal{B}} \mathcal{V})^{1-\alpha} \mathbb{E}_i^{v_k} \left[e^{\gamma \check{\tau}(\mathcal{B})} \mathcal{V} (X_{\check{\tau}(\mathcal{B})}) \right]^\alpha \\ &\leq (\max_{\mathcal{B}} \mathcal{V})^{1-\alpha} \mathcal{V}^\alpha(i) \end{aligned}$$

for all $i \in \mathcal{B}^c$. This gives (i). Similar argument also works for [Assumption 2.2\(b\)](#).

Next we consider (ii). Fix $i \in S$. Suppose, on the contrary, that $V_k(i) \rightarrow 0$, along some subsequence, as $k \rightarrow \infty$. Using a standard diagonalization argument and the bound in (i), we can find a further subsequence so that

$$\lambda_k \rightarrow \lambda_\circ, \quad v_k(j) \rightarrow v(j), \quad V_k(j) \rightarrow V(j)$$

for all $j \in S$, as $k \rightarrow \infty$. It is also evident from (i) that $V(j) \leq \kappa(\mathcal{V}(j))^\alpha$ for all j . We claim that

$$e^{c(j, v_k(j))} \sum_{z \in S} V_k(z) P(z|j, v_k(j)) \rightarrow e^{c(j, v(j))} \sum_{z \in S} V(z) P(z|j, v(j)) \quad (4.13)$$

as $k \rightarrow \infty$, for all j . Note that for any given $\epsilon > 0$, since \mathcal{V} is norm-like, we can find $z_1 \in S$ such that

$$\sum_{z \geq z_1} V_k(z) P(z|j, v_k(j)) \leq [\kappa \sup_{j \geq z_1} \mathcal{V}^{\alpha-1}(j)] \sum_{z \geq z_1} \mathcal{V}(z) P(z|j, v_k(j)) < \epsilon,$$

using [\(4.12\)](#) and [Assumption 2.2](#). Thus, applying [Assumption 2.1\(a\)](#), we get [\(4.13\)](#). Now passing the limit in [\(4.10\)](#) we have

$$e^{\lambda_\circ} V(j) = e^{c(j, v(j))} \sum_{z \in S} V(z) P(z|j, v(j)) \quad j \in S,$$

and $V(i) = 0$. On the other hand, since $\max_{\mathcal{B}}(\mathcal{V} - V_k) = 0$, we must have $\max_{\mathcal{B}}(\mathcal{V} - V) = 0$ implying V is positive at some point in \mathcal{B} . Applying the arguments of [Lemma 2.5](#) (see [\(2.36\)](#)) we get a contradiction. This proves (ii). \square

We also need the following uniqueness result.

Theorem 4.2. *Suppose that $(\rho, W) \in \mathbb{R}_+ \times \mathfrak{o}(\mathcal{V})$ be such that $W > 0$ and*

$$W(i) = \min_{u \in \mathbb{U}(i)} \left[e^{(c(i, u) - \rho)} \sum_{j \in S} W(j) P(j|i, u) \right] \quad i \in S. \quad (4.14)$$

Moreover, assume that $\rho \geq \lambda^$. Then we must have $\rho = \lambda^*$ and W is a scalar multiple of ψ^* .*

Proof. Let v^* be a minimizing selector of [\(2.9\)](#). Then for a finite set \mathcal{B} containing \mathcal{K} and satisfying

$$\begin{aligned} \max_{u \in \mathbb{U}(i)} c(i, u) - \lambda^* &< \alpha \gamma, \quad \text{under } \text{Assumption 2.2(a)}, \\ \max_{u \in \mathbb{U}(i)} c(i, u) - \lambda^* &< \alpha \ell(i), \quad \text{under } \text{Assumption 2.2(b)}, \end{aligned}$$

for all $i \in \mathcal{B}^c$, we have the representation

$$\psi^*(i) = \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \lambda^*)} \psi^*(X_{\check{\tau}(\mathcal{B})}) \right] \quad \forall i \in \mathcal{B}^c, \quad (4.15)$$

from [Lemma 2.5](#). Using [\(4.14\)](#) we get

$$W(i) \leq e^{(c(i, u) - \rho)} \sum_{j \in S} W(j) P(j|i, v(i)) \leq e^{(c(i, u) - \lambda^*)} \sum_{j \in S} W(j) P(j|i, v(i)).$$

Denoting τ_n , the first exit time from $\{1, 2, \dots, n\}$, and applying Dynkin's formula we obtain

$$W(i) \leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge \tau_n \wedge T-1} (c(X_t, v^*(X_t)) - \lambda^*)} W(X_{\check{\tau}(\mathcal{B}) \wedge \tau_n \wedge T}) \right],$$

for $i \in \mathcal{B}^c \cap \mathcal{D}_n$. In view of [Lemma 2.3](#), we can let $T \rightarrow \infty$ in the above to obtain

$$W(i) \leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B}) \wedge \tau_n-1} (c(X_t, v^*(X_t)) - \lambda^*)} W(X_{\check{\tau}(\mathcal{B}) \wedge \tau_n}) \right]. \quad (4.16)$$

Since $W \in \mathfrak{o}(\mathcal{V})$, it also follows from [\(2.4\)](#) that

$$\begin{aligned} \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\tau_n-1} (c(X_t, v^*(X_t)) - \lambda^*)} W(X_{\tau_n}) \mathbf{1}_{\{\tau_n < \check{\tau}(\mathcal{B})\}} \right] &\leq \sup_{j>n} \frac{W(j)}{\mathcal{V}(j)} \mathbb{E}_i^{v^*} \left[e^{\gamma \tau_n} \mathcal{V}(X_{\tau_n}) \mathbf{1}_{\{\tau_n < \check{\tau}(\mathcal{B})\}} \right] \\ &\leq \sup_{j>n} \frac{W(j)}{\mathcal{V}(j)} \mathcal{V}(i) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Thus, letting $n \rightarrow \infty$ in [\(4.16\)](#), we obtain

$$W(i) \leq \mathbb{E}_i^{v^*} \left[e^{\sum_{t=0}^{\check{\tau}(\mathcal{B})-1} (c(X_t, v^*(X_t)) - \lambda^*)} W(X_{\check{\tau}(\mathcal{B})}) \right] \quad i \in \mathcal{B}^c.$$

Now we can apply the arguments in [Lemma 2.7](#) (see [\(2.51\)](#) and the argument that follows) to obtain that W is a scalar multiple of ψ^* . Using [\(4.14\)](#) it then follows that $\rho = \lambda^*$ and this completes the proof. \square

Next we need the notion of twisted kernel. Consider the eigenpair (λ_k, V_k) and define the transition kernel

$$P^{(k)}(j|i) = \frac{V_k(j)P(j|i)}{\sum_{j \in S} V_k(j)P(j|i, v_k(i))} \quad j \in S. \quad (4.17)$$

Let $\mathbf{Y}^{(k)}$ be the Markov process associated to this kernel. Since $P(j|i, v_k(i))$ generates an irreducible Markov chain, $\mathbf{Y}^{(k)}$ is also irreducible. We denote by $\mathbb{E}^{(k)}[\cdot]$ the expectation operator with respect to the kernel $P^{(k)}$. For any nonnegative function g we then have

$$\mathbb{E}_i^{v_k} \left[e^{\sum_{t=0}^{m-1} (c_k(X_t) - \lambda_k)} V_k(X_m) g(X_m) \right] = V_k(i) \mathbb{E}_i^{(k)} [g(Y_m^{(k)})] \quad i \in S. \quad (4.18)$$

This can be proved by induction and Markov property. In particular, for $m = 1$, [\(4.18\)](#) follows from [\(4.10\)](#) and [\(4.17\)](#). Suppose [\(4.18\)](#) is holds for $m - 1$. Then

$$\begin{aligned} \mathbb{E}_i^{v_k} \left[e^{\sum_{t=0}^m (c_k(X_t) - \lambda_k)} V_k(X_{m+1}) g(X_{m+1}) \right] &= \mathbb{E}_i^{v_k} \left[e^{\sum_{t=0}^m (c_k(X_t) - \lambda_k)} \mathbb{E}_{X_m} [V_k(X_1) g(X_1)] \right] \\ &= \mathbb{E}_i^{v_k} \left[e^{\sum_{t=0}^{m-1} (c_k(X_t) - \lambda_k)} V_k(X_m) \mathbb{E}_{X_m}^{(k)} [g(Y_1^{(k)})] \right] \\ &= V_k(i) \mathbb{E}_i^{(k)} \left[\mathbb{E}_{Y_m^{(k)}}^{(k)} [g(Y_1^{(k)})] \right] = V_k(i) \mathbb{E}_i^{(k)} [g(Y_{m+1}^{(k)})]. \end{aligned}$$

This gives [\(4.18\)](#).

Let C be a finite set containing z_0, \mathcal{B} and \mathcal{K} (see [Assumption 2.2](#)). Let $\tilde{\mathcal{V}}_k = \frac{\mathcal{V}}{V_k}$. Using [\(4.10\)](#) and [\(2.4\)](#) see note that

$$\sum_{j \in S} \tilde{\mathcal{V}}_k(j) P^{(k)}(j|i) = \frac{\sum_{j \in S} \mathcal{V}(j) P(j|i, v_k(i))}{\sum_{j \in S} V_k(j) P(j|i, v_k(i))} \leq e^{-(1-\alpha)\gamma} \tilde{\mathcal{V}}_k(i) + \kappa_1 \mathbf{1}_C, \quad (4.19)$$

for some constant κ_1 , independent of k , which is possible by [Lemma 4.2\(ii\)](#). Similar estimate also possible under [Assumption 2.2\(b\)](#). We also note that since $V_k \leq \mathcal{V}$

$$\sum_{j \in S} V_k(j) P(j|i, v_k(i)) \leq \sum_{j \in S} \mathcal{V}(j) P(j|i, v_k(i)) \leq \kappa(i),$$

by [Assumption 2.2](#), for some constant $\kappa(i)$. Therefore, letting $\nu = \delta_{z_o}$, we see from [\(4.7\)](#) that

$$P^{(k)}(A|i) \geq \kappa_2 \nu(A) \quad \text{for } A \subset C, \text{ and } i \in C,$$

for some κ_2 , independent of k . This is possible due to [Lemma 4.2\(ii\)](#). Therefore, C is a small set for the chain $\mathbf{Y}^{(k)}$. Applying [[38](#), Theorem 2.3] we then obtain the following

Lemma 4.3. *For every $k \in \mathbb{N}$, the Markov chain $\mathbf{Y}^{(k)}$ has a unique stationary probability measure π_k and there are constant M and $r \in (0, 1)$, not depending on k , that satisfy*

$$\left\| \left(P^{(k)} \right)^n - \pi_k \right\|_{\tilde{\mathcal{V}}_k} \leq M \tilde{\mathcal{V}}_k(i) r^n \quad \text{for all } n \in \mathbb{N}, \quad (4.20)$$

when the chain starts from the initial state i .

In the above

$$\|\mu_1 - \mu_2\|_V := \sup_{|f| \leq V} |\mu_1(f) - \mu_2(f)|,$$

where $\mu_m(f) = \sum_{j \in S} f(j) \mu_m(j)$ for $m = 1, 2$. Recall from [\(4.12\)](#) that

$$\tilde{\mathcal{V}}_k(i) \geq \kappa^{-1} \mathcal{V}^{1-\alpha}(i) \quad i \in S.$$

Hence using [\(4.19\)](#), we get

$$\sum_{j \in S} \mathcal{V}^{1-\alpha}(j) \pi_k(j) \leq \kappa_3 \quad \text{for all } k \in \mathbb{N} \quad (4.21)$$

for some constant κ_3 . This of course, implies that $\{\pi_k\}$ is tight. We claim that every limit points of $\{\pi_k\}$ will have support in all of S . To see this, suppose along some subsequence, $\pi_k \rightarrow \pi$ as $k \rightarrow \infty$. In view of [Lemma 4.2](#) and compactness of action space, we can extract a further subsequence along which

$$V_k(i) \rightarrow V(i) > 0 \quad \text{and} \quad v_k(i) \rightarrow v(i) \quad \text{for all } i \in S,$$

as $k \rightarrow \infty$. This of course, implies (see [\(4.13\)](#))

$$P^{(k)}(j|i) = \frac{V_k(j) P(j|i, v_k(i))}{\sum_{j \in S} V_k(j) P(j|i, v_k(i))} \rightarrow \frac{V(j) P(j|i, v(i))}{\sum_{j \in S} V(j) P(j|i, v(i))} := \tilde{P}(j|i),$$

as $k \rightarrow \infty$. Therefore, for any bounded function $f : S \rightarrow \mathbb{R}$ supported on a finite set, we get

$$\begin{aligned} \sum_{j \in S} f(j) \pi(j) &= \lim_{k \rightarrow \infty} \sum_{j \in S} f(j) \pi_k(j) \\ &= \lim_{k \rightarrow \infty} \sum_{j \in S} \pi_k(j) \left[\sum_{z \in S} f(z) P^{(k)}(z|j) \right] \\ &= \sum_{j \in S} \pi(j) \left[\sum_{z \in S} f(z) \tilde{P}(z|j) \right], \end{aligned}$$

where in the last line we use tightness of $\{\pi_k\}$. By a limiting argument we see that

$$\sum_{j \in S} f(j) \pi(j) = \sum_{j \in S} \pi(j) \left[\sum_{z \in S} f(z) \tilde{P}(z|j) \right]$$

for all bounded function f . Thus π is a stationary distribution to the Markov process generated by the kernel \tilde{P} . Since \tilde{P} is irreducible, which follows from the irreducibility of $P(j|i, v(i))$, π must have support in all of S . Now we are ready to prove the following key lemma.

Lemma 4.4. *Grant [Assumptions 2.1](#) and [4.1](#). Then we have $\theta_k(i) \rightarrow 0$ as $k \rightarrow \infty$, for all $i \in S$.*

Proof. From (4.11) we see that for $k \in \mathbb{N}$

$$\theta_{k+1}(i)V_k(i) = V_k(i) - e^{c_{k+1}(i)-\lambda_k} \sum_{j \in S} V_k(j)P(j|i, v_{k+1}(i)) \quad i \in S. \quad (4.22)$$

Applying Dynkin's formula to (4.22) gives

$$\begin{aligned} V_k(i) &= \mathbb{E}_i^{v_{k+1}} \left[e^{\sum_{t=0}^T (c_{k+1}(X_t) - \lambda_k)} V_k(X_{T+1}) \right] + \sum_{t=0}^T \mathbb{E}_i^{v_{k+1}} \left[e^{\sum_{n=0}^{t-1} (c_{k+1}(X_n) - \lambda_k)} \theta_{k+1}(X_t) V_k(X_t) \right] \\ &\geq \sum_{t=0}^T \mathbb{E}_i^{v_{k+1}} \left[e^{\sum_{n=0}^{t-1} (c_{k+1}(X_n) - \lambda_k)} \theta_{k+1}(X_t) V_k(X_t) \right] \end{aligned} \quad (4.23)$$

for all $T \in \mathbb{N}$ and $i \in S$. Let us now define

$$h_{k+1}(i) = \theta_{k+1}(i) \frac{V_k(i)}{V_{k+1}(i)} \quad i \in S.$$

Then combining (4.18) and (4.23) we have

$$V_k(i) \geq V_{k+1}(i) \sum_{t=0}^T e^{(\lambda_{k+1} - \lambda_k)t} \tilde{\mathbb{E}}_i^{(k+1)} \left[h_{k+1}(Y_t^{(k+1)}) \right]. \quad (4.24)$$

Let

$$U_{k+1}(i) = \frac{V_k(i)}{V_{k+1}(i)} \quad i \in S.$$

Since

$$V_k(i) \geq e^{c_{k+1}(i) - \lambda_k} \sum_{j \in S} V_k(j)P(j|i, v_{k+1}(i)), \quad V_{k+1}(i) = e^{c_{k+1}(i) - \lambda_{k+1}} \sum_{j \in S} V_{k+1}(j)P(j|i, v_{k+1}(i)),$$

we obtain from (4.17)

$$U_{k+1}(i) \geq e^{\lambda_{k+1} - \lambda_k} \sum_{j \in S} U_{k+1}(j)P^{(k+1)}(j|i) \quad \text{for all } i \in S. \quad (4.25)$$

We now split the proof into two cases.

Case 1. Suppose that some $k \geq 0$ we have $\lambda_{k+1} = \lambda_k$. It then follows from (4.25) that $\{U_{k+1}(Y_n^{(k+1)})\}$ is a super-martingale. Since $\mathbf{Y}^{(k+1)}$ is recurrent, we must have U_{k+1} constant. From (4.11) we get $\theta_{k+1} = 0$ and

$$V_k(i) = e^{c_k(i) - \lambda_k} \sum_{j \in S} V_k(j)P(j|i, v_k(i)) = \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u) - \lambda_k} \sum_{j \in S} V_k(j)P(j|i, u) \right].$$

Hence v_k is also a minimizing selector. From Theorem 4.2 we then see that V_k is a scalar multiple of ψ^* which in turn, implies from Theorem 2.1 that $\lambda^* = \lambda_k = \lambda_{k+1} = \lambda_{k+2} = \dots$ and $\theta_j = 0$ for $j \geq k+1$.

Case 2. Suppose that the sequence $\{\lambda_k\}$ is strictly decreasing. Note from Lemma 4.2 that $\sup_k h_k(i) < \infty$ for all i . Let \mathcal{D} be any finite set. Then using (4.20) and (4.24) we have

$$U_{k+1}(i) \geq \sum_{t=0}^T e^{(\lambda_{k+1} - \lambda_k)t} \left[\pi_{k+1}(h_{k+1} \mathbf{1}_{\mathcal{D}}) - M\tilde{\mathcal{V}}(i)r^t \right].$$

Letting $T \rightarrow \infty$ we get

$$U_{k+1}(i) \geq [1 - e^{(\lambda_{k+1} - \lambda_k)}]^{-1} \pi_{k+1}(h_{k+1} \mathbf{1}_{\mathcal{D}}) - \kappa_4 \tilde{\mathcal{V}}(i),$$

for some constant κ_4 , not dependent on k . Since $\{U_k(i)\}$ is a bounded sequence, by [Lemma 2.3](#), and $\{\lambda_k\}$ is decreasing, it follows from above that

$$\lim_{k \rightarrow \infty} \pi_{k+1}(h_{k+1} \mathbb{1}_{\mathcal{D}}) = 0, \quad (4.26)$$

for every finite set \mathcal{D} . Now suppose that for some $i \in S$, $\limsup_{k \rightarrow \infty} \theta_k(i) > 0$. We choose $\mathcal{D} = \{i\}$. Since every subsequential limit of $\{\pi_k\}$ has support in S , $\liminf_{k \rightarrow \infty} V_k(i) > 0$ by [Lemma 4.2\(ii\)](#), this gives a contradiction to (4.26). Hence we must have $\lim_{k \rightarrow \infty} \theta_k(i) = 0$ for all $i \in S$. This completes the proof. \square

Now we are ready to complete the proof of [Theorem 4.1](#).

Proof of Theorem 4.1. Suppose that $\lim_{k \rightarrow \infty} \lambda_k = \rho$. It is obvious that $\rho \geq \lambda^*$. Using (4.11) we write

$$\theta_{k+1}(i)V_k(i) + \min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)-\lambda_k} \sum_{j \in S} V_k(j)P(j|i, u) \right] = V_k(i). \quad (4.27)$$

Using [Lemma 4.2](#) and a diagonalization argument we can find positive $V \in \mathfrak{o}(\mathcal{V})$ so that, along some subsequence, $V_k(i) \rightarrow V(i)$ for all $i \in S$. Passing the limit in (4.27) and using [Lemma 4.4](#) we obtain

$$\min_{u \in \mathbb{U}(i)} \left[e^{c(i,u)-\rho} \sum_{j \in S} V(j)P(j|i, u) \right] = V(i) \quad i \in S.$$

From [Theorem 4.2](#) we see that $\rho = \lambda^*$ and V is a scalar multiple of ψ^* . Since ψ^* is unique upto a normalization, (ii) follows. \square

4.2. Continuous time stable case. In this section we prove a PIA for the CTCMP we considered in [Section 3](#). Most of the statement and proofs in the section are continuous time analogue of [Section 4.1](#), therefore we mainly provide sketches for most of the results. We begin with the following assumption which we impose in this section, compare it with [Assumption 4.1](#).

Assumption 4.2. We suppose that [Assumption 3.4](#) holds for a norm-like function \mathcal{V} . Furthermore, in case of [Assumption 3.4\(b\)](#), we have $\max_{u \in \mathbb{U}(i)} c(i, u) \leq \eta \ell(i)$ for $i \in S$ and some $\eta \in (0, 1)$.

As before, we define the generalized Perron-Frobenius eigenvalue as follows

$$\lambda_1^c = \inf \{ \lambda \in \mathbb{R} : \exists \Psi > 0 \text{ satisfying } \min_{u \in \mathbb{U}(i)} \left[\sum_{j \in S} \Psi(j)q(j|i, u) + c(i, u)\Psi(i) \right] \leq \lambda \Psi(i) \forall i \in S \}, \quad (4.28)$$

and for every stationary Markov control v we also define

$$\lambda_1^c(v) = \inf \{ \lambda \in \mathbb{R} : \exists \Psi > 0 \text{ satisfying } \left[\sum_{j \in S} \Psi(j)P(j|i, v(i)) + c(i, v(i))\Psi(i) \right] \leq \lambda \Psi(i) \forall i \in S \}. \quad (4.29)$$

A claim analogous to [Lemma 4.1](#) holds true for CTCMP and under the setting of [Theorem 3.1](#) we also have $\lambda_1^c = \lambda^*$ and $\lambda_1^c(v) = \rho_v$ (see [Remark 3.1](#)). We now describe our PIA.

Algorithm 4.2. Policy iteration.

1. Initialization. Set $k = 0$ and select any $v_0 \in \mathfrak{U}_{\text{sm}}$.

2. Value determination. Let V_k be the unique principal eigenfunction satisfying $V_k(i_0) = 1$ and

$$\rho_k V_k(i) = \sum_{j \in S} V_k(j) q(j|i, v_k(i)) + c(i, v_k(i)) V_k(i) \quad i \in S. \quad (4.30)$$

Existence of a unique principal eigenfunction in (4.30) follows from [Remark 3.1](#) and [Lemma 3.5](#) which is based on [Lemma 2.7](#). We let $\lambda_k := \lambda_1^c(v_k) = \mathcal{E}_i(c, v_k) = \rho_k$.

3. Policy improvement. Choose any $v_{k+1} \in \mathfrak{U}_{\text{sm}}$ satisfying

$$v_{k+1}(i) \in \text{Arg min}_{u \in \mathfrak{U}(i)} \left[\sum_{j \in S} V_k(j) q(j|i, u) + c(i, u) V_k(i) \right], \quad i \in S.$$

We show that [Algorithm 4.2](#) converges.

Theorem 4.3. *Under [Assumptions 3.1 to 3.3](#) and [4.2](#) the following holds.*

- (i) *For all $k \in \mathbb{N}$, we have $\lambda_{k+1} \leq \lambda_k$ and $\lim_{k \rightarrow \infty} \lambda_k = \lambda^*$.*
- (ii) *V_k converges pointwise, as $k \rightarrow \infty$, to ψ^* where ψ^* is the unique solution to [\(3.7\)](#).*

We adapt the proof of [Theorem 4.1](#) with suitable modification. Our next lemma follows by adapting the arguments of [Lemma 4.2](#) and [Theorem 4.2](#) in a straightforward manner.

Lemma 4.5. *Grant the setting of [Theorem 4.3](#). Then the following hold.*

- (i) *There exists κ , independent of k , such that*

$$V_k(i) \leq \kappa (\mathcal{V}(i))^\alpha \quad \text{for all } i \in S, \quad (4.31)$$

where α is given by [\(4.8\)](#).

- (ii) *For every $i \in S$ we have $\inf_{k \in \mathbb{N}} V_k(i) > 0$.*
- (iii) *If for some $(\rho, W) \in \mathbb{R}_+ \times \mathfrak{o}(\mathcal{V})$ with $W > 0$ we have*

$$\rho W(i) = \min_{u \in \mathfrak{U}(i)} \left[\sum_{j \in S} W(j) q(j|i, u) + c(i, u) W(i) \right] \quad i \in S,$$

where $\rho \geq \lambda^*$. Then we must have $\rho = \lambda^*$ and W is a scalar multiple of ψ^* where ψ^* is given by [\(3.7\)](#).

As before, we denote by $c_k(i) = c(i, v_k(i))$. Define $\mathcal{V}_\kappa(i) = (\mathcal{V}(i))^\kappa$ where $\kappa \in (0, 1)$. Since $t \mapsto t^\kappa$ is concave in $(0, \infty)$, we observe from [Assumption 3.4](#) that

$$\begin{aligned} \sum_{j \in S} \mathcal{V}_\kappa(j) q(j|i, u) &= \sum_{j \neq i} (\mathcal{V}^\kappa(j) - \mathcal{V}^\kappa(i)) q(j|i, u) \\ &\leq \sum_{j \neq i} \kappa \mathcal{V}^{\kappa-1}(i) (\mathcal{V}(j) - \mathcal{V}(i)) q(j|i, u) \\ &\leq \begin{cases} \kappa \mathcal{V}^{\kappa-1}(i) \widehat{C} \mathbf{1}_\mathcal{K} - \kappa \gamma \mathcal{V}_\kappa(i), & \text{under (3.4),} \\ \kappa \mathcal{V}^{\kappa-1}(i) \widehat{C} \mathbf{1}_\mathcal{K} - \kappa \ell(i) \mathcal{V}_\kappa(i), & \text{under (3.5).} \end{cases} \end{aligned} \quad (4.32)$$

We fix $\kappa \in (\eta, 1)$ where η is given by [Assumption 4.2](#). For [\(3.4\)](#), we shall fix κ close to 1 so that $\|c\|_\infty < \kappa \gamma$. Also, note that we may choose $\alpha < \kappa$ in [\(4.8\)](#).

Let us now introduce the *twisted rate kernel*. For the eigenpair (λ_k, V_k) we define the kernel

$$q^{(k)}(j|i) = \frac{V_k(j)}{V_k(i)} q(j|i, v_k(i)) \quad \text{for } i \neq j, \quad \text{and} \quad q^{(k)}(i|i) = - \sum_{j \neq i} q^{(k)}(j|i).$$

From [\(4.30\)](#) we see that

$$-q^{(k)}(i|i) = \rho_k - c_k - q(i|i, v_k(i)) < \infty \quad i \in S.$$

Let $\tilde{\mathcal{V}}_k(i) = \frac{\mathcal{V}(i)}{V_k(i)}$ for $i \in S$. From [Assumption 3.4](#) it then follows that

$$\begin{aligned}
\sum_{j \in S} \tilde{\mathcal{V}}_k(j) q^{(k)}(j|i) &= \sum_{j \neq i} (\tilde{\mathcal{V}}_k(j) - \tilde{\mathcal{V}}_k(i)) \frac{V_k(j)}{V_k(i)} q(j|i, v_k(i)) \\
&= \frac{1}{V_k(i)} \sum_{j \in S} \mathcal{V}(j) q(j|i, v_k(i)) - \frac{\mathcal{V}(i)}{V_k^2(i)} \sum_{j \in S} V_k(j) q(j|i, v_k(i)) \\
&\leq \begin{cases} \frac{1}{V_k(i)} \widehat{C}_1 \mathbf{1}_{\mathcal{K}} - (\gamma - c_k(i) + \lambda_k) \tilde{\mathcal{V}}_k(i), & \text{by (3.4),} \\ \frac{1}{V_k(i)} \widehat{C}_1 \mathbf{1}_{\mathcal{K}} - (\ell(i) - c_k(i) + \lambda_k) \tilde{\mathcal{V}}_k(i), & \text{by (3.5),} \end{cases} \\
&\leq \begin{cases} \widehat{C}_1 \mathbf{1}_{\mathcal{K}} - (1 - \alpha) \gamma \tilde{\mathcal{V}}_k(i), \\ \widehat{C}_1 \mathbf{1}_{\mathcal{K}} - (1 - \alpha) \ell(i) \tilde{\mathcal{V}}_k(i), \end{cases} \tag{4.33}
\end{aligned}$$

for some constant \widehat{C}_1 , where we use [\(4.8\)](#). Using [\(4.33\)](#) and [\[40, Theorem 2.2.4\]](#) we find a non-explosive Markov process $\mathbf{Y}^{(k)}$ corresponding to the kernel $q^{(k)}$. Furthermore, since q is irreducible for every stationary Markov control, we have $\mathbf{Y}^{(k)}$ irreducible. Letting

$$\Phi_{\kappa, k}(i) = \frac{\mathcal{V}_\kappa(i)}{V_k(i)} \quad i \in S,$$

from [\(4.32\)](#) and [\(4.33\)](#) we obtain

$$\sum_{j \in S} \Phi_{\kappa, k}(j) q^{(k)}(j|i) \leq \begin{cases} \widehat{C}_2 \mathbf{1}_{\mathcal{K}} - (\kappa - \alpha) \gamma \Phi_{\kappa, k}(i), \\ \widehat{C}_2 \mathbf{1}_{\mathcal{K}} - (\kappa - \alpha) \ell(i) \Phi_{\kappa, k}(i), \end{cases} \tag{4.34}$$

for some constant \widehat{C}_2 . By [Assumption 4.2](#), $\tilde{\mathcal{V}}_k(i)/\Phi_{\kappa, k} = \mathcal{V}_{1-\kappa}(i) \rightarrow \infty$ as $i \rightarrow \infty$. Therefore, by [\[41, Theorem 3.13\]](#), $\mathbf{Y}^{(k)}$ is exponentially ergodic with a unique invariant measure π_k . Using [\(4.31\)](#) and [\(4.33\)](#) we get that

$$\sup_k \sum_{j \in S} \mathcal{V}^{1-\alpha}(j) \pi_k(j) \leq \kappa,$$

for some constant κ . Thus $\{\pi_k\}$ is tight. As before, see [Section 4.1](#), we next show that any subsequential limit of $\{\pi_k\}$ is supported on whole of S . Since we do not have an exact analogue of [Lemma 4.3](#) for CTCMP, we modify the argument a bit. Consider a subsequence of $\{\pi_k\}$ along which $\pi_k \rightarrow \pi$. Using a diagonalization argument and selecting a further subsequence, if required, we can assure that

$$V_k(i) \rightarrow V(i), \quad v_k(i) \rightarrow v(i) \quad \text{for all } i \in S.$$

Using [\(4.31\)](#) it is easily seen that

$$q^{(k)}(j|i) \rightarrow \tilde{q}(j|i), \quad \tilde{\mathcal{V}}_k(i) \rightarrow \tilde{\mathcal{V}}(i), \quad \Phi_{\kappa, k}(i) \rightarrow \Phi_\kappa(i) \quad \text{for all } i, j,$$

where

$$\tilde{\mathcal{V}}(i) = \frac{\mathcal{V}(i)}{V(i)}, \quad \Phi_\kappa(i) = \frac{\mathcal{V}_\kappa(i)}{V(i)}.$$

[\(4.33\)](#) and [\(4.34\)](#) holds true for the kernel \tilde{q} . It can also be easily checked that π is the invariant measure corresponding to the kernel \tilde{q} . Since \tilde{q} generates an irreducible Markov process, π must have its support in all of S . This proves the claim.

Let us now define the error term

$$\theta_{k+1}(i) = \lambda_k - c_{k+1}(i) - \frac{1}{V_k(i)} \sum_{j \in S} V_k(j) q(j|i, v_{k+1}(i)) \quad i \in S. \tag{4.35}$$

It follows from the definition (see step 3 of [Algorithm 4.2](#)) that $\theta_k \geq 0$. On the other hand,

$$\theta_{k+1}(i) \leq \lambda_k + \left(\sup_{u \in \mathbb{U}(i)} -q(i|i, u) \right) \leq \lambda_0 + \left(\sup_{u \in \mathbb{U}(i)} -q(i|i, u) \right) < \infty \quad i \in S.$$

Thus, $\{\theta_k\}$ is locally bounded in k . Next we show the following.

Lemma 4.6. *Grant the setting of [Theorem 4.3](#). Then we have $\lim_{k \rightarrow \infty} \theta_k(i) = 0$ for all $i \in S$.*

Proof. Suppose, on the contrary, that for some $i \in S$, we have $\limsup_{k \rightarrow \infty} \theta_k(i) > 0$. Passing to the subsequence we assume that $\theta_k(i) \rightarrow \tilde{\theta} > 0$. Now applying Dynkin's formula to [\(4.35\)](#) we have

$$V_{k-1}(i) \geq \int_0^T \mathbb{E}_i^{v_k} \left[e^{\int_0^t (c_k(X_s) - \lambda_{k-1}) ds} \theta_k(X_t) V_{k-1}(X_t) \right] dt, \quad (4.36)$$

for all $T > 0$. On the other hand, for $U_k(i) = \frac{V_{k-1}(i)}{V_k(i)}$, we have

$$\sum_{j \in S} U_k(j) q^{(k)}(j|i) \leq (\lambda_{k-1} - \lambda_k) U_k(i).$$

Thus, if $\lambda_{k-1} = \lambda_k$ the proof follows from the argument of [Lemma 4.4](#), Case 1. So we assume that $\{\lambda_k\}$ is strictly decreasing.

To this end, we need a continuous time counterpart of [\(4.18\)](#). Suppose that g is a non-negative function supported on a finite subset of S . Defining $u(t)(i) = \mathbb{E}_i^{(k)}[g(Y_t^{(k)})]$ we know that

$$\frac{du(t)(i)}{dt} = \sum_{j \in S} u(t)(j) q^{(k)}(j|i) \quad i \in S.$$

Using [\(4.30\)](#) this can be rewritten as

$$\frac{dV_k(i)u(t)(i)}{dt} = \sum_{j \in S} V_k(j)u(t)(j)q(j|i, v_k(i)) + (c_k(i) - \lambda_k)V_k(i)u(t)(i).$$

Thus, from Dynkin's formula, we obtain

$$V_k(i) \mathbb{E}_i^{(k)}[g(Y_t^{(k)})] = \mathbb{E}_i^{v_k} \left[e^{\int_0^t (c_k(i) - \lambda_k) ds} g(X_t) V_k(t) \right]. \quad (4.37)$$

By a standard approximation the above relation can be extended to all nonnegative functions g on S . Let $h_k(i) = \theta_k(i)V_{k-1}(i)/V_k(i)$. Using [\(4.36\)](#) and [\(4.37\)](#) we obtain

$$U_k(i) \geq \int_0^T e^{(\lambda_k - \lambda_{k-1})t} \mathbb{E}_i^{(k)}[\mathbb{1}_{\{i\}}(Y_t^{(k)}) h_k(Y_t^{(k)})].$$

Now we let $k \rightarrow \infty$, so that $q^{(k)} \rightarrow \tilde{q}$ and $\pi_k \rightarrow \pi$, along some subsequence. Since $\mathbf{Y}^{(k)}$ converges in distribution to $\tilde{\mathbf{Y}}$ where $\tilde{\mathbf{Y}}$ is the Markov process associated to \tilde{q} (this can be seen by adapting the arguments of [\[41, Lemma 5.8\]](#)) and, $U_k(i)$ is bounded above and $h_k(i)$ is bounded below (by [Lemma 4.5](#)), we get from above

$$\int_0^T \mathbb{E}_i^{(k)}[\mathbb{1}_{\{i\}}(\tilde{Y}_t)] \leq \kappa_1$$

for some κ_1 . But $\tilde{\mathbf{Y}}$ is exponentially ergodic [\[41, Theorem 3.13\]](#) with invariant measure π having support in i . Letting $T \rightarrow \infty$, we get a contradiction. \square

Now we can complete the proof of [Theorem 4.3](#).

Proof of [Theorem 4.3](#). The proof follows from [Lemmas 4.5](#) and [4.6](#) together with the arguments of [Theorem 4.1](#). \square

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