OPTIMAL SEMIGROUP REGULARITY FOR VELOCITY COUPLED ELASTIC SYSTEMS: A DEGENERATE FRACTIONAL DAMPING CASE

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ABSTRACT. In this note, we consider an abstract system of two damped elastic systems. The damping involves the average velocity and a fractional power of the principal operator, with power θ in $[0, 1]$. The damping matrix is degenerate, which makes the the regularity analysis more delicate. First, using a combination of the frequency domain method and multipliers technique, we prove the following regularity for the underlying semigroup:

- The semigroup is of Gevrey class δ for every $\delta > 1/2\theta$, for each θ in $(0, 1/2)$.
- The semigroup is analytic for $\theta = 1/2$.
- The semigroup is of Gevrey class δ for every $\delta > 1/2(1-\theta)$, for each θ in $(1/2, 1)$.

Next, we analyze the point spectrum, and derive the optimality of our regularity results. We also prove that the semigroup is not differentiable for $\theta = 0$ or $\theta = 1$. Those results strongly improve upon some recent results presented in [2].

CONTENTS

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1. Introduction

Responding to two conjectures of G. Chen and Russell [7], S. Chen and Triggiani [8, 9] considered the following abstract elastic system:

$$y_{tt} + Ay + By_t = 0 \text{ in } (0, \infty)$$

where A and B are self-adjoint positive definite operators on some Hilbert space H with

$$\exists \alpha_1 > 0 \text{ and } \alpha_2 > 0 : \ \alpha_1 A^\mu \le B \le \alpha_2 A^\mu, \text{ for some constant } \mu \in (0, 1].$$

They proved that the underlying semigroup is

(1) analytic for $\frac{1}{2} \le \mu \le 1$, but not analytic for $0 < \mu < \frac{1}{2}$, though differentiable, [8]
(2) of Gevrey class δ for all $\delta > \frac{1}{2\mu}$ for $0 < \mu < \frac{1}{2}$, [9]

In fact, [9] generalizes the work of Taylor [28] where the author discusses Gevrey semigroups, and illustrates his work with several examples including the case $B = 2\rho A^\mu$ for some positive constant ρ.
Later on, Liu and Liu proposed a method for proving analyticity or Gevrey class regularity for semigroups associated with elastic systems [21]; their method combines resolvent estimates using a contradiction argument and mulptipliers technique. Other closely related works include e.g. [1, 10, 15, 16, 17, 18, 19, 22]; those works discuss either regularity or stability issues, which are the main themes of the present contribution.
In a recent work [2], the authors consider a system of two damped abstract elastic systems in a Hilbert space, (see System (2.1) below). The damping involves a fractional power θ of the principal operator, and the average velocity, so that it is degenerate (see below, just after System (2.1), a brief explanation of why we use the term degenerate.) The power θ lies in the interval $[-1, 1]$. They prove the following regularity and stability results for the associated semigroup:

- The semigroup is not analytic for every θ in $(0, 1]$.
- The semigroup is eventually differentiable for each θ in $(0, 1)$.
- The semigroup is of Gevrey class δ for every $\delta > 1/2\theta$ for each θ in $(0, 1/4]$.
- The semigroup is of Gevrey class δ for every $\delta > (2\theta + 1)/3\theta$ for each θ in $(1/4, 1/2]$.
- The semigroup is exponentially stable for each θ in $[0, 1]$.
- The semigroup is polynomially stable of order $O(t^{\frac{1}{2\theta}})$, as t goes to infinity for each θ in $[-1, 0)$.
- Further, the resolvent estimate for the Gevrey class regularity in the case where θ in $(0, 1/4]$ is optimal, and so is the polynomial stability for each θ in $[-1, 0)$.

All of those results are established using resolvent estimates and multipliers technique, combined with semigroup stability characterization results e.g. [4, 14, 27], or regularity results [26, 28].
One notes that, unlike the case of a single equation where analyticity holds for every θ in $[1/2, 1]$, in [2] the authors did not get analyticity for any value of θ. Though they were able to prove Gevrey class regularity for θ in $(0, 1/2)$, one notices that they provide

two different Gevrey classes depending on whether θ lies in $(0, 1/4]$ or in $(1/4, 1/2]$; the Gevrey class for θ lying in $(0, 1/4]$ is in agreement with the case of a single similarly damped elastic system. However, the Gevrey class corresponding to θ in $(1/4, 1/2]$ is off by $(1-4\theta)/6\theta$. A natural question is then: why this discrepancy? Is this due to the degeneracy of the damping operator matrix, or to some technical difficulties? Besides, is the differentiablity of the semigroup the best regularity that one should expect for θ in $(1/2, 1)$?
The purpose of this note is to carefully examine those questions. The plan of the rest of our work is as follows: in Section 2, we discuss new regularity results. Section 3 deals with the eigenvalue asymptotics and the optimality of our regularity results. Details about eigenvalues computation and asymptotics are provided in Section 4. In section 5, we present some examples of application.

2. New regularity results

Let H be a complex Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and the induced norm $|.|$. We consider the following abstract system of coupled equations:

$$\begin{cases} u_{tt} = -aAu - \gamma A^\theta(u_t + w_t) \\ w_{tt} = -bAw - \gamma A^\theta(u_t + w_t), \\ u(0) = u_0, \quad u_t(0) = v_0, \quad w(0) = w_0, \quad w_t(0) = z_0 \end{cases} \tag{2.1}$$

where A is a self-adjoint, positive definite (unbounded) operator on a complex Hilbert space H, $a, b, \gamma > 0$, $a \neq b$, and $\theta \in [0, 1]$.
Notice that the matrix defining the damping is given by

$$D = \begin{pmatrix} \gamma & \gamma \\ \gamma & \gamma \end{pmatrix}$$

and it is a singular matrix; the associated quadratic form is degenerate, hence the terminology "degenerate fractional damping".
Set $V = D(A^{\frac{1}{2}})$, and assume that $V \subset H \subset V'$ with compact injections, where V' denotes the topological dual of V. We also assume that for any r, s in $\mathbb{R}$ with $r < s$, the embedding $D(A^s) \subset D(A^r)$ is compact.
Introduce the Hilbert space $\mathcal{H} = V \times H \times V \times H$, over the field $\mathbb{C}$ of complex numbers, equipped with the norm

$$||Z||^2 = a|A^{\frac{1}{2}}u|^2 + |v|^2 + b|A^{\frac{1}{2}}w|^2 + |z|^2, \quad \forall Z = (u, v, w, z) \in \mathcal{H}.$$

Throughout this note, we shall assume:

$$\exists a_0 > 0 : |u| \le a_0|A^{\frac{1}{2}}u|, \quad \forall u \in V. \tag{2.2}$$

Introduce the operator

$$\mathcal{A}_\theta = \begin{pmatrix} 0 & I & 0 & 0 \\ -aA & -\gamma A^\theta & 0 & -\gamma A^\theta \\ 0 & 0 & 0 & I \\ 0 & -\gamma A^\theta & -bA & -\gamma A^\theta \end{pmatrix} \tag{2.3}$$

with domain

$$D(\mathcal{A}_\theta) = \left\{(u,v,w,z) \in V^4; \quad \begin{array}{l} u \in D(A^{\frac{3}{2}-\tau}),\ aA^{\frac{3}{2}-\tau}u + \gamma A^{\frac{1}{2}}(v+z) \in D(A^{\tau-\frac{1}{2}}) \\ w \in D(A^{\frac{3}{2}-\tau}),\ bA^{\frac{3}{2}-\tau}w + \gamma A^{\frac{1}{2}}(v+z) \in D(A^{\tau-\frac{1}{2}}) \end{array} \right\},$$

where $\tau = \max\{\theta, \frac{1}{2}\}$. One easily checks that for every $Z = (u,v,w,z) \in D(\mathcal{A}_\theta)$,

$$\mathrm{Re}\,(\mathcal{A}_\theta Z, Z) = -\gamma |A^{\frac{\theta}{2}}(v+z)|^2 \le 0. \tag{2.4}$$

so that the operator $\mathcal{A}_\theta$ is dissipative. Further, the operator $\mathcal{A}_\theta$ is densely defined. It is straight-forward to show that $\mathcal{A}_\theta$ is bijective from $D(\mathcal{A}_\theta)$ to $\mathcal{H}$. Therefore, the Lumer-Phillips Theorem shows that the operator $\mathcal{A}_\theta$ generates a strongly continuous semigroup of contractions $(S_\theta(t))_{t\ge 0}$ on the Hilbert space $\mathcal{H}$. One also checks that

$$i\mathbb{R} \subset \rho(\mathcal{A}_\theta) \tag{2.5}$$

where $\rho(\mathcal{A}_\theta)$ denotes the resolvent set of $\mathcal{A}_\theta$. This shows that the semigroup $(S_\theta(t))_{t\ge 0}$ is strongly stable on the Hilbert space $\mathcal{H}$, thanks to the strong stability criterion of e.g. Arendt and Batty [3].

Remark 2.1. *We find it useful to give some helpful information about how the inclusion* (2.5) *is established. First, notice that for each $\theta \in [0,1]$ we have $D(\mathcal{A}_\theta)$ contained in $((D(A) + D(A^{\frac{3}{2}-\theta})) \times V)^2$; notice that when θ lies is in $[0,1/2]$, the sum $D(A) + D(A^{\frac{3}{2}-\theta})$ reduces to $D(A)$, while for θ in $(1/2,1]$, this sum simplifies to $D(A^{\frac{3}{2}-\theta})$. Thus, one readily deduces that for θ in $[0,1)$, $\mathcal{A}_\theta$ has a compact resolvent. Therefore the spectrum of $\mathcal{A}_\theta$ is discrete. Proving* (2.5) *reduces to showing that for any λ in $\mathbb{R}$, one has $Ker(i\lambda I - \mathcal{A}_\theta) = \{0_{\mathcal{H}}\}$, which is fairly easy. When $\theta = 1$, the operator $\mathcal{A}_\theta$ no longer has a compact resolvent; in this case, to prove* (2.5)*, not only do we have to show that $Ker(i\lambda I - \mathcal{A}_\theta) = \{0_{\mathcal{H}}\}$ for any λ in $\mathbb{R}$, which is easy, but we also have to show that for any λ in $\mathbb{R}$, the operator $i\lambda I - \mathcal{A}_\theta$ is surjective; we rely on the Fredholm alternative and the closed graph theorem to show the latter as follows: Set $\theta = 1$. Let λ in $\mathbb{R}$. Let $F = (f,g,h,\ell)$ in $\mathcal{H}$, we shall find $U = (u,v,w,z)$ in $D(\mathcal{A}_1)$ such that $(i\lambda I - \mathcal{A}_1)U = F$, which reduces to*

$$\begin{aligned} -\lambda^2 u + aAu + i\lambda\gamma A(u+w) &= g + i\lambda f + \gamma A(f+h) \\ -\lambda^2 w + bAw + i\lambda\gamma A(u+w) &= \ell + i\lambda h + \gamma A(f+h). \end{aligned} \tag{2.6}$$

Now, set

$$L(u,w) = (aAu + i\lambda\gamma A(u+w), bAw + i\lambda\gamma A(u+w)).$$

One readily checks that $L : V \times V \longrightarrow V' \times V'$ is an isomorphism. Furthermore, the operator L^{-1} is compact as $L^{-1}(V' \times V') = V \times V$ and the embedding $V \times V \subset H \times H$ is compact. Notice that the system (2.6) *may be rewritten*

$$-\lambda^2(u,w) + L(u,w) = (g + i\lambda f + \gamma A(f+h), \ell + i\lambda h + \gamma A(f+h)),$$

or

$$(u,w) - \lambda^2 L^{-1}(u,w) = L^{-1}(g + i\lambda f + \gamma A(f+h), \ell + i\lambda h + \gamma A(f+h)). \tag{2.7}$$

The Fredholm alternative, e.g. [5, Theorem VI.6, p. 92] *shows that solving* (2.7) *amounts to proving that the equation* $(u,w)-\lambda^2L^{-1}(u,w)=(0,0)$ *has the unique solution* $(u,w)=(0,0)$*, or equivalently that (0,0) is the unique solution of the equation* $L(u,w)-\lambda^2(u,w)=(0,0)$*. Taking the duality product between* $V'\times V'$ *and* $V\times V$ *on both sides of the latter equation with* (u,w)*, we derive*

$$-\lambda^2(|u|^2+|w|^2)+a|A^{\frac{1}{2}}u|^2+b|A^{\frac{1}{2}}w|^2+i\lambda\gamma|A^{\frac{1}{2}}(u+w)|^2=0. \tag{2.8}$$

Taking the imaginary part in (2.8)*, one deduces* $u=-w$*. Therefore, we have on the one hand*

$$-\lambda^2u+aAu=0 \tag{2.9}$$

and on the other hand $-\lambda^2w+bAw=0$*, which is equivalent to*

$$\lambda^2u-bAu=0, \tag{2.10}$$

since $u=-w$*.*
The combination of (2.9) *and* (2.10) *yields* $(a-b)Au=0$*, from which it readily follows* $u=0$*, since* $a\neq b$*. Therefore* $w=0$*. Thus, for* $\theta=1$*, combining the fact that* $i\lambda I-\mathcal{A}_1$ *is bijective and the closed graph theorem, we derive that* (2.5) *holds.*

Before stating our main result, we find it useful to recall the following definition, and result (adapted from [28, Theorem 4, p. 153]).

Definition Let $t_0\geq 0$ be a real number. A strongly continuous semigroup $T=(T(t))_{t\geq 0}$, defined on a Banach space X, is of Gevrey class $s>1$ for $t>t_0$, if $T(t)$ is infinitely differentiable for $t>t_0$, and for every compact set $K\subset(t_0,\infty)$ and each $\mu>0$, there exists a constant $C=C(\mu,K)>0$ such that

$$||T^{(n)}(t)||_{L(X)}\leq C\mu^n(n!)^s,\ \text{ for all }\ t\in K,\ \ n=0,1,2...$$

Theorem 2.1. *(*[28]*) Let* $T=(T(t))_{t\geq 0}$ *be a strongly continuous and bounded semigroup on a Hilbert space* X*. Suppose that the infinitesimal generator* A *of the semigroup* T *satisfies the following estimate, for some* $0<\alpha<1$*:*

$$\overline{\lim_{|\lambda|\to\infty}}|\lambda|^\alpha||(i\lambda I-A)^{-1}||_{\mathcal{L}(X)}<\infty. \tag{2.11}$$

Then $T=(T(t))_{t\geq 0}$ *is of Gevrey class* δ *for* $t>0$*, for every* $\delta>\frac{1}{\alpha}$*.*

Our main result reads:

Theorem 2.2. *The semigroup* $e^{\mathcal{A}_\theta t}$ *is*

(1) analytic when $\theta=\frac{1}{2}$*,*
(2) of Gevrey class of order δ *for* $t>0$*, for each* $\delta>\frac{1}{2\theta}$*, when* $\theta\in(0,\frac{1}{2})$*,*
(3) of Gevrey class of order δ *for* $t>0$*, for each* $\delta>\frac{1}{2(1-\theta)}$*, when* $\theta\in(\frac{1}{2},1)$*.*

Proof Suppose for some $\mu\in(0,1]$ and $\theta\in(0,1)$ without having any specific relations among them, the following is not true:

$$\overline{\lim_{\lambda\in\mathbb{R},\,|\lambda|\to\infty}}|\lambda|^\mu\|(i\lambda-\mathcal{A}_\theta)^{-1}\|<\infty.$$

Then there exists a sequence $\{(\lambda_n, U_n) \mid n \geq 1\} \subseteq \mathbb{R} \times \mathcal{D}(\mathcal{A}_\theta)$ with $U_n \equiv (u_n, v_n, w_n, z_n)^T$, and

$$\begin{cases} \lim_{n\to\infty} |\lambda_n| = \infty, \\ \|U_n\|_{\mathcal{H}}^2 = a\|A^{\frac{1}{2}}u_n\|^2 + \|v_n\|^2 + b\|A^{\frac{1}{2}}w_n\|^2 + \|z_n\|^2 = 1, \quad n \geq 1, \end{cases} \tag{2.12}$$

such that

$$\lim_{n\to\infty} |\lambda_n|^{-\mu} \|(i\lambda_n - \mathcal{A}_\theta)U_n\|_{\mathcal{H}} = 0. \tag{2.13}$$

Therefore, it follows

$$i\lambda_n^{-\mu+1}A^{\frac{1}{2}}u_n - \lambda_n^{-\mu}A^{\frac{1}{2}}v_n = o(1), \tag{2.14}$$

$$i\lambda_n^{-\mu+1}v_n + \lambda_n^{-\mu}(aAu_n + \gamma A^\theta(v_n + z_n)) = o(1), \tag{2.15}$$

$$i\lambda_n^{-\mu+1}A^{\frac{1}{2}}w_n - \lambda_n^{-\mu}A^{\frac{1}{2}}z_n = o(1), \tag{2.16}$$

$$i\lambda_n^{-\mu+1}z_n + \lambda_n^{-\mu}(bAw_n + \gamma A^\theta(v_n + z_n)) = o(1) \tag{2.17}$$

Hereafter $o(1)$, (respectively, $O(1)$) stands for a vector in H (or a quantity in $\mathbb{R}$) which goes to zero (respectively, is bounded) as $n \to \infty$. Without losing generality, we have assumed $\lambda_n > 0$ for all n.

Adding (2.14) and (2.16), as well as (2.15) and (2.17), respectively, leads to

$$i\lambda_n^{-\mu+1}A^{\frac{1}{2}}\xi_n - \lambda_n^{-\mu}A^{\frac{1}{2}}\eta_n = o(1), \tag{2.18}$$

$$i\lambda_n^{-\mu+1}\eta_n + \lambda_n^{-\mu}(aA\xi_n + b_1Aw_n + 2\gamma A^\theta\eta_n) = o(1) \tag{2.19}$$

where $b_1 = b - a$, $\xi_n = u_n + w_n$ and $\eta_n = v_n + z_n$.

It follows from the dissipativeness of $\mathcal{A}_\theta$, (see (2.4)), Cauchy-Schwarz inequality, (2.12), and (2.13):

$$\|\lambda_n^{-\mu/2}A^{\frac{\theta}{2}}\eta_n\| = o(1). \tag{2.20}$$

From now on, we proceed by cases, according to the interval of the parameter θ.

Case 1: $\mu = 2\theta$ for $\theta \in (0, \frac{1}{2}]$. Notice that $\mu = 1$ when $\theta = 1/2$.

Dividing (2.19) by $\lambda_n^{-\mu+1}$, we see that

$$\|\lambda_n^{-1}(aA\xi_n + b_1Aw_n + 2\gamma A^\theta\eta_n)\| = O(1).$$

On the other hand, by (2.12) and $\theta \leq \frac{1}{2}$, we also have

$$\|aA^{\frac{1}{2}}\xi_n + b_1A^{\frac{1}{2}}w_n + 2\gamma A^{\theta-\frac{1}{2}}\eta_n\| = O(1).$$

By interpolation,

$$\begin{aligned} &\|\lambda_n^{-1+\theta}(aA^{1-\frac{\theta}{2}}\xi_n + b_1A^{1-\frac{\theta}{2}}w_n + 2\gamma A^{\frac{\theta}{2}}\eta_n)\| \\ \leq\ & C\|\lambda_n^{-1}(aA\xi_n + b_1Aw_n + 2\gamma A^\theta\eta_n)\|^{1-\theta}\|aA^{\frac{1}{2}}\xi_n + b_1A^{\frac{1}{2}}w_n + 2\gamma A^{\theta-\frac{1}{2}}\eta_n\|^\theta \\ =\ & O(1). \end{aligned} \tag{2.21}$$

where, here and in the sequel, C denotes a generic positive constant independent of n. Next, we take inner product of (2.19) with $\lambda_n^{-1+2\theta}\eta_n$ to get

$$i\|\eta_n\|^2 + \langle \lambda_n^{-1+\theta}(aA^{1-\frac{\theta}{2}}\xi_n + b_1 A^{1-\frac{\theta}{2}}w_n + 2\gamma A^{\frac{\theta}{2}}\eta_n),\ \lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\rangle = o(1). \tag{2.22}$$

The inner product term in the above equality is $o(1)$ due to (2.20) and (2.21). Therefore, we obtain

$$\|\eta_n\| = o(1). \tag{2.23}$$

Furthermore, equation (2.22) can now be rewritten as

$$\begin{aligned} o(1) &= a\langle A^{\frac{1}{2}}\xi_n, \lambda_n^{-1}A^{\frac{1}{2}}\eta_n\rangle + b_1\langle \lambda_n^{-1+\theta}A^{1-\frac{\theta}{2}}w_n,\ \lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\rangle + 2\gamma\|\lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\|^2 \\ &= -ia\|A^{\frac{1}{2}}\xi_n\|^2 + o(1) + b_1\langle \lambda_n^{-1+\theta}A^{1-\frac{\theta}{2}}w_n,\ \lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\rangle + o(1) \end{aligned} \tag{2.24}$$

where we have replaced $\lambda_n^{-1}A^{\frac{1}{2}}\eta_n$ by $iA^{\frac{1}{2}}\xi_n + o(1)$ due to (2.18).

The inner product of (2.17) with $\lambda_n^{-1+2\theta}\eta_n$ leads to

$$i\langle z_n, \eta_n\rangle + b\langle \lambda_n^{-1+\theta}A^{1-\frac{\theta}{2}}w_n,\ \lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\rangle + \gamma\|\lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\|^2 = o(1).$$

Combining the last equality with (2.12), (2.20) and (2.23), we arrive at

$$\langle \lambda_n^{-1+\theta}A^{1-\frac{\theta}{2}}w_n,\ \lambda_n^{-\theta}A^{\frac{\theta}{2}}\eta_n\rangle = o(1). \tag{2.25}$$

Inserting (2.25) into (2.24) yields

$$\|A^{\frac{1}{2}}\xi_n\| = o(1). \tag{2.26}$$

To estimate the term $\|A^{\frac{1}{2}}w_n\|$, we take the inner product of (2.19) with $\lambda_n^{-1+2\theta}z_n$ to get

$$i\langle \eta_n, z_n\rangle + a\langle A^{\frac{1}{2}}\xi_n, \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + b_1\langle A^{\frac{1}{2}}w_n,\ \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + 2\gamma\langle \eta_n, \lambda_n^{-1}A^{\theta}z_n\rangle = o(1). \tag{2.27}$$

We replace the term $\lambda_n^{-1}A^{\frac{1}{2}}z_n$ in (2.27) by $iA^{\frac{1}{2}}w_n + o(1)$ due to (2.16). Applying (2.23), (2.26) and the fact $\theta \le \frac{1}{2}$, the above equality can be simplified to

$$\|A^{\frac{1}{2}}w_n\| = o(1). \tag{2.28}$$

Finally, we take the inner product of (2.17) with $\lambda_n^{-1+2\theta}z_n$ to get

$$\begin{aligned} o(1) &= i\|z_n\|^2 + b\langle A^{\frac{1}{2}}w_n, \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + \gamma\langle \eta_n, \lambda_n^{-1}A^{\theta}z_n\rangle \\ &= i\|z_n\|^2 + o(1). \end{aligned} \tag{2.29}$$

Combining (2.23), (2.26), (2.28), and (2.29), we arrive at a contradiction with $\|U_n\|_{\mathcal{H}} = 1$, thereby establishing the claimed analyticity and Gevrey class regularity.

Case 2: $\mu = 2(1-\theta)$ for $\theta \in (\frac{1}{2}, 1)$

Dividing (2.19) by $\lambda_n^{-\mu+1}$, we see that

$$\|\lambda_n^{-1}(aA\xi_n + b_1 Aw_n + 2\gamma A^{\theta}\eta_n)\| = O(1).$$

On the other hand, by (2.12) and $\frac{1}{2} < \theta$, we also have

$$\|aA^{1-\theta}\xi_n + b_1 A^{1-\theta}w_n + 2\gamma\eta_n\| = O(1).$$

By interpolation,

$$\begin{aligned}
&\|\lambda_n^{-\frac{1}{2}}(aA^{1-\frac{\theta}{2}}\xi_n + b_1 A^{1-\frac{\theta}{2}} w_n + 2\gamma A^{\frac{\theta}{2}}\eta_n)\| \\
\leq\ & C\|\lambda_n^{-1}(aA\xi_n + b_1 A w_n + 2\gamma A^{\theta}\eta_n)\|^{\frac{1}{2}} \|aA^{1-\theta}\xi_n + b_1 A^{1-\theta} w_n + 2\gamma\eta_n\|^{\frac{1}{2}} \\
=\ & O(1). \qquad (2.30)
\end{aligned}$$

Next, we take the inner product of (2.19) with $\lambda_n^{1-2\theta}\eta_n$ to get

$$i\|\eta_n\|^2 + \langle \lambda_n^{-\theta}(aA^{1-\frac{\theta}{2}}\xi_n + b_1 A^{1-\frac{\theta}{2}} w_n + 2\gamma A^{\frac{\theta}{2}}\eta_n),\ \lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\rangle = o(1). \qquad (2.31)$$

The inner product term in the above equality is $o(1)$ due to (2.20) and (2.30). Therefore, we obtain

$$\|\eta_n\| = o(1). \qquad (2.32)$$

Furthermore, (2.31) can now be rewritten as

$$\begin{aligned}
o(1) &= a\langle A^{\frac{1}{2}}\xi_n, \lambda_n^{-1}A^{\frac{1}{2}}\eta_n\rangle + b_1\langle \lambda_n^{-\theta} A^{1-\frac{\theta}{2}} w_n,\ \lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\rangle + 2\gamma\lambda_n^{1-2\theta}\|\lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\|^2 \\
&= -ia\|A^{\frac{1}{2}}\xi_n\|^2 + o(1) + b_1\langle \lambda_n^{-\theta} A^{1-\frac{\theta}{2}} w_n,\ \lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\rangle + o(1) \qquad (2.33)
\end{aligned}$$

where we have replaced $\lambda_n^{-1}A^{\frac{1}{2}}\eta_n$ by $iA^{\frac{1}{2}}\xi_n + o(1)$ due to (2.18).

The inner product of (2.17) with $\lambda_n^{1-2\theta}\eta_n$ leads to

$$i\langle z_n, \eta_n\rangle + b\langle \lambda_n^{-\theta} A^{1-\frac{\theta}{2}} w_n,\ \lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\rangle + \gamma\lambda_n^{1-2\theta}\|\lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\|^2 = o(1).$$

Combining the last equality with (2.20) and (2.32), we arrive at

$$\langle \lambda_n^{-\theta} A^{1-\frac{\theta}{2}} w_n,\ \lambda_n^{-(1-\theta)} A^{\frac{\theta}{2}}\eta_n\rangle = o(1). \qquad (2.34)$$

Inserting (2.34) into (2.33) yields

$$\|A^{\frac{1}{2}}\xi_n\| = o(1). \qquad (2.35)$$

To estimate the term $\|A^{\frac{1}{2}}w_n\|$, we take the inner product of (2.19) with $\lambda_n^{1-2\theta}z_n$ to get

$$\begin{aligned}
o(1) &= i\langle \eta_n, z_n\rangle + a\langle A^{\frac{1}{2}}\xi_n, \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + b_1\langle A^{\frac{1}{2}}w_n,\ \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + 2\gamma\langle \eta_n, \lambda_n^{-1}A^{\theta}z\rangle \\
&= o(1) - ib_1\|A^{\frac{1}{2}}w_n\|^2 + 2\gamma\langle \lambda_n^{-(1-\theta)}A^{\frac{\theta}{2}}\eta_n, \lambda_n^{-\theta}A^{\frac{\theta}{2}}z_n\rangle. \qquad (2.36)
\end{aligned}$$

Since

$$\|\lambda_n^{-\theta}A^{\frac{\theta}{2}}z_n\| \leq C\|\lambda_n^{-1}A^{\frac{1}{2}}z_n\|^{\theta}\|z_n\|^{1-\theta} = O(1),$$

then (2.36) implies that

$$\|A^{\frac{1}{2}}w_n\| = o(1). \qquad (2.37)$$

Finally, we take the inner product of (2.17) with $\lambda_n^{-1+2\theta}z_n$ to get

$$\begin{aligned}
o(1) &= i\|z_n\|^2 + b\langle A^{\frac{1}{2}}w_n, \lambda_n^{-1}A^{\frac{1}{2}}z_n\rangle + \gamma\langle \lambda_n^{-(1-\theta)}A^{\frac{\theta}{2}}\eta_n, \lambda_n^{-\theta}A^{\frac{\theta}{2}}z_n\rangle \\
&= i\|z_n\|^2 + o(1). \qquad (2.38)
\end{aligned}$$

Combining (2.32), (2.35), (2.37), and (2.38), we arrive at a contradiction with $\|U_n\|_{\mathcal{H}} = 1$. □

Remark 2.2. *In* [2, Theorem 1.1], *the auhors prove that for every* $\theta \in (1/2, 1]$, *and every* r *in* $(2(1-\theta), 1]$, *one has :*

$$\overline{\lim_{|\lambda|\to\infty}} |\lambda|^r \left\|(i\lambda I - \mathcal{A}_\theta)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \infty. \tag{2.39}$$

That equality shows that our resolvent estimate leading to the claimed Gevrey regularity class for θ *in* $(1/2, 1)$ *is optimal. Similarly, in* [2, Theorem 1.2] *they prove: for every* $\theta \in (0, 1/2)$ *and every* $r \in (2\theta, 1]$, *one has:*

$$\overline{\lim_{|\lambda|\to\infty}} |\lambda|^r \left\|(i\lambda I - \mathcal{A}_\theta)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \infty, \tag{2.40}$$

which shows the optimality of our resolvent estimate for θ *in* $(0, 1/2)$ *as well.*
These optimality results prove that the analyticity of the semigroup occurs for the only value $\theta = 1/2$. *Thus, in the case of interacting elastic systems with degenerate damping as in this contribution, analyticity of the underlying semigroup occurs only when the damping is structural; the degeneracy prevents analyticity for above structural damping mechanisms.*

A careful reader may have noticed that in Remark 2.2, we do not claim the optimality of the Gevrey class; this is due to the fact that we are using a sufficient condition for Gevrey regularity, namely, Theorem 2.1. In the next section, we shall discuss the eigenvalue problem, state asymptotic expressions for the eigenvalues, and use those to show that our analyticity or Gevrey class regularity results are indeed optimal. We shall also prove the nondifferentiability of the semigroup for the values $\theta = 0$ and $\theta = 1$.

3. Eigenvalue asymptotics and optimality of regularity results

Let $U = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}$ in $D(\mathcal{A}_\theta)$. We have

$$\mathcal{A}_\theta U = \begin{bmatrix} 0 & I & 0 & 0 \\ -aA & -\gamma A^\theta & 0 & -\gamma A^\theta \\ 0 & 0 & 0 & I \\ 0 & -\gamma A^\theta & -bA & -\gamma A^\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} v \\ -aAu - \gamma A^\theta v - \gamma A^\theta z \\ z \\ -\gamma A^\theta v - bAw - \gamma A^\theta z. \end{bmatrix}$$

Consequently, the eigenvalue system reads

$$(\lambda - A_\theta)U = \begin{bmatrix} \lambda u - v \\ \lambda v + aAu + \gamma A^\theta v + \gamma A^\theta z \\ \lambda w - z \\ \lambda z + bAw + \gamma A^\theta v + \gamma A^\theta z \end{bmatrix} = 0, \tag{3.1}$$

where a, b, and γ are positive constants, and $\theta \in [-1, 1]$. From the first three equations,

$$\begin{cases} v = & \lambda u \\ w = & -\dfrac{A^{-\theta}\left(aA + \gamma\lambda A^{\theta} + \lambda^2\right) u}{\gamma\lambda} \\ z = & -\dfrac{A^{-\theta}\left(aA + \gamma\lambda A^{\theta} + \lambda^2\right) u}{\gamma} \end{cases}.$$

Therefore, using the fourth equation, and denoting $\{\mu_n\}$ the sequence of eigenvalues of the operator A, we find

$$\lambda^4 + \beta_n\lambda^3 + c_n\lambda^2 + d_n\lambda + e_n = 0, \tag{3.2}$$

where

$$\beta_n = 2\gamma\mu_n^{\theta}, \quad c_n = (a+b)\mu_n, \quad d_n = a\gamma\mu_n^{\theta+1} + b\gamma\mu_n^{\theta+1}, \quad \text{and} \quad e_n = ab\mu_n^2.$$

From now on, we assume

$$a = 1, \quad b = 2, \quad \gamma = 1. \tag{3.3}$$

This assumption leads us to the simplified characteristic equation:

$$\lambda_n^4 + 2\mu_n^{\theta}\lambda_n^3 + 3\mu_n\lambda_n^2 + 3\mu_n^{1+\theta}\lambda_n + 2\mu_n^2 = 0. \tag{3.4}$$

Our main results for this section read

Theorem 3.1. *The solutions to the quartic characteristic equation* (3.4) *when* $\theta \in [-1, \frac{1}{2})$*,* $\theta = \frac{1}{2}$*, and* $\theta \in (\frac{1}{2}, 1]$ *are given by*

$$\lambda_{n,1} = \begin{cases} -\frac{1}{2}\mu_n^{\theta}(1+o(1)) + i\sqrt{2}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \psi_{n,1}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ -\frac{2}{3}\mu_n^{1-\theta}(1+o(1)) & \theta \in (\frac{1}{2}, 1]. \end{cases}$$

$$\lambda_{n,2} = \begin{cases} -\frac{1}{2}\mu_n^{\theta}(1+o(1)) - i\sqrt{2}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \psi_{n,2}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ -2\mu^{\theta}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases}$$

$$\lambda_{n,3} = \begin{cases} -\frac{1}{2}\mu_n^{\theta}(1+o(1)) + i\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \psi_{n,3}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ -\frac{1}{24}\mu_n^{1-\theta}(1+o(1)) + i\frac{\sqrt{6}}{2}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases}$$

$$\lambda_{n,4} = \begin{cases} -\frac{1}{2}\mu_n^{\theta}(1+o(1)) - i\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \psi_{n,4}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ -\frac{1}{24}\mu_n^{1-\theta}(1+o(1)) - i\frac{\sqrt{6}}{2}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases}$$

Here, $\psi_{n,1}$*,* $\psi_{n,2}$*,* $\psi_{n,3}$*, and* $\psi_{n,4}$ *are two pairs of complex conjugate numbers.*

Theorem 3.2. *Set*

$$m_\theta = \begin{cases} 1, & \theta = \frac{1}{2} \\ 2\theta, & 0 < \theta < \frac{1}{2} \\ 2(1-\theta), & \frac{1}{2} < \theta < 1. \end{cases} \tag{3.5}$$

(i) For every θ in $(0,1)$, there exists a sequence (λ_n) of eigenvalues of $\mathcal{A}_\theta$ such that for every $\varepsilon > 0$, one has

$$\overline{\lim_{n\to\infty}} \frac{Re\lambda_n}{|Im\lambda_n|^{m_\theta+\varepsilon}} = 0. \tag{3.6}$$

(ii) For $\theta = 0$ or $\theta = 1$, there exists a sequence (λ_n) of eigenvalues of $\mathcal{A}_\theta$ such that

$$\lim_{n\to\infty} Re\lambda_n = \ell, \quad \lim_{n\to\infty} |\lambda_n| = \infty, \tag{3.7}$$

for some real number ℓ.

The proof of Theorem 3.1 is very technical and will be dealt with in the next section. The proof of Theorem 3.2 follows at once from Theorem 3.1.
In particular, thanks to Theorem 3.2 and the following two results, one derives the optimality of our regularity results, as well as the nondifferentiability of the semigroup for $\theta = 0$ and $\theta = 1$.

Lemma 3.1. *([11, 23, 26, 28]) Let $T = (T(t))_{t\geq 0}$ be a strongly continuous and bounded semigroup on a Hilbert space X. Let $\mathcal{A}$ denote the infinitesimal generator $\mathcal{A}$ of the semigroup T.*

(i) The semigroup T is analytic if and only if there exist real numbers a, b and C, with $b > 0$ and $C > 0$ such that

$$\rho(\mathcal{A}) \supseteq \Sigma_{a,b} = \{\lambda \in \mathbb{C}; Re\lambda > a - b|Im\lambda|\},$$

and

$$||(\lambda - \mathcal{A})^{-1}|| \leq \frac{C}{1+|\lambda|}, \quad \forall \lambda \in \Sigma_{a,b}.$$

(ii) The semigroup T is of Gevrey class $\delta > 1$ if and only if for any positive real numbers b and τ, there are real constants a and C with $C > 0$ depending on b, τ and δ such that

$$\rho(\mathcal{A}) \supseteq \Sigma_b(\delta) = \{\lambda \in \mathbb{C}; Re\lambda > a - b|Im\lambda|^{\frac{1}{\delta}}\},$$

and

$$||(\lambda - \mathcal{A})^{-1}|| \leq C\left(1 + e^{-\tau Re\lambda}\right), \quad \forall \lambda \in \Sigma_b(\delta).$$

(iii) The semigroup T is differentiable if and only if for any positive real number b, there are real constants a and C with $C > 0$ depending on b such that

$$\rho(\mathcal{A}) \supseteq \Sigma_b = \{\lambda \in \mathbb{C}; Re\lambda > a - b\log|Im\lambda|\},$$

and

$$||(\lambda - \mathcal{A})^{-1}|| \leq C|Im\lambda|, \quad \forall \lambda \in \Sigma_b \text{ with } Re\lambda \leq 0.$$

Lemma 3.2. [19, Corollary 2.2] *Let $T = (T(t))_{t\geq 0}$ be a strongly continuous and bounded semigroup on a Hilbert space X. Let $\mathcal{A}$ denote the infinitesimal generator $\mathcal{A}$ of the semigroup T.*

(i) Suppose that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ contains a sequence (λ_n) such that

$$\lim_{n\to\infty} Re\lambda_n = \ell, \quad \lim_{n\to\infty} |\lambda_n| = \infty,$$

for some ℓ in $\mathbb{R}$. Then the semigroup T is not differentiable.

(ii) Let $\delta \geq 1$. Suppose that there exists a sequence (λ_n) in $\sigma(\mathcal{A})$ such that

$$\lim_{n\to\infty} \frac{Re\lambda_n}{|Im\lambda_n|^{\frac{1}{\delta}}} = 0.$$

Then the semigroup T is not of Gevrey class δ.

Remark 3.1. *In Theorem 2.2, we prove regularity results for System* (2.1). *As noted above, the combination of Theorem 3.2, Lemma 3.1 and Lemma 3.2 show that those regularity results are optimal. Accounting for the stability results established in* [2], *we can summarize regularity and stability results for System* (2.1) *in the following table.*

4. Point spectrum Analysis

In this section, our goal is to analyze the point spectrum by solving the quartic characteristic equation (3.4). To this end, first, we describe how to compute the analytic solution to a general quartic equation. Then, we describe how to solve (3.4).

4.1. **Solutions to a General Quartic Equation.** We begin by describing how to solve a general quartic equation:

$$\lambda^4 + \beta\lambda^3 + c\lambda^2 + d\lambda + e = 0 \tag{4.1}$$

analytically. Our procedure follows that described in [6]. To this end, we define

$$\begin{gathered} m = 8c - 3\beta^2, \quad s = 3\beta^4 + 16c^2 + 16\beta d - 16\beta^2 c - 64e, \\ r = -\beta^3 + 4\beta c - 8d, \quad l = -r^2. \end{gathered} \tag{4.2}$$

According to [6], we have the following cubic resolvent associated with (4.1).

$$j^3 + mj^2 + sj + l = 0. \tag{4.3}$$

To solve (4.1), we first need to solve (4.3). For this purpose, we define

$$h = 3s - m^2, \quad g = 2m^3 - 9ms + 27l, \quad P = h/3, \quad Q = g/27. \tag{4.4}$$

We further define the discriminant as:

$$\Delta = \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2. \tag{4.5}$$

TABLE 1. Summary of Stability and Regularity results of the Semigroup $e^{\mathcal{A}_\theta t}$

Region	Stability	Regularity
$\theta \in [-1, 0)$	polynomially stable of order $-\frac{1}{2\theta}$	not differentiable
$\theta = 0$	exponentially stable	not differentiable
$\theta \in (0, \frac{1}{2})$	exponentially stable	Gevrey class of order $\frac{1}{2\theta}$
$\theta = \frac{1}{2}$	exponentially stable	analytic
$\theta \in (\frac{1}{2}, 1)$	exponentially stable	Gevrey class of order $\frac{1}{2-2\theta}$
$\theta = 1$	exponentially stable	not differentiable

We also need to define:

$$\Phi_{\pm} = -\frac{Q}{2} \pm \sqrt{\Delta} = -\frac{Q}{2} \pm \sqrt{\left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^2}, \tag{4.6}$$

where, for any $\xi = |\xi| e^{i\theta}$, we consider $\xi^{\frac{1}{2}} = |\xi|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ and $\xi^{\frac{1}{3}} = |\xi|^{\frac{1}{3}} e^{i\frac{\theta}{3}}$. From (4.6), we have that

$$\Lambda = \Phi_+^{\frac{1}{3}} + \Phi_-^{\frac{1}{3}} = \frac{-Q}{\Phi_+^{\frac{2}{3}} - \Phi_+^{\frac{1}{3}}\Phi_-^{\frac{1}{3}} + \Phi_-^{\frac{2}{3}}}, \quad \Omega = \Phi_+^{\frac{1}{3}} - \Phi_-^{\frac{1}{3}} = \frac{2\sqrt{\Delta}}{\Phi_+^{\frac{2}{3}} + \Phi_+^{\frac{1}{3}}\Phi_-^{\frac{1}{3}} + \Phi_-^{\frac{2}{3}}}. \tag{4.7}$$

With (4.7) and m from (4.2), the solutions to the cubic resolvent can then be expressed via the Cardano's formula [24]:

$$j_0 = \Lambda - \frac{m}{3}, \quad j_{\pm} = -\frac{1}{2}\Lambda \pm i\frac{\sqrt{3}}{2}\Omega - \frac{m}{3}. \tag{4.8}$$

Furthermore, let $k_0 = j_0^{\frac{1}{2}}$, $k_+ = j_+^{\frac{1}{2}}$, and $k_- = j_-^{\frac{1}{2}}$, such that $k_0 k_+ k_- = r$, we can denote the four roots to (4.1) according to [6] as:

$$\begin{aligned}
\lambda_1 &= (-\beta + k_0 + k_+ + k_-)/4 \\
\lambda_2 &= (-\beta + k_0 - k_+ - k_-)/4 \\
\lambda_3 &= (-\beta - k_0 + k_+ - k_-)/4 \\
\lambda_4 &= (-\beta - k_0 - k_+ + k_-)/4.
\end{aligned} \tag{4.9}$$

4.2. **Compute the Roots of the Characteristic Equation** (3.4)**.** In this section, we describe how to solve (3.4) based on the the analytic solutions to a general quartic equation described in Section 4.1. To this end, for the sequence of eigenvalues of the operator A, $\{\mu_n\}$, recall that

$$\beta_n = 2\mu_n^{\theta}, \quad c_n = 3\mu_n, \quad d_n = 3\mu_n^{1+\theta}, \quad e_n = 2\mu_n^2, \tag{4.10}$$

where β_n, c_n, d_n, and e_n are directly associated with β, c, d, and e in (4.1), representing the coefficients of the quartic characteristic equation associated with μ_n. In particular, we use the subscript n to stress the dependency of these coefficients on the sequence of $\{\mu_n\}$ and we shall use the same convention subsequently for the quantities that are associated with $\{\mu_n\}$. So, from (4.2) and (4.4), we have that

$$m_n = 24\mu_n - 12\mu_n^{2\theta}, \quad s_n = 48\mu_n^{4\theta} - 96\mu_n^{2\theta+1} + 16\mu_n^2,$$
$$r_n = -8\mu_n^{3\theta}, \quad l_n = -64\mu_n^{6\theta}, \quad h_n = 288\mu_n^{2\theta+1} - 528\mu_n^2, \quad g_n = 24192\mu_n^3 - 19008\mu_n^{2\theta+2},$$
$$P_n = 96\mu_n^{2\theta+1} - 176\mu_n^2, \quad Q_n = 896\mu_n^3 - 704\mu_n^{2\theta+2},$$

that are associated with $\{\mu_n\}$. As a result, we also have Δ_n from (4.5), j_n from (4.3), $\Phi_{n,\pm}$ from (4.6), Λ_n and Ω_n from (4.7), $j_{n,0}$ and $j_{n,\pm}$ from (4.8), and $k_{n,0}$, $k_{n,\pm}$, $\lambda_{n,1}$, $\lambda_{n,2}$, $\lambda_{n,3}$, and $\lambda_{n,4}$ from (4.9) that are associated with $\{\mu_n\}$.

With these definitions, we introduce Lemma 4.1 and Lemma 4.2, which are concerned with the expression of Λ_n and Ω_n.

Lemma 4.1. *If $-\frac{Q_n}{2} = o(\sqrt{\Delta_n})$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \Lambda_n = \lim_{n\to\infty} -\frac{Q_n}{P_n}, \quad \lim_{n\to\infty} \Omega_n = \lim_{n\to\infty} \frac{6\sqrt{\Delta_n}}{P_n}. \tag{4.11}$$

If $\sqrt{\Delta_n} = o(-\frac{Q_n}{2})$ as $n \to \infty$, then

$$\lim_{n\to\infty} \Lambda_n = \lim_{n\to\infty} \frac{3Q_n}{P_n}, \quad \lim_{n\to\infty} \Omega_n = \lim_{n\to\infty} -\frac{2\sqrt{\Delta_n}}{P_n}. \tag{4.12}$$

If $\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$ as $n \to \infty$, then

$$\lim_{n\to\infty} \Lambda_n, \Omega_n = \left((\kappa_{n,1} + \kappa_{n,2})^{\frac{1}{3}} \pm (\kappa_{n,1} - \kappa_{n,2})^{\frac{1}{3}}\right) \mu_n^x (1 + o(1)), \tag{4.13}$$

where $\kappa_{n,1}$, $\kappa_{n,2}$, and x are quantities related to $-\frac{Q_n}{2}$ and $\sqrt{\Delta_n}$ via

$$-\frac{Q_n}{2} = \kappa_{n,1}\mu_n^{3x}(1 + o(1)), \quad \sqrt{\Delta_n} = \kappa_{n,2}\mu_n^{3x}(1 + o(1)). \tag{4.14}$$

In (4.14)*, $3x$ is the power of the leading term shared by $-\frac{Q_n}{2}$ and $\sqrt{\Delta_n}$ because we consider $\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$. We further assume that $\kappa_{n,1} \pm \kappa_{n,2} \neq 0$. We omit the discussion when $\kappa_{n,1} \pm \kappa_{n,2} = 0$ as it is not involved in our problem.*

Proof. We discuss our proof in three cases: $-\frac{Q_n}{2} = o(\sqrt{\Delta_n})$, $\sqrt{\Delta_n} = o\left(-\frac{Q_n}{2}\right)$, and $\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$.

- If $-\frac{Q_n}{2} = o(\sqrt{\Delta_n})$, then by (4.6), $\Phi_{n,\pm} = O(\sqrt{\Delta_n})$ and $\lim_{n\to\infty} \Phi_{n,\pm} = \lim_{n\to\infty} \pm\sqrt{\Delta_n} \Rightarrow \lim_{n\to\infty} \Phi_{n,+} = -\lim_{n\to\infty} \Phi_{n,-} \Rightarrow \lim_{n\to\infty} \Phi_{n,+}^{\frac{1}{3}} = -\lim_{n\to\infty} \Phi_{n,-}^{\frac{1}{3}}$. Recall that $\Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} = -\frac{P_n}{3}$. Therefore, we have $\lim_{n\to\infty} \Phi_{n,\pm}^{\frac{2}{3}} = \lim_{n\to\infty} \frac{P_n}{3}$. Therefore, by (4.7), we can show that (4.11) holds.
- If $\sqrt{\Delta_n} = o\left(-\frac{Q_n}{2}\right)$, then by (4.6), $\Phi_{n,\pm} = O(-\frac{Q_n}{2})$ and $\lim_{n\to\infty} \Phi_{n,\pm} = \lim_{n\to\infty} -\frac{Q_n}{2} \Rightarrow \lim_{n\to\infty} \Phi_{n,+}^{\frac{1}{3}} = \lim_{n\to\infty} \Phi_{n,-}^{\frac{1}{3}}$. By $\Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} = -\frac{P_n}{3}$, we have that $\lim_{n\to\infty} \Phi_{n,\pm}^{\frac{2}{3}} = -\lim_{n\to\infty} \frac{P_n}{3}$. Therefore, by (4.7), we can show that (4.12) holds.

• Finally, when $\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$, without loss of generality, we can write:

$$-\frac{Q_n}{2} = \kappa_{n,1}\mu_n^{3x}(1+o(1)), \quad \sqrt{\Delta_n} = \kappa_{n,2}\mu_n^{3x}(1+o(1)),$$

where we further assume that $\kappa_{n,1} \pm \kappa_{n,2} \neq 0$. Then

$$\Phi_{n,\pm} = (\kappa_{n,1} \pm \kappa_{n,2})\mu_n^{3x}(1+o(1)) \Rightarrow \Phi_{n,\pm}^{\frac{1}{3}} = (\kappa_{n,1} \pm \kappa_{n,2})^{\frac{1}{3}}\mu_n^{x}(1+o(1)).$$

Therefore, by (4.7), (4.13) also holds. □

While Lemma 4.1 provides a characterization of the leading term of Λ_n, later on, we will show that using just the leading term of Λ_n does not suffice to estimate the leading terms of $\lambda_{n,0}$ and $\lambda_{n,\pm}$ as desired. This motivates us to further characterize the lower order terms of Λ_n, yielding Lemma 4.2.

Lemma 4.2. *Let $\Lambda_n = Z_n + \omega_n$, where Z_n is the leading term of Λ_n as $n \to \infty$, and ω_n is the lower order term. i.e. $\omega_n = o(Z_n)$. Then*

$$\lim_{n\to\infty} \omega_n = \lim_{n\to\infty} -\frac{P_nZ_n + Q_n + Z_n^3}{P_n + 3Z_n^2}. \tag{4.15}$$

Proof. First, we define

$$x_{n,0} = \Lambda_n, \quad x_{n,\pm} = -\frac{1}{2}\Lambda_n \pm i\frac{\sqrt{3}}{2}\Omega_n.$$

By Cardano's formula, $x_{n,0}$, $x_{n,\pm}$ are solutions to $x_n^3 + P_nx_n + Q_n = 0$. We further define

$$y_{n,0} = \omega_n, \quad y_{n,\pm} = -\frac{\omega_n}{2} - \frac{3}{2}Z_n \pm i\frac{\sqrt{3}}{2}\Omega_n, \tag{4.16}$$

where $\Lambda_n = Z_n + \omega_n$, Z_n is the leading term of Λ_n, and $\omega_n = o(Z_n)$. Clearly,

$$x_{n,0} = Z_n + y_{n,0}, \quad x_{n,\pm} = Z_n + y_{n,\pm}.$$

Therefore, $y_{n,0}$ and $y_{n,\pm}$ are the three solutions for:

$$(y_n + Z_n)^3 + P_n(y_n + Z_n) + Q_n = 0 \Rightarrow y_n^3 + 3Z_ny_n^2 + (P_n + 3Z_n^2)y_n + P_nZ_n + Q_n + Z_n^3 = 0.$$

On the other hand, using (4.16), by Vieta's formula with respect to $y_{n,0}$, and $y_{n,\pm}$,

$$y_{n,0}y_{n,+} + y_{n,0}y_{n,-} + y_{n,+}y_{n,-} = \frac{3}{4}\Omega_n^2 + \frac{9}{4}Z_n^2 - \frac{3}{2}Z_n\omega_n - \frac{3}{4}\omega_n^2 = P_n + 3Z_n^2, \tag{4.17a}$$

$$y_{n,0} + y_{n,+} + y_{n,-} = \frac{3}{4}\Omega_n^2\omega_n + \frac{9}{4}Z_n^2\omega_n + \frac{3}{2}Z_n\omega_n^2 + \frac{1}{4}\omega_n^3 = -\left(P_nZ_n + Q_n + Z_n^3\right). \tag{4.17b}$$

If $-\frac{Q_n}{2} = o(\sqrt{\Delta_n})$, by (4.11), $Z_n = o(\Omega_n)$. From (4.17a) and (4.17b), as $n \to \infty$,

$$\lim_{n\to\infty} \frac{3}{4}\Omega_n^2 = \lim_{n\to\infty} P_n + 3Z_n^2, \quad \lim_{n\to\infty} \frac{3}{4}\Omega_n^2\omega_n = \lim_{n\to\infty} -\left(P_nZ_n + Q_n + Z_n^3\right),$$

which indicates that (4.15) is true.

If $\sqrt{\Delta_n} = o\left(-\frac{Q_n}{2}\right)$, by (4.12), $\Omega_n = o(Z_n)$. From (4.17a) and (4.17b), as $n \to \infty$,

$$\lim_{n\to\infty} \frac{9}{4} Z_n^2 = \lim_{n\to\infty} P_n + 3Z_n^2, \quad \lim_{n\to\infty} \frac{9}{4} Z_n^2 \omega_n = \lim_{n\to\infty} -\left(P_n Z_n + Q_n + Z_n^3\right),$$

which indicates that (4.15) is also true in this case.

Finally, if $\sqrt{\Delta_n} = O\left(-\frac{Q_n}{2}\right)$, by (4.13), $Z_n = O(\Omega_n)$. From (4.17a) and (4.17b), as $n \to \infty$,

$$\lim_{n\to\infty} \frac{3}{4}\Omega_n^2 + \frac{9}{4} Z_n^2 = \lim_{n\to\infty} P_n + 3Z_n^2, \quad \lim_{n\to\infty} \frac{3}{4}\Omega_n^2\omega_n + \frac{9}{4} Z_n^2 \omega_n = \lim_{n\to\infty} -\left(P_n Z_n + Q_n + Z_n^3\right),$$

which validates (4.15) in this case. □

With Section 4.1, Lemma 4.1 and Lemma 4.2, we are now ready to solve (3.4). From (4.5), the discriminant is

$$\Delta_n = \frac{45056}{3}\mu_n^{2\theta+5} - 56320\mu_n^{4\theta+4} + 32768\mu_n^{6\theta+3} - \frac{32768}{27}\mu_n^6. \tag{4.18}$$

We partition $\theta \in [-1, 1]$ into non-overlapping regions based on the leading terms of Δ_n: $\theta \in [-1, \frac{1}{2})$, $\theta = \frac{1}{2}$, and $\theta \in (\frac{1}{2}, 1]$. The expression of Δ_n can be given as:

$$\Delta_n = \begin{cases} -\frac{32768}{27}\mu_n^6(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ -\frac{263168}{27}\mu_n^6(1+o(1)), & \theta = \frac{1}{2}, \\ 32768\mu_n^{6\theta+3}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases}.$$

We describe how the leading terms of quantity is determined. As an illustrative example, we show how to determine the leading term of Δ_n when $\theta \in [-1, \frac{1}{2})$. Using the formula for Δ_n given in (4.18), there are four exponential terms in total with μ_n as the base. Therefore, it suffices to identify the largest exponent among the four exponential terms by solving the following optimization problem:

$$6 = \max_{\theta \in [-1, \frac{1}{2})} \{2\theta + 5, 4\theta + 4, 6\theta + 3, 6\}.$$

Our next goal is to identify the solutions to the characteristic equation (3.4) across all the regions of θ defined as aforementioned. These solutions have been summarized in Theorem 3.1.

The rest of this section is dedicated to prove Theorem 3.1. To this end, in Section 4.2.1, first, we will compute a detailed expression for the roots of the cubic resolvent $j_{n,0}$ and $j_{n,\pm}$ through the use of Cardano's formula, Lemma 4.1, and Lemma 4.2. Upon knowing j_n's, we can compute λ_n's, through the use of (4.9) in Section 4.2.2.

4.2.1. *Compute the Roots of the Cubic Resolvent.* In this section, we would like to compute the roots of the cubic resolvent (4.3). For this purpose, first, we compute the leading terms of Λ_n and Ω_n using Lemma 4.1. Then, we compute the lower order terms of Λ_n. Upon having sufficient knowledge about the expression of Λ_n and Ω_n, we will compute the roots to the cubic resolvent via the Cardano's formula (4.8).

By Lemma 4.1, to compute the leading terms of Λ_n and Ω_n, we need to know about the leading terms P_n, Q_n, and $\sqrt{\Delta_n}$. We would also need to know the relative order of $\sqrt{\Delta_n}$ and $-\frac{Q_n}{2}$. These quantities are given in Table 2.

TABLE 2. P_n, Q_n, and $\sqrt{\Delta_n}$

Region	P_n	Q_n	$\sqrt{\Delta_n}$	Relative Order
$\theta \in [-1, \frac{1}{2})$	$-176\mu_n^2(1+o(1))$	$896\mu_n^3(1+o(1))$	$\frac{128\sqrt{6}}{9}i\mu_n^3(1+o(1))$	$\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$
$\theta = \frac{1}{2}$	$-80\mu_n^2(1+o(1))$	$192\mu_n^3(1+o(1))$	$\frac{32\sqrt{771}}{9}i\mu_n^3(1+o(1))$	$\sqrt{\Delta_n} = O(-\frac{Q_n}{2})$
$\theta \in (\frac{1}{2}, 1]$	$96\mu_n^{1+2\theta}(1+o(1))$	$-704\mu_n^{2+2\theta}(1+o(1))$	$128\sqrt{2}i\mu_n^{\frac{3}{2}+3\theta}(1+o(1))$	$-\frac{Q_n}{2} = o(\sqrt{\Delta_n})$

With the quantities presented in Table 2, we can compute the leading terms of Λ_n and Ω_n. The results are summarized in Table 3. Note that in Table 3,

$$\tau_{n,\pm} = \sqrt[3]{-96 + \frac{32}{3}i\sqrt{\frac{257}{3}}} \pm \sqrt[3]{-96 - \frac{1}{3}32i\sqrt{\frac{257}{3}}}, \quad \tau_{n,+} \approx 7.33697, \quad \tau_{n,-} \approx -7.2688i.$$

For the convenience of applying the Cardano's formula, we also report m_n in Table 3.

Now that we know Λ_n and Ω_n, we would like to apply Cardano's formula in (4.8) to compute j_n's. We discuss the computation case by case according to the value of θ. We first discuss when $\theta \in [-1, \frac{1}{2})$. In this case, using (4.8) and Table 3, we notice that $j_{n,0} = o(\mu_n)$. This motivates us to use Lemma 4.2 to acquire lower order terms for Λ_n. From (4.15), Table 2, and Table 3, we have that

$$\Lambda_n = 8\mu_n - 4\mu_n^{2\theta} + 4\mu_n^{6\theta-2}(1+o(1)). \tag{4.19}$$

Therefore, by (4.8), (4.19), and Table 3, we have that

$$\begin{aligned} j_{n,0} &= 8\mu_n - 4\mu_n^{2\theta} + 4\mu_n^{6\theta-2}(1+o(1)) - \tfrac{1}{3}\cdot\left(24\mu_n - 12\mu_n^{2\theta}\right) = 4\mu_n^{6\theta-2}(1+o(1)), \\ j_{n,\pm} &= -(4 \pm 8\sqrt{2})\mu_n(1+o(1)) - \tfrac{1}{3}\cdot 24\mu_n(1+o(1)) = -(12 \pm 8\sqrt{2})\mu_n(1+o(1)). \end{aligned}$$

We then discuss the case when $\theta = \frac{1}{2}$. By (4.8) and Table 3, we have that

$$j_{n,0} = (\tau_{n,+} - 4)\mu_n(1+o(1)), \quad j_{n,\pm} = \left(-\frac{1}{2}\tau_{n,+} \pm i\frac{\sqrt{3}}{2}\tau_{n,-} - 4\right)\mu_n(1+o(1)).$$

Finally, we discuss the case when $\theta \in (\frac{1}{2}, 1]$. From (4.8) and Table 3, it is obvious that

$$j_{n,0} = 4\mu_n^{2\theta}(1+o(1)), \quad j_{n,\pm} = 4\mu_n^{2\theta}(1+o(1)).$$

TABLE 3. Λ_n, Ω_n, and m_n

Region	Λ_n	Ω_n	m_n
$\theta \in [-1, \frac{1}{2})$	$8\mu_n(1+o(1))$	$\frac{16\sqrt{6}}{3}i\mu_n(1+o(1))$	$24\mu_n - 12\mu_n^{2\theta}$
$\theta = \frac{1}{2}$	$\tau_{n,+}\mu_n(1+o(1))$	$\tau_{n,-}\mu_n(1+o(1))$	$12\mu_n$
$\theta \in (\frac{1}{2}, 1]$	$\frac{22}{3}\mu_n(1+o(1))$	$8\sqrt{2}\mu_n^{\theta+\frac{1}{2}}(1+o(1))$	$-12\mu_n^{2\theta} + 24\mu_n$

We need to compute lower order terms of j_n's. To this end, consider $z_n = j_n - 4\mu_n^{2\theta}$. Then, z_n is the solution to the cubic equation

$$\begin{aligned} &(z_n + 4\mu_n^{2\theta})^3 + m_n(z_n + 4\mu_n^{2\theta})^2 + s_n(z_n + 4\mu_n^{2\theta}) + l_n = 0 \\ \Rightarrow\quad & z_n^3 + 24\mu_n z_n^2 + (16\mu_n^2 + 96\mu_n^{1+2\theta})z_n + 64\mu_n^{2+2\theta} = 0. \end{aligned}$$

Examining the expression of z_n's in a similar fashion to that of j_n's, we have that

$$z_{n,0} = -\frac{2}{3}\mu_n(1+o(1)), \quad z_{n,\pm} = \pm i4\sqrt{6}\mu_n^{\frac{1}{2}+\theta}(1+o(1)).$$

As a result,

$$j_{n,0} = 4\mu_n^{2\theta} - \frac{2}{3}\mu_n(1+o(1)), \quad j_{n,\pm} = 4\mu_n^{2\theta} \pm i4\sqrt{6}\mu_n^{\frac{1}{2}+\theta}(1+o(1)).$$

We have finished the discussion of all three cases. To sum up, The roots to the cubic resolvent are

$$\begin{aligned} j_{n,0} &= \begin{cases} 4\mu_n^{6\theta-2}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ (\tau_{n,+} - 4)\mu_n(1+o(1)), & \theta = \frac{1}{2}, \\ 4\mu_n^{2\theta} - \frac{2}{3}\mu_n(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases} \\ j_{n,\pm} &= \begin{cases} -4(2\sqrt{2} \pm 3)\mu_n(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \left(-\frac{1}{2}\tau_{n,+} \pm i\frac{\sqrt{3}}{2}\tau_{n,-} - 4\right)\mu_n(1+o(1)), & \theta = \frac{1}{2}, \\ 4\mu_n^{2\theta} \pm i4\sqrt{6}\mu_n^{\theta+\frac{1}{2}}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases} \end{aligned} \tag{4.20}$$

4.2.2. *Compute the Roots of the Characteristic Equation Given the Roots to the Cubic Resolvent.* Now that we know the roots of the cubic resolvent as given in (4.20), we are ready to proceed to compute the roots of the characteristic equation (3.4). To this end, first, we compute k_n's. Then, we compute the λ_n's via (4.9).

From (4.20), the definition of k_n's, and using the fact that $r_n = -8\mu_n^{3\theta}$, we have that:

$$\begin{aligned} k_{n,0} &= \begin{cases} 2\mu_n^{3\theta-1}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \chi_{n,0}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ 2\mu_n^{\theta} - \frac{1}{6}\mu_n^{1-\theta}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases} \\ k_{n,\pm} &= \begin{cases} i2(\sqrt{2} \pm 1)\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in [-1, \frac{1}{2}), \\ \chi_{n,\pm}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta = \frac{1}{2}, \\ 2\mu_n^{\theta} \pm i\sqrt{6}\mu_n^{\frac{1}{2}}(1+o(1)), & \theta \in (\frac{1}{2}, 1]. \end{cases} \end{aligned} \tag{4.21}$$

Here, $\chi_{n,0}$ is the square root of $\tau_{n,+} - 4$, and $\chi_{n,\pm}$ are the square root of $-\frac{1}{2}\tau_{n,+} \pm i\frac{\sqrt{3}}{2}\tau_{n,-} - 4$ such that $\chi_{n,0}\chi_{n,+}\chi_{n,-} = r_n$.

With (4.9) and (4.21), the roots to the characteristic equation are given as in Theorem 3.1. □

5. SOME EXAMPLES OF APPLICATION

In this section, we provide a few concrete examples of application of our regularity results. To this end, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth enough boundary Γ. These examples and the abstract system investigated in this contribution are inspired by Haraux' work [12] on the simultaneous stabilization of wave equations with different speeds of propagation. The work of Haraux followed from discussions with J.L. Lions about simultaneous control of uncoupled elastic systems, [13].

5.1. **Interacting plates I : hinged plates.** Consider the system of coupled plate equations given by

$$\begin{aligned} & y_{tt} + a\Delta^2 y - \gamma\Delta(y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ & z_{tt} + b\Delta^2 z - \gamma\Delta(y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ & y = 0, \quad \Delta y = 0, \quad z = 0, \quad \Delta z = 0 \text{ on } \Gamma \times (0,\infty), \end{aligned} \tag{5.1}$$

with initial conditions

$$y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x),$$

where a, b and γ are positive constants with $a \neq b$.
Let A be the operator defined on the Hilbert space $L^2(\Omega)$ by: $A = \Delta^2$ with $D(A) = \{u \in H^4(\Omega) \cap H_0^1(\Omega); u = 0 = \Delta u \text{ on } \Gamma\}$.
It is well-known that $A^{\frac{1}{2}} = -\Delta$ with $D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$. Thus, the damping is structural in (5.1). Theorem 2.2 shows that the semigroup corresponding to System (5.1) is analytic on the energy space $\mathcal{H} = ((H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)))^2$.

5.2. **Interacting plates II: clamped plates.** Consider now the system of coupled plate equations given by

$$\begin{aligned} & y_{tt} + a\Delta^2 y - \gamma\Delta(y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ & z_{tt} + b\Delta^2 z - \gamma\Delta(y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ & y = 0, \quad \frac{\partial y}{\partial \nu} = 0, \quad z = 0, \quad \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \times (0,\infty), \end{aligned} \tag{5.2}$$

with initial conditions

$$y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x),$$

where a, b and γ are positive constants with $a \neq b$.
Let the operator A be defined on the Hilbert space $L^2(\Omega)$ now by: $A = \Delta^2$ with $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$.
As the work in [20] shows, we no longer have $A^{\frac{1}{2}} = -\Delta$, with $D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega)$. But the damping here is of structural type, and the semigroup associated with (5.2) can be shown to be analytic on the energy space $\mathcal{H} = (H_0^2(\Omega) \times L^2(\Omega)))^2$; the proof follows along the lines of the proof of Theorem 2.2.

5.3. **Interacting membranes.** We consider the following system

$$\begin{aligned} &y_{tt} - a\Delta y + \gamma(-\Delta)^\theta (y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ &z_{tt} - b\Delta z + \gamma(-\Delta)^\theta (y_t + z_t) = 0 \text{ in } \Omega \times (0,\infty), \\ &y = 0, \quad z = 0 \text{ on } \Gamma \times (0,\infty), \end{aligned} \tag{5.3}$$

with initial conditions

$$y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x),$$

where a, b and γ are positive constants with $a \neq b$.
Let A be the operator defined on the Hilbert space $L^2(\Omega)$ by: $A = -\Delta$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Theorem 2.2 shows that the semigroup corresponding to System (5.3) is analytic for $\theta = \frac{1}{2}$, and of Gevrey class δ for every $\delta > 2$, for $\theta = 1/4$ or $\theta = 3/4$, on the energy space $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$.

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